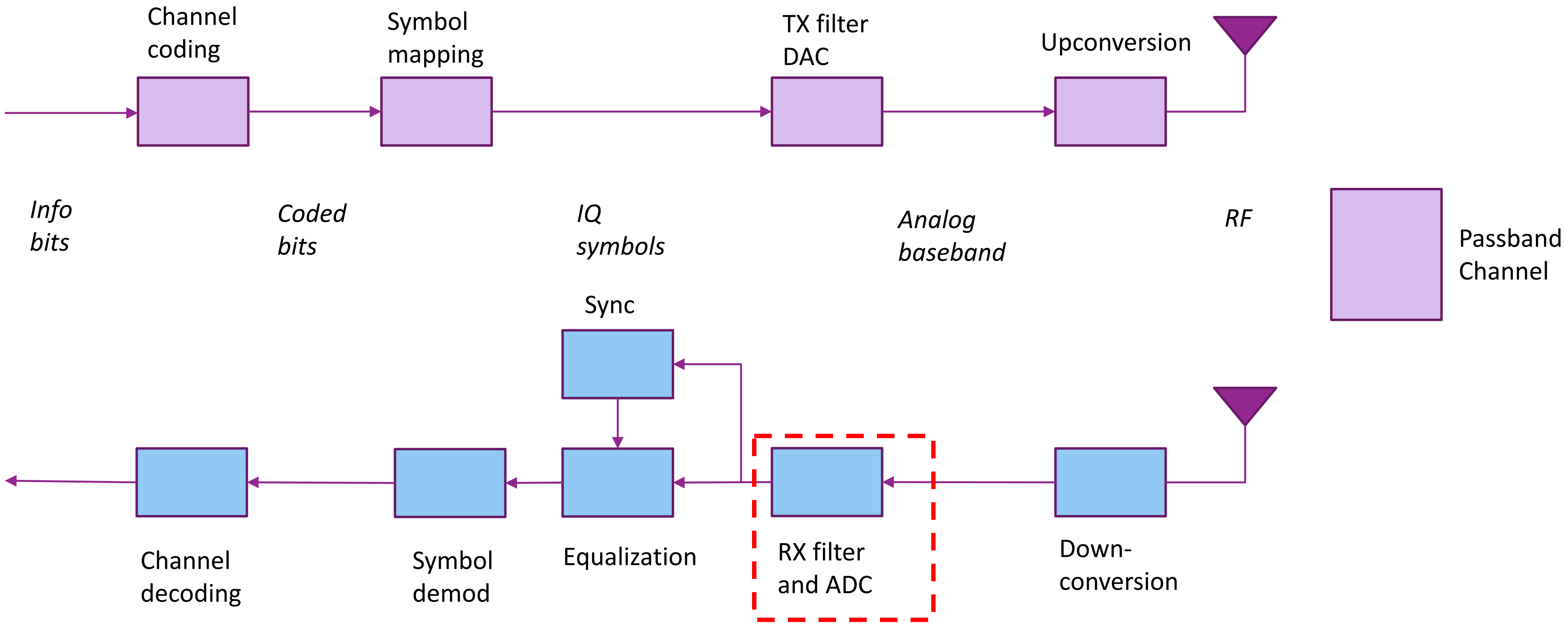
Unit 3: Receive Filtering

EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN

This Unit



Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

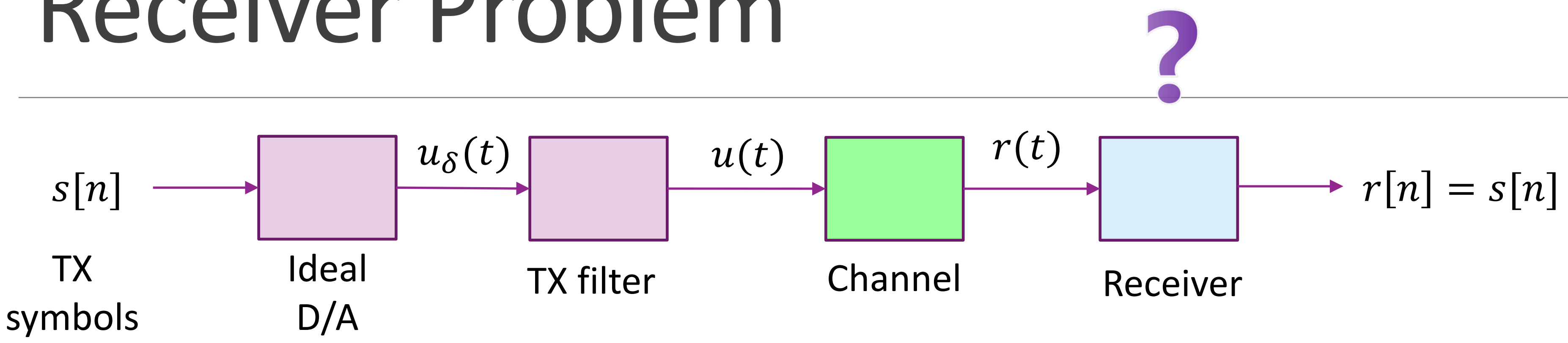
Learning Objectives

- Describe the steps in recovering symbols for a linearly modulated signal
  - Determine the matched filter response
- Determine MF for known gain and delay in the channel
- Compute the effective discrete-time channel given
  - Channel response, TX an RX filter
  - Time-domain or frequency-domain method
- Determine if there is ISI
- Compute the frequency response using digital RX filtering and downsampling
- Determine specifications on the digital and analog filters

Outline

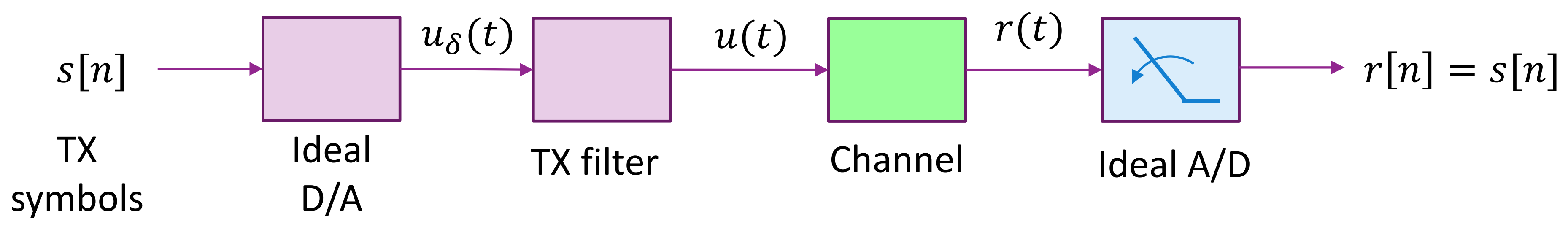
- ❑ Receiver filtering and sampling
- ❑ Perfect reconstruction with orthonormal modulation
- ❑ General channels: Time-domain analysis
- ❑ General channels: Frequency-domain analysis
- ❑ PSD Analysis
- ❑ Practical RX filter design
- ❑ Channel sounding

Receiver Problem

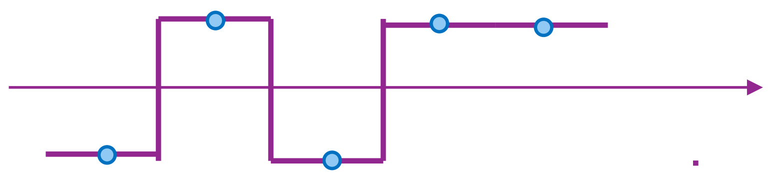


- Transmit steps so far:
  - Symbols $s[n]$
  - Linearly modulate: $u(t) = \sum s[n] p_{tx}(t - nT)$
  - Baseband equivalent channel $r(t) = h_{chan}(t) * u(t)$
- Question at the receiver: Can we recover the transmitted symbols?
- Want a mapping that input $r(t)$ to samples $r[n]$
- Ideally $r[n] = s[n]$

Simple Idea: Sampling

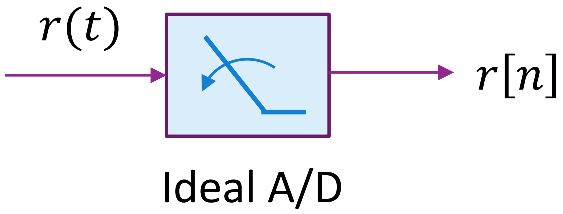


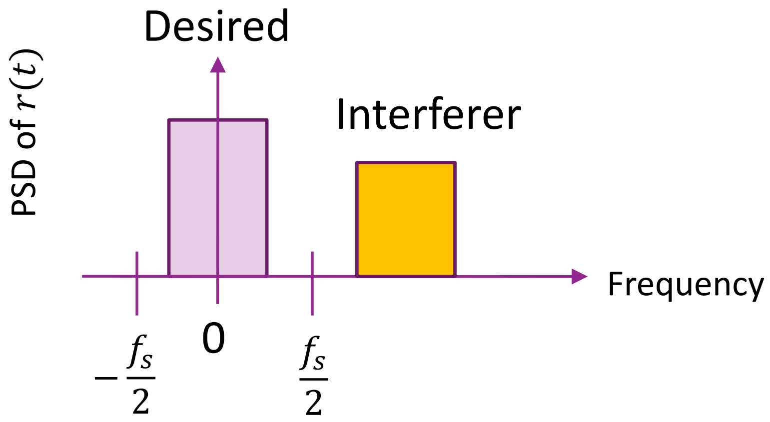
- Take samples with an ideal A/D: $r[n] = r(nT)$
- This could work: Example
  - Suppose $p_{tx}(t) = Rect\left(\frac{t}{T}\right)$ (Ideal zero-order-hold D/A)
  - $h_{chan}(t) = \delta(t)$ (no channel effect)
  - Then: $r(t) = u(t) = \sum_n s[n]\, Rect\left(\frac{t-nT}{T}\right)$
  - So, $r(nT) = s[n]$
  - Hence, if we sample at exactly the right time, we can recover $s[n]$



Problems with Ideal Sampling

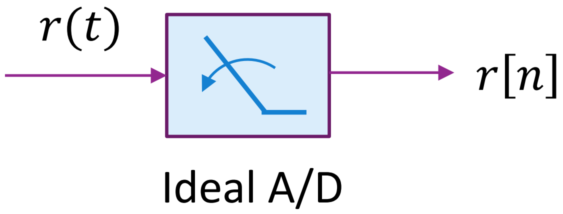
- Three problems in implementing an ideal A/D
- Problem 1: No circuit exactly samples at one instant
  - Most circuits integrate over some period
  - Ex: Charge fills a capacitor at the input to the A/D
- Problem 2: Out-of-band emissions ("blockers")
  - The received signal may contain signals at neighboring frequencies
  - Ex: Transmissions in other wireless channels
  - The system may not have control over these
  - Without filtering, these will be aliased into $r[n]$

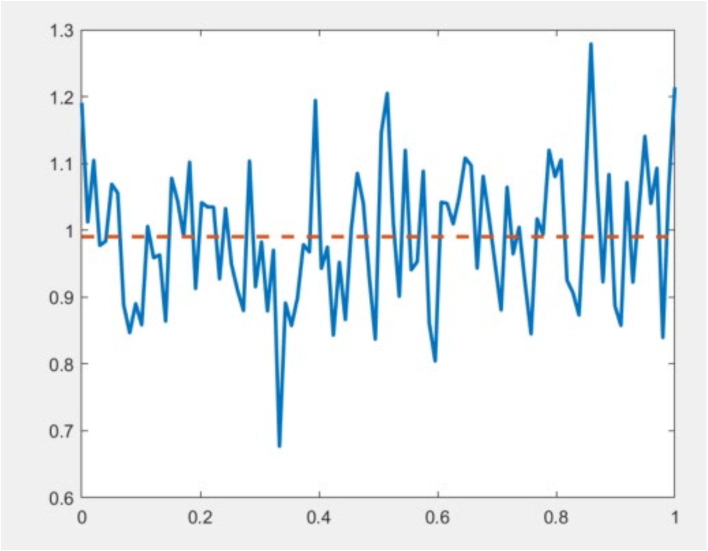




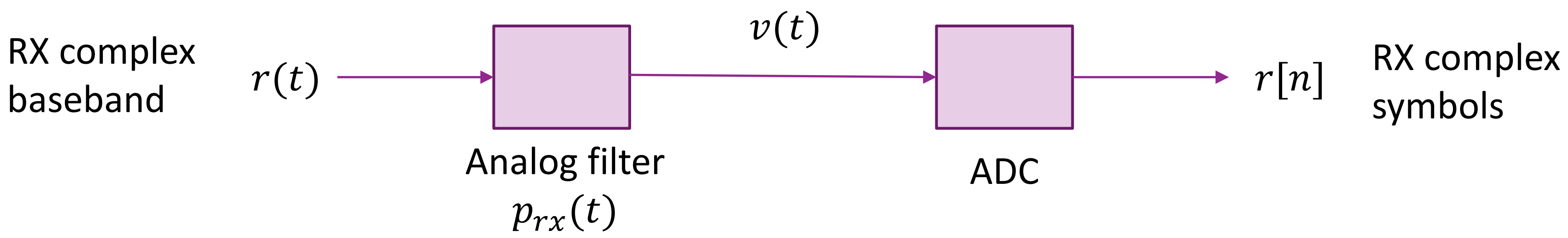
Problems with Ideal Sampling

- Three problems in implementing an ideal A/D
- Problem 3: No noise filtering
  - Suppose that in some symbol period:
    $$r(t) = s[n] + w(t), \qquad t \in [nT, (n+1)T)$$
  - $r(t) =$ "desired signal" + "noise".
  - Noise $w(t)$ will appears in any sample.
  - But suppose we average: $r_{avg}[n] = \frac{1}{T}\int_{nT}^{(n+1)T} r(t)dt = s[n] + v[n]$
  - Effective noise is $v[n] = \frac{1}{T}\int_{nT}^{(n+1)T} w(t)dt$
  - This will, in general, have a lower variance
  - We will describe this more next unit





Two Step Receiver



- Discussion motivates a two-step process
- Step 1: Receive filter: $v(t) = p_{rx}(t) * r(t)$
  - $p_{rx}(t)$ is the RX filter response
- Step 2: Sample $r[n] = v(nT)$
- Filter is useful to:
  - Model imperfections in the sampling
  - Filter out blockers. Anti-aliasing
  - Average out noise (more on this later)

Outline

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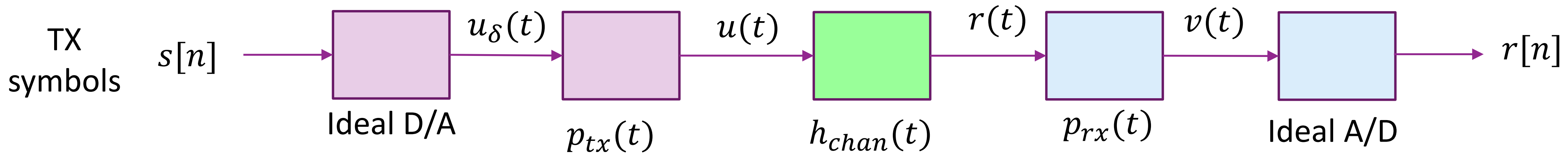
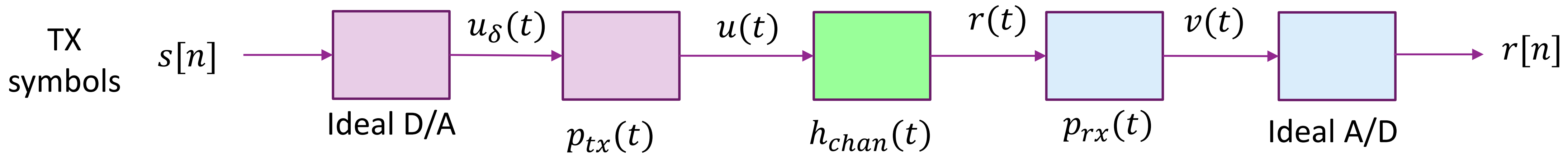
End-to-end TX and RX Chain so Far



- Steps:
  - Impulse D/A: $u_\delta(t) = \sum s[n]\delta(t-nT)$
  - TX pulse shape: $u(t) = u_\delta(t) * p_{tx}(t) = \sum s[n] p_{tx}(t-nT)$
  - Channel: $r(t) = u(t) * h_{chan}(t)$
  - RX filter: $v(t) = r(t) * p_{rx}(t)$
  - Sampling A/D: $r[n] = v(nT)$

Basic Questions



- Under what circumstances can we construct transmitted signals.
- That is, how do we select $p_{rx}(t)$ such that $r[n] = s[n]$?
- We first analyze this for a simple case:
  - Orthonormal pulse shapes
  - No channel impairments

Inner Products and Orthonormal Signals

- Let $f(t), g(t)$ be two complex-valued signals
- Definition 1: The inner product of f, g is:

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(t) g(t) dt$$

  - Here $f^*(t)$ =complex-conjugate of $f(t)$
- Definition 2: We say $f(t), g(t)$ are orthogonal if $\langle f, g \rangle = 0$
  - We will write this as $f \perp g$
- Definition 3: The signal energy is $\|f\|^2 := \langle f, f \rangle = \int_{-\infty}^{\infty} |f(t)|^2 dt$
- We will discuss this in much more detail in the next unit on signal spaces

Example Problem

- Suppose $f(t) = aRect\left(\frac{t}{T}\right), g(t) = (b + ct)Rect\left(\frac{t}{T}\right)$
  - Complex a, b, c with $a \neq 0$
- Compute $\langle f, g \rangle$
- When is $f \perp g$?
- Solution:
  - $\langle f, g \rangle = \int_{-\infty}^{\infty} f^*(t)g(t)dt = \int_{-T/2}^{T/2} a^*(b + ct)dt = a^*bT$
  - Therefore $f \perp g = 0 \Leftrightarrow \langle f, g \rangle = a^*bT = 0.$
  - Since $a, T \neq 0, f \perp g = 0 \Leftrightarrow b = 0$

Orthogonality in Frequency Domain

- Sometimes it is more convenient to evaluate inner products in frequency domain
- Parseval's Theorem: Let $f(t), g(t)$ be any two signals. Then:

$$\langle f, g \rangle = \int f^*(t) g(t) dt = \int F^*(f) G(f) df$$

- This is useful whenever the Fourier transforms $F(f), G(f)$ are simple to work out.

Example

- Suppose $f(t) = A\ sinc\left(\frac{t}{T}\right),\ g(t) = Bsinc\left(\frac{t-\tau}{T}\right)$
- Compute $\langle f, g\rangle$. When are they orthogonal?
- Solution:
  - Do this in frequency domain
  - $F(f) = AT\ Rect(fT),\ G(f) = BTRect(fT)e^{-2\pi i f\tau}$
  - From Parseval's Theorem: Let $f_0 = 1/(2T)$

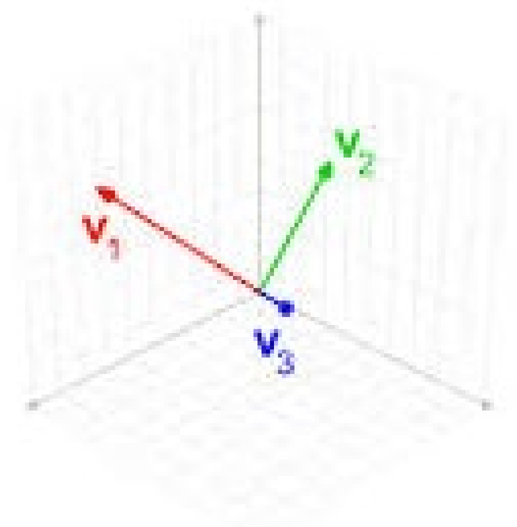
$$\langle f, g\rangle = \langle F, G\rangle = \int F^*(f)G(f)df = ABT^2 \int_{-f_0}^{f_0} e^{-2\pi i f\tau} df$$

$$= \frac{ABT^2}{2\pi i\tau}\left[e^{2\pi i f_0\tau} - e^{-2\pi i f_0\tau}\right] = ABTsinc(2f_0\tau) = ABT\ sinc\left(\frac{\tau}{T}\right)$$

  - $\langle f, g\rangle = 0$ when $\tau = kT$ for some integer k

Orthonormal Signals

- Let $\phi_n(t), n = 0,1,\ldots$ be a set of signals
  - This can be indexed from $n = -\infty$ to ∞ as well
- Definition: The set $\phi_n(\cdot)$ is orthonormal if:
  - $\|\phi_n\| = 1$ for all n (all signals have unit energy)
  - $\langle \phi_n, \phi_m \rangle = 0$ for all $n \neq m$ (different signals are orthogonal)



- This generalizes the concept of orthonormal vectors
- We will discuss orthonormal sets much more in signal space theory

Orthonormal Pulses and Matched Filtering

- Consider the linear modulation: $u(t) = \sum_n s[n] p_{tx}(t - nT)$
- Definition 1: We will say that the modulation is orthogonal if

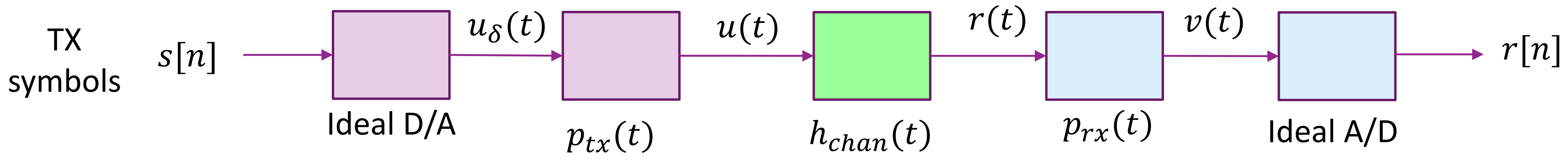
$$\phi_n(t) = p_{tx}(t - nT), \qquad n = \cdots, -2, -1, 0, 1, 2, \ldots$$

  is an orthonormal set.
  - We say $p_{tx}(t)$ is an orthonormal pulse at sample rate $\frac{1}{T}$.
- Definition 2: Given any transmit pulse, $p_{tx}(t)$, the matched filter RX pulse is:

$$p_{rx}(t) = p_{tx}^*(-t)$$

  - The pulse that is complex conjugate and flipped in time
  - Note that if TX filter is causal, RX filter is anti-causal

Reconstruction With Orthonormal Pulses



- Theorem: Suppose that:
  - $p_{tx}(t)$ is an orthonormal pulse at sample rate $\frac{1}{T}$
  - $h_{chan}(t) = \delta(t)$ (i.e., $r(t) = u(t)$ so there are no channel impairments)
  - $p_{rx}(t) = p_{tx}^*(-t)$ (RX uses matched filter)

  Then the receiver will exactly recover the TX samples in that $r[n] = s[n]$
- Theorem answers our question. We can reconstruct the RX samples
  - Under several assumptions

Proof of Reconstruction Theorem

- TX signal is: $u(t) = \sum_n s[n] p_{tx}(t - nT)$
- Since $h_{chan}(t) = \delta(t) \Rightarrow r(t) = h_{chan}(t) * u(t) = u(t) = \sum_n s[n] p_{tx}(t - nT)$
- RX filtered signal is: $v(t) = p_{rx}(t) * u(t) = \sum_n s[n] (p_{rx} * p_{tx})(t - nT)$
- Sampling is: $r[m] = v(mT) = \sum_n s[n] (p_{rx} * p_{tx})((m - n)T)$
- Now look at convolution:

$$(p_{rx} * p_{tx})(t) = \int p_{rx}(t - s) p_{tx}(s) ds = \int p_{tx}^*(s - t) p_{tx}(s) ds$$

- At $t = kT$: $(p_{rx} * p_{tx})(kT) = \int p_{tx}^*(s - kT) p_{tx}(s) ds$
- But $\phi_k(t) = p_{tx}(t - kT)$ is an orthonormal set
- So, $(p_{rx} * p_{tx})(kT) = \langle \phi_k, \phi_0 \rangle = \delta_k$
- Hence: $r[m] = v(mT) = \sum_n s[n] \delta_{m-n} = s[m]$

"Practical" Orthogonal Pulses

- There are two important "practical" orthonormal pulses
- Rectangles: $p_{tx}(t) = \frac{1}{\sqrt{T}} \mathrm{Rect}\left(\frac{t}{T}\right)$
  - Orthonormal since $p_{tx}(t - nT)$ and $p_{tx}(t - mT)$ do not overlap when $n \neq m$
  - Scaling by $\frac{1}{\sqrt{T}}$ ensures they are normalized
  - This can be achieved (with some scaling) by a zero-order hold ADC
- Sinc pulses: $p_{tx}(t) = \frac{1}{\sqrt{T}} \mathrm{Sinc}\left(\frac{t}{T}\right)$
  - Use similar frequency domain calculation as before to prove these are orthonormal
  - This would arise with ideal filtering at the TX and RX.
  - No filter is exactly ideal.
  - But, practical filters get quite close to this response.

In-Class Exercise

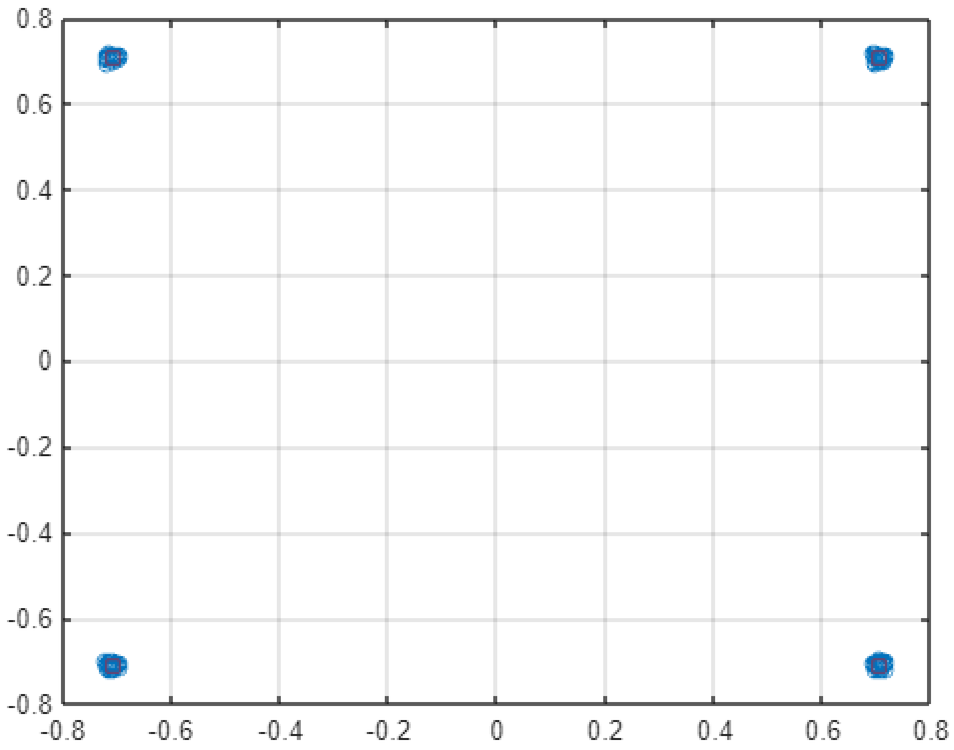
Simulating a Simple TX and RX Filter

We will first simulate a raised cosine TX and RX filter with no channel impairments.

First, generate a set of QAM symbols with the following parameters.

```
nsym = 128;              % number of symbols
nbitsPerSym = 2;         % num bits / sym (2=QPSK)
nbits = nbitsPerSym*nsym;  % num bits
fsym = 10e6;             % symbol rate
M = 2^nbitsPerSym;       % QAM order


% TODO:  Generate random bits and QPSK symbols
%     b = 
```



Outline

- Receiver filtering and sampling
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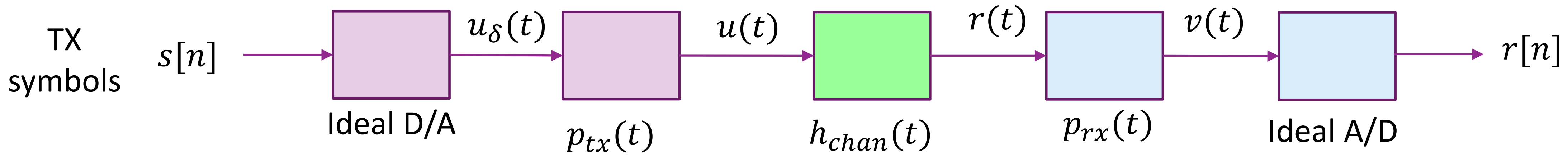
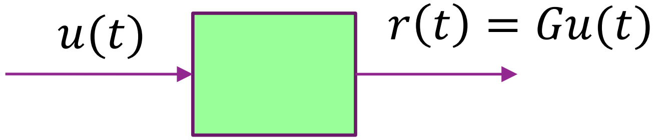
Modeling the End-to-End System



- We have seen so far that when:
  - Pulse shapes are matched
  - Modulation is orthonormal
  - No channel impairments
- Then: $r[n] = s[n]$
- What happens when these conditions fail?

Channel Gain

- Consider simple deviation: Channel gain $r(t) = Gu(t)$
  - Gain can be due to attenuation in wire, for example.
- Suppose, as before, that:
  - $p_{tx}(t - nT)$ are orthonormal for different n
  - $p_{rx}(t) = p_{tx}^{*}(-t)$, i.e. matched filter
- Then $r[n] = Gs[n]$
  - Proof on board
  - Simply scales symbols.
  - Can recover symbols from $r[n]/G$
- But, requires that gain is known. More on this later



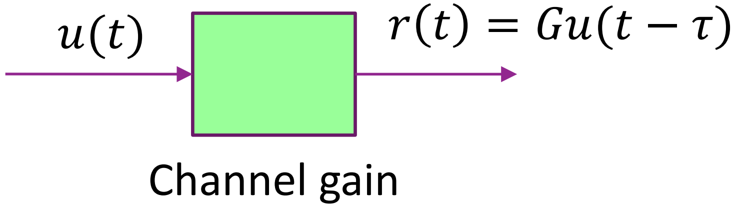
Channel gain

Channel Gain and With Known Delay

- Now consider gain and delay
  - $r(t) = Gu(t - \tau)$
- Then, $r(t) = \sum G s_n p_{tx}(t - \tau - nT)$
- Suppose gain and delay are known
- Use shifted and scaled receive filter

$$p_{rx}(t) = \frac{1}{G} p_{tx}^{*}(-t + \tau)$$

- Then: $r[n] = s[n]$
  - Proof on board
- RX filter is shifted to delay
  - Must know the gain and delay
  - Requires synchronization



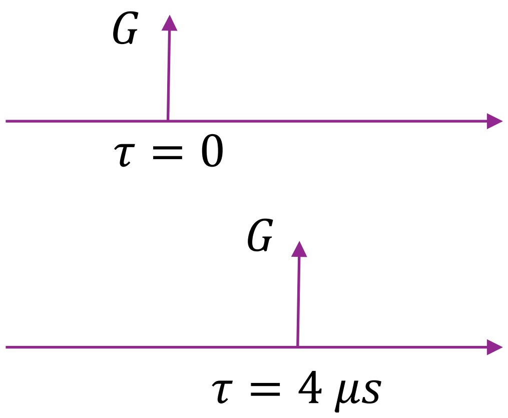
Integer Delays

- Suppose: $r(t) = Gs(t - \tau)$
  - Delay is an integer multiple of the sample period: $\tau = k_0 T$
- Suppose, as before:
  - TX uses orthonormal modulation
  - MF receiver $p_{rx}(t) = p_{rx}^*(-t)$ (but, not shifted and scaled)
- Then: $r[n] = Gs[n - k_0]$
  - Channel delay of $\tau = k_0 T \Rightarrow$ Symbol delay of k_0
  - Proof on board

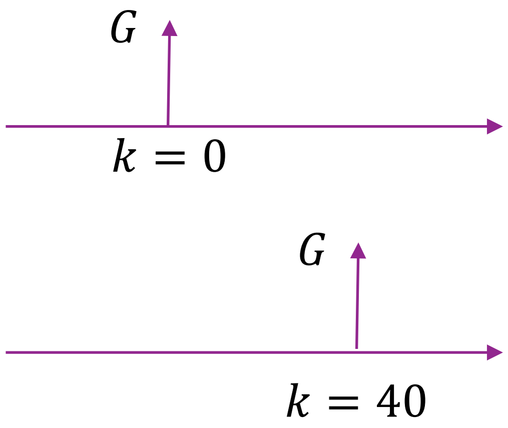
Integer Delays visualized

- Suppose TX uses orthonormal modulation and RX uses matched filter
- Suppose sample rate is $T = 0.1\ \mu s$ ($f = 10$ Msym/s)

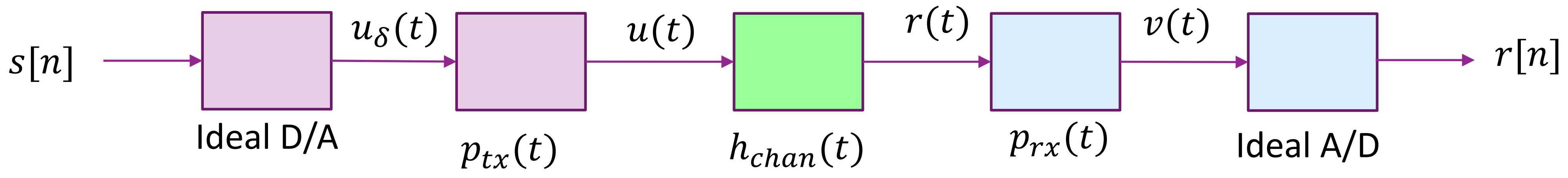
Baseband channel impulse response



Effective discrete-time channel



General case



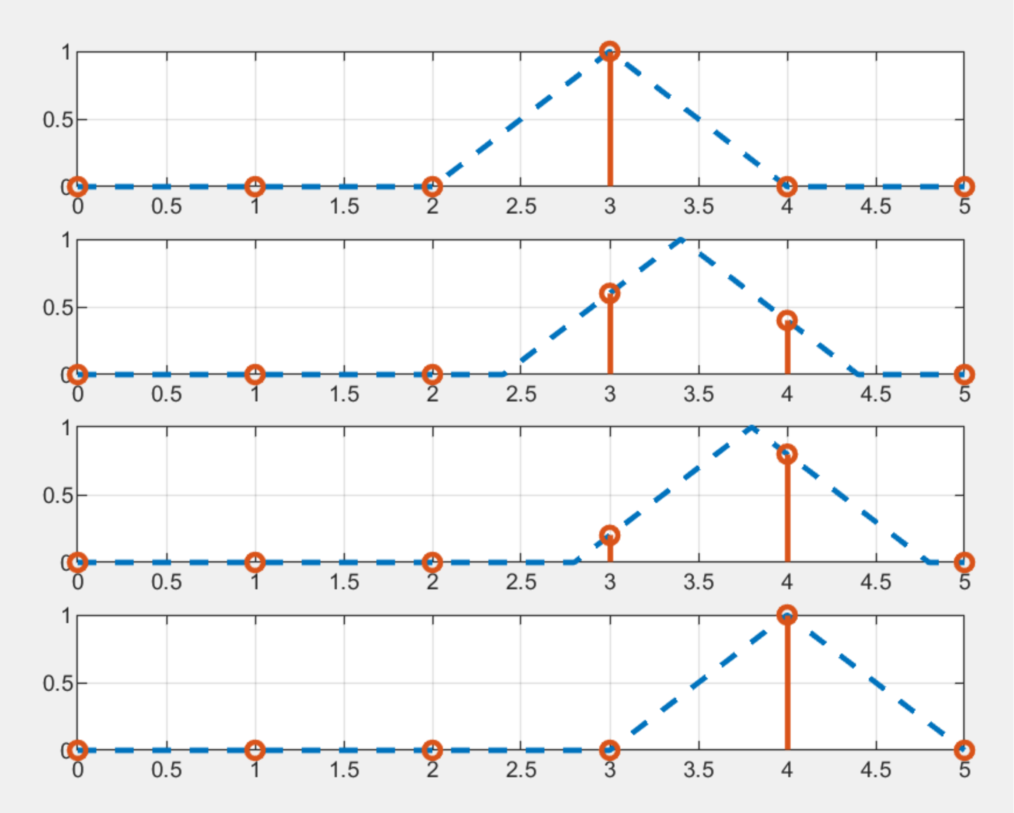
- Define channel impulse response with filtering $g(t) = p_{tx}(t) * h_{chan}(t) * p_{rx}(t)$
  - Represents path from DAC output to ADC input
- Theorem: Mapping from $r[n]$ to $s[n]$ is LTI with impulse response is $h[n] = g(nT)$,
- Receive symbols will be given by $r[n] = \sum_{k=-\infty}^{\infty} h[k]s[n-k]$

Example 1: Rectangular Pulse

- Suppose that $p_{tx}(t) = p_{rx}(t) = \; Rect(t/T)$
- If channel is $r(t) = Gu(t - \tau)$, what is effective DT channel $h[n]$
- Solution:
  - Impulse response is $h_{chan}(t) = \delta(t - \tau)$
  - Impulse response from $u_\delta(t) \mapsto v(t)$ is $g(t) = p_{tx}(t) * p_{rx}(t) * h_{chan}(t) = GT\ \mathrm{Tri}(\mathrm{t} - \tau)$
  - Then $h[n] = g(nT)$

Example 2 Illustrated

- Channel response with filtering
$$g(t) = GT\,Tri(t-\tau)$$
- Effective discrete-time channel:
$$h[n] = g(nT)$$
- Plot to right: $GT = 1, T = 1$
  - $g(t)$: Blue dashed
  - $h[n]$: Red stem
- When $\tau = kT$ is an integer:
  - $h[n] = \delta_{n-k}$
  - Single tap
- When $\tau \in (kT, (k+1)T)$
  - $h[n]$ has two taps



$\tau = 3$

$\tau = 3.4$

$\tau = 3.8$

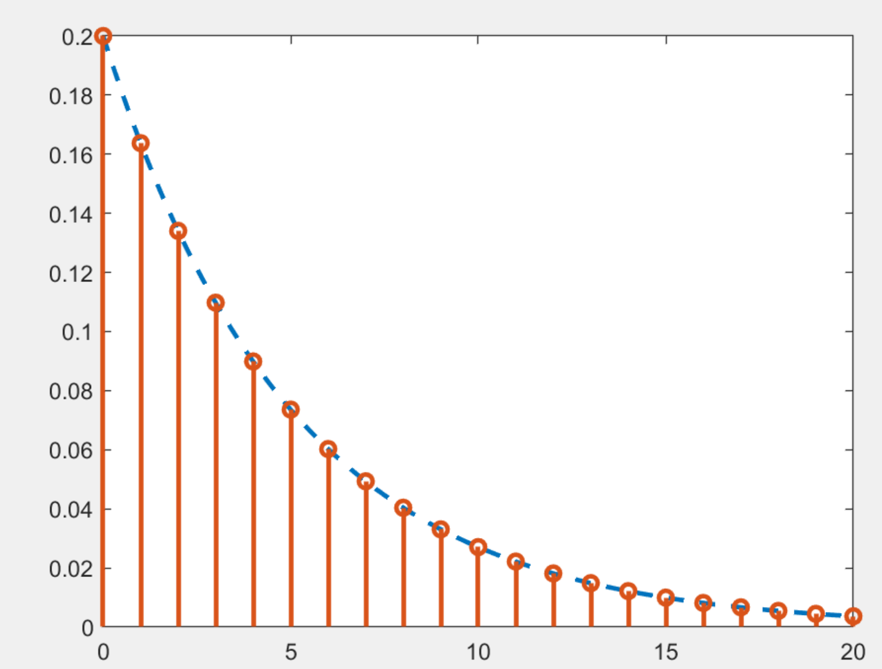
$\tau = 4$

Example 2: Exponential

- Suppose that $p_{tx}(t) = \delta(t), p_{rx}(t) = T\delta(t)$
- Baseband channel is: $\frac{dr(t)}{dt} = \frac{\alpha}{T}(u(t) - r(t))$
- Find filtered channel response $g(t)$ and effective discrete-time response $h[n]$
- Solution:
  - Take Laplace transform of differential eqn: $R(s) = \frac{c}{s+c}U(s),\ c = \frac{\alpha}{T}$
  - Using inverse Laplace transform: $r(t) = h_{chan}(t) * u(t)$ where $h_{chan}(t) = ce^{-ct}1_{[0,\infty)}(t)$
  - Filtered channel response: $g(t) = p_{tx}(t) * p_{rx}(t) * h_{chan}(t) = cTe^{-ct}1_{[0,\infty)}(t)$
  - Discrete-time response: $h[n] = g(nT) = cTe^{-cnT}1_{\{n\geq 0\}} = \alpha e^{-\alpha n}\ 1_{\{n\geq 0\}}$

Example 2: Exponential Illustrated

- Suppose that $p_{tx}(t) = \delta(t), p_{rx}(t) = T\delta(t)$
- Baseband channel is: $\frac{dr(t)}{dt} = \frac{\alpha}{T}(u(t) - r(t))$
- From previous slide:
  - Filtered channel response: $g(t) = cTe^{-ct}1_{[0,\infty)}(t)$
  - Discrete-time response: $h[n] = g(nT) = \alpha e^{-\alpha n}\, 1_{\{n \geq 0\}}$



Effective Discrete-Time Channel

- Lack of synchronization causes channel of the form

$$r[n] = \sum_{k=-\infty}^{\infty} h[k]s[n-k]$$

- $h[k]$ called the effective discrete-time channel
- Inter-symbol interference:
  - Whenever $h[k] \neq 0$ for $k \neq 0$
  - Other symbols interfere with one another
- ISI occurs for many reasons:
  - Lack of synchronization
  - Channel impairments

ISI and Equalization

- Effective discrete-time channel is:

$$r[n] = \sum_{k=-\infty}^{\infty} h[k]s[n-k]$$

- System has inter-symbol interference (ISI):
  - Multiple symbols $s[n-k]$ effect $r[n]$
- The receiver must undo this ISI.
- This process is called equalization
- We will discuss this later.

In-Class Exercise

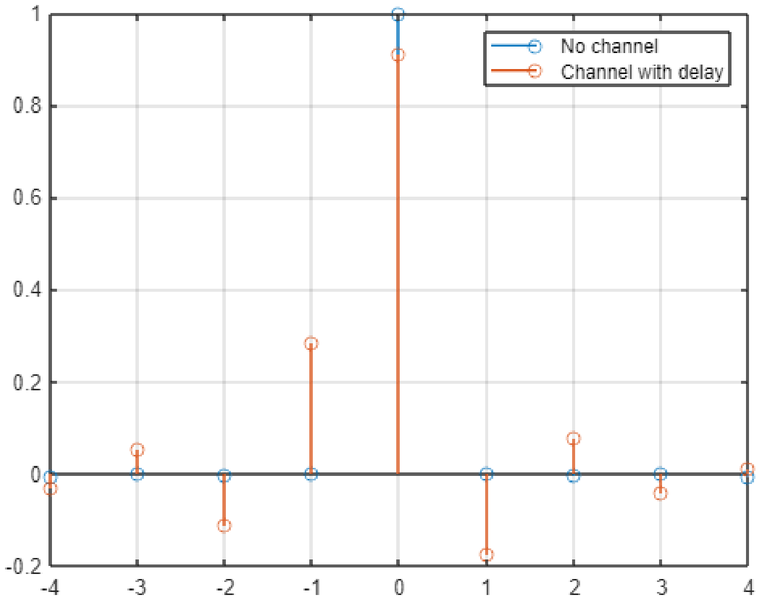
Computing an Effective Discrete-Time Channel

The baseband equivalent filter response is relative to some start time `t0` is:

```
hdis[n] = g(t0 + n*Tsym),   g(t) = ptx(t)*prx(t)*hchan(t)
```

`where Tsym` is the symbol period. We can approximately compute by performing the convolutions in discrete time,

```
hdis[n] = g[n*nov+1],   g[k] = ptx[k]*prx[k]*hchan[k]
```



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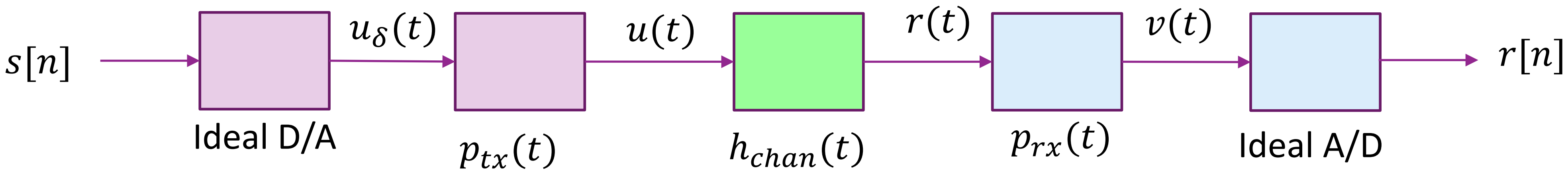
Digital Channel Frequency Response



- We saw that effective digital channel from $s[n] \mapsto u[n]$ is:

$$r[n] = \sum_k h[k]s[n-k], \quad h[k] = g(kT), \qquad g(t) = p_{rx}(t) * h_{chan}(t) * p_{tx}(t)$$

- Question: What is the frequency response?

$$R(\Omega) = H(\Omega)S(\Omega)$$

Frequency Response of DT Filter

- Fact from signals and systems: If $h[n] = g(nT)$

$$H(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G\left(\left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}\right)$$

- Shifted copies of $G(f)$
- Continuous frequency f mapped to DT frequency $\Omega = 2\pi f/f_s$, $f_s = 1/T$
- Can obtain coefficients from inverse DTFT:

$$h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(\Omega)e^{jn\Omega}d\Omega$$

Computing Effective DT Frequency Response
Summary

- Compute frequency response of channel with filtering $G(f) = P_{rx}(f)H_{chan}(f)P_{tx}(f)$
- Effective DT channel response is:

$$H(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G\left(\left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}\right)$$

  - Scale $G(f)$ vertically by $1/T$
  - Map continuous-time frequency f to $\Omega = 2\pi f T = \frac{2\pi f}{f_s}$
  - Create shifted versions every 2π
  - Shifted versions may overlap if there is aliasing

Bandlimited Channel

- Suppose one of P_{rx}, P_{tx} or G bandlimited to $|f| < \frac{1}{2T}$,

- Effective discrete-time channel reduces to
  - $H(\Omega) = \frac{1}{T} P_{rx}\left(\frac{\Omega}{2\pi T}\right) P_{tx}\left(\frac{\Omega}{2\pi T}\right) H_{chan}\left(\frac{\Omega}{2\pi T}\right)$ for $|\Omega| < \pi$

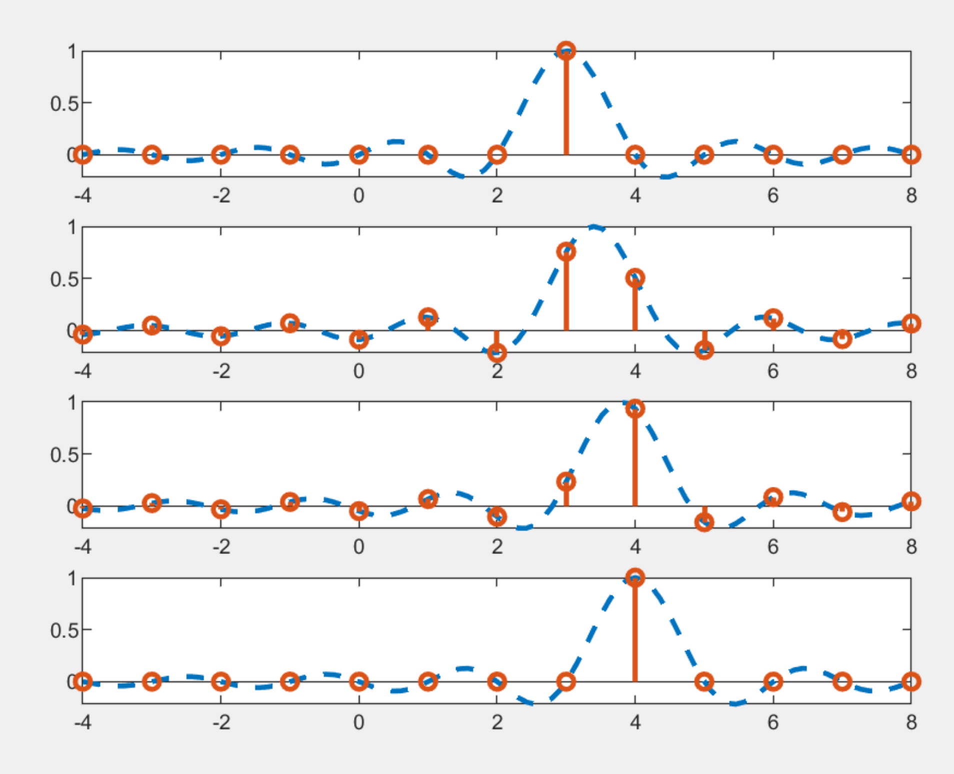
- If TX and RX filters are ideal low-pass:
  - $H(\Omega) = G\left(\frac{\Omega}{2\pi T}\right) P_{rx}(f) = P_{tx}(f) = \sqrt{T}\mathrm{Rect}(fT)$

Example: Sinc Pulses

- Suppose that
  - $p_{rx}(t) = p_{tx}(t) = \frac{1}{\sqrt{T}} sinc\left(\frac{t}{T}\right)$
  - $H_{chan}(f) = 1$ (no impairments)
- Then,
  - $P_{rx}(f) = P_{tx}(f) = \sqrt{T} rect(fT)$
  - $H(\Omega)$ =1
- Hence, $R(\Omega) = S(\Omega)$.
- Recover symbols exactly. No ISI

Example: Sinc Pulses with Delay

- Suppose that $p_{rx}(t) = p_{tx}(t) = \frac{1}{\sqrt{T}} sinc\left(\frac{t}{T}\right)$
- Channel has gain and delay: $r(t) = Gu(t - \tau)$
- $H_{chan}(f) = Ge^{-2\pi jf\tau}$
- Then,
  - $P_{rx}(f) = P_{tx}(f) = \sqrt{T} rect(fT)$
  - $H(\Omega) = H_{chan}\left(\frac{\Omega}{2\pi T}\right) = Ge^{-j\Omega\tau/T}$
- Similar calculation as before: $h[n] = sinc\left(\frac{\tau n}{T}\right)$



$\tau = 3$

$\tau = 3.4$

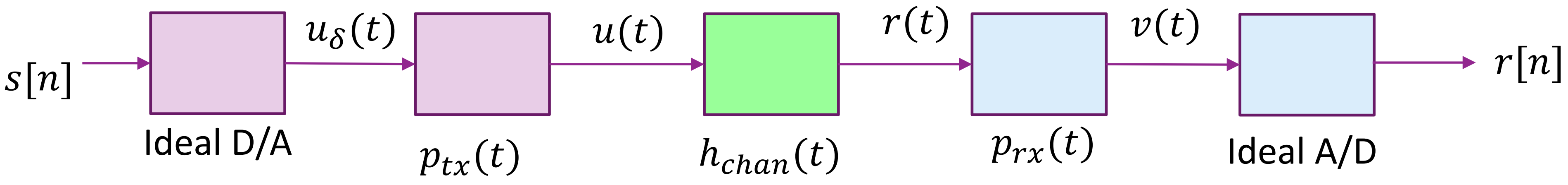
$\tau = 3.8$

$\tau = 4$

Flat Channels

- Suppose that
  - $P_{rx}(f) = P_{tx}(f) = \sqrt{T} rect(fT)$
  - $G(f) \approx G_0$ in $|f| < \frac{1}{2T}$
  - "Flat" over the channel.
- Then, effective discrete-time channel is
  - $H(f) = G_0$
  - $r[n] = G_0 s[n]$
  - No ISI
- Conclusion: When channel is "flat" over band, equalization is not needed

Numerically Computing the Channel



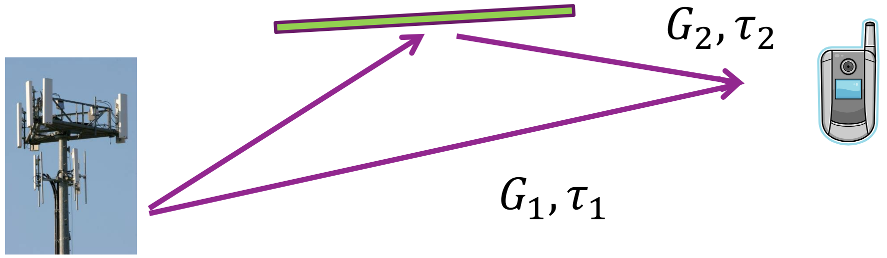
- For most real channels, we cannot analytically compute $h[n]$
  - Transmit and receive filters have complex frequency response
  - Channels can many taps at arbitrary delays
- But, we need $h[n]$ for proper simulation of communication systems
- Common solution: Approximately compute $h[n]$ numerically

Numeric Computation via Discretization

- Baseband channel frequency response with filtering: $G(f) = P_{rx}(f)P_{tx}(f)H_{chan}(f)$
- Effective discrete-time frequency response: $H(\Omega) = \frac{1}{T}G\left(\frac{\Omega}{2\pi T}\right)$
  - Assume no aliasing.
- Discrete-time impulse response: $h[n] = \frac{1}{2\pi}\int_0^{2\pi} H(\Omega)e^{i\Omega n}d\Omega$
- Evaluate integral by discretization:
  - Take N points
  - $h[n] \approx \frac{1}{N}\sum_{k=-N/2+1}^{N/2} H(\Omega_\mathrm{k})e^{\frac{i2\pi kn}{N}}, \quad \Omega_k = \frac{2\pi k}{N}$
  - Writing in terms of $G(f)$: $h[n] \approx \frac{1}{NT}\sum_{k=-N/2+1}^{N/2} G(f_k)e^{\frac{i2\pi kn}{N}}, \;\; f_k = \frac{k}{TN}$
  - Summation can be computed via an IFFT

Example: Two Path Channel

- Sample rate = $\frac{1}{T} = 8(120)(1.024) = 983.04$ MHz
  - Sample rate for an 8 channel 5G New Radio system at 120 sub-carrier spacing
- $p_{rx}(t) = p_{tx}(t) = \frac{1}{\sqrt{T}} \mathrm{Rect}\left(\frac{t}{T}\right)$ (zero-order hold ADCs)
- $h_{chan}(t)$ has two paths:
  - LOS path: Gain -10 dB, delay = 0
  - NLOS path: Gain -15 dB, delay = 10 ns
- In a wireless system, this would correspond to a path difference of $10(10)^{-9}3(10)^{8} = 3$ m



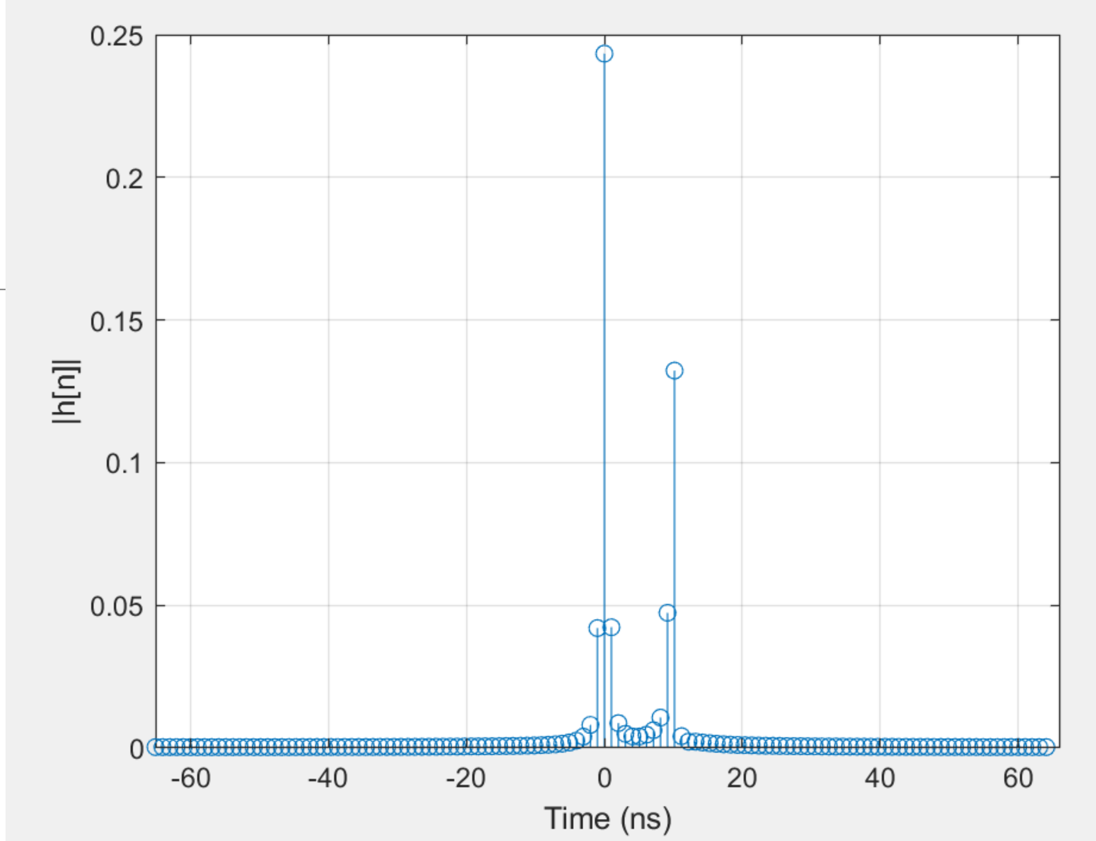
MATLAB Code

```
% Parameters
fsampMHz = 8*120*1.024;
Tsamp = 1/fsampMHz;

% Freq discretization points
npts = 128;
f = fsampMHz*(-npts/2+1:npts/2)'/npts;

% TX and RX filter
Prx = sqrt(Tsamp)*sinc(f*Tsamp);
Ptx = Prx;

% Channel
gaindB = [-10,-15];
dlyus = [0,0.01];
npath = length(gaindB);
Hchan = zeros(npts,1);
for ip = 1:npath
    Hchan = Hchan + 10^(0.05*gaindB(ip))*exp(-1i*2*pi*f*dlyus(ip));
end
```




```
% Compute discrete-time channel
G = Hchan.*Prx.*Ptx;
t = (-npts/2+1:npts/2)*Tsamp;
h = 1/Tsamp*ifft(G);
h = fftshift(h);
stem(t,abs(h));
```

Outline

- Receiver filtering and sampling
- Perfect reconstruction with orthonormal modulation
- General channels: Time-domain analysis
- General channels: Frequency-domain analysis
- PSD Analysis
- Practical RX filter design
- Channel sounding

PSD of a Sampled Signal

- Suppose that $y[n] = x(nT)$
- If there is no aliasing: $Y(\Omega) = \frac{1}{T} X\left(\frac{\Omega}{2\pi T}\right)$
- Now, look at auto-correlation:

$$R_y[m] = E(y[n]y^*[n-m]) = E\left(x(nT)x^*\big((n-m)T\big)\right) = R_x(mT)$$

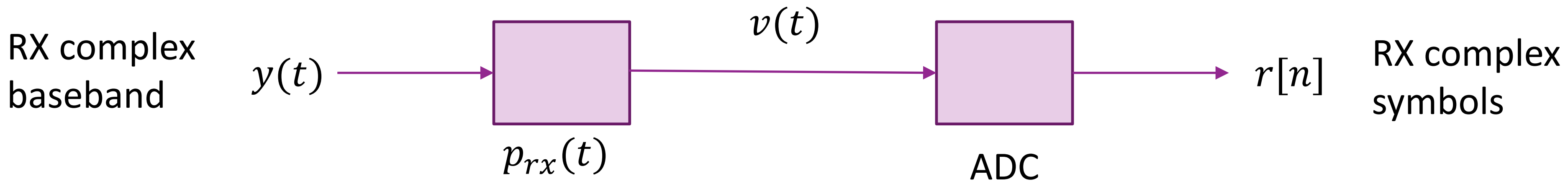
- Hence, if there is no aliasing

$$S_y(\Omega) = \frac{1}{T} S_x\left(\frac{\Omega}{2\pi T}\right)$$

- With aliasing:

$$S_y(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} S_x\left(\left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}\right)$$

PSD of the Receiver



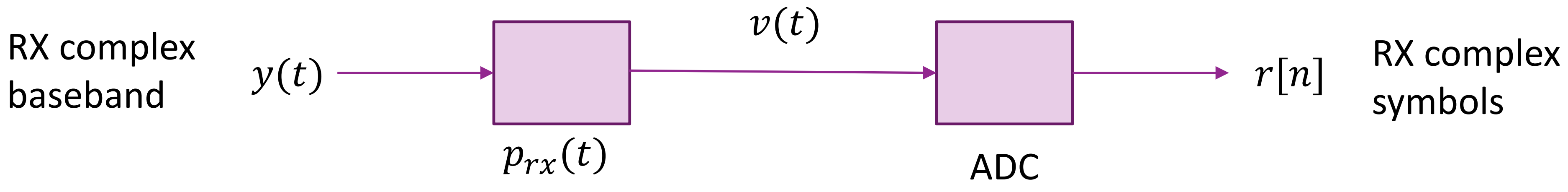
- We have $r[n] = v(nT)$
- By sampling result, discrete-time PSD is:

$$S_r(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_v(f_k) = \frac{1}{T}\sum_{k=-\infty}^{\infty} |P_{rx}(f_k)|^2 S_y(f_k)$$

  - $f_k = \left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}$

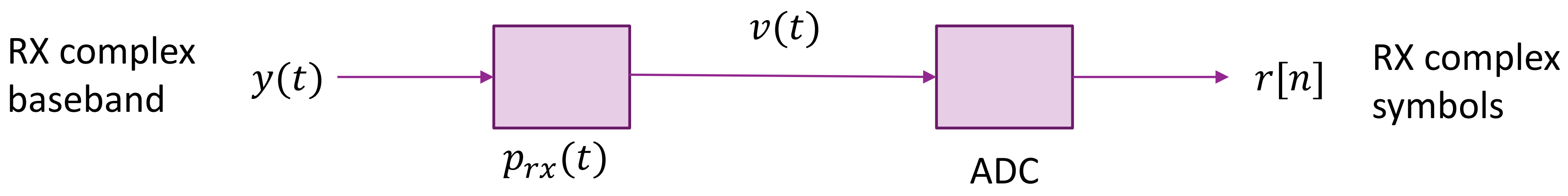
- Without aliasing: $S_r(\Omega) = \frac{1}{T}\left|P_{rx}\left(\frac{\Omega}{2\pi T}\right)\right|^2 S_y\left(\frac{\Omega}{2\pi T}\right)$

Units in RX Chain: Time Domain



- In time-domain: $v(t) = p_{rx}(t) * y(t), \;\; r[n] = v(nT)$
- Units of $|y(t)|^2$= power = energy / time
- We take convention: $|p_{rx}(t)|^2$= 1 / (time*sample)
- Then: $r[n] = \int r(t) p_{rx}(nT - t)dt = \sqrt{\frac{\text{Energy}}{\text{Time}}} \times \sqrt{\frac{1}{\text{Samples} \times \text{Time}}} \times \text{Time} = \sqrt{\frac{\text{Energy}}{\text{Sample}}}$
- Hence: $|r[n]|^2 = \frac{\text{Energy}}{\text{Sample}}$

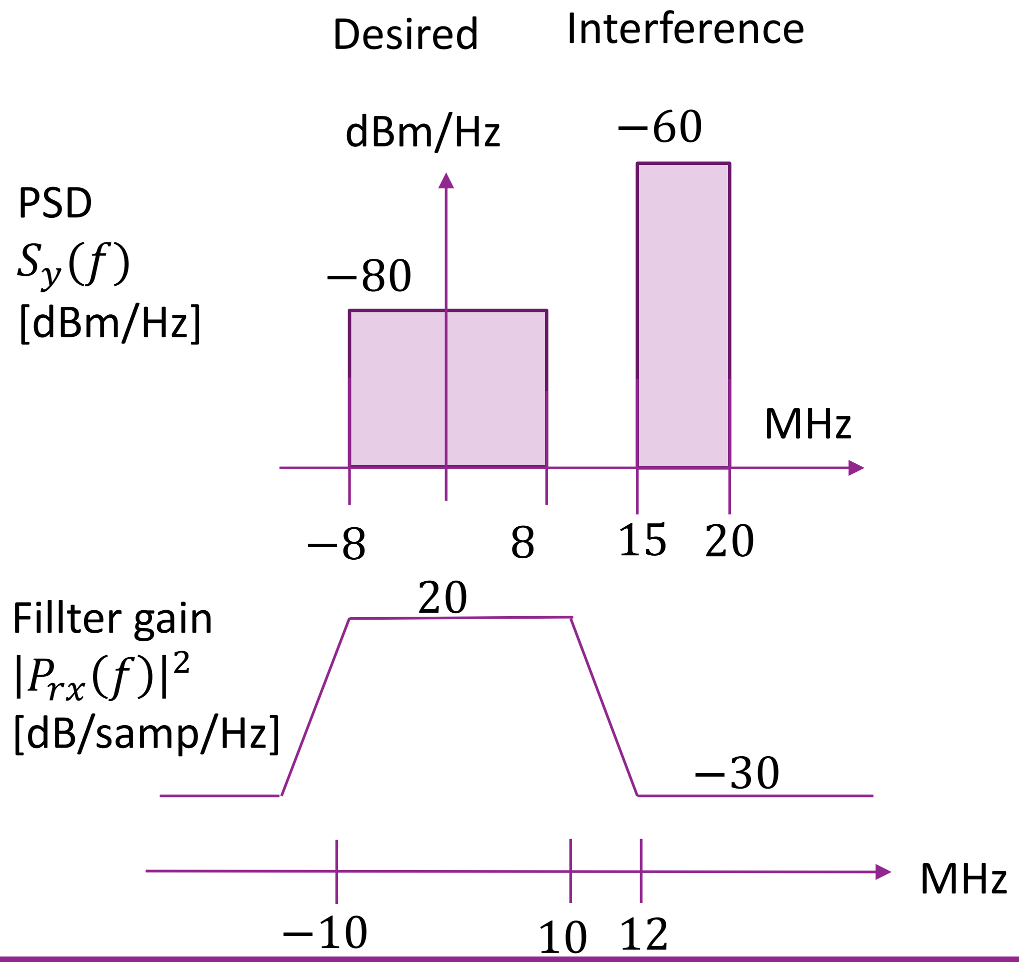
Units in RX Chain: PSD



- $S_r(\Omega) = \frac{1}{T}\left|P_{rx}\left(\frac{\Omega}{2\pi T}\right)\right|^2 S_y\left(\frac{\Omega}{2\pi T}\right)$
- Units of $S_y(f)$= power/Hz = energy
- Units of $|p_{rx}(t)|^2 = 1/(samples \times time)$
- Units of $|P_{rx}(f)|^2 = 1/(samples \times Hz)$
- Units of $S_r(\Omega) = \frac{1}{Time} \times Energy \times \frac{Time}{Sample} = \text{Energy per radian}$

Sample problem (Solution on board)

- Received PSD and filter as shown
- Two components:
  - Desired signal
  - Nearby adjacent carrier signal
- Assume they are uncorrelated
  - Hence powers add.
- Draw discrete-time PSD $S_r(\Omega)$
  - Sample rate = 80 Msamples / sec
  - Sample rate = 20 Msamples / sec



Outline

- Receiver filtering and sampling
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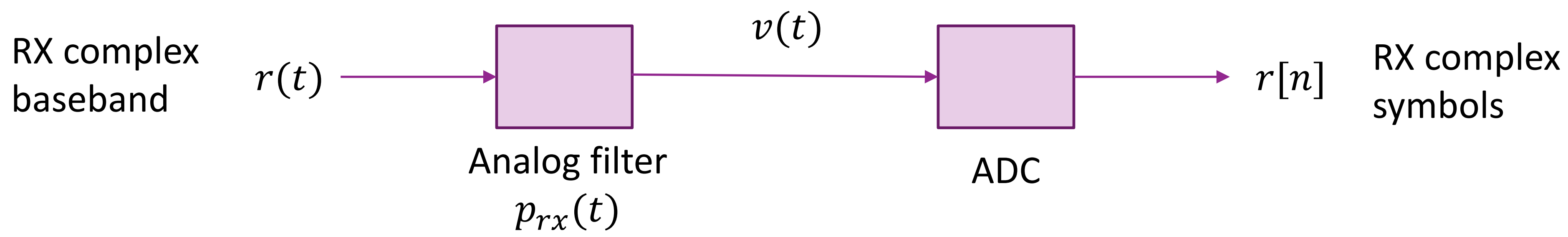
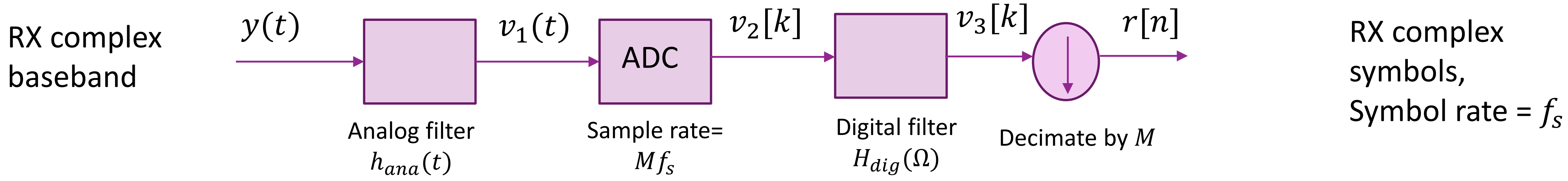
Problems with Analog Filtering



- Up to now, we have considered two stage filtering
  - Filtering: $v(t) = p_{rx}(t) * r(t)$
  - Sampling / ADC: $r[n] = v(nT)$
- Problem: Filtering is performed in analog
  - Desire sharp filters to remove close adjacent carrier
  - Difficult to design sharp filters

Typical Digital Implementation of RX Filtering



- Use combination of analog and digital filtering in four steps:
- Step 1. Analog filtering $v_1(t) = h_{ana}(t) * y(t)$
- Step 2. Sample at M times symbol rate: $v_2[k] = v_1(kT/M)$
  - M = oversampling ratio
- Step 3. Digitally filter: $v_3[k] = h_{dig}[k] * v_2[k]$
- Step 4. Decimate: $r[n] = v_3[nM]$.
  - Takes one every M samples

Frequency Domain Analysis

- Step 1: $V_1(f) = H_{ana}(f)Y(f)$
- Step 2: Sampling at $\frac{T}{M}$: $V_2(\Omega) = \frac{M}{T}\sum_{k=-\infty}^{\infty} V_1\left(\left(\frac{\Omega}{2\pi} + k\right)\frac{M}{T}\right)$
- Step 3: Digital filtering: $V_3(\Omega) = V_2(\Omega)H_{dig}(\Omega)$
- Step 4: Decimate: $R(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} V_3\left(\frac{\Omega+2\pi m}{M}\right)$

- Assuming no aliasing: Effective RX pulse shape is $P_{rx}(f) = H_{ana}(f)H_{dig}\left(\frac{2\pi f}{f_s M}\right)$

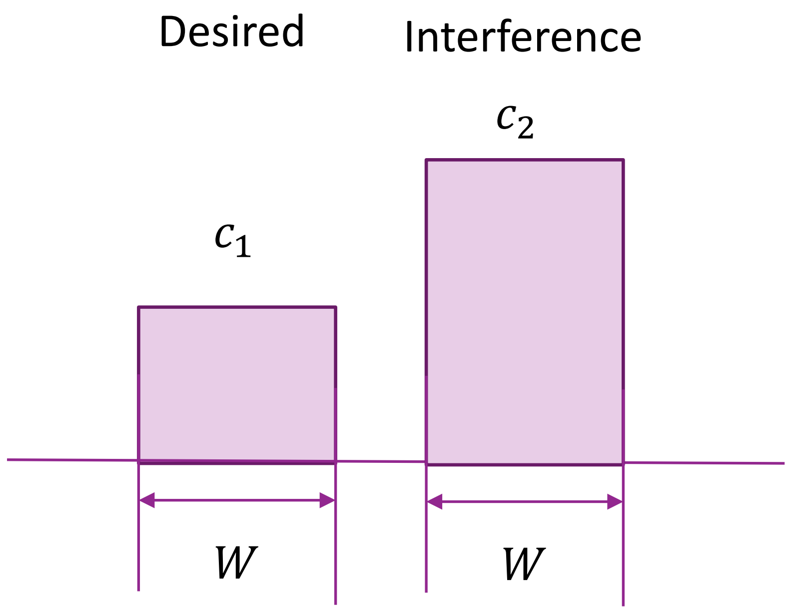
Frequency Domain Analysis: PSD

- Step 1: $S_1(f) = |H_{ana}(f)|^2 S_y(f)$
- Step 2: Sampling at $\frac{T}{M}$: $S_2(\Omega) = \frac{M}{T}\sum_{k=-\infty}^{\infty} S_1\left(\left(\frac{\Omega}{2\pi} + k\right)\frac{M}{T}\right)$
- Step 3: Digital filtering: $S_3(\Omega) = S_2(\Omega)\left|H_{dig}(\Omega)\right|^2$
- Step 4: Decimate: $R(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} S_3\left(\frac{\Omega+2\pi m}{M}\right)$

- Assuming no aliasing: Effective RX pulse shape is $P_{rx}(f) = H_{ana}(f)H_{dig}\left(\frac{2\pi f}{f_s M}\right)$

Sample problem (Solution on board)

- Received passband signal $S_y(f)$ as shown
- Two components:
  - Desired signal
  - Nearby adjacent carrier signal
- Draw the following:
  - Complex baseband $R(f)$ after mixing at f_{c1}
  - Response after analog filtering with cutoff $|f| \leq 20$ MHz
  - Sampling at 40 Ms/s
  - Digital filtering with $|\Omega| \leq \pi/2$
  - Downsampling by 2



$f_{c1} = 2.3$ GHz
$W = 12$ MHz

$f_{c2} = 2.5$ GHz
$W = 12$ MHz

Filter Design

- Effective pulse shape: $P_{rx}(f) = H_{ana}(f)H_{dig}\left(\frac{2\pi f}{f_s M}\right)$
- Want $P_{rx}(f)$ to be low-pass with cutoff $f = \frac{1}{T}$
- Typical design for analog filter
  - $H_{ana}(f)$ passband up to $\frac{1}{2T}$, Stopband $\frac{2M-1}{2T}$
  - Removes images before sampling
  - Large transition region. Easy to design
- Design spec for digital filter
  - Low pass with digital cut-off frequency π/M
  - Typically very sharp to remove close adjacent carrier

Summary of Sampling Relations

Operation	Time domain	Frequency-domain	PSD
Ideal DAC	$u(t) = \sum_n s[n]p(t - nT)$	$U(f) = P(f)S\left(\frac{2\pi f}{f_s}\right)$	$S_u(f) = \frac{1}{T}S_s(2\pi fT)\|P(f)\|^2$
Digital upsampling	$s[n] = \begin{cases} x[k] & n = kM \\ 0 & else \end{cases}$	$S(\Omega) = X(M\Omega)$	$S_s(\Omega) = \frac{1}{M}S_x(M\Omega)$
Ideal ADC	$r[n] = v(nT)$	$R(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} V(f_k)$ $f_k = \left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}$	$S_r(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} S_v(f_k)$ $f_k = \left(\frac{\Omega}{2\pi} + k\right)\frac{1}{T}$
Ideal ADC. No aliasing	$r[n] = v(nT)$	$R(\Omega) = \frac{1}{T}V\left(\frac{\Omega}{2\pi T}\right)$	$S_r(\Omega) = \frac{1}{T}S_v\left(\frac{\Omega}{2\pi T}\right)$
Digital decimation	$r[n] = x(nM)$	$R(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(\frac{\Omega+2\pi m}{M}\right)$	$S_r(\Omega) = \frac{1}{M}\sum_{m=0}^{M-1} S_x\left(\frac{\Omega+2\pi m}{M}\right)$
Digital decimation. No aliasing	$r[n] = x(nM)$	$R(\Omega) = \frac{1}{M}X\left(\frac{\Omega}{M}\right)$	$S_r(\Omega) = \frac{1}{M}S_x\left(\frac{\Omega}{M}\right)$

Summary of Units

Operation	Quantity	Time domain units	PSD units
TX symbols	$s[n]$	$\|s[n]\|^2$: Energy per sample	$S_S(\Omega)$: Energy per sample per radian
TX modulation	$u(t) = \sum p_{tx}(t - nT)s[n]$	$\|p_{tx}(t)\|^2$: Samples / sec	$\|P_{tx}(f)\|^2$: Samp / Hz = Samp x sec
		$\|u(t)\|^2$: Energy / sec = power	$S_u(f)$: Power / Hz = Energy
RX signal	$y(t)$	$\|y(t)\|^2$: Energy / sec = power	$S_y(f)$: Power / Hz = Energy
RX filtered & sampled	$v(t) = p_{rx}(t) * y(t)$ $r[n] = v(nT)$	$\|p_{rx}(t)\|^2$: 1/(samp x time)	$\|P_{rx}(f)\|^2$: 1/(samp x Hz)=time/samp
		$\|r[n]\|^2$: Energy per sample	$S_r(\Omega)$: Energy per sample per radian

Example RX Filtering Circuit

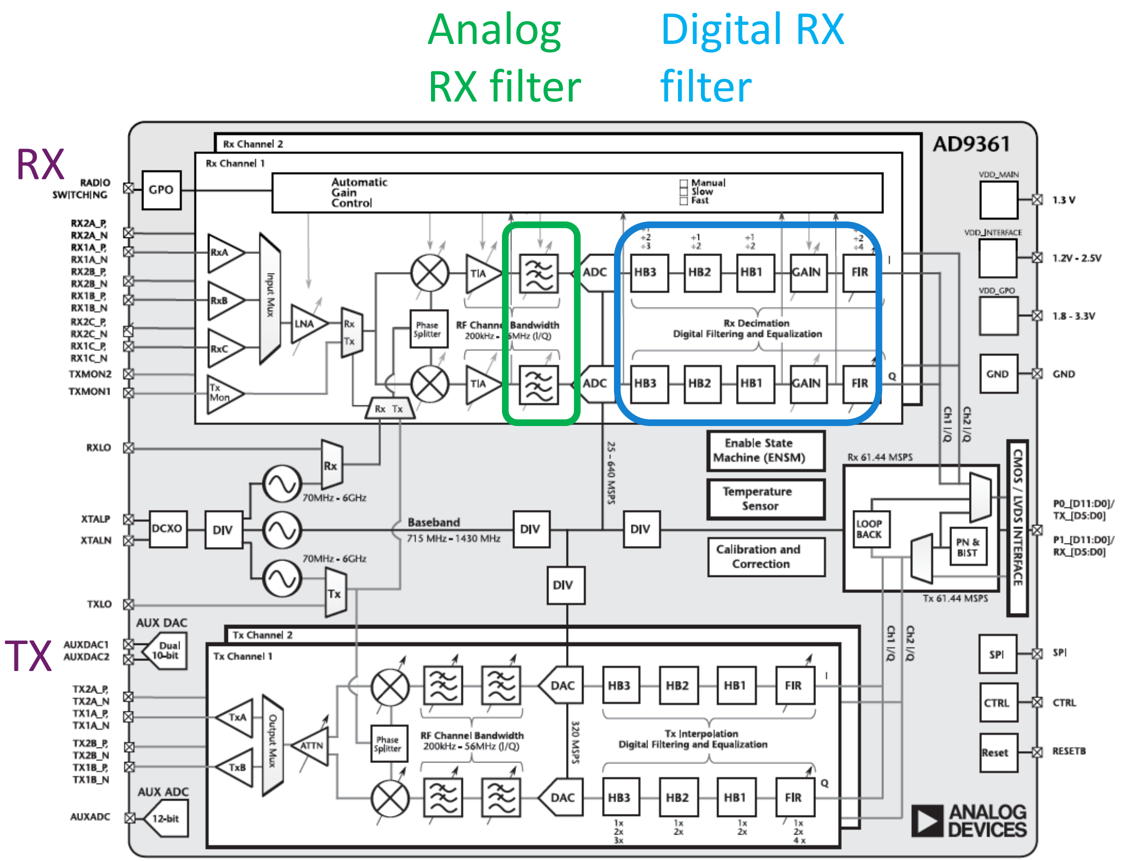


Figure 1.8 Integrated ZIF architecture used in the Pluto SDR.

- Analog Devices AD9361 Wideband TXCR
- RX filtering performed in two stages:
- Analog filtering (before ADC)
  - Depends on channel bandwidth
- Digital filtering:
  - Multiple stages of interpolation
  - Programmable depending on sample rate

Outline

- Receiver filtering and sampling
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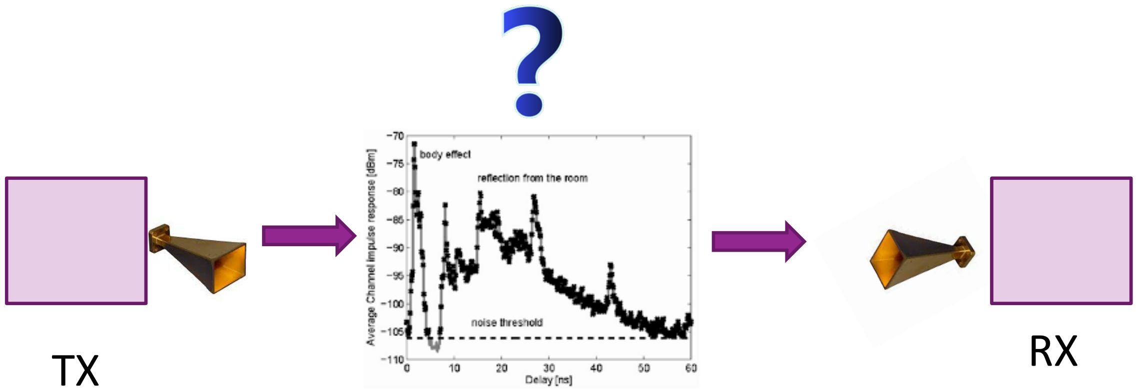
Channel Sounder

- Measure channel
- Useful for:
  - Wireless propagation analysis
  - Measure multipath components, signal strengths, directions of arrival
  - But, also good for debugging any front-end



TX

RX

Channel impulse response

MmWave and sub-THz channel sounder by Rappaport lab
https://wireless.engineering.nyu.edu/mmwave-5g-and-6g-channel-sounder/

FFT-Based Channel Sounding

- Effective discrete-time channel: $r[n] = h[n] * s[n]$
- Select period N
- Repeatedly transmit $s[n] = \sum_{k=0}^{N-1} S_k e^{2\pi ikn/N} = \mathrm{FFT}(S_k)$
- Receiver will get: $\mathrm{r}[n] = \sum_{k=0}^{N-1} R_k e^{2\pi ikn/N}$, $R_k = S_k H\left(\frac{2\pi k}{N}\right)$
- Recover $R_k = IFFT(r[n])$
- Get channel response: $H\left(\frac{2\pi k}{N}\right) = \frac{R_k}{S_k} = H_{chan}\left(\frac{k}{NT}\right) P_{tx}\left(\frac{k}{NT}\right) P_{rx}\left(\frac{k}{NT}\right)$
- Learn $P_{tx}\left(\frac{k}{NT}\right), P_{rx}\left(\frac{k}{NT}\right)$ from calibration
- Estimate $H_{chan}\left(\frac{k}{NT}\right)$
- Provides a discrete estimate of the channel.
- More in the lab