Unit 1. Passband Modulation

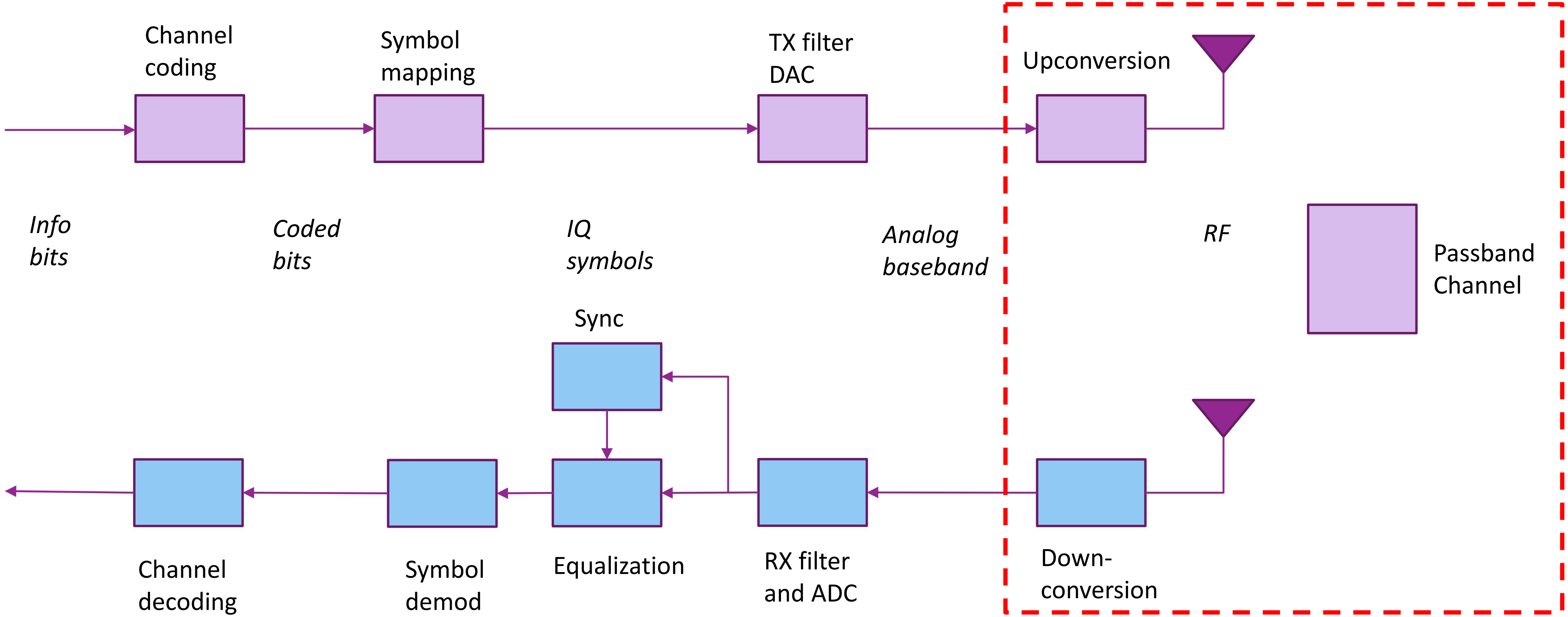
EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN

Learning Objectives

- ❑ Determine if a system is real passband or baseband
- ❑ Mathematically describe upconversion and downconversion
  - In time-domain and frequency-domain
- ❑ Compute simple continuous-time Fourier transforms (Review)
- ❑ Select parameters and analyze low-pass filter in down conversion
- ❑ Determine if a signal is a power or energy signal
  - Convert power in dBm
- ❑ Compute the effective baseband filter given a passband filter
- ❑ Model impairments such as time and frequency offsets

This Unit



Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

Outline

- Time-Domain Relationships
- Fourier Transform Review
- Frequency-Domain Relationships
- Power and Energy Spectra
- Baseband equivalent filters
- Practical up and down-conversion circuits
- Wireless channels

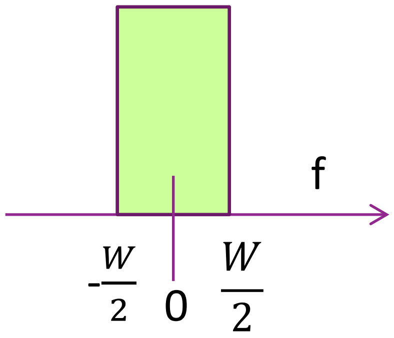
Signals in Communications

- Signal: Any quantity that varies in time
  - Can be continuous time $x(t)$
  - Or discrete time $x[n]$
  - Real or complex valued

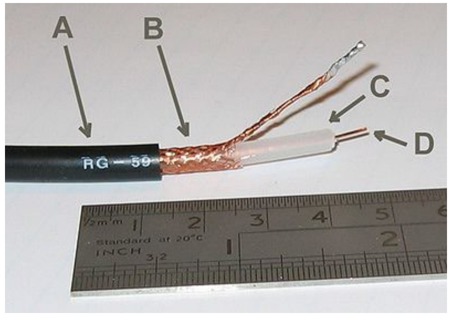
- Signals in communications:
  - $v(t)$ = Voltage at a particular point / place in a circuit (relative to ground)
  - $E_z(t)$ = Electric field strength in a particular direction
    Note: electric field is a vector quantity $E(t) = [E_x(t), E_y(t), E_z(t)]$
  - A digital sample of a signal
  - An intermediate value used in processing a signal

Real Baseband Systems

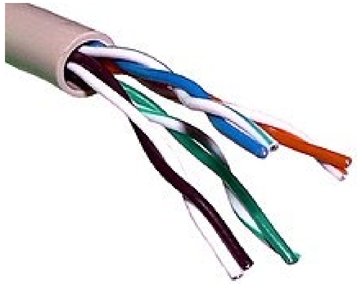
- Real baseband communication systems:
  - Communicate with lowpass real-valued signals
  - $X(f) \approx 0$ for $|f| \leq \frac{W}{2}$



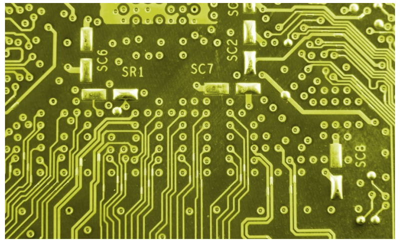
- Examples



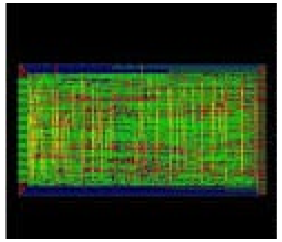
Coaxial cable



Twisted pair



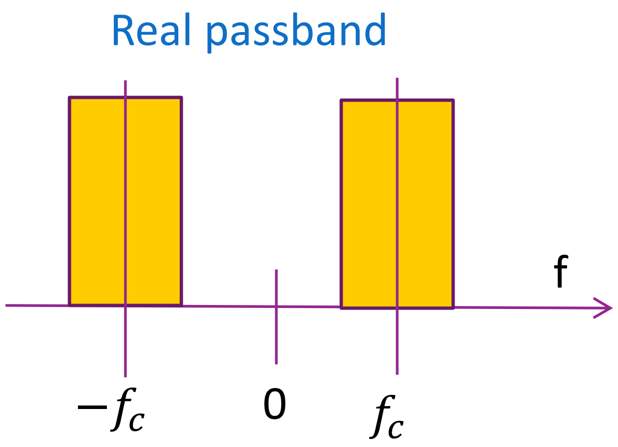
PCB traces
e.g., microstrip or stripline



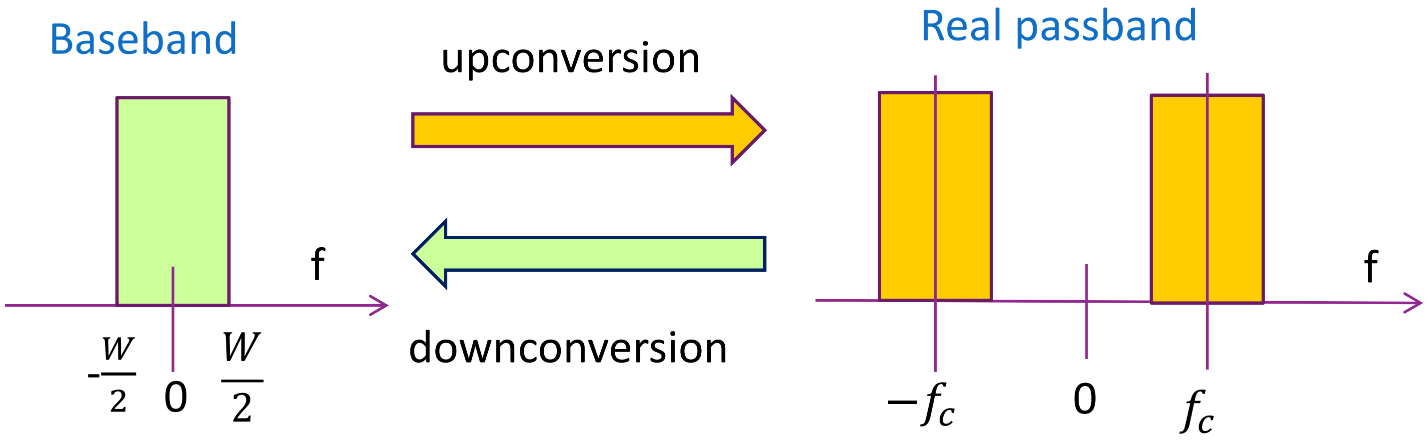
ASIC metal traces

Real Passband Communications

- Real passband communication systems
  - Transmit around a carrier frequency f_c
  - f_c is sometimes called the center frequency
  - $X(f) \neq 0$ for $|f - f_c| < W$ and $|f + f_c| < W$
- Mostly radio frequency communication
  - Often wireless
  - Transmissions are restricted to bandwidth
  - Also, RF propagation is limited to certain bands
  - RF communication also occurs over cables



Up- and Downconversion



- Up and downconversion: Shift center frequency of signals
- Used for all passband communications systems
  - Information occurs or is processed in baseband
  - Transmitted and received in real passband

Upconversion in Time Domain

- Baseband signals: $u_i(t)$ and $u_q(t)$,
  - Also called "in-phase" and "quadrature" (I and Q)
  - Real-valued. Typically, bandlimited to $|f| < \frac{W}{2}$ ($\frac{W}{2}$ = Single-sided bandwidth)
  - Sometimes called the "cosine" and "sine" part.
- Carrier frequency f_c: Also called the "center" frequency
- Create real passband signal:

$$u_p(t) = u_i(t)\cos(2\pi f_c t) - u_q(t)\sin(2\pi f_c t)$$

- Upconversion is also called modulation
  - But we will use that term for something later.

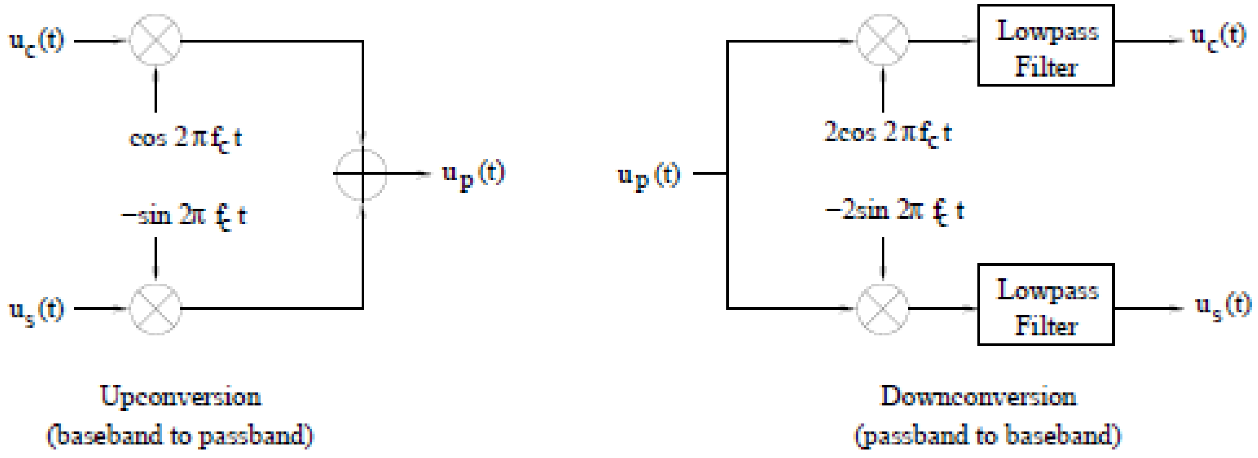
Downconversion

- Can recover I part from multiplication by sinusoid:
- Recovery of the I part:
  - $v_i(t) = 2u_p(t)\cos(2\pi f_c t) = u_i(t) +$ high freq terms
  - $u_i(t) = \mathrm{LPF}(v_i(t))$
- Recovery of the Q part:
  - $v_q(t) = -2u_p(t)\sin(2\pi f_c t) = u_q(t) +$ high freq terms
  - $u_q(t) = \mathrm{LPF}(v_q(t))$
- Can derive relations using

$$\sin(2x) = 2\sin(x)\cos(x) \quad 2\cos^2(x) = 1 + \cos(2\mathrm{x})$$

- Note gain of 2 and sign.

Up and Downconversion Block Diagram



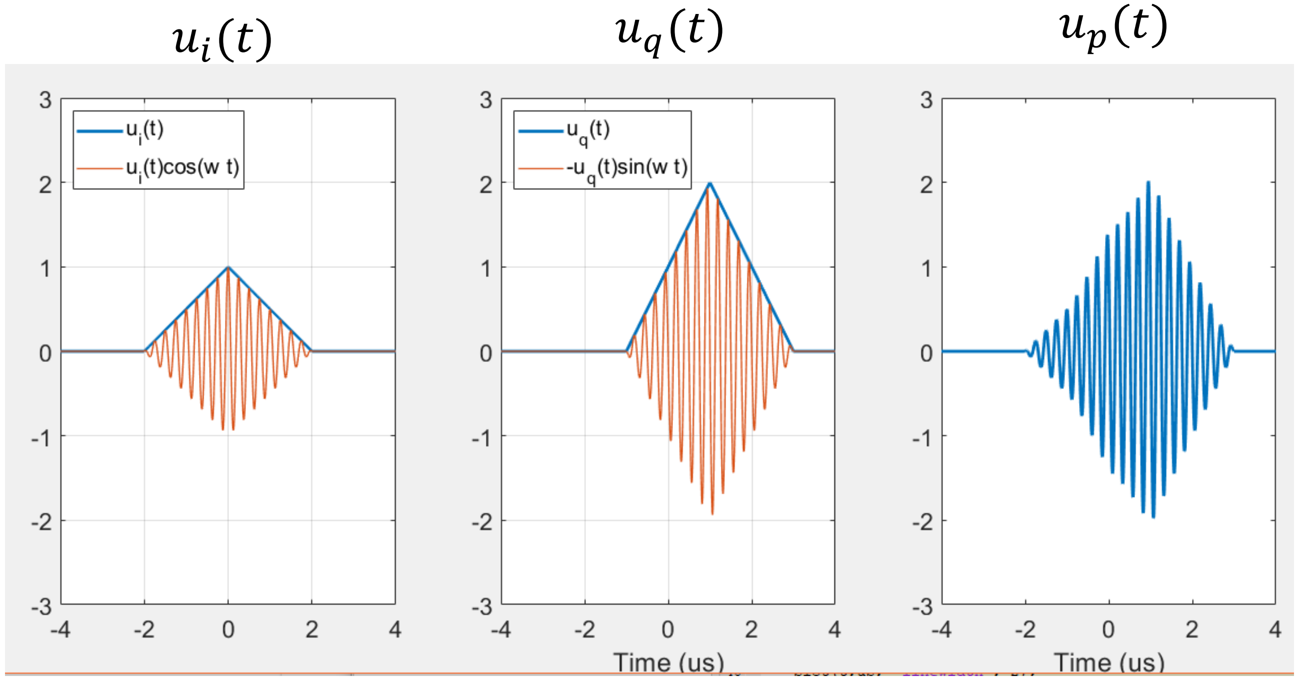
- Fig. 2.28 from Madhow
- Implementation with multipliers

Example

- Suppose
  - $u_i(t) = Tri\left(\frac{t}{T}\right),\ u_q(t) = 2Tri\left(\frac{t}{T} - 0.5\right), Tri(s) := \max(0{,}1 - |s|)$
  - $T = 2\ \mu s,\ f_c = \frac{8}{T} = 4$ MHz

```
% Create the baseband signals
nt = 1024;
T = 2.0;
t = linspace(-2*T,2*T,nt)';
f0 = 8/T;
ui = max(1-abs(t/T),0);
uq = 2*max(1-abs(t/T-0.5),0);


% Modulate the I and Q components
uicos = ui.*cos(2*pi*f0*t);
uqsin = -uq.*sin(2*pi*f0*t);
up = uicos + uqsin;
```



Actual IQ Mixer

http://www.markimicrowave.com/Mixers/IQ_Quadrature-IF_Double-Balanced/IQ-0318.aspx

- LO = “local oscillator” = square or sine wave at f_c
- I1, I2 = I and Q inputs.
  - Generally, lowpass
- RF = passband output centered at f_c

Datasheet	RF [GHz]	LO [GHz]	IF [MHz]	Conversion Loss [dB]	Image Rejection [dB]	Amplitude Deviation [dB]	Phase Deviation [Degrees]	Isolations L-R [dB]	Isolations L-I [dB]
IQ-0318	3 to 18	3 to 18	DC to 500	7	22	0.75	10	40	20

Complex Baseband Notation

- Computations are often done in complex domain
- Complex baseband signal: $u(t) = u_i(t) + ju_q(t)$
- Upconversion: Real passband is $u_p(t) = Re\left[u(t)e^{2\pi j f_c t}\right]$
- Down-conversion:
  - $v(t) = 2u_p(t)e^{-j\omega_c t} = u(t) +$ High freq terms
  - $u(t) = H_{LPF}(v(t))$

Sample Problem (Soln on Board)

- Suppose that $T = 1\ \mu s$ and

$$u(t) = \begin{cases} 1+j & t \in [0,T) \\ 1-j & t \in [T,2T) \\ 0 & \text{else} \end{cases}$$

- What are $u_i(t)$ and $u_q(t)$? Draw them.
- Write an equation for $u_p(t)$ with a carrier frequency $f_c = 4\ \text{MHz}$
- Draw $u_p(t)$ for $t \in [1,2]\ \mu s$.

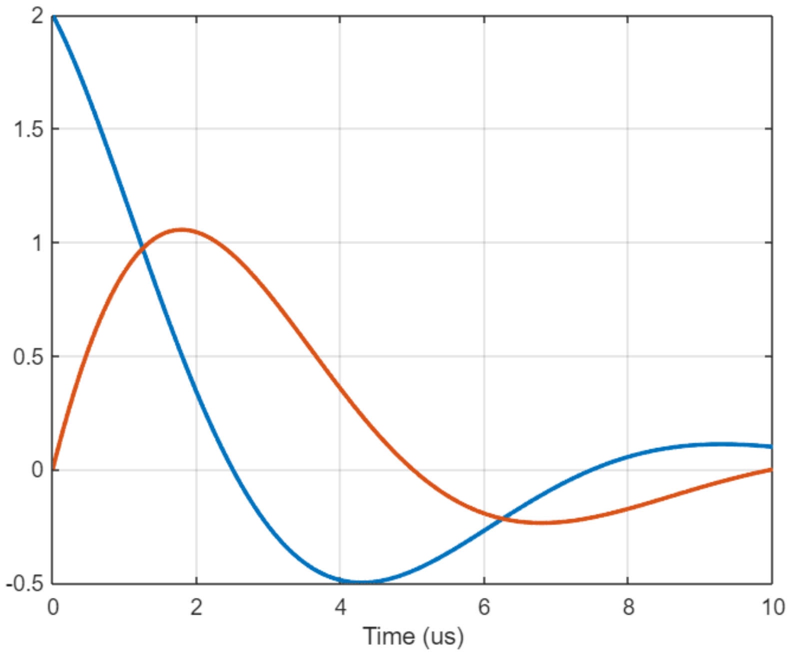
In-Class Exercise

Passband Up- and Down-Conversion In-Class Exercises

Up and Down-Conversion in Time-Domain

In this exercise, we will perform simple up and down-conversion. Normally, up and down-conversion are
numerically. We start with a simple complex baseband time-domain signal:

```
u = u0*exp((1i*w-alpha)*t)
```



- See passbandInClass.mlx MATLAB live script
- Fill in TODO sections

Outline

- ❑Time-Domain Relationships
- ❑Fourier Transform Review
- ❑Frequency-Domain Relationships
- ❑Power and Energy Spectra
- ❑Baseband equivalent filters
- ❑Practical up and down-conversion circuits
- ❑Wireless channels

Complex Exponential

- Continuous-time complex exponential signal:

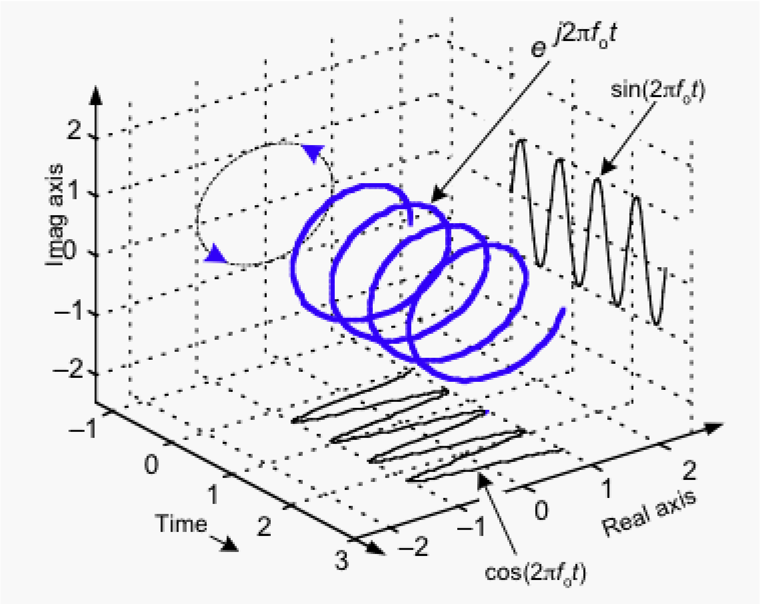
$$\boxed{x(t) = A\exp(2\pi i\, ft) = A\exp(i\omega t)}$$

  - A = complex magnitude
  - f = frequency in Hz, $\omega = 2\pi f$ = angular frequency in rad/s

- If we write $A = re^{i\theta}$, $r = |A|$ then:

$$\boxed{x(t) = r\exp(\omega t + \theta)}$$

  - r = magnitude
  - θ = phase



Fourier Transform

- $s(t)$: real or complex continuous-time signal
- Fourier Transform: Expresses signal as sum / integral of complex exponentials
- Time-domain to frequency domain

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-2\pi i f t}\, dt$$

- Inverse Fourier transform:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{2\pi i f t}\, df$$

- Represents signals in their frequency components

FT in Angular Frequency

- Angular frequency: $\omega = 2\pi f$
- Fourier Transform: time-domain to frequency domain

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t}\, dt$$

- Inverse Fourier transform:

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)e^{i\omega t}\, df$$

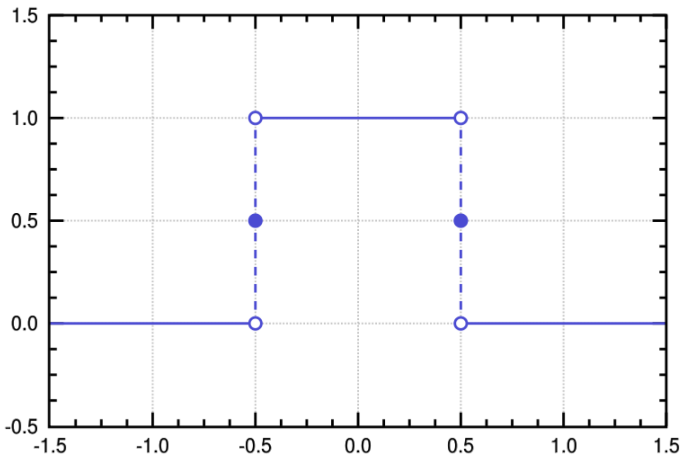
- Note scaling
- Some texts use other scalings

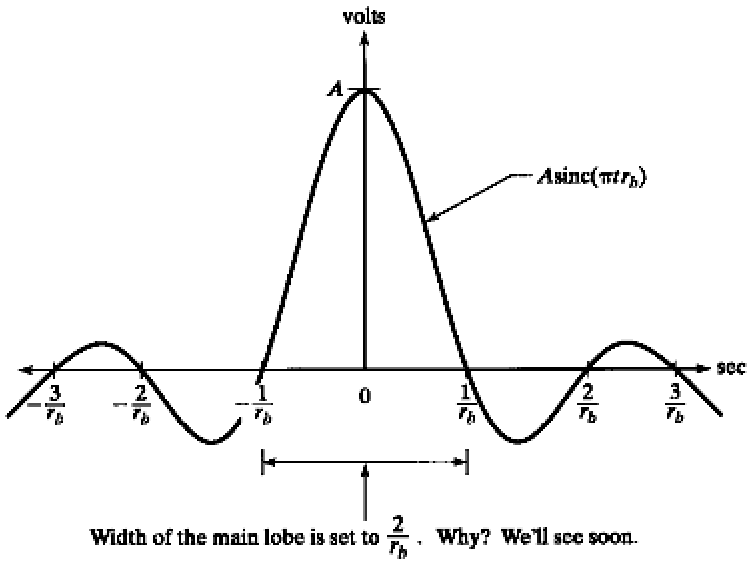
Rect and Sinc

- $rect(at) \leftrightarrow \frac{1}{|a|} sinc\left(\frac{f}{a}\right) = \frac{\sin(\pi f/|a|)}{\pi f}$
- $sinc(at) \leftrightarrow \frac{1}{|a|} rect\left(\frac{f}{a}\right)$

Height = $\frac{1}{|a|}$

Main lobe f $= \pm|a|$

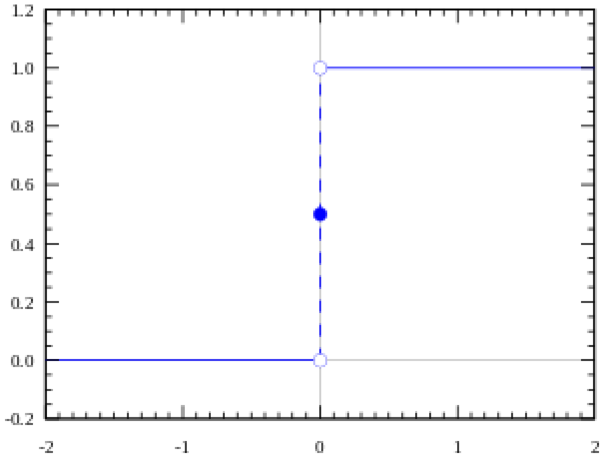




Unit Steps

- Unit step

$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$



- FT:
  - $e^{-\alpha t} u(t) \leftrightarrow \frac{1}{\alpha + 2\pi i f}, \; Re(\alpha) > 0$
  - $u(t) \leftrightarrow \frac{1}{2}\delta(f) + \frac{1}{2\pi i f}$

Table (From WikiPedia)

	Function	Fourier transform unitary, ordinary frequency
	$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$
201	$\operatorname{rect}(ax)$	$\frac{1}{\lvert a\rvert} \cdot \operatorname{sinc}\left(\frac{\xi}{a}\right)$
202	$\operatorname{sinc}(ax)$	$\frac{1}{\lvert a\rvert} \cdot \operatorname{rect}\left(\frac{\xi}{a}\right)$
203	$\operatorname{sinc}^2(ax)$	$\frac{1}{\lvert a\rvert} \cdot \operatorname{tri}\left(\frac{\xi}{a}\right)$
204	$\operatorname{tri}(ax)$	$\frac{1}{\lvert a\rvert} \cdot \operatorname{sinc}^2\left(\frac{\xi}{a}\right)$
205	$e^{-ax} u(x)$	$\frac{1}{a + 2\pi i \xi}$
206	$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{(\pi \xi)^2}{\alpha}}$
207	$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2 + 4\pi^2 \xi^2}$

$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$
1	$\delta(\xi)$
$\delta(x)$	1
e^{iax}	$\delta\left(\xi - \frac{a}{2\pi}\right)$
$\cos(ax)$	$\frac{\delta\left(\xi - \frac{a}{2\pi}\right) + \delta\left(\xi + \frac{a}{2\pi}\right)}{2}$
$\sin(ax)$	$\frac{\delta\left(\xi - \frac{a}{2\pi}\right) - \delta\left(\xi + \frac{a}{2\pi}\right)}{2i}$

$\operatorname{sgn}(x)$	$\frac{1}{i\pi\xi}$
$u(x)$	$\frac{1}{2}\left(\frac{1}{i\pi\xi} + \delta(\xi)\right)$
$\sum_{n=-\infty}^{\infty} \delta(x - nT)$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\xi - \frac{k}{T}\right)$

Other Properties

- $s(t-a) \leftrightarrow e^{-2\pi i a f} S(f)$
- $e^{2\pi i a t} s(t) \leftrightarrow S(f-a)$
- $s(at) \leftrightarrow S(f/a)/|a|$
- $d^n s(t)/dt^n \leftrightarrow (2\pi i f)^n S(f)$
- $t^n s(t) \leftrightarrow d^n S(f)/df^n$
- $s^*(t) \leftrightarrow S^*(-f)$
- $s(t) \leftrightarrow S(f) \Rightarrow S(t) \leftrightarrow s(-f)$

Problems

2.10 Determine the Fourier transform of each of the following signals (α is positive).

1. $x(t) = \frac{1}{1+t^2}$
2. $\Pi(t-3) + \Pi(t+3)$
3. $\Lambda(2t+3) + \Lambda(3t-2)$
4. $\text{sinc}^3 t$
5. $t\,\text{sinc}\,t$
6. $t \cos 2\pi f_0 t$
7. $e^{-\alpha|t|} \cos(\beta t)$
8. $t e^{-\alpha t} \cos(\beta t)$

- Solutions on board

Sampled Complex Exponentials

- Often need to process digitally sampled complex exponentials
  - E.g. MATLAB, digital circuits, …
- Continuous-time signal: $x(t) = A\exp(2\pi i f t) = A\exp(i\omega t)$
- Sampled signal:

$$x[n] = x(nT), \qquad T = 1/f_s$$

- Resulting discrete-time signal:

$$x[n] = A\exp(2\pi i\, \nu\, n) = A\exp(i\Omega n)$$

  - $\nu = \frac{f}{f_s}$: Normalized digital frequency [cycles / sample]. Typically $[-0.5, 0.5]$
  - $\Omega = 2\pi\nu = \frac{2\pi f}{f_s}$: Digital angular frequency [rads / sample]. Typically $[-\pi, \pi]$
- Will discuss sampling much more later

Computing the FT with the FFT

- Often approximately compute FFT on the sampled signal
- Continuous-time $x(t)$. Want to compute FT $X(f)$
- Take sampled signal $x_d[n] = x(nT), n = 0, \ldots, N-1$
- Compute FFT (Fast Fourier Transform):

$$X_d[k] = \sum_{n=0}^{N-1} x_d[n] e^{2\pi i k n/N}$$

- Then, if $X(f)$ is bandlimited to $|f| < 1/(2T)$ then:

$$\boxed{X\left(\frac{k}{T}\right) = T X_d[k'], \qquad k' = k \bmod N, \qquad k = -\frac{N}{2}+1, \ldots, \frac{N}{2}}$$

- Obtain sampled version of the FT from the FFT
- Note scaling factor T and modulo operation

In-Class Exercise

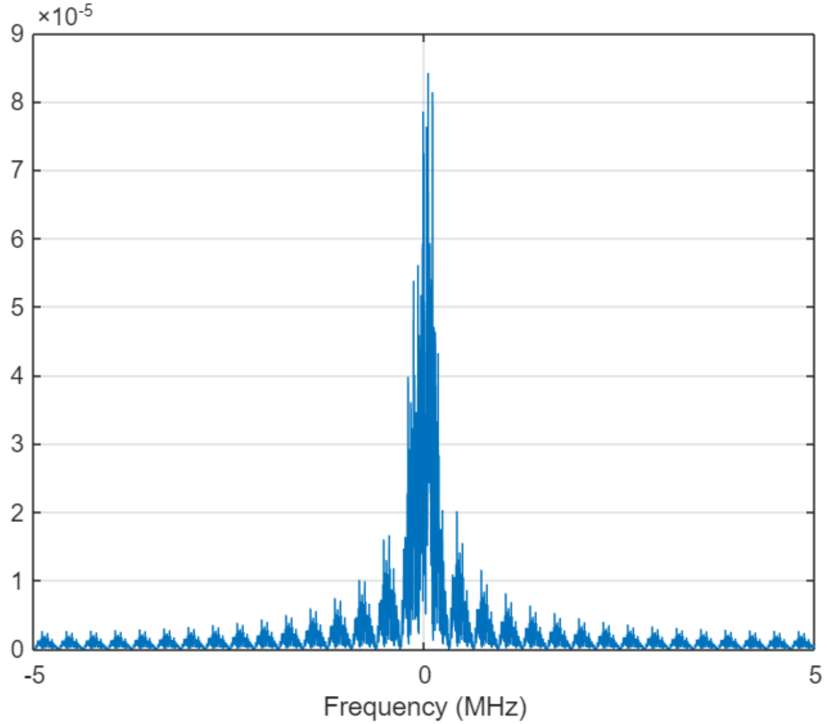
Computing a Fourier Transform with an FFT

In this exercise, you will see how to approximately compute the Fourier Transform of a sampled signal with the FFT. First, we create a sampled-signal.

- Generate random complex symbols, `sym[k]=exp(1i*theta[k])` where `theta[k]` is uniform in `[0,2*pi]` and `k=1,...,nsym`
- Create an up-sampled version of sym, denoted u: `u[n]=sym[k]` for `n = k*sampPerSym-1,...,(k+1)*sampsPerSym`.
- Plot the `real(u) vs.` time in micro-seconds.

`So, u` is a signal that randomly changes phases every `sampsPerSym`.

```
sampPerSym = 2^5;   % number of samples per symbol
nsym = 2^7;         % number of symbols
nsamp = sampPerSym*nsym;  % total number of samples
fs = 10e6;          % sample rate in Hz
```



Outline

- Time-Domain Relationships
- Fourier Transform Review
- Frequency-Domain Relationships
- Power and Energy Spectra
- Baseband equivalent filters
- Practical up and down-conversion circuits
- Wireless channels

Up-Conversion of a Complex Exponential

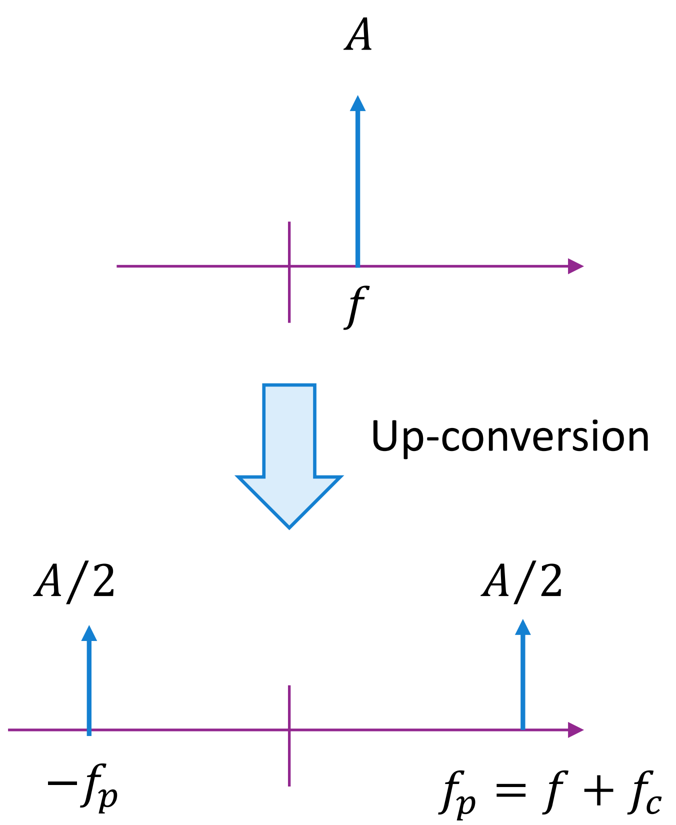
- Suppose complex baseband is $u(t) = A\exp(2\pi i f t)$
- Up-conversion:

$$u_p(t) = Real\left(\, u(t)\, e^{2\pi i\, f_c t}\,\right) = A\cos(2\pi i f_p t)$$

  - Shifts f to $f_p = f + f_c$
- Can write as $u_p(t) = \frac{A}{2}\left[e^{2\pi i f_p t} + e^{-2\pi i f_p t}\right]$
- If $u(t) = A\exp(2\pi i f t + \theta)$ then

$$u_p(t) = A\cos(2\pi i(f + f_c)t + \theta)$$

  - Phase is unchanged
- Down-conversion reverses the process



Down-Conversion of a Complex Exponential

- From previous slide, after upconversion:

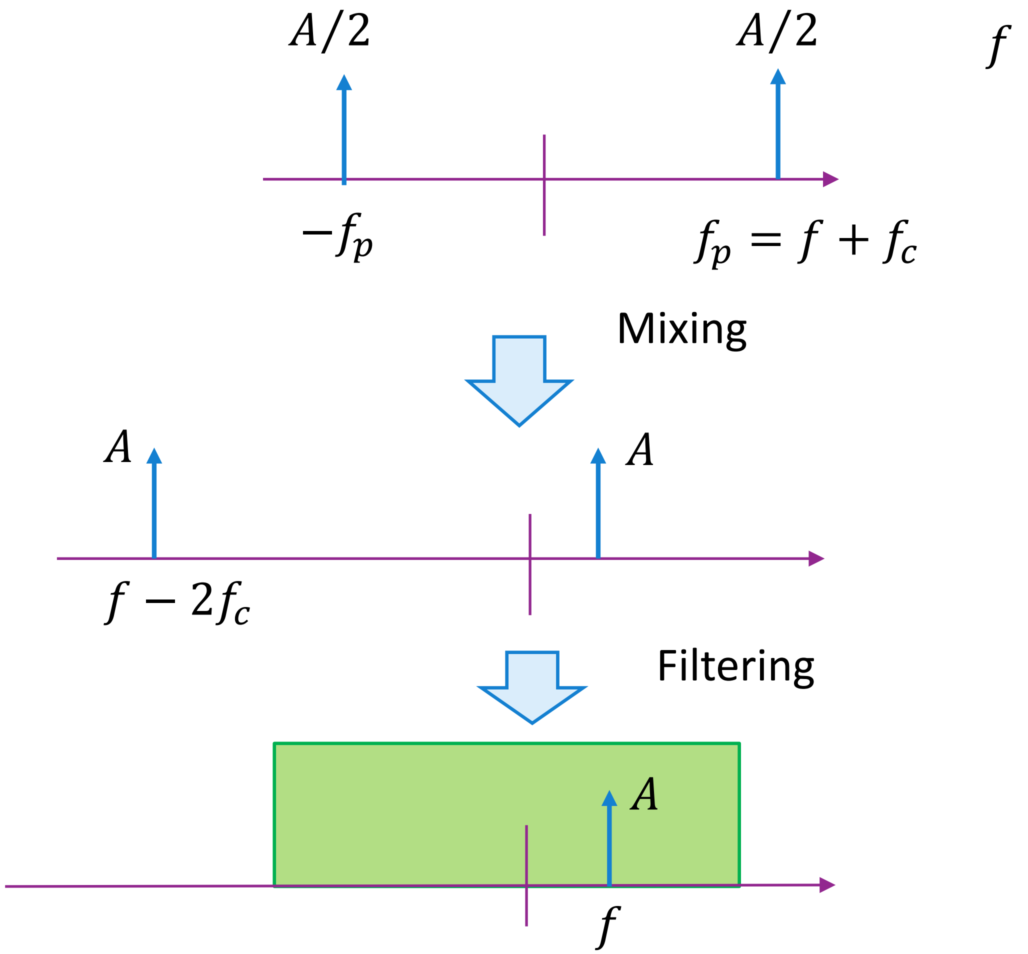
$$u_p(t) = \frac{A}{2}\left[e^{2\pi i f_p t} + e^{-2\pi i f_p t}\right]$$

- Mixing:

$$v(t) = 2u_p(t)e^{2\pi i f_c t}$$
$$= Ae^{2\pi i f t} + Ae^{-2\pi i (f+2f_c)t}$$

- Filtering:

$$u(t) = LPF\big(v(t)\big) = Ae^{2\pi i f t}$$

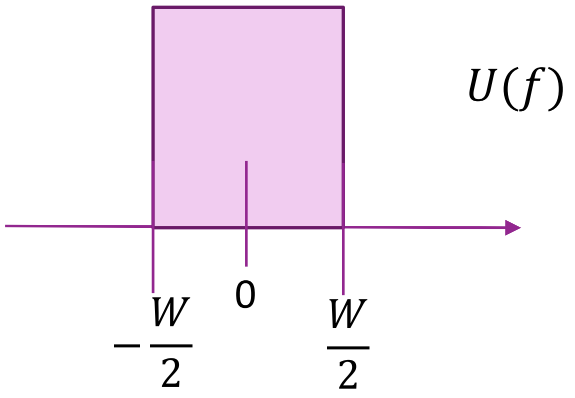


Example

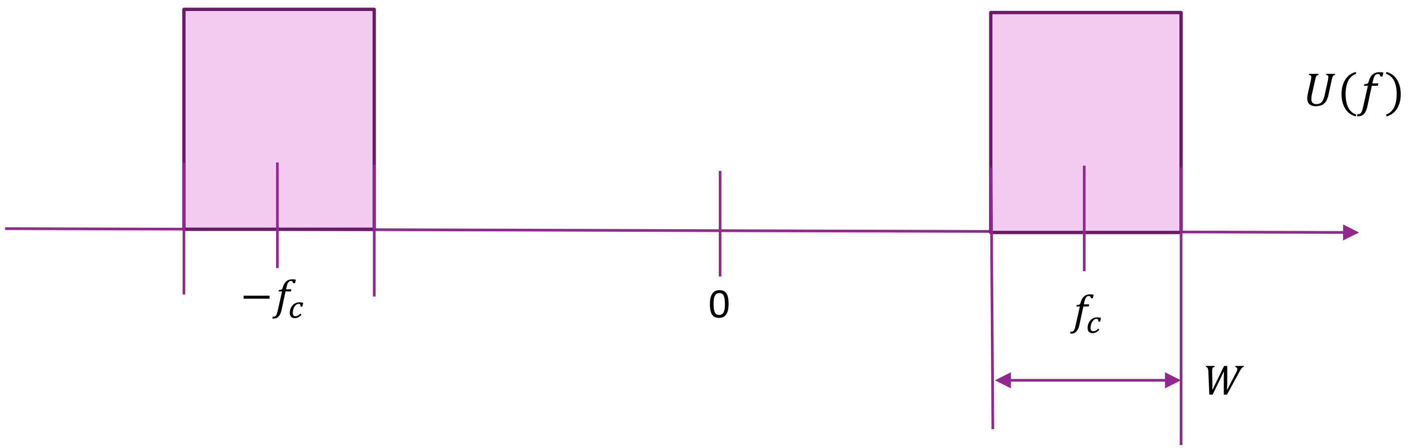
- Suppose the digital complex baseband signal is: $u[n] = A\exp(2\pi i v n)$
  - $v = 0.1,\ A = 4$
- Suppose sampling rate is $f_s = 20$ MHz and carrier is $f_c = 2$ GHz
- What is the continuous-time complex baseband signal?
- Ans: $u(t) = A\exp(2\pi i\, f t)\,,\ \ f = v f_s = (0.1)(20) = 2$ MHz
- What is the real passband signal after up-conversion?
- Ans: $u_p(t) = A\exp(2\pi i\, f_p t)\,,\ \ f_p = f + f_c = 2.002$ GHz
  - Be careful with units

Bandwidth Terminology

- Now consider more complex signals
- Baseband signals
  - Centered around $f = 0$, complex
  - $\frac{W}{2} =$ single sided bandwidth
  - $W =$ two sided bandwidth
  - Band-limited to $|f| \leq \frac{W}{2}$

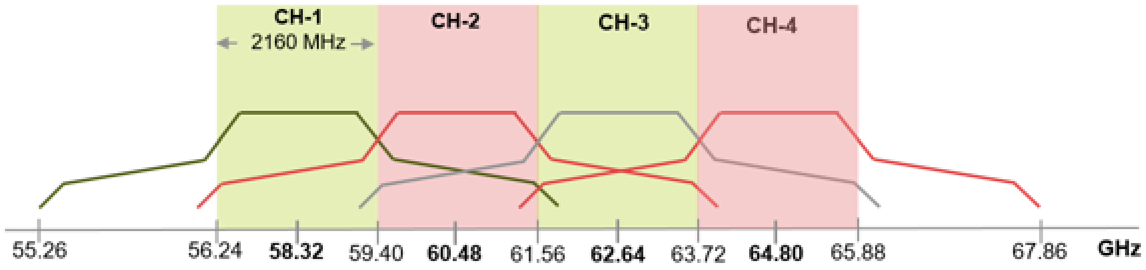


- Passband signals
  - Centered around $f = f_c$, real
  - $W =$ bandwidth (per side or image)
  - Band-limited to $|f - f_c| \leq \frac{W}{2}$



Importance of Bandwidth

- Data rate generally scales linearly in bandwidth
  - If the transmit power and bandwidth increase by $N \Rightarrow$ the communication rate increases by N
  - We will see this in detail later
- Ex: Compare GSM (2G) and LTE (4G)
  - Single channel of GSM system = 200 kHz
  - Single channel of LTE = 20 MHz
  - If power scales sufficiently, LTE would in general have 100x data rate
  - LTE, in fact, can have even more capacity due to other improvements
- Figure to the right: 802.11ad channels
  - The channels are > 2 GHz



Frequency Domain Relationships
Baseband to Passband

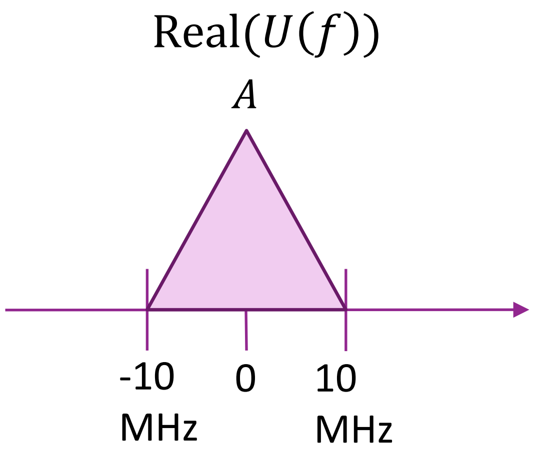
- Suppose that $U(f)$ is bandlimited to $\left[-\frac{W}{2}, \frac{W}{2}\right]$ and $f_c > W$

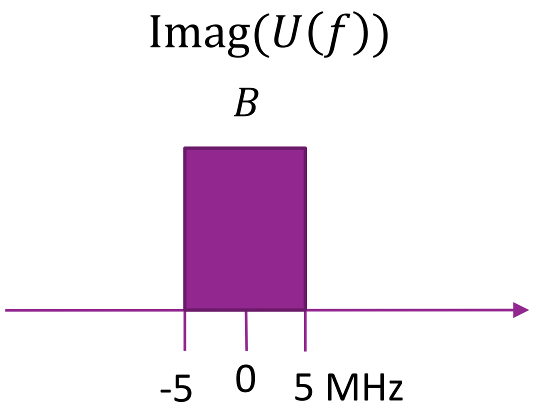
$$U_p(f) = \frac{1}{2}[U(f - f_c) + U^*(-f - f_c)]$$

- Use notation:
  - $U^+(f) \coloneqq \frac{1}{2}U(f - f_c)$: This is $U(f)$ shifted to the right by f_c and scaled by $\frac{1}{2}$
  - $U^-(f) \coloneqq \frac{1}{2}U^*(-f - f_c)$: Flip $U^+(f)$ around y axis and take negative of the imaginary part
- Proof:
  - Let $c(t) = u(t)e^{2\pi j f_c t} \leftrightarrow C(f) = U(f - f_c)$
  - $u_p(t) = Re\big(c(t)\big) = \frac{1}{2}\big(c(t) + c^*(t)\big)$
  - Now use conjugate symmetry $c^*(t) \leftrightarrow C^*(-f)$

Example Problem

- Suppose baseband signal is as drawn:





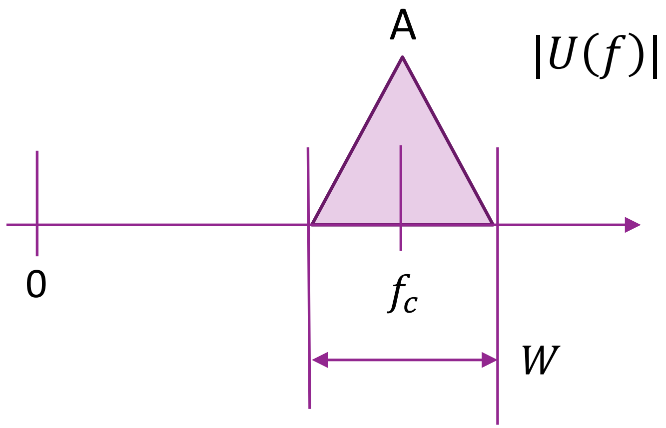
- What is the:
  - Single-sided bandwidth?
  - Two-sided bandwidth?
- Write an equation for $u_i(t)$
  - You do not need to evaluate the integral.
- Draw the passband frequency response if $f_c = 2$ GHz
  - Draw both the positive and negative images

Frequency Domain Relationships
Passband to Baseband

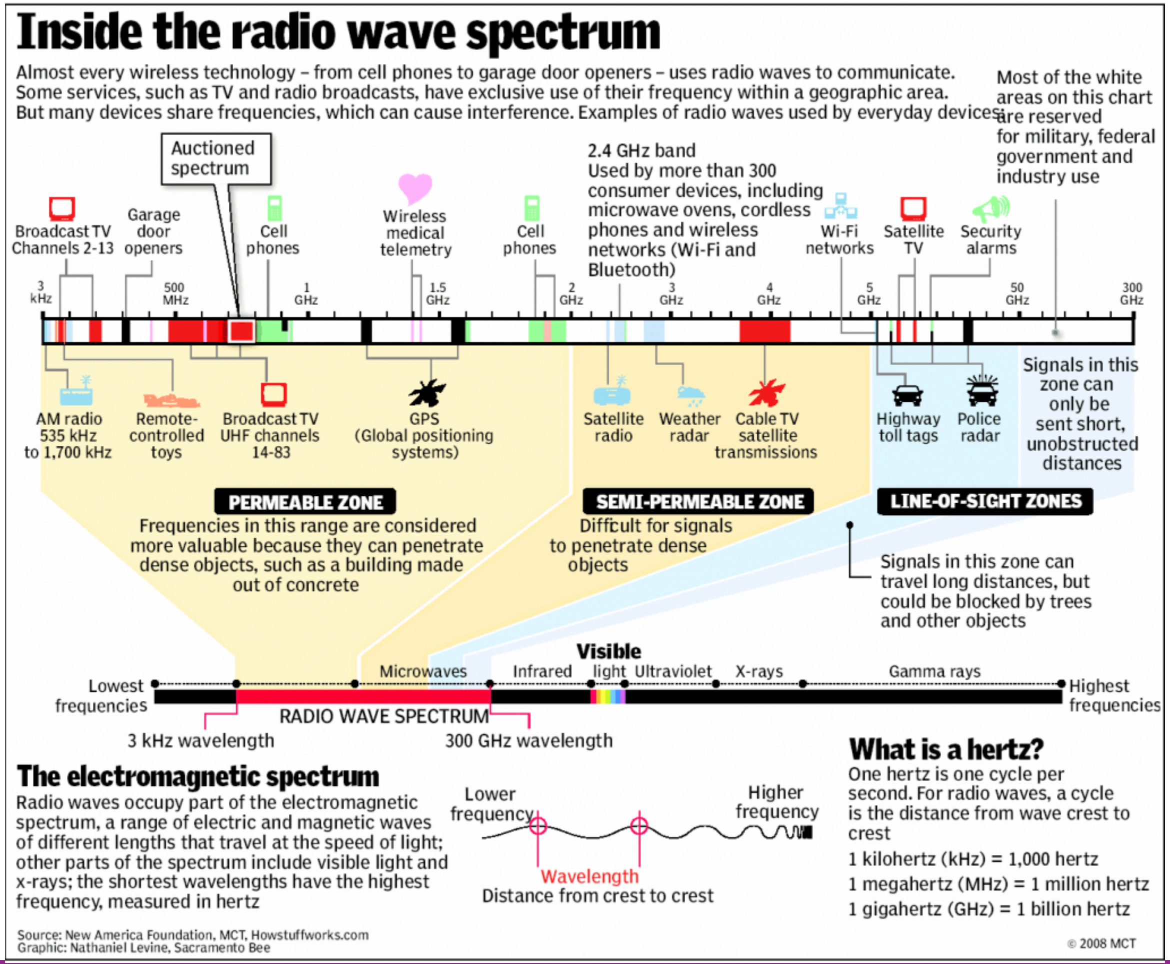
- Downcoversion in time-domain: $v(t) = 2u_p(t)e^{-j\omega_c t}, \; u(t) = h_{LPF}(t) * v(t)$
- In frequency-domain: $U(f) = 2U_p(f + f_c)H_{LPF}(f)$
  - Shift to left, scale by 2 and filter
- Ideal filtering:
  - Suppose $U_p(f)$ has bandwidth W around f_c
  - Then typically have: $H_{LPF}(f) = 1$ for $|f| \le \frac{W}{2}$ and $H_{LPF}(f) = 0$ for $|f| > f_c - \frac{W}{2}$
  - $U(f) = 2U_p(f + f_c)1_{\{|f| \le W\}}$
  - Shift to the left and remove left image.
- Pictures on board

Example Problem

- Suppose right image of passband is as shown:
  - $W = 4$ MHz, $f_c = 800$ MHz
- Assume a LPF $H_{LPF}(f) = Rect\left(\frac{f}{f_0}\right)$
- Draw magnitude spectrum of down-converted signal
  - When $f_0 = 5$ MHz
  - When $f_0 = 3$ MHz
- What range of values f_0 will:
  - Keep the low-pass component
  - Reject the high frequency component
- Solution on board



Radio Spectrum

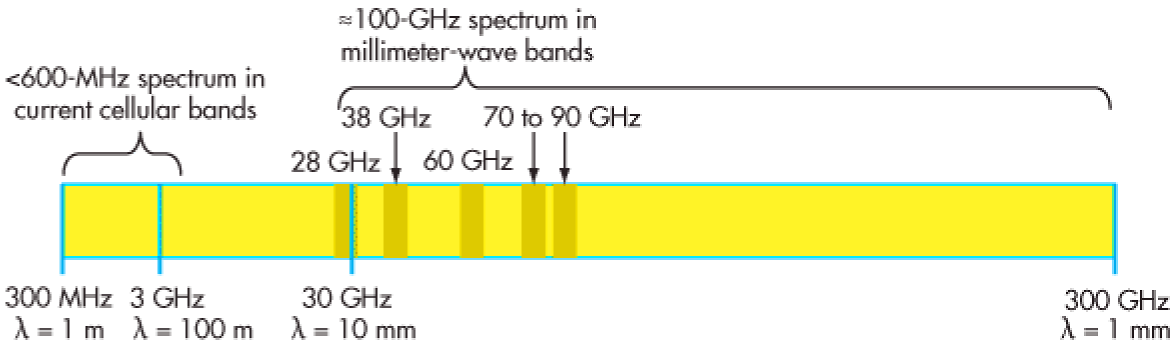


Bandwidth and Center Frequencies Examples

System	Duplex	Center freq (MHz)	Bandwidth
GSM	FDD	GSM-850: 824-849 (UL), 869-894 (DL) GSM-900: 890-914 (UL), 935-959 (DL) GSM-1800: 1710–1784(UL), 1805.2–1879(DL) GSM-1900: 1850–1910(UL), 1930–1990(DL)	200 kHz per channel
UMTS	FDD	GSM + other bands ~2100 and ~1900	5 MHz per carrier
LTE	Mostly FDD	Mostly in 2100 to 2600 MHz	1.4 to 20 MHz, 10 MHz typical
802.11abg			20 MHz
802.11n	TDD	2.4 GHz (ISM band) and 5 GHz (U-NII band)	20, 40 MHz
802.11ac			20-160 MHz
802.11ad	TDD	60 GHz (millimeter wave spectrum)	2.16 GHz

Millimeter Wave

- New bands for 5G
  - 100x more bandwidth than conventional bands below 6 GHz
  - 5G systems today are operating in 28 GHz and 38 GHz



Outline

- Time-Domain Relationships
- Fourier Transform Review
- Frequency-Domain Relationships
- Power and Energy Spectra
- Baseband equivalent filters
- Practical up and down-conversion circuits
- Wireless channels

Energy and Power Signals

- Instantaneous power: $|x(t)|^2$
  - Why squared?
- Energy:
  - $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
  - Signal is called an "energy signal" if $E_x < \infty$
- Power:
  - $P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$
  - Energy per unit time
  - Signal is called a "power signal" if limit P_x exists and is finite

Power of a Periodic Signal
Time-Domain Method

- Suppose $x(t)$ is periodic, period T
- Theorem: $x(t)$ is a power signal and power can be computed from any one period

$$P_x = \frac{1}{T}\int_0^T |x(t)|^2 dt = \frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2 dt$$

  - Proof on board

Example: Done on board

- Suppose that $x(t)$ has period T

$$x(t) = a + bt, t \in [0, T]$$

  - a, b are real
- Draw $x(t)$
- What is P_x
- What if a, b were complex?

Power of a Periodic Signal
Fourier Series Method

- Suppose that $x(t)$ is periodic with period T
- Then has Fourier Series

$$x(t) = \sum c_n e^{2\pi j f_n t}, \qquad f_n = n/T$$

- Theorem: Power of $x(t)$ is:

$$P_x = \sum |c_n|^2$$

- Note that if $x(t) = \sum g(t - nT)$, then $c_n = G(f_n)$
  - Can compute power from Fourier transform of $g(t)$

Example: On board

- Suppose $T = 10\ \mu s$,

$$x(t) = \sum g(t - nT), \qquad g(t) = \begin{cases} 2 & t \in [0, T/4) \\ -1 & t \in [T/4\,, T) \\ 0 & \text{else} \end{cases}$$

- Draw $x(t)$
- What is the FT $G(f)$?
- What is the FT $X(f)$?
- What is the power of $\mathrm{x(t)}$?
- What fraction of power of $\mathrm{x(t)}$ is in the $|f| \leq 250$ kHz?

Energy Density

- Energy of signal: $E_x = \int |x(t)|^2 dt$

- From Parseval's identity: $E_x = \int |X(f)|^2 df$
  - Can compute energy in frequency-domain

- Energy density: $G_x(f) = |X(f)|^2$
  - Density of energy around frequency f

Power Spectral Density (PSD)

- Three equivalent ways to define PSD
- Definition 1: via windowing in time
- Definition 2: via filtering
- Definition 3: via auto-correlation for a random process
  - More advanced.
  - We will cover this in the next unit

PSD: Time-Windowing Definition

- Let $x(t)$ be a power signal
- Define windowed signal:

$$x_T(t) = \begin{cases} x(t) & |t| \le T \\ 0 & |t| > T \end{cases}$$

- PSD is defined as:

$$S_x(f) \coloneqq \lim_{T\to\infty} \frac{1}{2T} |X_T(f)|^2$$

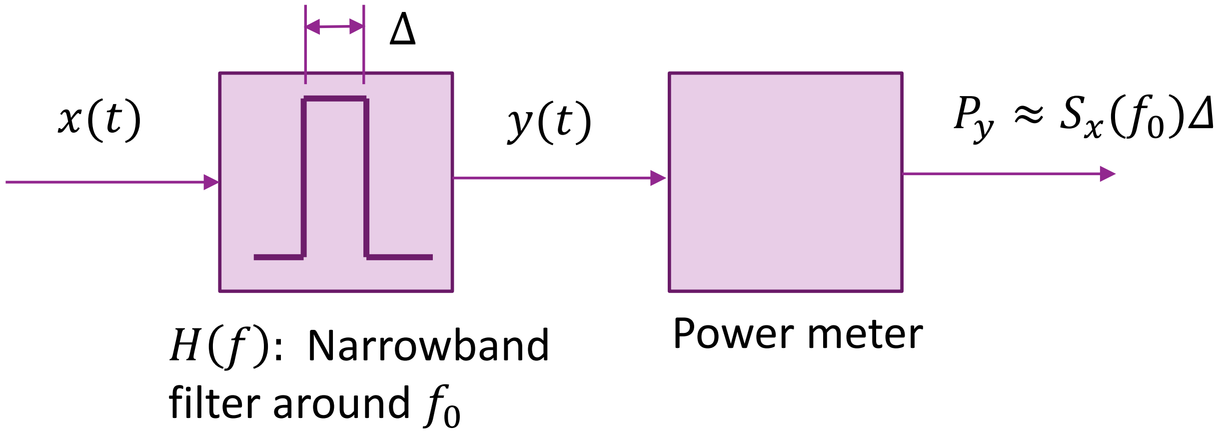
- Similar to energy signal, but with averaging over time.
- Can show power is given by:

$$P_x = \int_{-\infty}^{\infty} S_x(f)df$$

  - $S_x(f)$ represents power per unit frequency

PSD: Filtering Definition

- Let $x(t)$ be a power signal
- Select frequency f_0 to measure PSD
- Filter with narrowband filter
  - $y(t) = h(t) * x(t)$
  - $H(f) = 1$ for $|f - f_0| \leq \Delta/2$
- Measure power P_y



- PSD at f_0 is defined as

$$S_x(f_0) := \lim_{\Delta \to 0} \frac{1}{\Delta} P_y$$

- Can show this is equivalent to window definition
- Reveals how much power is in a certain frequency

Spectrum Analyzer

- Measures PSD in real time
- Uses averaging of FT
  - But proper averaging is quite tricky
- Lab 2: Use MATLAB function pwelch

Units

- Energy signals:
  - E_x: Joules
  - $G(f) = |X(f)|^2$: Joules / Hz

- Power signals (much more common):
  - P_x: Joules / sec = Watts
  - $S_x(f)$: Watts / Hz = Joules

Power: Linear and decibel scale

- Receive or transmit antenna energy per unit time
  - Measured in Watts (W) or mW
  - Power values in W or mW called *linear scale*
  - Use notation $P_{|W}$ or $P_{|mW}$ when units need to be specified

- Power often measured in dB scale:
  - $P_{|dBW} = 10\log_{10}(P_{|W} / 1W)$
  - $P_{|dBm} = 10\log_{10}(P_{|mW} / 1mW)$

- Example: P = 250 mW (typical max mobile transmit power)
  - $P_{|dBW} = 10\log_{10}(0.25W / 1W)$
  - $P_{|dBm} = 10\log_{10}(250mW / 1mW)$

Some important dB values

- Some conversions don't need a calculator:
  - 10log10(2) = 3 [Most important: Doubling power = 3dB]
  - 10log10(3) =4.7 ~5
  - 10log10(10) = 10

- You can cascade these.
- Ex: If the power is increased by 50 in linear scale, what is the increase in dB? Answer:

$$
\begin{aligned}
&10\log_{10}(50) = 10\log_{10}(10^2/2) \\
&= 2\times 10\log_{10}(10) - 10\log_{10}(2) = 2\times 10 - 3 = 17 \text{ dB}
\end{aligned}
$$

PSD and Linear Filters

- Suppose $y(t) = h(t) * x(t)$
- Then: $Y(f) = H(f)X(f)$
  - $S_y(f) = |H(f)|^2 S_x(f)$
- Transfer function $|H(f)|^2$ power gain at frequency f:

$$|H(f)|^2 = \frac{S_y(f)}{S_x(f)} \frac{\text{output power at } f}{\text{input power at } f}$$

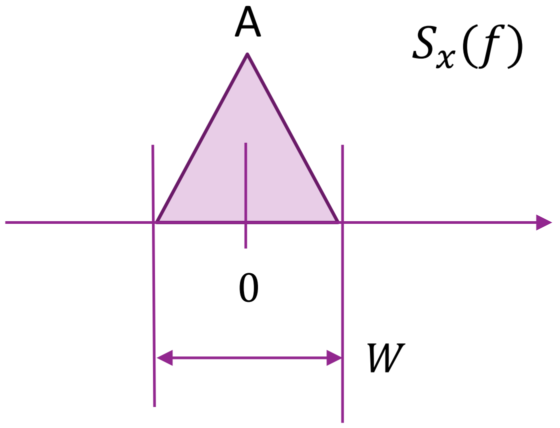
  - Dimensionless quantity
  - Often expressed in dB

Typical Wireless Power Transmit Levels

- 100 kW = 80 dBm: Typical FM radio transmission with 50 km radius
- 1 kW = 60 dBm: Microwave oven element (most of this doesn't escape)
- ~300 W = 55 dBm: Geostationary satellite
- 250 mW = 24 dBm: Cellular phone maximum power (class 2)
- 200 mW = 23 dBm: WiFi access point
- 32 mW = 15 dBm: WiFi transmitter in a laptop
- 4 mW = 6 dBm: Bluetooth 10 m range
- 1 mW = 0 dBm: Bluetooth, 1 m range

Example 1:

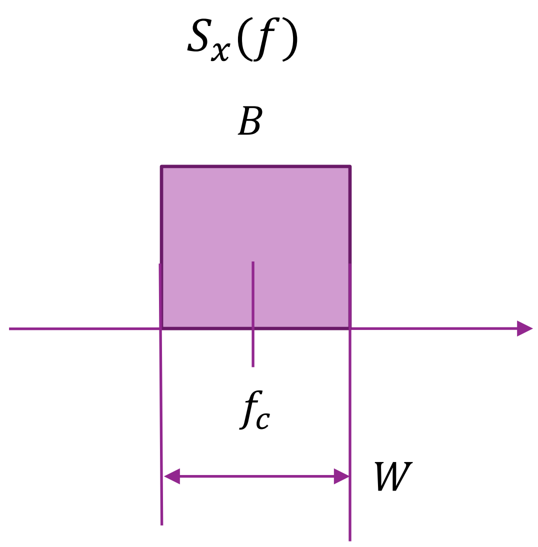
- $S_x(f)$ is as shown.
- What is the power (in linear scale) in terms of W, A?
- Suppose the power is $P_x = 20$ dBm, $W = 20$ MHz, $f_c = 2$ GHz
  - What is A?
  - What are the units of A?



- Answer: on board

Example 2

- $S_x(f)$ is shown for $f > 0$. Assume $x(t)$ is real. $W = 20$ MHz, B=$2(10)^{-8}$ mW/Hz, $f_c = 2$ GHz
- What is P_x? (Linear and in dBm).
- Suppose $y(t) = h(t) * x(t)$ with $H(f) = f_0/(2\pi jf + f_0)$
- What is $S_y(f)$? Draw it.
- Assuming $f_c \gg f_0$ what is P_y?
- What is the attenutation in dB?



In-Class Exercise

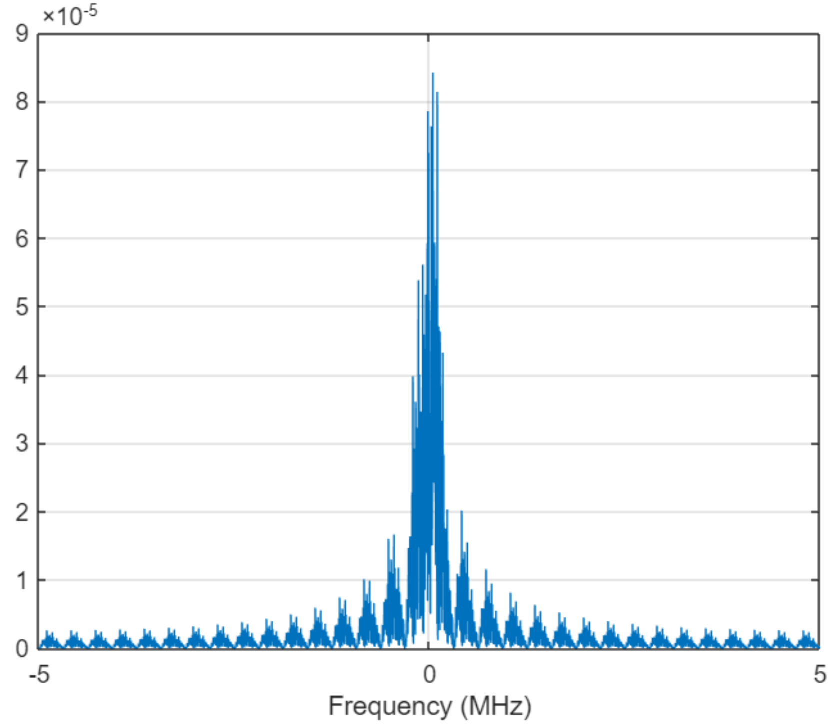
Computing a Fourier Transform with an FFT

In this exercise, you will see how to approximately compute the Fourier Transform of a sampled signal with the FFT. First, we create a sampled-signal.

- Generate random complex symbols, `sym[k]=exp(1i*theta[k])` where `theta[k]` is uniform in `[0,2*pi]` and `k=1,...,nsym`
- Create an up-sampled version of sym, denoted u: `u[n]=sym[k]` for `n = k*sampPerSym-1,...,(k+1)*sampsPerSym`.
- Plot the `real(u) vs.` time in micro-seconds.

`So, u` is a signal that randomly changes phases every `sampsPerSym`.

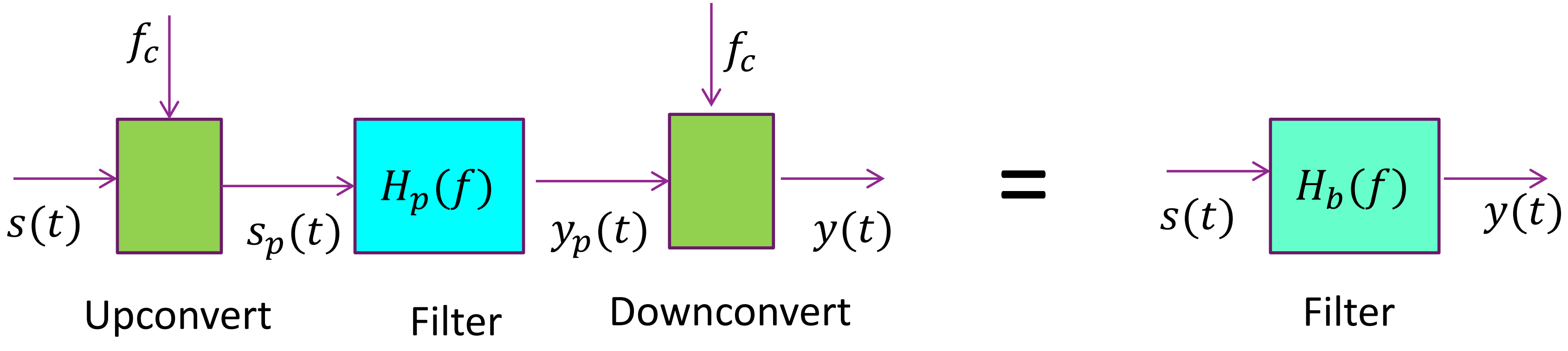
```
sampPerSym = 2^5;   % number of samples per symbol
nsym = 2^7;         % number of symbols
nsamp = sampPerSym*nsym;  % total number of samples
fs = 10e6;          % sample rate in Hz
```



Outline

- ❑Time-Domain Relationships
- ❑Fourier Transform Review
- ❑Frequency-Domain Relationships
- ❑Power and Energy Spectra
- ❑Baseband equivalent filters
- ❑Practical up and down-conversion circuits
- ❑Wireless channels

Filtering



- Filtering at passband equivalent to complex baseband filter
- Assuming downconversion filter is ideal (see next slide):
  - $H_b(f) = H_p(f + f_c)$ for $|f| \le \frac{W}{2}$
  - Simply shift $H_p(f)$ to the left by f_c.

Proof of Result

- Using the conversions from passband:
- Downconversion formula: $Y(f) = 2Y_p(f + f_c)H_{LPF}(f)$
- Filtering in passband: $Y(f) = 2H(f + f_c)U_p(f + f_c)H_{LPF}(f)$
- Using upconversion formula:

$$Y(f) = H(f + f_c)\{U^*(-f - 2f_c) + U(f)\}H_{LPF}(f)$$

- Assume:
  - $U(f)H_{LPF}(f) \approx U(f)$ Filtering preserves baseband image
  - $U^*(-f - 2f_c)H_{LPF}(f) \approx 0$ Filtering removes image around $-2f_c$
- Then $Y(f) = H(f + f_c)U(f)$

Delay

- Important special case: Suppose that $h_p(t) = A\delta(t - \tau)$
  - A = gain
  - $\tau =$ delay
- Passband frequency response is: $H_p(f) = Ae^{-2\pi j f \tau}$
- Baseband frequency response:

$$H_b(f) = H_p(f + f_c) = Ae^{-2\pi j (f_c + f)\tau}$$

- Equivalent impulse response:

$$h_b(t) = Ae^{-2\pi j f_c \tau}\delta(t - \tau)$$

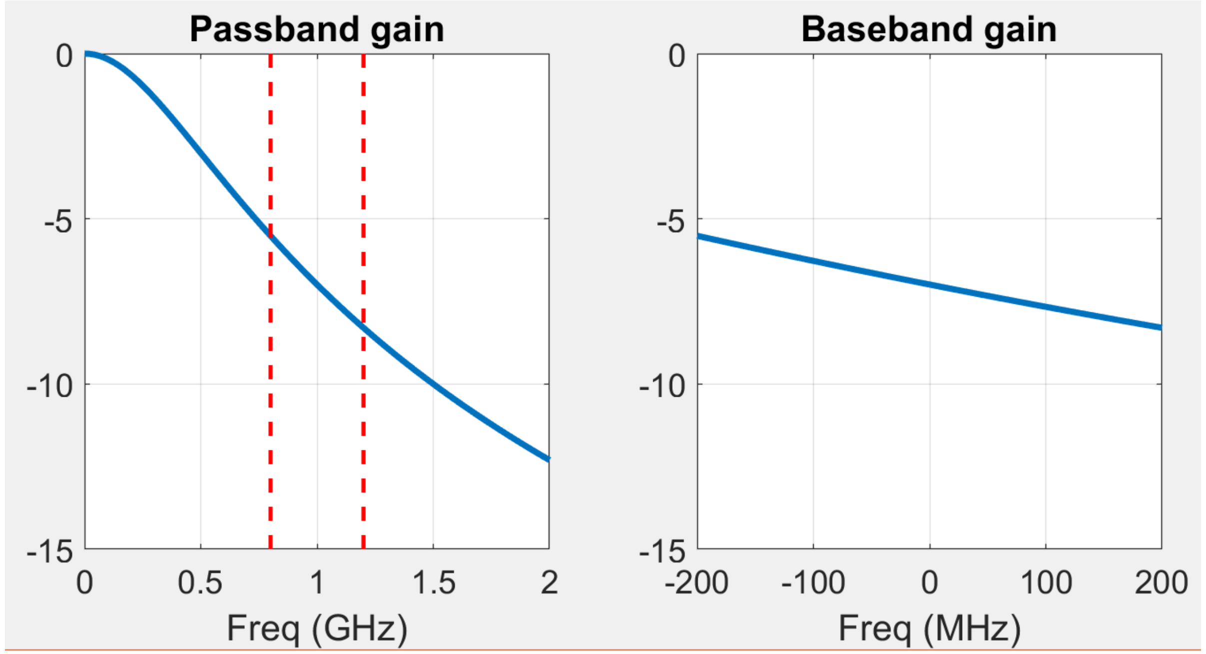
- Delay adds a constant phase rotation

Example: First Order Filter

- Passband: $H_p(\omega) = \frac{1}{1+j\omega/\omega_0}$
- Effective baseband: $H(\omega) = \frac{1}{1+j(\omega+\omega_c)/\omega_0}$
- Observe baseband response is:
  - Almost flat
  - Not symmetric around $f = 0$

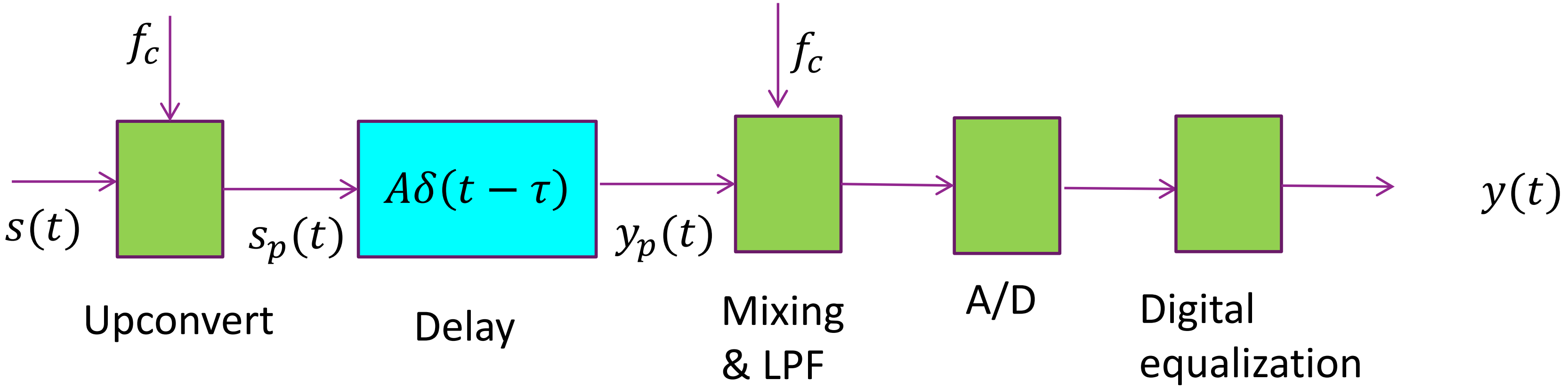
```
fp = 1e9*linspace(0,2,128)';
Hp = freqs(G0,[1/w0 1], 2*pi*fp);
plot(fp/1e9, 20*log10(abs(Hp)), 'Linewidth', 3);
```

```
fb = linspace(-2e8,2e8,128)';
Hb = freqs(G0, [1/w0, 1+1i*fc/f0], 2*pi*fb);
plot(fb/1e6, 20*log10(abs(Hb)), 'Linewidth', 3);
```



Passband cutoff freq $f_0 = 0.5$ GHz
Carrier freq $f_c = 1$ GHz

Delay and Synchronization



- Two methods to compensate for delay at the RX

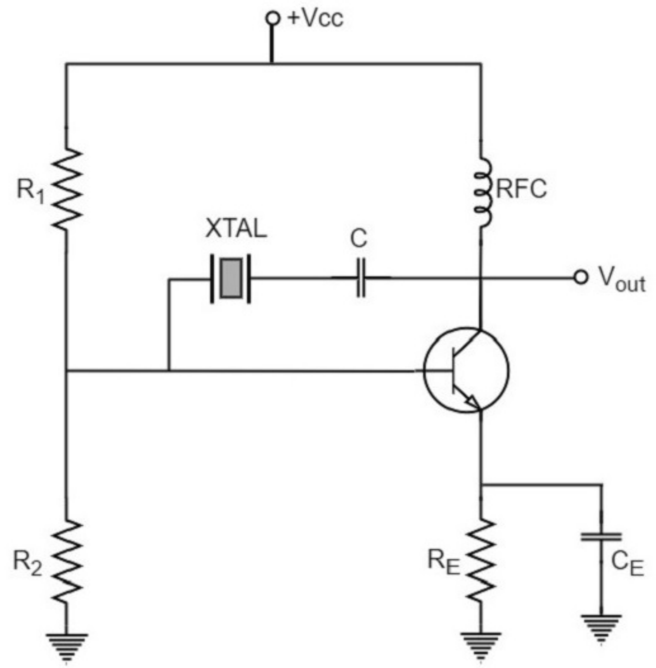
- Method 1: Correct in analog by adjusting phase of LO
- Method 2: Correct digital by inverting the gain $Ae^{2\pi j f_c \tau}$
  - This is a special case of equalization

Outline

- Time-Domain Relationships
- Fourier Transform Review
- Frequency-Domain Relationships
- Power and Energy Spectra
- Baseband equivalent filters
- Practical up and down-conversion circuits
- Wireless channels

Local Oscillator Crystal Source

- We saw that the mixer requires a sinusoidal input
  - Called the local oscillator (LO)
  - Sinusoidal input frequency should = desired carrier frequency
- The LO is often generated by a crystal
- Crystal has a resonant frequency
- A circuit then amplifies resonant frequency
- Resonant frequency is sometimes voltage controlled
  - VCXO (Voltage-controlled crystal oscillator)
  - Allows tunability

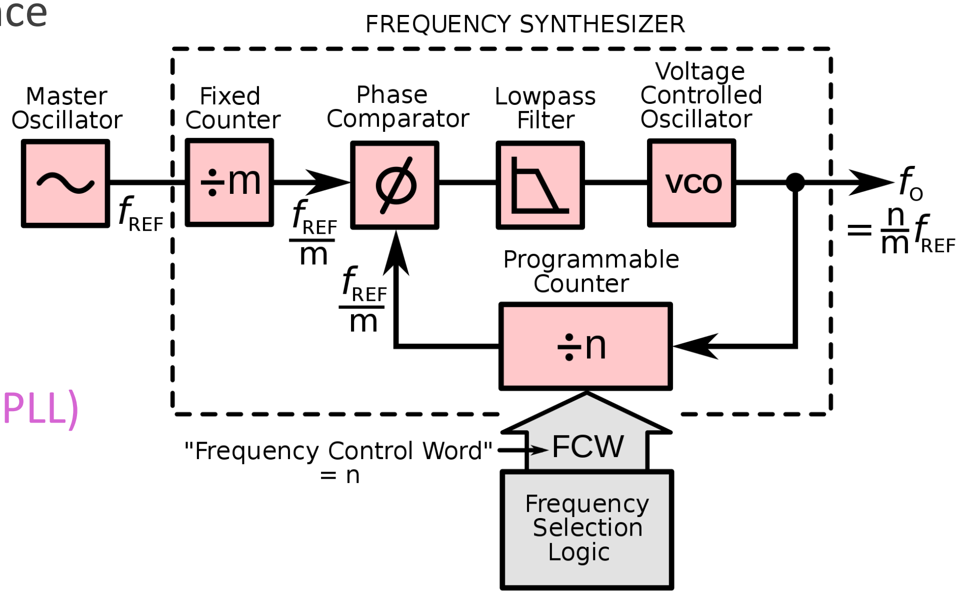


Frequency Synthesis

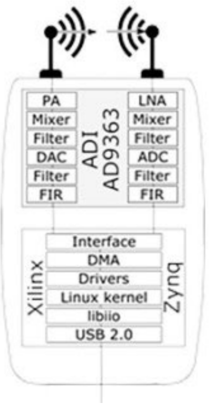
- The crystal oscillator may not directly produce desired carrier frequency
  - Frequency is constrained by resonant properties of the crystal
- Frequency synthesizer
  - Creates a desired LO at carrier frequency from the crystal reference
- Typically creates a rational multiple

$$f_0 = \frac{n}{m} f_{REF}$$

  - f_{REF} = Frequency from crystal
  - f_0 = Output frequency (usually the carrier)
  - Performed by a clock multiplier, divider and phased locked loop (PLL)



Example: The ADALM-Pluto



- The SDR we use in the lab
- The mixing is performed in ADI 9363 chip
- Simple architecture
  - But basic steps similar in more complex devices
  - Good to look at as a starting point
- Let's look inside!

Example Up and Down-Conversion Circuit

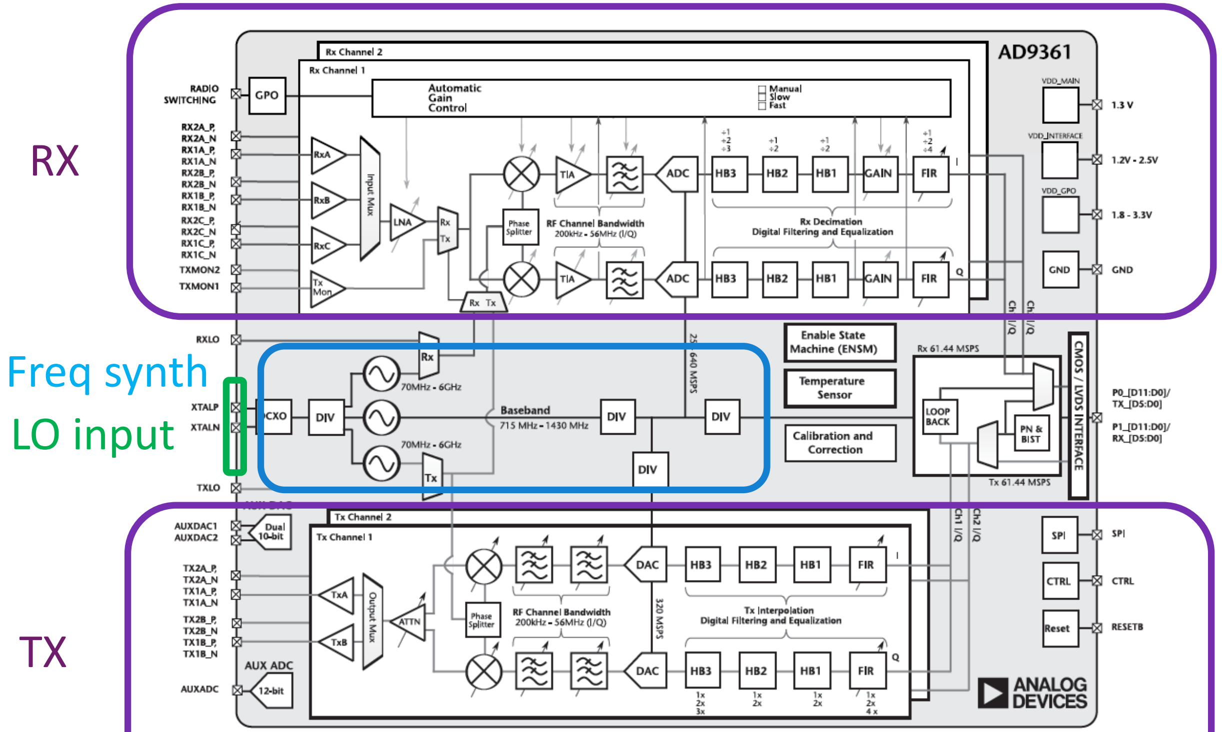
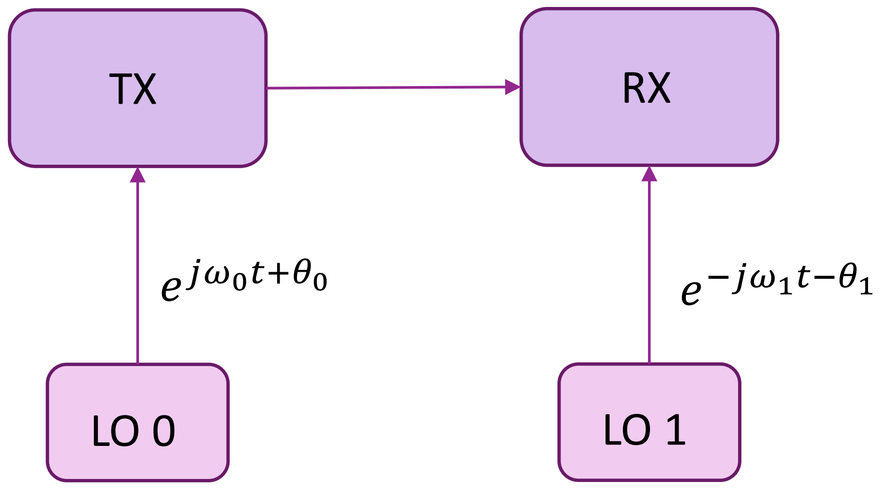


Figure 1.8 Integrated ZIF architecture used in the Pluto SDR.

- ❑Analog Devices AD9361 Wideband TXCR
  - Single integrated circuit
- ❑External local oscillator (LO) from crystal
  - Followed by frequency synthesizer
- ❑Receiver (RX):
  - Left to right
  - LNA, tunable mixer, ADC, Filters
- ❑Transmitter (TX):
  - Right to left
  - Filters, FAC, tunable mixer, power amplifier

Frequency Errors

- Two sources of errors in actual mixers
  - Gains are not matched at TX and RX
  - Frequencies are not matched at TX and RX (used different crystal sources)
- Suppose
  - Upconversion: $u_p(t) = Re\left(u_{TX}(t)e^{j\omega_0 t+\theta_0}\right)$
  - Downcoversion: $u_{RX}(t) = G u_p(t)e^{-(j\omega_1 t+\theta_1)}$ + LPF
- Then:
  - $u_{RX}(t) = G u_{TX}(t)e^{j((\omega_0-\omega_1)t+(\theta_0-\theta_1))}$
- Causes a carrier frequency offset (CFO)



Parts Per Million

- Oscillator error often measured in parts per million (ppm):

$$\Delta(\text{ppm}) := \frac{|f_c - f_c'|}{f_c}(10)^6$$

  - f_c = desired carrier frequency
  - $f_c{}'$ = actual carrier frequency

- Example:
  - f_c = 2.5 GHz, $\Delta = 10$ ppm (typ value for low-cost oscillator)
  - Then,

$$|f_c - f_c'| = (2.5)(10)^9(10)(10)^6 = 25 \text{ kHz}$$

  - Very large frequency shift!

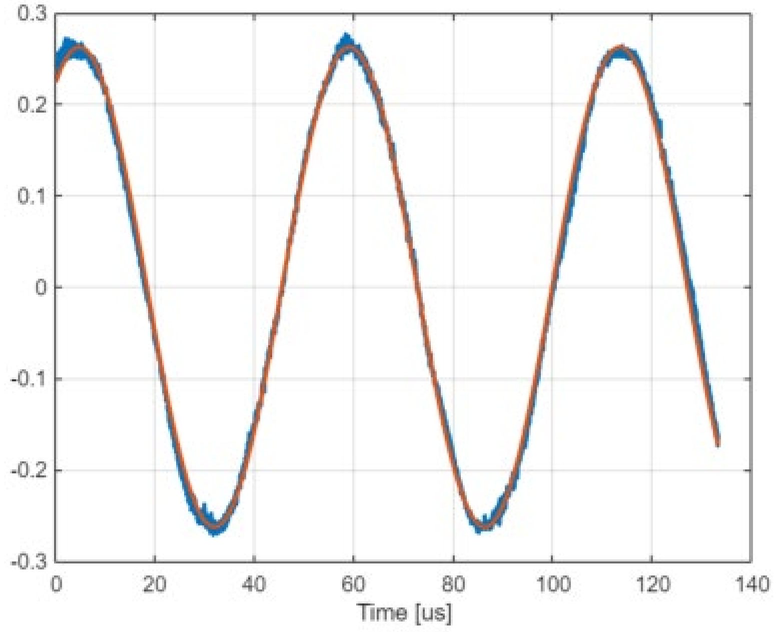
Lab Preview

Frequency Estimation and Carrier Frequency Offset

Table of Contents

Complex exponentials are the most fundamental signals for all frequency domain analysis of linear system. In this lab you will learn to:

- Send a complex exponential or continuous-wave (CW) signal through the SDR
- Estimate the complex gain and frequency of a sinusoid via (1) correlation method, (2) FFT method
- Estimate the carrier frequency offset
- Save data for files for offline processing



- Lab 2 in the [SDR lab github]

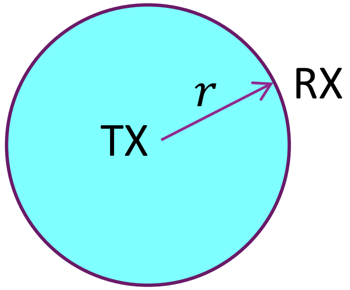
Outline

- ❑Time-Domain Relationships
- ❑Fourier Transform Review
- ❑Frequency-Domain Relationships
- ❑Power and Energy Spectra
- ❑Baseband equivalent filters
- ❑Practical up and down-conversion circuits
- ❑Wireless channels

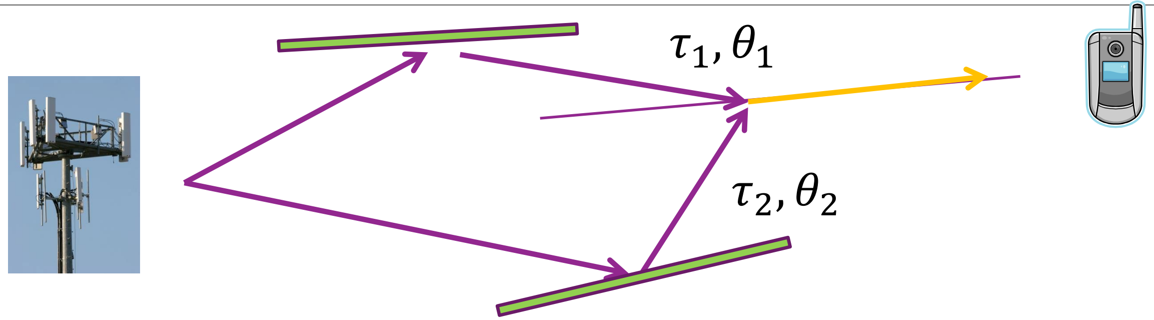
Free Space Wireless Channels

- Free space propagation
  - No obstacles
  - Isotropic (equal power in all directions)
- Power decreases as r^{-2}
  - $\Rightarrow$ Gain = Ar^{-1} for some A
- Delay is $\tau = r/c, c = 3(10)^8$ m/s
- Hence, baseband channel is:

$$h(t) = \frac{A}{r} e^{\frac{j2\pi f_c r}{c}} \delta(t - r/c)$$



Multipath

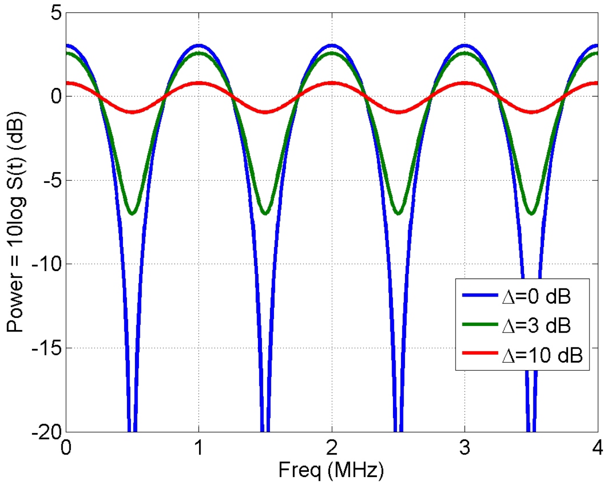


- Wireless signals can arrive in many directions
  - Reflections, diffraction, transmission, …
- Each path will have different gain and delay
- Receiver sees the combined total

$$h(t) = \sum_{k=1}^{K} h_k e^{-2\pi j f_c \tau_k} \delta(t - \tau_k)$$

  - h_k = complex gain of each path

Two-Path Example



Magnitude response

$$S(f) = |H(f)|^2 = \left|h_1 e^{2\pi i f \tau_1} + h_2 e^{2\pi i f \tau_2}\right|^2$$

Plot shows:

$$\tau_2 - \tau_1 = 1\ \mu s,\ \ |h_1|^2 + |h_2|^2 = 1,\ \ |h_2|^2 = 10^{0.1\Delta}|h_1|^2$$

- Rate of variation in frequency depends on delay spread: $\tau_2 - \tau_1$
- Size of variation depends on spread of path gains:
  - Average $S(t) = |h_1|^2 + |h_2|^2$
  - Min $S(t) = (|h_1| - |h_2|)^2$, Max $S(t) = (|h_1| + |h_2|)^2$

Example Problem (On board)

- A wireless channel has 2 paths:
  - Path 1: Power gain of -80 dB, travels 100m
  - Path 2: Power gain of -83 dB, travels 120m
- What are the amplitude gains of the two paths, h_1, h_2?
- What are the two delays of the paths: τ_1, τ_2?
- What is the average, minimum and maximum power?