Unit 10: Convolutional Codes

EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN
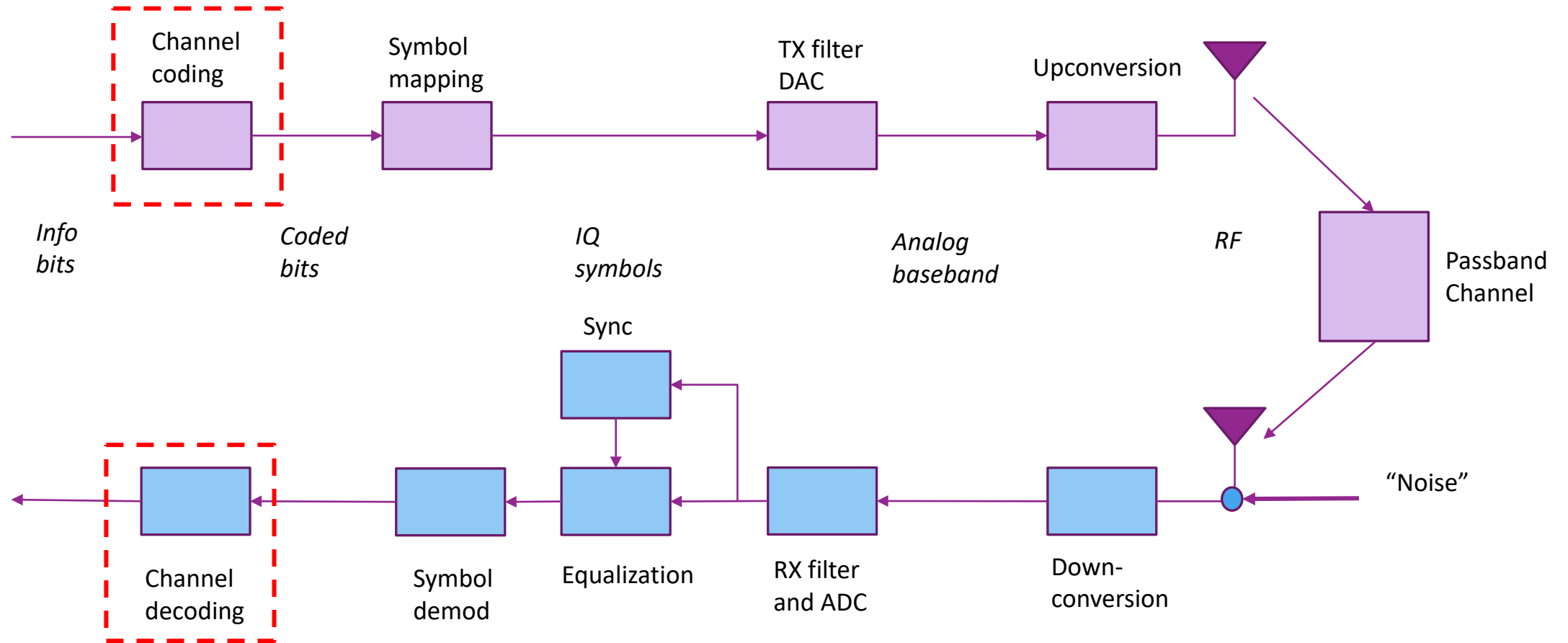
Learning Objectives

❑Encode data using a convolutional code for given a generator polynomial

❑Compute the rate of the code including tail bits

❑Represent the code via a trellis diagram and finite state machine

❑Compute the branch metrics of a code for a memoryless channel

❑Decode the code via the Viterbi algorithm

Graduate vs. Undergraduate

❑Unit 9 on basic codes is skipped for the graduate students

❑However, a few of the Unit 9 slides are included here for completeness

❑Grad students who would like a review are encouraged to look at Unit 9

This Unit

Channel coding

Symbol mapping

TX filter DAC

Upconversion

Info bits

Coded bits

IQ symbols

Analog baseband

RF

Passband Channel

Sync

Channel decoding

Symbol demod

Equalization

RX filter and ADC

Down-conversion

"Noise"

# Outline

➡️ Convolutional codes: encoding and representations

☐ Tree, trellis and state diagrams

☐ Decoding with branch metrics

☐ Viterbi decoding

Convolutional Codes History

❑Generates a stream of coded bits from uncoded bits
  ◦ Block codes form by terminating the stream

❑Output stream created by binary FIR filters

❑Developed originally by Elias (1955)
  ◦ Key challenge was decoders. Much study in 1960s.
  ◦ Practical, optimal decoders developed by Viterbi,1967.

❑Can perform within 2-3 dB of Shannon limit.

❑Instrumental in Pioneer Space program (along with RS codes)

❑Most widely-used code in industry today: WiFi, cellular
  ◦ In the mid 1990s, longer block length cellular codes were replaced with Turbo codes,
  ◦ But, these use convolutional codes as basis
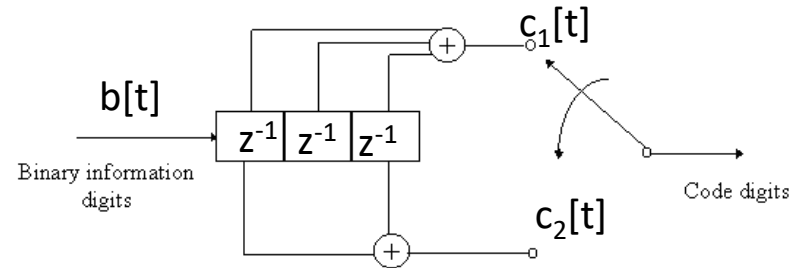
Convolutional Codes

❑Encode data through parallel binary (usu. FIR) filters

❑Example convolutional code:
- Rate = ½ (two output bits $(c_1[t], c_2[t])$ for each input bit b[t].
- Constraint length K=3 (size of shift register)

b[t]

Binary information digits

z⁻¹ | z⁻¹ | z⁻¹

$c_1[t]$

$c_2[t]$

Code digits
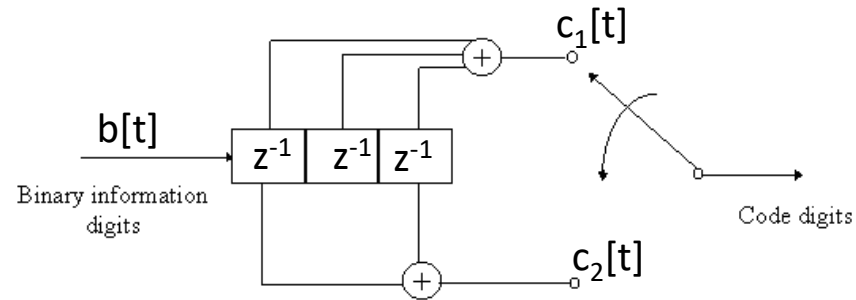
$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

Convolutional Codes
Block Implementation

$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

❑ To implement as block code:
◦ Start with L input bits b[0],b[1],…,b[L-1]
◦ Append K-1 zero b[L]=b[L+1]=…=b[L+K-2]=0 (called tail bits)
◦ Generate output bits from each branch
  ◦ $c_j[0]$, $c_j[1]$, … , $c_j[L+K-2]$, j=1,…,N where N = number of branches
◦ Final codeword is concatenation of branch output
  ◦ **c**=($c_1[0]$, $c_1[1]$, … , $c_1[L+K-2]$,…, $c_N[0]$, $c_N[1]$, … , $c_N[L+K-2]$)

❑ Effective rate: $R = \dfrac{L}{N(L+K-1)} \approx \dfrac{1}{N}$ for large L
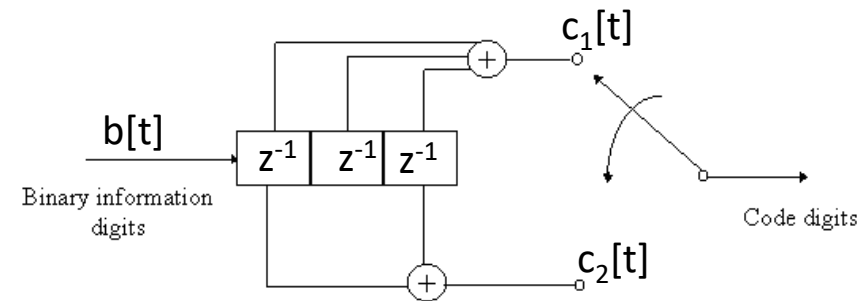
Convolutional Codes
Encoding Example

❑Encode message **b** = [1 0 1]

❑Branch outputs **c**1=[11011] c2=[11001]

❑Final output **c**=[11,10,00,10,11] (interleaved)
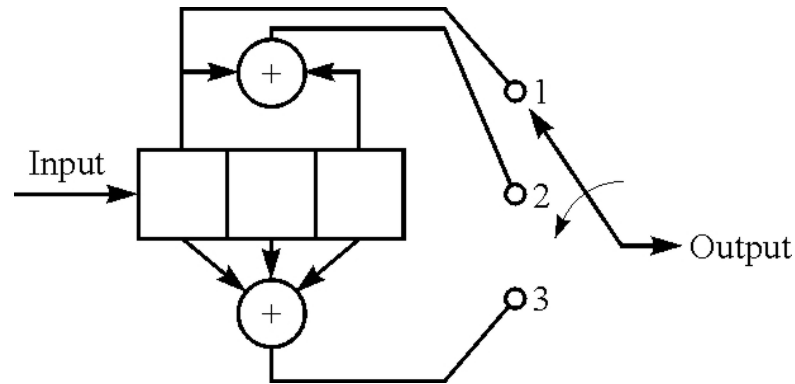
❑Rate = 3/10

t	b[t]	c1[t]	c2[t]
0	1	1	1
1	0	1	0
2	1	0	0
3	0	1	0
4	0	1	1

Tail bits

$$c_1[t] = b[t] + b[t-1] + b[t-2]$$
$$c_2[t] = b[t] + b[t-2]$$

$c_1[t]$

b[t]

z^{-1} z^{-1} z^{-1}

Binary information digits

Code digits

$c_2[t]$

Generator Polynomials: Binary Form

Rate 1/3, K=3 example:

$$c_1[t] = b[t]$$
$$c_2[t] = b[t] + b[t-1]$$
$$c_2[t] = b[t] + b[t-1] + b[t-2]$$

❑Code polynomials: Binary vector of filter coefficients

g_1 = [100]

g_2 = [101]

g_3 = [111]

Generator Polynomials: Octal Form

❑For large constraint lengths (large K), binary form is inefficient
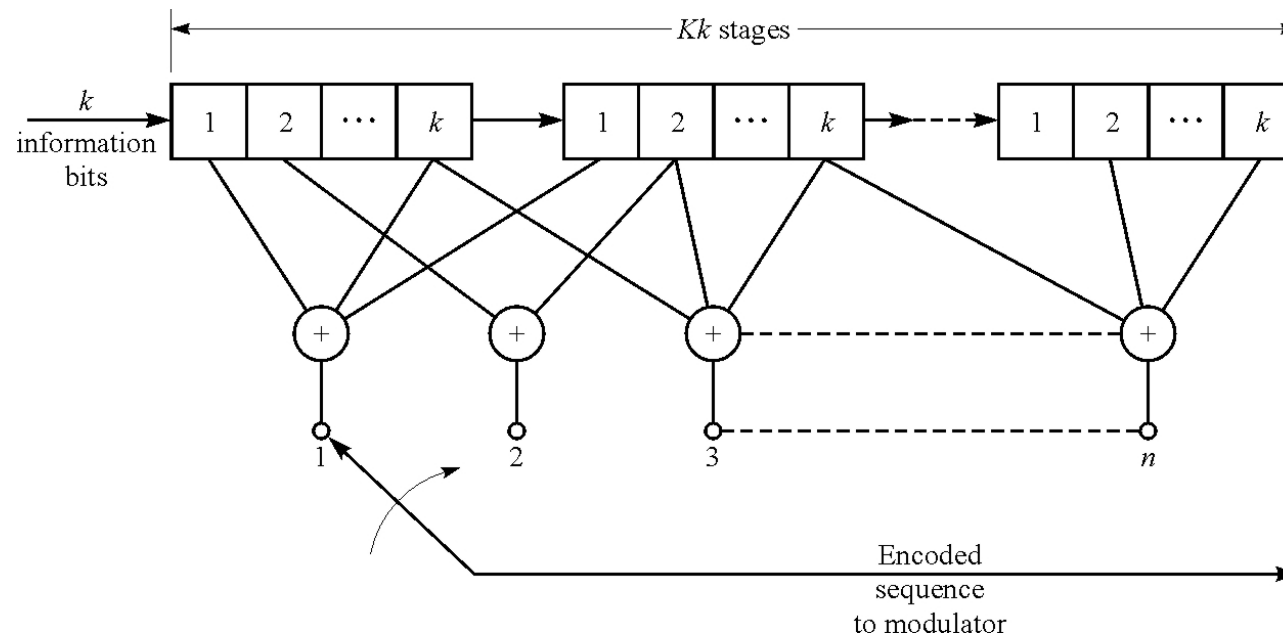
◦ Engineers often use octal form

◦ Base 8, each digit 0…7

◦ Each digit represents three bits

❑Example:

g_1 = [1 011]　　　　➡　　　g_1 = [13]

g_2 = [1 101]　　　　　　　　g_2 = [15]

g_3 = [1 010]　　　　　　　　g_3 = [12]

Binary　　　　　　　　　　　　Octal

Multiple Inputs

❑Examples up to now are R=1/n

❑Can extend to rate R=k/n
  ◦ Add k bits at a time

In-Class Exercise

Convolution Code In-Class Exercises

Building an Encoder

We will look at a simple R=1/2 convolution code:

$$c(t,1) = b(t) + b(t-1) + b(t-2)$$
$$c(t,2) = b(t) + b(t-2)$$

This encoder can be represented with a generator matrix G as shown.

```
G = [1 1 1; 1 0 1];
nin = 1;                % number of input streams
nout = size(G,1);       % number of output streams
conLen = size(G,2);     %cconstraint length
```

We will build a simple encoder using the convolution method. The convolution method in MATLAB uses real values, so we will use a modulo to convert back to .

Outline

❑Convolutional codes: encoding and representations

Tree, trellis and state diagrams

❑Decoding with branch metrics

❑Viterbi decoding

Convolutional Codes as State Machines

❑Convolutional codes have memory
- ◦ Stored in contents of shift registers
- ◦ Each input bit changes memory contents
- ◦ Output bits are a function of memory and input

❑Three common ways to represent evolution of the memory:
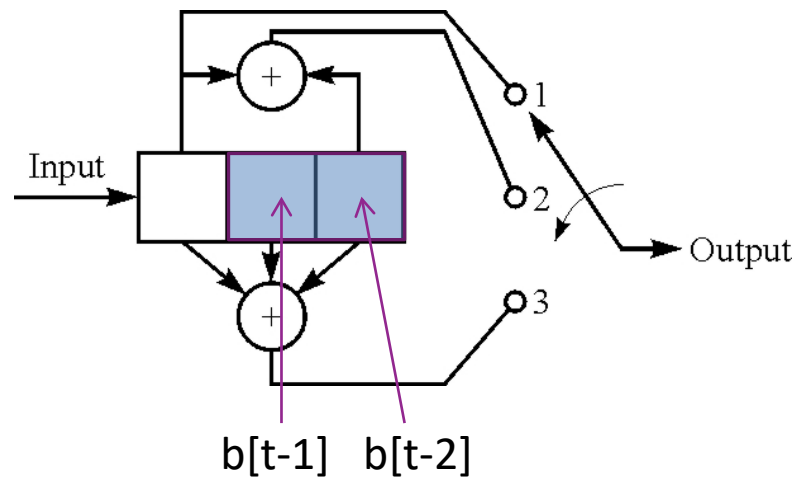- ◦ Tree diagram
- ◦ Trellis diagram
- ◦ State diagram

Encoder States

❑ State of the encoder determined by contents of shift register:
$$x[t] = (b[t-1], \dots, b[t-K])$$

❑ Only need to look at most recent K-1 bits not including the current bit
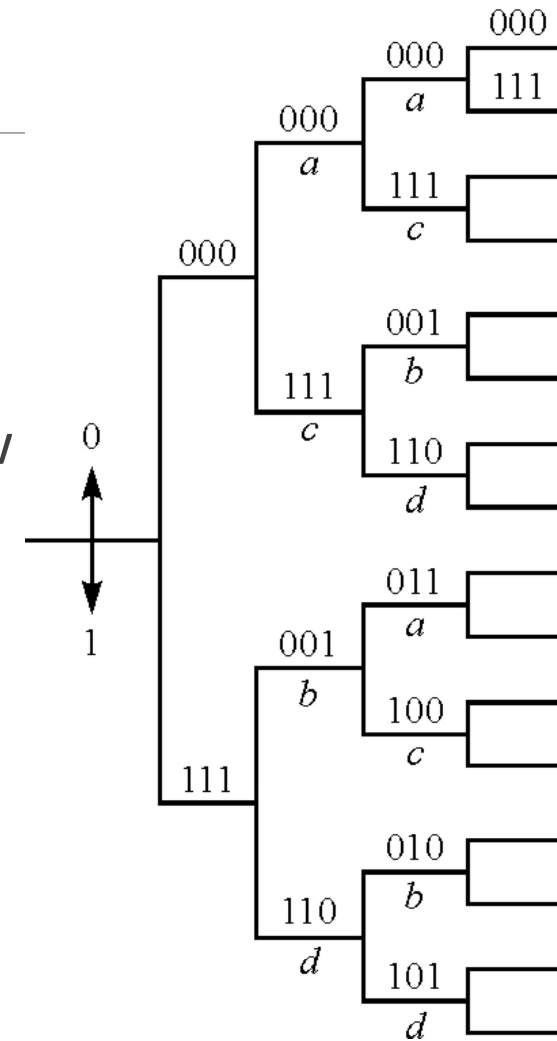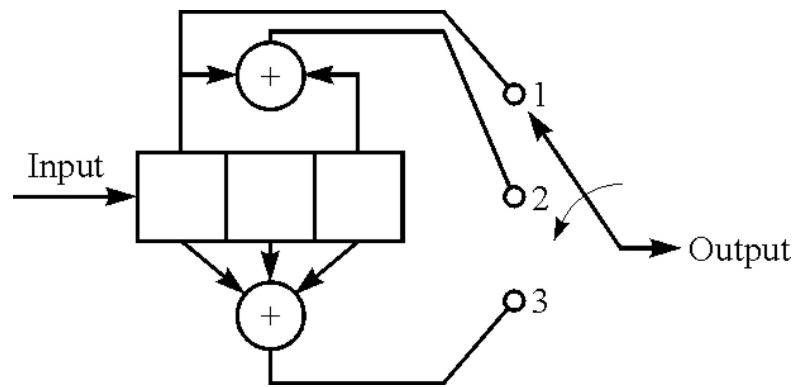 ◦ Last bit will be shifted out
  ◦ There are 2^{K-1} states

b[t-1]   b[t-2]

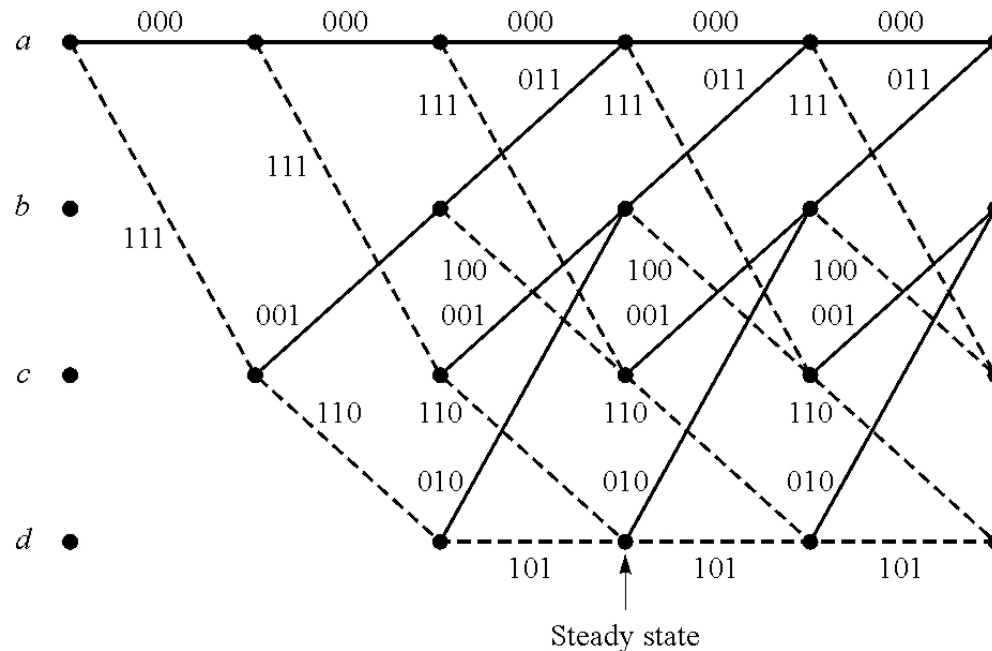| State label	$b[t-1]$	$b[t-2]$
a	0	0
b	0	1
c	1	0
d	1	1

Tree Diagram

❑ Two branches for each input bit 0 or 1.

❑ Branches labeled by output bits (n bits)
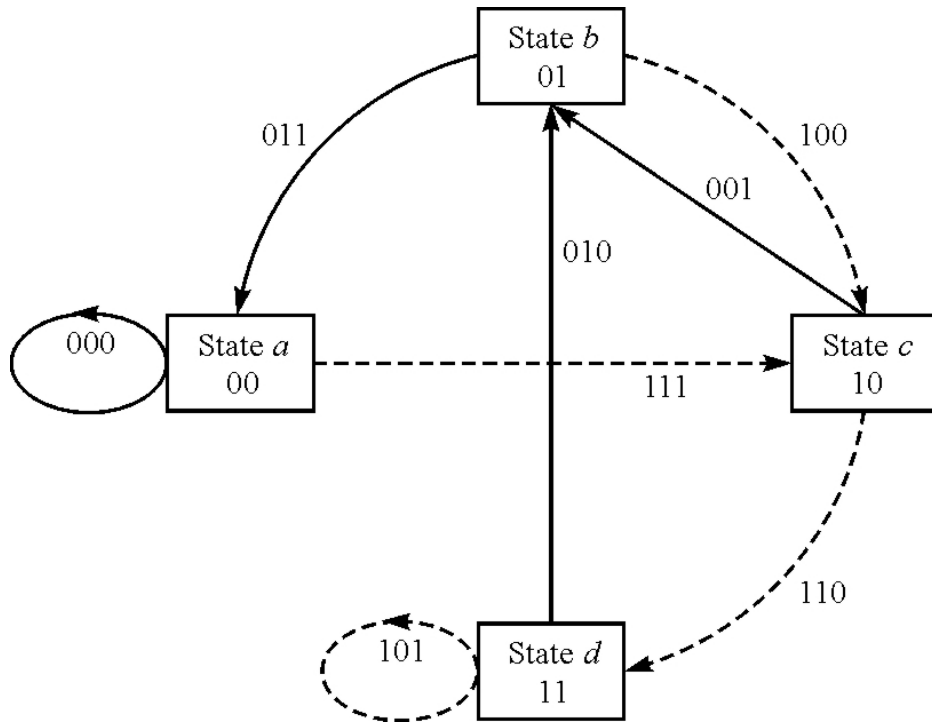
❑ Difficult to use since branches infinitely grow

Trellis Diagram

❑Show trajectory through the states

❑Solid lines: paths with b[t]=0, Dash lines: b[t]=1

Labels are the outputs along each path

State Diagram

- ❏ One node per state
- ❏ Solid lines:
  - ◦ Transitions with b[t]=0
- ❏ Dashed lines:
  - ◦ Transitions with b[t]=1
- ❏ Labels indicate outputs on each state

State Machine Functional Description

❑Finite state machine:
- $x[t]$: Sequence of states, $x[t] \in \{0, \ldots, 2^{K-1} - 1\}$
- $b[t]$: Input sequence
- $c[t]$: Output sequence

❑Iterative generating sequence:

$$x[t + 1] = f(x[t], b[t]) \quad \longleftarrow \text{State function}$$
$$c[t] = h(x[t], b[t]) \quad \longleftarrow \text{Output function}$$

❑Initial condition: $x[0]$

In-Class Exercise

Representing the State Transition Function

We wish to represent the convolutional encoder as a finite state machine. To this end, we will represent the states by a single decimal number:

$$x(t) = 2*b(t-2) + b(t-1) + 1$$

This state runs from $x(t) = 1, \ldots,$ nstate where nstate $= 2^{\{convLen-1\}}$

To represent the state transition, we create a matrix xnextMat

- xnextMat(i,1)= the value of $x(t+1)$ when $b(t)=0$ and $x(t)=i$
- xnextMat(i,2) = the value of $x(t+1)$ when $b(t)=1$ and $x(t)=i$

```
ninVal = 2^nin;          % number of input values per time step
nstate = 2^(conLen-1);   % number of states

% Intiialize matrix
xnextMat = zeros(nstate,ninVal);

% TODO:  Write a for loop to fill in the xnextMat matrix
```

We next create a matrix to represent the output:

- coutMat(k,i,j) = output $c(t,k)$ when $x(t)=i$ and $b(t)=j+1$

Outline

❑Convolutional codes: encoding and representations

❑Tree, trellis and state diagrams

➡❑Decoding with branch metrics

❑Viterbi decoding

ML Estimation

❑ Assume the following dimensions:
- N outputs per time steps: $c[t] = (c_1[t], \ldots, c_N[t])$
- T time steps (including tail bits!)

❑ Channel model:
- For each bit $c_i[t]$, we make some observation $r_i[t]$
- Output is probabilistic $P(r_i[t]|c_i[t])$
- Assume all outputs are independent:

$$P(\boldsymbol{r}|\boldsymbol{c}) = \prod_{t=0}^{T-1} \prod_{i=1}^{N} P(r_i[t]|c_i[t])$$

❑ Find ML estimate:

$$\hat{c} = \arg \max_{c} P(r|c)$$

- Maximum over all sequences

LLR Review

☐ Before studying convolutional codes, consider simple code:

- ◦ Codeword has N output bits: $\boldsymbol{c} = (c_1, \ldots, c_N)$
- ◦ Receive N symbols $r = (r_1, \ldots, r_N)$. Each r_i may be binary, discrete, real-valued or complex-valued
- ◦ Assume a memoryless channel:

$$P(r|c) = \prod_{i=1}^{N} P(r_i|c_i)$$

☐ Define LLR: $L_i = \log \dfrac{P(r_i|c_i = 1)}{P(r_i|c_i=0)}$

☐ Theorem: The ML codeword maximizes the LLR sum value function (se Unit 9)

$$\hat{c} = \arg \max_{c} J(c), \qquad J(c) = \sum_{i=0}^{N-1} c_i L_i$$

Example 1: Binary Symmetric Channel

❏ Binary symmetric channel: Output $r_i = 0$ or 1

❏ Error probability $p < 1/2$

$$P(r_i|c_i) = \begin{cases} 1-p & \text{if } c_i = r_i \text{ (no error)} \\ p & \text{if } c_i \neq r_i \text{ (error)} \end{cases}$$

❏ Branch metric: $L_i = \begin{cases} A & \text{if } c_i = r_i \text{ (no error)} \\ -A & \text{if } c_i \neq r_i \text{ (error)} \end{cases}, \quad A = \log\dfrac{1-p}{p} > 0$

❏ Observe $c_i L_i = A(1_{\{r_i=c_i\}} + 1 - r_i)$

❏ Hence maximizing value function:

$$\arg\max_c J(c) = \arg\max_c \sum [1_{\{r_i=c_i\}} + 1 - r_i] = \arg\max_c \sum [1_{\{r_i=c_i\}}]$$

$$= \arg\max_c \#(correct\ bits)$$

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Example 2: QPSK Channel

❑ Binary modulation (on a real or imaginary dimension):

$$r_i = s_i + w_i, \qquad w_i \sim N(0, N_0/2), \qquad s_i = \begin{cases} A & c_i = 1 \\ -A & c_i = 0 \end{cases}$$

❑ Gaussian distribution: $\log p(r_i|s_i) = -\frac{1}{N_0}(r_i - s_i)^2 - \frac{1}{2}\log\frac{N_0}{2}$

❑ LLR is: $L_i = \log p(r_i|s_i = A) - \log p(r_i|s_i = -A) = -\frac{1}{N_0}[(r_i - A)^2 - (r_i + A)^2] = \frac{4A}{N_0}r_i$

❑ LLR sum value function:

$$J(c) = \sum_i L_i c_i = \frac{4A}{N_0}\sum_i r_i c_i$$

Branch Metric

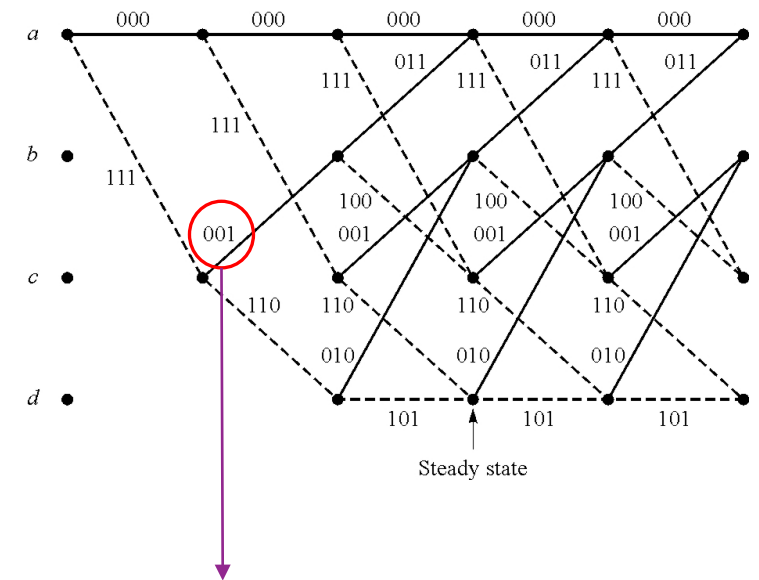□ Log-likelihood ratio: $L_i[t] = \log \dfrac{P(r_i[t]|c_i[t] = 1)}{P(r_i[t]|c_i[t] = 0)}$

○ Depends on received symbol $r_i[t]$

□ Each branch has an output set of bits $c[t] = (c_1[t], \dots, c_N[t])$

□ Define the branch metric:

$$\mu_t(x_{t+1}, x_t) = \sum_{i=1}^{N} c_i[t]\, L_i[t]$$

○ $c_i[t]$ is the coded outputs on the branch

□ Describes likelihood that sequence passed through the branch

○ Branches with higher branch metrics are more likely

$c[t] = (0,0,1) \Rightarrow$
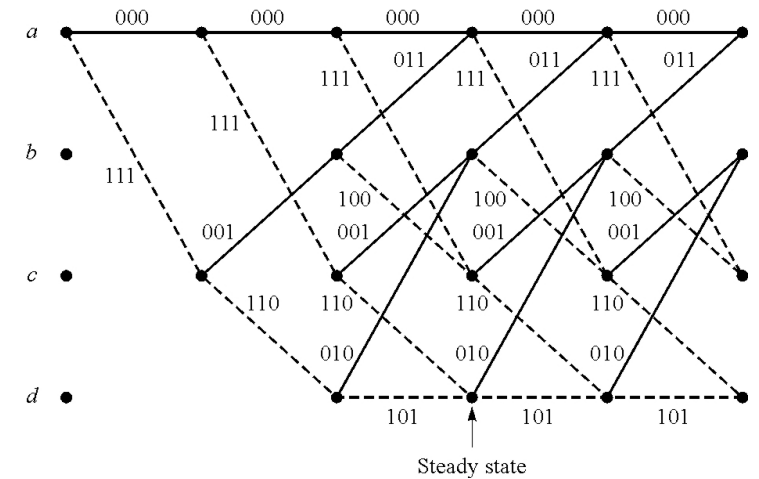$\mu_t = 0L_1[t] + 0L_2[t] + L_3[t] = L_3[t]$

Branch Metric Maximization

❑Each branch in trellis has a branch metric $\mu_t(x_{t+1}, x_t) = \sum_{i=1}^{N} c_i[t] L_i[t]$

❑We can write LLR sum value function as:

$$J(c) = \sum_{t=0}^{T-1} \sum_{i=1}^{N} c_i[t] L_i[t] = \sum_{t=0}^{T-1} \mu_t(x_{t+1}, x_t)$$

❑Interpretation:
◦ Each code sequence is a path in the trellis
◦ Value function is the sum of branch metrics on the path

Conclusion: ML estimate is the codeword with the highest total path

Example

□Consider toy Trellis (not a real conv code)

□Input: $b[t], t = 0,1,2$

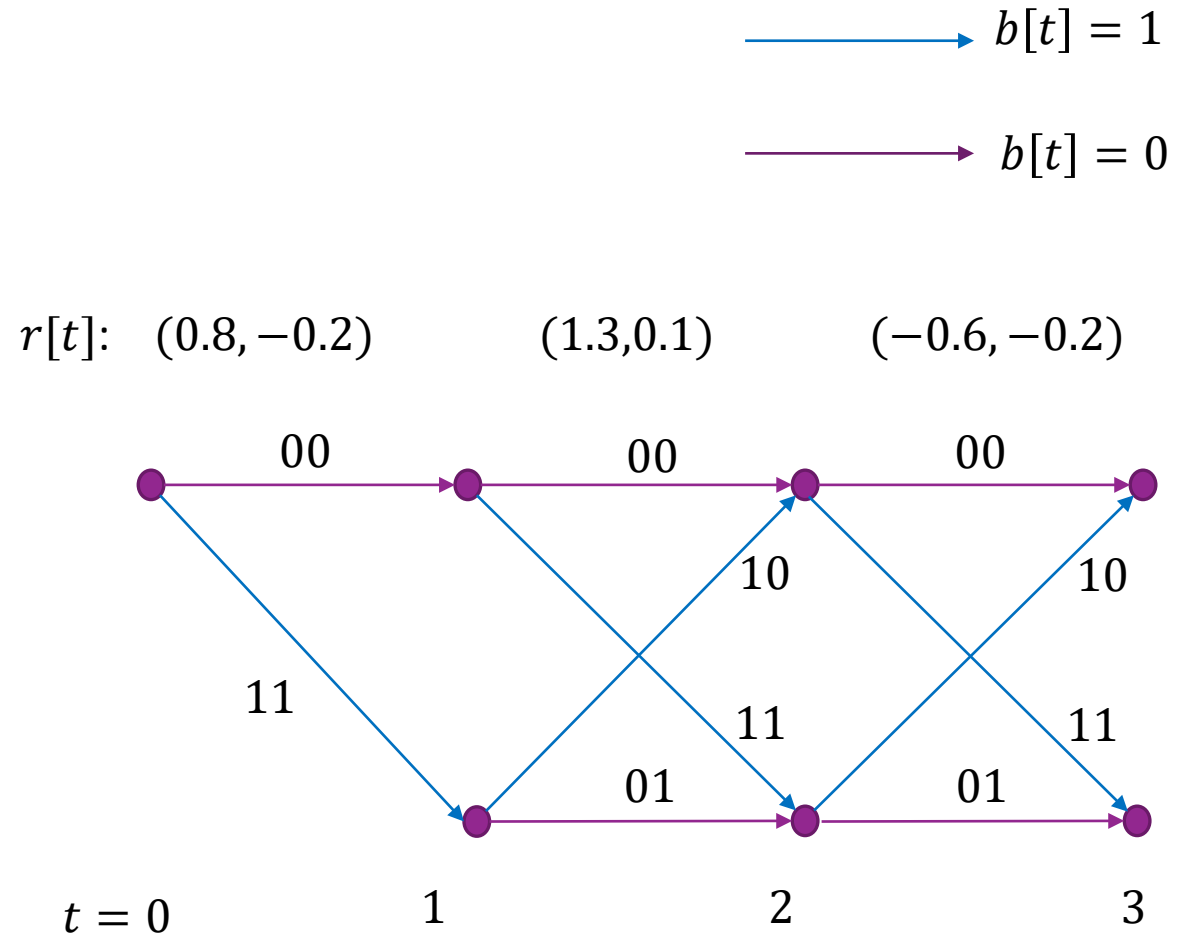□Output: $c[t] = (c_1[t], c_2[t])$
  ◦ Shown on branches

□ RX symbols: $r[t] = (r_1[t], r_2[t])$
  ◦ Values shown above Trellis

□Assume QPSK channel value function:

$$J(c) = \sum_t \sum_i r_i c_i$$

  ◦ Constant is ignored

$b[t] = 1$

$b[t] = 0$

$r[t]$: $(0.8, -0.2)$ $(1.3, 0.1)$ $(-0.6, -0.2)$

00   00   00

10   10

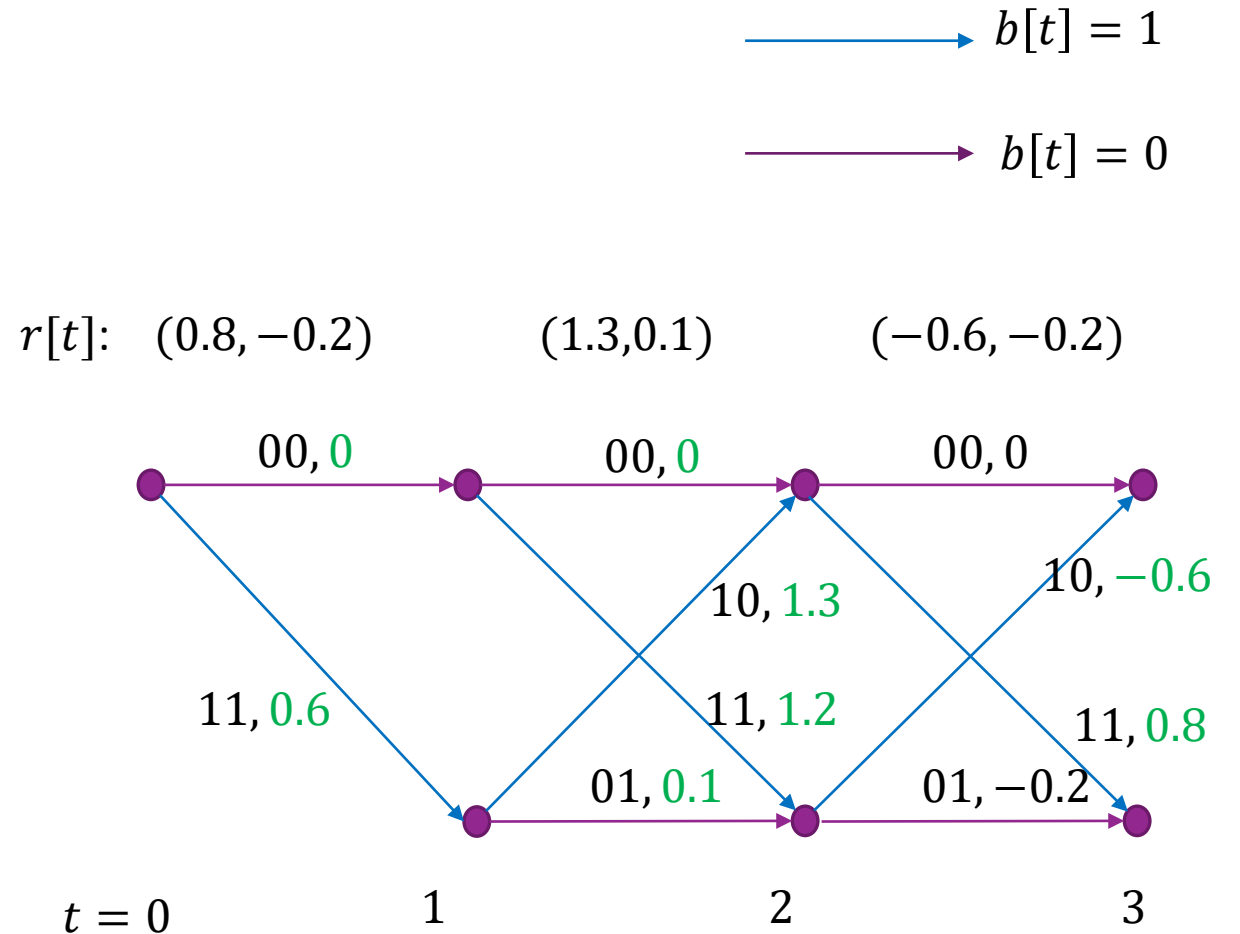11

11   11

01   01

$t = 0$ 1   2   3

Step 1: Compute Branch Metrics

☐ For QPSK channel, branch metric is

$$\mu_t(x_{t+1}, x_t) = \sum_i r_i[t]c_i[t]$$

☐ On each branch we have labeled:
(Output Bits, branch)$= (c[t], \mu_t)$

$b[t] = 1$

$b[t] = 0$

$r[t]$: $(0.8, -0.2)$ $(1.3, 0.1)$ $(-0.6, -0.2)$

$00, 0$ $00, 0$ $00, 0$

$10, -0.6$

$10, 1.3$

$11, 0.6$ $11, 1.2$ $11, 0.8$

$01, 0.1$ $01, -0.2$

$t = 0$ 1 2 3

Step 2: Find the Maximizing Path

❑ For this simple Trellis, we can do this manually

❑ Path is shown with red nodes 🔴

❑ Solution:
  ◦ Coded sequence: (11), (10), (00)
  ◦ Bits: (1,1,0)

$b[t] = 1$

$b[t] = 0$
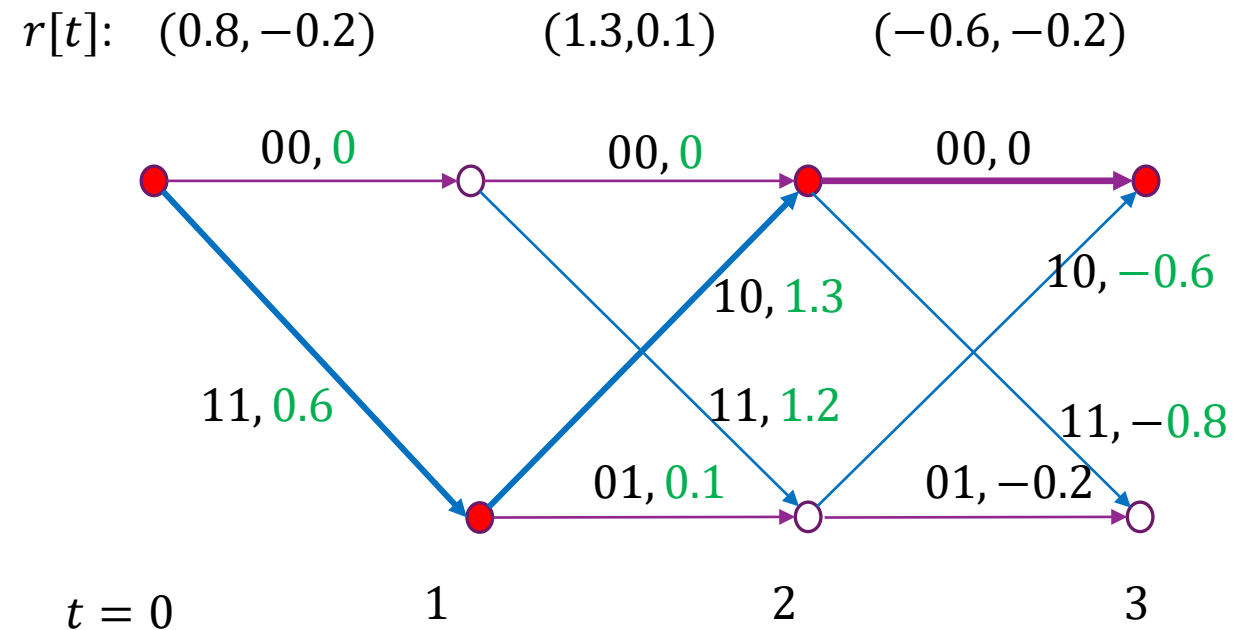
$r[t]:$ $(0.8, -0.2)$ $(1.3, 0.1)$ $(-0.6, -0.2)$

$00, 0$ $00, 0$ $00, 0$

$10, 1.3$

$10, -0.6$

$11, 0.6$ $11, 1.2$ $11, -0.8$

$01, 0.1$ $01, -0.2$

$t = 0$ 1 2 3

Outline

❑Convolutional codes: encoding and representations

❑Tree, trellis and state diagrams

❑Decoding with branch metrics

➡❑Viterbi decoding

Viterbi Algorithm by Example

❑Iterative solution for finding maximum path in a graph

❑Key idea: Find the max path one time step at a time

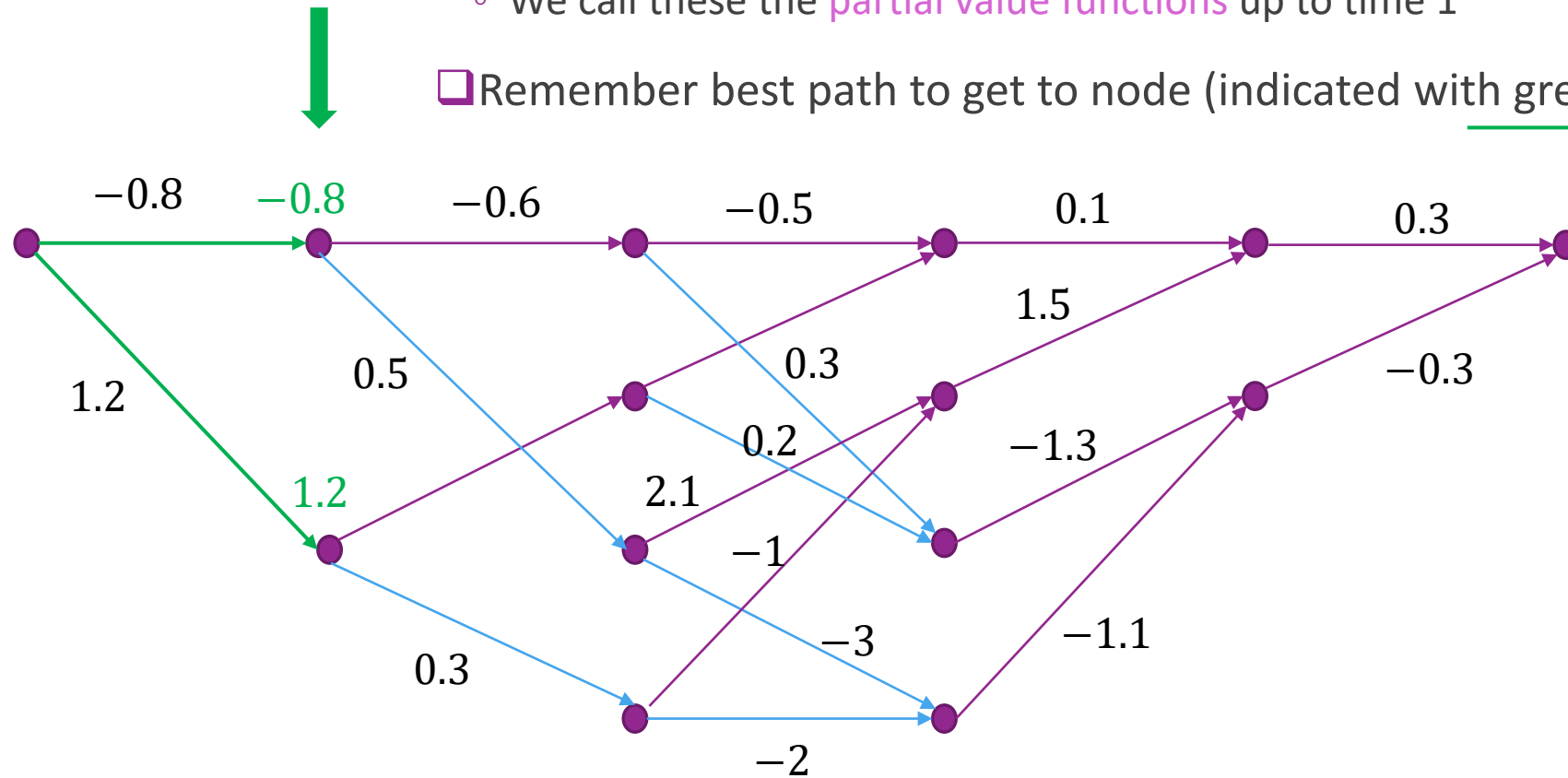❑We will illustrate by example: Shown are the branch metrics
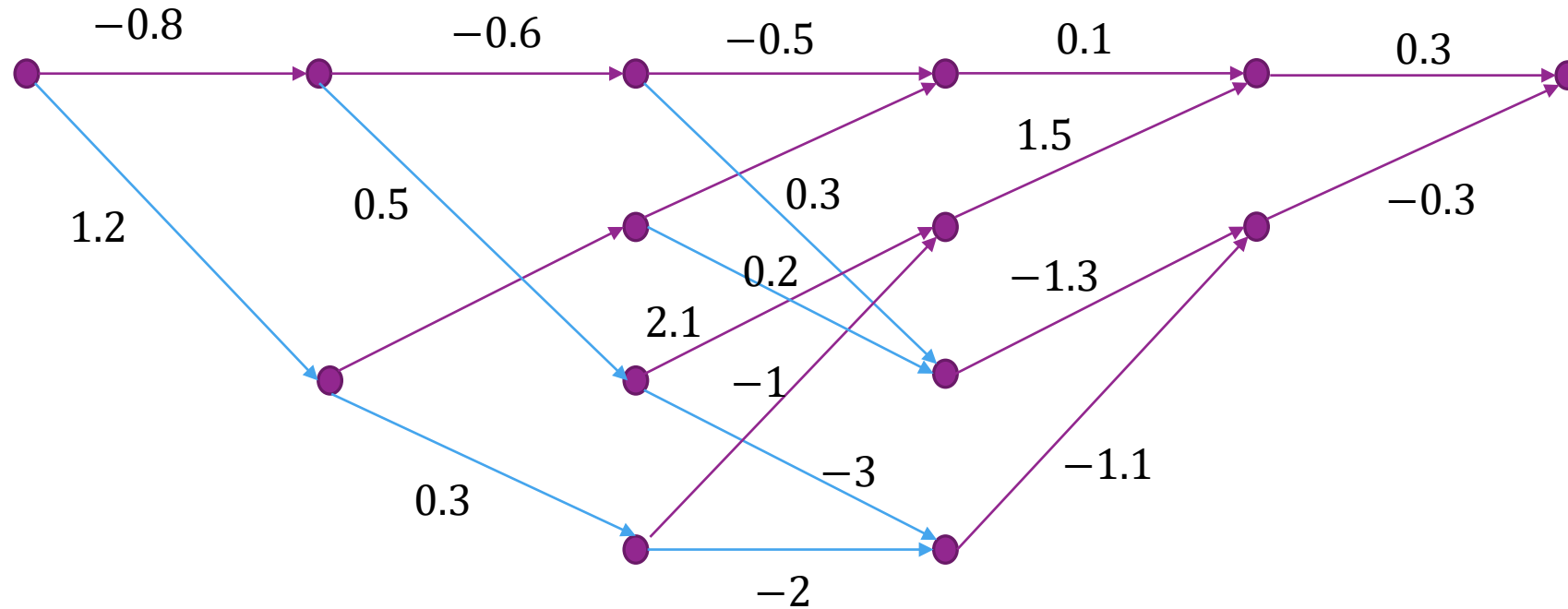
Ex: Time 1

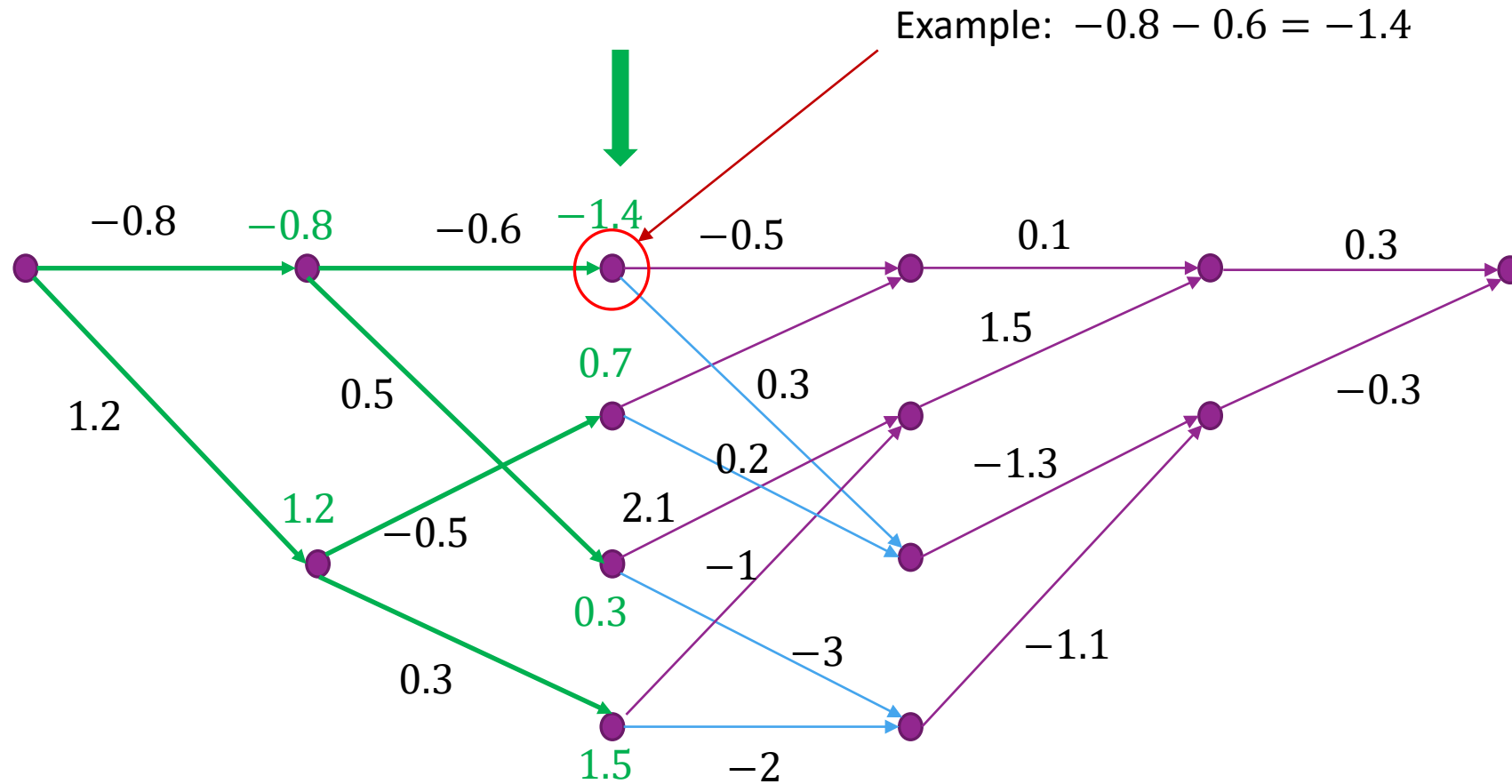☐ Find total value to get to each node at time 1
  ◦ We call these the partial value functions up to time 1

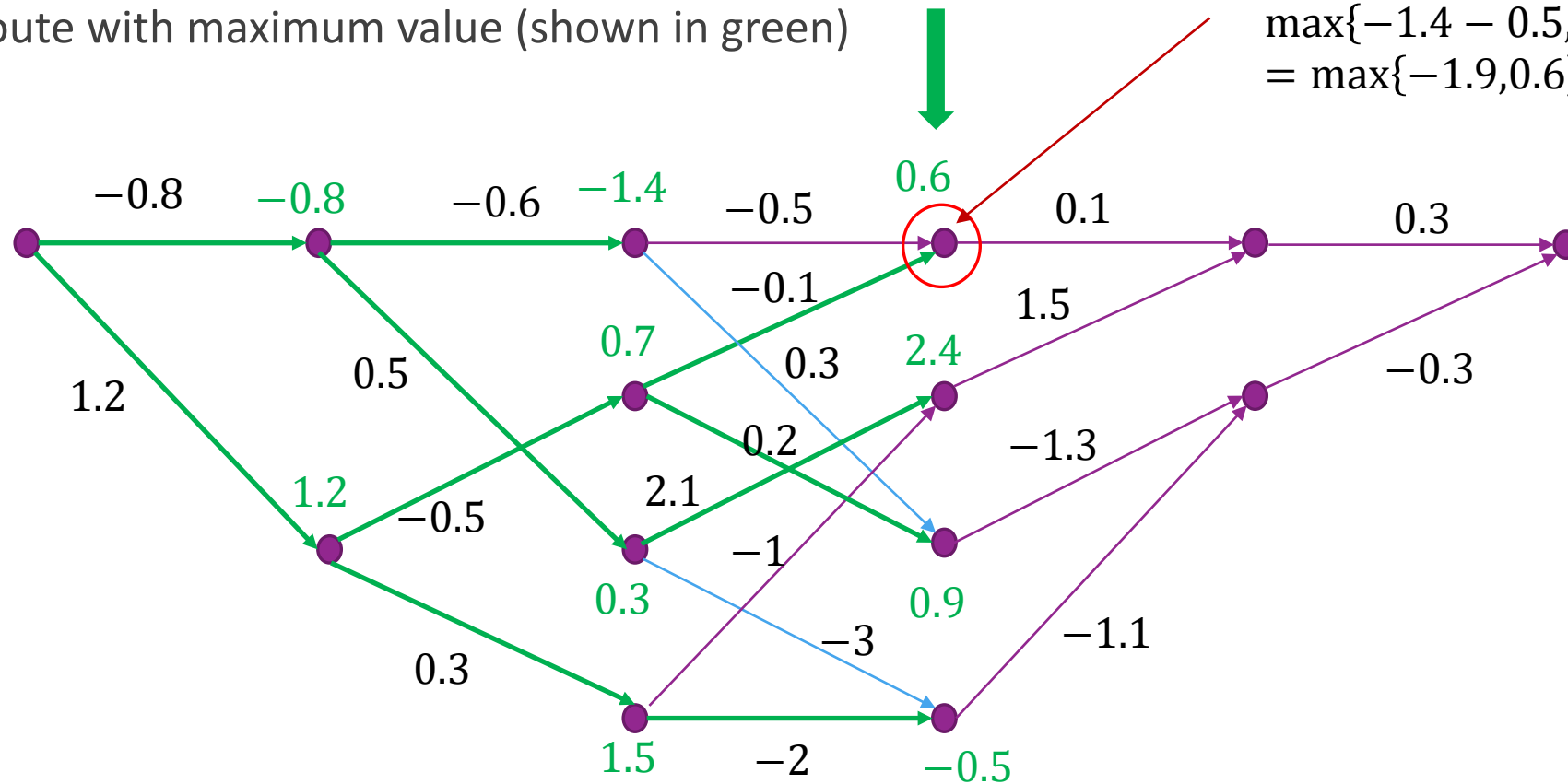☐ Remember best path to get to node (indicated with green line)

Ex: Time 2

☐Add value from Time 1 + branch
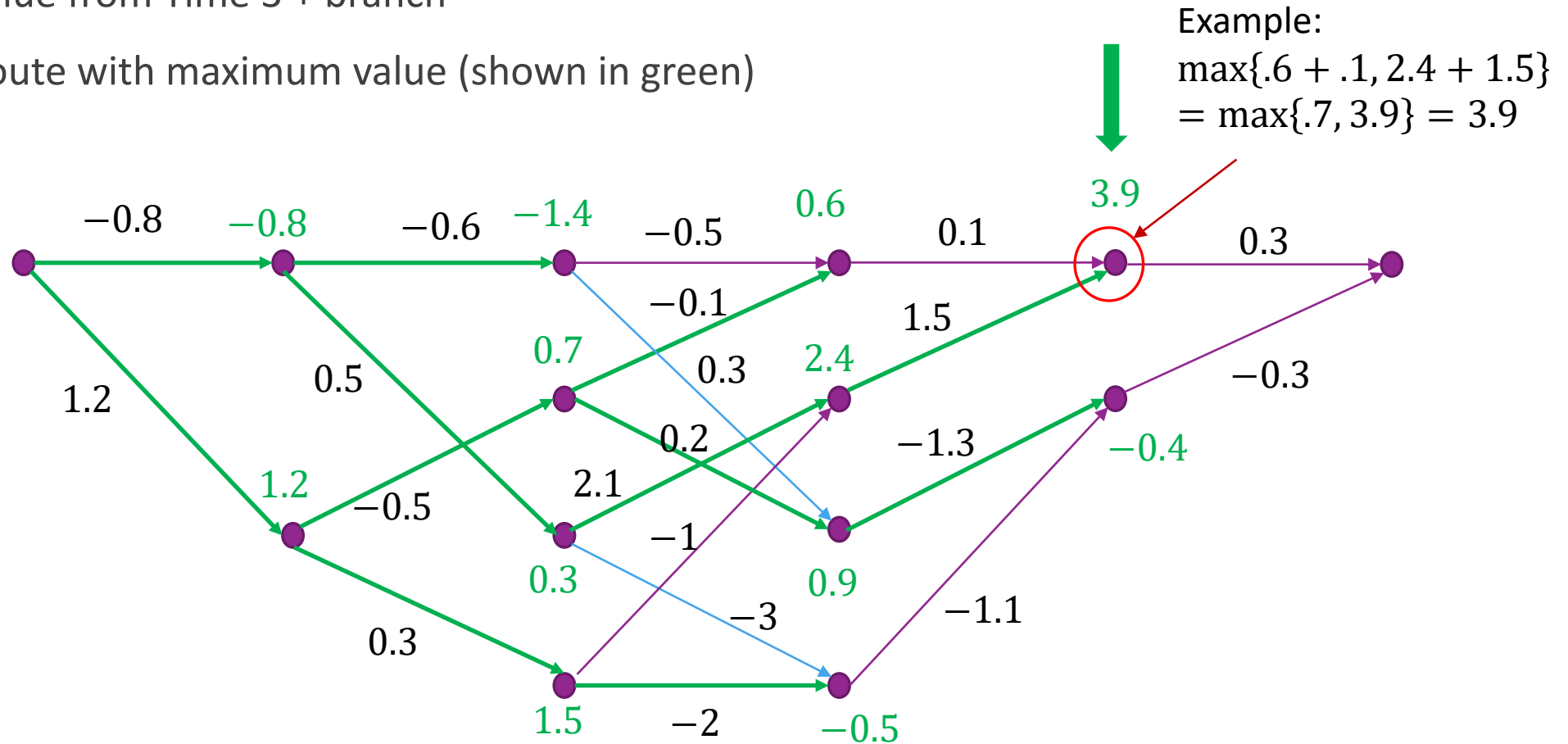
Example: $-0.8 - 0.6 = -1.4$

Ex: Time 3

☐ Add value from Time 2 + branch

☐ Take route with maximum value (shown in green)

Example:
$\max\{-1.4 - 0.5, 0.7 - 0.1\}$
$= \max\{-1.9, 0.6\} = 0.6$

Ex: Time 4

☐ Add value from Time 3 + branch

☐ Take route with maximum value (shown in green)

Example:
$\max\{.6 + .1, 2.4 + 1.5\}$
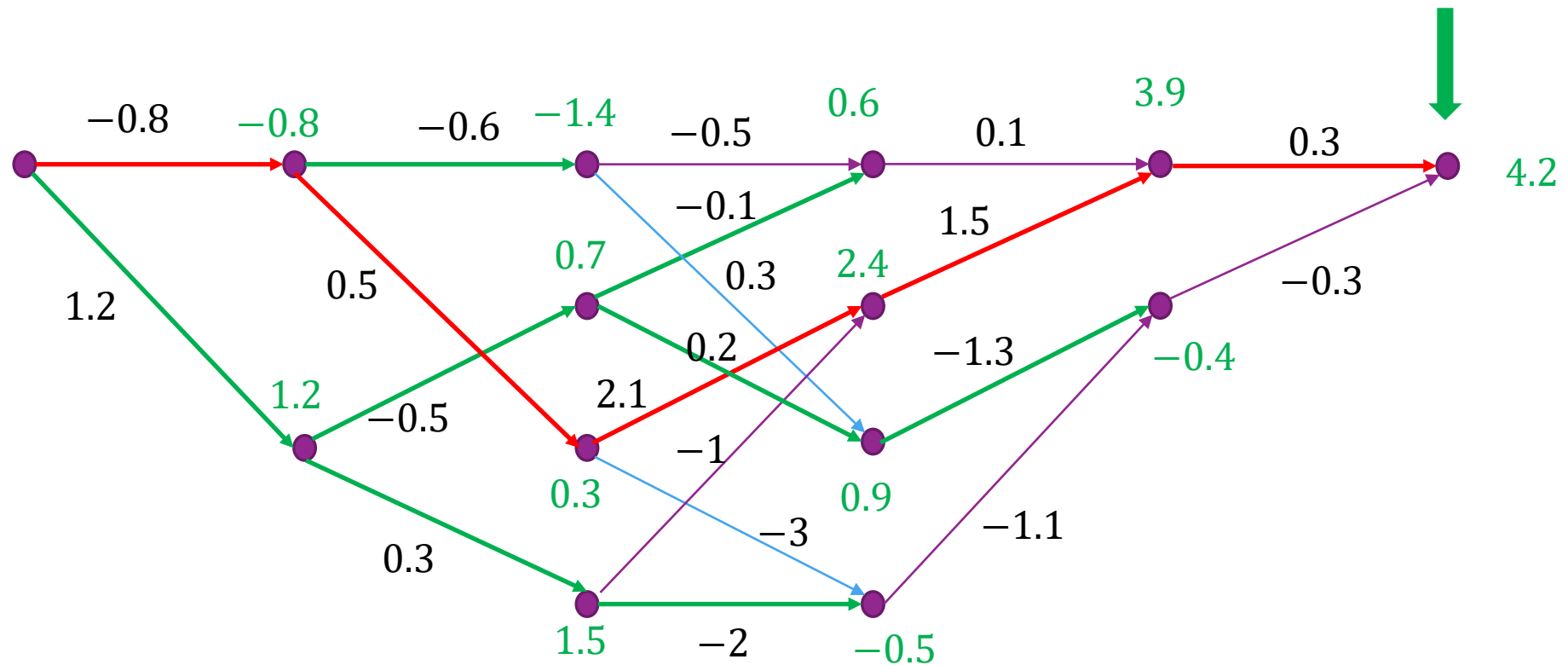$= \max\{.7, 3.9\} = 3.9$

Ex: Time 5

□ Add value from Time 4 + branch

□ Take route with maximum value (shown in green)
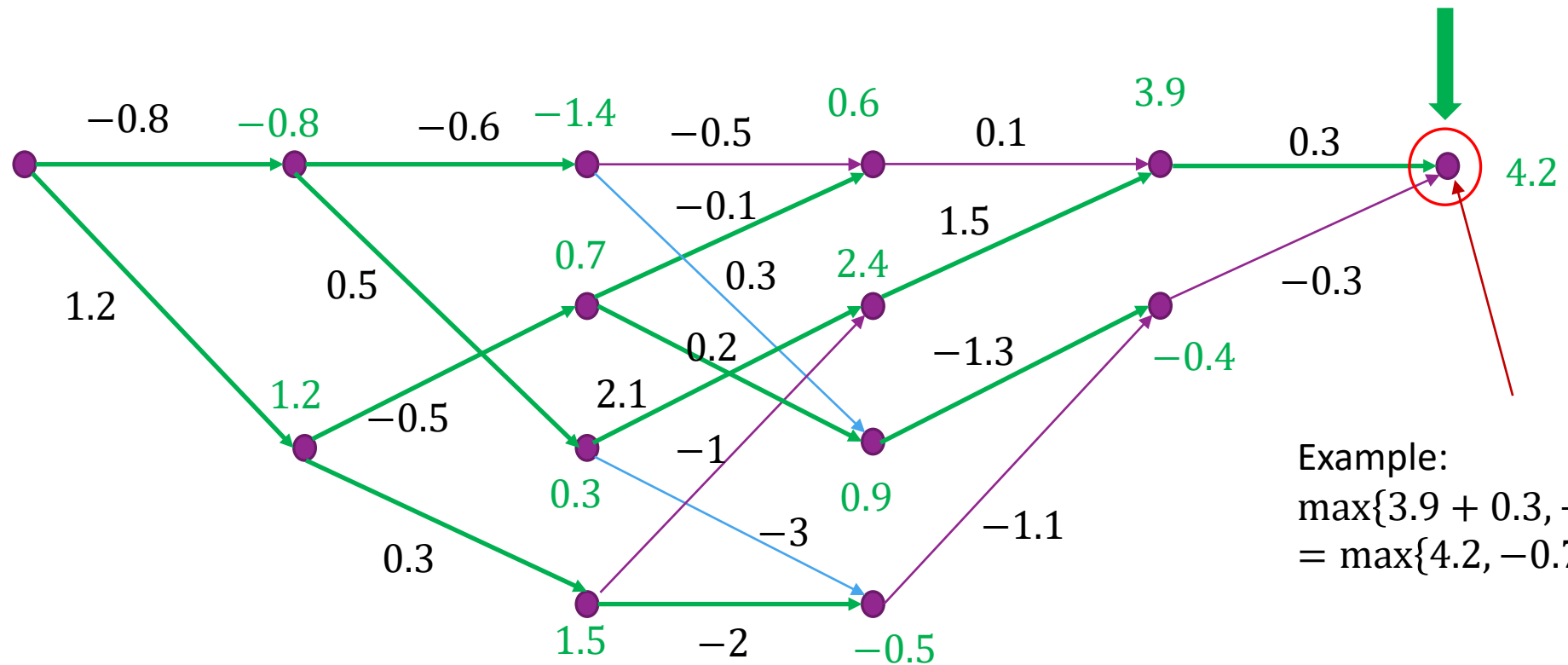
Example:
$$\max\{3.9 + 0.3, -0.4 - 0.3\}$$
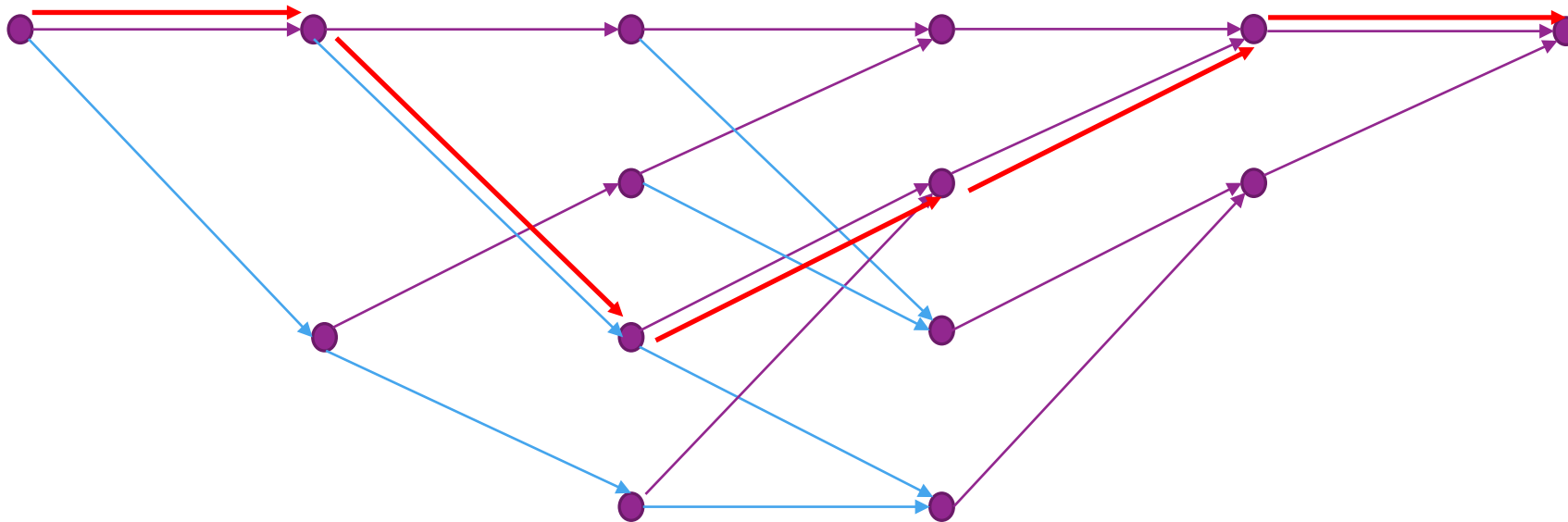$$= \max\{4.2, -0.7\} = 4.2$$

Ex: Trace back path

☐Trace back maximum path. Shown in red

Ex: Read off bits

❑ Read off the bits from the maximum path

❑ ML input bits = (0,1,0,0,0)

Complexity

❑ Update of each node requires maxima over 2^k branches

❑ There are $2^{k(K-1)}$ states, so complexity / time is $O(2^{kK})$

❑ Total complexity is $O(T2^{kK})$

❑ Storage is also $O(T2^{kK})$. (From the paths)

❑ Summary:
  ◦ Viterbi algorithm is linear in block length.
  ◦ Can have very long block lengths (often T in the 1000s)
  ◦ But, complexity is exponential in constraint length
  ◦ Practical decoders limited to $K = 7$ or $K = 9$.
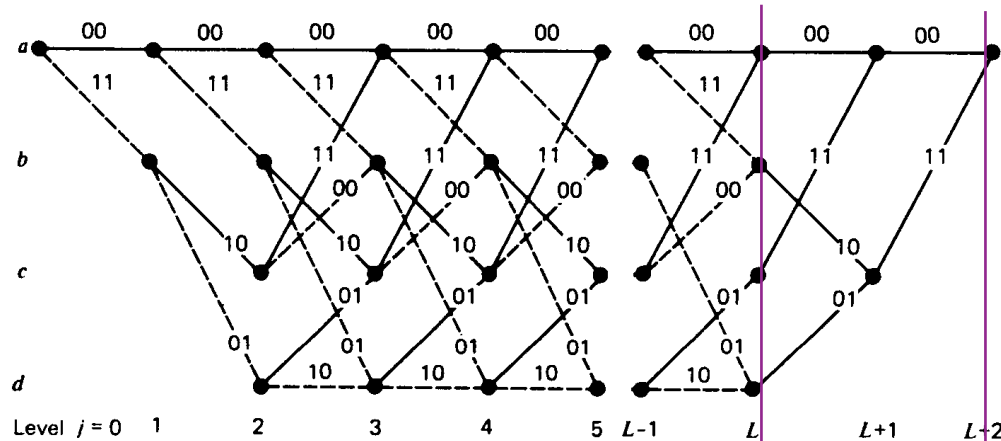
Terminating the Trellis

Figure 11.15 Trellis for the convolutional encoder of Fig. 11.13a.

During tail bits only paths with zero bit inputs.

- ❑ Recall tail bits are zero: $\boldsymbol{b}(L) = \cdots = \boldsymbol{b}(L + K - 2) = 0$.

- ❑ Limits the paths at the end of the trellis

- ❑ Viterbi algorithm should only be done on the zero path.

- ❑ Very important to not forget the bits at the end.

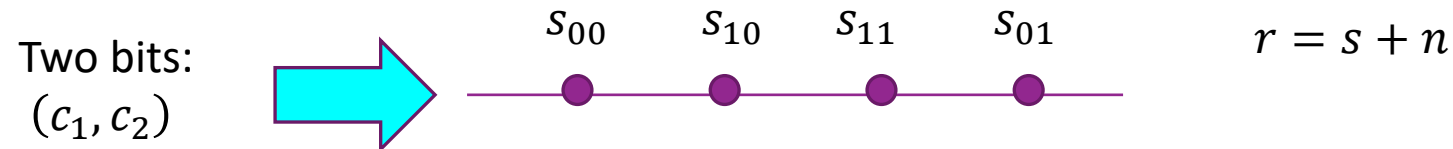- ❑ Otherwise, final bits are not protected.

Pruning the Path Memory

❑ In current algorithm, path $P_t(x_t)$ grows to full block length.

❑ Adds storage: Storage is O$\left(T2^{kK}\right)$. Linear in *T*

❑ Adds delay. No bits can be determined until code is fully decoded.

❑ Many practical implementations:
- ◦ Store some finite length δ of each surviving path.
- ◦ Can make decision on bit input $\boldsymbol{b}(t)$ after $\mathbf{r}(t+\delta)$.
- ◦ Rule of thumb: Good performance if $\delta \geq 5K$.
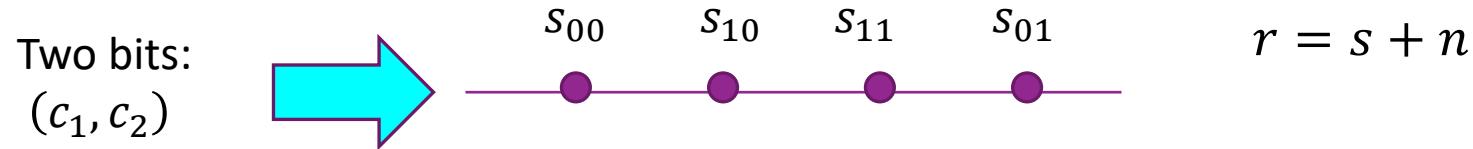
Rate Matching Convolutional Codes

❏ Convolutional codes have limited rates, usu. ½ or 1/3.

❏ Obtain other rates through
  ◦ Puncturing: Remove coded bits to increase rate
  ◦ Repetition: Repeat coded bits to decrease rate

❏ Puncture / repeat pattern is important (see Proakis)
  ◦ Try to spread out modified bits

❏ For punctured bits, set corresponding LLRs to zero
  ◦ Viterbi decoder just ignores those bits

❏ For repeated bits, add the corresponding LLRs
  ◦ Viterbi decoder will increase confidence on that branch

High Order Constellations

❑Higher order constellations (eg. 16- or 64-QAM)

❑Each constellation is a point is a function of multiple bits.

❑Likelihood does not factorize
 ◦ Each symbol r(t) depends on multiple bits

❑Example 4-PAM (or one dimension of 16-QAM):
 ◦ Each symbol likelihood depends on two bits: $p(r|c_1, c_2)$

Two bits:
(c_1, c_2)

s_{00} s_{10} s_{11} s_{01}

$r = s + n$

High Order Constellations

Two bits:
(c_1, c_2)
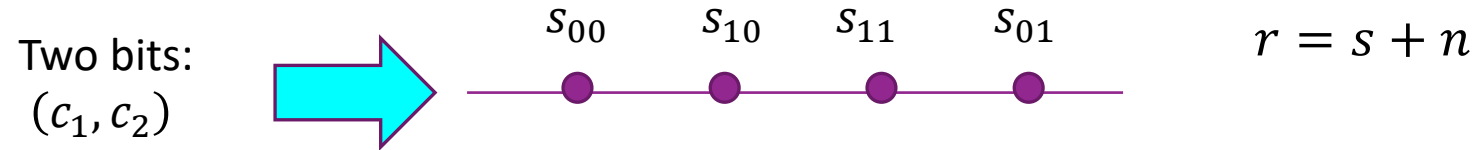
s_{00} s_{10} s_{11} s_{01}

$r = s + n$

❑To create LLRs for individual bits use total probability rule:

$$p(r|c_1) = \frac{1}{2}\big(p(r|c_1, c_2 = 0) + p(r|c_1, c_2 = 1)\big)$$
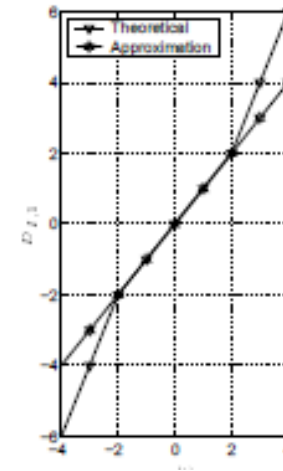
❑Resulting bitwise LLR:

$$LLR \text{ for } c_1 = \log\frac{p(r|c_1, c_2 = 1,0) + p(r|c_1, c_2 = 1,1)}{p(r|c_1, c_2 = 0,0) + p(r|c_1, c_2 = 0,1)}$$
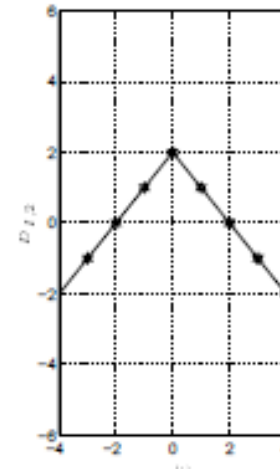
High Order Constellations

Two bits:
(c_1, c_2)

s_{00} s_{10} s_{11} s_{01}

$r = s + n$

LLR for c2   LLR for c1

❑ LLRs can have irregular shapes

❑ Not simple linear function as in BPSK / QPSK case

❑ Often use approximations

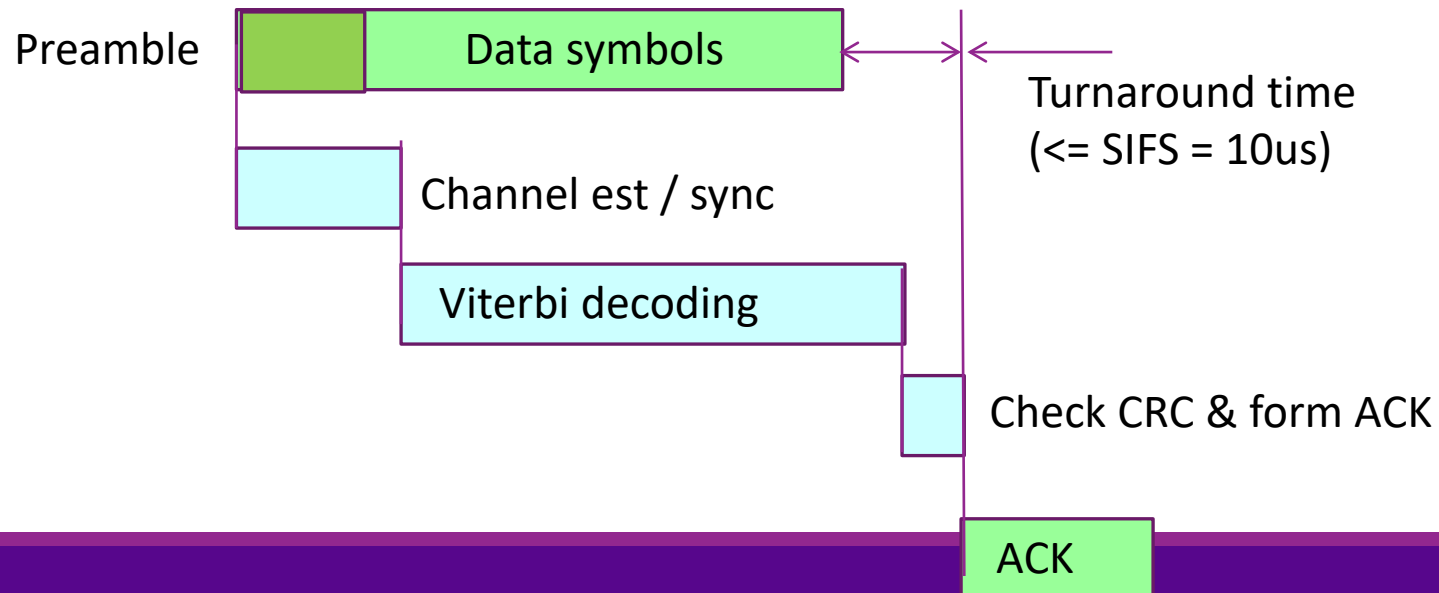❑ More info: Caire, Taricco and Biglieri, "Bit-Interleaved Coded Modulation," 1998.

Convolutional Codes in WiFi

❑ 802.11 uses R=1/2 K=7 code.

❑ Length adjusted to packet size

❑ Higher rates (R=2/3 and ¾) achieved through puncturing

❑ Enables decoding as data arrives for ACK fast turnaround

Preamble | Data symbols

Turnaround time
(<= SIFS = 10us)

Channel est / sync

Viterbi decoding

Check CRC & form ACK

ACK

Convolutional Codes in LTE

❑Convolutional codes in LTE used for:
  ◦ Control channels (payload typ 20-40 bits +CRC), and
  ◦ Short (< 128 bit) data frames

❑Larger payloads encoded with turbo codes (discussed later)

❑Uses rate=1/3 base convolutional code with K=7.

❑Higher rates achieved via puncturing

❑Control channels use a sophisticated technique called tail biting to reduce loss on the tail bits
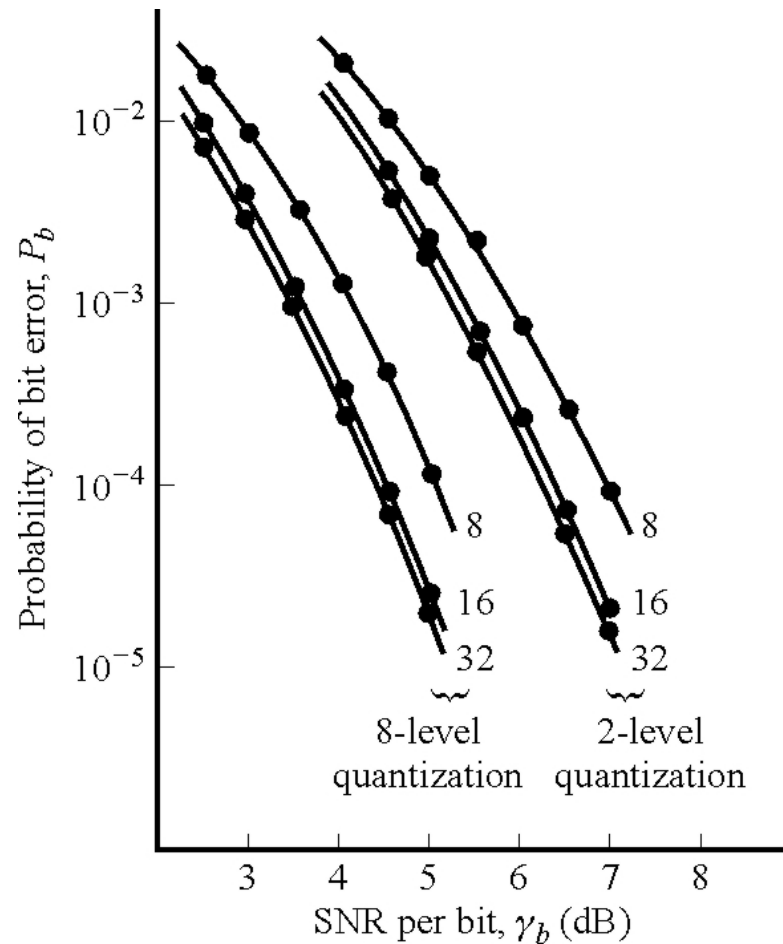
Decoding Performance
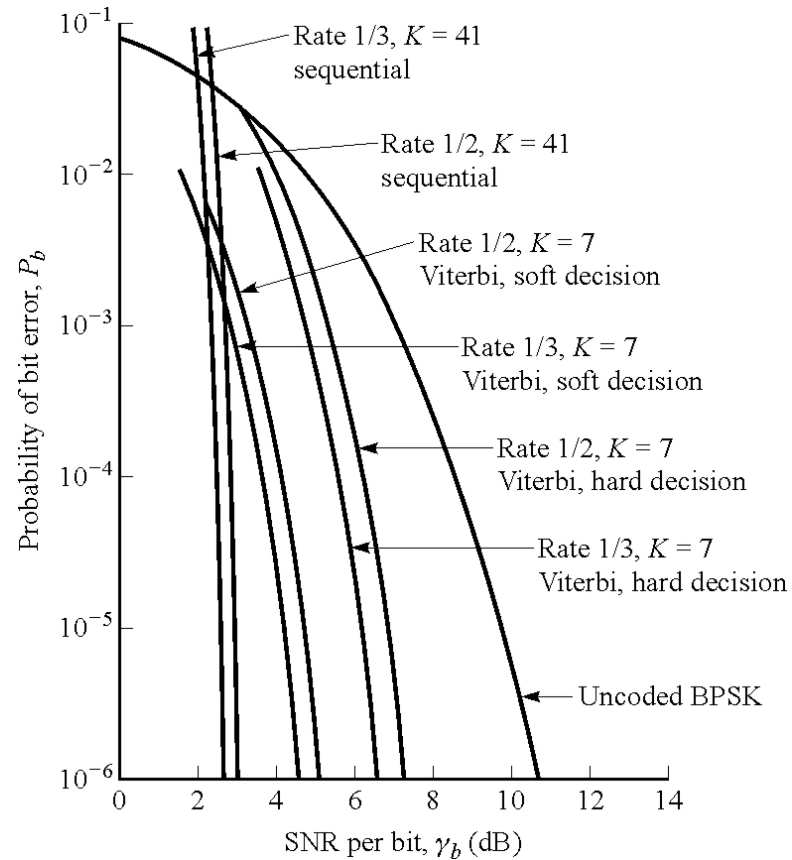
- Hard decision loses approximately 1.5 to 2 dB

- Constraint length K=7 is sufficient for very sharp error performance

Different Constraint Lengths
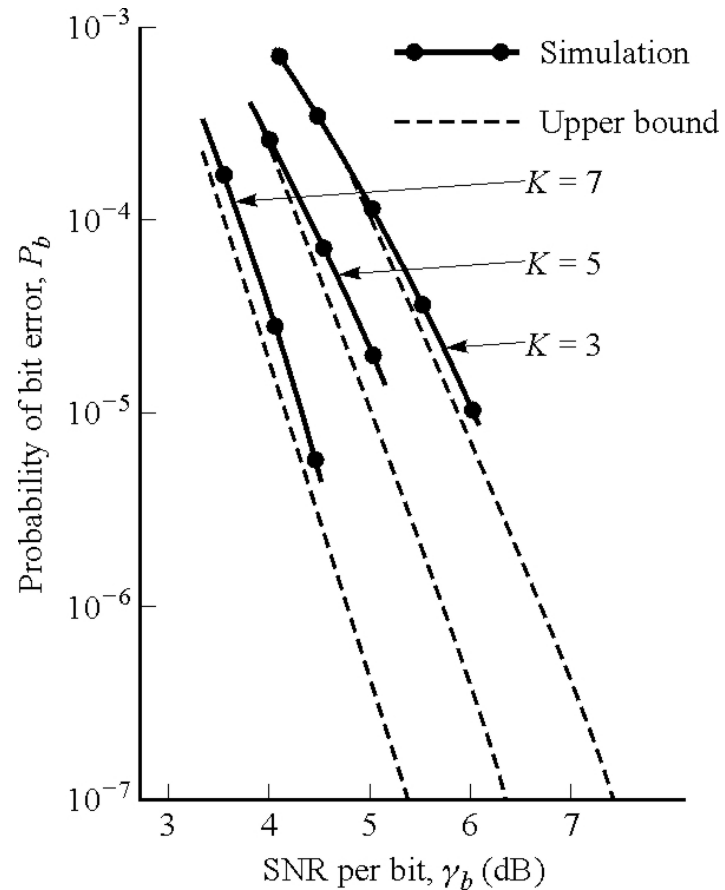
- ❏ Approximate 1 dB improvement between K=3, 5 and 7

- ❏ Higher constraint lengths become computationally intractable

- ❏ Recall decoding complexity is exponential in K

LLR Quantization

- Minimal gains after 16 bit quantization of branch metrics

- Recall HD is equivalent to 1 bit quantization

- Most commercial implementations use 6-bit LLRs

MATLAB Convolution Tools

Demo Convolutional Code

The Viterbi decoder built in the in-class exercise is very slow. Fortunately, MATLAB has excellent tools for encoding and decoding

- Perform convolutional encoding with the MATLAB comm toolbox commands
- Extract LLRs from the qamdemod command
- Perform convolution decoding from the soft LLRs
- Measure the block error rate as a funciton of the SNR
-

We illustrate with a simple convolutional coder over an AWGN channel. We use a standard constraint length K=7 code:

```
% Create the convolutional encoder and decoder with constraint length K=7
trellis = poly2trellis(7,[171 133]);
convEnc = comm.ConvolutionalEncoder('TrellisStructure', trellis, 'TerminationMethod','Terminated');
convDec = comm.ViterbiDecoder('TrellisStructure', trellis, 'TerminationMethod','Terminated');

% Modulation
bitsPerSym = 2;    % QPSK
M = 2^bitsPerSym;

% Number of information bits per block
nbits = 1000;
```

❑ MATLAB comm toolbox
  ◦ General convolution encoders & decoders
  ◦ Efficient implementation
  ◦ Excellent for testing

❑ See demo

In-Class Exercise

Building a Simple Viterbi Decoder

We conclude with building a simple decoder. This decoder is slow but illustrates the ideas.

To start, let's encode some random data.

```
% Number of bits
nbits = 100;

% TODO:  Generate random bits
%    b = ...


% TODO:  Create output bits.
%    c = ...
```