Unit 10: Convolutional Codes

EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN

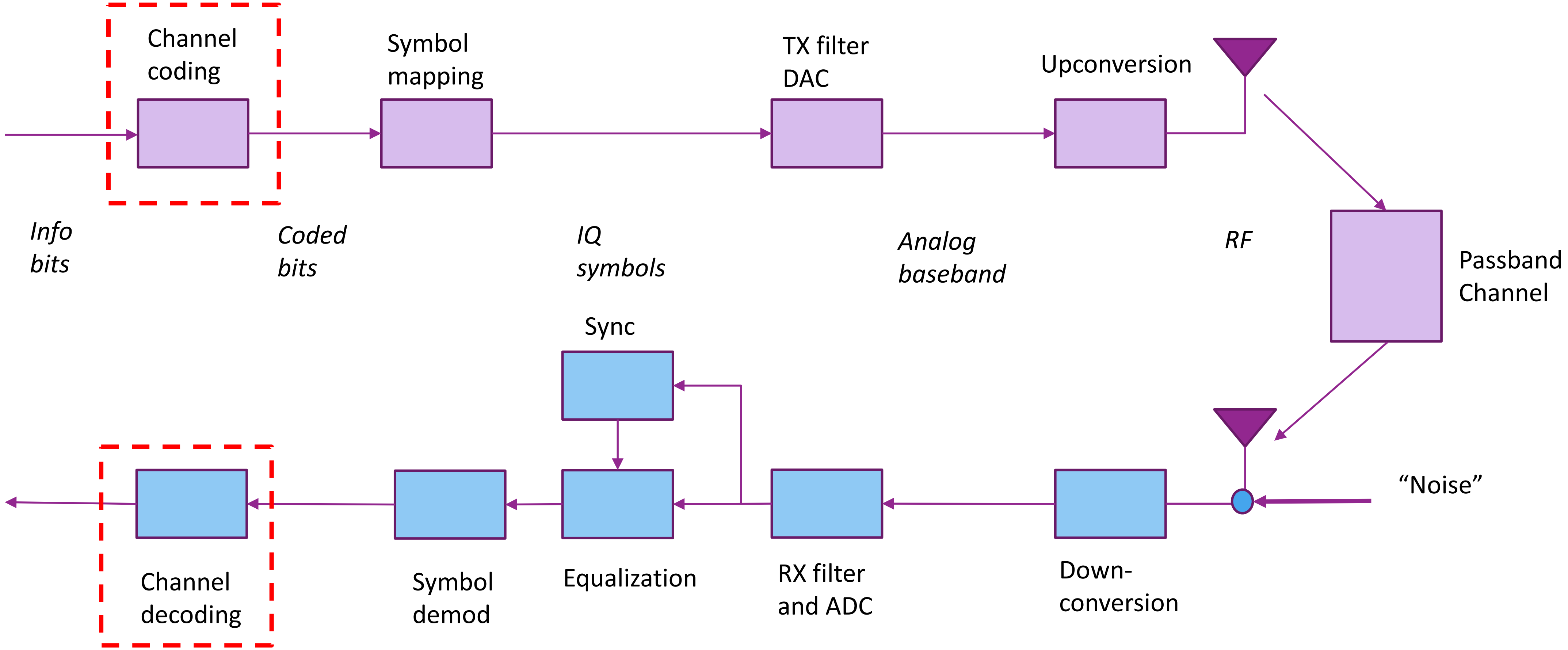
Learning Objectives

- Encode data using a convolutional code for given a generator polynomial
- Compute the rate of the code including tail bits
- Represent the code via a trellis diagram and finite state machine
- Compute the branch metrics of a code for a memoryless channel
- Decode the code via the Viterbi algorithm

Graduate vs. Undergraduate

- Unit 9 on basic codes is skipped for the graduate students
- However, a few of the Unit 9 slides are included here for completeness
- Grad students who would like a review are encouraged to look at Unit 9

This Unit



Channel
coding
Symbol
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TX filter
DAC
Upconversion
Info
bits
Coded
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Analog
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RF
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Sync
"Noise"
Channel
decoding
Symbol
demod
Equalization
RX filter
and ADC
Down-
conversion

Outline

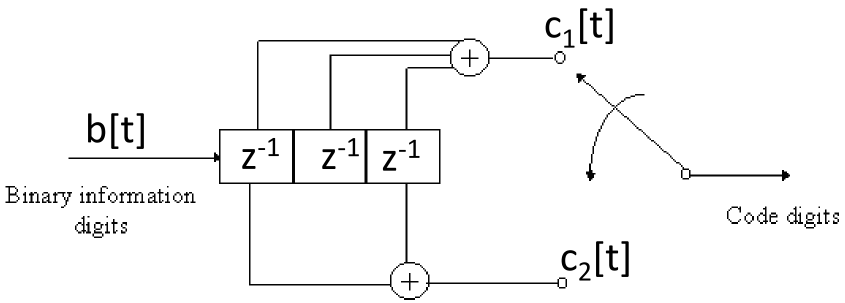
- Convolutional codes: encoding and representations
- Tree, trellis and state diagrams
- Decoding with branch metrics
- Viterbi decoding

Convolutional Codes History

- Generates a stream of coded bits from uncoded bits
  - Block codes form by terminating the stream
- Output stream created by binary FIR filters
- Developed originally by Elias (1955)
  - Key challenge was decoders. Much study in 1960s.
  - Practical, optimal decoders developed by Viterbi,1967.
- Can perform within 2-3 dB of Shannon limit.
- Instrumental in Pioneer Space program (along with RS codes)
- Most widely-used code in industry today: WiFi, cellular
  - In the mid 1990s, longer block length cellular codes were replaced with Turbo codes,
  - But, these use convolutional codes as basis

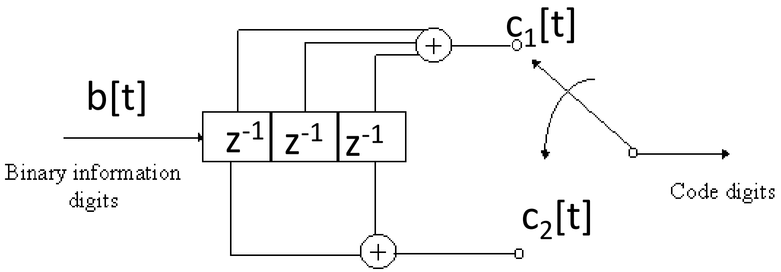
Convolutional Codes

- Encode data through parallel binary (usu. FIR) filters
- Example convolutional code:
  - Rate = ½ (two output bits ($c_1[t], c_2[t]$) for each input bit b[t].
  - Constraint length K=3 (size of shift register)



$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

Convolutional Codes
Block Implementation



$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

- To implement as block code:
  - Start with L input bits b[0],b[1],…,b[L-1]
  - Append K-1 zero b[L]=b[L+1]=…=b[L+K-2]=0 (called tail bits)
  - Generate output bits from each branch
    - $c_j[0], c_j[1], \ldots, c_j[L+K-2]$, j=1,…,N where N = number of branches
  - Final codeword is concatenation of branch output
    - **c**=($c_1[0], c_1[1], \ldots, c_1[L+K-2], \ldots, c_N[0], c_N[1], \ldots, c_N[L+K-2]$)
- Effective rate: $R = \frac{L}{N(L+K-1)} \approx \frac{1}{N}$ for large L

Convolutional Codes

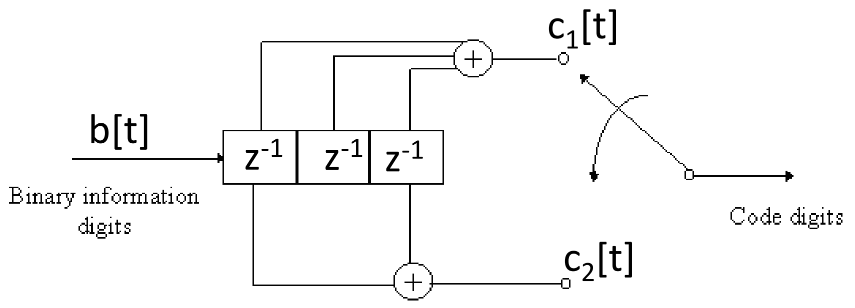
Encoding Example

- Encode message **b** = [1 0 1]
- Branch outputs **c**1=[11011] c2=[11001]
- Final output **c**=[11,10,00,10,11] (interleaved)
- Rate = 3/10

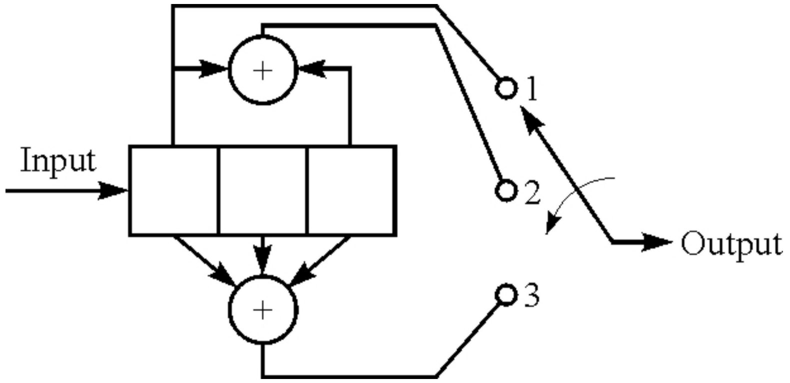
t	b[t]	c1[t]	c2[t]
0	1	1	1
1	0	1	0
2	1	0	0
3	0	1	0
4	0	1	1

Tail bits

$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$



Generator Polynomials: Binary Form



Rate 1/3, K=3 example:

$$c_1[t] = b[t]$$
$$c_2[t] = b[t] + b[t-1]$$
$$c_2[t] = b[t] + b[t-1] + b[t-2]$$

- Code polynomials: Binary vector of filter coefficients

$$g_1 = [100]$$
$$g_2 = [101]$$
$$g_3 = [111]$$

Generator Polynomials: Octal Form

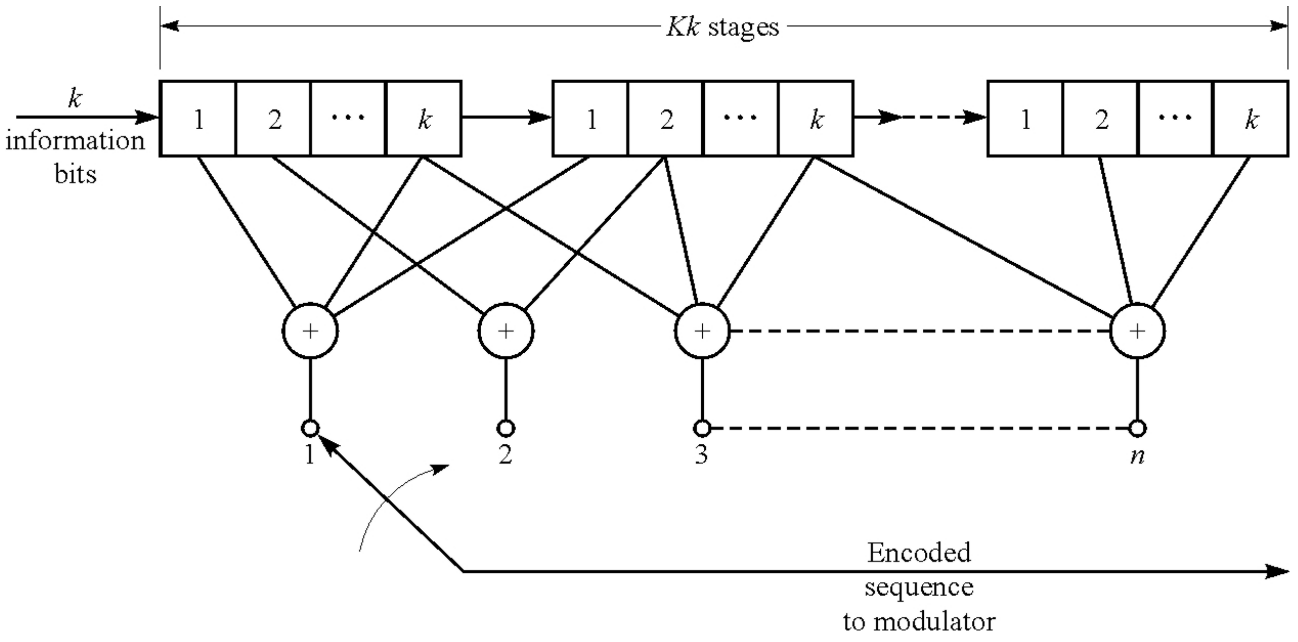
- For large constraint lengths (large K), binary form is inefficient
  - Engineers often use octal form
  - Base 8, each digit 0…7
  - Each digit represents three bits
- Example:

$$g_1 = [1\ 011] \quad\rightarrow\quad g_1 = [13]$$
$$g_2 = [1\ 101] \quad\rightarrow\quad g_2 = [15]$$
$$g_3 = [1\ 010] \quad\rightarrow\quad g_3 = [12]$$

Binary → Octal

Multiple Inputs

- Examples up to now are R=1/n
- Can extend to rate R=k/n
  - Add k bits at a time



In-Class Exercise

Convolution Code In-Class Exercises

Building an Encoder

We will look at a simple R=1/2 convolution code:

```
c(t,1) = b(t) + b(t-1) + b(t-2)
c(t,2) = b(t) + b(t-2)
```

This encoder can be represented with a generator matrix `G` as shown.

```
G = [1 1 1; 1 0 1];
nin = 1;                % number of input streams
nout = size(G,1);       % number of output streams
conLen = size(G,2);     %cconstraint length
```

We will build a simple encoder using the convolution method. The convolution method in MATLAB uses real values, so we will use a modulo to convert back to .

Outline

- Convolutional codes: encoding and representations
- Tree, trellis and state diagrams
- Decoding with branch metrics
- Viterbi decoding

Convolutional Codes as State Machines

- Convolutional codes have memory
  - Stored in contents of shift registers
  - Each input bit changes memory contents
  - Output bits are a function of memory and input

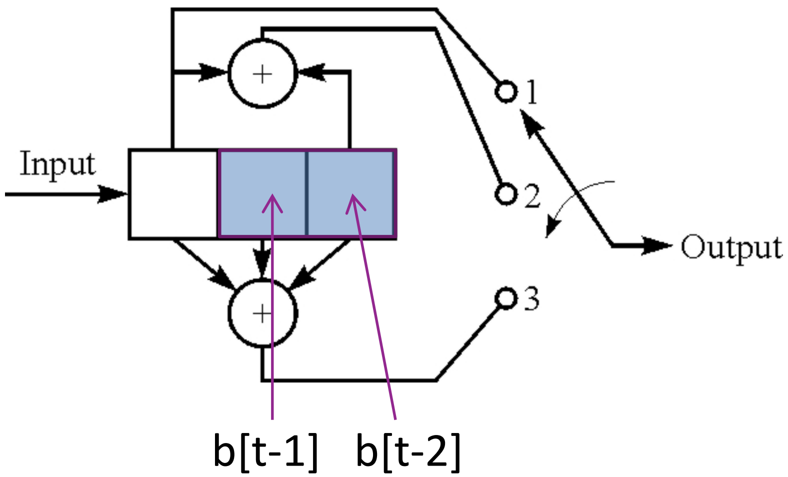
- Three common ways to represent evolution of the memory:
  - Tree diagram
  - Trellis diagram
  - State diagram

Encoder States

- State of the encoder determined by contents of shift register:

$$x[t] = (b[t-1], \dots, b[t-K])$$

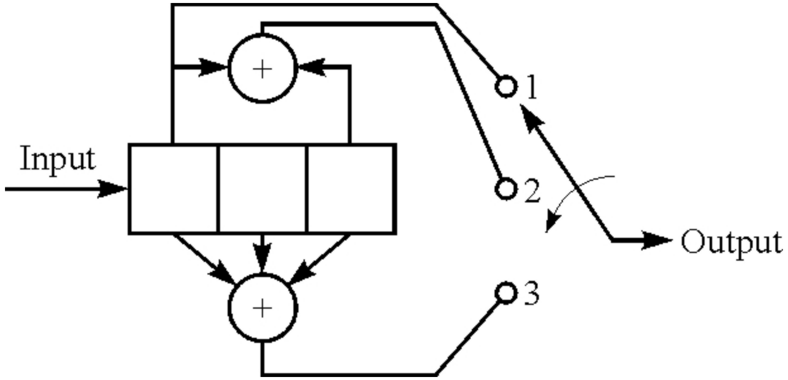
- Only need to look at most recent K-1 bits not including the current bit
  - Last bit will be shifted out
  - There are 2^{K-1} states

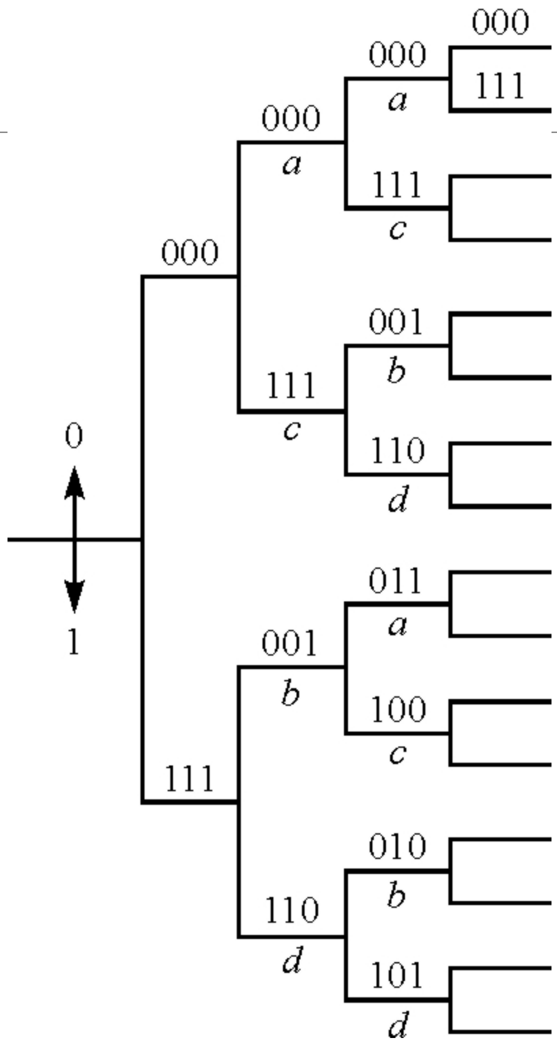


State label	$b[t-1]$	$b[t-2]$
a	0	0
b	0	1
c	1	0
d	1	1

Tree Diagram

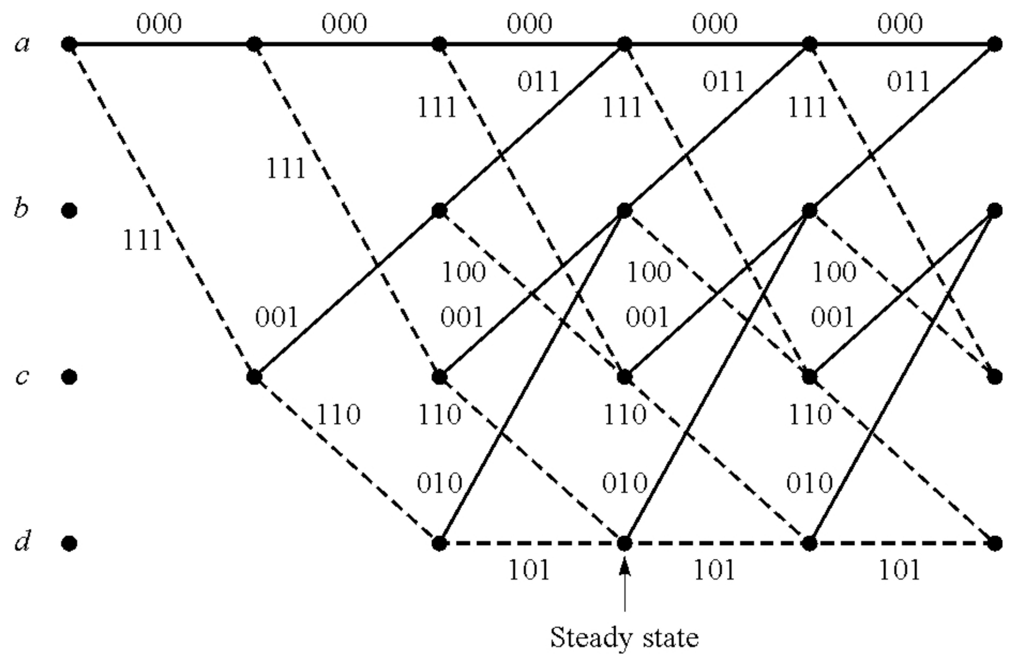
- ❑Two branches for each input bit 0 or 1.
- ❑Branches labeled by output bits (n bits)
- ❑Difficult to use since branches infinitely grow





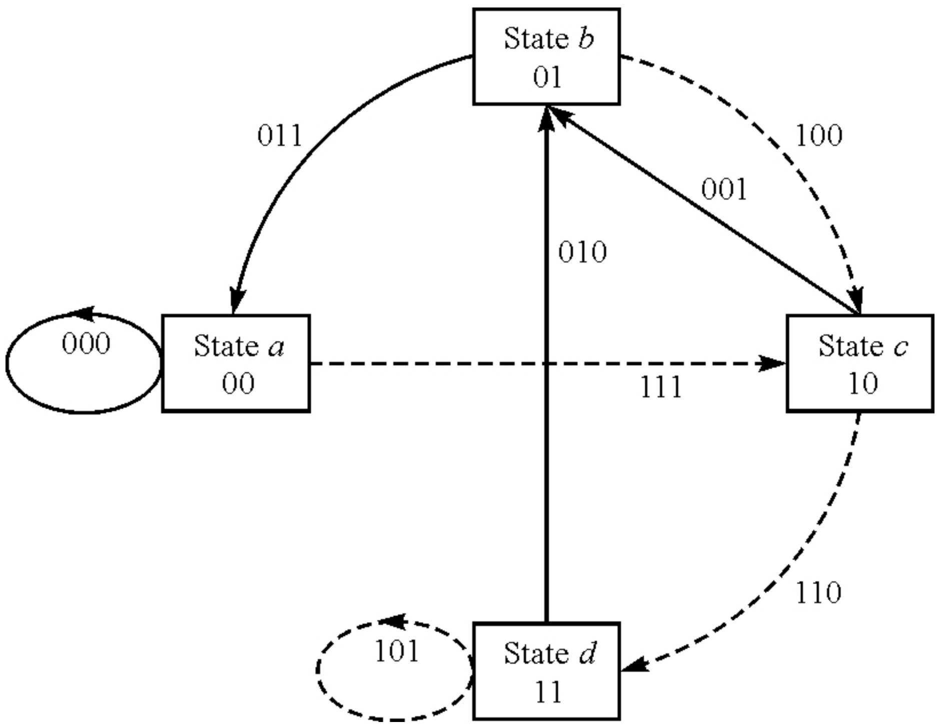
Trellis Diagram

- Show trajectory through the states
- Solid lines: paths with b[t]=0, Dash lines: b[t]=1



Labels are the outputs along each path

State Diagram



- One node per state
- Solid lines:
  - Transitions with b[t]=0
- Dashed lines:
  - Transitions with b[t]=1
- Labels indicate outputs on each state

State Machine Functional Description

- Finite state machine:
  - $x[t]$: Sequence of states, $x[t] \in \{0, \ldots, 2^{K-1} - 1\}$
  - $b[t]$: Input sequence
  - $c[t]$: Output sequence
- Iterative generating sequence:

$$x[t+1] = f(x[t], b[t])$$
$$c[t] = h(x[t], b[t])$$

State function

Output function

- Initial condition: $x[0]$

In-Class Exercise

Representing the State Transition Function

We wish to represent the convolutional encoder as a finite state machine. To this end, we will represent the states by a single decimal number:

```
x(t) = 2*b(t-2) + b(t-1) + 1
```

This state runs from `x(t) = 1,...,nstate` where `nstate = 2^{convLen-1)`

To represent the state transition, we create a matrix `xnextMat`

- `xnextMat(i,1)`= the value of `x(t+1)` when `b(t)=0` and `x(t)=i`
- `xnextMat(i,2)` = the value of `x(t+1)` when `b(t)=1` and `x(t)=i`

```
ninVal = 2^nin;         % number of input values per time step
nstate = 2^(conLen-1);  % number of states

% Intiialize matrix
xnextMat = zeros(nstate,ninVal);

% TODO:  Write a for loop to fill in the xnextMat matrix
```

We next create a matrix to represent the output:

- `coutMat(k,i,j) =` output `c(t,k)` `when x(t)=i and b(t)=j+1`

Outline

- ❑Convolutional codes: encoding and representations
- ❑Tree, trellis and state diagrams
- ❑Decoding with branch metrics
- ❑Viterbi decoding

ML Estimation

- Assume the following dimensions:
  - N outputs per time steps: $c[t] = (c_1[t], \ldots, c_N[t])$
  - T time steps (including tail bits!)
- Channel model:
  - For each bit $c_i[t]$, we make some observation $r_i[t]$
  - Output is probabilistic $P(r_i[t]|c_i[t])$
  - Assume all outputs are independent:

$$P(\boldsymbol{r}|\boldsymbol{c}) = \prod_{t=0}^{T-1} \prod_{i=1}^{N} P(r_i[t]|c_i[t])$$

- Find ML estimate:

$$\hat{c} = \arg\max_{c} P(r|c)$$

  - Maximum over all sequences

LLR Review

- Before studying convolutional codes, consider simple code:
  - Codeword has N output bits: $\boldsymbol{c} = (c_1, \dots, c_N)$
  - Receive N symbols $r = (r_1, \dots, r_N)$. Each r_i may be binary, discrete, real-valued or complex-valued
  - Assume a memoryless channel:

$$P(r|c) = \prod_{i=1}^{N} P(r_i|c_i)$$

- Define LLR: $L_i = \log \frac{P(r_i|c_i = 1)}{P(r_i|c_i=0)}$
- Theorem: The ML codeword maximizes the LLR sum value function (se Unit 9)

$$\hat{c} = \arg\max_{c} J(c), \qquad J(c) = \sum_{i=0}^{N-1} c_i L_i$$

Example 1: Binary Symmetric Channel

- Binary symmetric channel: Output $r_i = 0$ or 1
- Error probability $p < 1/2$

$$\mathrm{P}(r_i|c_i) = \begin{cases} 1-p & \text{if } c_i = r_i \text{ (no error)} \\ p & \text{if } c_i \neq r_i \text{ (error)} \end{cases}$$

- Branch metric:

$$L_i = \begin{cases} A & \text{if } c_i = r_i \text{ (no error)} \\ -A & \text{if } c_i \neq r_i \text{ (error)} \end{cases}, \qquad A = \log \frac{1-p}{p} > 0$$

- ML estimate value function:

$$J(c) = \sum_{i=0}^{N-1} c_i L_i = A \sum_{i=0}^{N-1} c_i 1_{\{c_i = r_i\}} = A(\#\text{ correct bits})$$

  - Maximizes number of correct bits

Example 1: Binary Symmetric Channel

- Binary symmetric channel: Output $r_i = 0$ or 1
- Error probability $p < 1/2$

$$\mathrm{P}(r_i|c_i) = \begin{cases} 1-p & \text{if } c_i = r_i \text{ (no error)} \\ p & \text{if } c_i \neq r_i \text{ (error)} \end{cases}$$

- Branch metric: $L_i = \begin{cases} A & \text{if } c_i = r_i \text{ (no error)} \\ -A & \text{if } c_i \neq r_i \text{ (error)} \end{cases}$, $A = \log \frac{1-p}{p} > 0$
- Observe $c_i L_i = A(1_{\{r_i = c_i\}} + 1 - r_i)$
- Hence maximizing value function:

$$\arg\max_c J(c) = \arg\max_c \sum [1_{\{r_i=c_i\}} + 1 - r_i] = \arg\max_c \sum [1_{\{r_i=c_i\}}]$$

$$= \arg\max_c \#(correct\ bits)$$

Example 2: QPSK Channel

- Binary modulation (on a real or imaginary dimension):

$$r_i = s_i + w_i, \qquad w_i \sim N(0, N_0/2), \qquad s_i = \begin{cases} A & c_i = 1 \\ -A & c_i = 0 \end{cases}$$

- Gaussian distribution: $\log p(r_i|s_i) = -\frac{1}{N_0}(r_i - s_i)^2 - \frac{1}{2}\log\frac{N_0}{2}$
- LLR is: $L_i = \log p(r_i|s_i = A) - \log p(r_i|s_i = -A) = -\frac{1}{N_0}[(r_i - A)^2 - (r_i + A)^2] = \frac{4A}{N_0} r_i$
- LLR sum value function:

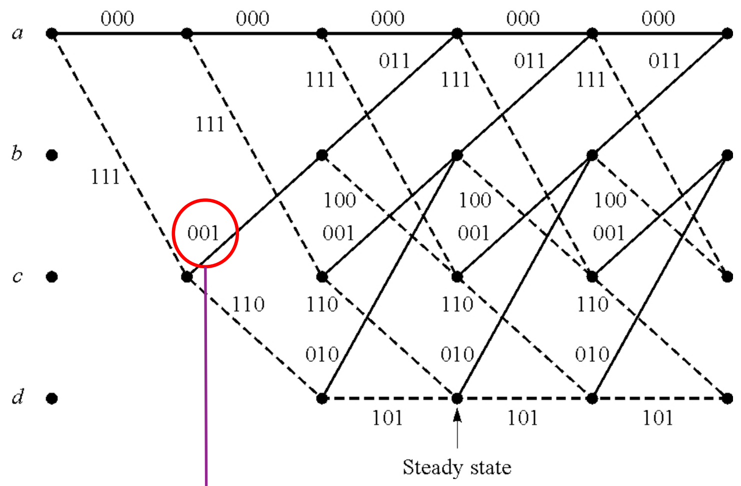
$$J(c) = \sum_i L_i c_i = \frac{4\mathrm{A}}{N_0} \sum_i r_i c_i$$

Branch Metric

- Log-likelihood ratio: $L_i[t] = \log \frac{P\left(r_i[t] \middle| c_i[t] = 1\right)}{P\left(r_i[t] \middle| c_i[t] = 0\right)}$
  - Depends on received symbol $r_i[t]$
- Each branch has an output set of bits $c[t] = (c_1[t], \ldots, c_N[t])$
- Define the branch metric:

$$\mu_t(x_{t+1}, x_t) = \sum_{i=1}^{N} c_i[t]\, L_i[t]$$

  - $c_i[t]$ is the coded outputs on the branch
- Describes likelihood that sequence passed through the branch
  - Branches with higher branch metrics are more likely

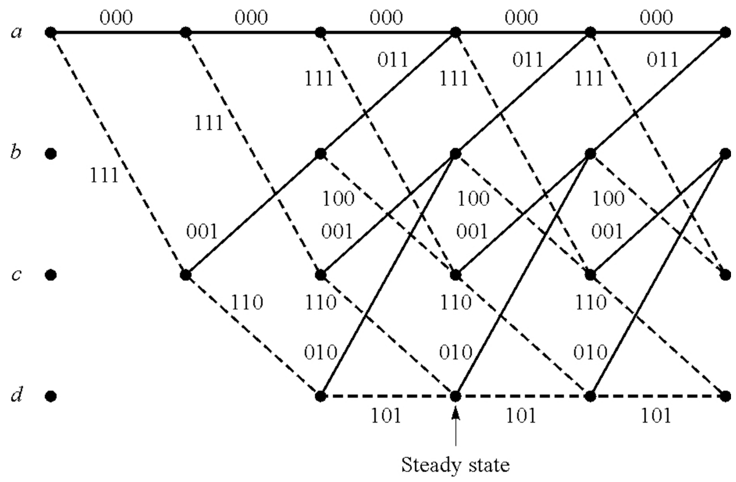


$c[t] = (0,0,1) \Rightarrow$
$\mu_t = 0L_1[t] + 0L_2[t] + L_3[t] = L_3[t]$

Branch Metric Maximization

- Each branch in trellis has a branch metric $\mu_t(x_{t+1}, x_t) = \sum_{i=1}^{N} c_i[t]\, L_i[t]$
- We can write LLR sum value function as:

$$J(c) = \sum_{t=0}^{T-1} \sum_{i=1}^{N} c_i[t]\, L_i[t] = \sum_{t=0}^{T-1} \mu_t(x_{t+1}, x_t)$$



- Interpretation:
  - Each code sequence is a path in the trellis
  - Value function is the sum of branch metrics on the path

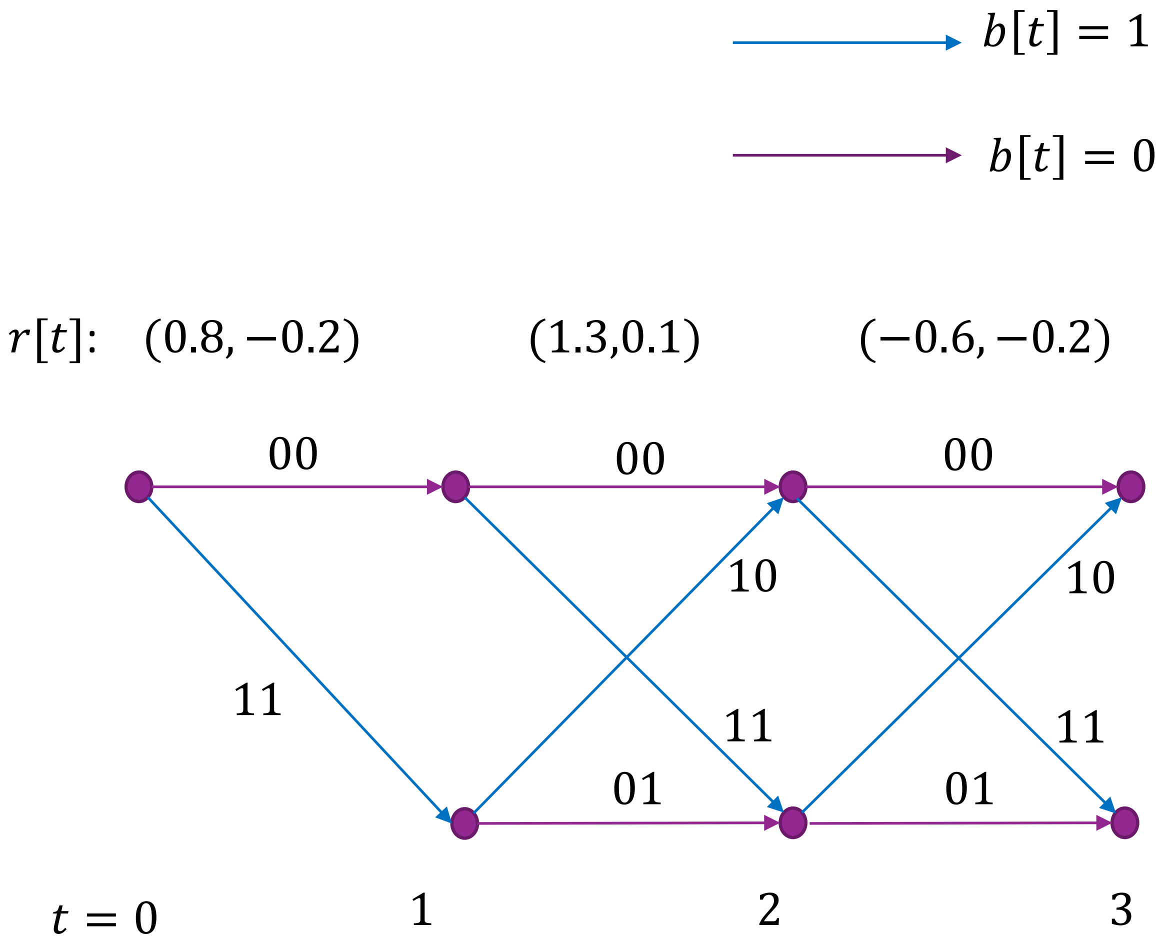
> *Conclusion: ML estimate is the codeword with the highest total path*

Example

- Consider toy Trellis (not a real conv code)
- Input: $b[t], t = 0,1,2$
- Output: $c[t] = (c_1[t], c_2[t])$
  - Shown on branches
- RX symbols: $r[t] = (r_1[t], r_2[t])$
  - Values shown above Trellis
- Assume QPSK channel value function:

$$J(c) = \sum_t \sum_i r_i c_i$$

  - Constant is ignored



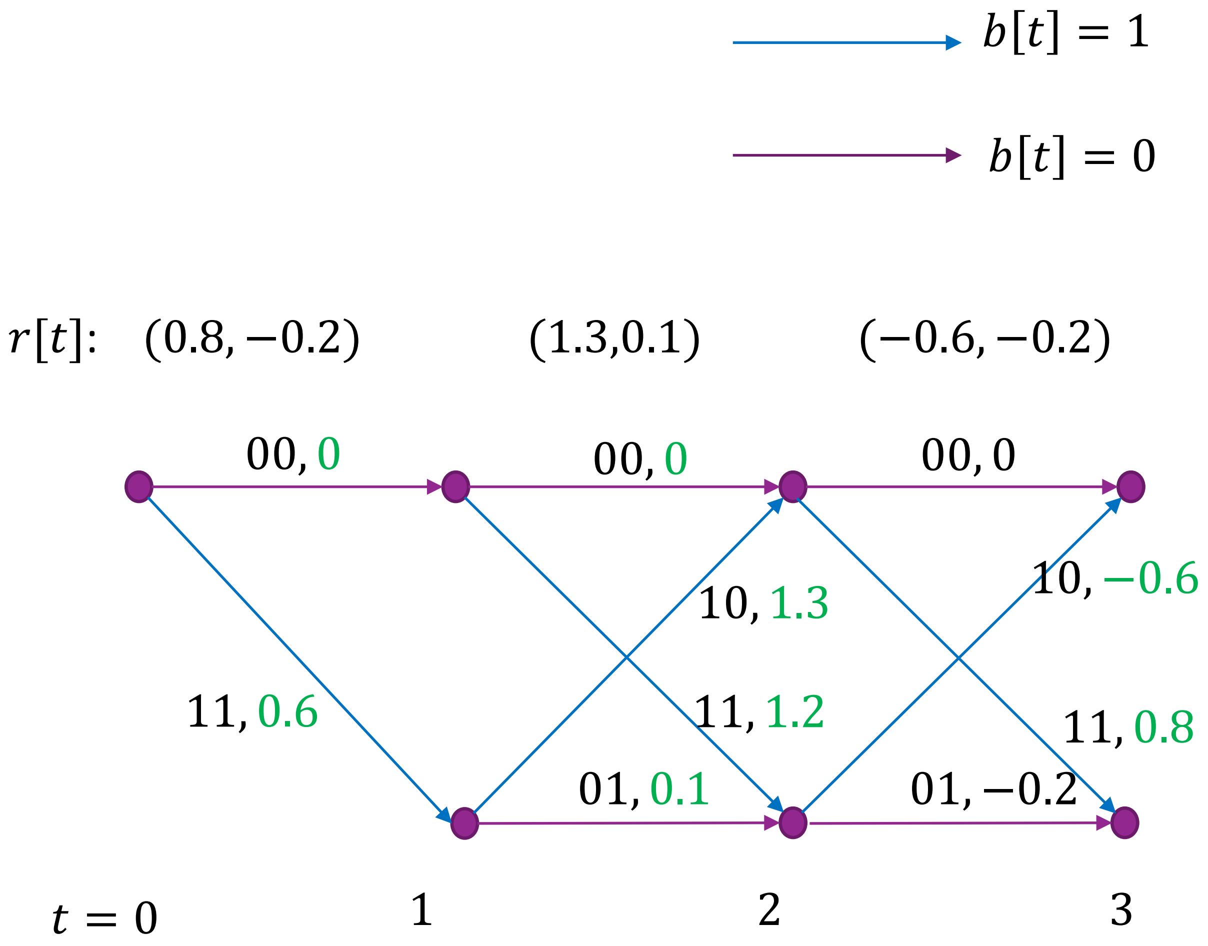
Step 1: Compute Branch Metrics

- For QPSK channel, branch metric is

$$\mu_t(x_{t+1}, x_t) = \sum_i r_i[t] c_i[t]$$

- On each branch we have labeled:

  (Output Bits, branch)= $(c[t], \mu_t)$

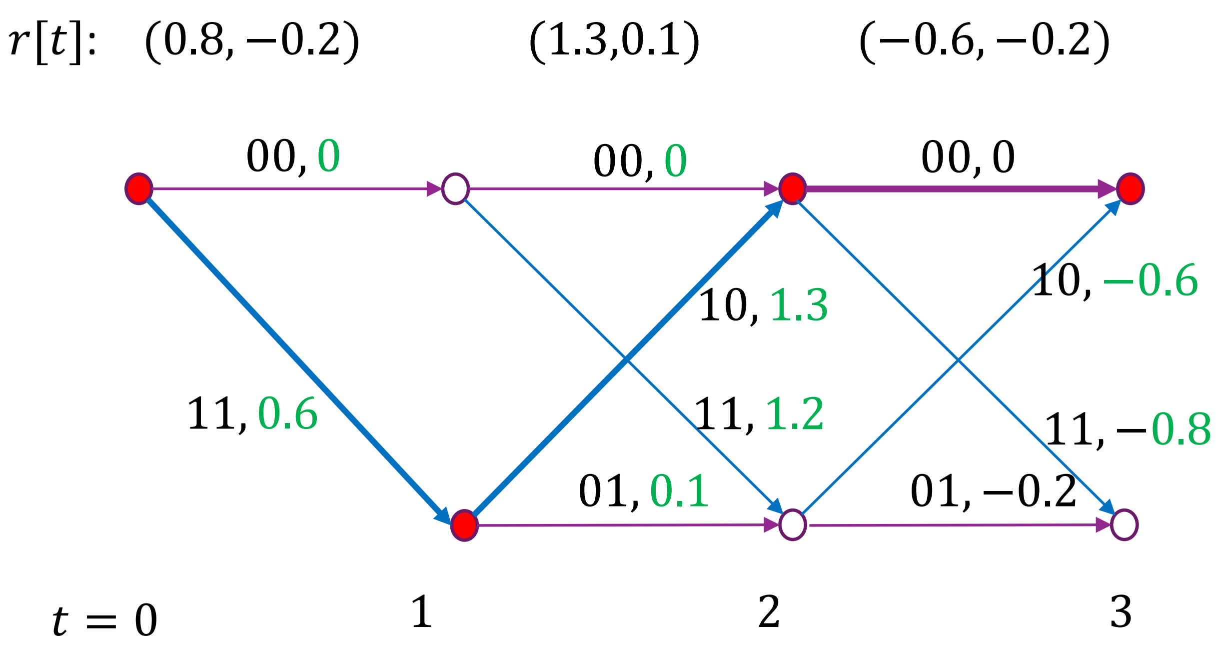


Step 2: Find the Maximizing Path

- For this simple Trellis, we can do this manually
- Path is shown with red nodes
- Solution:
  - Coded sequence: (11), (10), (00)
  - Bits: (1,1,0)

$b[t] = 1$

$b[t] = 0$



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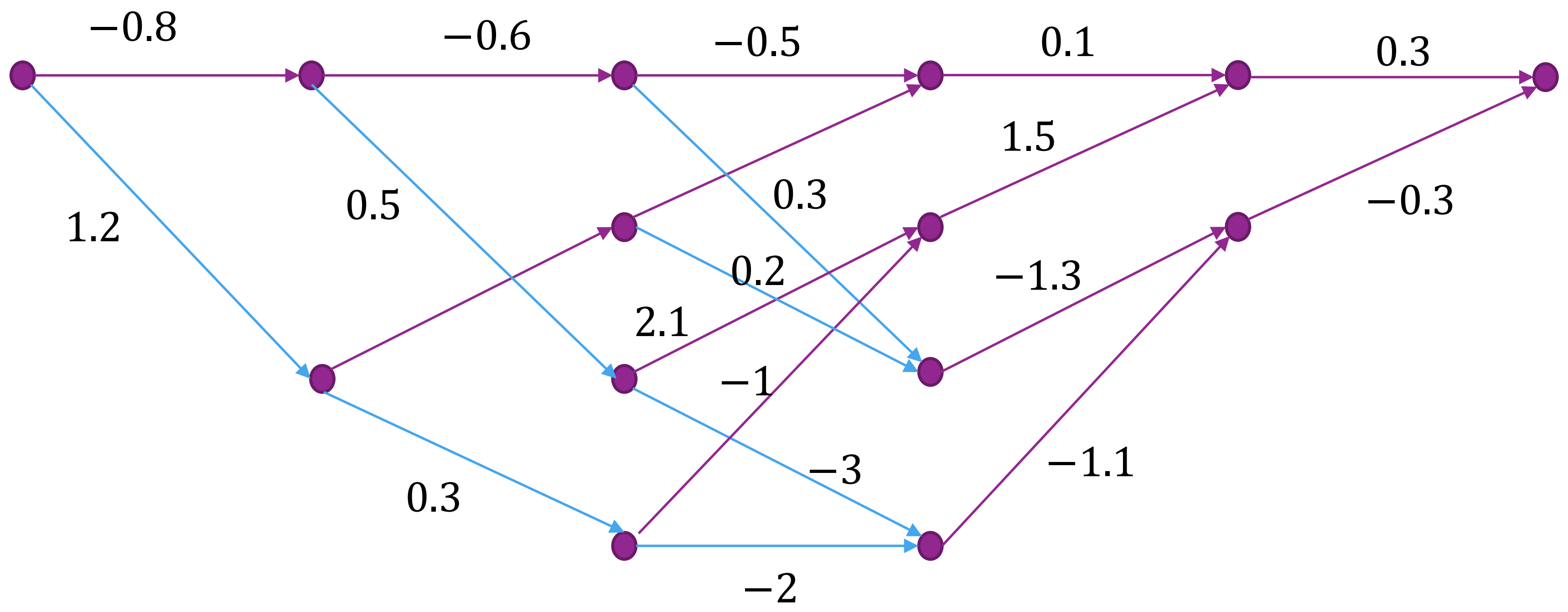
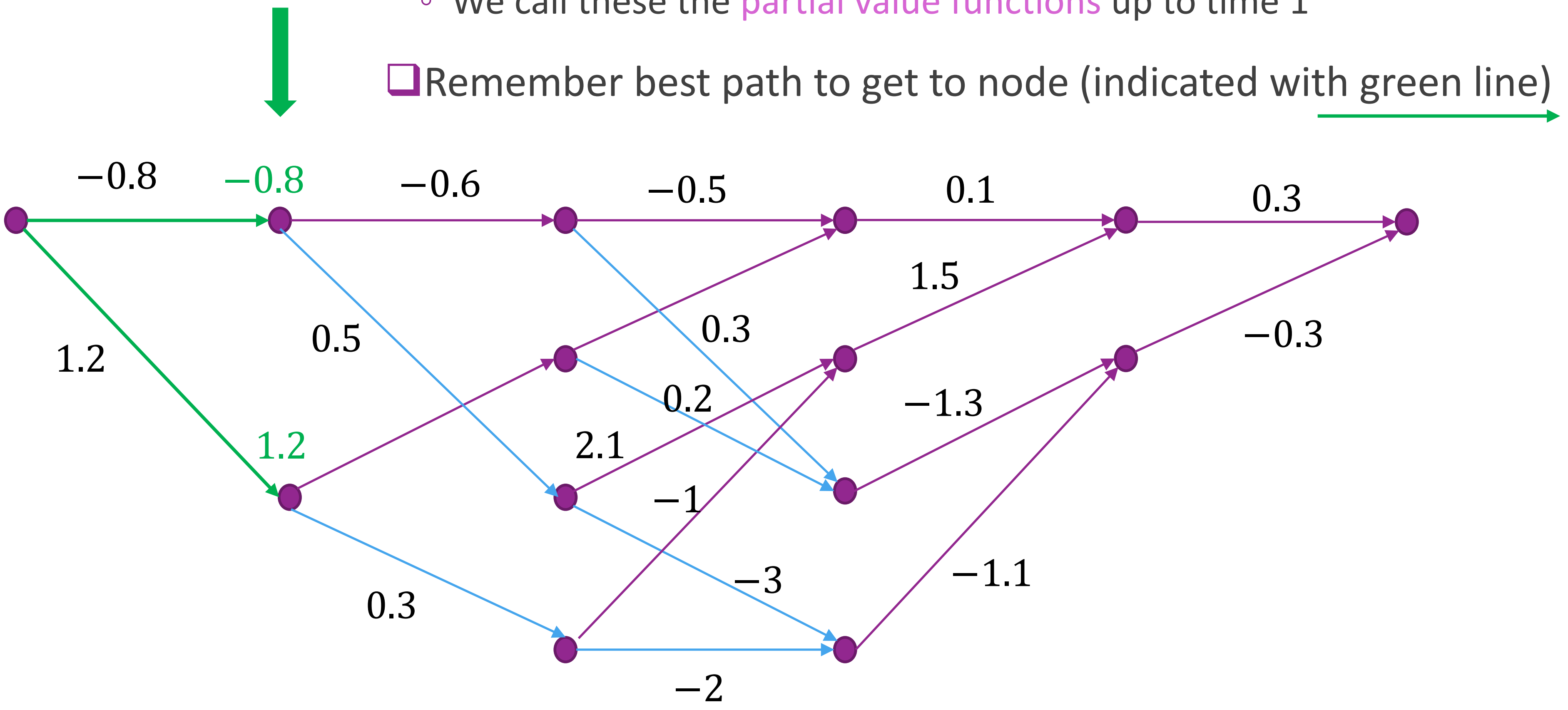
Viterbi Algorithm by Example

- Iterative solution for finding maximum path in a graph
- Key idea: Find the max path one time step at a time
- We will illustrate by example: Shown are the branch metrics



Ex: Time 1

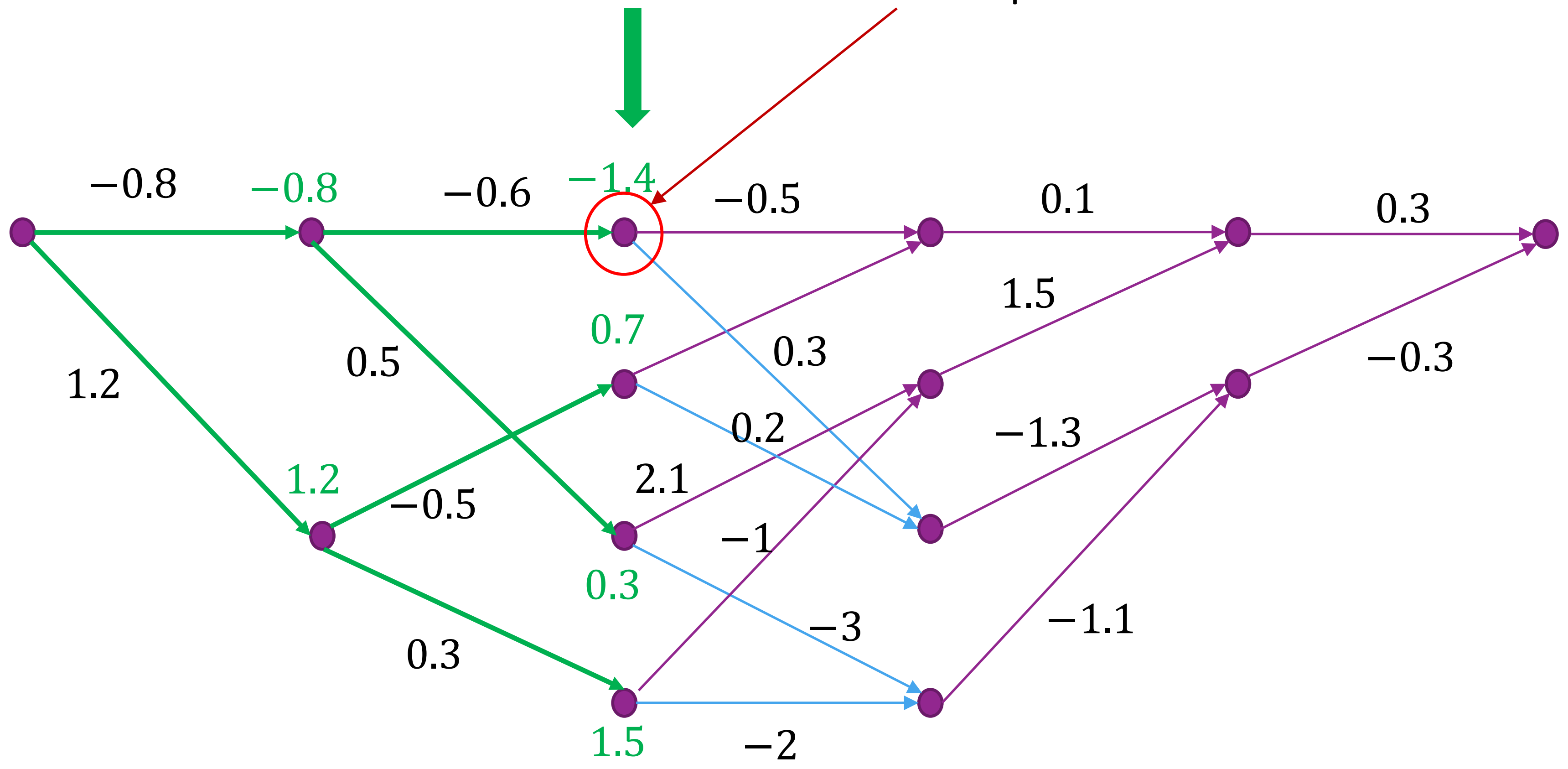
- Find total value to get to each node at time 1
  - We call these the partial value functions up to time 1
- Remember best path to get to node (indicated with green line)



Ex: Time 2

- Add value from Time 1 + branch

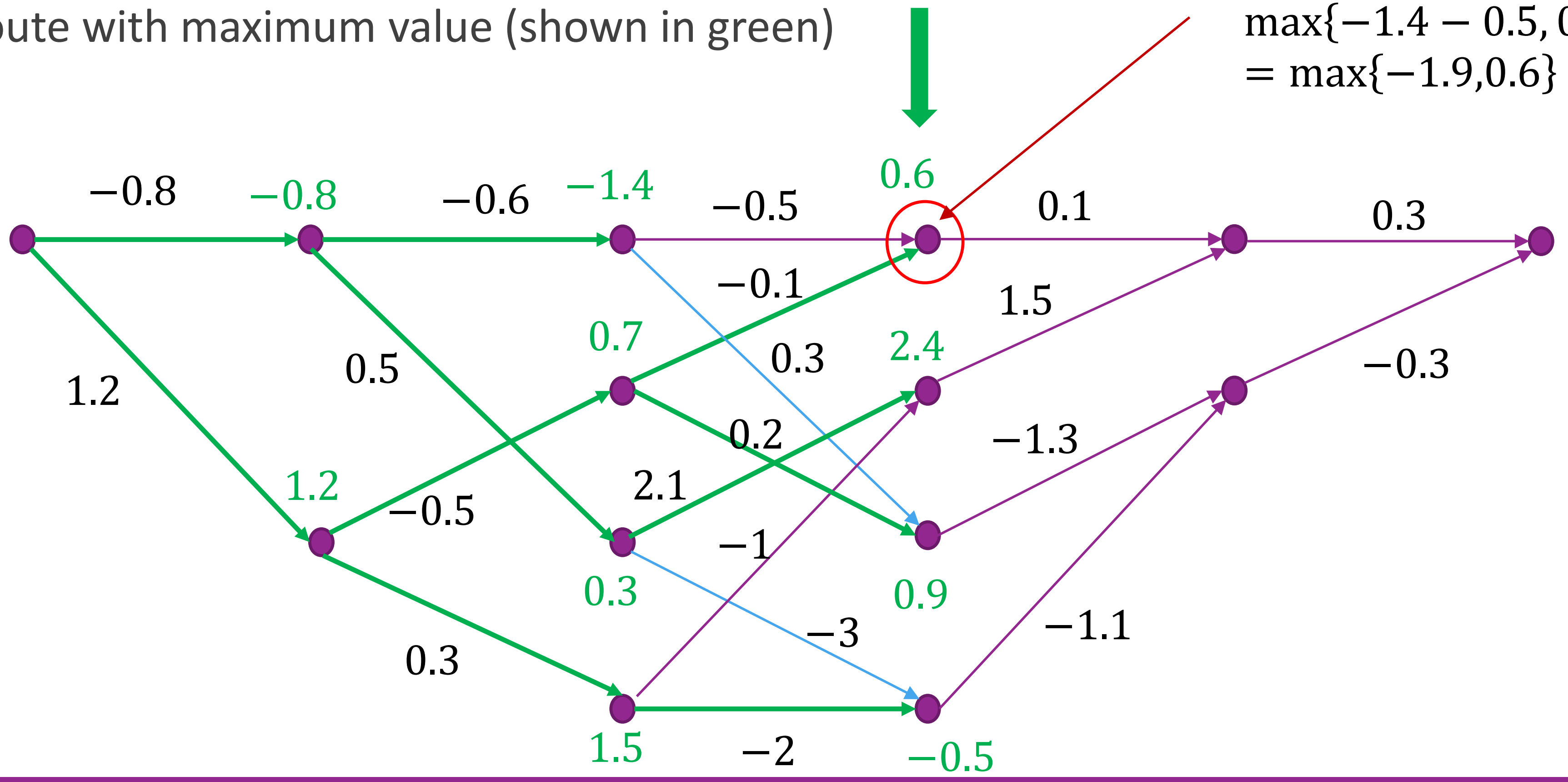
Example: $-0.8-0.6=-1.4$



Ex: Time 3

- Add value from Time 2 + branch
- Take route with maximum value (shown in green)

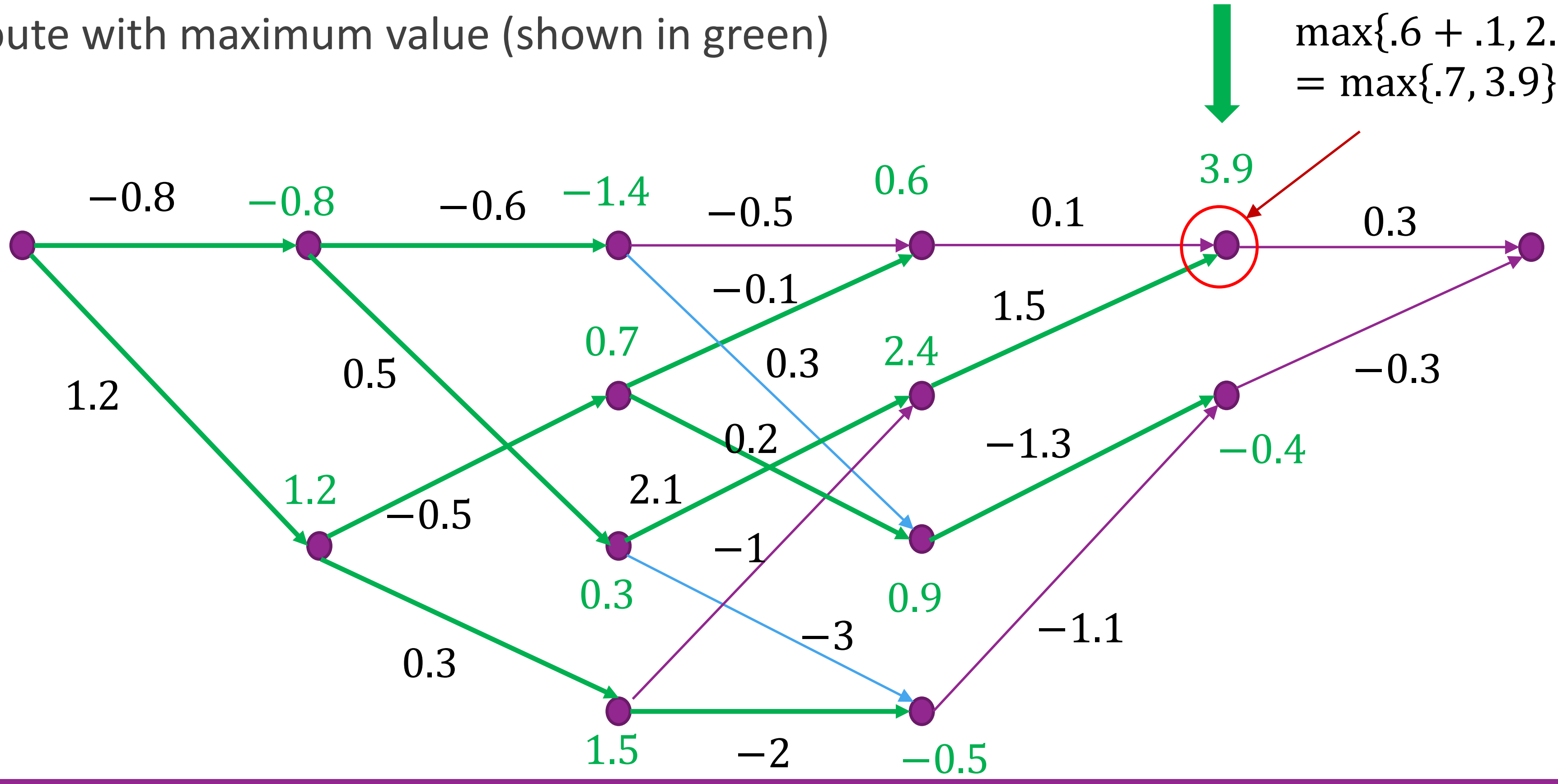
Example:
$\max\{-1.4 - 0.5, 0.7 - 0.1\}$
$= \max\{-1.9, 0.6\} = 0.6$



Ex: Time 4

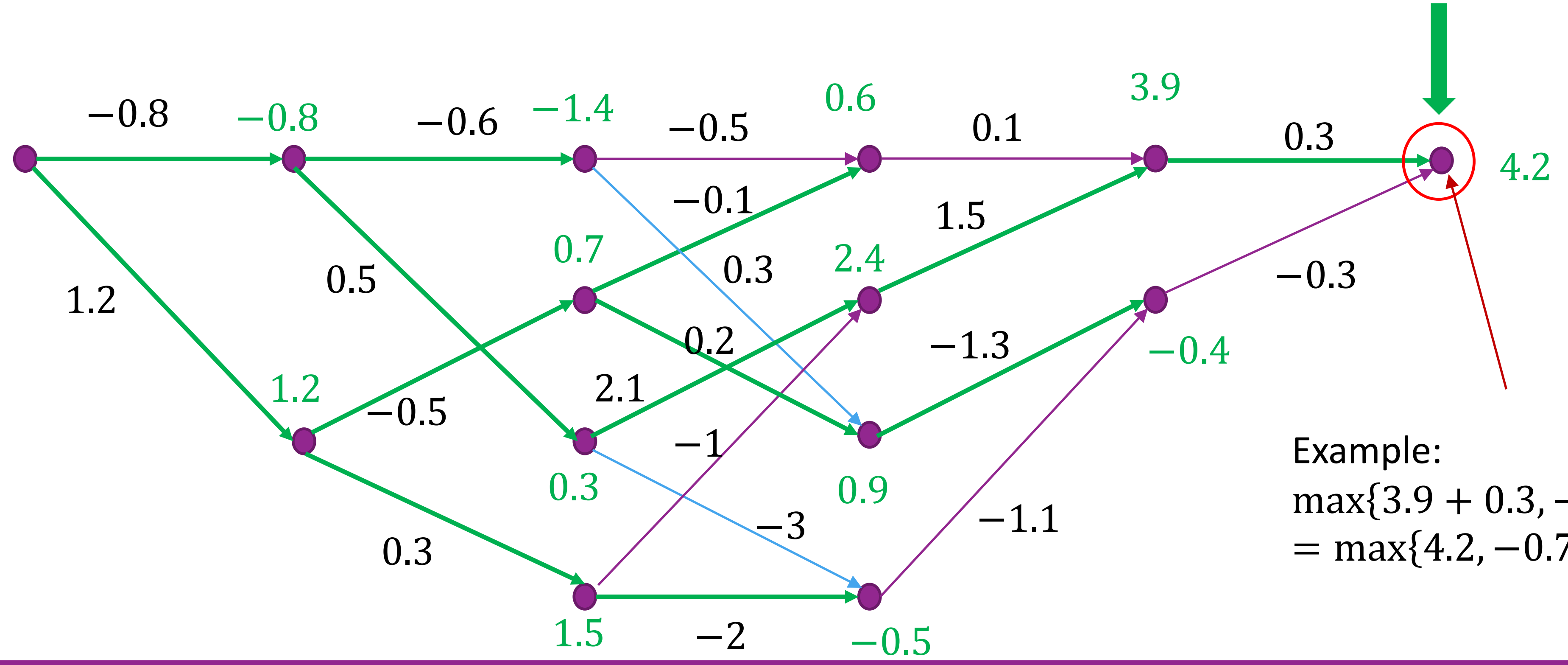
- ❑Add value from Time 3 + branch
- ❑Take route with maximum value (shown in green)

Example:
$\max\{.6 + .1, 2.4 + 1.5\}$
$= \max\{.7, 3.9\} = 3.9$



Ex: Time 5

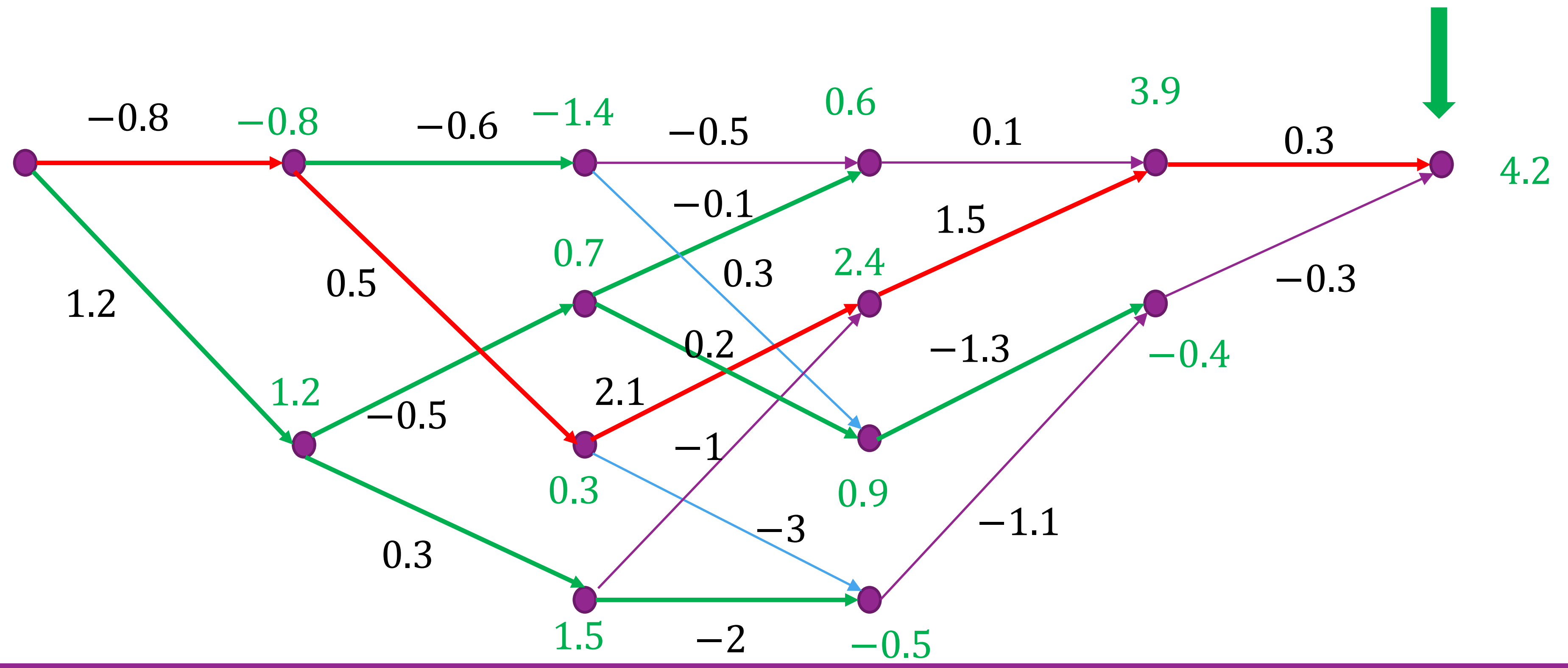
- Add value from Time 4 + branch
- Take route with maximum value (shown in green)



Example:
$\max\{3.9 + 0.3, -0.4 - 0.3\}$
$= \max\{4.2, -0.7\} = 4.2$

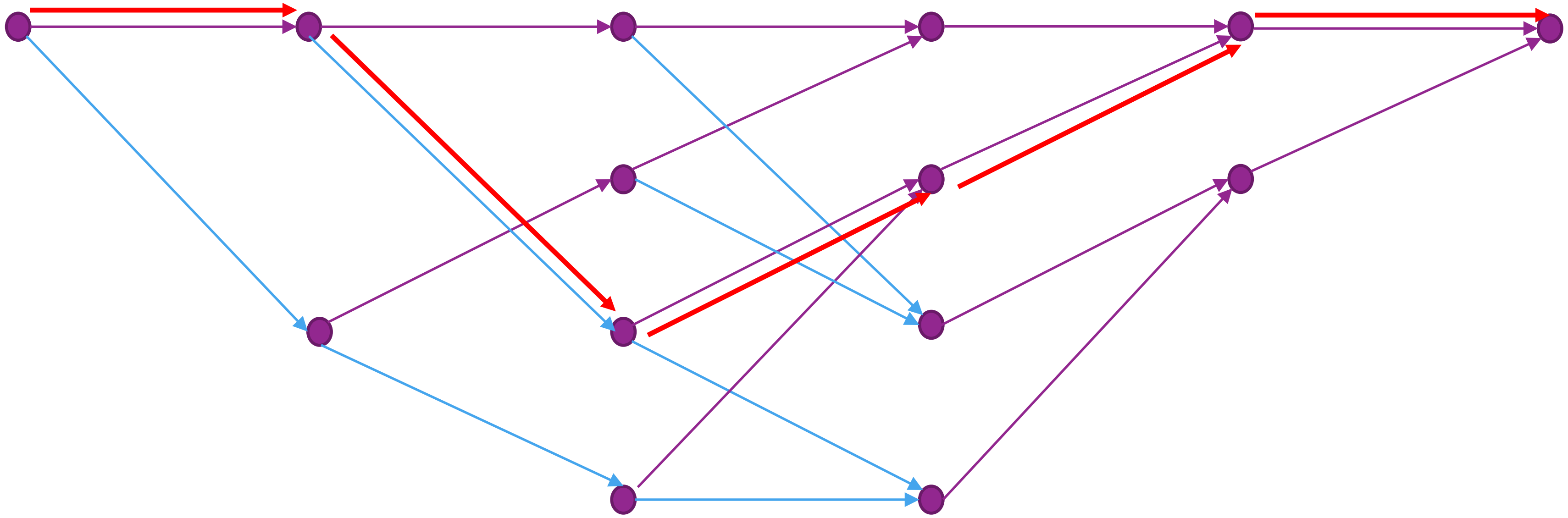
Ex: Trace back path

- Trace back maximum path. Shown in red



Ex: Read off bits

- Read off the bits from the maximum path
- ML input bits = $(0,1,0,0,0)$



Complexity

- Update of each node requires maxima over 2^k branches
- There are $2^{k(K-1)}$ states, so complexity / time is $O(2^{kK})$
- Total complexity is $O(T2^{kK})$
- Storage is also $O(T2^{kK})$. (From the paths)
- Summary:
  - Viterbi algorithm is linear in block length.
  - Can have very long block lengths (often T in the 1000s)
  - But, complexity is exponential in constraint length
  - Practical decoders limited to $K = 7$ or $K = 9$.

Terminating the Trellis

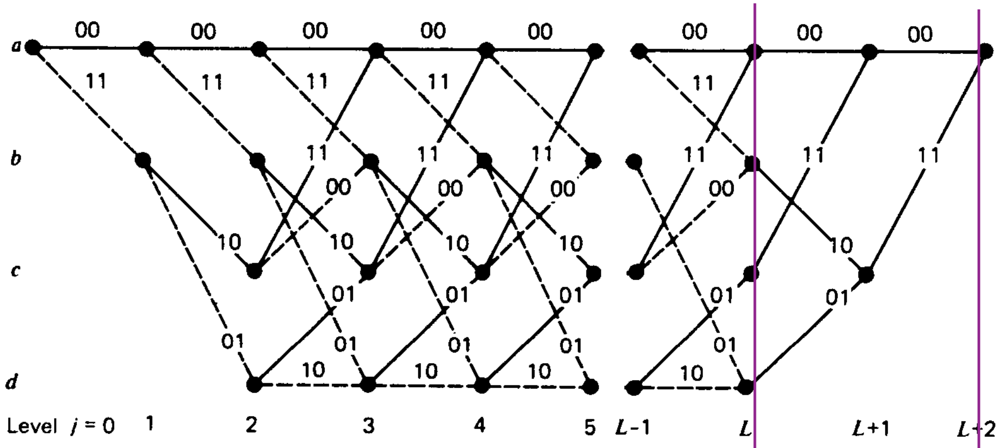


Figure 11.15 Trellis for the convolutional encoder of Fig. 11.13*a*.

During tail bits only paths with zero bit inputs.

- Recall tail bits are zero: $\boldsymbol{b}(L) = \cdots = \boldsymbol{b}(L + K - 2) = 0.$
- Limits the paths at the end of the trellis
- Viterbi algorithm should only be done on the zero path.
- Very important to not forget the bits at the end.
- Otherwise, final bits are not protected.

Pruning the Path Memory

- In current algorithm, path $P_t(x_t)$ grows to full block length.
- Adds storage: Storage is O$\left(T2^{kK}\right)$. Linear in T
- Adds delay. No bits can be determined until code is fully decoded.
- Many practical implementations:
  - Store some finite length δ of each surviving path.
  - Can make decision on bit input $\boldsymbol{b}(t)$ after $\mathbf{r}(t+\delta)$.
  - Rule of thumb: Good performance if $\delta \geq 5K$.

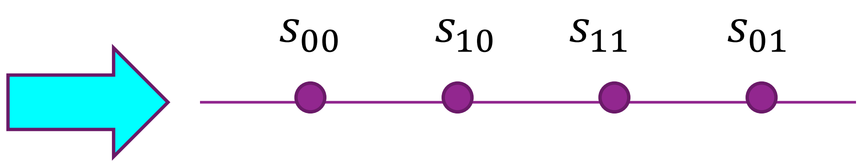
Rate Matching Convolutional Codes

- Convolutional codes have limited rates, usu. ½ or 1/3.
- Obtain other rates through
  - Puncturing: Remove coded bits to increase rate
  - Repetition: Repeat coded bits to decrease rate
- Puncture / repeat pattern is important (see Proakis)
  - Try to spread out modified bits
- For punctured bits, set corresponding LLRs to zero
  - Viterbi decoder just ignores those bits
- For repeated bits, add the corresponding LLRs
  - Viterbi decoder will increase confidence on that branch

High Order Constellations

- Higher order constellations (eg. 16- or 64-QAM)
- Each constellation is a point is a function of multiple bits.
- Likelihood does not factorize
  - Each symbol r(t) depends on multiple bits
- Example 4-PAM (or one dimension of 16-QAM):
  - Each symbol likelihood depends on two bits: $p(r|c_1, c_2)$

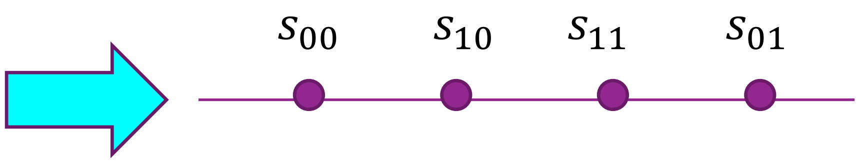
Two bits:
(c_1, c_2)



$r = s + n$

High Order Constellations

Two bits:
(c_1, c_2)



$$r = s + n$$

- To create LLRs for individual bits use total probability rule:

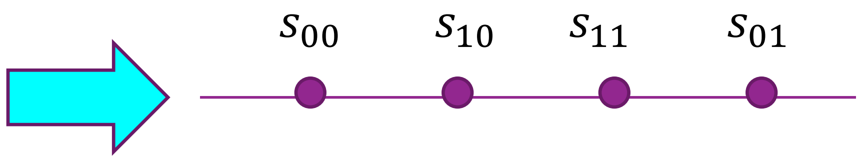
$$p(r|c_1) = \frac{1}{2}\big(p(r|c_1, c_2 = 0) + p(r|c_1, c_2 = 1)\big)$$

- Resulting bitwise LLR:

$$LLR \text{ for } c_1 = \log \frac{p(r|c_1, c_2 = 1{,}0) + p(r|c_1, c_2 = 1{,}1)}{p(r|c_1, c_2 = 0{,}0) + p(r|c_1, c_2 = 0{,}1)}$$

High Order Constellations

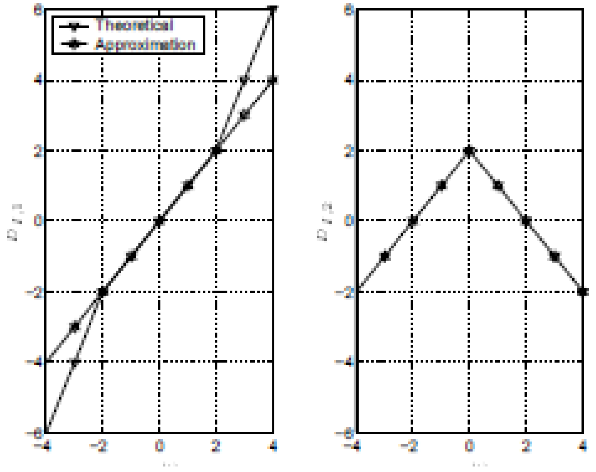
Two bits:
(c_1, c_2)



$r = s + n$

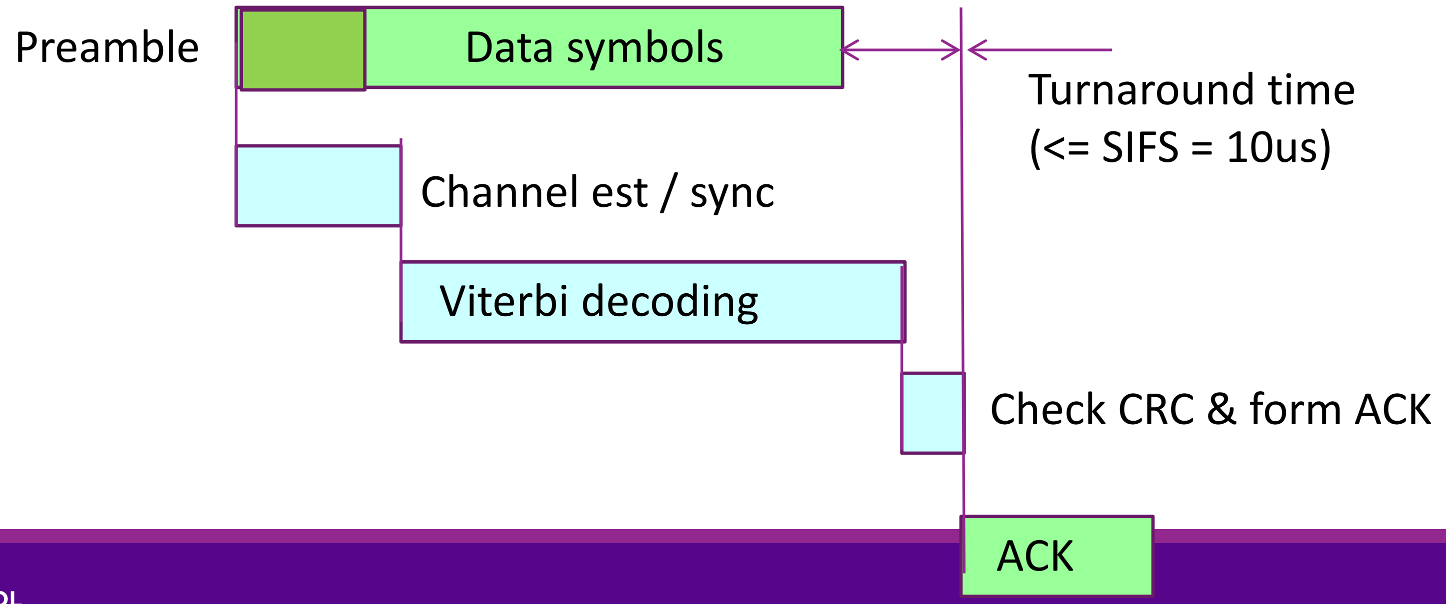
- LLRs can have irregular shapes
- Not simple linear function as in BPSK / QPSK case
- Often use approximations
- More info: Caire, Taricco and Biglieri, “Bit-Interleaved Coded Modulation," 1998.

LLR for c2 LLR for c1



Convolutional Codes in WiFi

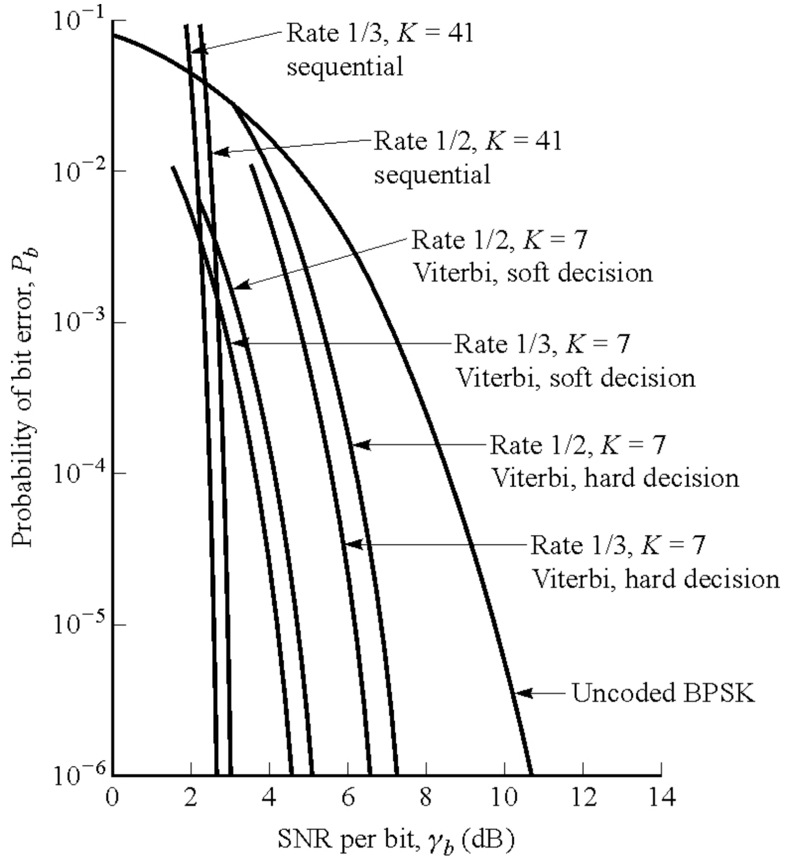
- 802.11 uses R=1/2 K=7 code.
- Length adjusted to packet size
- Higher rates (R=2/3 and ¾) achieved through puncturing
- Enables decoding as data arrives for ACK fast turnaround



Convolutional Codes in LTE

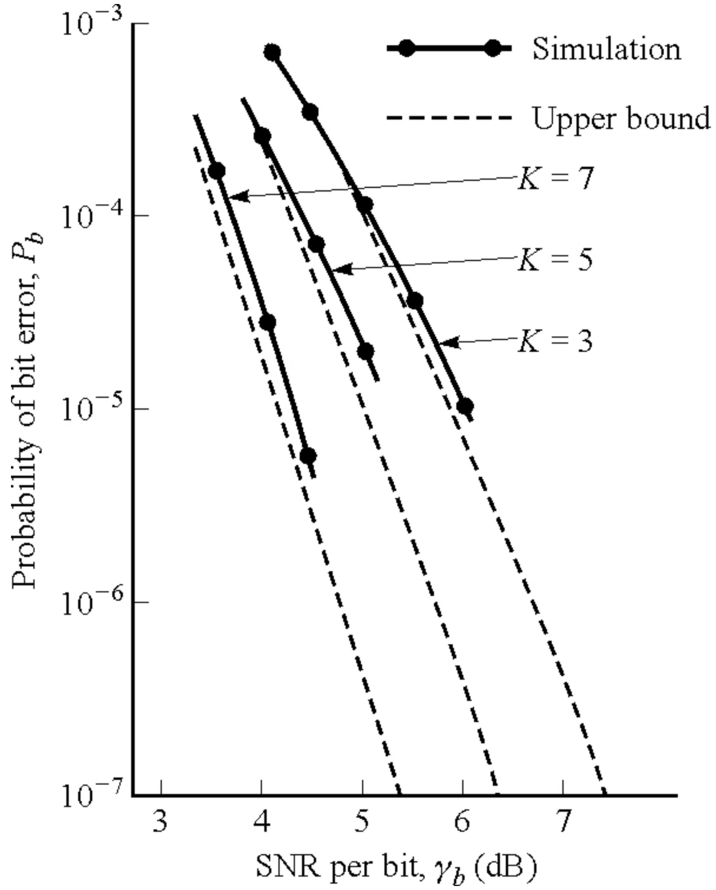
- Convolutional codes in LTE used for:
  - Control channels (payload typ 20-40 bits +CRC), and
  - Short (< 128 bit) data frames
- Larger payloads encoded with turbo codes (discussed later)
- Uses rate=1/3 base convolutional code with K=7.
- Higher rates achieved via puncturing
- Control channels use a sophisticated technique called tail biting to reduce loss on the tail bits

Decoding Performance



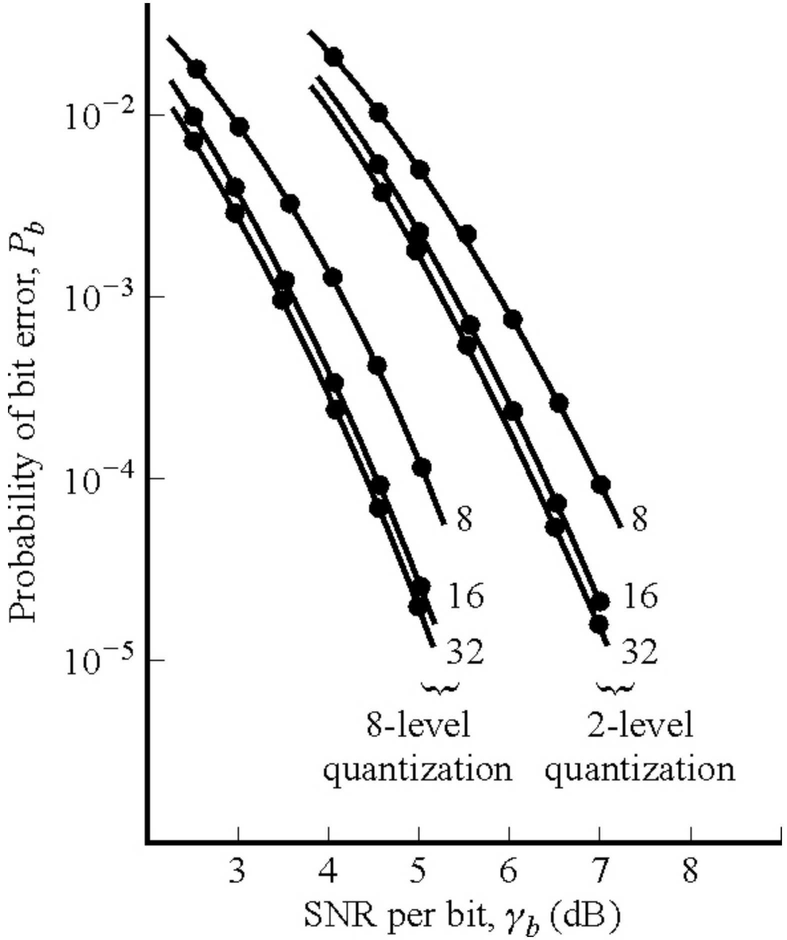
- Hard decision loses approximately 1.5 to 2 dB
- Constraint length K=7 is sufficient for very sharp error performance

Different Constraint Lengths



- Approximate 1 dB improvement between K=3, 5 and 7
- Higher constraint lengths become computationally intractable
- Recall decoding complexity is exponential in K

LLR Quantization



- Minimal gains after 16 bit quantization of branch metrics
- Recall HD is equivalent to 1 bit quantization
- Most commercial implementations use 6-bit LLRs

MATLAB Convolution Tools

Demo Convolutional Code

The Viterbi decoder built in the in-class exercise is very slow. Fortunately, MATLAB has excellent tools for encoding and decoding

- Perform convolutional encoding with the MATLAB comm toolbox commands
- Extract LLRs from the qamdemod command
- Perform convolution decoding from the soft LLRs
- Measure the block error rate as a funciton of the SNR
-

We illustrate with a simple convolutional coder over an AWGN channel. We use a standard constraint length K=7 code:

```
% Create the convolutional encoder and decoder with constraint length K=7
trellis = poly2trellis(7,[171 133]);
convEnc = comm.ConvolutionalEncoder('TrellisStructure', trellis, 'TerminationMethod','Terminated');
convDec = comm.ViterbiDecoder('TrellisStructure', trellis, 'TerminationMethod','Terminated');

% Modulation
bitsPerSym = 2;   % QPSK
M = 2^bitsPerSym;

% Number of information bits per block
nbits = 1000;
```

- MATLAB comm toolbox
  - General convolution encoders & decoders
  - Efficient implementation
  - Excellent for testing
- See demo

In-Class Exercise

Building a Simple Viterbi Decoder

We conclude with building a simple decoder. This decoder is slow but illustrates the ideas.

To start, let's encode some random data.

```
% Number of bits
nbits = 100;

% TODO:  Generate random bits
%   b = ...


% TODO:  Create output bits.
%   c = ...
```