Problems: Synchronization and Detection

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1. *Matched Filter SNR.* Suppose we wish to detect a synchronization signal x_n, $n = 0, 1, \ldots, N-1$ from received complex baseband samples of the form,

$$r_n = hx_n + w_n, \quad w_n \sim \mathcal{C}N(0, N_0), \quad n = 0, 1, \ldots, N-1,$$

where h is the complex channel gain and w_n is AWGN noise.

   (a) Describe the matched filter detector for the signal. You need only describe the MF *without* normalization by $\|\mathbf{r}\|^2$.

   (b) What is the SNR of the detector output as a function of h, x_n and N_0?

   (c) Suppose the signal is received at -100 dBm and the noise power density (including the noise figure) is -170 dBm/Hz. How long should the signal be transmitted such that the SNR of the MF detector is 20 dB?

   (d) Write an expression for the probability of missed detection as a function of the matched filter threshold and the SNR. Your expression will be a double integral (an integral over the complex plane). You do not need to solve the integral. But, you need to precisely describe the limits of integral.

2. *Matched Filter with Carrier Offset.* Similar to the previous problem, consider the MF detector for a signal x_n. However, suppose that there is carrier offset so that the received samples are given by,

$$r_n = he^{in\theta}x_n + w_n, \quad w_n \sim \mathcal{C}N(0, N_0), \quad n = 0, 1, \ldots, N-1,$$

where h is the complex channel gain, w_n is AWGN noise and θ is the phase rotation per sample. The phase rotation is due to carrier offset. Assume $|x_n|$ is constant (e.g. QPSK).

   (a) Suppose you run the MF detector assuming there is no carrier offset (i.e. $\theta = 0$). What is the degradation in SNR as a function of θ and N.

   (b) Suppose the carrier frequency is 2.5 GHz, the oscillator error is 10 ppm. What is the maximum duration of the transmitted signal up to which the SNR is degrated by at most 3 dB.

   (c) If the carrier offset were known, how would you modify the MF detector?

3. *Normalized MF.* We saw in class that the normalized MF for detecting a vector $\mathbf{x}$ from RX samples $\mathbf{r} = h\mathbf{x} + \mathbf{w}$, computes a statistic:

$$\rho := \frac{|\mathbf{x}^*\mathbf{r}|^2}{\|\mathbf{x}\|^2\|\mathbf{r}\|^2}.$$

The statistic $\rho \in [0, 1]$ is the fraction of energy in the direction of the synchronization signal $\mathbf{x}$. Assume the signal is length N.

(a) (This part is bonus). Read about the Beta distribution. Use any facts you can find on the Internet or elsewhere to show that when $\mathbf{r}$ is only white noise (i.e. there is no signal), ρ is distributed as,

$$\rho = 1 - U^{1/N}, \quad U \sim \mathrm{Unif}(0,1).$$

(b) Suppose the MF detects the signal when $\rho > t$. Using the result from part (a), find the threshold t as a function of N and the false alarm target.

(c) Write MATLAB code to run a short simulation to plot the missed detection probability as a function of the SNR for $N = 128$ and a false alarm target of 10^{-6}. Use any assumptions as necessary.

4. *Multiple delay hypotheses.* In the 5G NR standard, each base station cell periodically transmits a primary synchronization signal (PSS) that the mobile (UE) can detect. For 5G mmWave systems, the PSS bandwidth is typically 15.24 MHz (127 subcarriers at 120 kHz subcarrier spacing). The UE runs a matched filter (MF) to detect the PSS.

   (a) If the MF oversampled by a factor of two, how many delay hypotheses are there per second?

   (b) Suppose that to overcome carrier offset, the PSS MF detector tries several frequency hypotheses at each delay hypotheses. Suppose the carrier frequency is 37 GHz, the oscillator error is up to 10 ppm, and the maximum tolerable frequency error for the detector is 200 kHz. How many frequency hypotheses are needed in each delay hypothesis.

   (c) If the average number of false alarms should be less than one false alarm per second, what should the FA target be? Make any reasonable assumptions.