Problem: Information Theory and Capacity

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1. *Entropy of an exponential.* Find the relative entropy of an exponential distributed X with $\mathbb{E}(X) = 1/\lambda$.

2. *Mutual information on a discrete set.* Suppose that X is discrete uniform on $\{0, 1, \ldots, N-1\}$ for some $N > 0$. Let $Y = X + W$ where

$$P(W = 1) = 1 - P(W = 0) = p$$

for some $p > 0$.

   (a) Given $Y = y$ for $y > 0$, we know $X = y$ or $y - 1$. Find $P(X = y|Y = y)$ and $P(X = y - 1|Y = y)$.

   (b) Find the conditional entropy $H(X|Y = y)$ for $y > 0$.

   (c) Find the conditional entropy $H(X|Y = y)$ for $y = 0$.

   (d) Find the conditional entropy $H(X)$.

   (e) Find the mutual information $I(X; Y)$.

3. *AWGN Capacity.* Suppose that a signal is transmitted on a bandwidth $B = 100\,\mathrm{MHz}$, transmit power $P_t = 30\,\mathrm{dBm}$, path loss $L = 103\,\mathrm{dB}$, and noise PSD (including noise figure) of $N_0 = -170\,\mathrm{dBm/Hz}$.

   (a) What is the SNR per Hz, γ_s?

   (b) What is the Shannon capacity C?

   (c) Suppose that the system achieves a rate $R = 0.5C$. What is the E_b/N_0 in dB.

4. *Mutual information with a binary modulated exponential.* Suppose that $X \in \{0, 1\}$ is an equiprobable bit and we observe Y that has a conditional exponential distribution

$$p(y|X = i) = \lambda_i \exp(-\lambda_i y), \quad y \geq 0,$$

for values λ_0 and λ_1 with $\lambda_0 > \lambda_1$. We wish to compute the mutual information $I(X; Y)$.

   (a) Find the conditional entropy $h(Y|X)$. You can the results from Problem 1.

   (b) Find the PDF of Y, $p(y)$.

   (c) Find an expression for the relative entropy $h(Y)$ and the mutual information $I(Y; X)$. This expression will have an integral. You do not need to evaluate it.

Bits (c_1, c_2)	TX symbol s
00	$s_1 = -B$
01	$s_2 = -A$
11	$s_3 = A$
10	$s_4 = B$

Table 1: Problem : Bit to symbol mapping.

(d) Use MATLAB to compute and plot $I(X;Y)$ for $\lambda_0 = 1$ and $\lambda_1 = \lambda_0/\gamma$ where γ is in the range $\gamma \in [1, 50]$. You can interpret γ as a SNR since it is the ratio of the two exponential levels. To perform the numerical integration, you can use the MATLAB function `integral`. Although the integral is over $y \in [0, \infty)$, you may need to run it over a finite range to obtain good results.

5. *Numerically computing mutual information for a discrete channel.* In this problem, we show how to compute the mutual information numerically. As a completely toy example, suppose that $X \in \{0, 1, \ldots, N_x - 1\}$ is uniform and $Y \in \{0, 1, \ldots, N_y - 1\}$ with conditional PMF

$$P(y|x) = \frac{1}{Z(x)} exp(-\lambda|y - x|)$$

for some λ. The constant $Z(x)$ is for normalization. Complete the following MATLAB code to numerically compute and plot $H(Y)$, $H(Y|X)$ and $I(X;Y)$ for $N_x = 32$, $N_y = 128$, and $\lambda \in [0.5, 4]$.

```
% Parameters
nx = 32;
ny = 128;
lamTest = linspace(0.5,4,10);
nlam = length(lamTest);


for i = 1:nlam
    lam = lamTest(i);
    % TODO:
    %   Hyx = ...
    %   Hy = ...
    %   mi(i) = ...
end
```

6. *Bitwise LLR.* Suppose two bits (c_1, c_2) are mapped to one of four real symbol $s \in \{s_1, \ldots, s_4\}$ as shown in Table 1 for some $B > A > 0$. Assume the bits are equiprobable. The symbol s is transmitted through a real AWGN channel $r = s + w$ where $w \sim \mathcal{N}(0, \sigma^2)$.

(a) What is the posterior probability of $P(s = s_i|r)$ for any of the symbols $s = s_i$? Leave your answer as an expression in terms of the r, σ^2 and the values s_j.

(b) What are the bit-wise LLRs for c_1 and c_2:

$$L_1(r) = \log \frac{p(r|c_1 = 1)}{p(r|c_1 = 0)}, \quad L_2(r) = \log \frac{p(r|c_2 = 1)}{p(r|c_2 = 0)}.$$

(c) Use MATLAB to plot $L_1(r)$ and $L_2(r)$ vs. r for $r \in (-6, 6)$ with $A = 1$, $B = 4$ and $\sigma^2 = 4$.