Unit 7: Synchronization and Matched Filtering

EL-GY 6013: DIGITAL COMMUNICATIONS

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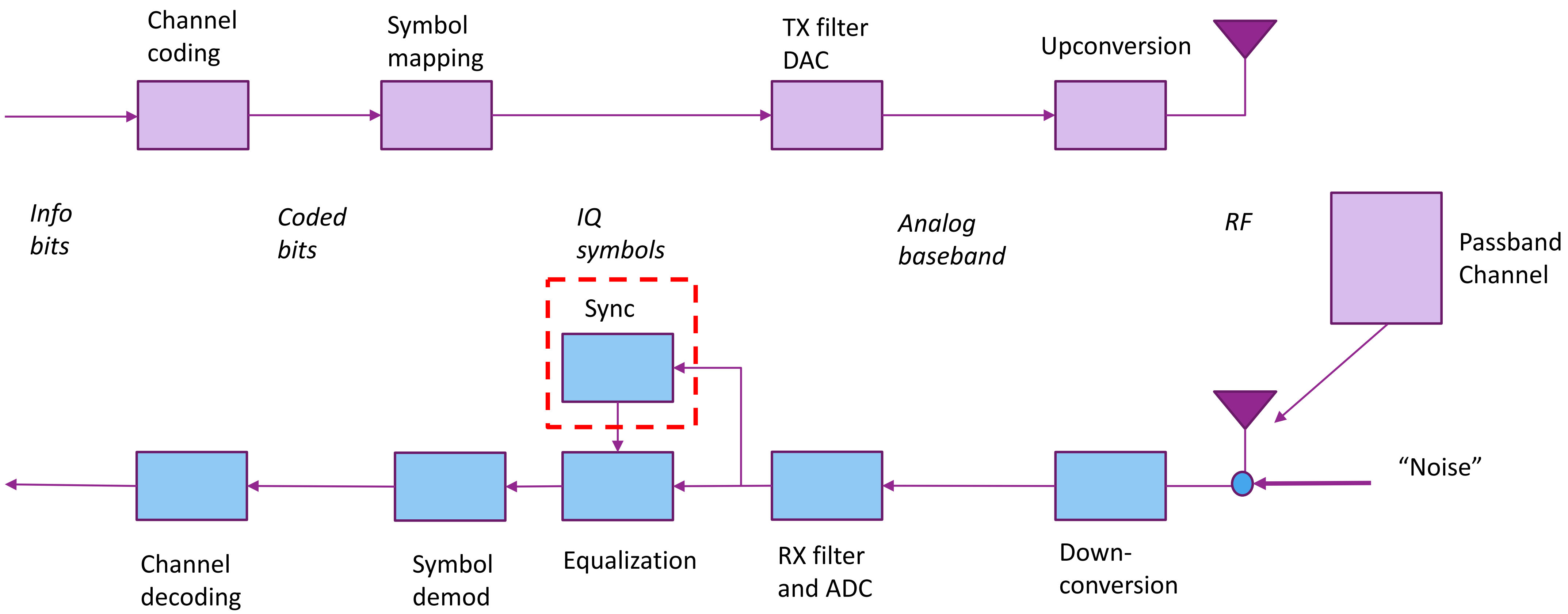
Learning Objectives

- ❑ Describe the synchronization mechanisms in common commercial standards
- ❑ Formulate binary decision tasks as hypothesis testing problems
- ❑ Compute the LRT detector for a hypothesis testing problem
- ❑ Compute error probabilities and optimize the threshold
- ❑ Formulate signal detection as a hypothesis test
- ❑ Describe and analyze the matched filter detector
- ❑ Analyze various non-idealities including clock offset, auto-correlation and multi-path
- ❑ Simulate the MF detector for real systems

This Unit

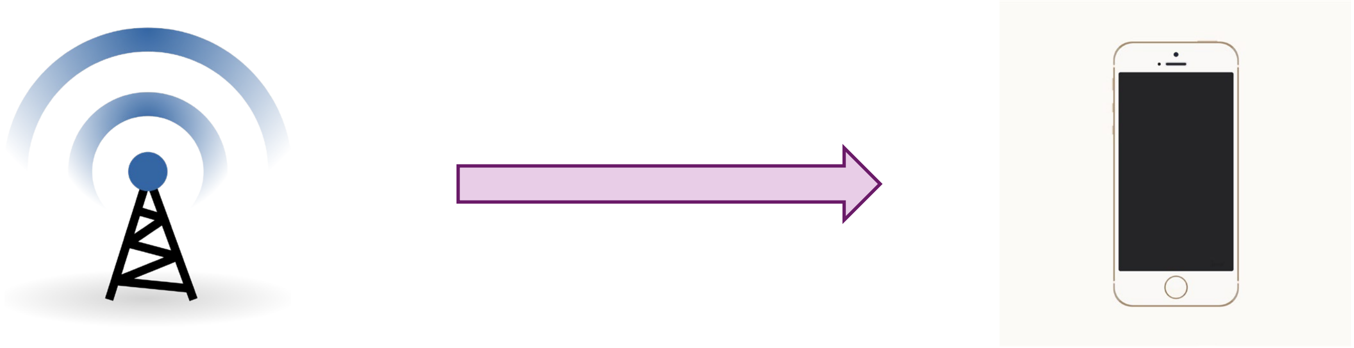


Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
"Noise"
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

Outline

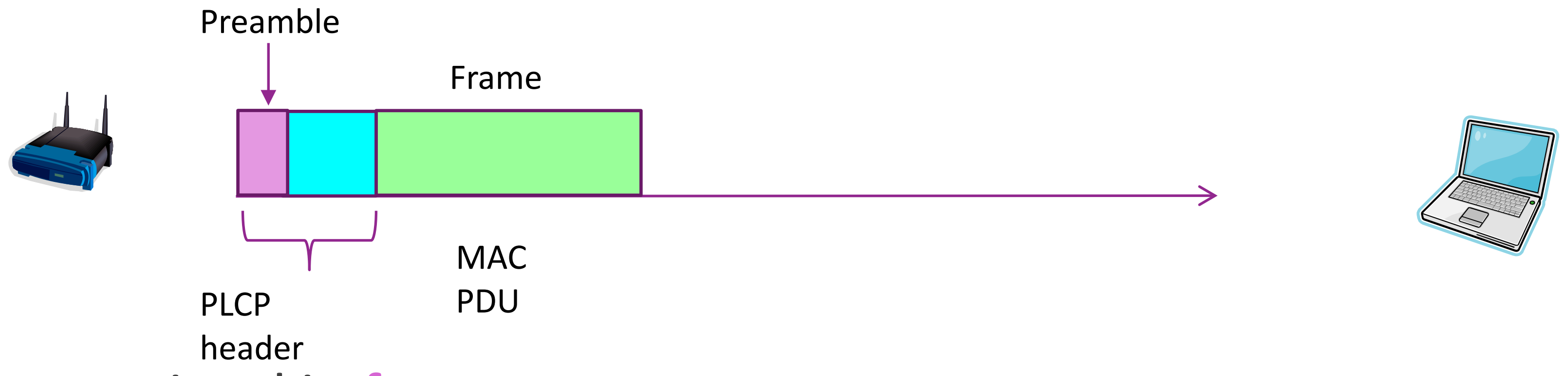
- Detection and Synchronization Problem
- Hypothesis Testing
- Match Filtering for Detection
- Match Filtering Convolution
- Matched Filtering as a Likelihood Ratio Test

Synchronization and Detection Problem



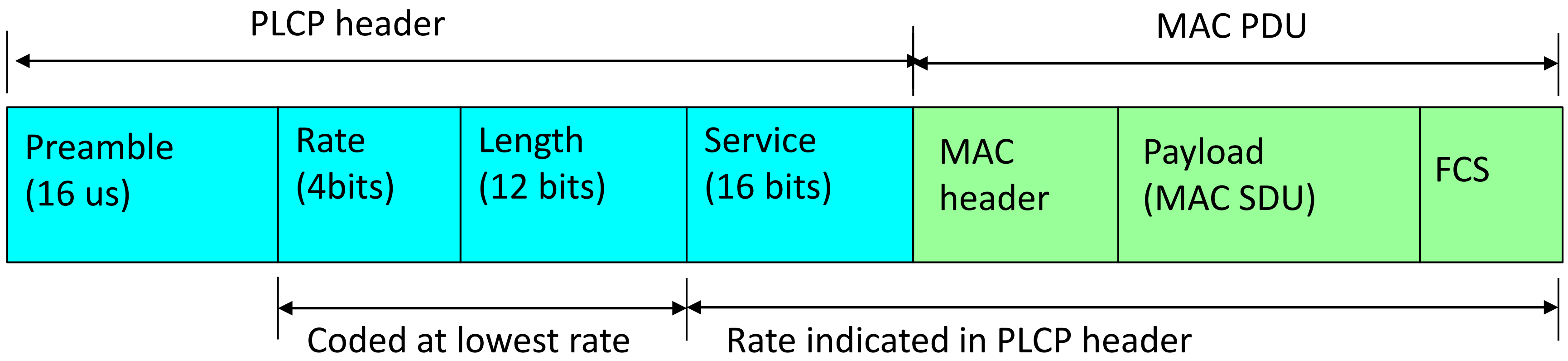
- Two key problems in most communication receivers:
  - Detect if a transmitter is present
  - Synchronize to the transmitter
- Basic first step in any communication process
- Assumes the transmitter broadcasts a signal
- Receiver must detect and synchronize to it

Ex 1: 802.11g Transmission



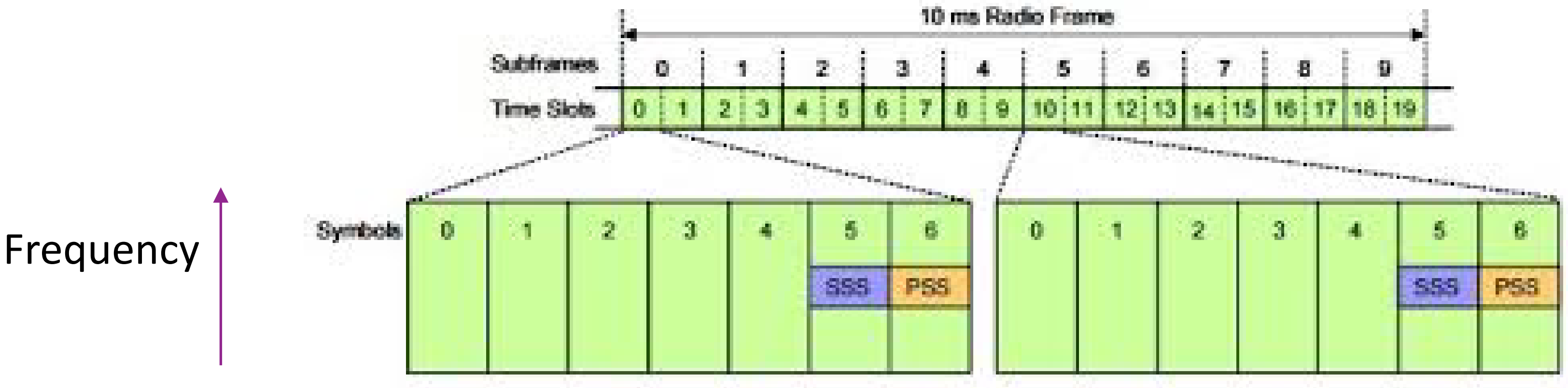
- ❑ All data is transmitted in frames
- ❑ Frames may arrive at any time
- ❑ Each frame begins with known preamble
  - Common to all frames
- ❑ RX station listens for preamble to detect:
  - Presence of frame.
  - If frame is present, determines timing delay of the remaining frame

802.11g PLCP Header Details



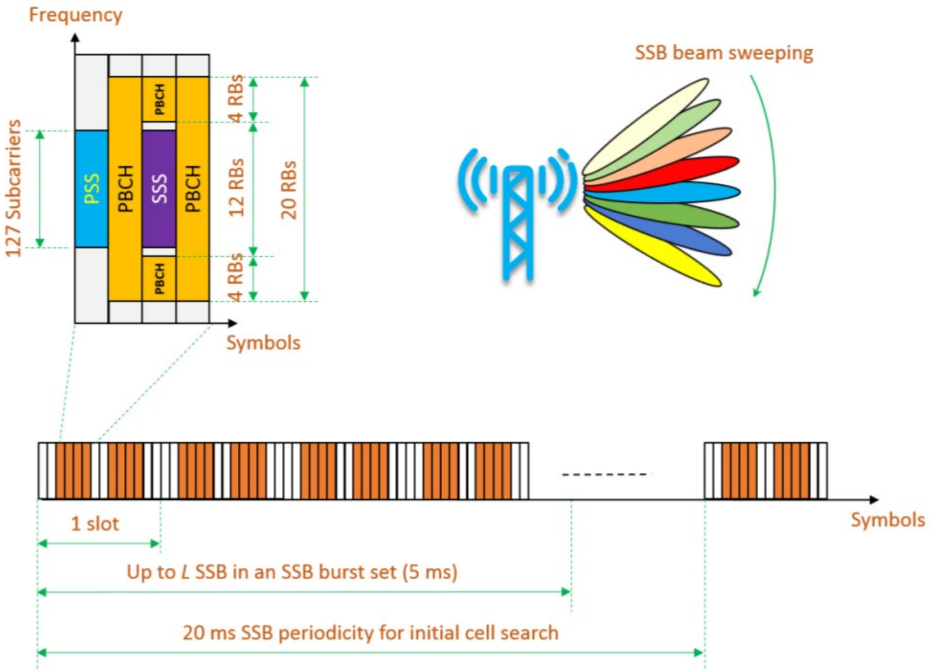
- PLCP header details:
  - Preamble: Used for initial detection, synchronization, channel estimation
  - Rate: Signals MCS for service bits & MAC PDU
  - Length: Number of OFDM symbols in frame
  - Service: Scrambler sync
- MAC header: Contains MAC layer control info
  - Segmentation, MAC addresses, …
- MAC FCS: frame check sum (used to detect errors)

Ex 2: LTE Downlink Primary Sync Signal (PSS)



- ❑Each cell transmits periodically PSS
  - Narrowband, short (71.4 us x 1.08 MHz)
  - One of 3 PSS signals
- ❑Once PSS is detected, mobile (UE) knows frame timing
  - Decodes subsequent signals SSS, broadcast, …

Ex. 3. 5G New Radio Beam Sweeping



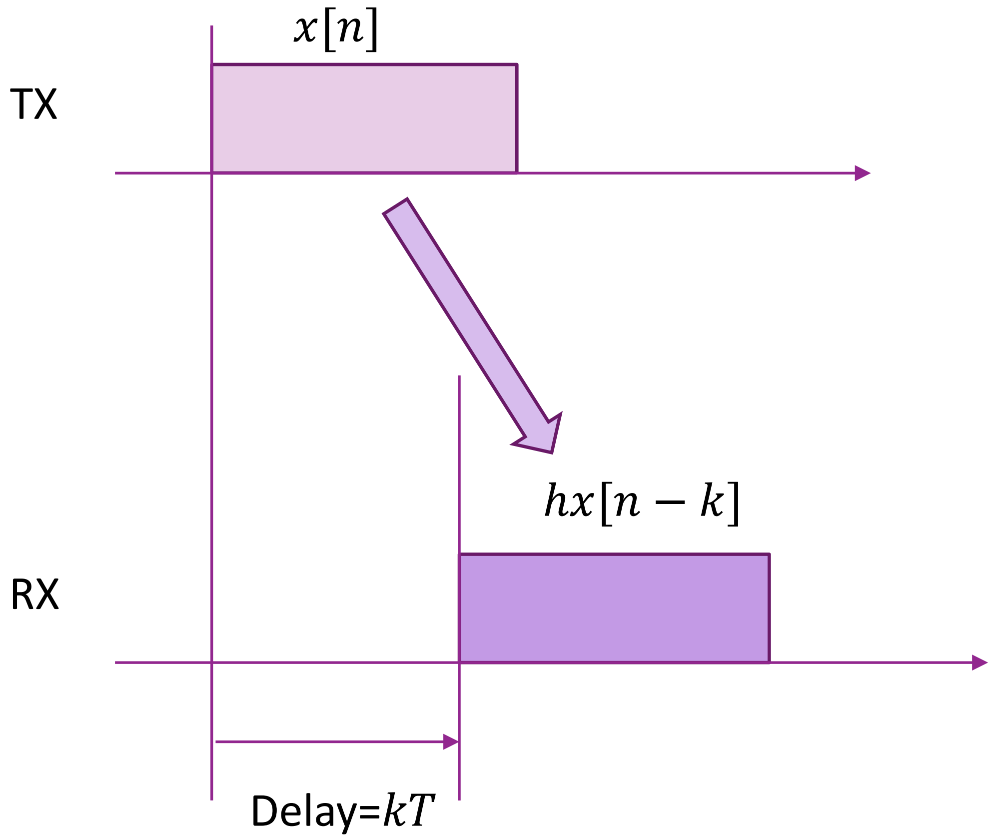
- Directional synchronization for mmWave
- Transmit multiple SS Burst
  - One in each direction
- MmWave typically use 120 kHz subcarrier spacing
- With 120 kHz SCS:
  - SSB = 4 OFDM symbols = 35.7 μs
  - Each SSB, contains a PSS
  - PSS time duration = 1 OFDM symbol = 8.92 μs
  - Bandwidth = 127 SC = 15.24 MHz
  - Up to 64 SS Bursts / burst period
  - Typical SSB periodicity = 20 ms
  -

Simple Synchronization Model

- TX sends a preamble / synchronization signal
  - $x[n],\ n = 0,1,2,\ldots,N-1$
  - Complex baseband samples.
  - Sample rate $\frac{1}{T}$
- If signal is present at RX:

$$y[n] = hx[n-k] + w[n]$$

  - h: Complex channel gain
  - k: Integer delay
- Problem detect if signal is present or not.
  - If so, what is the delay
- For now, we assume:
  - Integer delays, no multipath
  - Will address these issues later



Outline

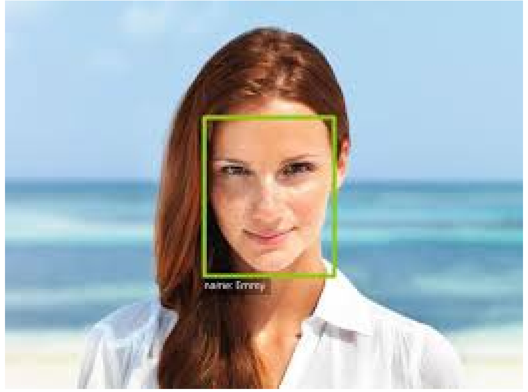
- Detection and Synchronization Problem
- Hypothesis Testing
- Match Filtering for Detection
- Match Filtering Convolution
- Matched Filtering as a Likelihood Ratio Test

Hypothesis Testing

- Classic problem in statistics or decision theory
- Observe data $\boldsymbol{y}$
- Two possible hypotheses for data
  - H0: Null hypothesis
  - H1: Alternate hypothesis
- Model statistically:
  - $p(\boldsymbol{y}|H_i), i = 0,1$
  - Assume some distribution for each hypothesis
  - Each density is the likelihood of $\boldsymbol{y}$
- Problem: Determine which hypothesis is true given data y

Applications

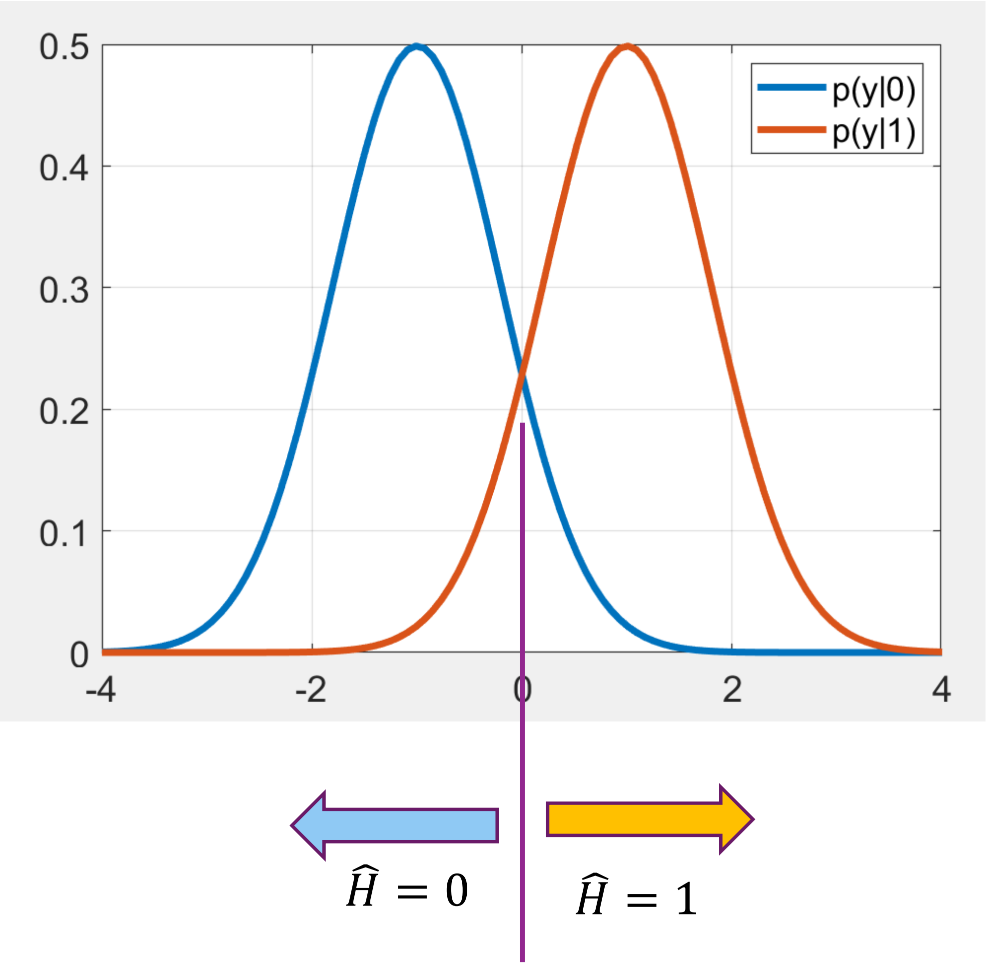
- ❑ Many applications
- ❑ Pattern recognition:
  - Does this image contain a face or not?
  - Is this person X?
- ❑ Detection:
  - Is the transmitted bit 0 or 1?
- ❑ This lecture: Is a signal present or not?



Simple Example

- Scalar Gaussian
  - H_0: $y = -A + w$
  - H_1: $y = A + w$,
  - $w \sim N(0, \sigma^2)$
- In this case:
  - $p(y|H_0) = N(y|-A, \sigma^2)$
  - $p(y|H_1) = N(y|A, \sigma^2)$
- Saw this earlier in BPSK transmissions
- Max likelihood detector from earlier
  - Selects the most likely hypothesis
  - In this case

$$\widehat{H} = \arg\max_{j} p(y|H = j) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



Types of Errors

- For binary detection problems, there are two errors:
  - Type I error (False alarm): Decide H1 when H0
  - Type II error (Missed detection): Decide H0 when H1
- In many problems, the consequences of these errors is different
- Example: Medical diagnosis
  - False alarm: You tell the patient he is ill, when he is fine
  - Missed detection: You miss the illness
  - Consequences are different
- Given detector, we define two error probabilities:
  - False alarm probability: $P_{FA} = P(\widehat{H} = 1|H = 0)$
  - Missed detection probability: $P_{MD} = P(\widehat{H} = 0|H = 1)$

Likelihood Ratio Test

- We can tradeoff the error probabilities with a likelihood ratio test:
- Likelihood ratio test (LRT)

$$\hat{H} = 1 \Leftrightarrow \frac{p(x|H_1)}{p(x|H_0)} \geq \gamma$$

  - γ is an adjustable threshold
  - Increasing $\gamma \Rightarrow$ Lowers P_{FA}, but lowers P_D
- Often performed in log domain

$$\hat{H} = 1 \Leftrightarrow L^*(x) = \log \frac{p(x|H_1)}{p(x|H_0)} \geq \gamma'$$

- Note that $\gamma = 0$ corresponds to maximum likelihood detector

Gaussian Example

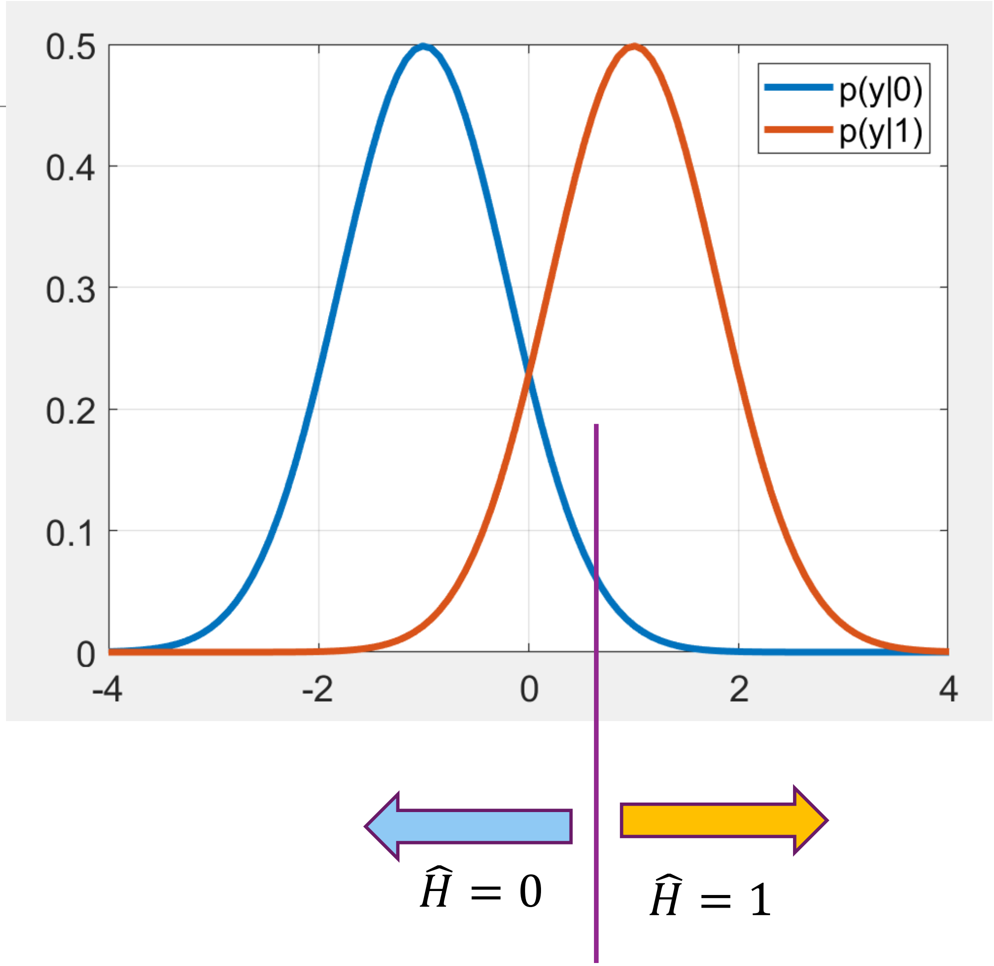
- Scalar Gaussian case:
  - $p(y|H_0) = N(y|-A, \sigma^2) = C \exp(-\frac{(y+A)^2}{2\sigma^2})$
  - $p(y|H_1) = N(y|A, \sigma^2) = C \exp(-\frac{(y-A)^2}{2\sigma^2})$
- Log likelihood ratio:

$$L(y) \coloneqq \ln \frac{p(y|H_1)}{p(y|H_0)}$$
$$= \frac{1}{2\sigma^2}[(y+A)^2 - (y-A)^2] = \frac{2Ay}{\sigma^2}$$

- LRT: $\hat{H} = 1$ if and only if

$$L(y) \geq \gamma \Leftrightarrow y \geq t = \frac{\gamma\sigma^2}{2A}$$

  - t is an adjustable threshold

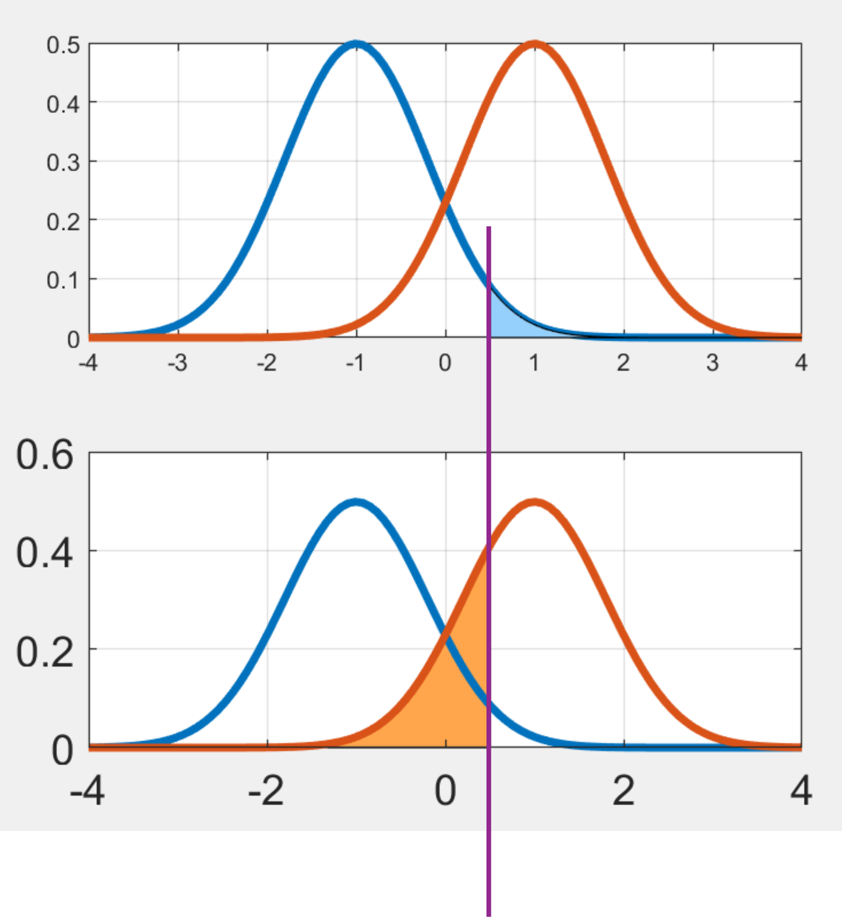


Computing Error Probabilities

- From previous slide, LRT detector is:

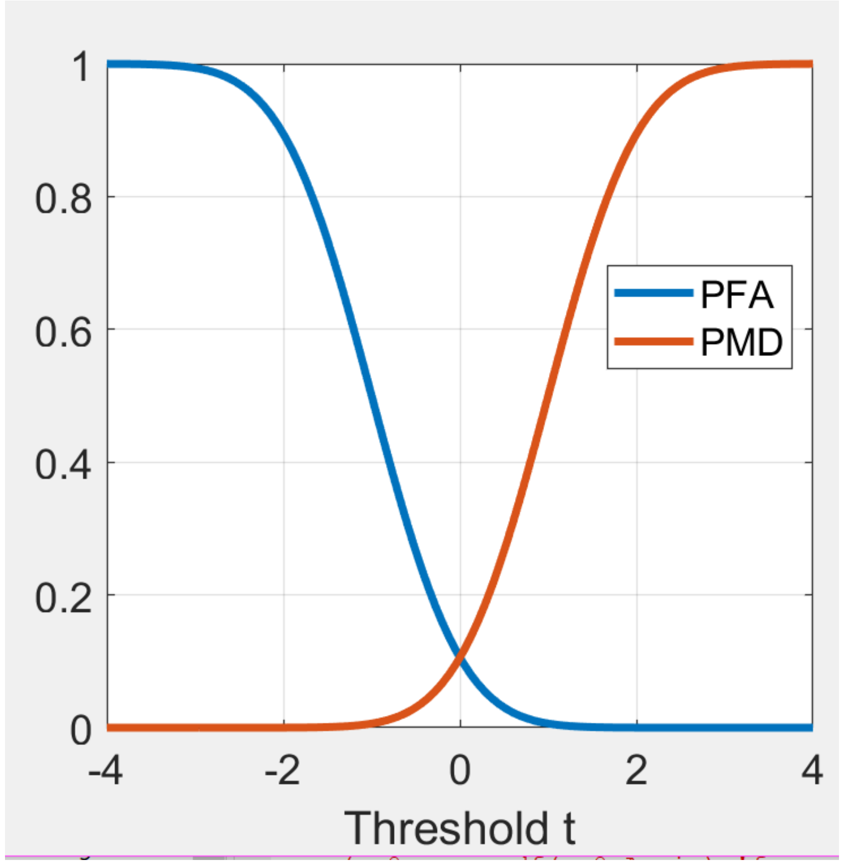
$$\hat{H} = \begin{cases} 1 & y \geq t \\ 0 & y < t \end{cases}$$

- FA probability:
  - $P_{FA} = P\left(\hat{H} = 1 \middle| H = 0\right) = P(y \geq t | H = 0) = \int_t^{\infty} p(y|0)dy$
  - This is the area under the curve (blue)
  - For Gaussian: $P_{FA} = Q\left(\frac{t+A}{\sigma}\right)$
- MD probability
  - $P_{MD} = P\left(\hat{H} = 0 \middle| H = 1\right) = P(y < t | H = 1) = \int_{-\infty}^{t} p(y|1)dy$
  - This is the area under the curve (orange)
  - For Gaussian: $P_{MD} = 1 - Q\left(\frac{t-A}{\sigma}\right)$



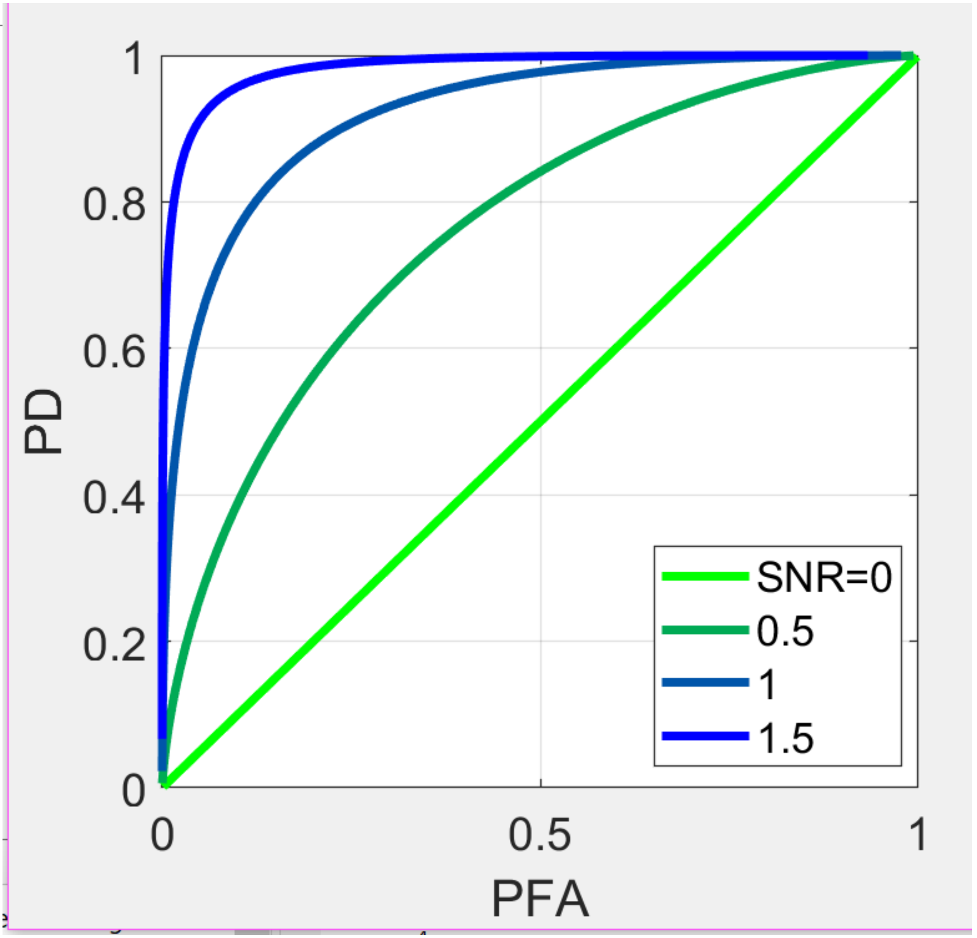
Tradeoff

- Tradeoff between P_{FA} and P_{MD}
  - $P_{FA} = Q\left(\frac{t+A}{\sigma}\right)$
  - $P_{MD} = 1 - Q\left(\frac{t-A}{\sigma}\right)$
- Increasing threshold t:
  - Decreases false alarms
  - But increases missed detections
- Selection of optimal threshold
  - Depends on the application
  - What are the relative costs of these errors?



ROC Curve

- Receiver operating characteristic
- Plot of P_D vs. P_{FA}
- Trace out: $(P_{FA}(\gamma), P_D(\gamma))$
- Random guessing achieves:
$$P_D + P_{FA} = 1$$
- Higher the line is better



Neyman-Pearson Theorem

- Theorem: Suppose that an LRT obtains $P_{FA} = \alpha$.
  Then any other test with P_{FA} will have a P_D less than or equal to the LRT.

- LRT is the most powerful test
- Obtains best P_{FA} vs. P_D performance

In Class Exercise

Synchronization In-Class Exercises

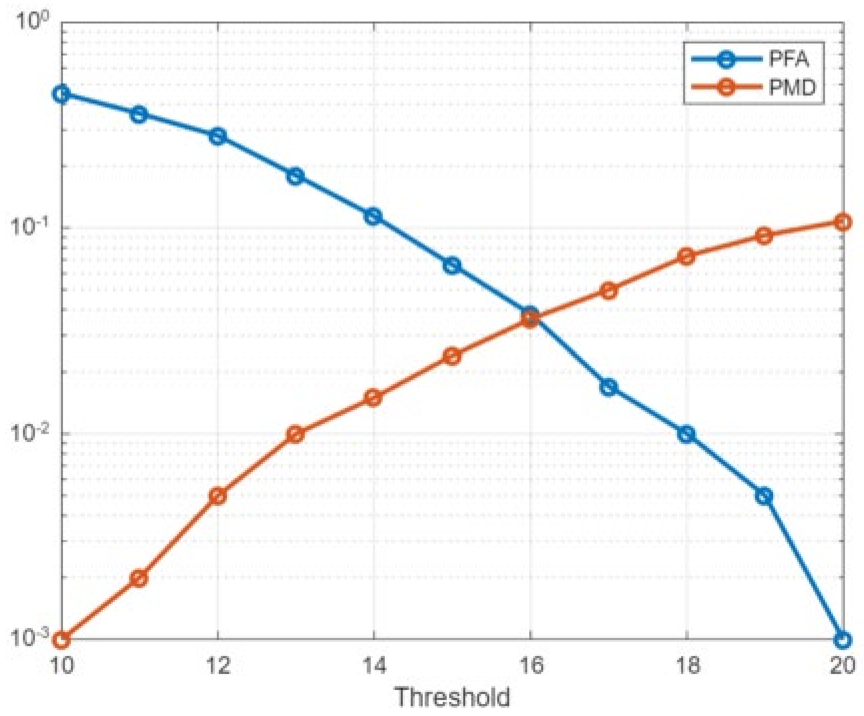
Hypothesis Testing for Poisson Random Variables

We will simulate hypothesis testing for discriminating between two Poisson distributions. This type of detector occurs in optical systems where the receiver counts the number of photons. The unknown variable is x=0 or 1. We receive a discrete random variable y with conditional probability:

- `P(y|x=0)` is Poisson with rate `lam0`. `P(x=0)=p0`
- `P(y|x=1)` is Poisson with rate `lam1`. `P(x=1)=p1`

The parameters are below

```
lam0 = 10;  % Rate when x=0
lam1 = 20;  % Rate when x=1
p0 = 0.8;   % P(x=0)
p1 = 1-p0;   % P(x=1)
```



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Simple Synchronization Model

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  - Sample rate $\frac{1}{T}$
- If signal is present at RX:

$$y[n] = hx[n-k] + w[n]$$

  - h: Complex channel gain
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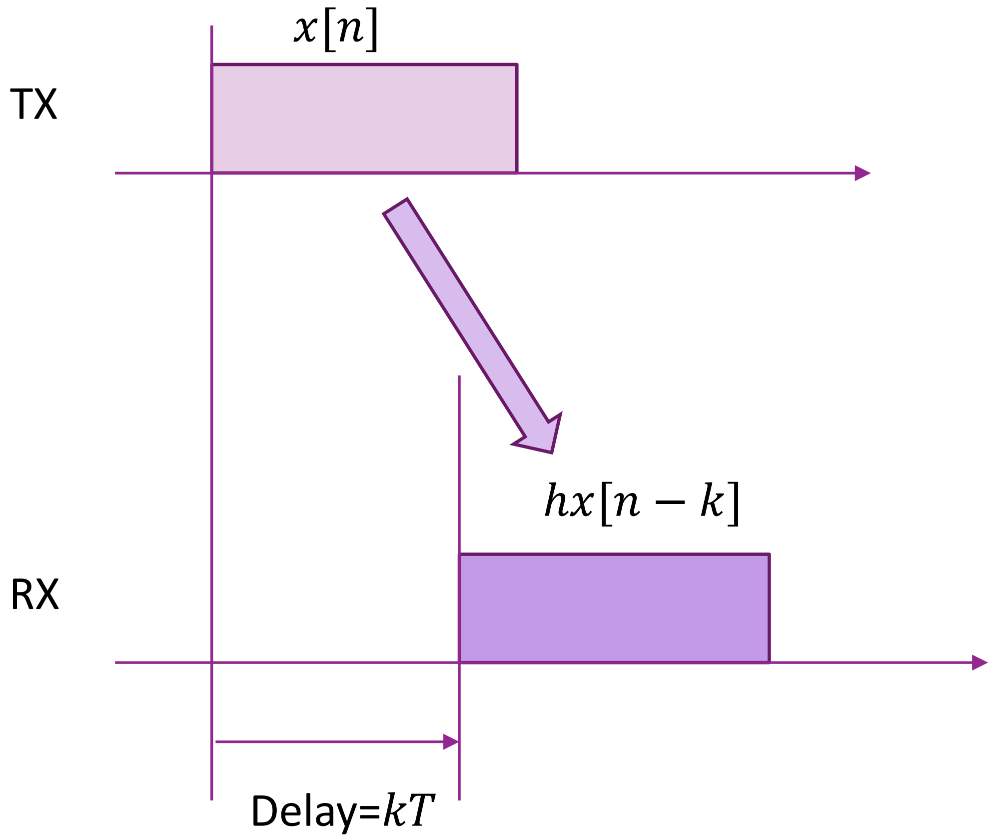
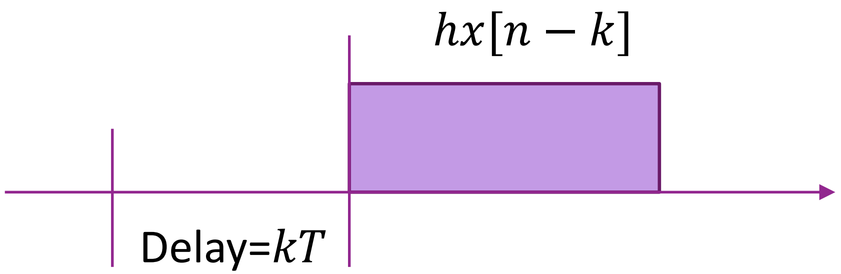


Detect as a Hypothesis Test

- At each delay k, we consider two hypotheses:
- H_1: Signal is present:

$$r[n] = hx[n-k] + w[n],$$

  - h is a complex, baseband channel gain
  - Recall that we are assuming a single path channel (for now)



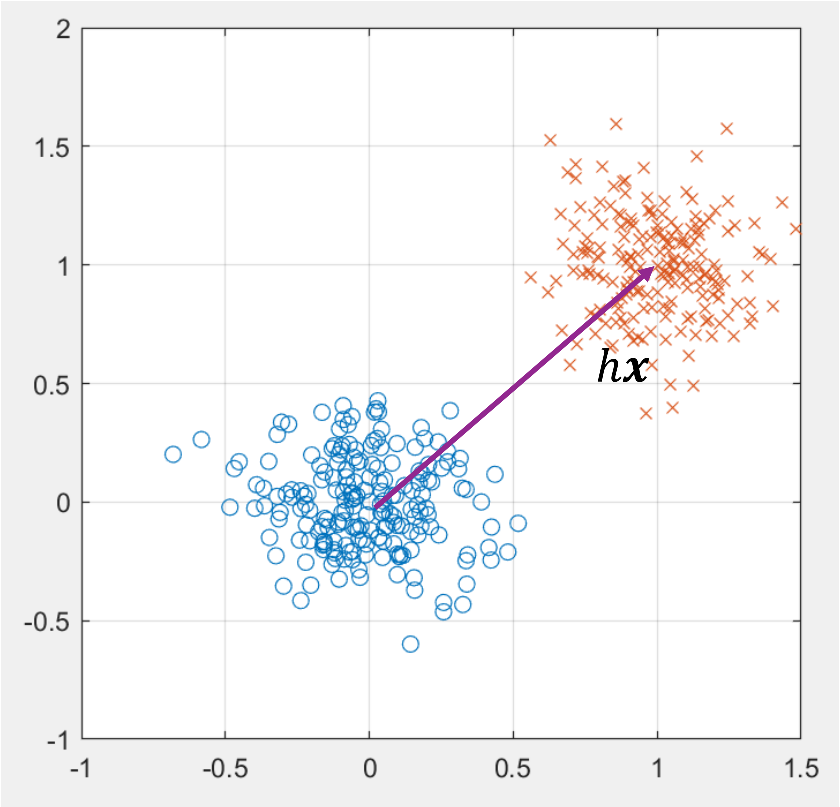
- H_0: Signal is absent:

$$r[n] = w[n]$$

- In both cases, assume $w[n]$ is white noise:
  - $w[n] \sim CN(0, N_0)$

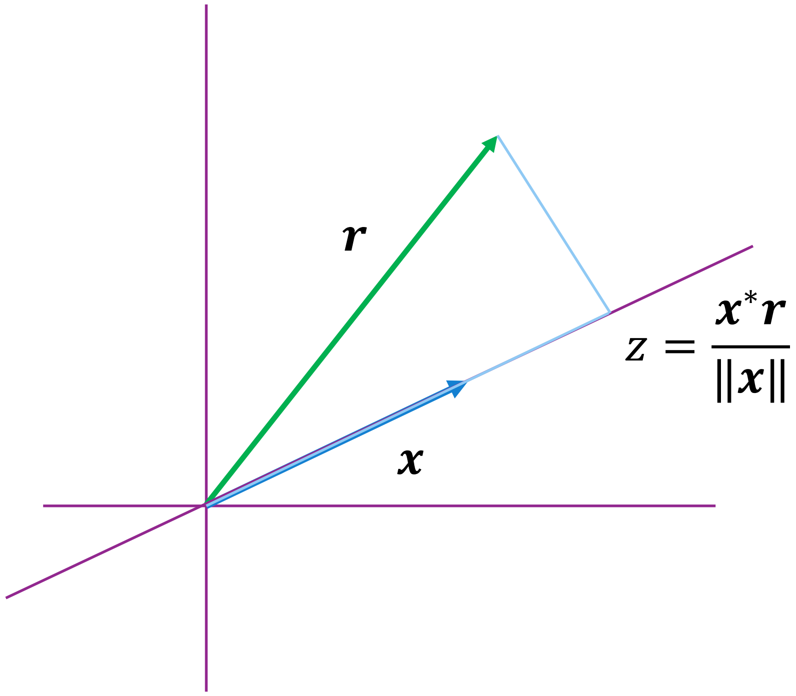
Hypothesis Test in Vector Form

- Without loss of generality, consider delay $k = 0$
- Let r be the vector of RX samples:
  - $\boldsymbol{r} = \left[r[0], \ldots, r[N-1]\right]^T$
- Write two hypotheses in vector form:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Geometrically:



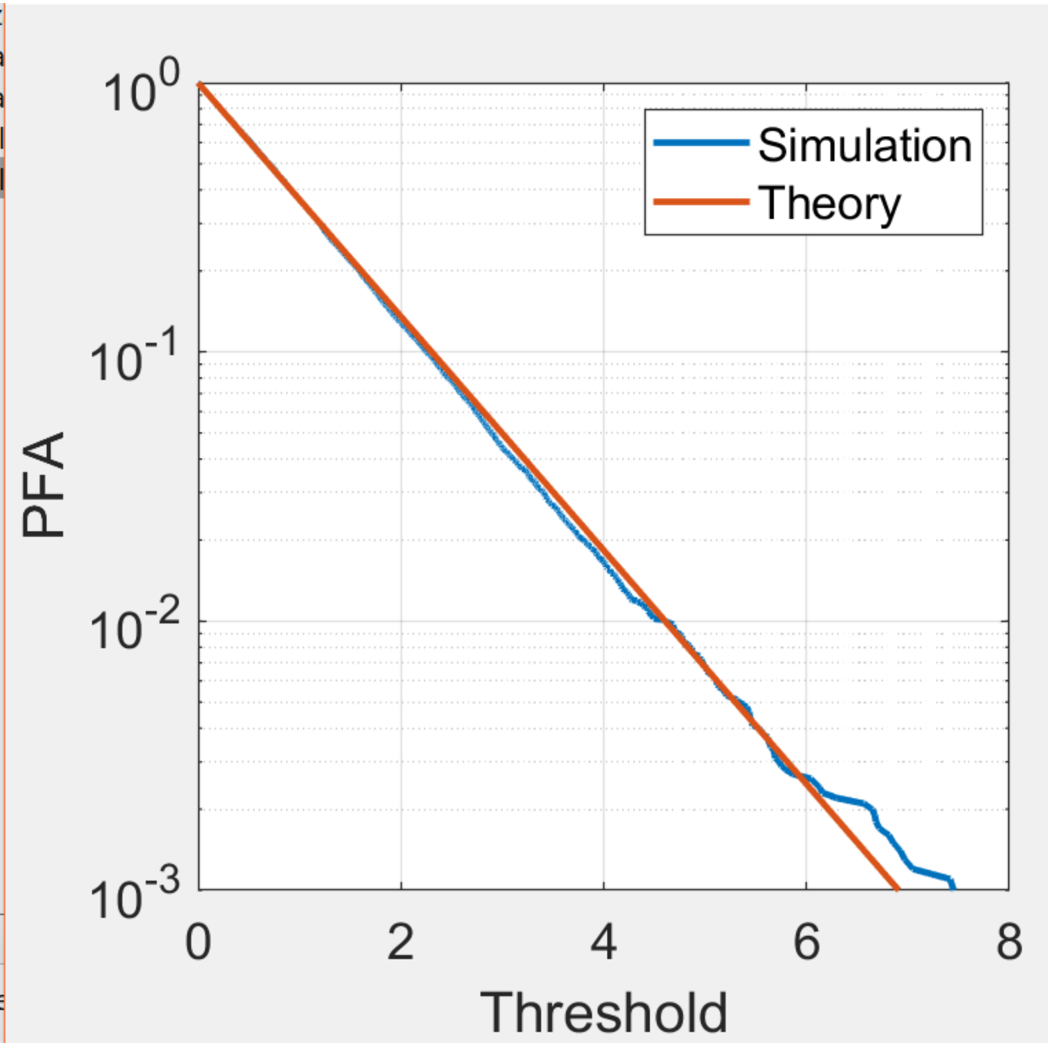
Match Filter Detector

- Hypotheses:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Match filter energy detector:
  - Project RX signal to TX waveform
  $$z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|}$$
  - Measure energy: $y = |z|^2$
  - Declare $\widehat{H} = \begin{cases} 1 & y \geq t \\ 0 & y < t \end{cases}$
  - t is a threshold
- Later we will show this is the optimal hypothesis test



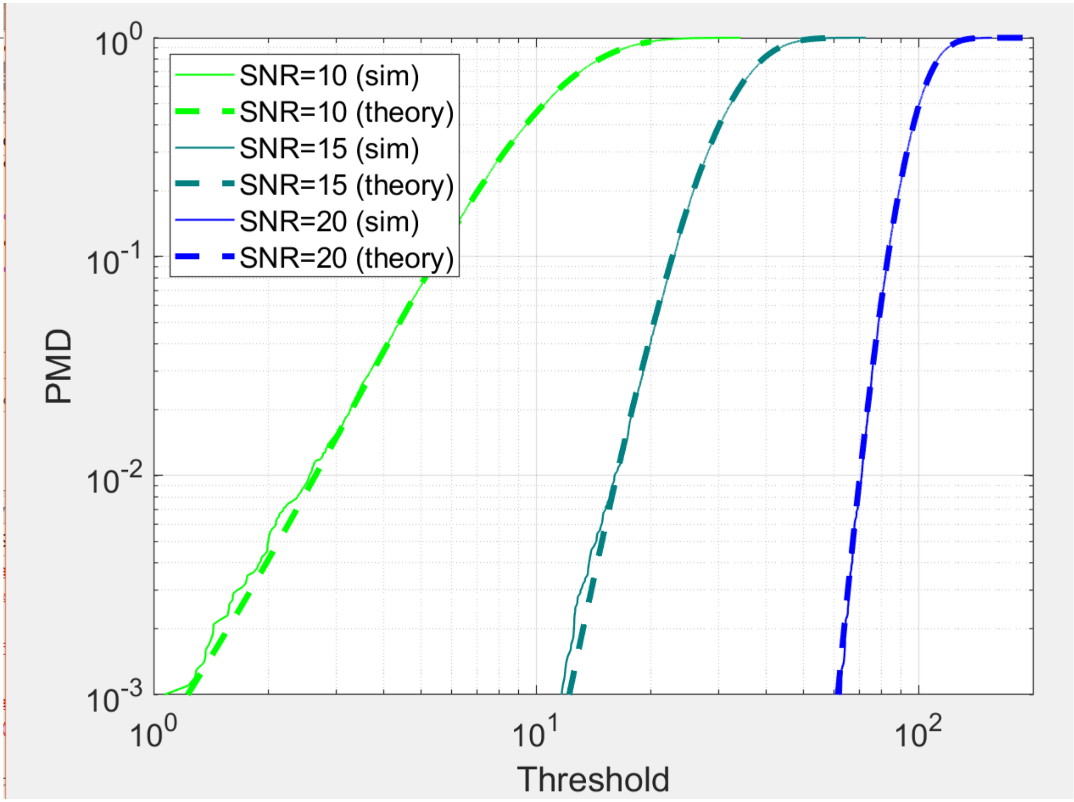
False Alarm

- False alarm
  - Under H_0: $\boldsymbol{r} = \boldsymbol{w}$, $\boldsymbol{w} \sim CN(0, N_0 \boldsymbol{I})$
  - Statistic $z = \frac{\boldsymbol{x}^* \boldsymbol{r}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}^* \boldsymbol{w}}{\|\boldsymbol{x}\|}$
  - This is a linear function of a Gaussian
  - $E(z) = \frac{\boldsymbol{x}^* E(\boldsymbol{w})}{\|\boldsymbol{x}\|} = 0,$
  - $E|z|^2 = \frac{\boldsymbol{x}^* E(\boldsymbol{w}\boldsymbol{w}^*)\boldsymbol{x}}{\|\boldsymbol{x}\|^2} = N_0 \frac{\boldsymbol{x}^* \boldsymbol{x}}{\|\boldsymbol{x}\|^2} = N_0$
  - Hence, $z \sim CN(0, N_0)$
  - Hence $y = |z|^2$ is exponential with $E(y) = N_0$
  - $P_{FA} = P(y \geq t) = e^{-t/N_0}$



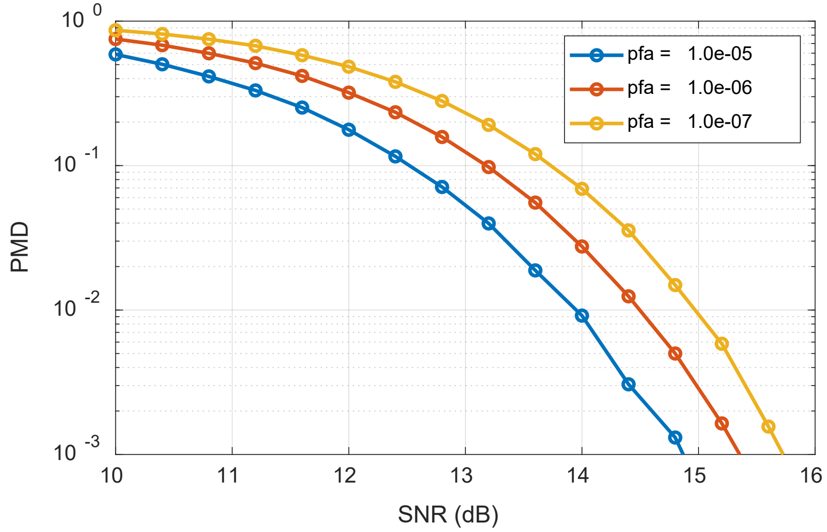
Missed Detection

- Under H_0: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w},\ \boldsymbol{w} \sim CN(0, N_0\boldsymbol{I})$
- Statistic $z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|} = \frac{\boldsymbol{x}^*\boldsymbol{w}}{\|\boldsymbol{x}\|}$
- Similar to FA calculation: $z \sim CN(A, N_0), A = h\|\boldsymbol{x}\|$
- Can show: $y = |z|^2 \sim \frac{N_0}{2} v$
  - v is a non-central chi squared with 2 degrees of freedom
  - Non-centrality parameter $\lambda = \frac{2|h|^2\|x\|^2}{N_0} = 2\ SNR$



Simulation

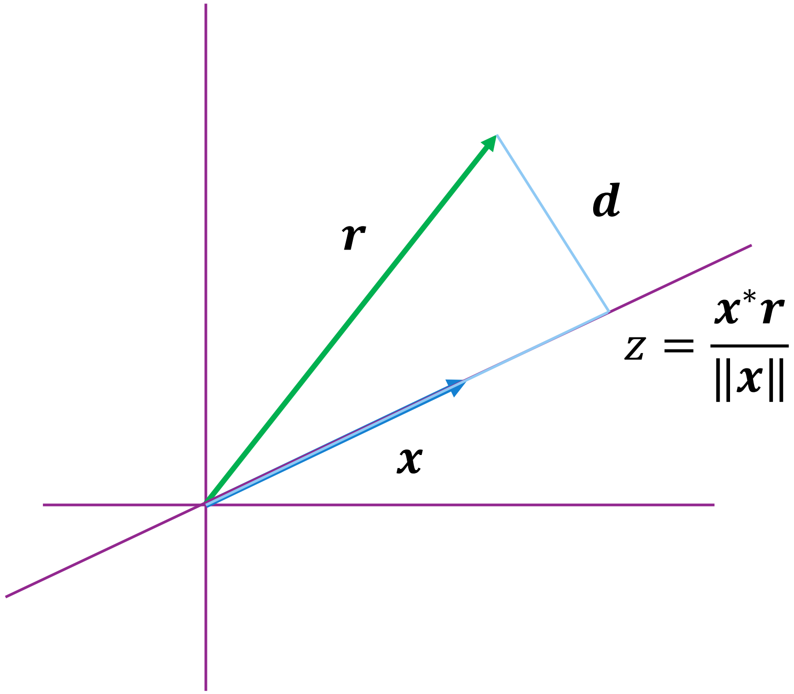
```
% FA targets to test
pfaTest = [1e-5,1e-6,1e-7];
nfa = length(pfaTest);
legstr = cell(nfa,1);
for ifa = 1:nfa
    % Compute FA target
    pfaTgt = pfaTest(ifa);
    t = -log(pfaTgt);

    % Measure PMD
    ntest = 1e5;
    snrTestTheory = linspace(10,18,21)';
    nsnr = length(snrTestTheory);
    pmdTheory = zeros(nsnr,1);

    for isnr = 1:nsnr
        snr = snrTestTheory(isnr);
        A = 10.^(0.05*snr);
        z = A + (randn(ntest,1)+1i*randn(ntest,1))/sqrt(2);
        rho = abs(z).^2;
        pmdTheory(isnr) = mean(rho < t);
    end

    semilogy(snrTestTheory, pmdTheory, 'o-', 'Linewidth', 2);
    hold on;

    legstr{ifa} = sprintf('pfa = %9.1e', pfaTgt);
end
```



- Theoretically calculated threshold based on PFA target
- Simulate PMD based on SNR

Noise Estimation

- Threshold $t = -N_0 \ln P_{FA}^{TGT}$
- Requires we know noise energy N_0
- How do we estimate this?
- Suppose $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$
- Consider residual signal: $\boldsymbol{d} = \boldsymbol{r} - z\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}, z = \frac{\boldsymbol{x}^*\boldsymbol{r}}{\|\boldsymbol{x}\|}$
  - Component of $\boldsymbol{r}$ not spanned by $\boldsymbol{x}$
  - $\boldsymbol{d}$ is the projection of $\boldsymbol{w}$ onto an $N-1$ dim space
  - Can show that $E\|\boldsymbol{d}\|^2 = (N-1)N_0$
  - Take noise estimate: $\widehat{N}_0 = \frac{1}{N-1}\|\boldsymbol{d}\|^2$



Noise Estimation 2

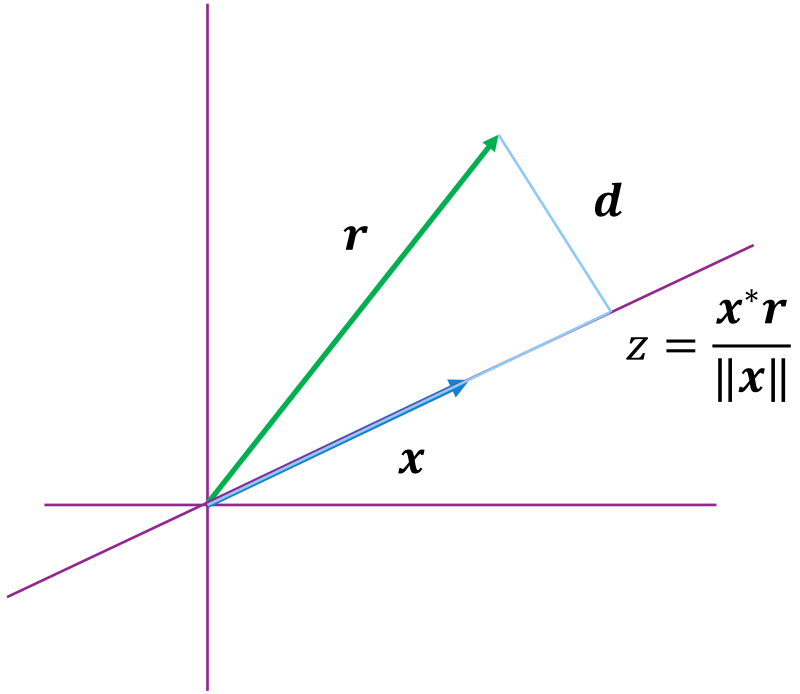
- Use threshold with estimate noise $t = -\widehat{N}_0 \ln P_{FA}^{TGT}$
- Detector takes $\widehat{H} = 1$ if

$$|z|^2 \geq t = -\widehat{N}_0 \ln P_{FA}^{TGT} = c\|\boldsymbol{d}\|^2,$$

  - $c = -\frac{\ln P_{FA}^{TGT}}{N-1}$
  - But $\|\boldsymbol{d}\|^2 = \|\boldsymbol{r}\|^2 - |z|^2$
  - So test is equivalent to:

$$\frac{|z|^2}{\|\boldsymbol{r}\|^2 - |z|^2} \geq c \Leftrightarrow \rho = \frac{|z|^2}{\|\boldsymbol{r}\|^2} \geq \frac{c}{1+c} = \gamma$$

  - Note $\rho = \frac{|z|^2}{\|\boldsymbol{r}\|^2} = \frac{|\boldsymbol{x}^*\boldsymbol{r}|^2}{\|\boldsymbol{r}\|^2\|\boldsymbol{x}\|^2}$ =fraction of energy in direction $\boldsymbol{x}$
- Conclusion: With noise estimation MF is equivalent to



In Class Exercise

Signal Detection

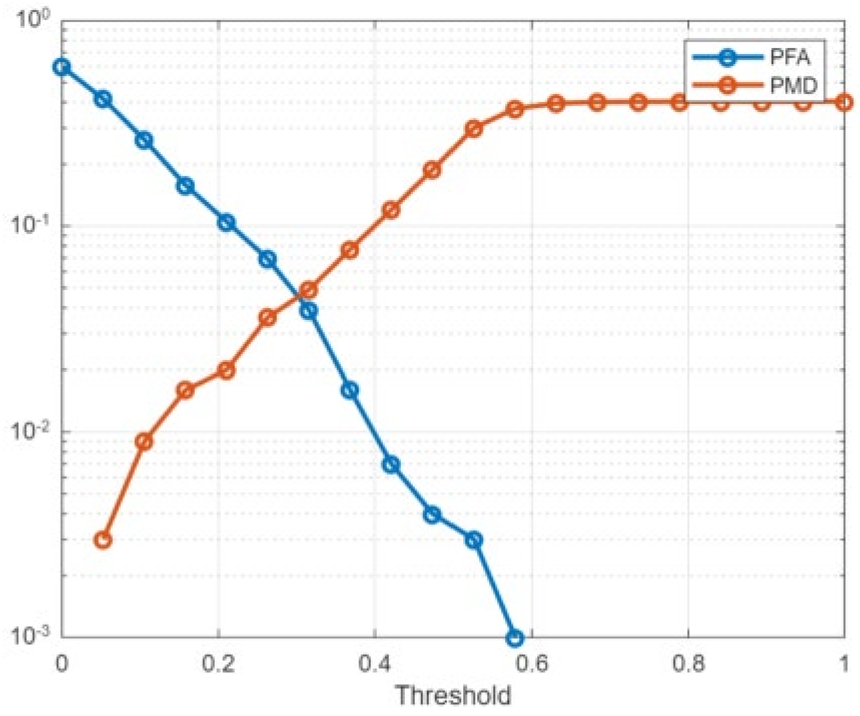
In this section, we will simulate a simple signal detection. We suppose we have a complex baseband signal of length `ns`. The unknown variable is `u=0` or `1` depending if a signal is present. The RX signal, r, can then be modeled as:

```
r(j) = u*h*x(j) + w(j),   j = 1,..., ns
```

where the channel gain and noise are modeled as complex Gaussians:

```
w(j) ~ CN(0,wvar),   h ~ CN(0,hvar)
```

```
p0 = 0.6;   % P(u=0)=1-P(u=1)
ns = 8;     % Signal length
snr = 10;   % hvar/wvar
hvar = 1;                  % Mean channel gain
wvar = db2pow(-snr);       % Noise energy per sample
ntrial = 1000;             % Number of trials to test
x0 = exp(1i*rand(ns,1)*2*pi);  % Random true signal
```



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Match Filtering with Unknown Delay

- Synchronization signal $x[n], n = 0,1, \ldots, N-1$
- RX signal at delay k:
  - $r[n] = hx[n-k] + w[n]$
- Problem: Detect if signal is present. If so, what is the delay k?
- Match filter (without normalization) at delay k is:

$$z[k] = \sum_n r[n+k]x^*[n]$$

- Hypothesis test:
  - $|z[k]|^2 \geq t \Rightarrow$ Detect signal at delay at k

Further Analysis Details

- We need to examine three key practical issues that degrade performance

- Preamble auto-correlation
- Multi-path
- Carrier offset

Match Filtering as a Convolution

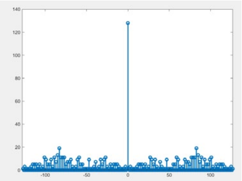
- Match filter (without normalization) at delay k is:

$$z[k] = \sum_n r[n+k]x^*[n]$$

- Define adjoint signal: $\tilde{x}[n] = x^*[-n]$
  - Complex conjugate and time reversal
- MF output can be computed via a convolution:
  - $z[k] = \sum_n r[n+k]x^*[n] = \sum_n r[n+k]\tilde{x}[-n] = \sum_n r[k-n]\tilde{x}[n] = (r * \tilde{x})[k]$

$r[n]$

$z[n]$



Detected signal locations

Filter with $\tilde{x}[n]$

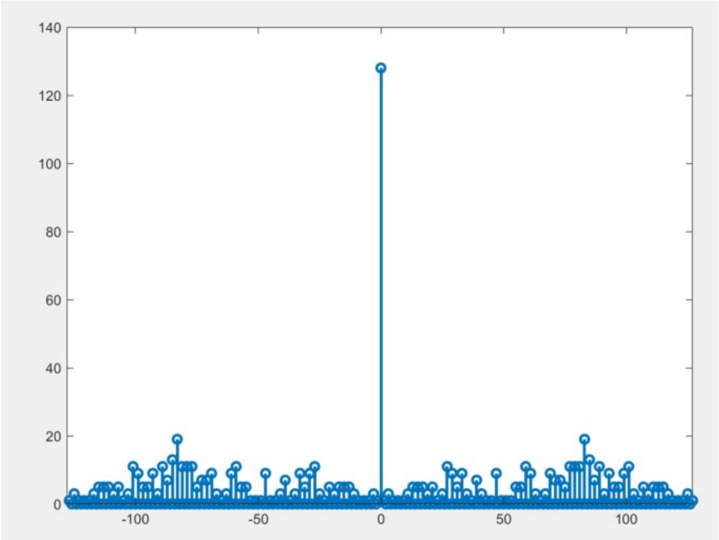
Peak detection

Signal Auto-Correlation

- Consider what happens with no noise:
  - $r[n] = hx[n-k_0],\ k_0 =$ "True" delay
- Run match filter: $z[k] = (r * \tilde{x})[k]$
- Can show output is: $z[k] = hR_x[k-k_0]$
  - $R_x[\ell] =$ autocorrelation of transmitted signal
  - $R_x[\ell] = \sum_n x[n]x^*[n-\ell]$
- Since we want $z[k]$ small for $k \neq k_0$, we want:

$$R_x[\ell] \approx 0 \ \text{ for } \ell \neq 0$$

- Many sequences with low auto-correlation
  - Golay, Walsh, ….



Auto-correlation of Golay 128 sequence
Used in 802.11ad preamle

Multipath

- Up to now we have assumed that there is a single path:

$$r[n] = hx[n - k_0]$$

- But, in reality there is often multipath:

$$r[n] = \sum_k h[k]x[n - k]$$

  - Due to multi-path in channel and pulse shape filtering

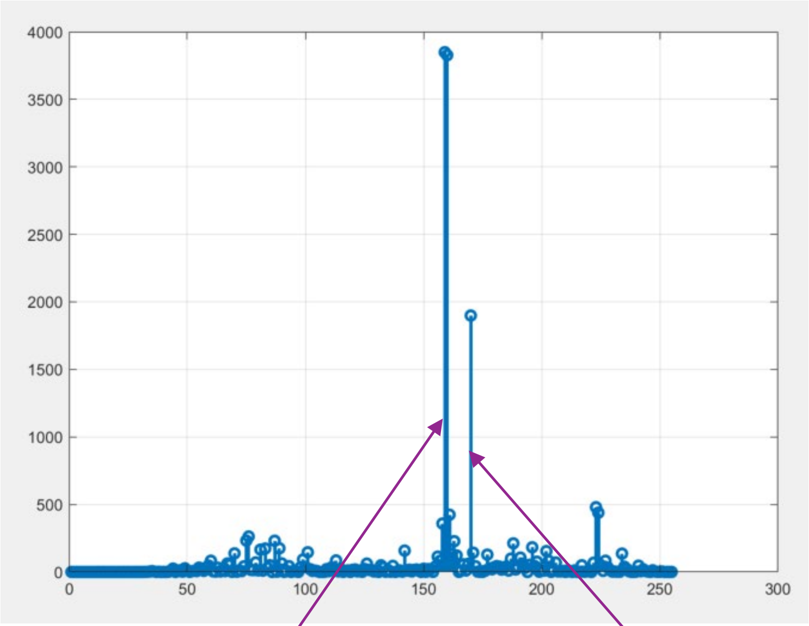
- Match filter has delayed copies of auto-correlation:

$$z[n] = \sum h[k]R_x[n - k]$$

  - One peak in MF output for each path

Ex: Two path channel

$$h[n] = sinc(n - 0.5)$$
$$+0.5sinc(n - 10.2)$$



Path at $k = 0.5$ Path at $k = 10.2$

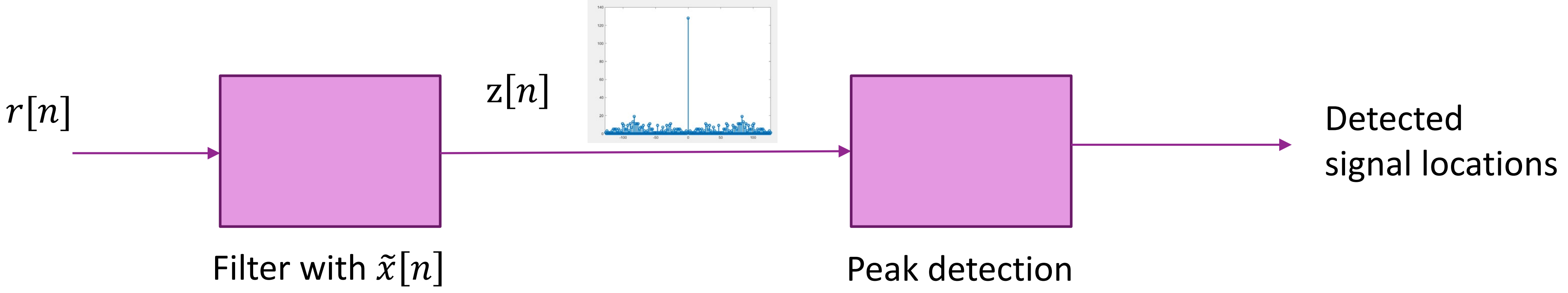
Frequency Offsets

- When initially searching for a preamble, there may be a significant carrier offset
- Causes a phase rotation in samples:

$$r[n] = e^{i\theta n} h x[n-k] + w[n]$$

  - θ is the phase rotation per sample
  - $\theta = \Delta f\ T,\ \Delta f =$frequency error, $T =$sampling rate
- Must integrate over range where phase does not change significantly
  - Pre-amble length must be $N \ll \frac{1}{\Delta f T}$
- Example: Suppose the carrier offset =10 ppm, $f_c = 60$ GHz and $\frac{1}{T} = 1.76$ Gs/s
  - $\Delta f T = \frac{(10)^{-5}(60)(10)^9}{1.76(10)^9} = 3.4(10)^{-4} \Rightarrow \frac{1}{\Delta f\ T} \approx 2.9(10)^3$ samples
  - In time duration, this is $\frac{1}{\Delta f} = 1.67$ us
  - A very short time before the signal is completely rotated

Detailed Simulation Example

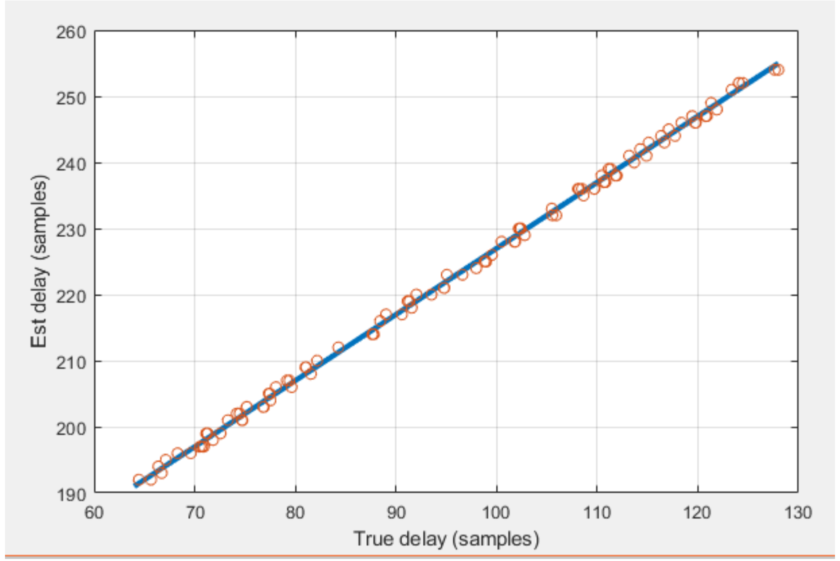


- Transmit 128 length Golay pre-amble
- Filter through channel with single (possibly fractional) delay
  - $r[n] = h[n] * x[n] + w[n], \;\; h[n] = sinc(n - \frac{\tau}{T})$
- Set threshold for FA target of 10^{-3} per 1000 samples
- Measure MD probability as a function of the SNR

Calibration

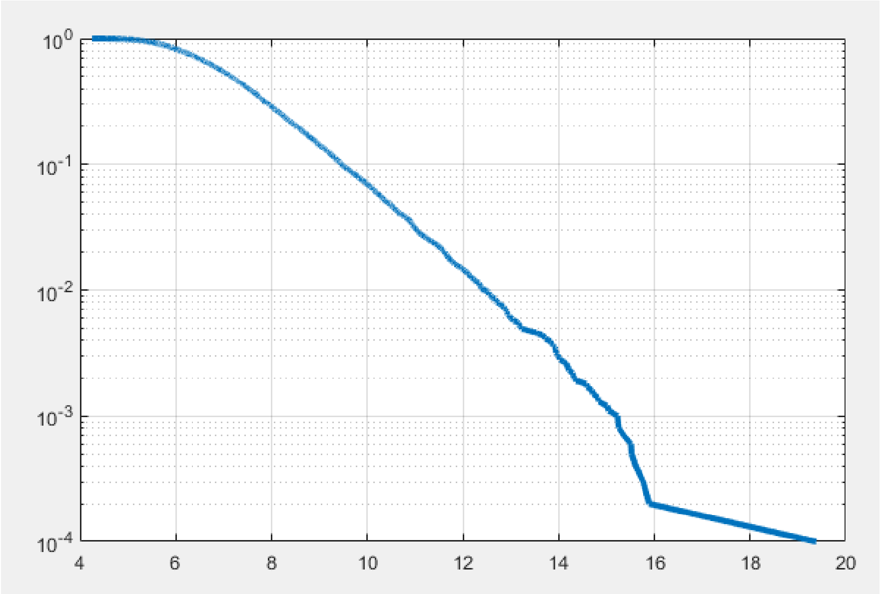
- Need to calibrate the FA probability and delay offset

Estimated Delay



True Delay

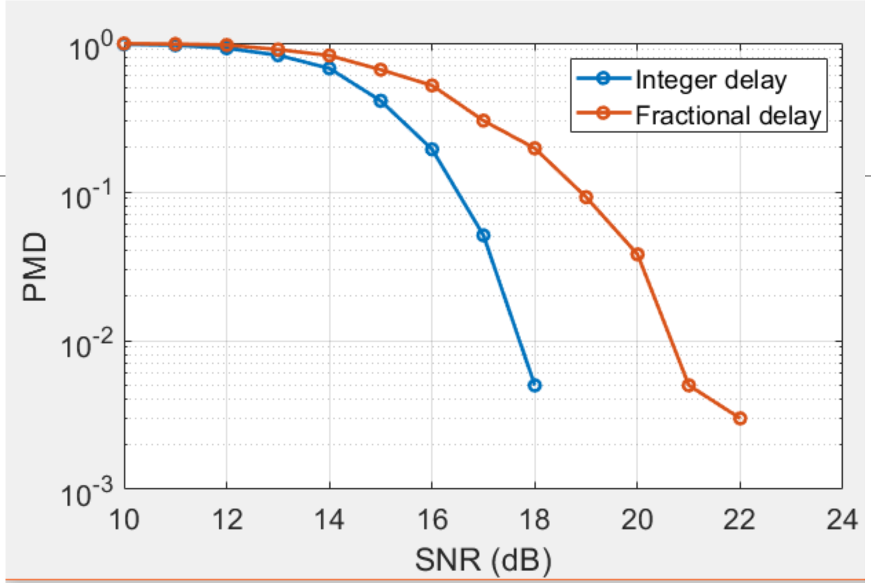
PFA



Threshold

Missed Detection

```
for isnr = 1:nsnr
    % Get the SNR
    snr = snrTest(isnr);
    wvar = 10.^(-0.1*snr)*npre;

    dly0 = unifrnd(64,128,ntest,1);
    dlyEst = zeros(ntest,1);
    rhoMax = zeros(ntest,1);

    for it = 1:ntest
        % Create a random delay
        gain = exp(1i*2*pi*rand(1));
        x = delaysig(xpre,gain,dly0(it),nsamp);

        % Add noise
        w = (randn(nsamp,1) + 1i*randn(nsamp,1))*sqrt(wvar/2);
        r = x + w;

        % Estimate the delay
        [rhom, im, ~] = predetect(r,xpre,maxdly);
        rhoMax(it) = rhom;
        dlyEst(it) = im - dlyOff;

    end

    I = (rhoMax > tfa);
    pmd(isnr) = 1-mean(I);
    dlyerr(isnr) = sqrt( mean((dlyEst(I) - dly0(I)).^2) );
    fprintf(1,'SNR = %12.4e PMD=%12.4e dly=%12.4e\n', ...
        snr, pmd(isnr), dlyerr(isnr));
end
```



- ❑Loss of about 3dB with fractional delay offset
- ❑Signal energy is split in two samples
- ❑Need to use over-sampling to compensate
  - See lab

Outline

- ❑Detection and Synchronization Problem
- ❑Hypothesis Testing
- ❑Match Filtering for Detection
- ❑Match Filtering Convolution
- ❑Matched Filtering as a Likelihood Ratio Test

Likelihood Ratio Test

- In vector form:
  - H_1: $\boldsymbol{r} = h\boldsymbol{x} + \boldsymbol{w}$ [Signal present]
  - H_0: $\boldsymbol{r} = \boldsymbol{w}$ [Signal absent]
- Likelihoods:
  - $p(r|H_0, \sigma^2) = \frac{C}{\sigma^{2N}} \exp\left(-\frac{\|\boldsymbol{r}\|^2}{\sigma^2}\right)$,
  - $p(r|H_1,, \sigma^2, h) = \frac{C}{\sigma^{2N}} \exp\left(-\frac{\|\boldsymbol{r}-h\boldsymbol{x}\|^2}{\sigma^2}\right)$
  - Cannot apply regular LRT since parameters are unknown
  - GLRT
  - $\Lambda_0(r, \sigma^2) \coloneqq -\frac{1}{N} \ln p(r|H_0) = \frac{1}{N} \ln \sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2}$

Generalized Likelihood Ratio Test

- Null hypothesis
  - $\Lambda_0(r, \sigma^2) := -\frac{1}{N}\ln p(r|H_0) = \frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2}$
  - $\overline{\Lambda}_0(r) := \min_{\sigma^2} \frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}\|^2}{N\sigma^2} = \frac{1}{N}\ln\frac{\|\boldsymbol{r}\|^2}{N} + 1$
- Present hypothesis:
  - $\Lambda_1(r, \sigma^2, h) := -\frac{1}{N}\ln p(r|H_1) = \frac{1}{N}\ln\sigma^2 + \frac{\|\boldsymbol{r}-h\boldsymbol{x}\|^2}{N\sigma^2}$
  - Minimize over h: $\min_{\mathrm{h}}\|\boldsymbol{r} - h\boldsymbol{x}\|^2 = \|\boldsymbol{r}\|^2 - \frac{|\boldsymbol{x}^*\boldsymbol{r}|^2}{\|\boldsymbol{x}\|^2}$
  - $\overline{\Lambda}_1(r) := \min_{\sigma^2, h} \ln p(r|H_1) = \frac{1}{N}\ln\frac{1}{N}\left[\|\boldsymbol{r}\|^2 - \frac{|\boldsymbol{x}^*\boldsymbol{h}|^2}{\|\boldsymbol{x}\|^2}\right] + 1$
- GLRT: $L(r) := \overline{\Lambda}_1(r) - \overline{\Lambda}_0(r) = -\ln[1-\rho]$, $\rho = \frac{|\boldsymbol{x}^*\boldsymbol{h}|^2}{\|\boldsymbol{x}\|^2\|\boldsymbol{r}\|^2}$
- Details in clas