

# Unit 1. Passband Modulation

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EL-GY 6013: DIGITAL COMMUNICATIONS

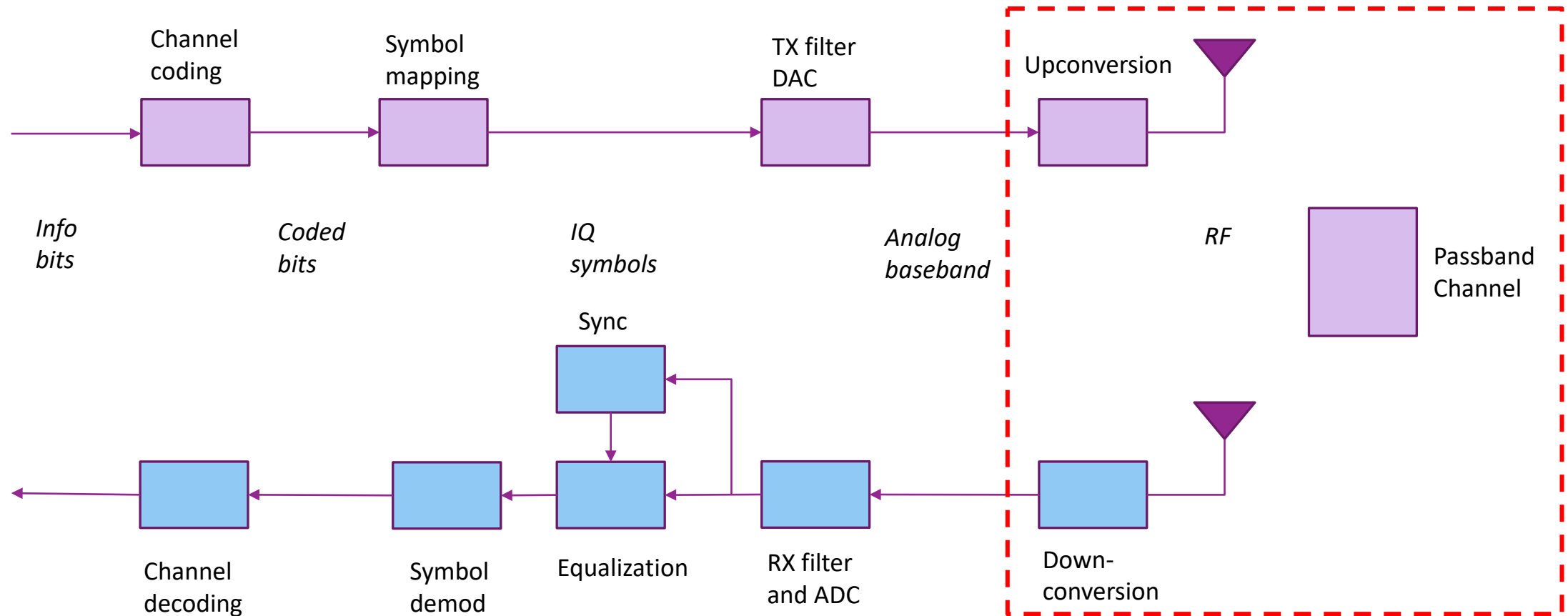
PROF. SUNDEEP RANGAN

# Learning Objectives

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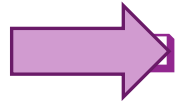
- ❑ Determine if a system is real passband or baseband
- ❑ Mathematically describe upconversion and downconversion
  - In time-domain and frequency-domain
- ❑ Compute simple continuous-time Fourier transforms (Review)
- ❑ Select parameters and analyze low-pass filter in down conversion
- ❑ Determine if a signal is a power or energy signal
  - Convert power in dBm
- ❑ Compute the effective baseband filter given a passband filter
- ❑ Model impairments such as time and frequency offsets

# This Unit



# Outline

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## Time-Domain Relationships

- ☐ Fourier Transform Review
- ☐ Frequency-Domain Relationships
- ☐ Power and Energy Spectra
- ☐ Baseband equivalent filters
- ☐ Practical up and down-conversion circuits
- ☐ Wireless channels



# Signals in Communications

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□ **Signal:** Any quantity that varies in time

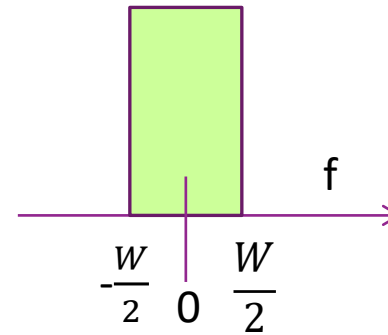
- Can be continuous time  $x(t)$
- Or discrete time  $x[n]$
- Real or complex valued

□ **Signals in communications:**

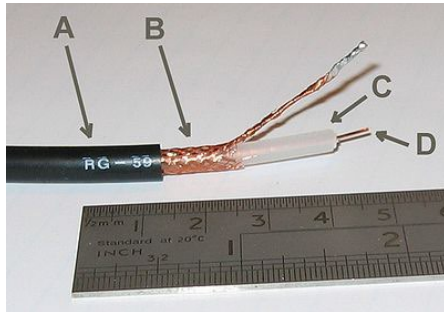
- $v(t)$  = Voltage at a particular point / place in a circuit (relative to ground)
- $E_z(t)$  = Electric field strength in a particular direction  
Note: electric field is a vector quantity  $E(t) = [E_x(t), E_y(t), E_z(t)]$
- A digital sample of a signal
- An intermediate value used in processing a signal

# Real Baseband Systems

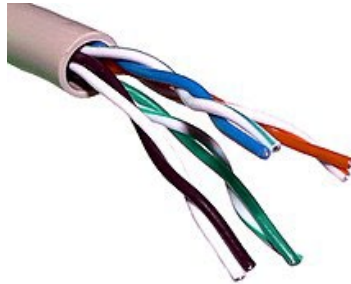
- Real baseband communication systems:
  - Communicate with lowpass real-valued signals
  - $X(f) \approx 0$  for  $|f| \leq \frac{W}{2}$



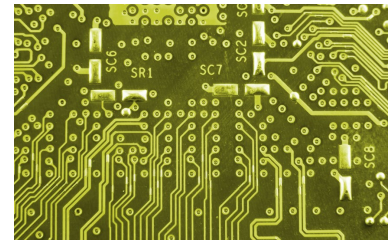
## Examples



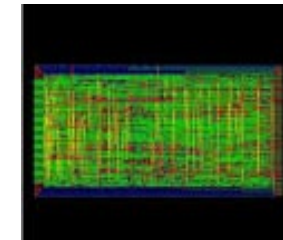
Coaxial cable



Twisted pair



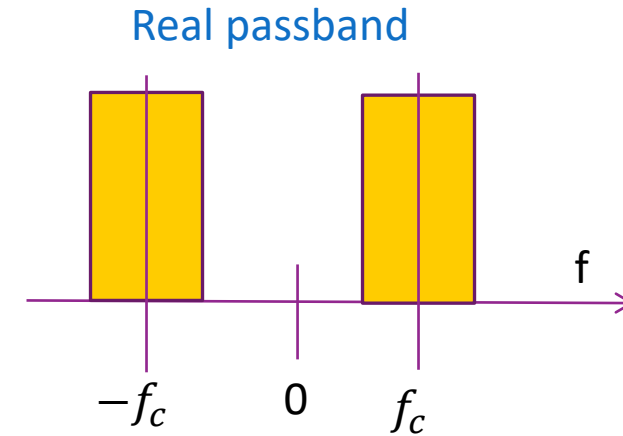
PCB traces  
e.g., microstrip or  
stripline



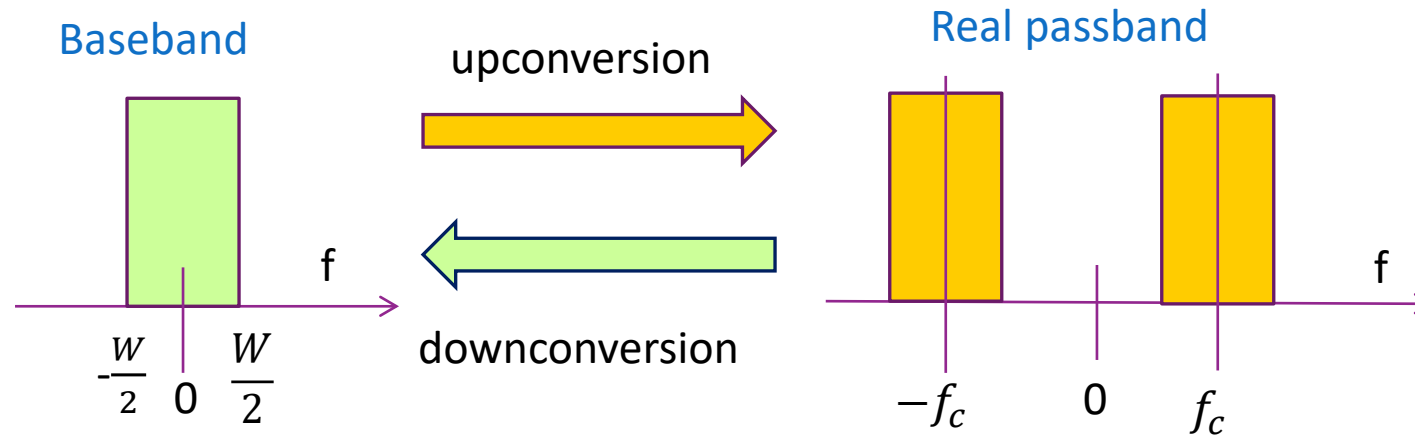
ASIC metal  
traces

# Real Passband Communications

- Real passband communication systems
  - Transmit around a **carrier frequency**  $f_c$
  - $f_c$  is sometimes called the **center frequency**
  - $X(f) \neq 0$  for  $|f - f_c| < W$  and  $|f + f_c| < W$
- Mostly radio frequency communication
  - Often wireless
  - Transmissions are restricted to bandwidth
  - Also, RF propagation is limited to certain bands
  - RF communication also occurs over cables



# Up- and Downconversion



- ❑ Up and downconversion: Shift center frequency of signals
- ❑ Used for all passband communications systems
  - Information occurs or is processed in **baseband**
  - Transmitted and received in **real passband**



# Upconversion in Time Domain

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❑ **Baseband signals:**  $u_i(t)$  and  $u_q(t)$ ,

- Also called “in-phase” and “quadrature” (I and Q)
- Real-valued. Typically, bandlimited to  $|f| < \frac{W}{2}$  ( $\frac{W}{2}$  = Single-sided bandwidth)
- Sometimes called the “cosine” and “sine” part.

❑ Carrier frequency  $f_c$ : Also called the “center” frequency

❑ Create **real passband signal**:

$$u_p(t) = u_i(t) \cos(2\pi f_c t) - u_q(t) \sin(2\pi f_c t)$$

❑ Upconversion is also called modulation

- But we will use that term for something later.

# Downconversion

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❑ Can recover I part from multiplication by sinusoid:

❑ Recovery of the I part:

- $v_i(t) = 2u_p(t) \cos(2\pi f_c t) = u_i(t) + \text{high freq terms}$
- $u_i(t) = \text{LPF}(v_i(t))$

❑ Recovery of the Q part:

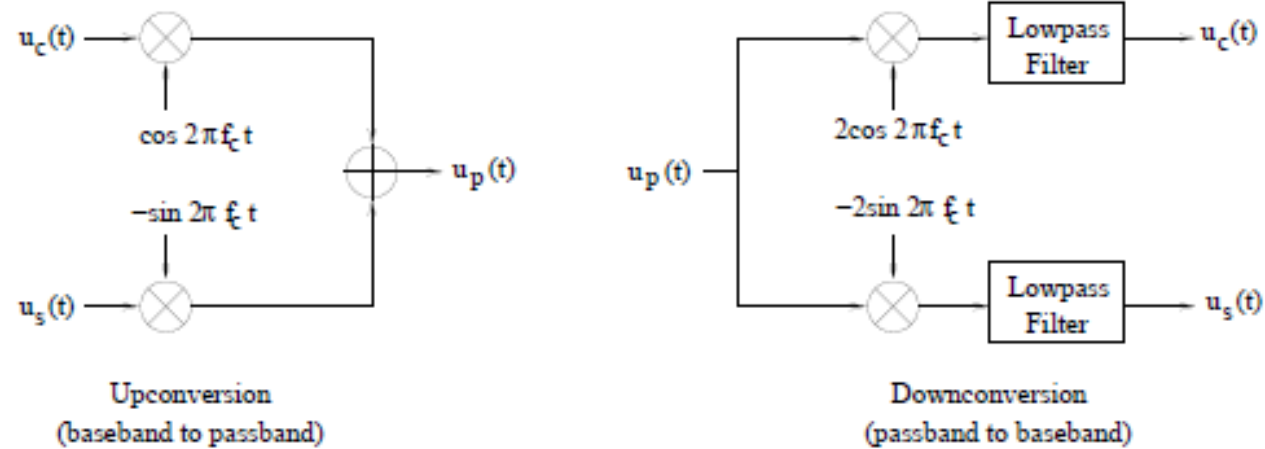
- $v_q(t) = -2u_p(t) \sin(2\pi f_c t) = u_q(t) + \text{high freq terms}$
- $u_q(t) = \text{LPF}(v_q(t))$

❑ Can derive relations using

$$\sin(2x) = 2 \sin(x) \cos(x) \quad 2\cos^2(x) = 1 + \cos(2x)$$

❑ Note gain of 2 and sign.

# Up and Downconversion Block Diagram



- Fig. 2.28 from Madhow
- Implementation with multipliers

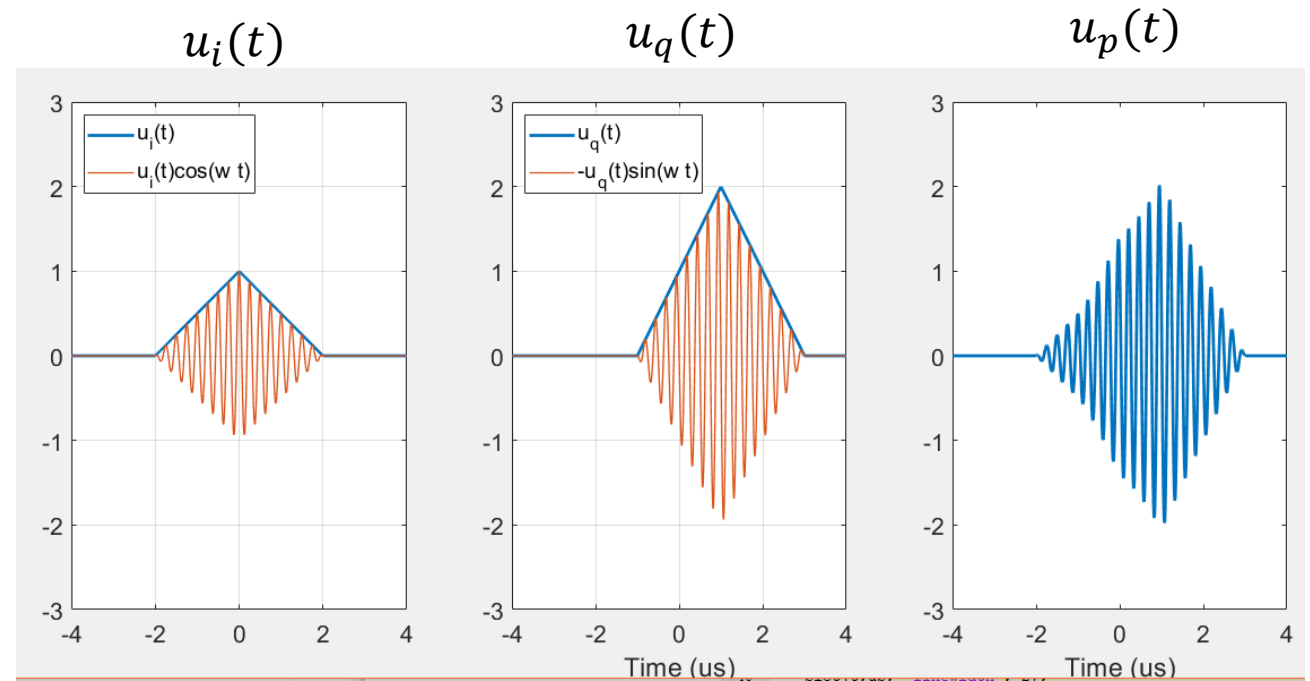
# Example

□ Suppose

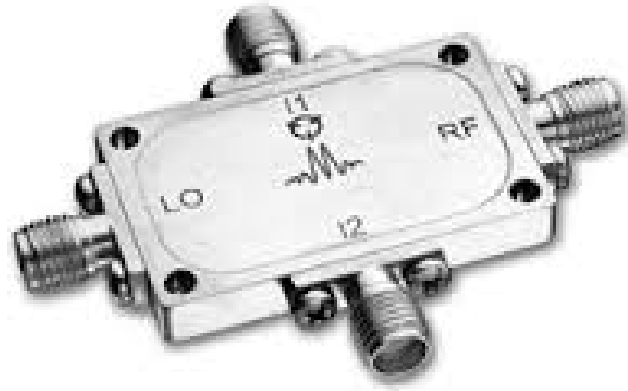
- $u_i(t) = \text{Tri}\left(\frac{t}{T}\right)$ ,  $u_q(t) = 2\text{Tri}\left(\frac{t}{T} - 0.5\right)$ ,  $\text{Tri}(s) := \max(0, 1 - |s|)$
- $T = 2 \mu\text{s}$ ,  $f_c = \frac{8}{T} = 4 \text{ MHz}$

```
% Create the baseband signals
nt = 1024;
T = 2.0;
t = linspace(-2*T, 2*T, nt)';
f0 = 8/T;
ui = max(1-abs(t/T), 0);
uq = 2*max(1-abs(t/T-0.5), 0);

% Modulate the I and Q components
uicos = ui.*cos(2*pi*f0*t);
uqsin = -uq.*sin(2*pi*f0*t);
up = uicos + uqsin;
```



# Actual IQ Mixer



- ❑ LO = “local oscillator” = square or sine wave at  $f_c$
- ❑ I1, I2 = I and Q inputs.
  - Generally, lowpass
- ❑ RF = passband output centered at  $f_c$

[http://www.markimicrowave.com/Mixers/IQ\\_Quadrature-IF\\_Double-Balanced/IQ-0318.aspx](http://www.markimicrowave.com/Mixers/IQ_Quadrature-IF_Double-Balanced/IQ-0318.aspx)

Datasheet	RF [GHz]	LO [GHz]	IF [MHz]	Conversion Loss [dB]	Image Rejection [dB]	Amplitude Deviation [dB]	Phase Deviation [Degrees]	Isolation L-R [dB]	Isolation L-I [dB]
<a href="#">IQ-0318</a>	3 to 18	3 to 18	DC to 500	7	22	0.75	10	40	20

# Complex Baseband Notation

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- ❑ Computations are often done in **complex domain**
- ❑ **Complex baseband signal**:  $u(t) = u_i(t) + ju_q(t)$
- ❑ **Upconversion**: Real passband is  $u_p(t) = \text{Re}[u(t)e^{2\pi j f_c t}]$
- ❑ **Down-conversion**:
  - $v(t) = 2u_p(t)e^{-j\omega_c t} = u(t) + \text{High freq terms}$
  - $u(t) = H_{LPF}(v(t))$

# Sample Problem (Soln on Board)

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□ Suppose that  $T = 1 \mu s$  and

$$u(t) = \begin{cases} 1 + j & t \in [0, T) \\ 1 - j & t \in [T, 2T) \\ 0 & \text{else} \end{cases}$$

□ What are  $u_i(t)$  and  $u_q(t)$ ? Draw them.

□ Write an equation for  $u_p(t)$  with a carrier frequency  $f_c = 4 \text{ MHz}$

□ Draw  $u_p(t)$  for  $t \in [1, 2] \mu s$ .

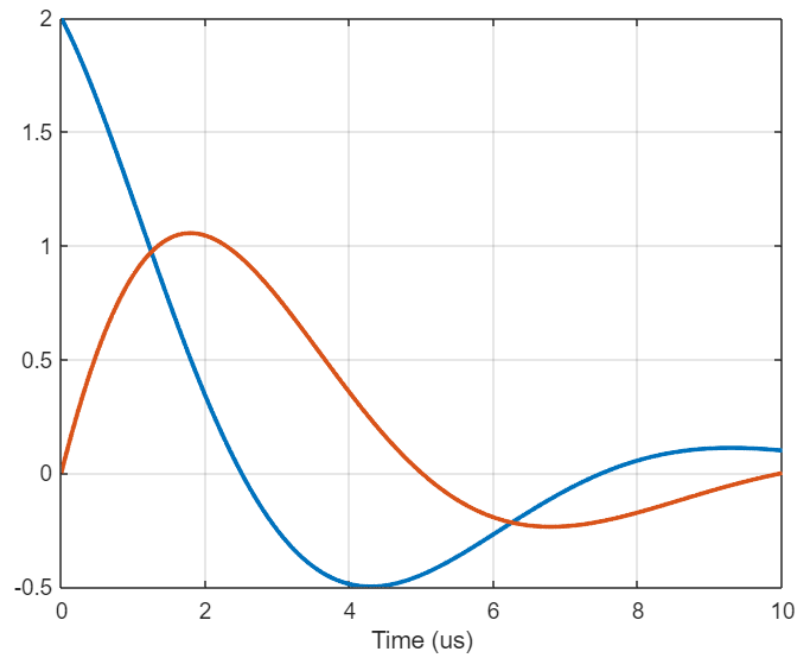
# In-Class Exercise

## Passband Up- and Down-Conversion In-Class Exercises

### Up and Down-Conversion in Time-Domain

In this exercise, we will perform simple up and down-conversion. Normally, up and down-conversion are numerically. We start with a simple complex baseband time-domain signal:

$$u = u_0 \exp((1i * \omega - \alpha) * t)$$



- ☐ See passbandInClass.mlx MATLAB live script
- ☐ Fill in TODO sections



# Outline

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☐ Time-Domain Relationships

 ☐ Fourier Transform Review

☐ Frequency-Domain Relationships

☐ Power and Energy Spectra

☐ Baseband equivalent filters

☐ Practical up and down-conversion circuits

☐ Wireless channels

# Complex Exponential

□ Continuous-time complex exponential signal:

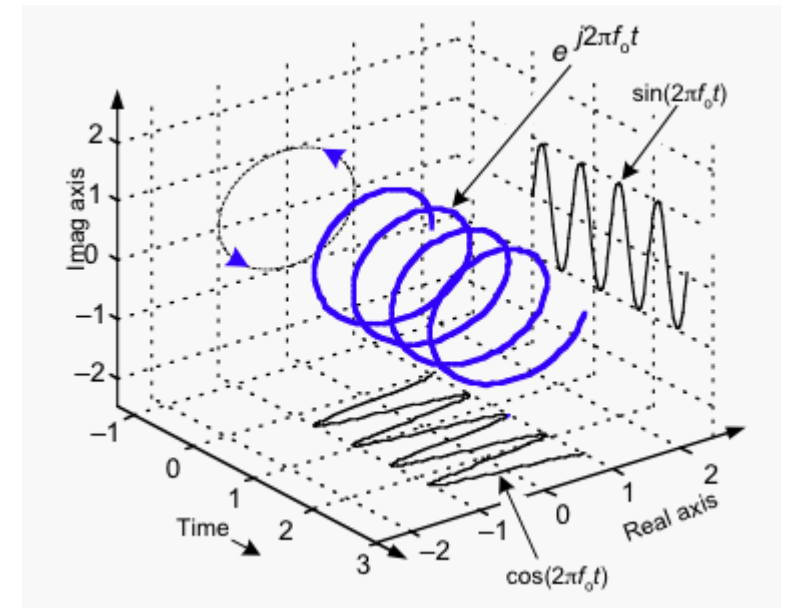
$$x(t) = A \exp(2\pi i f t) = A \exp(i\omega t)$$

- $A$  = complex magnitude
- $f$  = frequency in Hz,  $\omega = 2\pi f$  = angular frequency in rad/s

□ If we write  $A = r e^{i\theta}$ ,  $r = |A|$  then:

$$x(t) = r \exp(\omega t + \theta)$$

- $r$  = magnitude
- $\theta$  = phase



# Fourier Transform

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□  $s(t)$ : real or complex continuous-time signal

□ **Fourier Transform**: Expresses signal as sum / integral of complex exponentials

□ Time-domain to frequency domain

$$S(f) = \int_{-\infty}^{\infty} s(t) e^{-2\pi i f t} dt$$

□ Inverse Fourier transform:

$$s(t) = \int_{-\infty}^{\infty} S(f) e^{2\pi i f t} df$$

□ Represents signals in their frequency components

# FT in Angular Frequency

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□ Angular frequency:  $\omega = 2\pi f$

□ Fourier Transform: time-domain to frequency domain

$$S(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t} dt$$

□ Inverse Fourier transform:

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)e^{i\omega t} df$$

□ Note scaling

□ Some texts use other scalings

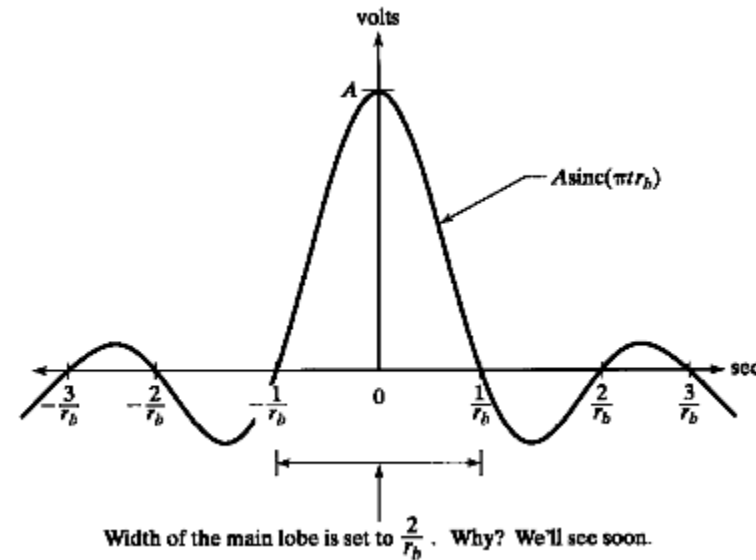
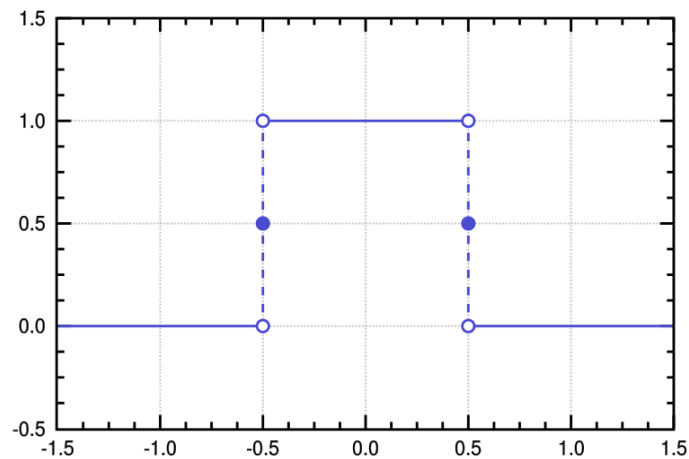
# Rect and Sinc

$$\square \text{rect}(at) \leftrightarrow \frac{1}{|a|} \text{sinc}\left(\frac{f}{a}\right) = \frac{\sin(\pi f/|a|)}{\pi f}$$

$$\square \text{sinc}(at) \leftrightarrow \frac{1}{|a|} \text{rect}\left(\frac{f}{a}\right)$$

$$\text{Height} = \frac{1}{|a|}$$

$$\text{Main lobe } f = \pm|a|$$



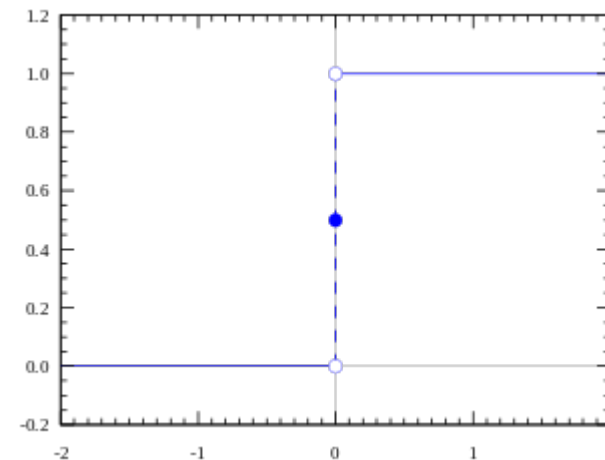
# Unit Steps

## Unit step

$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases}$$

## FT:

- $e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha + 2\pi i f}, \operatorname{Re}(\alpha) > 0$
- $u(t) \leftrightarrow \frac{1}{2}\delta(f) + \frac{1}{2\pi i f}$



# Table (From WikiPedia)

	Function	Fourier transform unitary, ordinary frequency
	$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$
201	$\text{rect}(ax)$	$\frac{1}{ a } \cdot \text{sinc}\left(\frac{\xi}{a}\right)$
202	$\text{sinc}(ax)$	$\frac{1}{ a } \cdot \text{rect}\left(\frac{\xi}{a}\right)$
203	$\text{sinc}^2(ax)$	$\frac{1}{ a } \cdot \text{tri}\left(\frac{\xi}{a}\right)$
204	$\text{tri}(ax)$	$\frac{1}{ a } \cdot \text{sinc}^2\left(\frac{\xi}{a}\right)$
205	$e^{-ax} u(x)$	$\frac{1}{a + 2\pi i \xi}$
206	$e^{-\alpha x^2}$	$\sqrt{\frac{\pi}{\alpha}} \cdot e^{-\frac{(\pi \xi)^2}{\alpha}}$
207	$e^{-a x }$	$\frac{2a}{a^2 + 4\pi^2 \xi^2}$

$f(x)$	$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$
1	$\delta(\xi)$
$\delta(x)$	1
$e^{iax}$	$\delta\left(\xi - \frac{a}{2\pi}\right)$
$\cos(ax)$	$\frac{\delta\left(\xi - \frac{a}{2\pi}\right) + \delta\left(\xi + \frac{a}{2\pi}\right)}{2}$
$\sin(ax)$	$\frac{\delta\left(\xi - \frac{a}{2\pi}\right) - \delta\left(\xi + \frac{a}{2\pi}\right)}{2i}$

$\text{sgn}(x)$	$\frac{1}{i\pi\xi}$
$u(x)$	$\frac{1}{2} \left( \frac{1}{i\pi\xi} + \delta(\xi) \right)$
$\sum_{n=-\infty}^{\infty} \delta(x - nT)$	$\frac{1}{T} \sum_{k=-\infty}^{\infty} \delta\left(\xi - \frac{k}{T}\right)$

# Other Properties

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$$\square s(t - a) \leftrightarrow e^{-2\pi i a f} S(f)$$

$$\square e^{2\pi i a t} s(t) \leftrightarrow S(f - a)$$

$$\square s(at) \leftrightarrow S(f/a)/|a|$$

$$\square d^n s(t)/dt^n \leftrightarrow (2\pi i f)^n S(f)$$

$$\square t^n s(t) \leftrightarrow d^n S(f)/df^n$$

$$\square s^*(t) \leftrightarrow S^*(-f)$$

$$\square s(t) \leftrightarrow S(f) \Rightarrow S(t) \leftrightarrow s(-f)$$



# Problems

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2.10 Determine the Fourier transform of each of the following signals ( $\alpha$  is positive).

1.  $x(t) = \frac{1}{1+t^2}$

2.  $\Pi(t-3) + \Pi(t+3)$

3.  $\Lambda(2t+3) + \Lambda(3t-2)$

4.  $\text{sinc}^3 t$

5.  $t \text{sinc } t$

6.  $t \cos 2\pi f_0 t$

7.  $e^{-\alpha|t|} \cos(\beta t)$

8.  $te^{-\alpha t} \cos(\beta t)$

□ Solutions on board

# Sampled Complex Exponentials

❑ Often need to process digitally sampled complex exponentials

- E.g. MATLAB, digital circuits, ...

❑ Continuous-time signal:  $x(t) = A \exp(2\pi i f t) = A \exp(i\omega t)$

❑ Sampled signal:

$$x[n] = x(nT), \quad T = 1/f_s$$

❑ Resulting discrete-time signal:

$$x[n] = A \exp(2\pi i \nu n) = A \exp(i\Omega n)$$

- $\nu = \frac{f}{f_s}$ : Normalized digital frequency [cycles / sample]. Typically  $[-0.5, 0.5]$
- $\Omega = 2\pi\nu = \frac{2\pi f}{f_s}$ : Digital angular frequency [rads / sample]. Typically  $[-\pi, \pi]$

❑ Will discuss sampling much more later

# Computing the FT with the FFT

- ❑ Often approximately compute FFT on the sampled signal

- ❑ Continuous-time  $x(t)$ . Want to compute FT  $X(f)$

- ❑ Take sampled signal  $x_d[n] = x(nT), n = 0, \dots, N - 1$

- ❑ Compute FFT (Fast Fourier Transform):

$$X_d[k] = \sum_{n=0}^{N-1} x_d[n] e^{2\pi i k n / N}$$

- ❑ Then, if  $X(f)$  is bandlimited to  $|f| < 1/(2T)$  then:

$$X\left(\frac{k}{T}\right) = T X_d[k'], \quad k' = k \bmod N, \quad k = -\frac{N}{2} + 1, \dots, \frac{N}{2}$$

- ❑ Obtain sampled version of the FT from the FFT

- ❑ Note scaling factor  $T$  and modulo operation

# In-Class Exercise

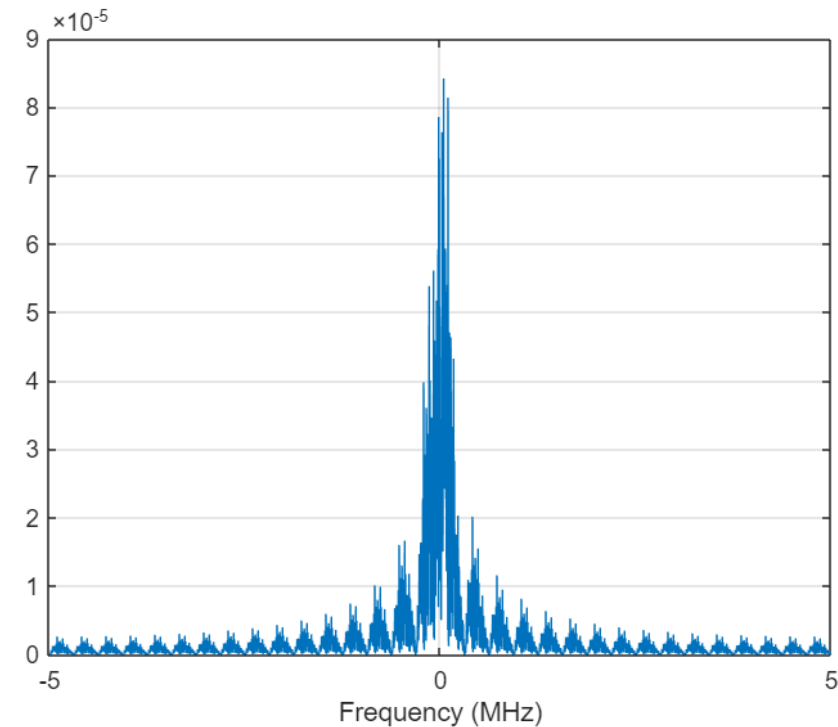
## Computing a Fourier Transform with an FFT

In this exercise, you will see how to approximately compute the Fourier Transform of a sampled signal with the FFT. First, we create a sampled-signal.

- Generate random complex symbols,  $\text{sym}[k] = \exp(1i \cdot \text{theta}[k])$  where  $\text{theta}[k]$  is uniform in  $[0, 2\pi]$  and  $k=1, \dots, \text{nsym}$
- Create an up-sampled version of  $\text{sym}$ , denoted  $u$ :  $u[n] = \text{sym}[k]$  for  $n = k \cdot \text{sampPerSym} - 1, \dots, (k+1) \cdot \text{sampPerSym}$ .
- Plot the  $\text{real}(u)$  vs. time in micro-seconds.

So,  $u$  is a signal that randomly changes phases every  $\text{sampPerSym}$ .

```
sampPerSym = 2^5; % number of samples per symbol
nsym = 2^7; % number of symbols
nsamp = sampPerSym*nsym; % total number of samples
fs = 10e6; % sample rate in Hz
```



# Outline

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☐ Time-Domain Relationships

☐ Fourier Transform Review

 ☐ Frequency-Domain Relationships

☐ Power and Energy Spectra

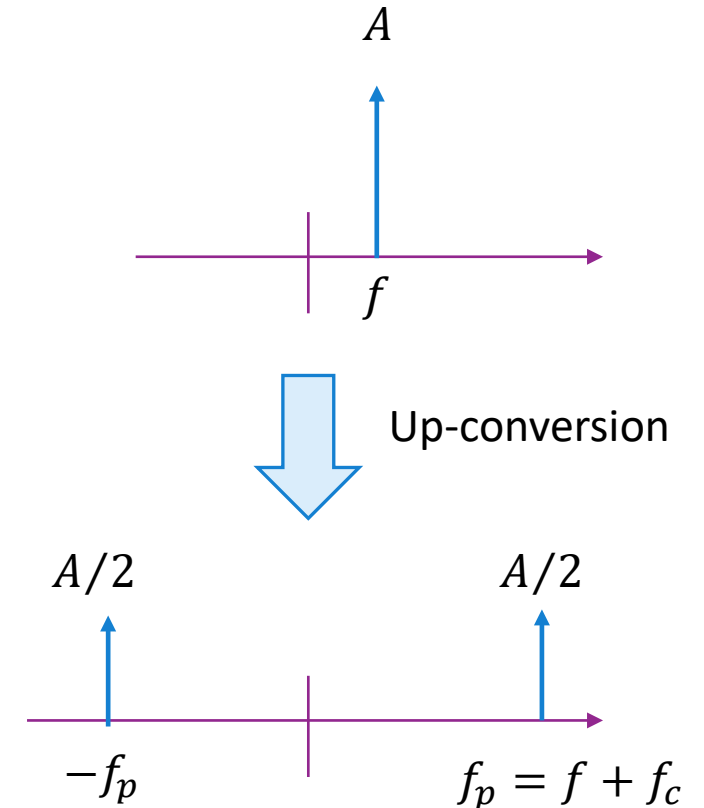
☐ Baseband equivalent filters

☐ Practical up and down-conversion circuits

☐ Wireless channels

# Up-Conversion of a Complex Exponential

- Suppose complex baseband is  $u(t) = A \exp(2\pi i f t)$
- Up-conversion:
$$u_p(t) = \text{Real}(u(t) e^{2\pi i f_c t}) = A \cos(2\pi i f_p t)$$
  - Shifts  $f$  to  $f_p = f + f_c$
- Can write as  $u_p(t) = \frac{A}{2} [e^{2\pi i f_p t} + e^{-2\pi i f_p t}]$
- If  $u(t) = A \exp(2\pi i f t + \theta)$  then
$$u_p(t) = A \cos(2\pi i (f + f_c) t + \theta)$$
  - Phase is unchanged
- Down-conversion reverses the process



# Down-Conversion of a Complex Exponential

From previous slide, after upconversion:

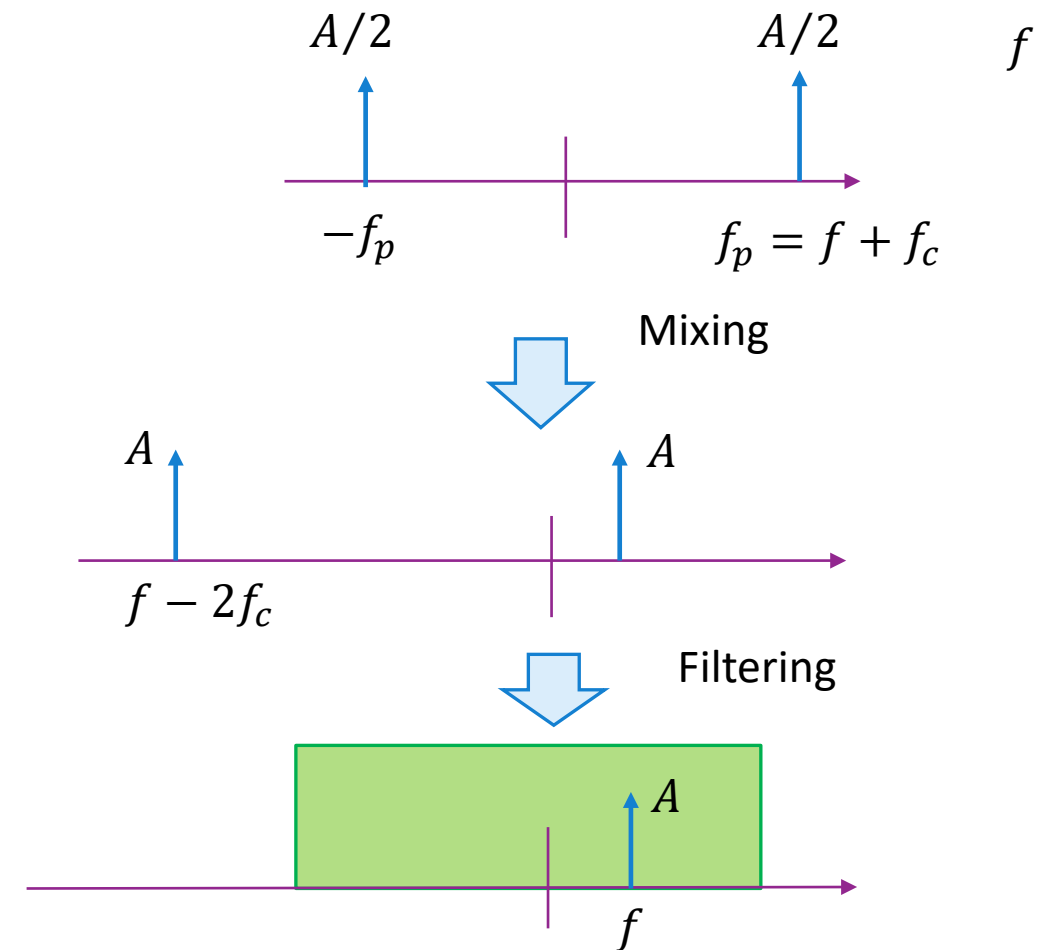
$$u_p(t) = \frac{A}{2} [e^{2\pi i f_p t} + e^{-2\pi i f_p t}]$$

Mixing:

$$\begin{aligned} v(t) &= 2u_p(t)e^{2\pi i f_c t} \\ &= Ae^{2\pi i f t} + Ae^{-2\pi i (f+2f_c)t} \end{aligned}$$

Filtering:

$$u(t) = LPF(v(t)) = Ae^{2\pi i f t}$$



# Example

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- ❑ Suppose the digital complex baseband signal is:  $u[n] = A \exp(2\pi i \nu n)$ 
  - $\nu = 0.1$ ,  $A = 4$
- ❑ Suppose sampling rate is  $f_s = 20$  MHz and carrier is  $f_c = 2$  GHz
- ❑ What is the continuous-time complex baseband signal?
- ❑ Ans:  $u(t) = A \exp(2\pi i f t)$ ,  $f = \nu f_s = (0.1)(20) = 2$  MHz
- ❑ What is the real passband signal after up-conversion?
- ❑ Ans:  $u_p(t) = A \exp(2\pi i f_p t)$ ,  $f_p = f + f_c = 2.002$  GHz
  - Be careful with units

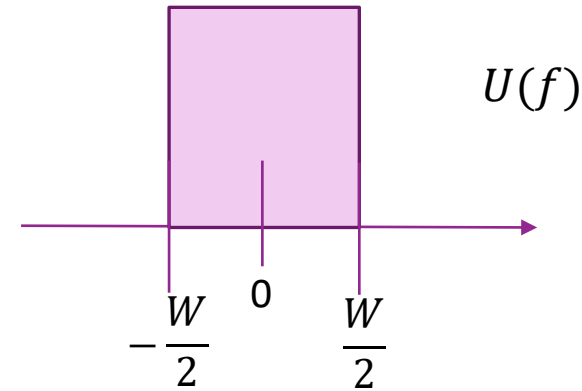


# Bandwidth Terminology

□ Now consider more complex signals

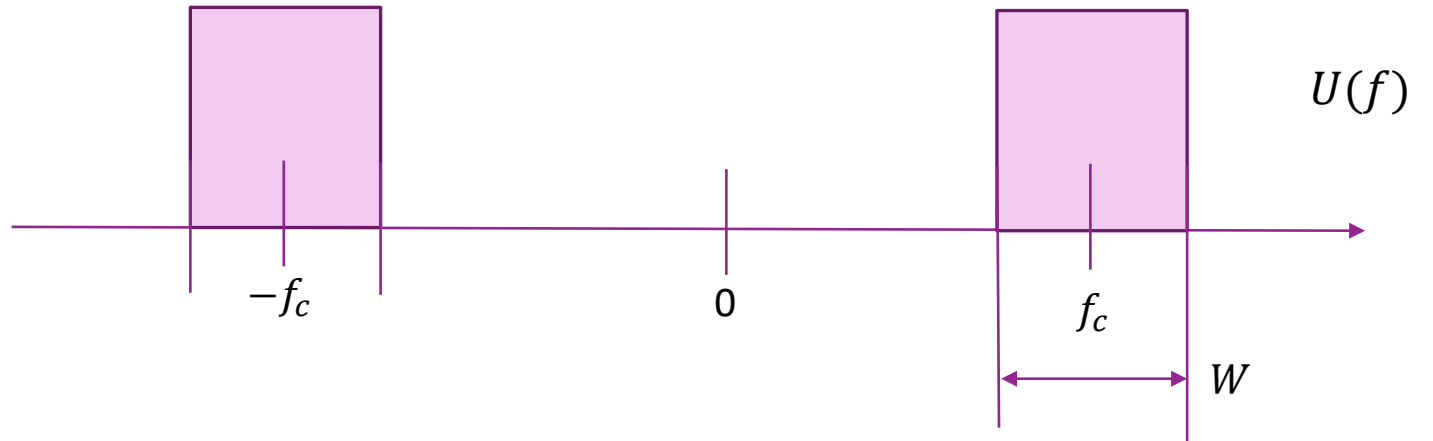
□ Baseband signals

- Centered around  $f = 0$ , complex
- $\frac{W}{2}$  = single sided bandwidth
- $W$  = two sided bandwidth
- Band-limited to  $|f| \leq \frac{W}{2}$



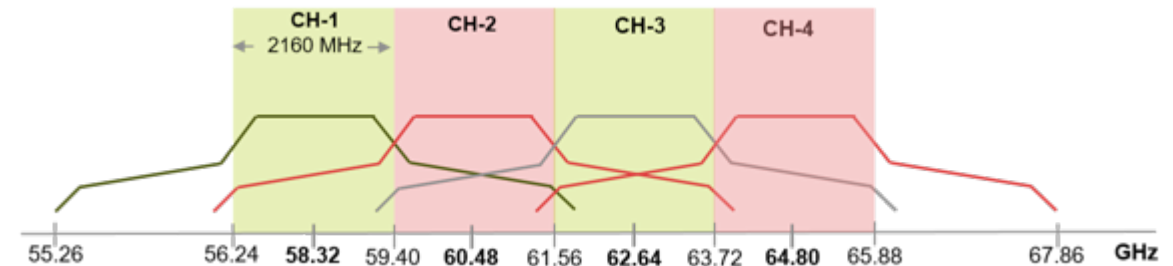
□ Passband signals

- Centered around  $f = f_c$ , real
- $W$  = bandwidth (per side or image)
- Band-limited to  $|f - f_c| \leq \frac{W}{2}$



# Importance of Bandwidth

- ❑ Data rate generally scales linearly in bandwidth
  - If the transmit power and bandwidth increase by  $N \Rightarrow$  the communication rate increases by  $N$
  - We will see this in detail later
- ❑ Ex: Compare GSM (2G) and LTE (4G)
  - Single channel of GSM system = 200 kHz
  - Single channel of LTE = 20 MHz
  - If power scales sufficiently, LTE would in general have 100x data rate
  - LTE, in fact, can have even more capacity due to other improvements
- ❑ Figure to the right: 802.11ad channels
  - The channels are > 2 GHz



# Frequency Domain Relationships

## Baseband to Passband

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□ Suppose that  $U(f)$  is bandlimited to  $\left[-\frac{W}{2}, \frac{W}{2}\right]$  and  $f_c > W$

$$U_p(f) = \frac{1}{2} [U(f - f_c) + U^*(-f - f_c)]$$

□ Use notation:

- $U^+(f) := \frac{1}{2} U(f - f_c)$ : This is  $U(f)$  shifted to the right by  $f_c$  and scaled by  $\frac{1}{2}$
- $U^-(f) := \frac{1}{2} U^*(-f - f_c)$ : Flip  $U^+(f)$  around  $y$  axis and take negative of the imaginary part

□ Proof:

- Let  $c(t) = u(t)e^{2\pi j f_c t} \leftrightarrow C(f) = U(f - f_c)$
- $u_p(t) = \text{Re}(c(t)) = \frac{1}{2}(c(t) + c^*(t))$
- Now use conjugate symmetry  $c^*(t) \leftrightarrow C^*(-f)$

# Example Problem

□ Suppose baseband signal is as drawn:

□ What is the:

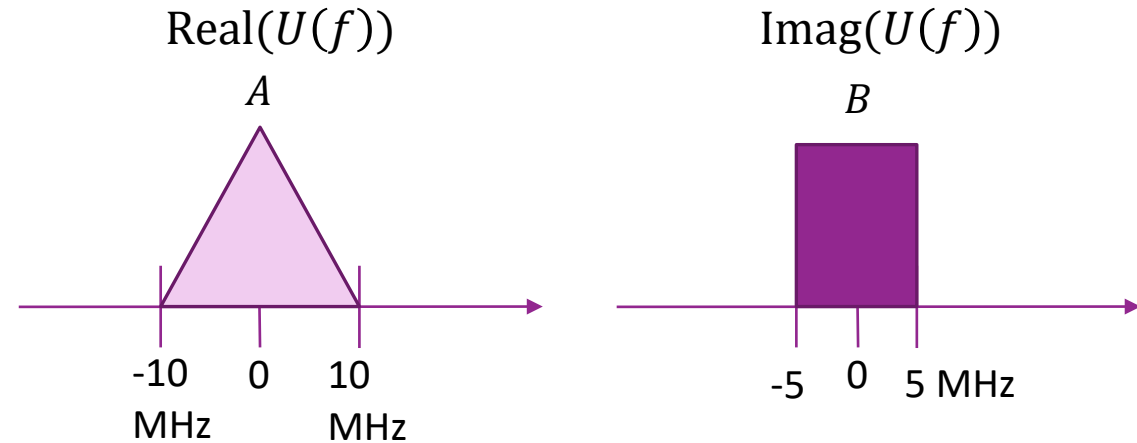
- Single-sided bandwidth?
- Two-sided bandwidth?

□ Write an equation for  $u_i(t)$

- You do not need to evaluate the integral.

□ Draw the passband frequency response if  $f_c = 2$  GHz

- Draw both the positive and negative images



# Frequency Domain Relationships

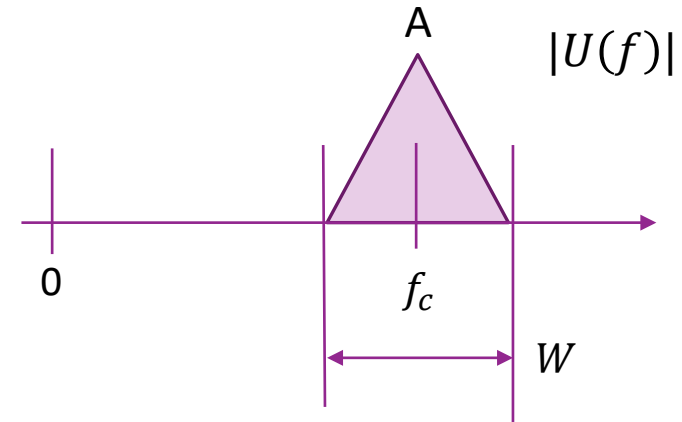
## Passband to Baseband

---

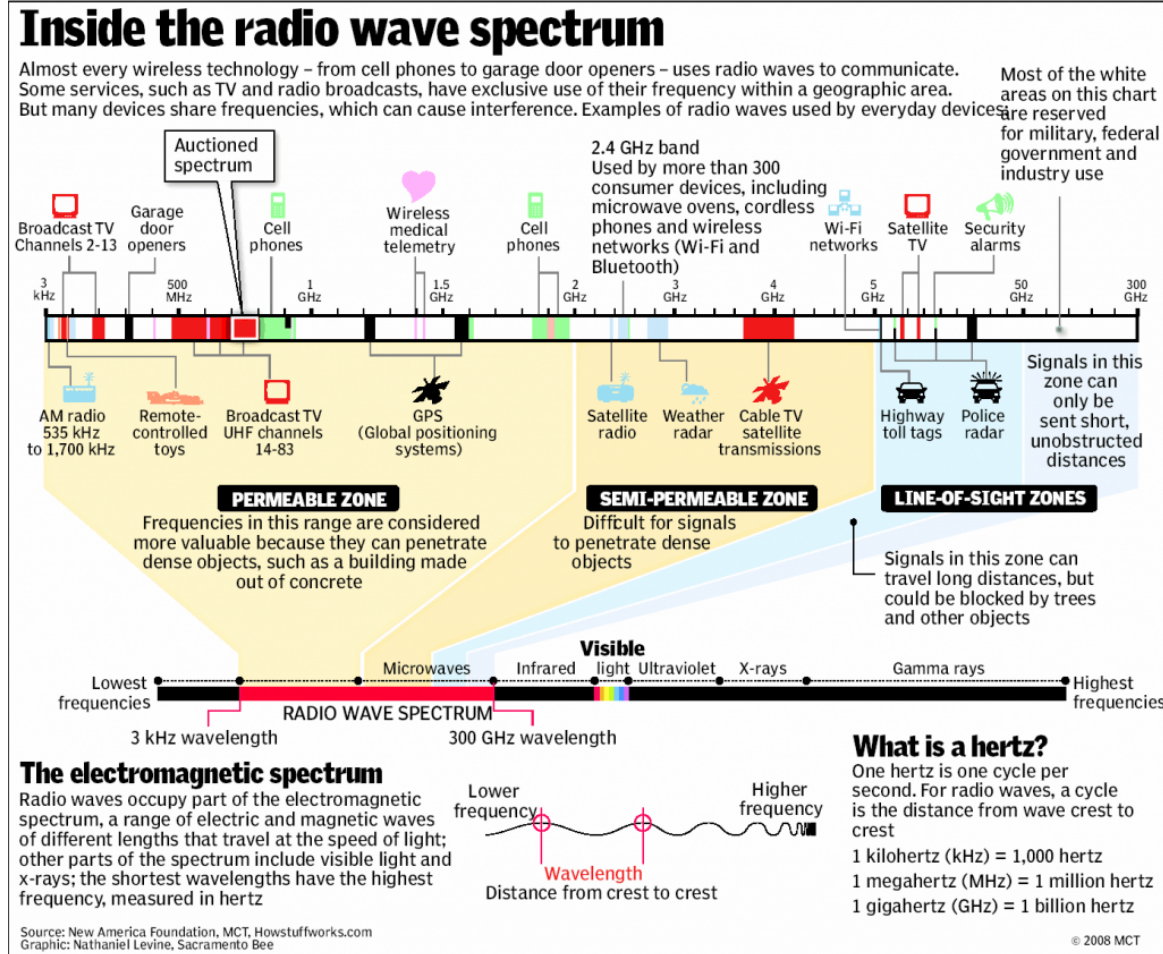
- ❑ Downconversion in time-domain:  $v(t) = 2u_p(t)e^{-j\omega_c t}$ ,  $u(t) = h_{LPF}(t) * v(t)$
- ❑ In frequency-domain:  $U(f) = 2U_p(f + f_c)H_{LPF}(f)$ 
  - Shift to left, scale by 2 and filter
- ❑ Ideal filtering:
  - Suppose  $U_p(f)$  has bandwidth  $W$  around  $f_c$
  - Then typically have:  $H_{LPF}(f) = 1$  for  $|f| \leq \frac{W}{2}$  and  $H_{LPF}(f) = 0$  for  $|f| > \frac{W}{2}$
  - $U(f) = 2U_p(f + f_c)1_{\{|f| \leq W\}}$
  - Shift to the left and remove left image.
- ❑ Pictures on board

# Example Problem

- Suppose right image of passband is as shown:
  - $W = 4$  MHz,  $f_c = 800$  MHz
- Assume a LPF  $H_{LPF}(f) = \text{Rect}\left(\frac{f}{f_0}\right)$
- Draw magnitude spectrum of down-converted signal
  - When  $f_0 = 5$  MHz
  - When  $f_0 = 3$  MHz
- What range of values  $f_0$  will:
  - Keep the low-pass component
  - Reject the high frequency component
- Solution on board



# Radio Spectrum



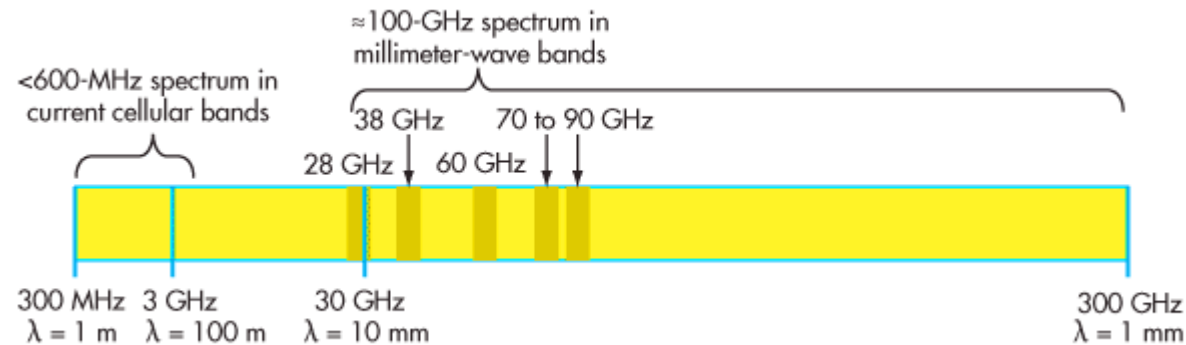
# Bandwidth and Center Frequencies Examples

System	Duplex	Center freq (MHz)	Bandwidth
GSM	FDD	GSM-850: 824-849 (UL), 869-894 (DL) GSM-900: 890-914 (UL), 935-959 (DL) GSM-1800: 1710–1784(UL), 1805.2–1879(DL) GSM-1900: 1850–1910(UL), 1930–1990(DL)	200 kHz per channel
UMTS	FDD	GSM + other bands ~2100 and ~1900	5 MHz per carrier
LTE	Mostly FDD	Mostly in 2100 to 2600 MHz	1.4 to 20 MHz, 10 MHz typical
802.11abg	TDD	2.4 GHz (ISM band) and 5 GHz (U-NII band)	20 MHz
802.11n			20, 40 MHz
802.11ac			20-160 MHz
802.11ad	TDD	60 GHz (millimeter wave spectrum)	2.16 GHz

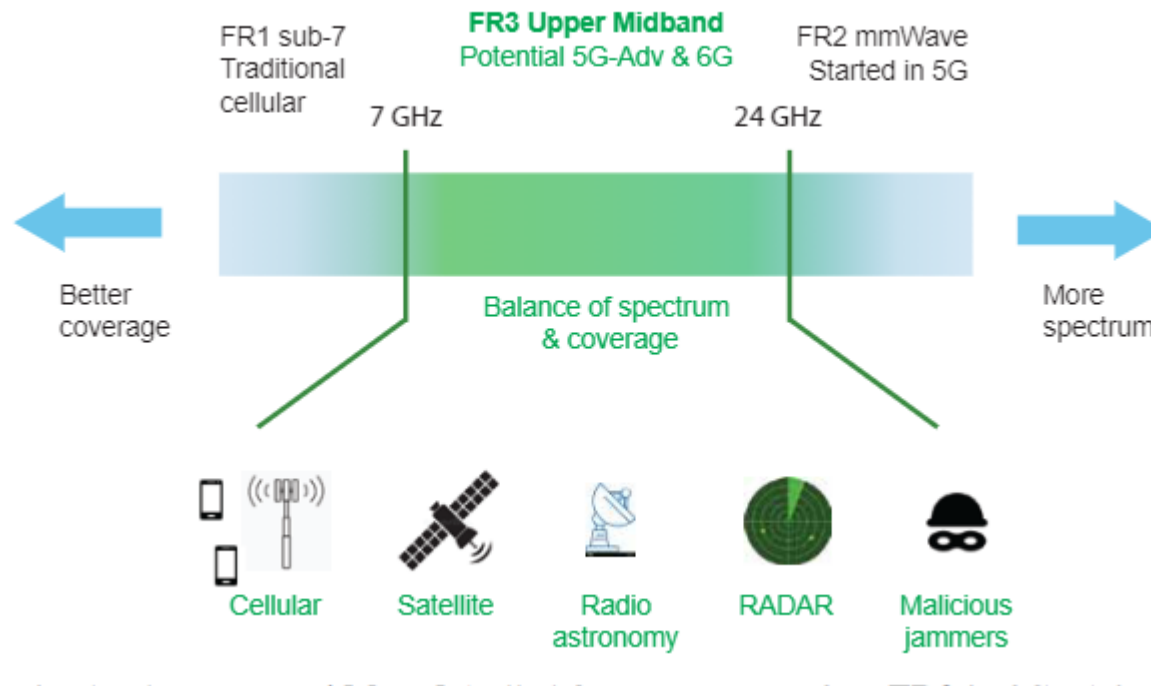


# Millimeter Wave

- New bands for 5G
  - 100x more bandwidth than conventional bands below 6 GHz
  - 5G systems today are operating in 28 GHz and 38 GHz



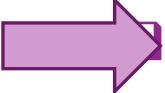
# Upper Mid-Band



- ❑ New frequency band of interest
- ❑ Balance of coverage and spectrum
- ❑ Many research challenges
  - Spectrum sharing
  - Wide bandwidth
  - Resiliency

# Outline

---

- ☐ Time-Domain Relationships
- ☐ Fourier Transform Review
- ☐ Frequency-Domain Relationships
-  ☐ Power and Energy Spectra
- ☐ Baseband equivalent filters
- ☐ Practical up and down-conversion circuits
- ☐ Wireless channels

# Energy and Power Signals

---

□ Instantaneous power:  $|x(t)|^2$

- Why squared?

□ Energy:

- $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$
- Signal is called an “energy signal” if  $E_x < \infty$

□ Power:

- $P_x = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$
- Energy per unit time
- Signal is called a “power signal” if limit  $P_x$  exists and is finite

# Power of a Periodic Signal

## Time-Domain Method

---

□ Suppose  $x(t)$  is periodic, period  $T$

□ **Theorem:**  $x(t)$  is a power signal and power can be computed from any one period

$$P_x = \frac{1}{T} \int_0^T |x(t)|^2 dt = \frac{1}{T} \int_{t_0}^{t_0+T} |x(t)|^2 dt$$

- Proof on board

# Example: Done on board

---

□ Suppose that  $x(t)$  has period  $T$

$$x(t) = a + bt, t \in [0, T]$$

- $a, b$  are real

□ Draw  $x(t)$

□ What is  $P_x$

□ What if  $a, b$  were complex?

# Power of a Periodic Signal

## Fourier Series Method

---

□ Suppose that  $x(t)$  is periodic with period  $T$

□ Then has Fourier Series

$$x(t) = \sum c_n e^{2\pi j f_n t}, \quad f_n = n/T$$

□ Theorem: Power of  $x(t)$  is:

$$P_x = \sum |c_n|^2$$

□ Note that if  $x(t) = \sum g(t - nT)$ , then  $c_n = G(f_n)$

- Can compute power from Fourier transform of  $g(t)$

# Example: On board

---

□ Suppose  $T = 10 \mu s$ ,

$$x(t) = \sum g(t - nT), \quad g(t) = \begin{cases} 2 & t \in [0, T/4) \\ -1 & t \in [T/4, T) \\ 0 & \text{else} \end{cases}$$

□ Draw  $x(t)$

□ What is the FT  $G(f)$ ?

□ What is the FT  $X(f)$ ?

□ What is the power of  $x(t)$ ?

□ What fraction of power of  $x(t)$  is in the  $|f| \leq 250 \text{ kHz}$ ?



# Energy Density

---

□ Energy of signal:  $E_x = \int |x(t)|^2 dt$

□ From Parseval's identity:  $E_x = \int |X(f)|^2 df$

- Can compute energy in frequency-domain

□ Energy density:  $G_x(f) = |X(f)|^2$

- Density of energy around frequency  $f$

# Power Spectral Density (PSD)

---

- ❑ Three equivalent ways to define PSD
- ❑ Definition 1: via windowing in time
- ❑ Definition 2: via filtering
- ❑ Definition 3: via auto-correlation for a random process
  - More advanced.
  - We will cover this in the next unit

# PSD: Time-Windowing Definition

---

□ Let  $x(t)$  be a power signal

□ Define windowed signal:

$$x_T(t) = \begin{cases} x(t) & |t| \leq T \\ 0 & |t| > T \end{cases}$$

□ PSD is defined as:

$$S_x(f) := \lim_{T \rightarrow \infty} \frac{1}{2T} |X_T(f)|^2$$

□ Similar to energy signal, but with averaging over time.

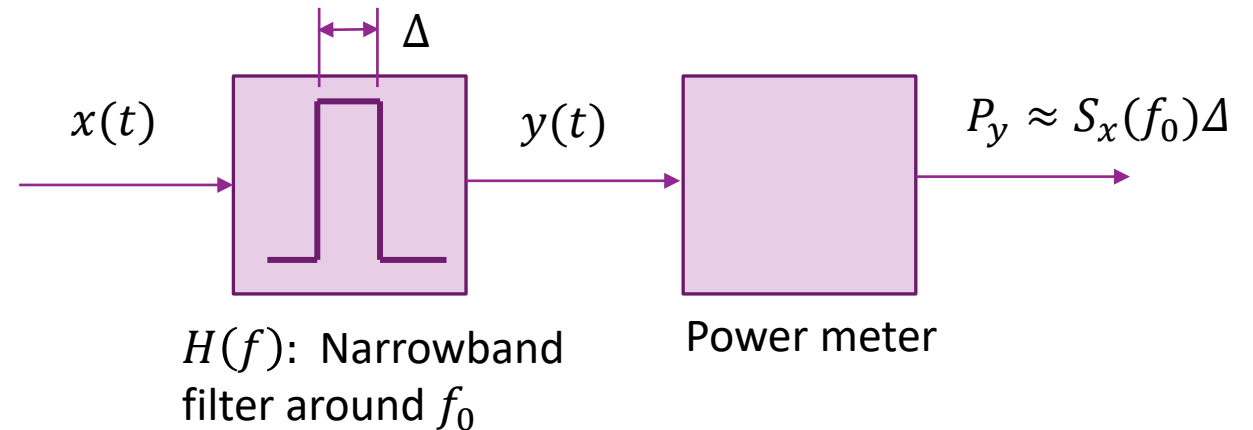
□ Can show power is given by:

$$P_x = \int_{-\infty}^{\infty} S_x(f) df$$

- $S_x(f)$  represents power per unit frequency

# PSD: Filtering Definition

- ❑ Let  $x(t)$  be a power signal
- ❑ Select frequency  $f_0$  to measure PSD
- ❑ Filter with narrowband filter
  - $y(t) = h(t) * x(t)$
  - $H(f) = 1$  for  $|f - f_0| \leq \Delta/2$
- ❑ Measure power  $P_y$



- ❑ PSD at  $f_0$  is defined as
$$S_x(f_0) := \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} P_y$$
- ❑ Can show this is equivalent to window definition
- ❑ Reveals how much power is in a certain frequency

# Spectrum Analyzer

---



- ❑ Measures PSD in real time
- ❑ Uses averaging of FT
  - But proper averaging is quite tricky
- ❑ Lab 2: Use MATLAB function pwelch

# Units

---

## □ Energy signals:

- $E_x$ : Joules
- $G(f) = |X(f)|^2$ : Joules / Hz

## □ Power signals (much more common):

- $P_x$ : Joules / sec = Watts
- $S_x(f)$ : Watts / Hz = Joules

# Power: Linear and decibel scale

---

## □ Receive or transmit antenna energy per unit time

- Measured in Watts (W) or mW
- Power values in W or mW called *linear scale*
- Use notation  $P_{|W}$  or  $P_{|mW}$  when units need to be specified

## □ Power often measured in dB scale:

- $P_{|dBW} = 10\log_{10}(P_{|W} / 1W)$
- $P_{|dBm} = 10\log_{10}(P_{|mW} / 1mW)$

## □ Example: $P = 250 \text{ mW}$ (typical max mobile transmit power)

- $P_{|dBW} = 10\log_{10}(0.25W / 1W)$
- $P_{|dBm} = 10\log_{10}(250mW / 1mW)$

# Some important dB values

---

□ Some conversions don't need a calculator:

- $10\log_{10}(2) = 3$  [Most important: Doubling power = 3dB]
- $10\log_{10}(3) = 4.7 \sim 5$
- $10\log_{10}(10) = 10$

□ You can cascade these.

□ Ex: If the power is increased by 50 in linear scale, what is the increase in dB? Answer:

$$\begin{aligned} 10\log_{10}(50) &= 10\log_{10}(10^2 / 2) \\ &= 2 \times 10\log_{10}(10) - 10\log_{10}(2) = 2 \times 10 - 3 = 17 \text{ dB} \end{aligned}$$



# PSD and Linear Filters

---

□ Suppose  $y(t) = h(t) * x(t)$

□ Then:  $Y(f) = H(f)X(f)$

- $S_y(f) = |H(f)|^2 S_x(f)$

□ Transfer function  $|H(f)|^2$  power gain at frequency  $f$ :

$$|H(f)|^2 = \frac{S_y(f)}{S_x(f)} \frac{\text{output power at } f}{\text{input power at } f}$$

- Dimensionless quantity
- Often expressed in dB

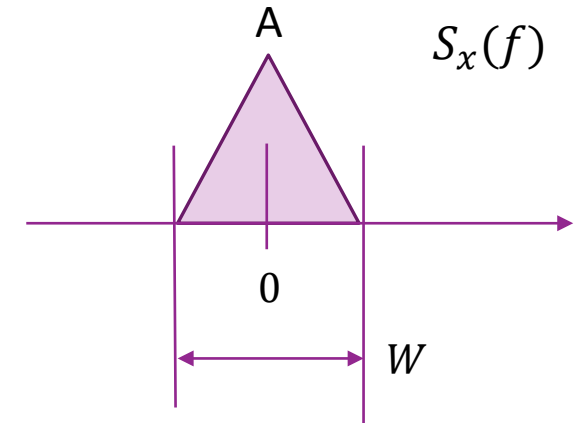
# Typical Wireless Power Transmit Levels

---

- ❑ 100 kW = 80 dBm: Typical FM radio transmission with 50 km radius
- ❑ 1 kW = 60 dBm: Microwave oven element (most of this doesn't escape)
- ❑ ~300 W = 55 dBm: Geostationary satellite
- ❑ 250 mW = 24 dBm: Cellular phone maximum power (class 2)
- ❑ 200 mW = 23 dBm: WiFi access point
- ❑ 32 mW = 15 dBm: WiFi transmitter in a laptop
- ❑ 4 mW = 6 dBm: Bluetooth 10 m range
- ❑ 1 mW = 0 dBm: Bluetooth, 1 m range

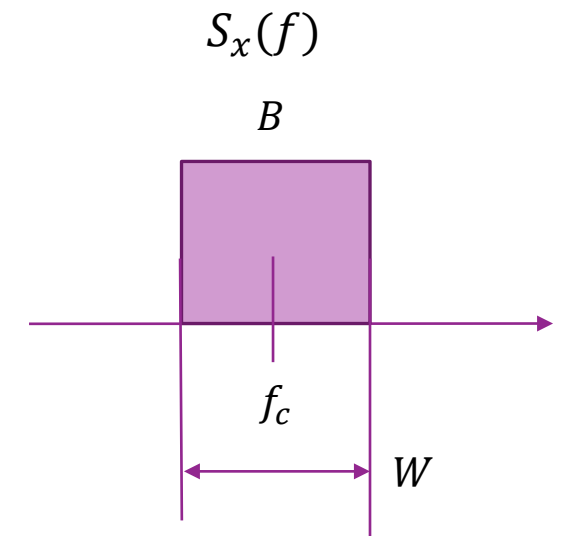
# Example 1:

- $S_x(f)$  is as shown.
- What is the power (in linear scale) in terms of  $W, A$ ?
- Suppose the power is  $P_x = 20$  dBm,  $W = 20$  MHz,  $f_c = 2$  GHz
  - What is  $A$ ?
  - What are the units of  $A$ ?
  
- Answer: on board



# Example 2

- $S_x(f)$  is shown for  $f > 0$ . Assume  $x(t)$  is real.  $W = 20$  MHz,  $B = 2(10)^{-8}$  mW/Hz,  $f_c = 2$  GHz
- What is  $P_x$ ? (Linear and in dBm).
- Suppose  $y(t) = h(t) * x(t)$  with  $H(f) = f_0 / (2\pi jf + f_0)$
- What is  $S_y(f)$ ? Draw it.
- Assuming  $f_c \gg f_0$  what is  $P_y$ ?
- What is the attenuation in dB?



# In-Class Exercise

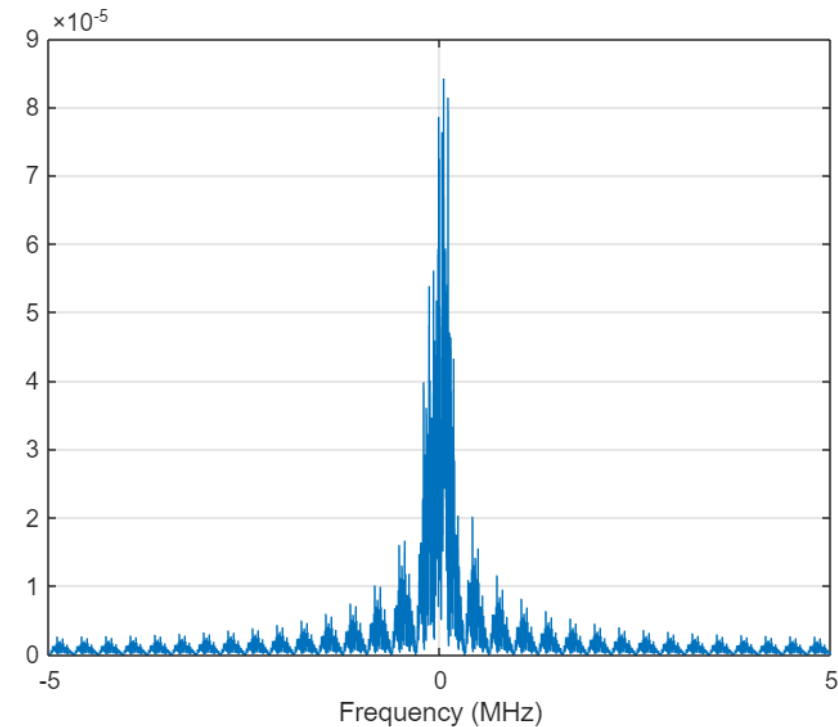
## Computing a Fourier Transform with an FFT

In this exercise, you will see how to approximately compute the Fourier Transform of a sampled signal with the FFT. First, we create a sampled-signal.

- Generate random complex symbols,  $\text{sym}[k] = \exp(1i \cdot \text{theta}[k])$  where  $\text{theta}[k]$  is uniform in  $[0, 2\pi]$  and  $k=1, \dots, \text{nsym}$
- Create an up-sampled version of  $\text{sym}$ , denoted  $u$ :  $u[n] = \text{sym}[k]$  for  $n = k \cdot \text{sampPerSym} - 1, \dots, (k+1) \cdot \text{sampPerSym}$ .
- Plot the  $\text{real}(u)$  vs. time in micro-seconds.

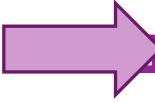
So,  $u$  is a signal that randomly changes phases every  $\text{sampPerSym}$ .

```
sampPerSym = 2^5; % number of samples per symbol
nsym = 2^7; % number of symbols
nsamp = sampPerSym*nsym; % total number of samples
fs = 10e6; % sample rate in Hz
```

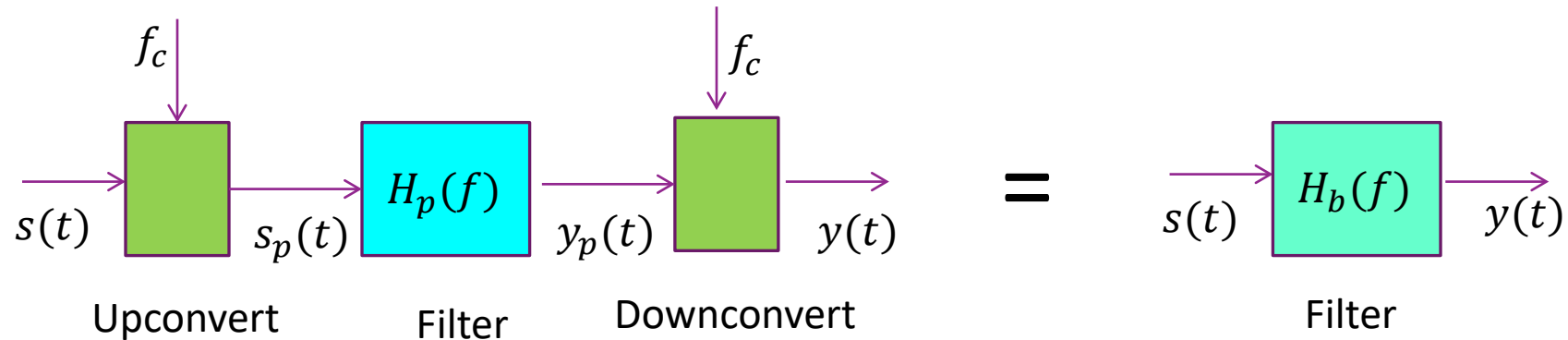


# Outline

---

- ☐ Time-Domain Relationships
- ☐ Fourier Transform Review
- ☐ Frequency-Domain Relationships
- ☐ Power and Energy Spectra
-  ☐ Baseband equivalent filters
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- ☐ Wireless channels

# Filtering



- Filtering at passband equivalent to complex baseband filter
- Assuming downconversion filter is ideal (see next slide):
  - $H_b(f) = H_p(f + f_c)$  for  $|f| \leq \frac{W}{2}$
  - Simply shift  $H_p(f)$  to the left by  $f_c$ .

# Proof of Result

---

□ Using the conversions from passband:

□ Downconversion formula:  $Y(f) = 2Y_p(f + f_c)H_{LPF}(f)$

□ Filtering in passband:  $Y(f) = 2H(f + f_c)U_p(f + f_c)H_{LPF}(f)$

□ Using upconversion formula:

$$Y(f) = H(f + f_c)\{U^*(-f - 2f_c) + U(f)\}H_{LPF}(f)$$

□ Assume:

- $U(f)H_{LPF}(f) \approx U(f)$  Filtering preserves baseband image
- $U^*(-f - 2f_c)H_{LPF}(f) \approx 0$  Filtering removes image around  $-2f_c$

□ Then  $Y(f) = H(f + f_c)U(f)$



# Delay

---

❑ Important special case: Suppose that  $h_p(t) = A\delta(t - \tau)$

- $A$  = gain
- $\tau$  = delay

❑ Passband frequency response is:  $H_p(f) = Ae^{-2\pi jf\tau}$

❑ Baseband frequency response:

$$H_b(f) = H_p(f + f_c) = Ae^{-2\pi j(f_c + f)\tau}$$

❑ Equivalent impulse response:

$$h_b(t) = Ae^{-2\pi jf_c\tau}\delta(t - \tau)$$

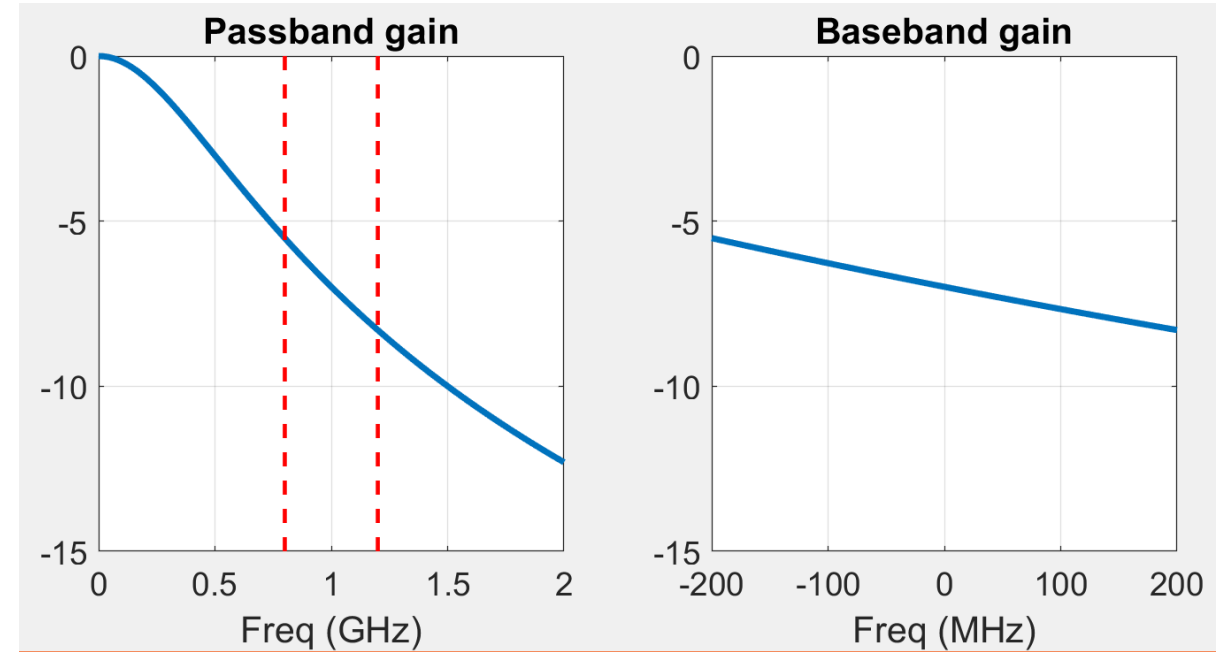
❑ Delay adds a constant phase rotation

# Example: First Order Filter

- Passband:  $H_p(\omega) = \frac{1}{1+j\omega/\omega_0}$
- Effective baseband:  $H(\omega) = \frac{1}{1+j(\omega+\omega_c)/\omega_0}$
- Observe baseband response is:
  - Almost flat
  - Not symmetric around  $f = 0$

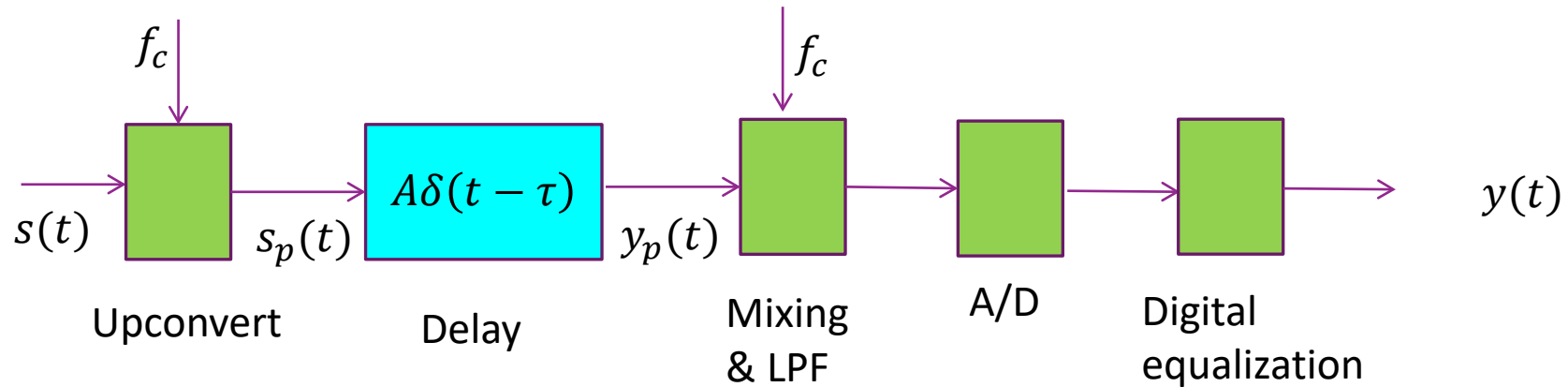
```
fp = 1e9*linspace(0,2,128)';  
Hp = freqs(G0,[1/w0 1], 2*pi*fp);  
plot(fp/1e9, 20*log10(abs(Hp)), 'Linewidth', 3);
```

```
fb = linspace(-2e8,2e8,128)';  
Hb = freqs(G0, [1/w0, 1+1i*fc/f0], 2*pi*fb);  
plot(fb/1e6, 20*log10(abs(Hb)), 'Linewidth', 3);
```



Passband cutoff freq  $f_0 = 0.5$  GHz  
Carrier freq  $f_c = 1$  GHz

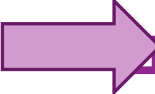
# Delay and Synchronization



- ❑ Two methods to compensate for delay at the RX
  - ❑ Method 1: Correct in analog by adjusting phase of LO
  - ❑ Method 2: Correct digital by inverting the gain  $Ae^{2\pi j f_c \tau}$ 
    - This is a special case of [equalization](#)

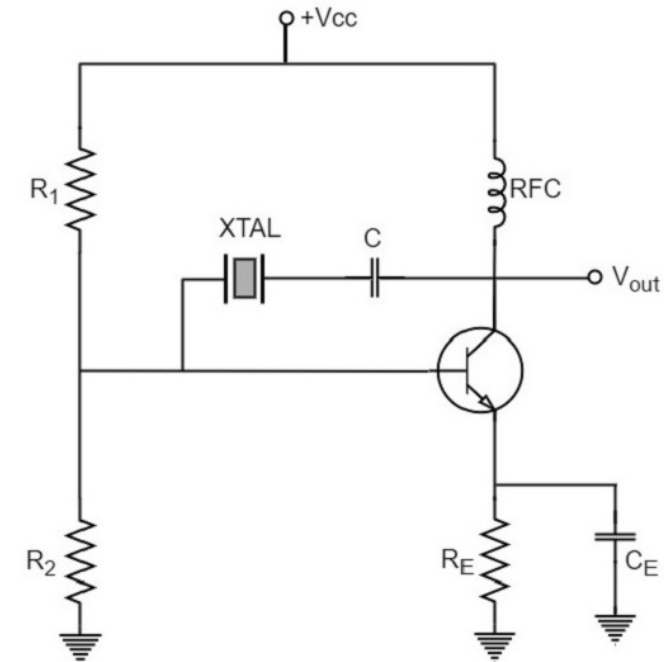
# Outline

---

- ☐ Time-Domain Relationships
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# Local Oscillator Crystal Source

- ❑ We saw that the mixer requires a sinusoidal input
  - Called the **local oscillator (LO)**
  - Sinusoidal input frequency should = desired carrier frequency
- ❑ The LO is often generated by a **crystal**
- ❑ Crystal has a **resonant frequency**
- ❑ A circuit then amplifies resonant frequency
- ❑ Resonant frequency is sometimes **voltage controlled**
  - VCXO (Voltage-controlled crystal oscillator)
  - Allows tunability



# Frequency Synthesis

❑ The crystal oscillator may not directly produce desired carrier frequency

- Frequency is constrained by resonant properties of the crystal

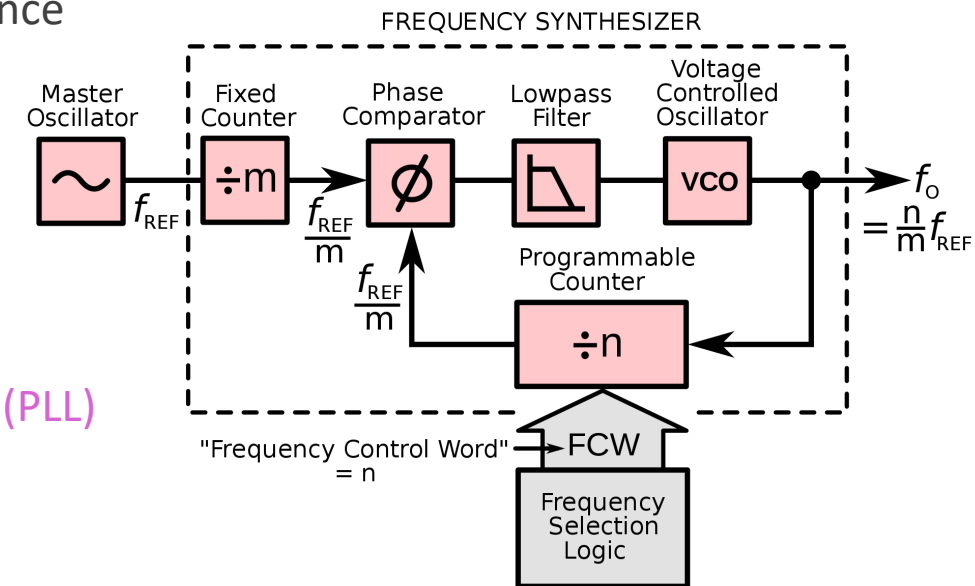
❑ Frequency synthesizer

- Creates a desired LO at carrier frequency from the crystal reference

❑ Typically creates a rational multiple

$$f_0 = \frac{n}{m} f_{REF}$$

- $f_{REF}$  = Frequency from crystal
- $f_0$  = Output frequency (usually the carrier)
- Performed by a clock multiplier, divider and **phased locked loop (PLL)**

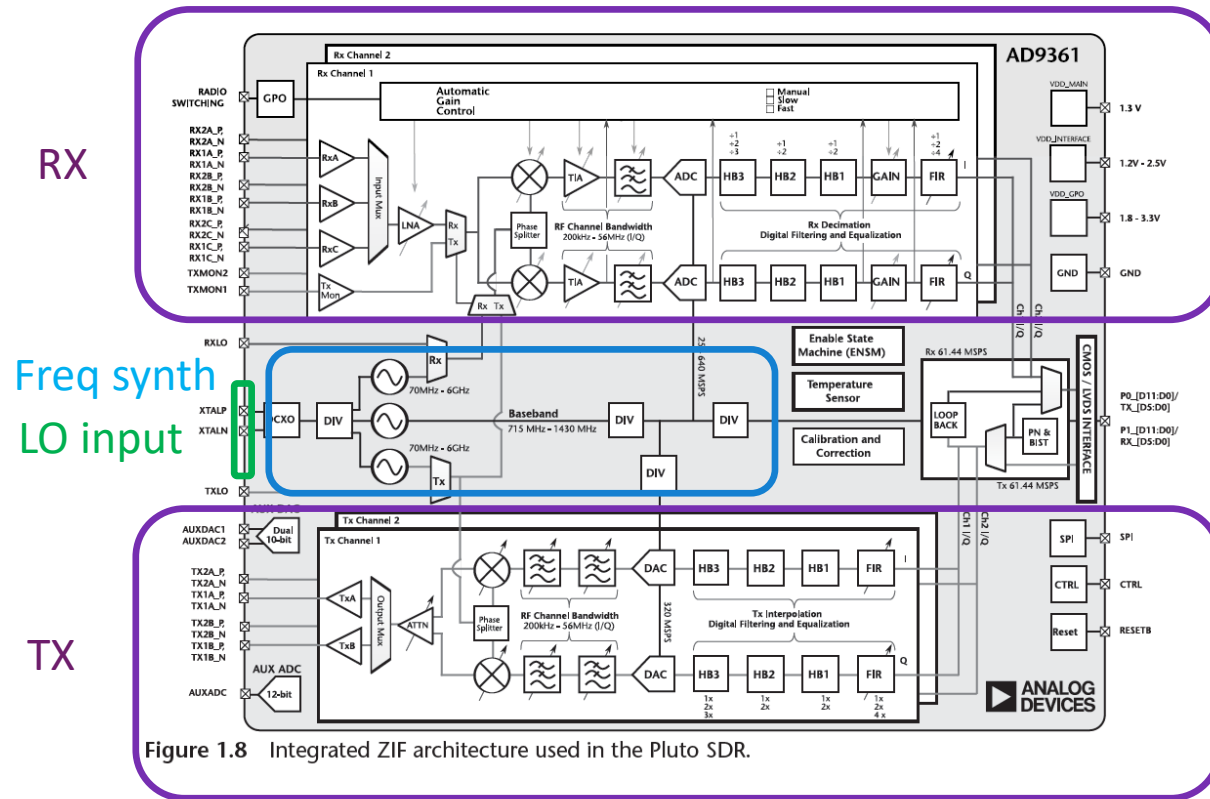


# Example: The ADALM-Pluto



- ❑ The SDR we use in the lab
- ❑ The mixing is performed in ADI 9363 chip
- ❑ Simple architecture
  - But basic steps similar in more complex devices
  - Good to look at as a starting point
- ❑ Let's look inside!

# Example Up and Down-Conversion Circuit

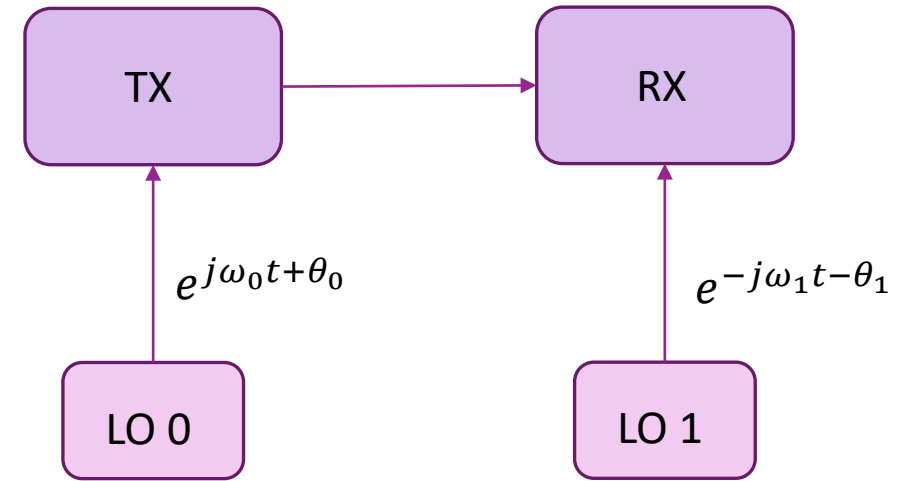


- ❑ Analog Devices AD9361 Wideband TXCR
  - Single integrated circuit
- ❑ External **local oscillator** (LO) from crystal
  - Followed by **frequency synthesizer**
- ❑ **Receiver (RX):**
  - Left to right
  - LNA, tunable mixer, ADC, Filters
- ❑ **Transmitter (TX):**
  - Right to left
  - Filters, FAC, tunable mixer, power amplifier



# Frequency Errors

- ❑ Two sources of errors in actual mixers
  - Gains are not matched at TX and RX
  - Frequencies are not matched at TX and RX (used different crystal sources)
- ❑ Suppose
  - Upconversion:  $u_p(t) = \text{Re}(u_{TX}(t)e^{j\omega_0 t + \theta_0})$
  - Downconversion:  $u_{RX}(t) = G u_p(t) e^{-j(\omega_1 t + \theta_1)} + \text{LPF}$
- ❑ Then:
  - $u_{RX}(t) = G u_{TX}(t) e^{j((\omega_0 - \omega_1)t + (\theta_0 - \theta_1))}$
- ❑ Causes a carrier frequency offset (CFO)



# Parts Per Million

---

- ❑ Oscillator error often measured in **parts per million** (ppm):

$$\Delta(\text{ppm}) := \frac{|f_c - f_c'|}{f_c} (10)^6$$

- $f_c$  = desired carrier frequency
- $f_c'$  = actual carrier frequency

- ❑ Example:

- $f_c = 2.5 \text{ GHz}$ ,  $\Delta = 10 \text{ ppm}$  (typ value for low-cost oscillator)
- Then,

$$|f_c - f_c'| = (2.5)(10)^9(10)(10)^6 = 25 \text{ kHz}$$

- Very large frequency shift!

# Lab Preview

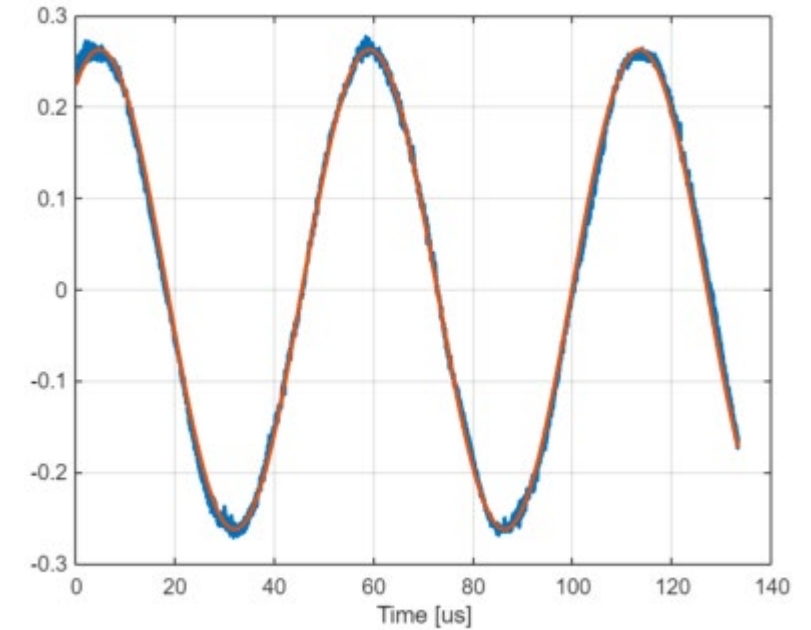
## Frequency Estimation and Carrier Frequency Offset

### Table of Contents

Create the TX and RX objects  
Sending a Continuous Wave  
Capture and Plot the Sinusoid  
Estimating the RX Frequency and CFO via the Correlation Method  
Estimating the Amplitude of the Complex Exponential via Least Squares  
Estimating the RX Frequency using an FFT  
Advanced Topics

Complex exponentials are the most fundamental signals for all frequency domain analysis of linear system. In this lab you will learn to:

- Send a complex exponential or continuous-wave (CW) signal through the SDR
- Estimate the complex gain and frequency of a sinusoid via (1) correlation method, (2) FFT method
- Estimate the carrier frequency offset
- Save data for files for offline processing



□ Lab is passband unit

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 Wireless channels

# Free Space Wireless Channels

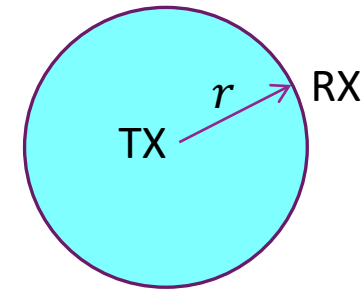
- ❑ Free space propagation
  - No obstacles
  - Isotropic (equal power in all directions)

- ❑ Power decreases as  $r^{-2}$ 
  - $\Rightarrow \text{Gain} = Ar^{-1}$  for some  $A$

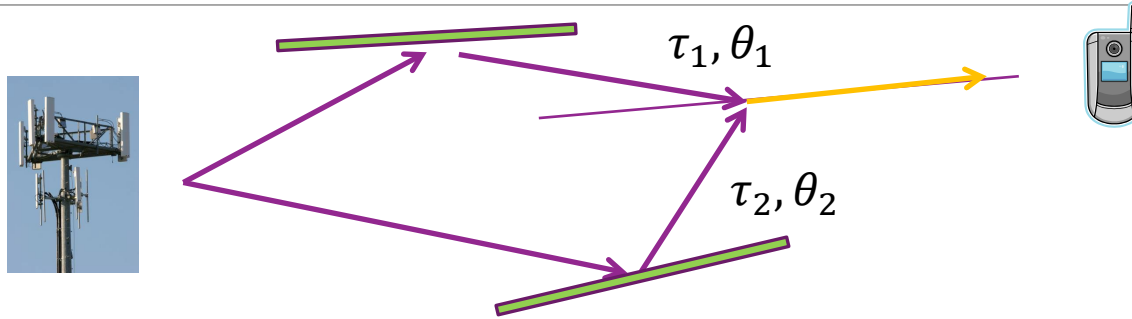
- ❑ Delay is  $\tau = r/c$ ,  $c = 3(10)^8$  m/s

- ❑ Hence, baseband channel is:

$$h(t) = \frac{A}{r} e^{\frac{j2\pi f_c r}{c}} \delta(t - r/c)$$



# Multipath



❑ Wireless signals can arrive in many directions

- Reflections, diffraction, transmission, ...

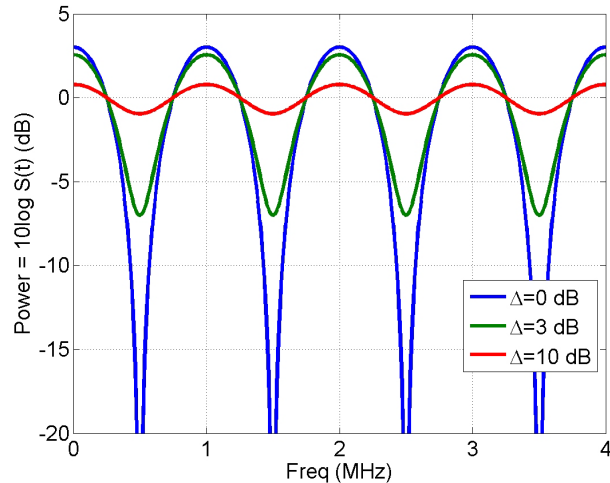
❑ Each path will have different gain and delay

❑ Receiver sees the combined total

$$h(t) = \sum_{k=1}^K h_k e^{-2\pi j f_c \tau_k} \delta(t - \tau_k)$$

- $h_k$  = complex gain of each path

# Two-Path Example



Magnitude response

$$S(f) = |H(f)|^2 = |h_1 e^{2\pi i f \tau_1} + h_2 e^{2\pi i f \tau_2}|^2$$

Plot shows:

$$\tau_2 - \tau_1 = 1 \mu s, |h_1|^2 + |h_2|^2 = 1, |h_2|^2 = 10^{0.1\Delta} |h_1|^2$$

□ Rate of variation in frequency depends on delay spread:  $\tau_2 - \tau_1$

□ Size of variation depends on spread of path gains:

- Average  $S(f) = |h_1|^2 + |h_2|^2$
- Min  $S(f) = (|h_1| - |h_2|)^2$ , Max  $S(f) = (|h_1| + |h_2|)^2$

# Example Problem (On board)

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- ❑ A wireless channel has 2 paths:
  - Path 1: Power gain of -80 dB, travels 100m
  - Path 2: Power gain of -83 dB, travels 120m
- ❑ What are the amplitude gains of the two paths,  $h_1, h_2$ ?
- ❑ What are the two delays of the paths:  $\tau_1, \tau_2$ ?
- ❑ What is the average, minimum and maximum power?