Problems: Synchronization and Detection

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1. *Hypothesis Testing:* Suppose that we wish to detect a binary variable $u = 0, 1$ from an observation y with conditional probabilities:

$$p(y|u=0) = \begin{cases} 1/2 & \text{if } |y| \leq 1, \\ 0 & \text{else} \end{cases}, \quad p(y|u=1) = \begin{cases} 3y^2/2 & \text{if } |y| \leq 1, \\ 0 & \text{else} \end{cases}.$$

   (a) Draw $p(y|u=0)$ and $p(y|u=1)$.

   (b) Find an expression for $\widehat{u}$, the ML detector for u from y.

   (c) Find the probability of false alarm $P_{\text{FA}} = \mathbb{P}(\widehat{u} = 1|u = 0)$.

   (d) Find the probability of missed detection $P_{\text{MD}} = \mathbb{P}(\widehat{u} = 0|u = 1)$.

2. *Simple Match Filter.* Suppose the target sequence is $\boldsymbol{x} = [1, 0, 1]^\intercal$, and we are given two RX vectors:

$$\boldsymbol{r}^{(1)} = [2, 0, 2]^\intercal, \quad \boldsymbol{r}^{(2)} = [3, -2, 3]^\intercal.$$

   (a) Find the un-normalized MF $z = \boldsymbol{x}^*\boldsymbol{r}$ for both RX vectors $\boldsymbol{r} = \boldsymbol{r}^{(i)}$. Which RX signal results in a higher $|z|$?

   (b) Find the squared correlation coefficient $\rho^2 = |\boldsymbol{x}^*\boldsymbol{r}|^2/\|\boldsymbol{x}\|^2\|\boldsymbol{r}\|^2$ for both RX vectors. Which RX signal results in a higher ρ^2?

3. *Matched Filter SNR.* A RX signal is received at $P_{\text{rx}} = -110\,\text{dBm}$ and the noise PSD (including noise figure) is $N_0 = -170\,\text{dBm/Hz}$.

   (a) What is the integration time needed to obtain an SNR of 20 dB?

   (b) If the sample rate is $f = 100\,\text{MHz}$, how many samples should the receiver use?

4. *Matched Filter with Convolution.* We are given an RX sequence $\boldsymbol{r}$ and target sequence $\boldsymbol{x}$:

$$\boldsymbol{r} = [r_0, \ldots, r_5] = [0, 2, -2, 3, 1, 1], \quad \boldsymbol{x} = [x_0, x_1] = [1, -1].$$

   You can compute the following with MATLAB:

   (a) Find unnormalized MF output

$$z_k = \sum_{n=0}^{N-1} x_n^* r_{n+k}, \quad k = 0, 1, \ldots, N - M - 1,$$

   where N and M are the lengths of $\boldsymbol{x}$ and $\boldsymbol{r}$, respectively.

(b) Find the squared correlation coefficient:

$$\rho_k^2 = \frac{|r_k|^2}{\|\boldsymbol{x}\|^2 E_k}, \quad E_k = \sum_{n=0}^{N-1} |r_{n+k}|^2.$$

(c) We believe that

$$r_n = h x_{n-k} + w_n,$$

for some noise w_n, channel gain h and delay k. Using the normalized MF output, what is the best estimate for the delay k?

5. *MF with a Rectangle.* Suppose that we have a target signal

$$x(t) = \mathrm{Rect}(t/a), \quad a = 2 \text{ ms},$$

in an RX signal $r(t)$. We compute the normalized MF:

$$z(t) = \frac{1}{\|x\|} \int x^*(s) r(t+s) ds. \tag{1}$$

(a) What are the limits of the integral in (1)?

(b) Suppose that $r(t)$ is

$$r(t) = c x(t - \tau),$$

where $c = 2\sqrt{\mathrm{mW}}$ and $\tau = 3\,\mathrm{ms}$. Draw $z(t)$. Label the axes. What are the units of $z(t)$?

6. *MF with an Interfering Signal.* As in the previous problem, suppose the target signal is

$$x(t) = \mathrm{Rect}(t/a), \quad a = 1 \text{ ms}.$$

The received signal is

$$r(t) = c_1 x(t - \tau_1) + c_2 w(t - \tau_2),$$

where $w(t)$ is an interfering signal in a triangular shape:

$$w(t) = \max\{1 - |t|/a, 0\}.$$

Assume $c_1 = 1\sqrt{\mathrm{mW}}$, $c_2 = 3\sqrt{\mathrm{mW}}$, $a = 1\,\mathrm{ms}$, $\tau_1 = 1\,\mathrm{ms}$ and $\tau_2 = 3\,\mathrm{ms}$.

(a) Draw $r(t)$. Label the axes.

(b) Let $z(t)$ be the normalized MF: $z(t) = \frac{1}{\|x\|} \int x^*(s) r(t+s)\, ds$. Compute $z(t)$ for $t = 1\,\mathrm{ms}$ and $t = 3\,\mathrm{ms}$.

(c) Let $\rho^2(t)$ be the squared correlation,

$$\rho^2(t) = \frac{|z(t)|^2}{E_r(t)}, \quad E_r(t) = \int_{-a/2}^{a/2} |r(t+s)|^2\, ds.$$

Find $\rho^2(t)$ for $t = 1\,\mathrm{ms}$ and $t = 3\,\mathrm{ms}$.

7. *Matched Filter Distribution*: Suppose we wish to detect a synchronization signal x_n, $n = 0, 1, \ldots, N-1$ from received complex baseband samples of the form,

$$r_n = hx_n + w_n, \quad w_n \sim \mathcal{C}N(0, N_0), \quad n = 0, 1, \ldots, N-1,$$

where h is the complex channel gain and w_n is AWGN noise. We use a matched filter (without normalization):

$$z = \boldsymbol{x}^* \boldsymbol{r} = \sum_{n=1}^{N} x_n^* r_n.$$

For a given h and $\boldsymbol{x}$:

(a) Show z is a complex Gaussian.

(b) Find the mean and variance of z.

(c) Find the SNR in z as a function of h, x_n and N_0?

8. *Matched Filter Probability of Missed Detection*: In this problem, we will show how to write an expression for the misdetection probability for a MF. If you did the calculations correctly in the previous problem correctly, you should have found that the MF output is:

$$z = h\|\boldsymbol{x}\|^2 + v, \quad v = \mathcal{CN}(0, \sigma^2), \quad \sigma^2 = \|\boldsymbol{x}\|^2 N_0. \tag{2}$$

Suppose we detect a signal when

$$|z| > t$$

for some threshold t. Then, the probability of misdetection is

$$P_{\text{MD}} = \mathbb{P}(|z| \leq t).$$

We will compute this probability for a given h and $\boldsymbol{x}$.

(a) Let $h = |h|e^{i\theta}$ where θ is the angle of h. Write

$$w = \frac{e^{-i\theta}}{\sigma} v. \tag{3}$$

Show $w \sim \mathcal{CN}(0, 1)$.

(b) Show that the missed detection event $|z| \leq t$ occurs if and only if:

$$|z| \leq t \Longleftrightarrow |\sqrt{\gamma} + w| < \alpha, \tag{4}$$

where γ is the SNR and α is the normalized threshold:

$$\gamma = \frac{|h|^2 \|\boldsymbol{x}\|^2}{N_0}, \quad \alpha = \frac{t}{\sigma}. \tag{5}$$

(c) Write $w = a + ib$ for real a and b. Write the event (4) as $(a, b) \in M$ for some region M. You should get that region M is a circle whose center and radius can be written in terms of γ and α.

(d) We can write the

$$P_{\text{MD}} = \mathbb{P}((a,b) \in M).$$

Using the fact that a, b are independent Gaussians, find a function $\phi(a)$ and limits $a_{\min}$, $a_{\max}$ such that

$$\mathbb{P}((a,b) \in M|a) = \begin{cases} 1 - 2Q(\phi(a)) & \text{if } a \in [a_{\min}, a_{\max}], \\ 0 & \text{else} \end{cases} \tag{6}$$

The function $\phi(a)$ and limits $a_{\min}$, $a_{\max}$ will depend on a, γ and α.

(e) Find an expression for P_{MD} by integrating the $\mathbb{P}((a,b) \in M|a)$ over a. You do not need to solve this integral. But, you should state the limits correctly.

9. *Matched Filter with Carrier Offset.* Suppose x_n is a target signal and r_n is the received signal

$$r_n = he^{in\theta}x_n + w_n, \quad w_n \sim \mathcal{C}N(0, N_0), \quad n = 0, 1, \ldots, N-1,$$

where h is the complex channel gain, w_n is AWGN noise and θ is unknown phase rotation per sample. The phase rotation is due to carrier offset. Assume $|x_n| = E_x$ for all n and some E_x. We run the un-normalized MF:

$$z = \sum_{n=0}^{N-1} x_n^* r_n.$$

(a) What is the SNR, $\gamma(\theta)$, as a function of the carrier offset θ?

(b) What is the degradation in SNR, meaning the ratio, $\frac{\gamma(\theta)}{\gamma(0)}$, as a function of θ and N?

(c) Suppose the carrier frequency is 2.5 GHz, the oscillator error is 10 ppm, the sample rate is $f_s = 20$ MHz, and we sample for $T = 10\ \mu s$. What is the degradation in SNR in dB?

(d) If the carrier offset were known, how would you modify the MF detector?

10. *FA for the Correlation Detector.* Consider the correlation squared:

$$\rho^2 := \frac{|\boldsymbol{x}^*\boldsymbol{r}|^2}{\|\boldsymbol{x}\|^2\|\boldsymbol{r}\|^2} \in [0, 1],$$

which is the fraction of energy in the target direction $\boldsymbol{x}$ of a RX signal $\boldsymbol{r}$. Assume the signals are length N. In this problem, we will show how to compute the FA probability when $\boldsymbol{r}$ is just noise meaning $\boldsymbol{r} \sim \mathcal{CN}(0, N_0\boldsymbol{I})$ for some variance N_0.

(a) Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ be any orthonormal basis of $\mathbb{C}^N$ where the first basis vector is $\boldsymbol{u}_1 = \boldsymbol{x}/\|\boldsymbol{x}\|$ and the other vectors are orthogonal to $\boldsymbol{u}_1$. Write $\boldsymbol{r}$ in the basis:

$$\boldsymbol{r} = \sum_{n=1}^{N} v_n \boldsymbol{u}_n, \quad v_n = \boldsymbol{u}_n^* \boldsymbol{r}.$$

Show that

$$\rho^2 = \frac{|v_1|^2}{\sum_n |v_n|^2}.$$

(b) Show that if $\boldsymbol{r} \sim \mathcal{CN}(0, N_0\boldsymbol{I})$ then $\boldsymbol{v} \sim \mathcal{CN}(0, N_0\boldsymbol{I})$, meaning that the coefficients v_n are i.i.d. $\mathcal{CN}(0, N_0)$.

(c) Read about the chi-squared distribution. Let A and B be

$$A = \frac{2}{N_0}|v_1|^2, \quad B = \frac{2}{N_0}\sum_{n=2}^{N}|v_n|^2.$$

Use any results from the Internet to show that $A \sim \chi(2)$ and $B \sim \chi(2(N-1))$, where $\chi(m)$ is the Chi-squared distribution with m degrees of freedom.

(d) Read about the Beta distribution. Use any facts you can find on the Internet or elsewhere to show that

$$\rho^2 \sim \text{Beta}(1, N-1).$$

(e) Show that when $\boldsymbol{r}$ is AWGN noise (i.e. there is no signal), ρ^2 is distributed as,

$$\rho^2 = 1 - U^{1/(N-1)}, \quad U \sim \text{Unif}(0,1).$$

11. *FA Simulation for the Correlation Detector.* Consider the correlation squared:

$$\rho^2 := \frac{|\boldsymbol{x}^*\boldsymbol{r}|^2}{\|\boldsymbol{x}\|^2\|\boldsymbol{r}\|^2} \in [0,1],$$

which is the fraction of energy in the target direction $\boldsymbol{x}$ of a RX signal $\boldsymbol{r}$. We declare a signal as detected when:

$$\rho \geq t$$

for some threshold t. Assume the signals are length N.

(a) The previous problem shows that when $\boldsymbol{r}$ is AWGN noise, ρ^2 is distributed as

$$\rho^2 = 1 - U^{1/(N-1)}, \quad U \sim \text{Unif}(0,1).$$

What is false alarm probability in terms of N and t.

(b) Suppose $N = 128$ and we have a FA target of $P_{\text{FA}} = 10^{-6}$, what is the threshold t.

(c) Write MATLAB code to run a short simulation to plot the missed detection probability as a function of the SNR for $N = 128$ and a false alarm target of 10^{-6}. Use at least $(10)^5$ trials per SNR and test the SNR in the range 5 dB to 20dB in 1 dB steps. Use any assumptions as necessary.

12. *Multiple delay hypotheses.* In the 5G NR standard, each base station cell periodically transmits a primary synchronization signal (PSS) that the mobile (UE) can detect. For 5G mmWave systems, the PSS bandwidth is typically 15.24 MHz (127 subcarriers at 120 kHz subcarrier spacing). The UE runs a matched filter (MF) to detect the PSS.

(a) If the MF oversampled by a factor of two, how many delay hypotheses are there per second?

(b) Suppose that to overcome carrier offset, the PSS MF detector tries several frequency hypotheses at each delay hypotheses. Suppose the carrier frequency is 37 GHz, the oscillator error is up to 10 ppm, and the maximum tolerable frequency error for the detector is 200 kHz. How many frequency hypotheses are needed in each delay hypothesis.

(c) If the average number of false alarms should be less than one false alarm per second, what should the FA target be? Make any reasonable assumptions.