Unit 7: Synchronization and Matched Filtering

EL-GY 6013: DIGITAL COMMUNICATIONS

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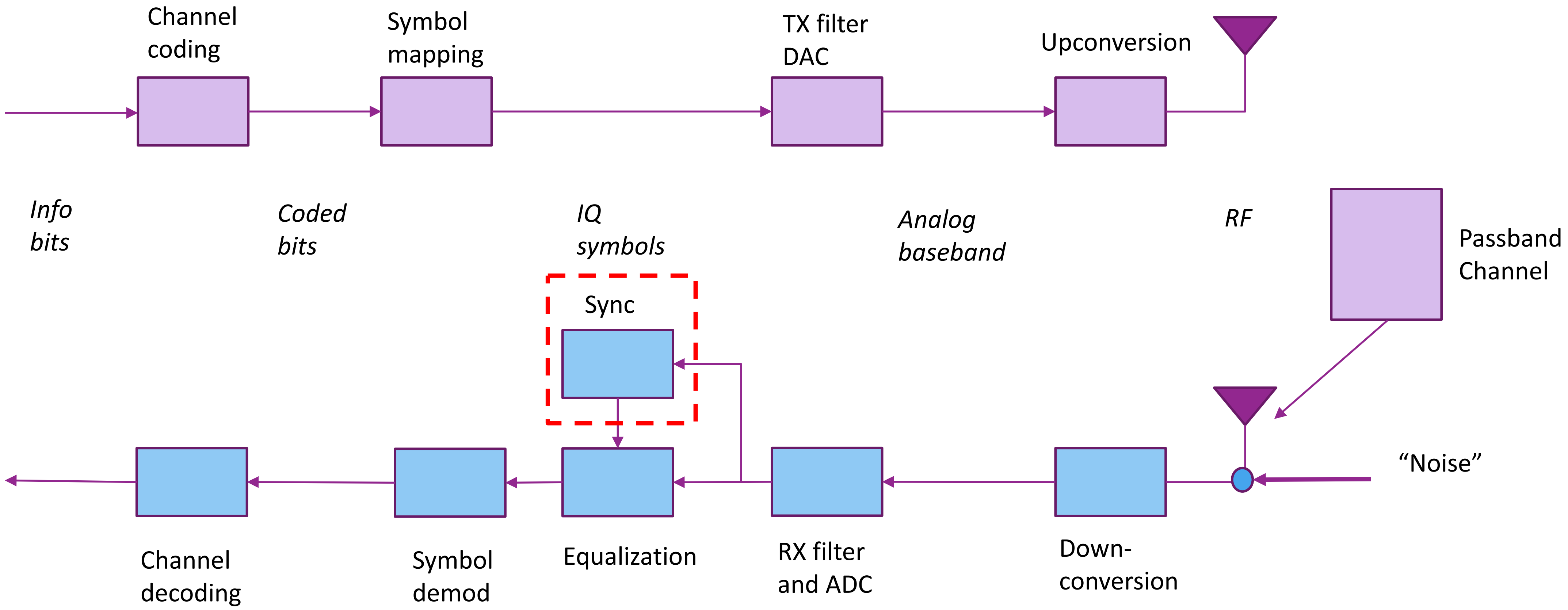
Learning Objectives

- ❑Describe the synchronization mechanisms in common commercial standards
- ❑Formulate binary decision tasks as hypothesis testing problems
- ❑Compute the LRT detector for a hypothesis testing problem
- ❑Compute error probabilities and optimize the threshold
- ❑Formulate signal detection as a hypothesis test
- ❑Describe and analyze the matched filter detector
- ❑Analyze various non-idealities including clock offset, auto-correlation and multi-path
- ❑Simulate the MF detector for real systems

This Unit



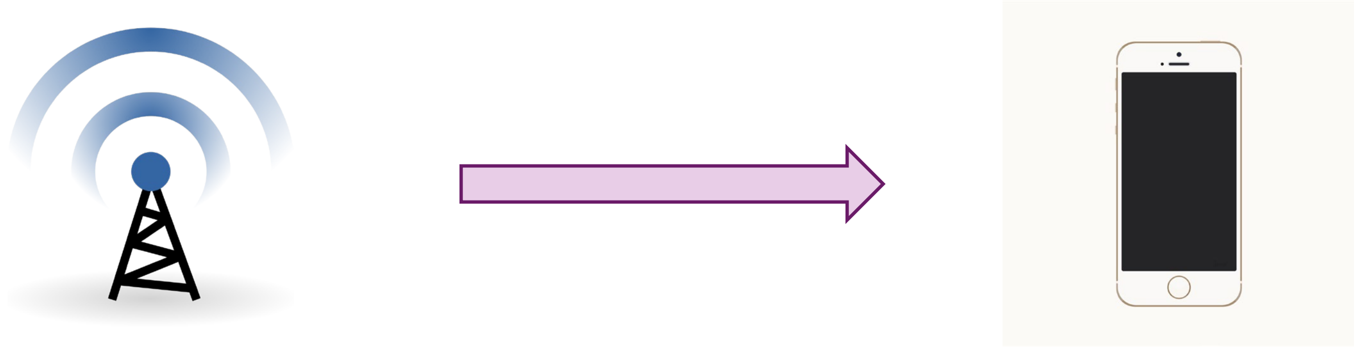
Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
"Noise"
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

Outline

- Detection and Synchronization Problem
- Hypothesis Testing
- Match Filtering for Detection at a Known Delay
- Match Filter SNR and Error Probabilities
- Match Filtering Convolution with an Unknown Signal Delay
- Automatic Gain Control (AGC)

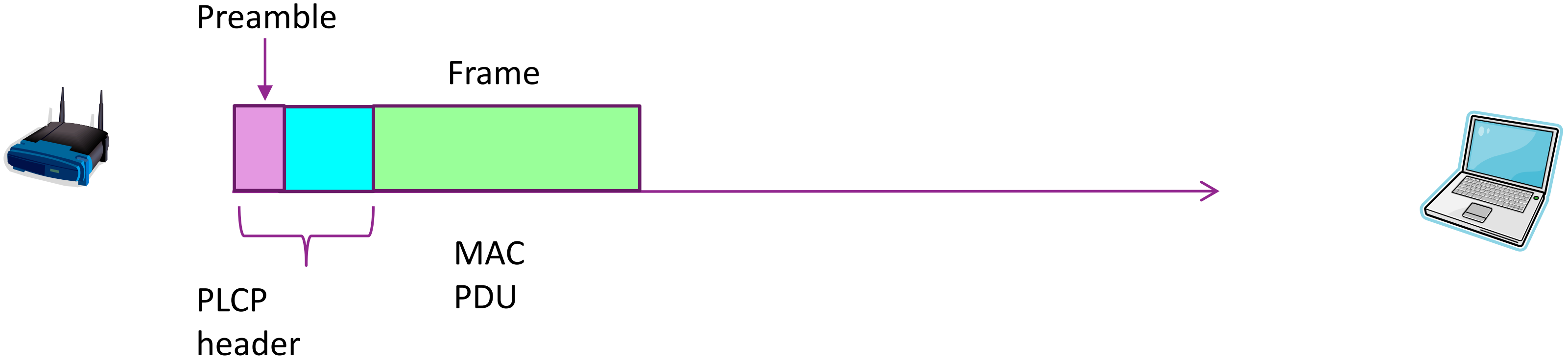
- Appendix 1. Error Probability Calculation Details
- Appendix 2. Matched Filtering as a Generalized Likelihood Ratio Test

Synchronization and Detection Problem



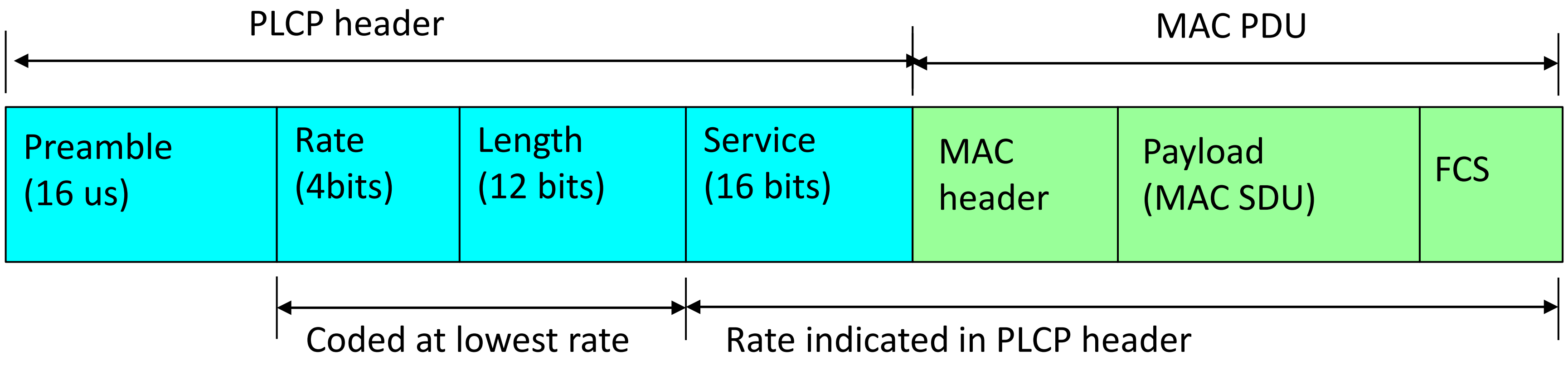
- Two key problems in most communication receivers:
  - Detect if a transmitter is present
  - Synchronize to the transmitter
- Basic first step in any communication process
- Assumes the transmitter broadcasts a signal
- Receiver must detect and synchronize to it

Ex 1: 802.11g Transmission



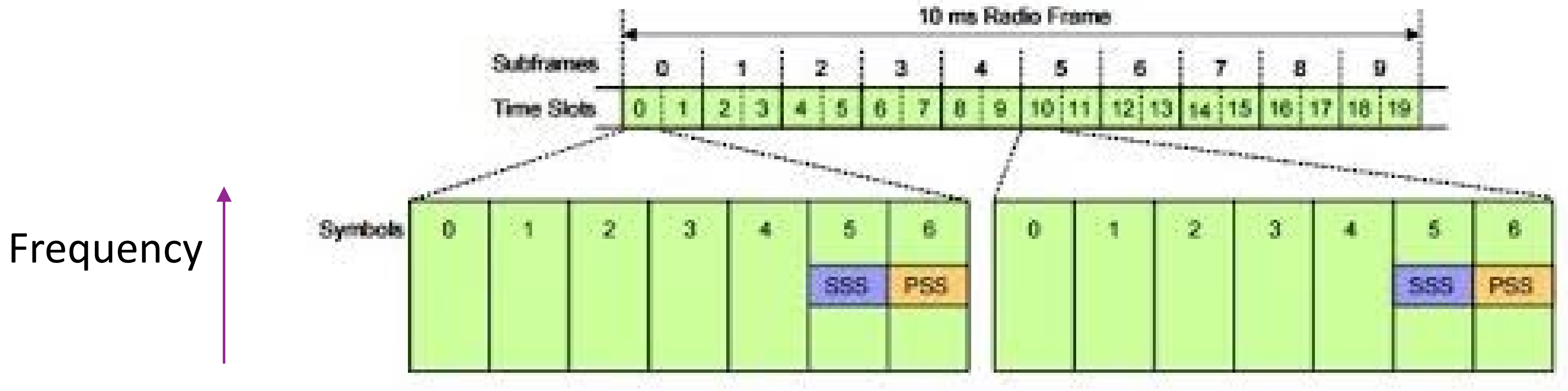
- All data is transmitted in frames
- Frames may arrive at any time
- Each frame begins with known preamble
  - Common to all frames
- RX station listens for preamble to detect:
  - Presence of frame.
  - If frame is present, determines timing delay of the remaining frame

802.11g PLCP Header Details



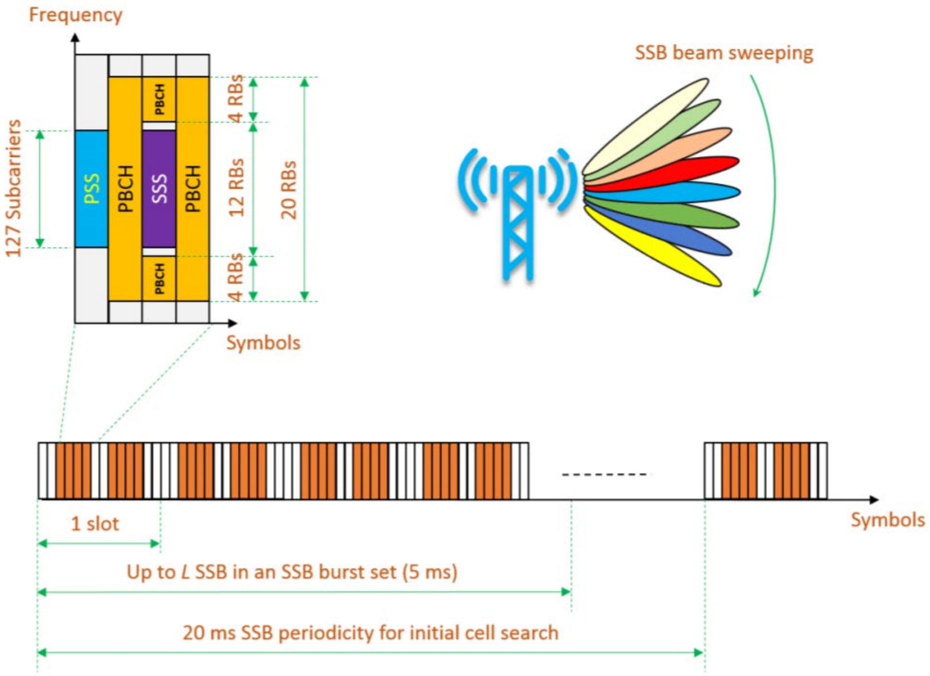
- PLCP header details:
  - Preamble: Used for initial detection, synchronization, channel estimation
  - Rate: Signals MCS for service bits & MAC PDU
  - Length: Number of OFDM symbols in frame
  - Service: Scrambler sync
- MAC header: Contains MAC layer control info
  - Segmentation, MAC addresses, …
- MAC FCS: frame check sum (used to detect errors)

Ex 2: LTE Downlink Primary Sync Signal (PSS)



- ❑ Each cell transmits periodically PSS
  - Narrowband, short (71.4 us x 1.08 MHz)
  - One of 3 PSS signals
- ❑ Once PSS is detected, mobile (UE) knows frame timing
  - Decodes subsequent signals SSS, broadcast, …

Ex. 3. 5G New Radio Beam Sweeping



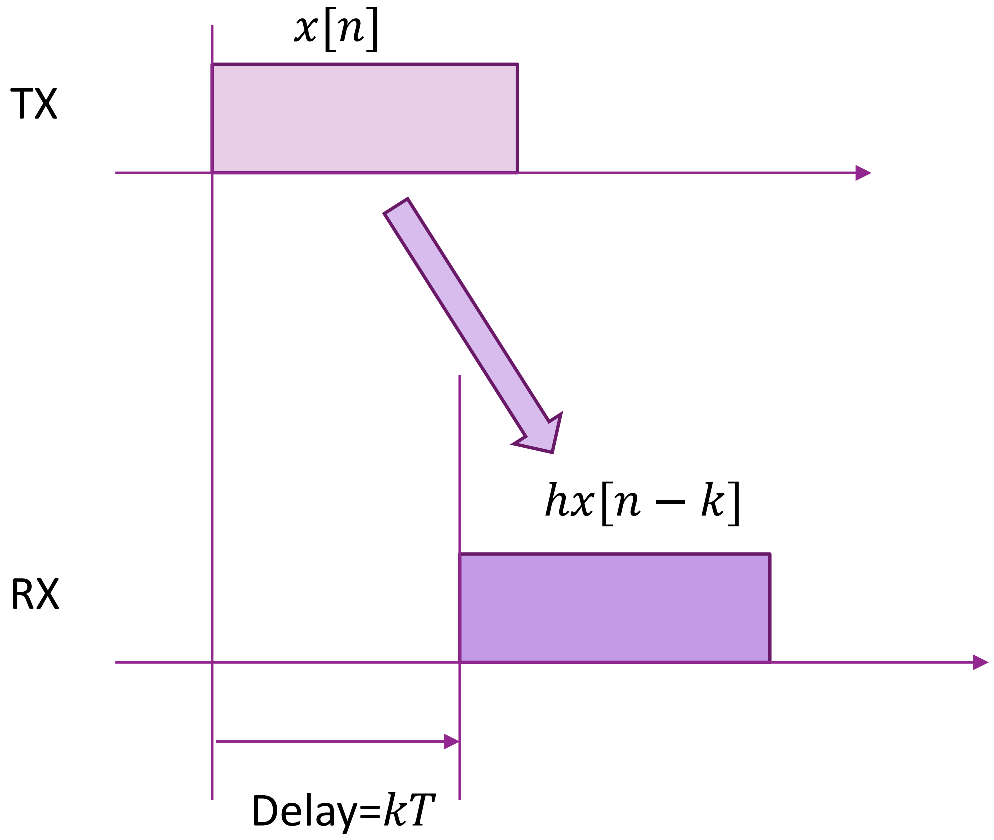
- Directional synchronization for mmWave
- Transmit multiple SS Burst
  - One in each direction
- MmWave typically use 120 kHz subcarrier spacing
- With 120 kHz SCS:
  - SSB = 4 OFDM symbols = 35.7 μs
  - Each SSB, contains a PSS
  - PSS time duration = 1 OFDM symbol = 8.92 μs
  - Bandwidth = 127 SC = 15.24 MHz
  - Up to 64 SS Bursts / burst period
  - Typical SSB periodicity = 20 ms

Simple Synchronization Model

- TX sends a preamble / synchronization signal
  - $x[n],\ n = 0,1,2,\ldots,N-1$
  - Complex baseband samples.
  - Sample rate $\frac{1}{T}$
- If signal is present at RX:

$$y[n] = hx[n-k] + w[n]$$

  - h: Complex channel gain
  - k: Integer delay
- Problem detect if signal is present or not.
  - If so, what is the delay
- For now, we assume:
  - Integer delays, no multipath
  - Will address these issues later



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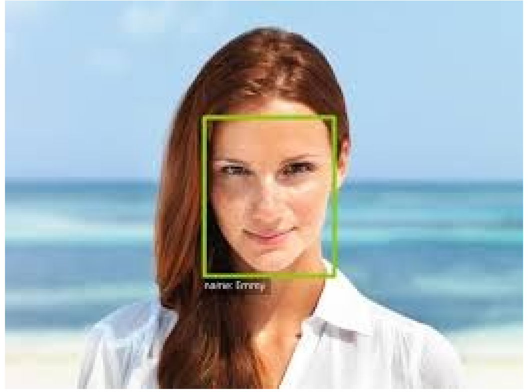
Hypothesis Testing

- Classic problem in statistics or decision theory
- Observe data $\boldsymbol{y}$
- Two possible hypotheses for data
  - H0: Null hypothesis
  - H1: Alternate hypothesis
- Model statistically:
  - $p(\boldsymbol{y}|H_i), i = 0,1$
  - Assume some distribution for each hypothesis
  - Each density is the likelihood of $\boldsymbol{y}$
- Problem: Determine which hypothesis is true given data y

Applications

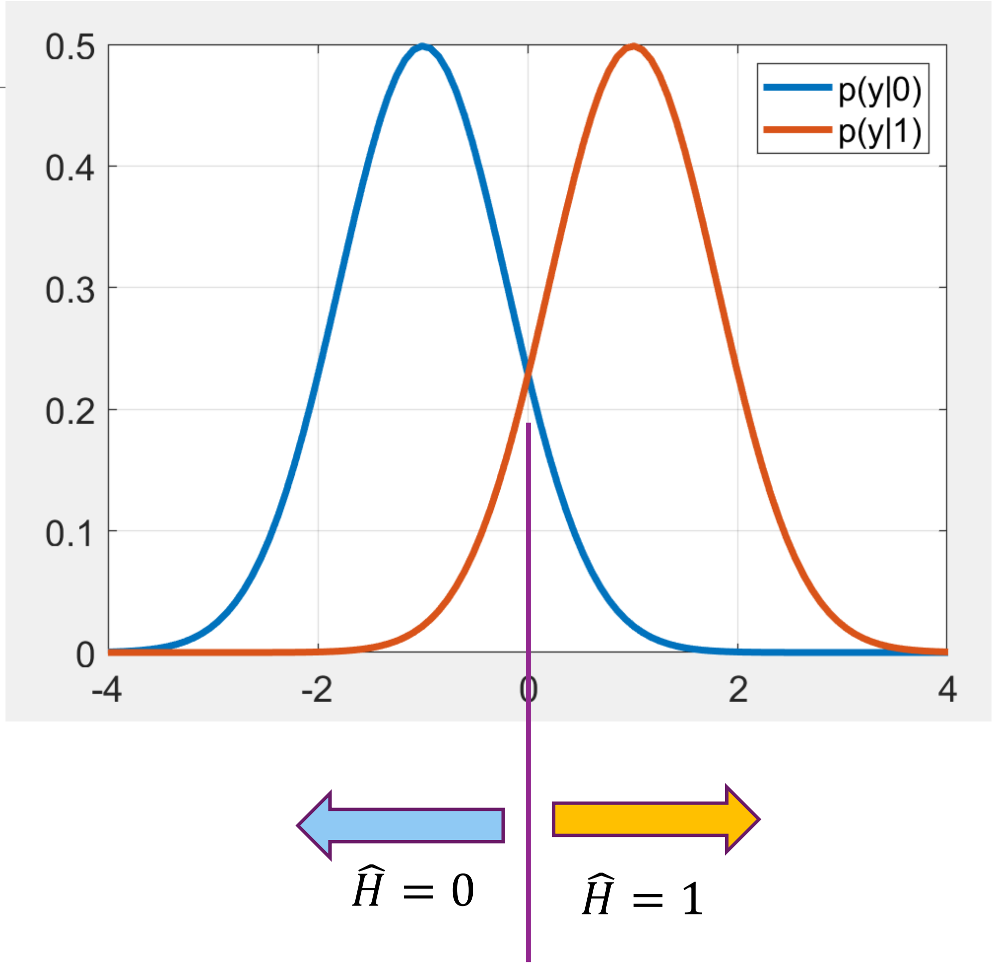
- ❑ Many applications
- ❑ Pattern recognition:
  - Does this image contain a face or not?
  - Is this person X?
- ❑ Detection:
  - Is the transmitted bit 0 or 1?
- ❑ This lecture: Is a signal present or not?



Simple Example

- Scalar Gaussian
  - H_0: $y = -A + w$
  - H_1: $y = A + w$,
  - $w \sim N(0, \sigma^2)$
- In this case:
  - $p(y|H_0) = N(y| - A, \sigma^2)$
  - $p(y|H_1) = N(y|A, \sigma^2)$
- Saw this earlier in BPSK transmissions
- Max likelihood detector from earlier
  - Selects the most likely hypothesis
  - In this case

$$\widehat{H} = \arg\max_{j} p(y|H = j) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Types of Errors

- For binary detection problems, there are two errors:
  - Type I error (False alarm): Decide H1 when H0
  - Type II error (Missed detection): Decide H0 when H1
- In many problems, the consequences of these errors is different
- Example: Medical diagnosis
  - False alarm: You tell the patient he is ill, when he is fine
  - Missed detection: You miss the illness
  - Consequences are different
- Given detector, we define two error probabilities:
  - False alarm probability: $P_{FA} = P(\widehat{H} = 1|H = 0)$
  - Missed detection probability: $P_{MD} = P(\widehat{H} = 0|H = 1)$

Likelihood Ratio Test

- We can tradeoff the error probabilities with a likelihood ratio test:
- Likelihood ratio test (LRT)

$$\hat{H} = 1 \Leftrightarrow \frac{p(x|H_1)}{p(x|H_0)} \geq \gamma$$

  - γ is an adjustable threshold
  - Increasing $\gamma \Rightarrow$ Lowers P_{FA}, but lowers P_D
- Often performed in log domain

$$\hat{H} = 1 \Leftrightarrow L^*(x) = \log \frac{p(x|H_1)}{p(x|H_0)} \geq \gamma'$$

- Note that $\gamma = 0$ corresponds to maximum likelihood detector

Gaussian Example

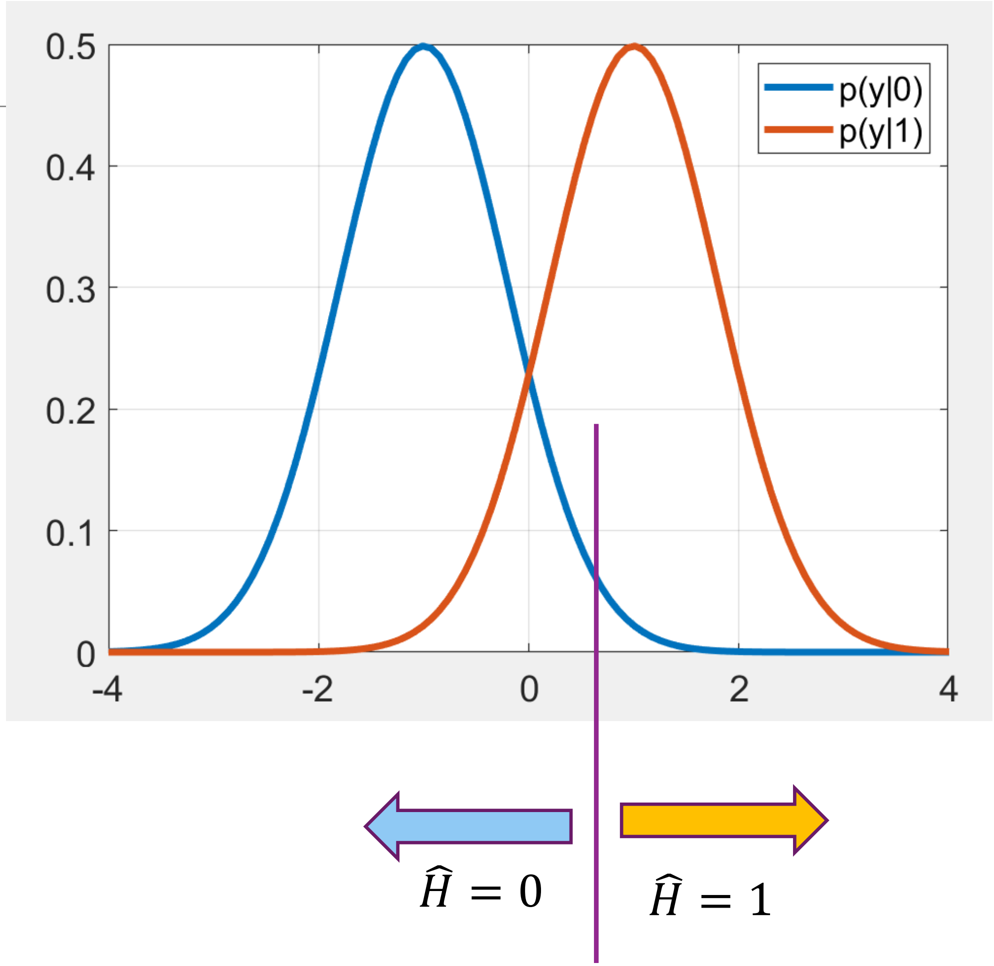
- Scalar Gaussian case:
  - $p(y|H_0) = N(y|-A, \sigma^2) = C \exp(-\frac{(y+A)^2}{2\sigma^2})$
  - $p(y|H_1) = N(y|A, \sigma^2) = C \exp(-\frac{(y-A)^2}{2\sigma^2})$
- Log likelihood ratio:

$$L(y) \coloneqq \ln \frac{p(y|H_1)}{p(y|H_0)}$$
$$= \frac{1}{2\sigma^2}[(y+A)^2 - (y-A)^2] = \frac{2Ay}{\sigma^2}$$

- LRT: $\hat{H} = 1$ if and only if

$$L(y) \geq \gamma \Leftrightarrow y \geq t = \frac{\gamma\sigma^2}{2A}$$

  - t is an adjustable threshold

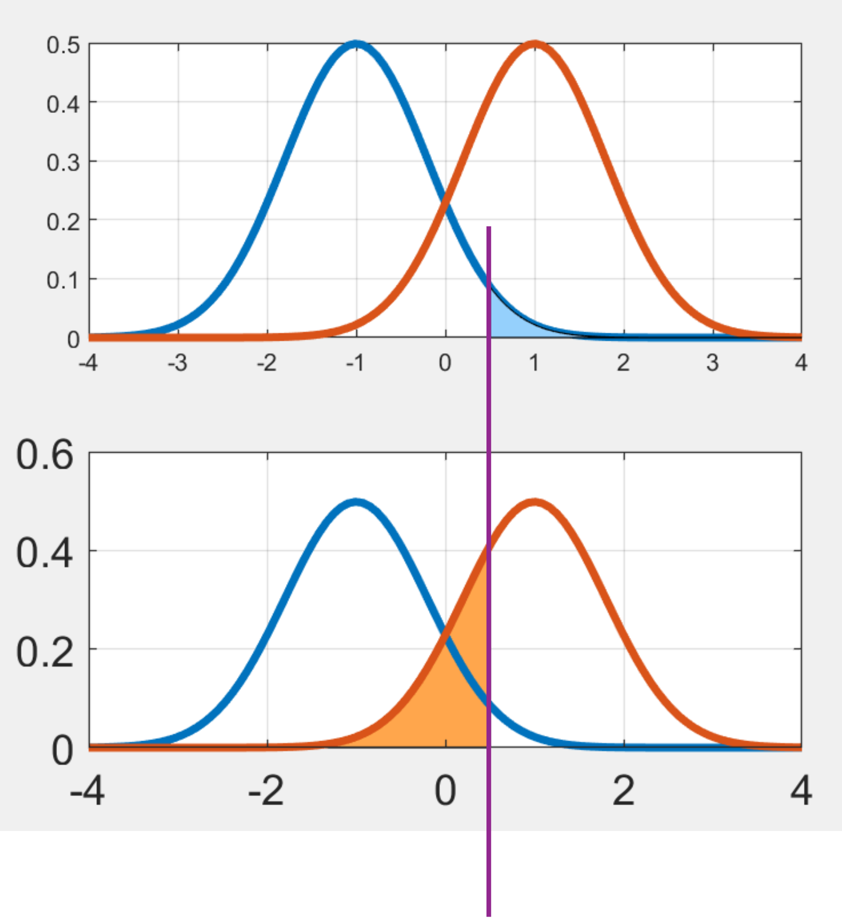


Computing Error Probabilities

- From previous slide, LRT detector is:

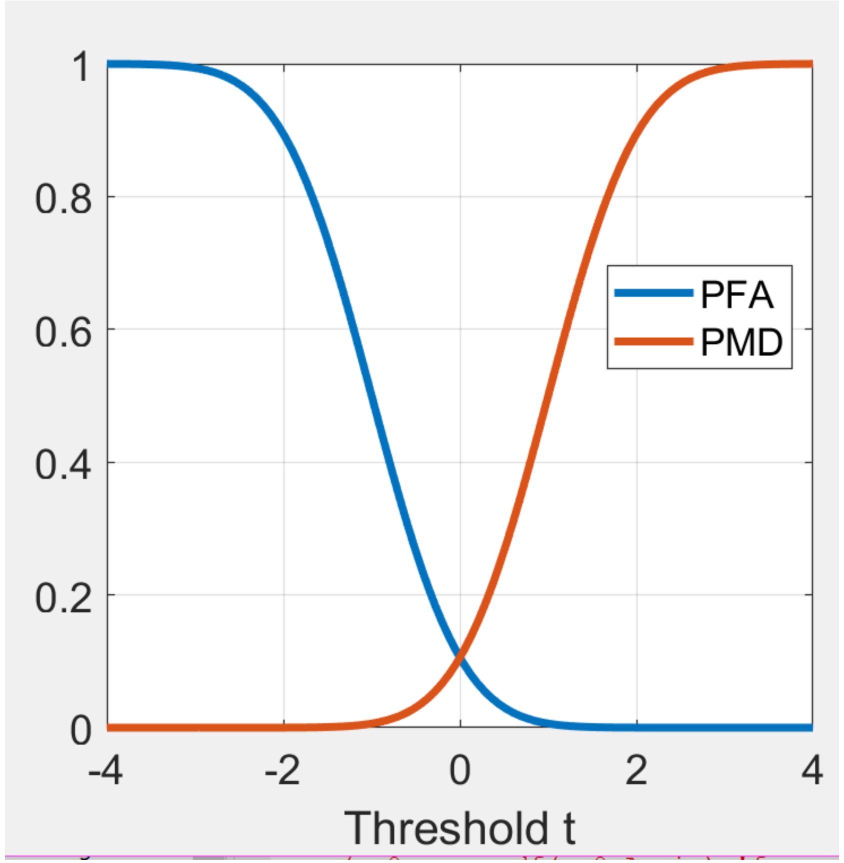
$$\hat{H} = \begin{cases} 1 & y \geq t \\ 0 & y < t \end{cases}$$

- FA probability:
  - $P_{FA} = P\left(\hat{H} = 1 \middle| H = 0\right) = P(y \geq t | H = 0) = \int_t^{\infty} p(y|0) dy$
  - This is the area under the curve (blue)
  - For Gaussian: $P_{FA} = Q\left(\frac{t+A}{\sigma}\right)$
- MD probability
  - $P_{MD} = P\left(\hat{H} = 0 \middle| H = 1\right) = P(y < t | H = 1) = \int_{-\infty}^{t} p(y|1) dy$
  - This is the area under the curve (orange)
  - For Gaussian: $P_{MD} = 1 - Q\left(\frac{t-A}{\sigma}\right)$



Tradeoff

- Tradeoff between P_{FA} and P_{MD}
  - $P_{FA} = Q\left(\frac{t+A}{\sigma}\right)$
  - $P_{MD} = 1 - Q\left(\frac{t-A}{\sigma}\right)$
- Increasing threshold t:
  - Decreases false alarms
  - But increases missed detections
- Selection of optimal threshold
  - Depends on the application
  - What are the relative costs of these errors?

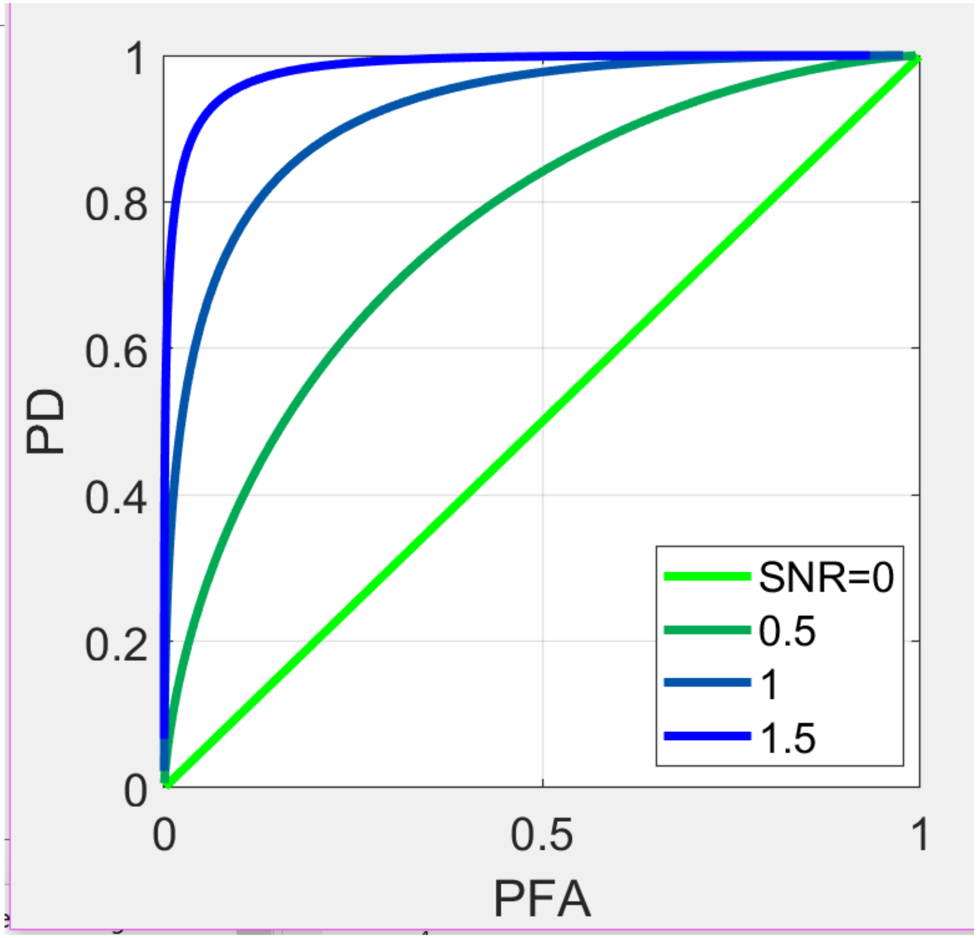


ROC Curve

- ❑Receiver operating characteristic
- ❑Plot of P_D vs. P_{FA}
- ❑Trace out: $(P_{FA}(\gamma), P_D(\gamma))$
- ❑Random guessing achieves:

$$P_D + P_{FA} = 1$$

- ❑Higher the line is better



Neyman-Pearson Theorem

- Theorem: Suppose that an LRT obtains $P_{FA} = \alpha$.
  Then any other test with P_{FA} will have a P_D less than or equal to the LRT.

- LRT is the most powerful test
- Obtains best P_{FA} vs. P_D performance

In Class Exercise

Synchronization In-Class Exercises

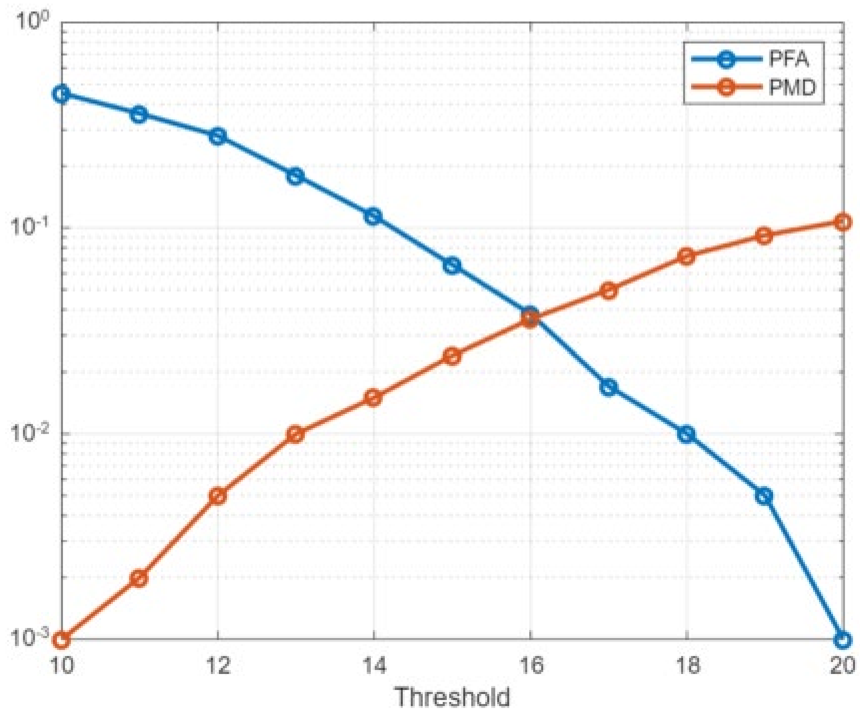
Hypothesis Testing for Poisson Random Variables

We will simulate hypothesis testing for discriminating between two Poisson distributions. This type of detector occurs in optical systems where the receiver counts the number of photons. The unknown variable is x=0 or 1. We receive a discrete random variable y with conditional probability:

- `P(y|x=0)` is Poisson with rate `lam0`. `P(x=0)=p0`
- `P(y|x=1)` is Poisson with rate `lam1`. `P(x=1)=p1`

The parameters are below

```
lam0 = 10;  % Rate when x=0
lam1 = 20;  % Rate when x=1
p0 = 0.8;   % P(x=0)
p1 = 1-p0;   % P(x=1)
```



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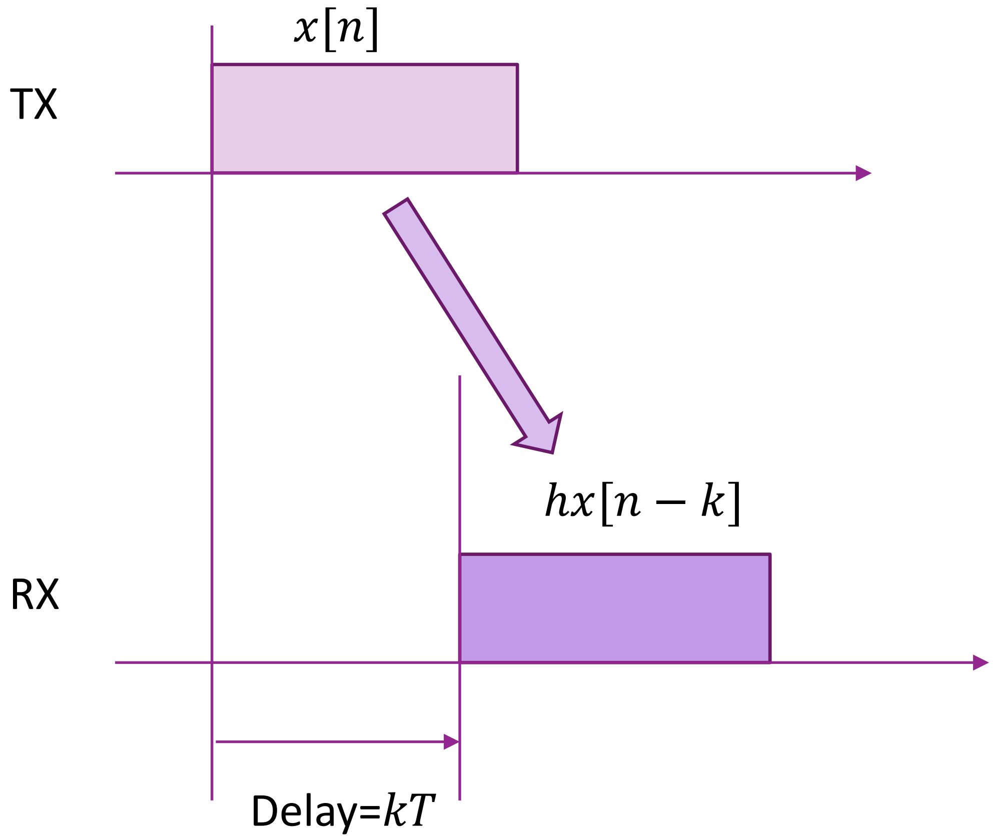
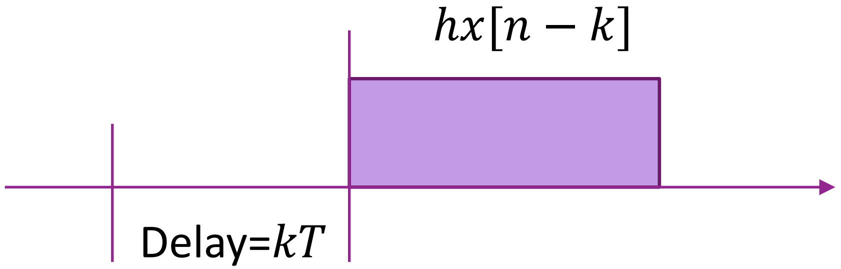


Detect as a Hypothesis Test

- ❑ At each delay k, we consider two hypotheses:
- ❑ H_1: Signal is present:

$$r[n] = hx[n-k] + w[n],$$

  - h is a complex, baseband channel gain
  - Recall that we are assuming a single path channel (for now)



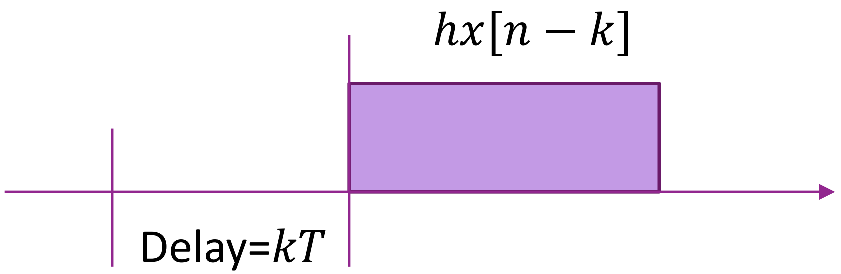
- ❑ H_0: Signal is absent:

$$r[n] = w[n]$$

- ❑ In both cases, assume $w[n]$ is white noise:
  - $w[n] \sim CN(0, N_0)$

Detection Problem with a Known Delay

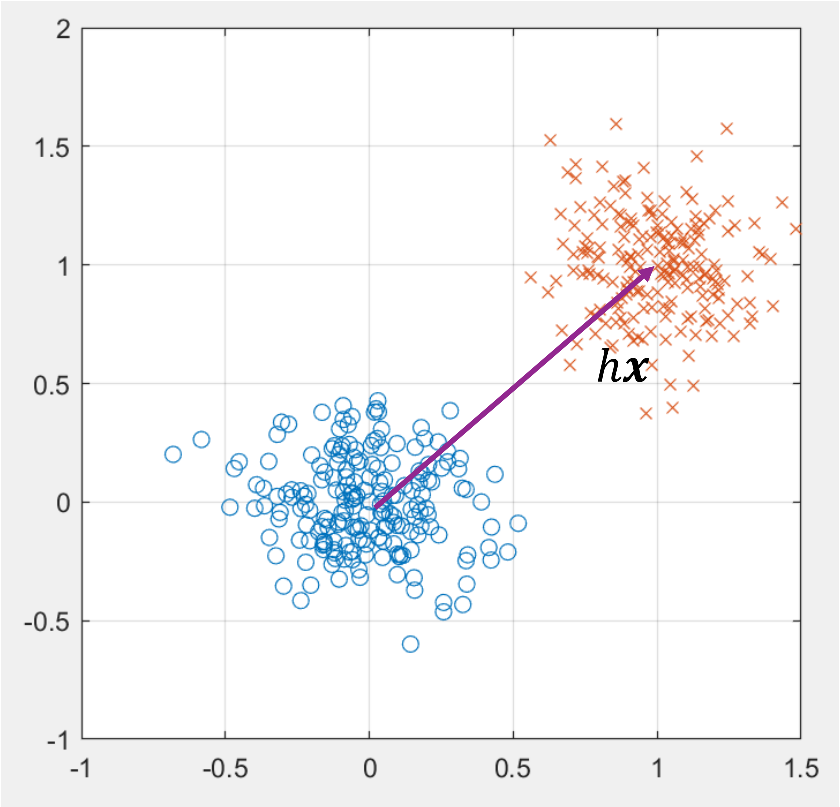
- Given :
  - RX signal $r[n],\ n = 0, \dots, N-1$
  - Target signal $x[m],\ m = 0, \dots, M-1$
  - Delay hypothesis: k



- Which "hypothesis" is more likely?
  - Signal is present: $r[n] = hx[n-k] + w[n]$ or
  - Signal is absent: $r[n] = w[n]$
- Channel gain h is not known
- Next section, we will also learn the delay k

Hypothesis Test in Vector Form

- Without loss of generality, assume:
  - Delay $k = 0$
  - Target signal length $M = N$
- Define vectors:
  - $\boldsymbol{r} = [r[0], \ldots, r[N-1]]^T$
  - $\boldsymbol{x} = [x[0], \ldots, x[N-1]]^T$
- Write two hypotheses in vector form:
  - H_1: $\quad \boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\quad \boldsymbol{r} = \boldsymbol{w}$ [Signal absent]

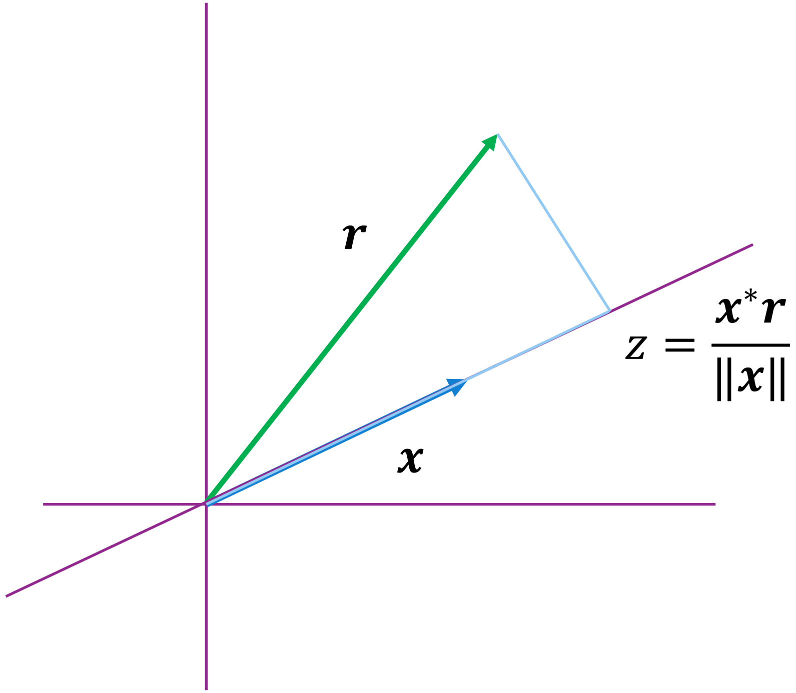


Projection

- Hypotheses in vector form:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Projection coefficient of $\boldsymbol{r}$ onto $\boldsymbol{x}$ is:

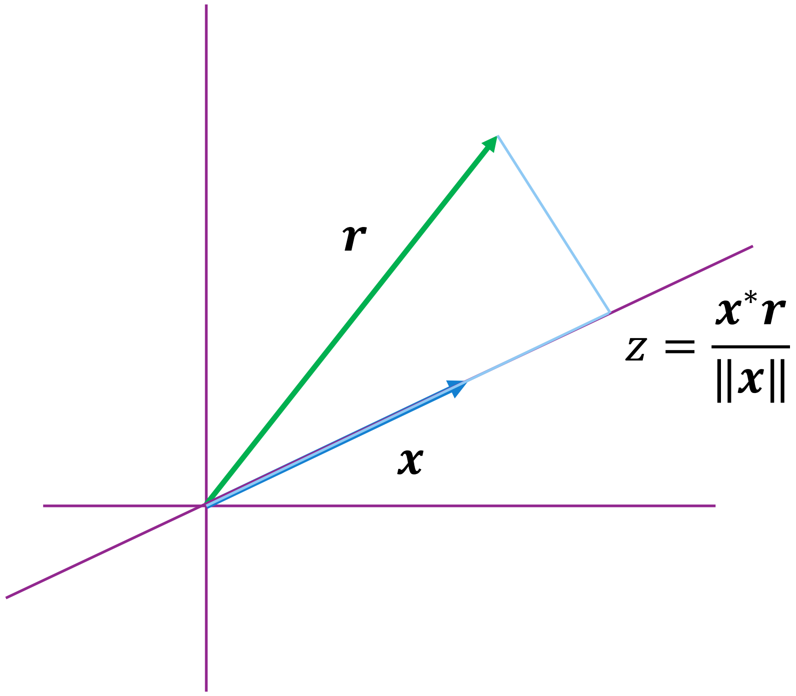
$$\boxed{z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|}}$$

  - For signal detection z is called the (scaled) matched filter
- Key idea: Declare signal present when $|z|$ is large
- Signal is detected if there is sufficient energy in target signal space



Match Filter Detector

- Given two hypotheses:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Match filter energy detector:
  - Compute scaled MF detector: $z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|}$
  - Measure energy: $y = |z|^2$
  - Declare $\widehat{H} = \begin{cases} 1 & y \geq t \\ 0 & y < t \end{cases}$
- Value t is a threshold



MF Example

- Suppose target signal is $\boldsymbol{x} = [1, -1, 1]$
  - Target signal norm is: $\|\boldsymbol{x}\|^2 = 1^2 + 1^2 + 1^2 = 3 \Rightarrow \|\boldsymbol{x}\| = \sqrt{3}$
- Suppose that threshold is $t = 10$
- Case 1: RX signal is $\boldsymbol{r} = [2, -3, 3]$
  - $|z|^2 = \frac{|x^* r|^2}{\|\boldsymbol{x}\|^2} = \frac{1}{3}(2 + 3 + 3)^2 \approx 21.3 > t$
  - Signal is detected!
- Case 2: RX signal is $\boldsymbol{r} = [2, 3, 3]$
  - $|z|^2 = \frac{|x^* r|^2}{\|\boldsymbol{x}\|^2} = \frac{1}{3}(2 - 3 + 3)^2 \approx 1.3 < t$
  - No signal is detected

MF Normalization

- Up to now, we have used the scaled or normalized MF:

$$z = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|}$$

- Often, we will use the un-normalized MF:

$$z = \boldsymbol{x}^* \boldsymbol{r}$$

- Detection method is the same, just the threshold changes:

$$\frac{|\boldsymbol{x}^* \boldsymbol{r}|}{\|\boldsymbol{x}\|} \geq t \Leftrightarrow |\boldsymbol{x}^* \boldsymbol{r}| \geq t\|\boldsymbol{x}\|$$

MF in Continuous-Time

- Target signal: $x(t)$
- RX signal: $r(t)$
- Un-normalized MF is: $z = \int x^*(t)r(t)dt$
- Normalized MF is: $z = \frac{1}{\|x\|}\int x^*(t)r(t)dt, \quad \|x\|^2 = \int |x(t)|^2 dt$
- Example: $x(t) = Rect\left(\frac{t}{a}\right),\ r(t) = t$ for $t \in \left[-\frac{a}{2}, \frac{a}{2}\right]$
- Then:

$$z = \int_{-\frac{a}{2}}^{\frac{a}{2}} t\, dt = \left[\frac{t^2}{2}\right]_{t=a/2} - \left[\frac{t^2}{2}\right]_{t=-\frac{a}{2}} = 0$$

Units

- Target signal: $|x(t)|^2$ is any units, say W
- Then: $\|x\|^2 = \int |x(t)|^2 dt = W \times$ sec
- Say RX signal $|r(t)|^2$ has units W
- Then, $\frac{x(t)}{\|x\|}$ has units $\frac{1}{\sqrt{sec}}$
- Hence, normalized MF output squared is:

$$|z|^2 = \frac{1}{\|x\|^2} \left| \int x^*(t) r(t) dt \right|^2 = \frac{1}{W \times sec} [W \times sec]^2 = W \times secs = J$$

  - $|z|^2$ has output of energy

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SNR of the MF Detector

- Suppose signal is present: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}, \ \boldsymbol{w} \sim CN(0, N_0\boldsymbol{I})$
- Compute scaled MF detector: $z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|}$
- Then: $z = \frac{1}{\|\boldsymbol{x}\|}\boldsymbol{x}^*(h\boldsymbol{x} + \boldsymbol{w}) = h\|\boldsymbol{x}\| + v, \quad v = \frac{\boldsymbol{x}^*\boldsymbol{w}}{\|\boldsymbol{x}\|}$
- Signal energy: $E_{sig} = |h|^2\|\boldsymbol{x}\|^2$
- Noise energy: $E|v|^2 = \frac{1}{\|\boldsymbol{x}\|^2}E|\boldsymbol{x}^*\boldsymbol{w}|^2 = \frac{\|\boldsymbol{x}\|^2}{\|\boldsymbol{x}\|^2}N_0 = N_0$
- SNR of the MF detector output:

$$\boxed{\gamma = \frac{E_{sig}}{E|v|^2} = \frac{|h|^2\|\boldsymbol{x}\|^2}{N_0}}$$

SNR and RX Power

- From previous slide, SNR is $\gamma = \frac{E_{sig}}{E|v|^2} = \frac{|h|^2 \|\boldsymbol{x}\|^2}{N_0}$
- E_{sig} = Total RX energy in time window
- N_0 = Noise PSD
- Example:
  - Suppose RX power is $P_{rx} = -100$ dBm, integration time $T = 4\mu s$, $N_0 = -170$ dBm/Hz
  - In linear scale $E_{sig} = P_{rx}T$ so $\gamma = \frac{P_{rx}T}{N_0}$
  - In dB: $\gamma = P_{rx} + 10\log_{10} T - N_0$
  - $10\log_{10} T = 10\log_{10}(4(10)^{-6}) = 2(3) - 6(10) = -54$
  - Therefore: $\gamma = P_{rx} + 10\log_{10} T - N_0 = -10 - 54 + 170 = 16$ dB

Error Probabilities

- Consider normalized MF: $z = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|}$ where we detect signal if $|z|^2 \geq t$
- It can be shown (see Appendix 1)
- Probability of false alarm: $P_{FA} = \exp(-t/N_0)$
  - Decreases with threshold t
- Probability of missed detection:
  - Complicated expression of SNR γ and threshold t (see Appendix 1)
  - Decreases with γ and increases with threshold t

Selecting the Threshold

- From previous slide: $P_{FA} = \exp(-t/N_0)$
- Set threshold to $t = -N_0 \log P_{FA}^{tgt}$
  - P_{FA}^{tgt} =target FA probability = Maximum allowable FA rate
- Then, P_{MD} will depend on the SNR
- Typical FA probabilities are very low: $P_{FA}^{tgt} = 10^{-9}$ to 10^{-7}
- As a result, SNR for detection is often high: γ
  - Can be $\gamma \geq 10$ to 20 dB

Simulation

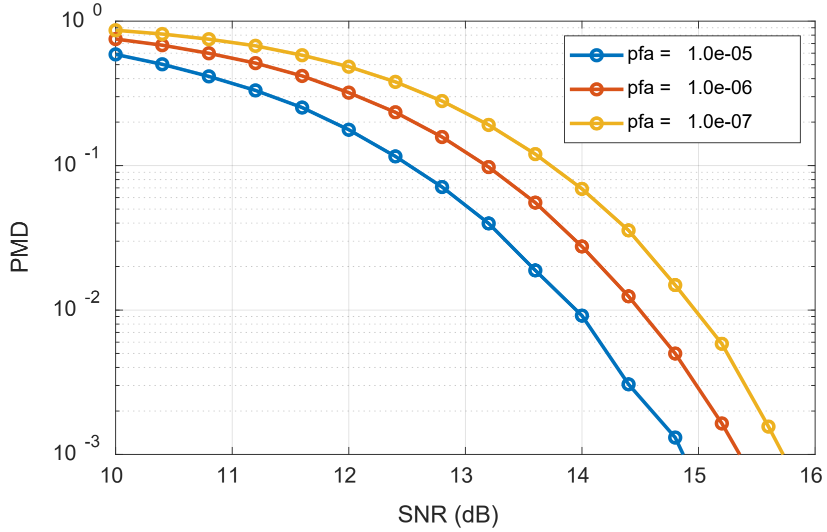
```
% FA targets to test
pfaTest = [1e-5,1e-6,1e-7];
nfa = length(pfaTest);
legstr = cell(nfa,1);
for ifa = 1:nfa
    % Compute FA target
    pfaTgt = pfaTest(ifa);
    t = -log(pfaTgt);

    % Measure PMD
    ntest = 1e5;
    snrTestTheory = linspace(10,18,21)';
    nsnr = length(snrTestTheory);
    pmdTheory = zeros(nsnr,1);

    for isnr = 1:nsnr
        snr = snrTestTheory(isnr);
        A = 10.^(0.05*snr);
        z = A + (randn(ntest,1)+1i*randn(ntest,1))/sqrt(2);
        rho = abs(z).^2;
        pmdTheory(isnr) = mean(rho < t);
    end

    semilogy(snrTestTheory, pmdTheory, 'o-', 'Linewidth', 2);
    hold on;

    legstr{ifa} = sprintf('pfa = %9.1e', pfaTgt);
end
```

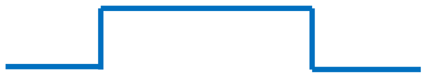


- ❑Theoretically calculated threshold based on PFA target
- ❑Simulate PMD based on SNR

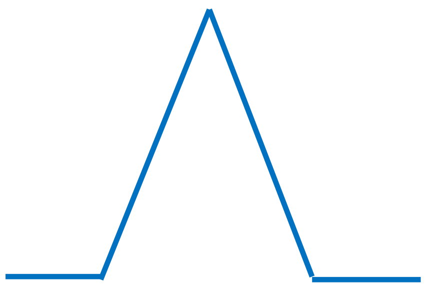
Problems with MF

- Consider normalized MF: $z = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|} = \|\boldsymbol{r}\| \cos\theta$
- Problem 1: FA threshold requires knowledge of N_0
  - Threshold $t = -N_0 \ln P_{FA}^{TGT}$
- Problem 2: Any signal r with $\|\boldsymbol{r}\|$ can make z large
  - Any high-power signal can trigger a detection
- Example

Target $\boldsymbol{x}$



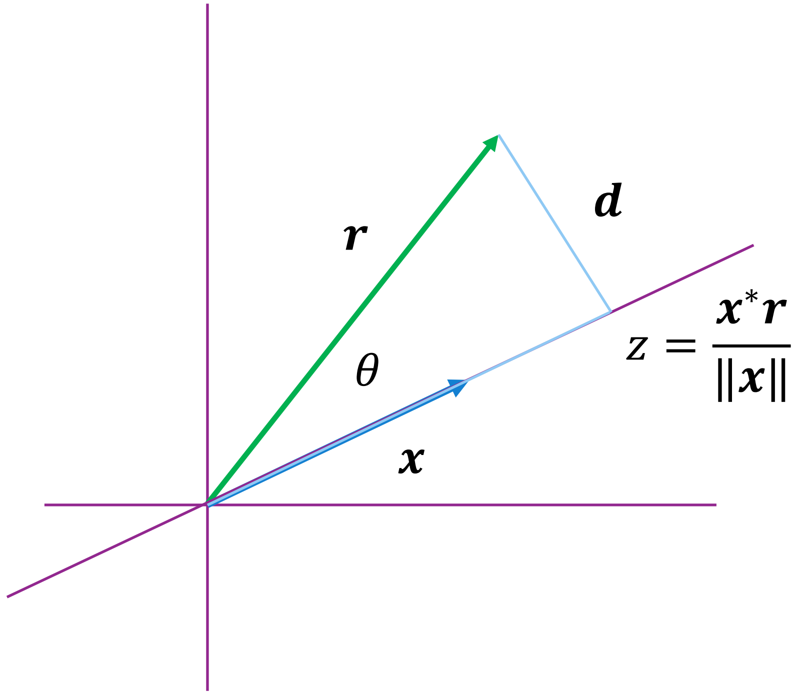
RX $\boldsymbol{r}$ matches target
$\boldsymbol{r} \approx h\boldsymbol{x}$ for some h
But z is low



RX $\boldsymbol{r}$ does not match target well
$\boldsymbol{r} \neq h\boldsymbol{x}$ for any h
But z is high

Correlation Coefficient Method

- Correlation coefficient method:
  - Compute $\rho = \frac{z}{\|\boldsymbol{r}\|} = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{r}\|\|\boldsymbol{x}\|} = \cos\theta$
  - Signal is detected if $\rho^2 \geq t$ for some $t \in [0,1]$
- Property 1:
  - ρ^2 = Represents fraction of energy of $\boldsymbol{r}$ in span of $\boldsymbol{x}$
  - Signals do not trigger detection just by being large
- Property 2:
  - FA alarm target does not depend on N_0
  - Suppose $\boldsymbol{r} = CN(0, N_0\boldsymbol{I})$
  - Distribution does not depend on scaling N_0
  - Can show (HW) $t = 1 - P_{FA}^{N}$,



SNR Estimate

- Given vectors $\boldsymbol{r}$ and $\boldsymbol{x}$, can show that the best linear estimate of

$$\boldsymbol{r} = \alpha \boldsymbol{x} + \boldsymbol{d}, \qquad \alpha = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|^2}, \qquad \|\boldsymbol{d}\|^2 = \|\boldsymbol{r}\|^2 (1 - \rho^2)$$

- Hence, SNR estimate is:

$$\gamma = \frac{|\alpha|^2 \|\boldsymbol{x}\|^2}{\|\boldsymbol{d}\|^2} = \frac{|\boldsymbol{x}^* \boldsymbol{r}|^2}{\|\boldsymbol{r}\|^2 \|\boldsymbol{x}\|^2 (1 - \rho^2)} = \frac{\rho^2}{1 - \rho^2}$$

- Thus, correlation coefficient provides an estimate of the SNR:

$$\gamma = \frac{\rho^2}{1 - \rho^2}$$

In Class Exercise

Signal Detection

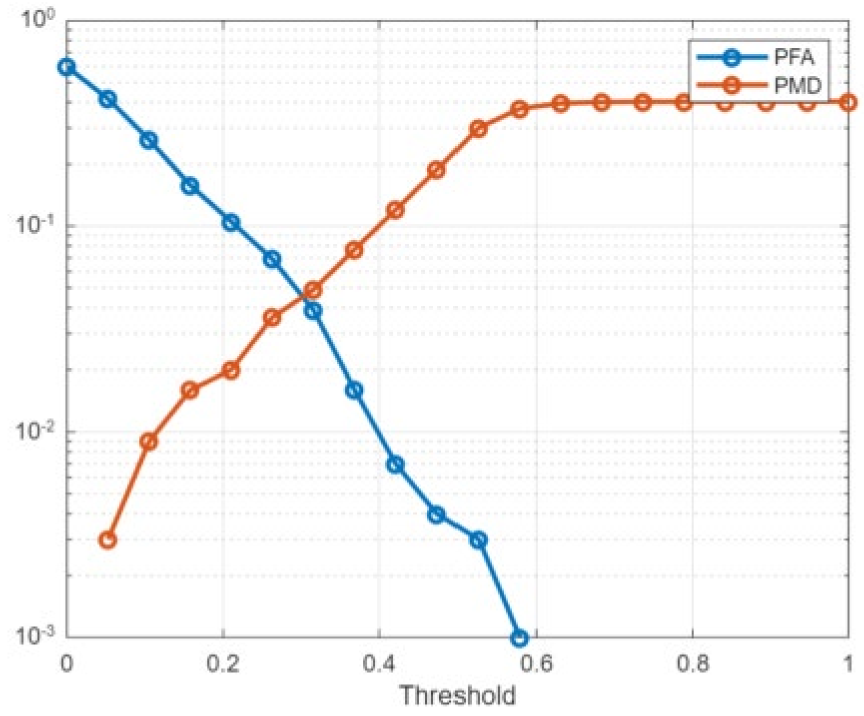
In this section, we will simulate a simple signal detection. We suppose we have a complex baseband signal of length `ns`. The unknown variable is `u=0` or `1` depending if a signal is present. The RX signal, r, can then be modeled as:

```
r(j) = u*h*x(j) + w(j),    j = 1,..., ns
```

where the channel gain and noise are modeled as complex Gaussians:

```
w(j) ~ CN(0,wvar),    h ~ CN(0,hvar)
```

```
p0 = 0.6;    % P(u=0)=1-P(u=1)
ns = 8;      % Signal length
snr = 10;    % hvar/wvar
hvar = 1;                   % Mean channel gain
wvar = db2pow(-snr);        % Noise energy per sample
ntrial = 1000;              % Number of trials to test
x0 = exp(1i*rand(ns,1)*2*pi);  % Random true signal
```



Outline

- Detection and Synchronization Problem
- Hypothesis Testing
- Match Filtering for Detection at a Known Delay
- Match Filter SNR and Error Probabilities
- Match Filtering Convolution with an Unknown Signal Delay
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- Appendix 1. Error Probability Calculation Details
- Appendix 2. Matched Filtering as a Generalized Likelihood Ratio Test

Match Filtering with Unknown Delay

- Synchronization signal $x[n], n = 0,1, \ldots, N-1$
- RX signal at delay k:
  - $r[n] = hx[n-k] + w[n]$
- Problem: Detect if signal is present. If so, what is the delay k?
- Match filter (without normalization) at delay k is:

$$z[k] = \sum_n r[n+k]x^*[n]$$

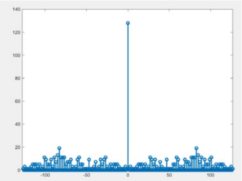
- Hypothesis test:
  - $|z[k]|^2 \geq t \Rightarrow$ Detect signal at delay at k

Match Filtering as a Convolution

- Match filter (without normalization) at delay k is:

$$z[k] = \sum_n r[n+k]x^*[n]$$

- Define adjoint signal: $\tilde{x}[n] = x^*[-n]$
  - Complex conjugate and time reversal
- MF output can be computed via a convolution:
  - $z[k] = \sum_n r[n+k]x^*[n] = \sum_n r[n+k]\tilde{x}[-n] = \sum_n r[k-n]\tilde{x}[n] = (r * \tilde{x})[k]$



$r[n]$ → Filter with $\tilde{x}[n]$ → $z[n]$ → Peak detection → Detected signal locations

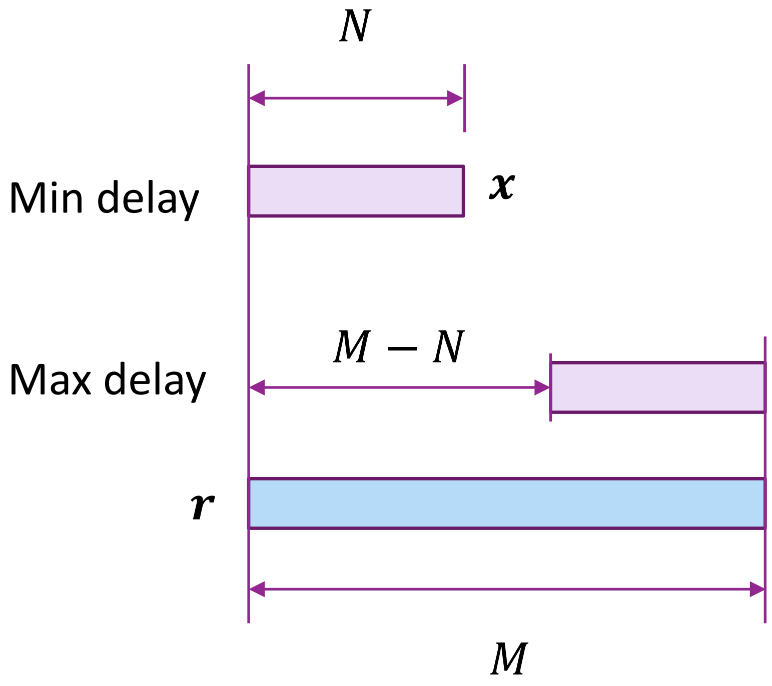
Boundary Conditions

- Match filter (without normalization) is:

$$z[k] = \sum_n r[n+k]x^*[n]$$

- Suppose:
  - Target $x[n]$ has N samples
  - RX signal $r[m]$ has M samples
- Then, we can test up to $K = M - N$ hypotheses:
  - $z[k],\ k = 0, \dots, M - N + 1$
- Compute in MATLAB with “valid” mode
  - If $\boldsymbol{x}$ and $\boldsymbol{r}$ are column vectors

```
z = conv(r,flipud(conj(x)),"valid");
```



Correlation Coefficient Method

- To compute correlation coefficient:
  - Compute un-normalized MF: $z[k] = \sum_{n=0}^{N-1} r[n+k]x^*[n]$
  - Moving average RX energy: $E_r[k] = \sum_{n=0}^{N-1} |r[n+k]|^2$
  - Signal energy: $E_x = \sum_{n=0}^{N-1} |x[n]|^2$
- Then, correlation coefficient squared is:

$$\rho^2[k] = \frac{|z[k]|^2}{E_r[k]\, E_x}$$

```
nx = length(x);
xadj = flipud(conj(x));
z = conv(r, xadj, "valid");
Er = conv(abs(r).^2, ones(nx,1), "valid");
Ex = sum(abs(x).^2);
rhosq = abs(z).^2./Er/Ex;
```

- Can be perform with two parallel convolutions

Further Analysis Details

- We need to examine three key practical issues that degrade performance

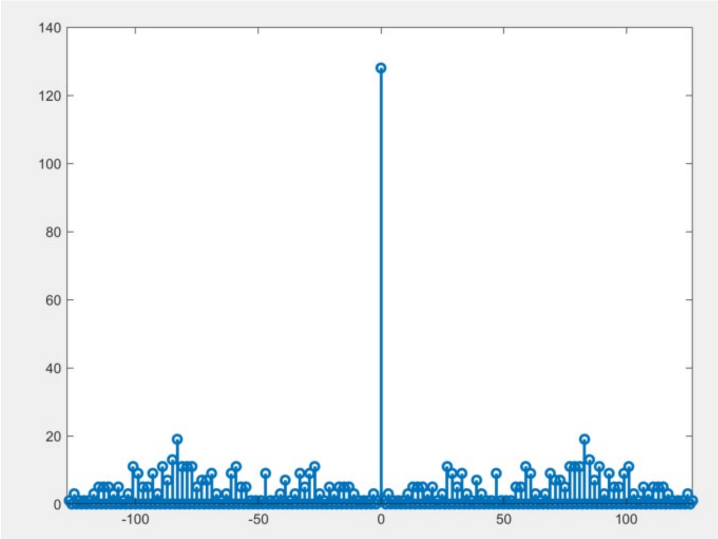
- Preamble auto-correlation
- Multi-path
- Carrier offset

Signal Auto-Correlation

- Consider what happens with no noise:
  - $r[n] = hx[n-k_0],\ k_0 =$ “True” delay
- Run match filter: $z[k] = (r * \tilde{x})[k]$
- Can show output is: $z[k] = hR_x[k-k_0]$
  - $R_x[\ell] =$autocorrelation of transmitted signal
  - $R_x[\ell] = \sum_n x[n]x^*[n-\ell]$
- Since we want $z[k]$ small for $k \neq k_0$, we want:

$$R_x[\ell] \approx 0 \text{ for } \ell \neq 0$$

- Many sequences with low auto-correlation
  - Golay, Walsh, ….



Auto-correlation of Golay 128 sequence
Used in 802.11ad preamle

Multipath

- Up to now we have assumed that there is a single path:

$$r[n] = hx[n-k_0]$$

- But, in reality there is often multipath:

$$r[n] = \sum_k h[k]x[n-k]$$

  - Due to multi-path in channel and pulse shape filtering

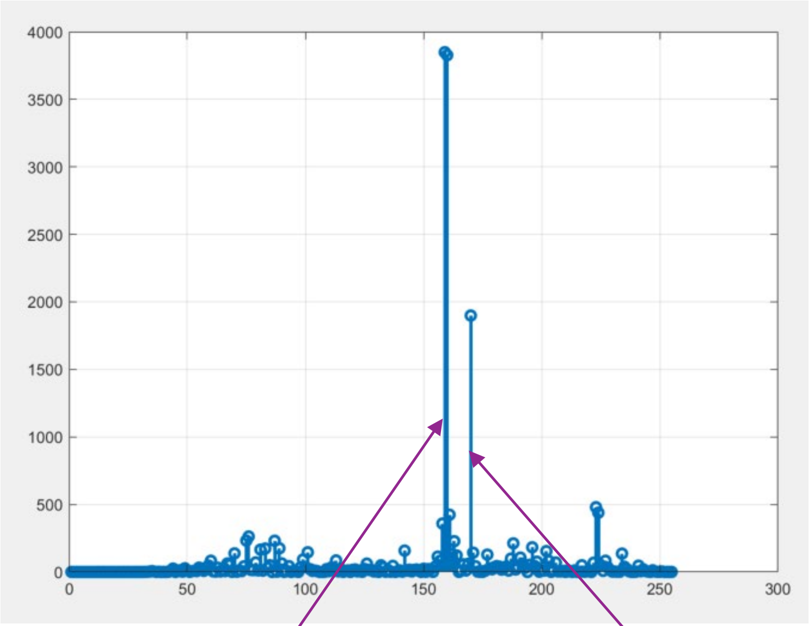
- Match filter has delayed copies of auto-correlation:

$$z[n] = \sum h[k]R_x[n-k]$$

  - One peak in MF output for each path

Ex: Two path channel

$$h[n] = sinc(n-0.5) + 0.5sinc(n-10.2)$$

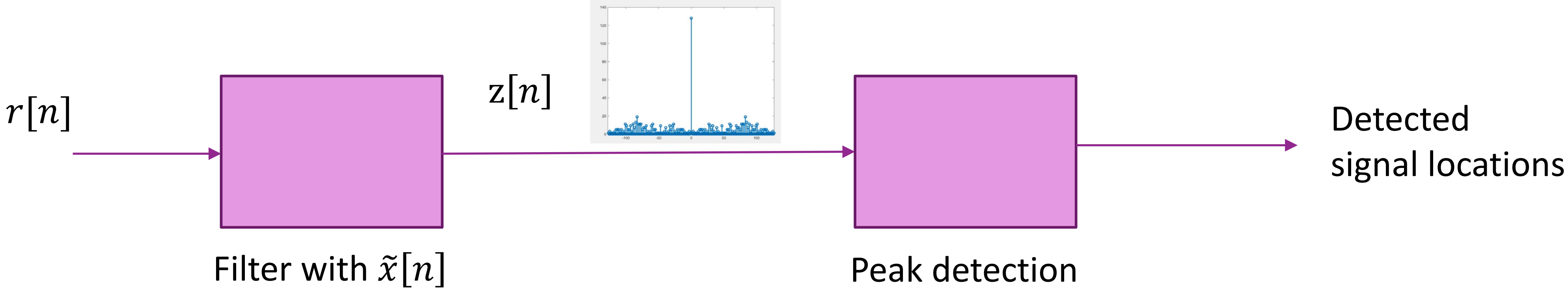


Path at $k = 0.5$ Path at $k = 10.2$

Frequency Offsets

- When initially searching for a preamble, there may be a significant carrier offset
- Causes a phase rotation in samples:
$$r[n] = e^{i\theta n} h x[n-k] + w[n]$$
  - θ is the phase rotation per sample
  - $\theta = \Delta f\ T,\ \Delta f$ =frequency error, T =sampling rate
- Must integrate over range where phase does not change significantly
  - Pre-amble length must be $N \ll \frac{1}{\Delta f T}$
- Example: Suppose the carrier offset =10 ppm, $f_c = 60$ GHz and $\frac{1}{T} = 1.76$ Gs/s
  - $\Delta f T = \frac{(10)^{-5}(60)(10)^9}{1.76(10)^9} = 3.4(10)^{-4} \Rightarrow \frac{1}{\Delta f\ T} \approx 2.9(10)^3$ samples
  - In time duration, this is $\frac{1}{\Delta f} = 1.67$ us
  - A very short time before the signal is completely rotated

Detailed Simulation Example

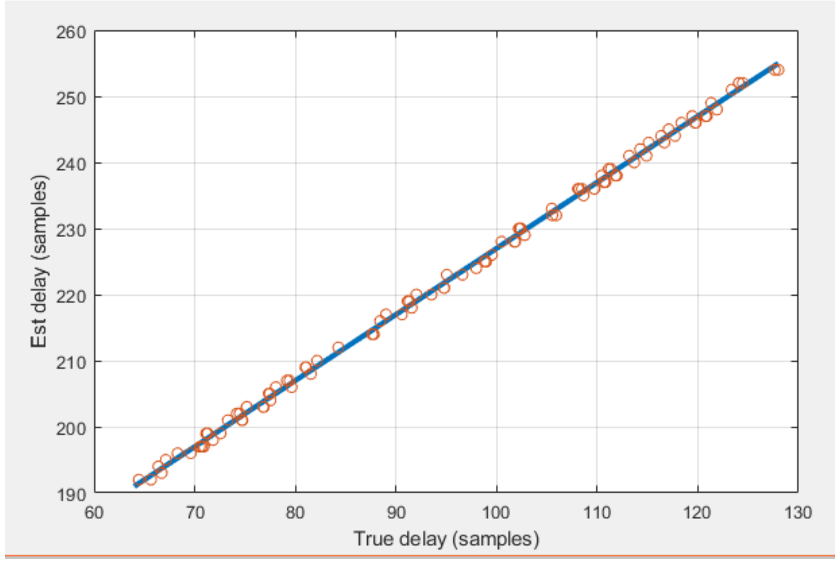


- Transmit 128 length Golay pre-amble
- Filter through channel with single (possibly fractional) delay
  - $r[n] = h[n] * x[n] + w[n], \quad h[n] = sinc(n - \frac{\tau}{T})$
- Set threshold for FA target of 10^{-3} per 1000 samples
- Measure MD probability as a function of the SNR

Calibration

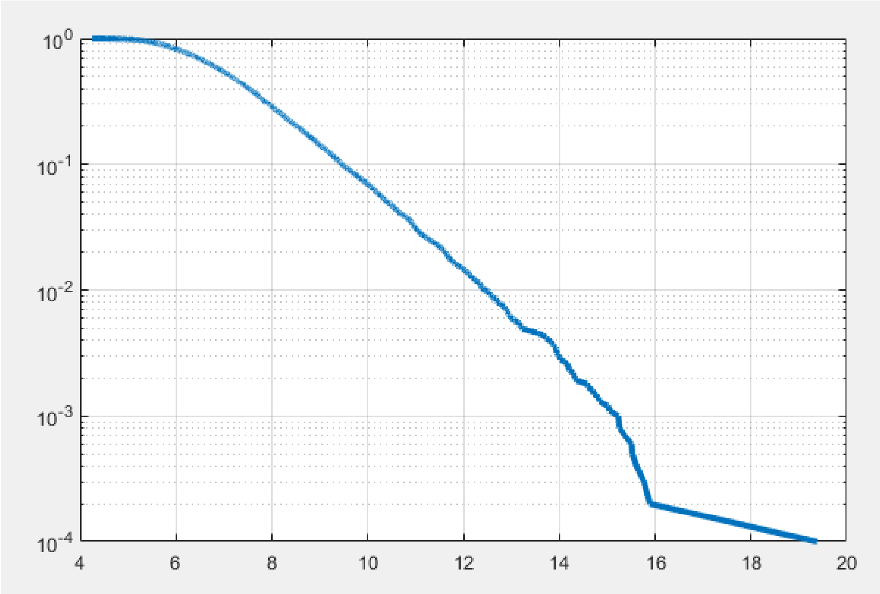
- Need to calibrate the FA probability and delay offset

Estimated Delay



True Delay

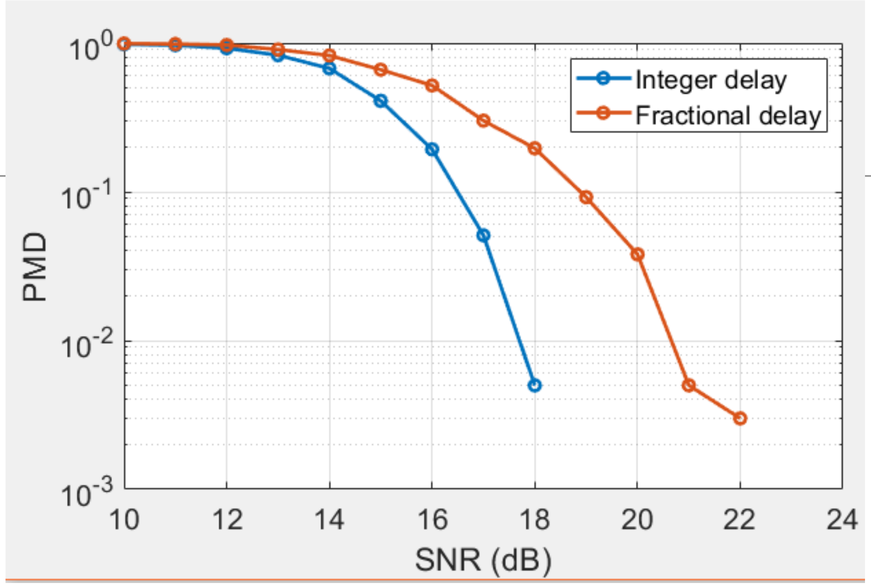
PFA



Threshold

Missed Detection

```
for isnr = 1:nsnr
    % Get the SNR
    snr = snrTest(isnr);
    wvar = 10.^(-0.1*snr)*npre;

    dly0 = unifrnd(64,128,ntest,1);
    dlyEst = zeros(ntest,1);
    rhoMax = zeros(ntest,1);

    for it = 1:ntest
        % Create a random delay
        gain = exp(1i*2*pi*rand(1));
        x = delaysig(xpre,gain,dly0(it),nsamp);

        % Add noise
        w = (randn(nsamp,1) + 1i*randn(nsamp,1))*sqrt(wvar/2);
        r = x + w;

        % Estimate the delay
        [rhom, im, ~] = predetect(r,xpre,maxdly);
        rhoMax(it) = rhom;
        dlyEst(it) = im - dlyOff;

    end

    I = (rhoMax > tfa);
    pmd(isnr) = 1-mean(I);
    dlyerr(isnr) = sqrt( mean((dlyEst(I) - dly0(I)).^2) );
    fprintf(1,'SNR = %12.4e PMD=%12.4e dly=%12.4e\n', ...
        snr, pmd(isnr), dlyerr(isnr));
end
```



- Loss of about 3dB with fractional delay offset
- Signal energy is split in two samples
- Need to use over-sampling to compensate
  - See lab

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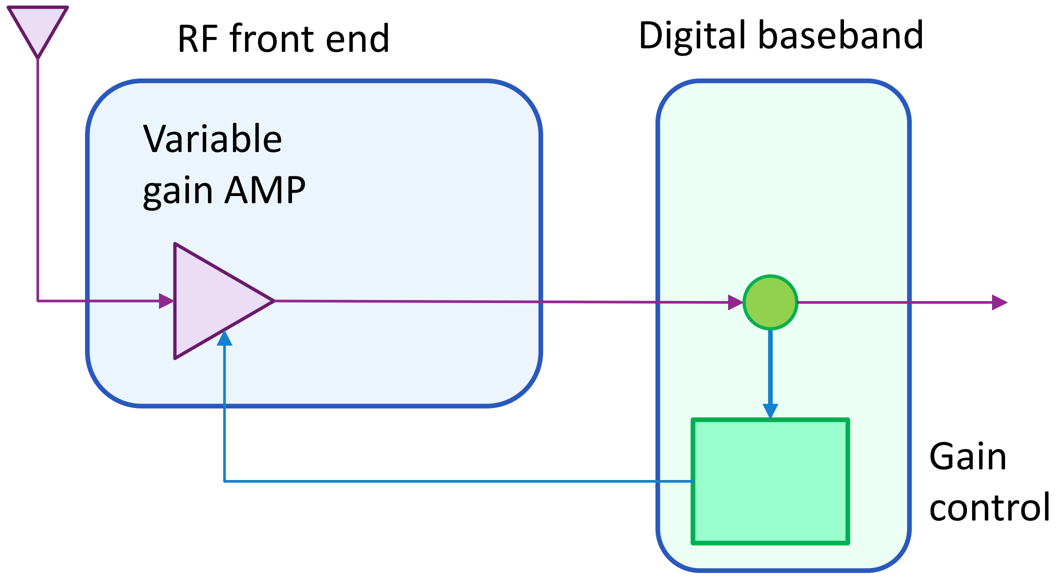
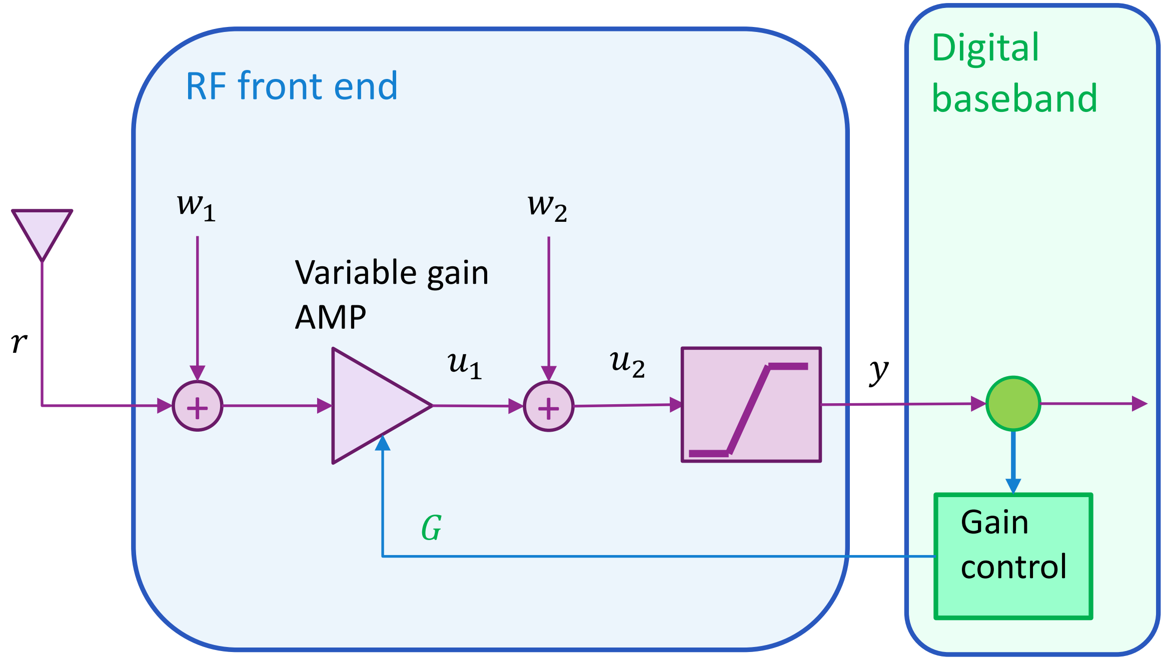
Automatic Gain Control



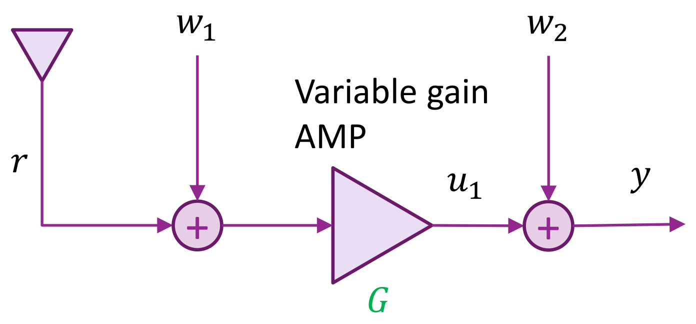
- AGC goal: Bring RX signal to "correct" level
- Tradeoff of two factors
- Want high gain: Overcome noise after AMP
- But gain too high ⇒ saturates RX
- AGC finds optimal level

Mathematical Model



- RF front-end
  - Linear gain with noise:
    $$u_1(t) = \sqrt{G}r(t) + w_1(t)$$
  - Noise after AMP
    $$u_2(t) = u_1(t) + w_2(t)$$
  - Memoryless nonlinearity:
    $$y(t) = \phi(u_2(t))$$
  - $w_1(t)$ and $w_2(t)$ have PSD N_1 and N_2
- Digital baseband
  - Measures RX power or SNR
  - Controls gain G

Analysis with No Nonlinearity

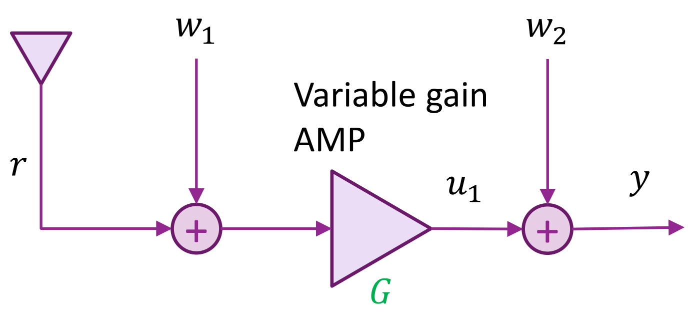


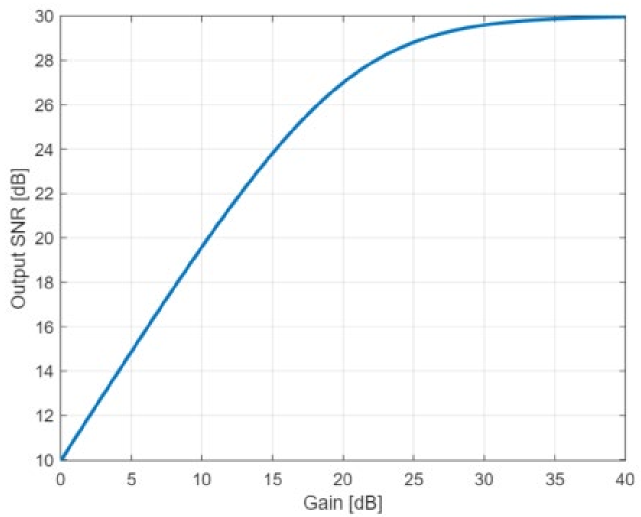
- With no nonlinearity:
  - $y = \sqrt{G}(r + w_1) + w_2 = \sqrt{G}r + v$
  - $v = \sqrt{G}w_1 + w_2$
- PSD of signal component $\sqrt{G}r$ in output: GS_r
- PSD of noise in output: $S_v = GN_1 + N_2$
- Output SNR:

$$\gamma_{out} = \frac{GS_R}{GN_1 + N_2} = \frac{\gamma_1}{1 + \frac{\gamma_1}{G\gamma_2}}$$

  - $\gamma_1 = \frac{S_r}{N_1}, \quad \gamma_2 = \frac{S_r}{N_2}$

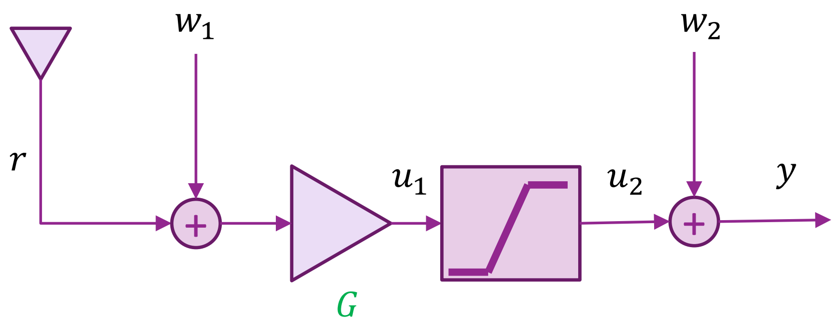
Analysis with No Nonlinearity Example

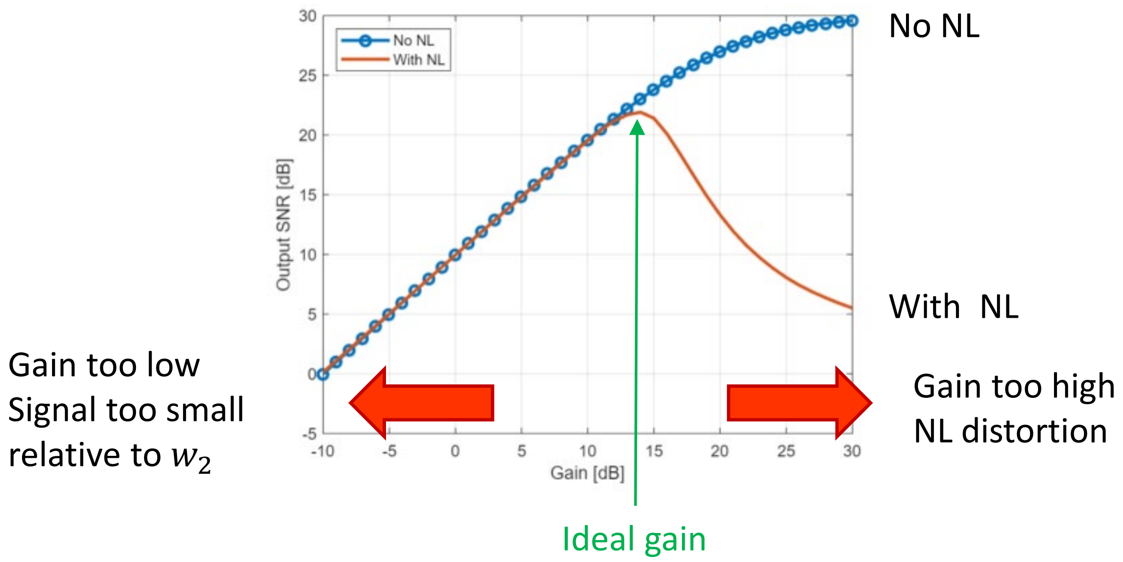




- Output SNR from previous slide: $\gamma_{out} = \frac{\gamma_1}{1+\frac{\gamma_1}{G\gamma_2}}$
- Fig to left: $\gamma_1 = 30$ dB, $\gamma_2 = 10$ dB
- Observations with no nonlinearity:
  - Increasing gain always improve output SNR
  - Saturates at $\gamma_{out} \to \gamma_1$ as $G \to \infty$
- Design principle:
  - Have high SNR (low noise) in first stage
  - Increase gain to overcome noise in later stage
- Typical RF design starts with a low noise amplifier (LNA)

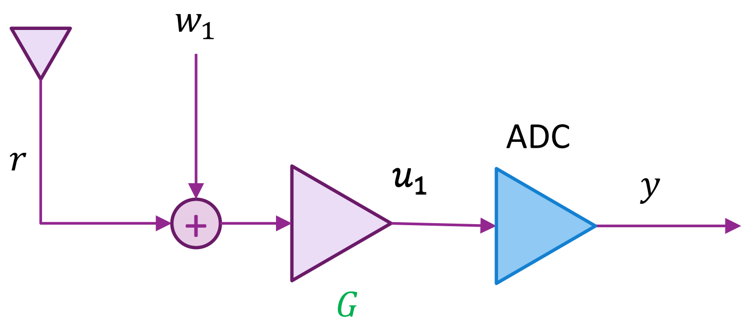
Example with Saturation Nonlinearity

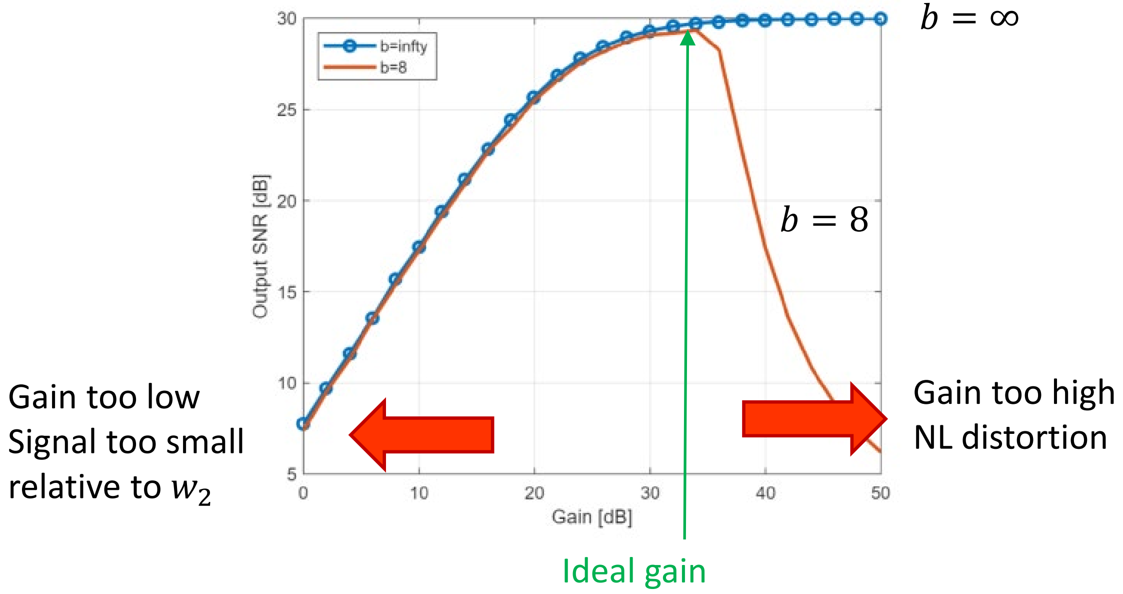




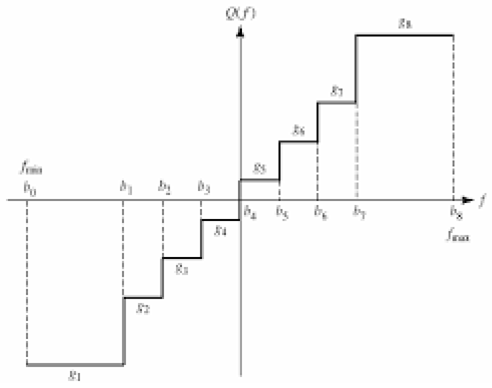
- With high gain G :
  - Signal u_1 becomes large
  - May saturate component $\Rightarrow$ distortion
- Saturation occurs in many RF components
  - E.g., mixer, LNA
- Example:
  - $\gamma_1 = \frac{E|r|^2}{E|w_1|^2} = 30$ dB, , $\gamma_2 = \frac{E|r|^2}{E|w_2|^2} = 10$ dB
  - Nonlinearity is a saturation: Clips at $\pm A$
  - Saturation level $\frac{|A|^2}{E|w_1|^2} = 50$ dB
- We see ideal gain trades off gain and NL distortion

Example with ADC

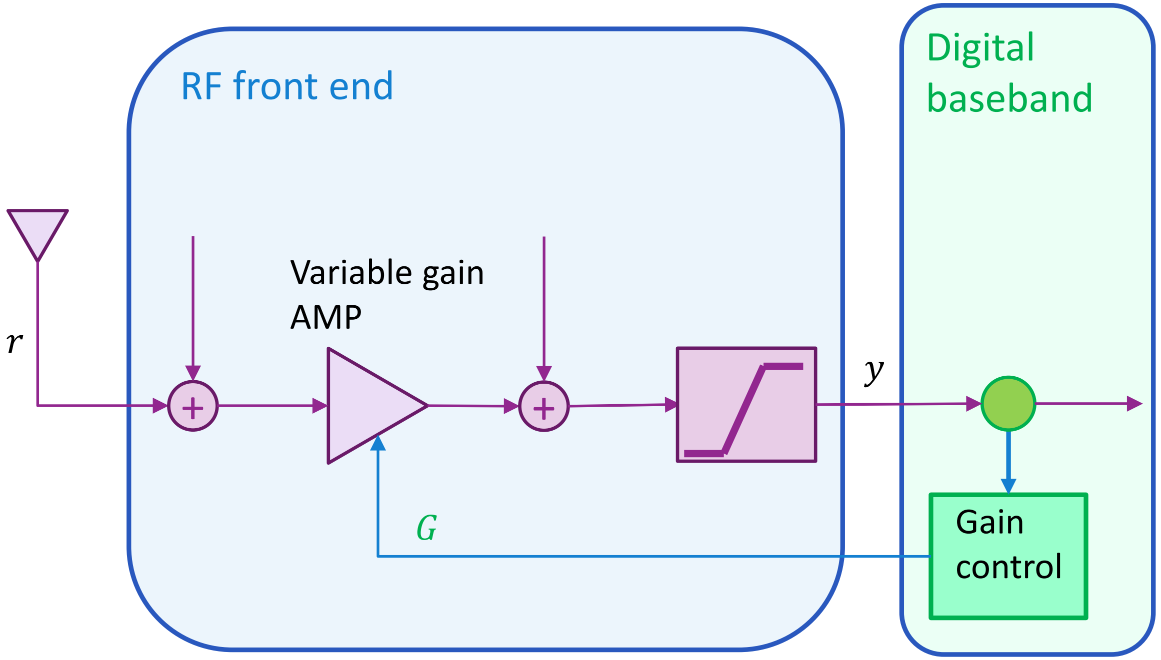




- Example ADC:
  - Similar to quantization + saturation
  - Saturates at $y \in [-2^{b-1}, 2^{b-1} - 1]$ with b bits
- With finite bit width
  - Set gain to not overflow ADC

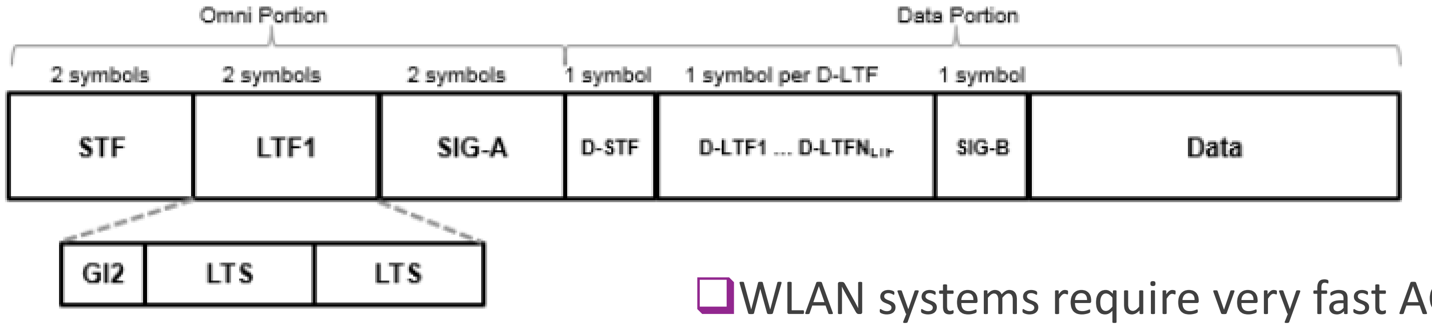


Practical Gain Control



- **Measure RX power**
  - $E_{rx}[n] = \sum_{i=0}^{L-1} |y[n-i]|^2$
  - RX power over last L symbols
- **Adjust gain:**
  - $E_{rx}[n] > t \Rightarrow$ Decrease gain
  - $E_{rx}[n] \leq t \Rightarrow$ Increase gain
- **Threshold t**
  - Set to ensure signal is not saturating

Gain Control in 802.11



- WLAN systems require very fast AGCs
- STF (short training field). Duration is $8\ \mu s$
  - Used for detection *and* AGC
  - Typically set AGC in first ~2 μs
  - Perform detection in remaining $6\ \mu s$
- Cannot detect STF until AGC has run
  - Otherwise, signal will be too low in SNR or too distorted
- AGC is implemented in hardware

The 9361 Integrated Circuit

RX

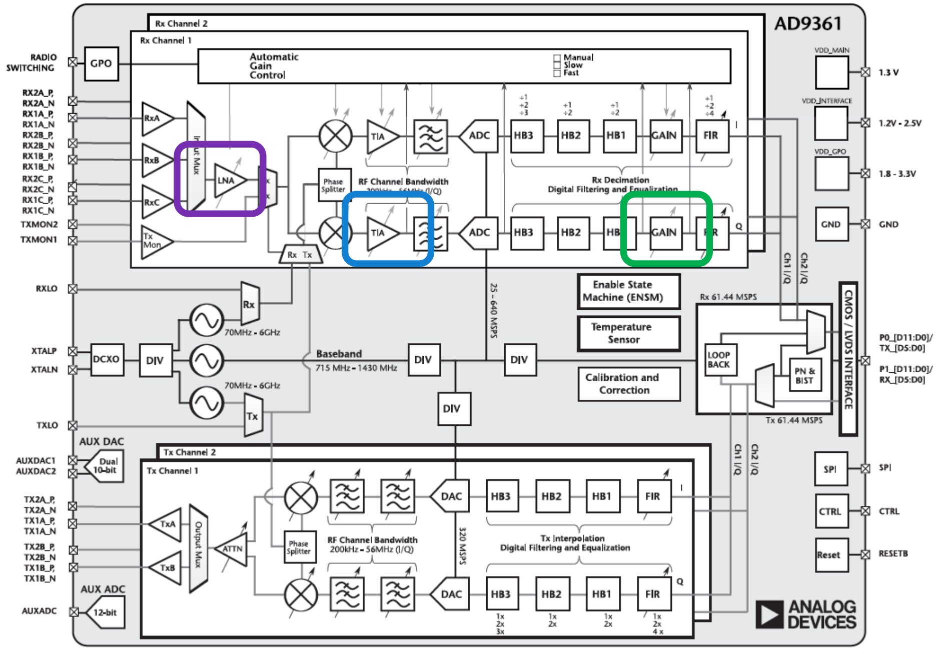


Figure 1.8 Integrated ZIF architecture used in the Pluto SDR.

TX

- Gain performed in at least three stages
  - LNA gain control at RF
  - TIA (transimpedance AMP) gain at IF
  - Digital gain during digital filtering
- Single gain setting selects all three gains
  - Ensure all intermediate points do not overflow

AGC in the Pluto

sdrrx

Create receiver System object for radio hardware

`GainSource` — **Gain source**
`'AGC Slow Attack'` (default) | `'AGC Fast Attack'` | `'Manual'`

Gain source, specified as one of the following:

- `'AGC Slow Attack'` — For signals with slowly changing power levels
- `'AGC Fast Attack'` — For signals with rapidly changing power levels
- `'Manual'` — For setting the gain manually with the `Gain` property

Data Types: `char` | `string`

`Gain` — **Radio receiver gain**
`10` (default) | scalar

Radio receiver gain in dB, specified as a scalar from `-4` to `71`.

- MATLAB interface gain control
- AGC Fast Attack
  - AGC automatically performed in HW
  - Ideal for WLAN with fast gain
  - But host has no visibility to gain selected
- Manual
  - User can manually select gain
  - We will use this in lab
  - But it is very slow
  - Just for education

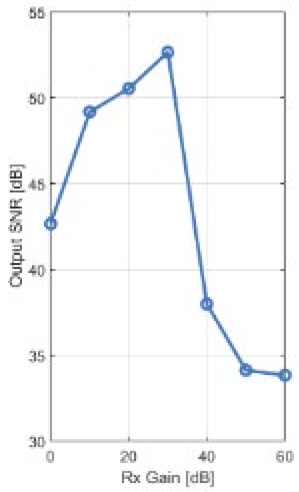
SDR Lab

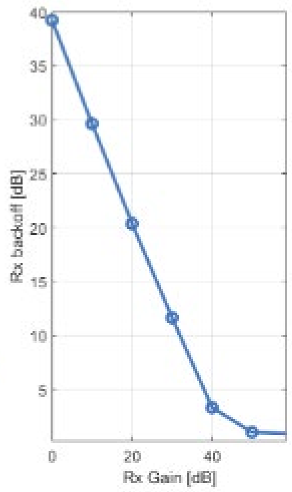
Lab 5: Gain Control and Building a Simple AGC

Gain control is a fundamental operation in a receiver to adjust signal levels to a correct value. In this lab, you will learn to:

- Mathematically model a receiver with variable gain and saturation
- Simulate the effect of gain and nonlinearities on the output SNR
- Manually control the gain on the SDR
- Measure the RX backoff of a received signal
- Build a simple AGC to maintain a target RX backoff
- Measure the RX power with gain

https://github.com/sdrangan/sdrlab/tree/main/lab05_gain





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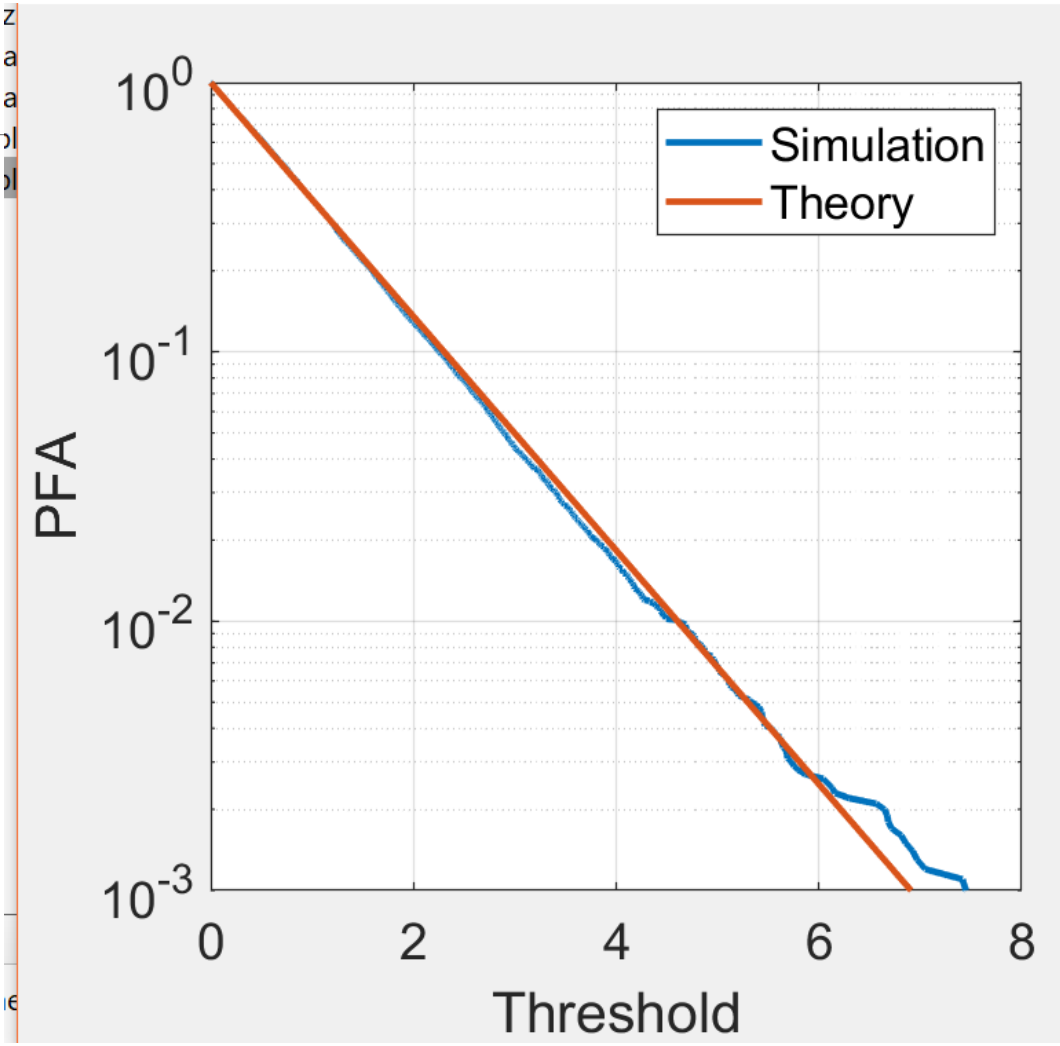
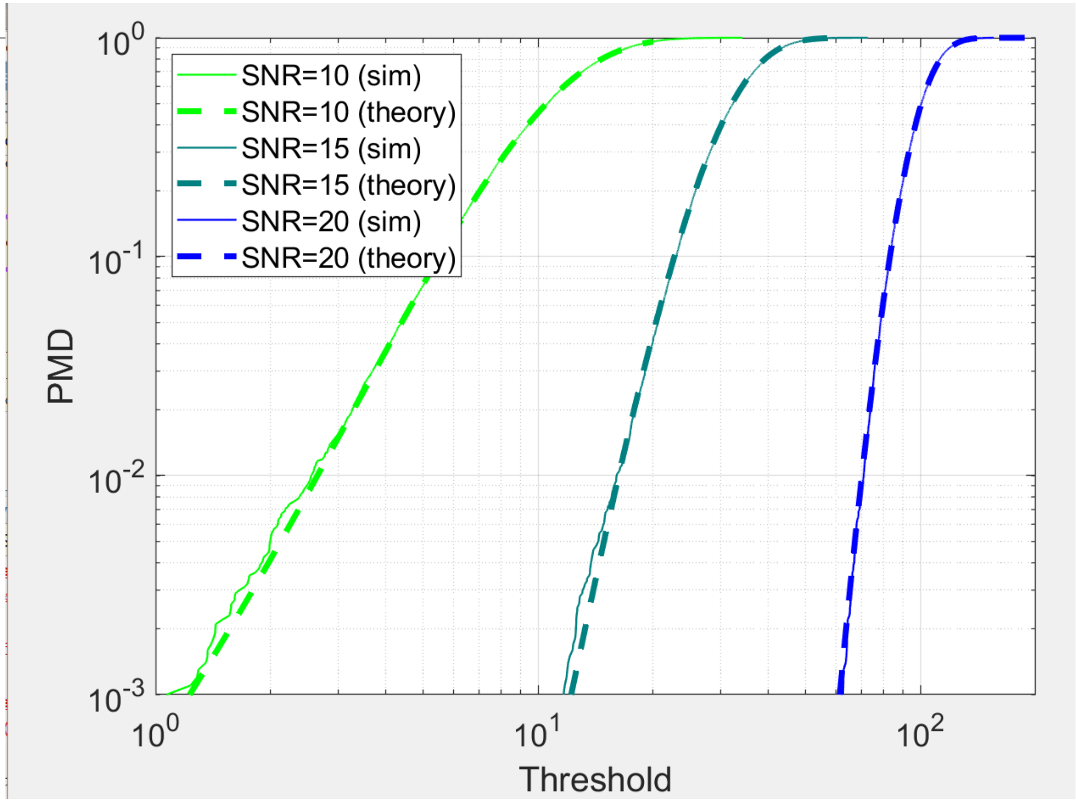
False Alarm

- False alarm
  - Under H_0: $\boldsymbol{r} = \boldsymbol{w}$, $\boldsymbol{w} \sim CN(0, N_0 \boldsymbol{I})$
  - Statistic $z = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}^* \boldsymbol{w}}{\|\boldsymbol{x}\|}$
  - This is a linear function of a Gaussian
  - $E(z) = \frac{\boldsymbol{x}^* E(\boldsymbol{w})}{\|\boldsymbol{x}\|} = 0,$
  - $E|z|^2 = \frac{\boldsymbol{x}^* E(\boldsymbol{w}\boldsymbol{w}^*)\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = N_0 \frac{\boldsymbol{x}^* \boldsymbol{x}}{\|\boldsymbol{x}\|^2} = N_0$
  - Hence, $z \sim CN(0, N_0)$
  - Hence $y = |z|^2$ is exponential with $E(y) = N_0$
  - $P_{FA} = P(y \geq t) = e^{-t/N_0}$



Missed Detection

- Under H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w},\ \boldsymbol{w} \sim CN(0, N_0\boldsymbol{I})$
- Statistic $z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|} = A + \frac{\boldsymbol{x}^*\boldsymbol{w}}{\|\boldsymbol{x}\|}, A = h\|\boldsymbol{x}\|$
- Similar to FA calculation: $z \sim CN(A, N_0)$,
- Can show: $y = |z|^2 \sim \frac{N_0}{2} v$
  - v is a non-central chi squared with 2 degrees of freedom
  - Non-centrality parameter $\lambda = \frac{2|h|^2\|\boldsymbol{x}\|^2}{N_0} = 2\ SNR$



Outline

- Detection and Synchronization Problem
- Hypothesis Testing
- Match Filtering for Detection at a Known Delay
- Match Filter SNR and Error Probabilities
- Match Filtering Convolution with an Unknown Signal Delay
- Automatic Gain Control (AGC)

- Appendix 1. Error Probability Calculation Details
- Appendix 2. Matched Filtering as a Generalized Likelihood Ratio Test

Likelihood Ratio Test

- In vector form:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Likelihoods:
  - $p(r|H_0, \sigma^2) = \frac{C}{\sigma^{2N}} \exp\left(-\frac{\|\boldsymbol{r}\|^2}{\sigma^2}\right),$
  - $p(r|H_1, , \sigma^2, h) = \frac{C}{\sigma^{2N}} \exp\left(-\frac{\|\boldsymbol{r} - h\boldsymbol{x}\|^2}{\sigma^2}\right)$
  - Cannot apply regular LRT since parameters are unknown
  - GLRT
  - $\Lambda_0(r, \sigma^2) \coloneqq -\frac{1}{N} \ln p(r|H_0) = \frac{1}{N} \ln \sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2}$

Generalized Likelihood Ratio Test

- Null hypothesis
  - $\Lambda_0(r,\sigma^2) := -\frac{1}{N}\ln p(r|H_0) = \frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2}$
  - $\overline{\Lambda}_0(r) := \min_{\sigma^2}\frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2} = \frac{1}{N}\ln\frac{\|\boldsymbol{r}\|^2}{N} + 1$
- Present hypothesis:
  - $\Lambda_1(r,\sigma^2,h) := -\frac{1}{N}\ln p(r|H_1) = \frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}-h\boldsymbol{x}\|^2}{N\sigma^2}$
  - Minimize over h: $\min_{\mathrm{h}}\|\boldsymbol{r}-h\boldsymbol{x}\|^2 = \|\boldsymbol{r}\|^2 - \frac{|\boldsymbol{x}^*\boldsymbol{r}|^2}{\|\boldsymbol{x}\|^2}$
  - $\overline{\Lambda}_1(r) := \min_{\sigma^2,h}\ln p(r|H_1) = \frac{1}{N}\ln\frac{1}{N}\left[\|\boldsymbol{r}\|^2 - \frac{|\boldsymbol{x}^*\boldsymbol{h}|^2}{\|\boldsymbol{x}\|^2}\right] + 1$
- GLRT: $L(r) := \overline{\Lambda}_1(r) - \overline{\Lambda}_0(r) = -\ln[1-\rho]\,,\ \rho = \frac{|\boldsymbol{x}^*\boldsymbol{h}|^2}{\|\boldsymbol{x}\|^2\|\boldsymbol{r}\|^2}$
- Details in class