Unit 8: Equalization

EL-GY 6013: DIGITAL COMMUNICATIONS

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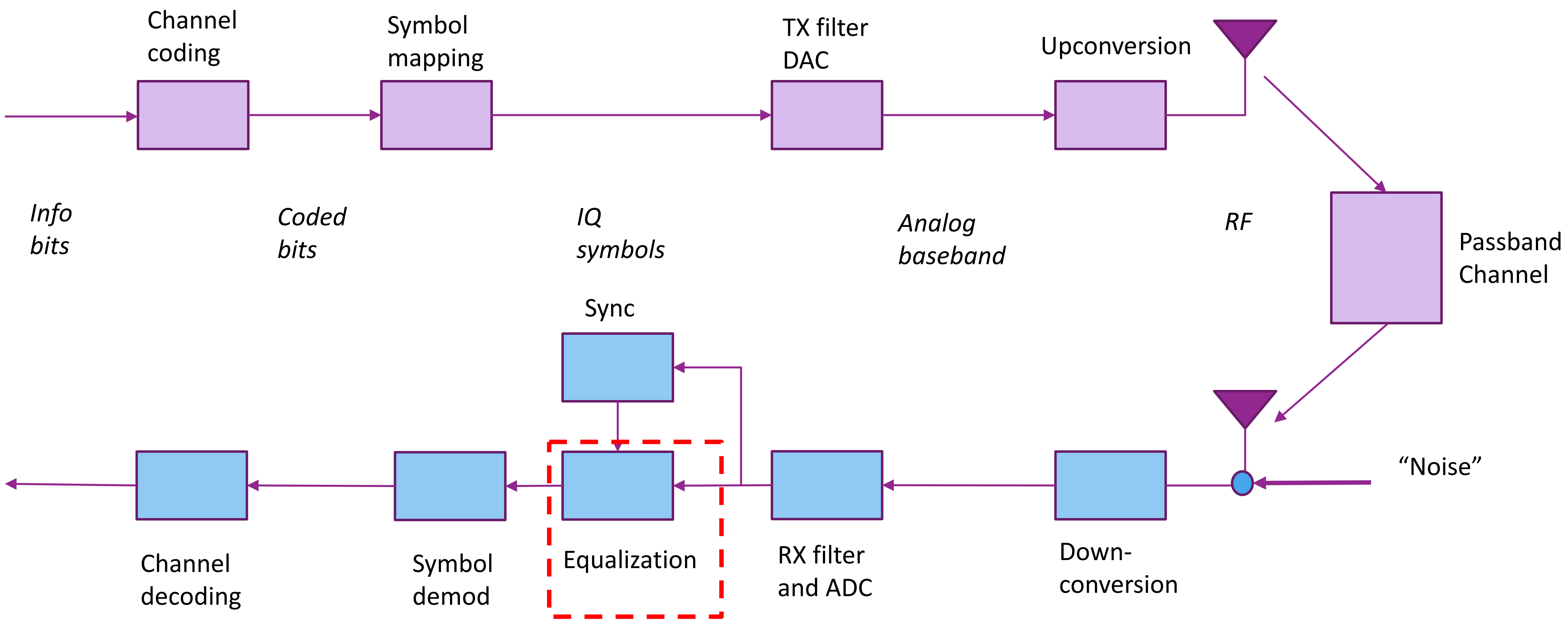
Learning Objectives

- ❑ Determine when equalization is necessary
  - Describe physical mechanisms for multi-path and ISI
- ❑ Describe and implement a simple OFDM TX and RX
- ❑ Select parameters for an OFDM system (CP, FFT size, …)
- ❑ Describe the parameters for commonly used commercial OFDM systems
  - Wireless LAN, LTE, 5G NR, …
- ❑ Write the OFDM channel gain given a discrete-time multipath channel
- ❑ Describe the synchronization and channel estimation process for OFDM
- ❑ Compute the bias and variance for simple channel estimators

This Unit


Channel coding
Symbol mapping
TX filter DAC
Upconversion
Info bits
Coded bits
IQ symbols
Analog baseband
RF
Passband Channel
Sync
"Noise"
Channel decoding
Symbol demod
Equalization
RX filter and ADC
Down-conversion

Outline

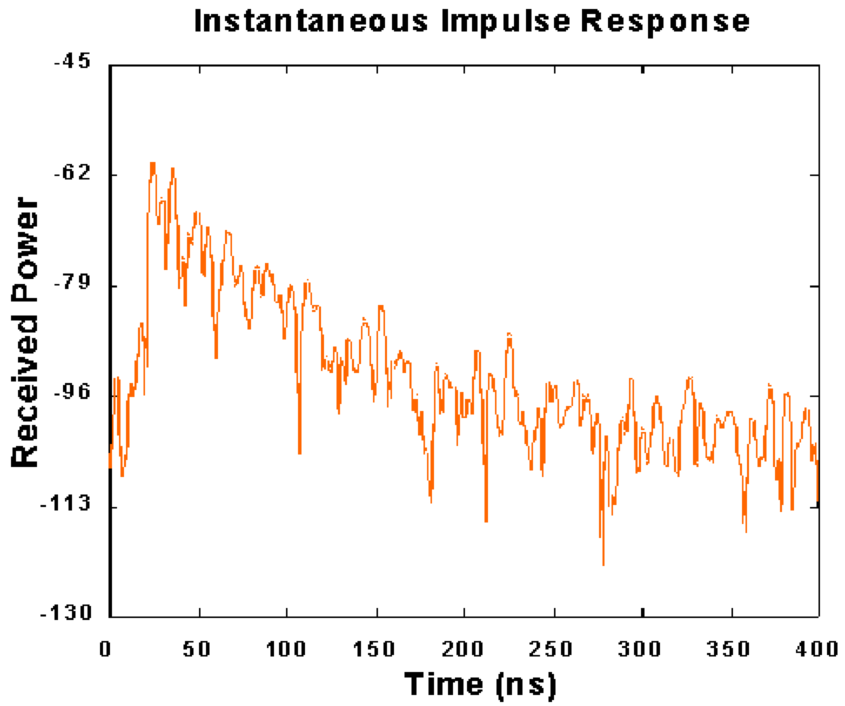
- What is Equalization?
- Time-Domain Equalization for Single Carrier Systems
- OFDM TX and RX
- OFDM Channel
- OFDM Synchronization and Channel Estimation

Multi-Path and ISI

- Effective discrete-time channels often have many taps

$$r[n] = \sum_k h[k]u[n-k] + w[n]$$

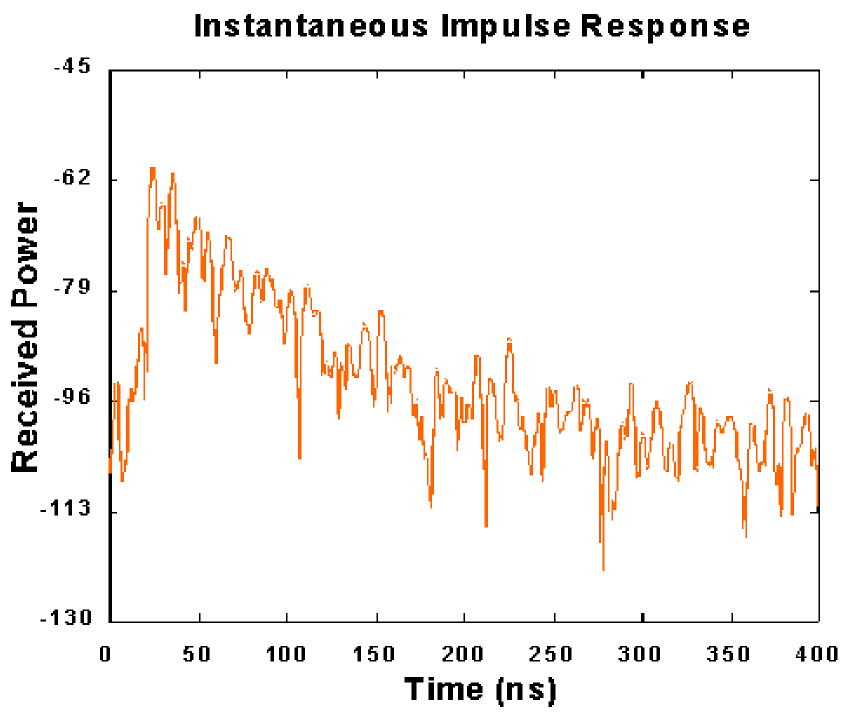
- Two causes:
  - Multi-path: Physical channel has many paths (Reflections, scattering, …)
  - Pulse shaping (TX and RX filter spread samples over time)
- Causes inter-symbol interference (ISI):
  - Each TX symbol received over many symbols



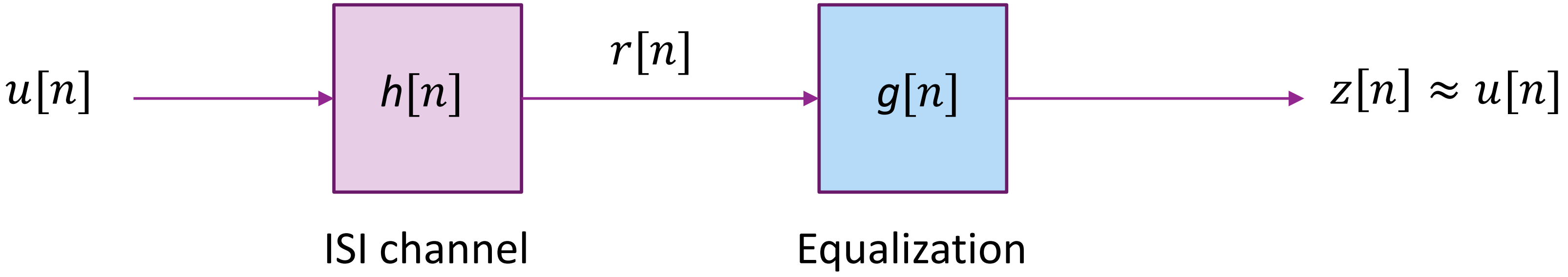
Multi-path indoor channel measured at UC Berkeley

Equalization

- Equalization: Any method to overcome ISI
- This lecture we look at two main methods
- Time-domain equalization
  - Used in single carrier systems
- Frequency-domain equalization
  - Used in OFDM systems



Linear Equalization



- Discrete-time ISI channel:

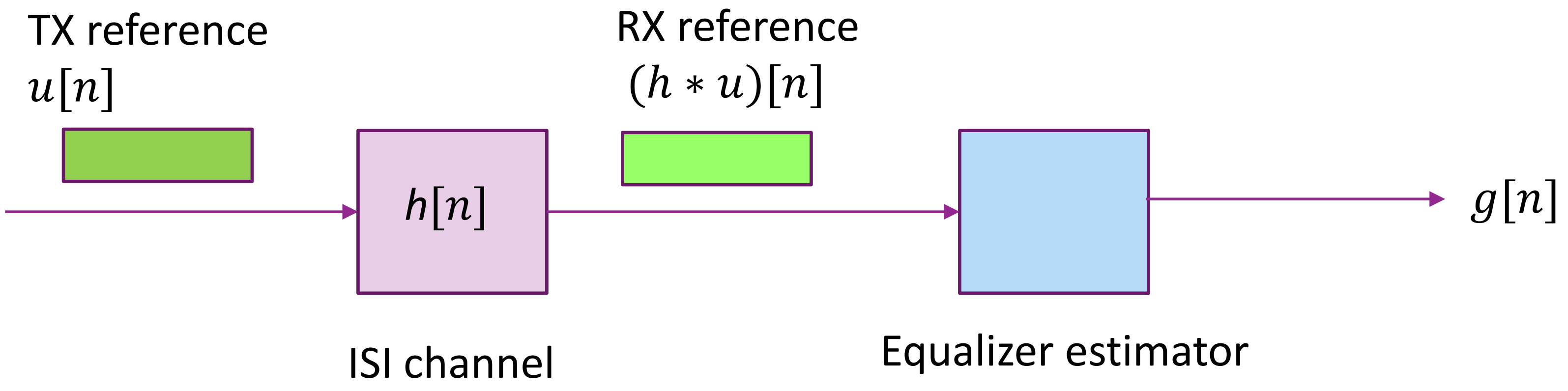
$$r[n] = \sum_k h[k]u[n-k] + w[n]$$

- Linear equalizers: Find a filter $g[k]$ to approximately invert the channel:

$$z[n] = \sum_k g[k]r[n-k] \approx u[n]$$

- In frequency-domain: $G(\Omega) \approx \frac{1}{H(\Omega)}$

Training Symbols



- All linear equalizers require training or reference signals
- TX sends a known reference sequence $u[n]$
- RX sees $r[n] = h[n] * u[n] + w[n]$
- Estimates channel and/or equalizer for $r[n]$

Outline

- ❑What is Equalization?
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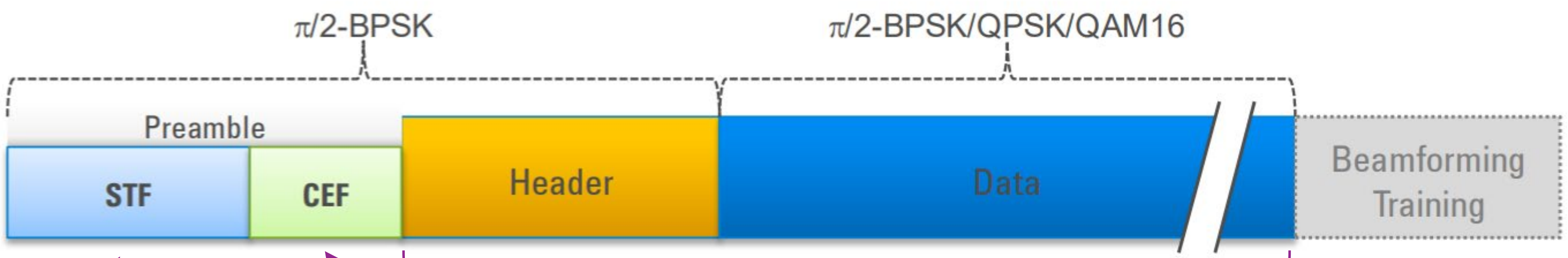
- ❑OFDM Synchronization and Channel Estimation

802.11 ad Preamble for 60 GHz WiFi

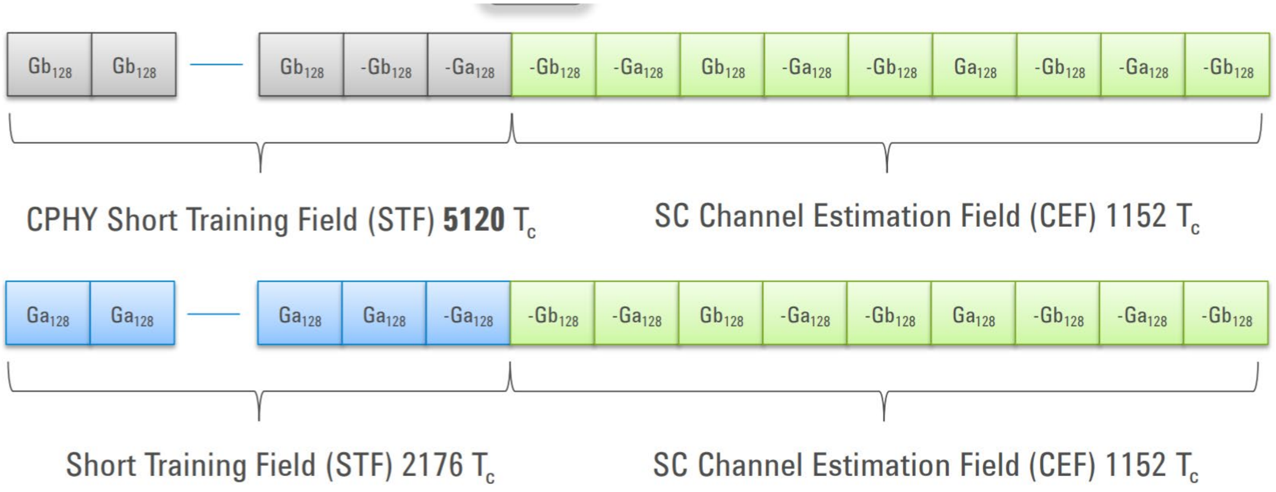


Apply equalizer

- STF: Short training field
  - AGC, detection of packet
  - coarse estimate of timing
  - Process as previous lecture

- CEF: Channel estimation field
  - Train equalizer

802.11ad Preamble Details



Control packets

Longer STF.

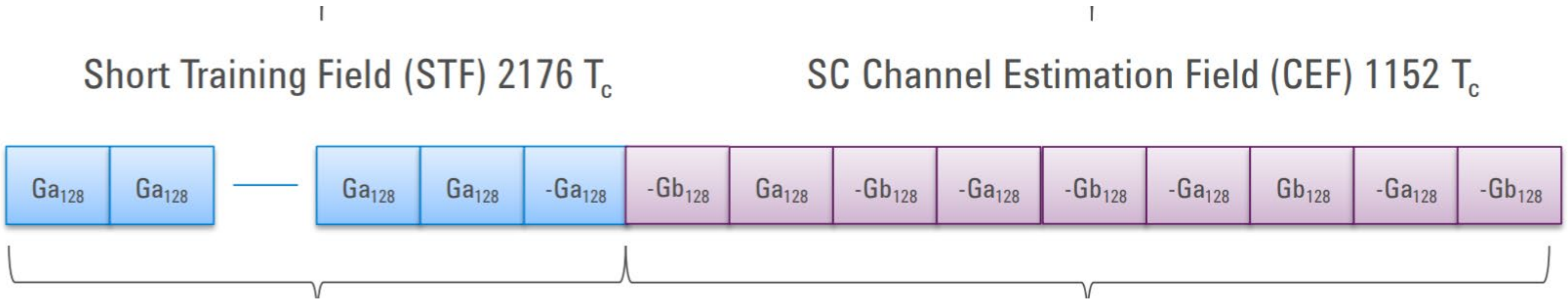
Need to be decoded by all stations. No directional gain

Data packets

Shorter STF

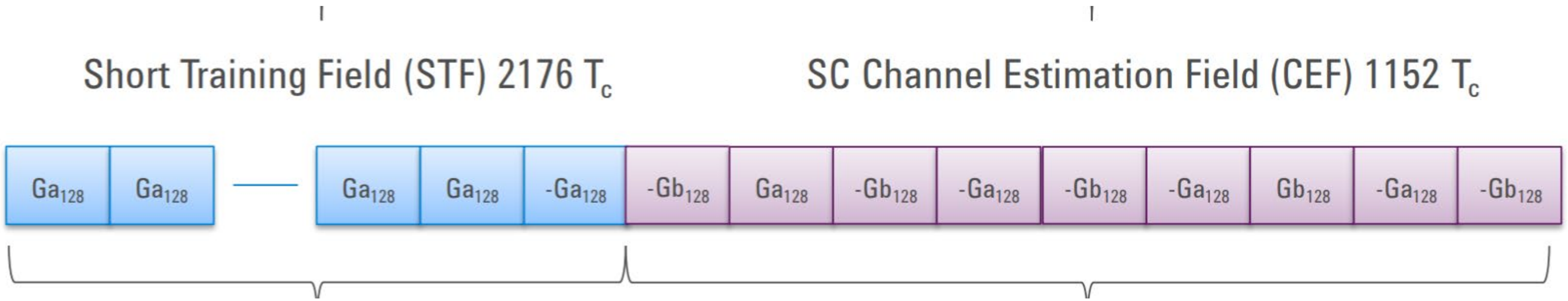
- Based on complementary Golay codes $[Ga, Gb]$
- Have low auto-correlation
- $\frac{1}{T_c} = 1.76$ Gsamp/S
- For data packets: STF = 1.23 μs, CE= 0.654 μs

STF Detection



- MF detector: Correlate with STF
  - Gives good performance but expensive
- Simpler delay detector

Estimation from the CEF



- Known sequence
- Linear estimation problem

Linear Estimation

- Discussed on board in class
- Linear estimation
- Matrix form
- Least squares solution

- This is not on the exam

Outline

- ❑What is Equalization?
- ❑Time-Domain Equalization for Single Carrier Systems
- ❑OFDM TX and RX
- ❑OFDM Channel
- ❑OFDM Synchronization and Channel Estimation

OFDM

- Basic problem: How do we make equalization easy?

- Recall from earlier lectures:
  - For narrowband channels, the channel response in frequency-domain is approximately flat
  - In time-domain, the channel is a single tap
  - No equalization necessary, once we have good synchronization

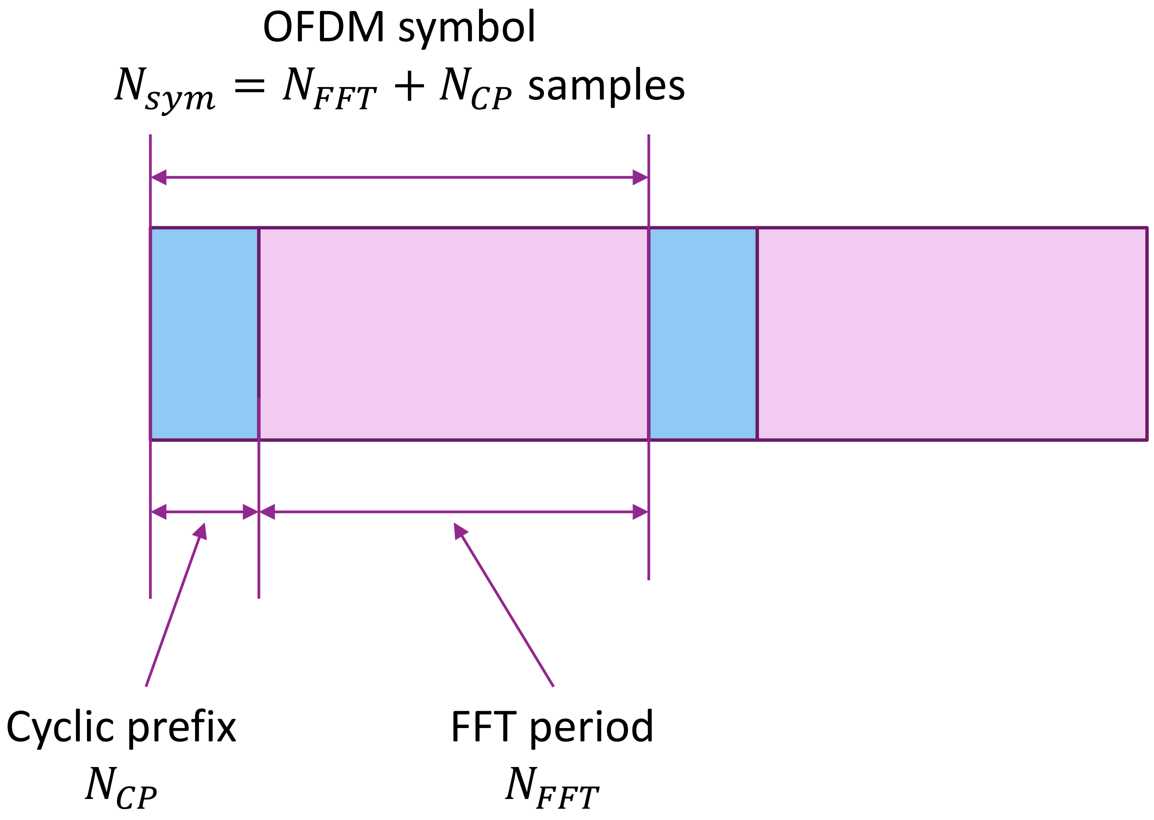
- The OFDM concept:
  - Divide a wideband channel into a set of narrowband channels
  - Each narrowband channel is modulated onto a sub-carrier
  - The frequency of the subcarriers are uniformly spaced

History of OFDM

- First proposed by Chang, Bell Labs,1966
- Many applications:
  - Early digital TV, LER, Paris 1988
  - ETSI Digital Audio Broadcast, 1995
  - ETSI Digital Video Broadcast, DVB-T, 1997
  - DSL, 1998
- Wireless LANs:
  - 802.11a, 1999 and 802.11g, 2002
  - 802.11n, 2004
- Fourth Generation Cellular standards
  - Flash-OFDM, 2001
  - WiMax, LTE
- 5G New Radio
  - Also based on OFDM for now.

OFDM Symbol Structure in Time-Domain

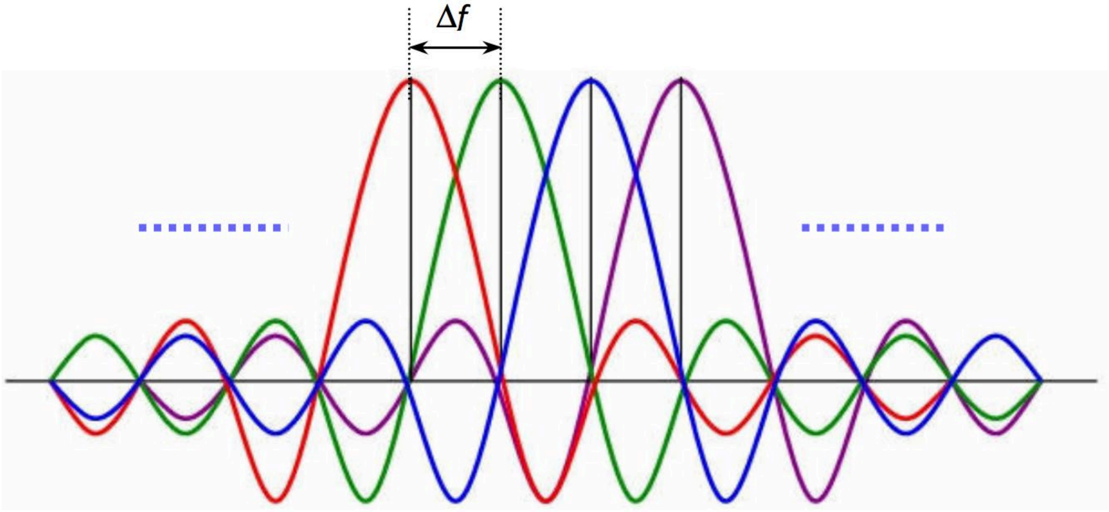
- Time is divided into OFDM symbols
  - Each symbol divided into samples
  - $\frac{1}{T}$= sample rate



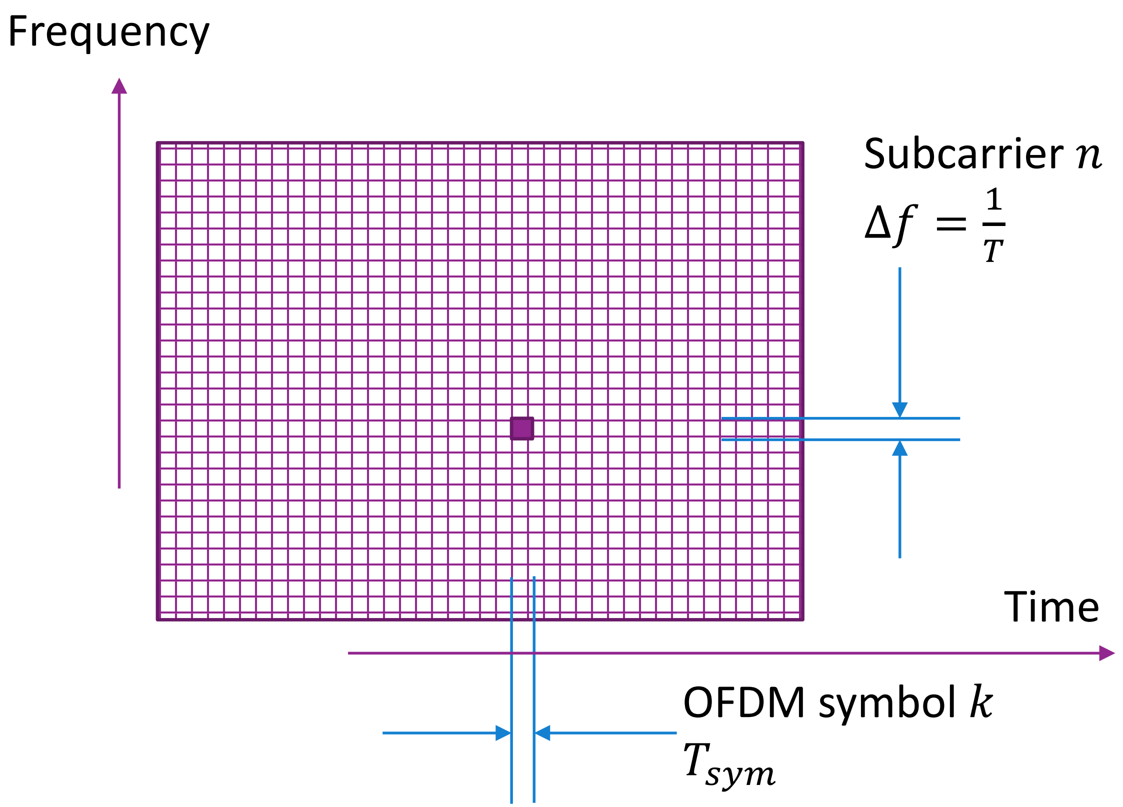
- Each symbol has:
  - FFT period, N_{FFT} samples
  - CP period, N_{CP} samples
  - N_{FFT} = number samples in FFT period
  - N_{CP} = number samples in cyclic prefix
  - $N_{sym} = N_{FFT} + N_{CP}$ = samples per symbol
  - $T_{sym} = TN_{sym}$ =symbol period

OFDM in Frequency Domain

- Frequency is divided into subcarriers
  - N_{FFT} subcarriers
  - Spaced at $\Delta f = \frac{1}{TN_{FFT}}$
- Subcarriers are divided into:
  - N_{SC} occupied subcarriers
  - $N_{FFT} - N_{SC}$ null subcarriers
- Subcarrier frequencies:
  - Digital frequency: $\Omega_n = \frac{2\pi n}{N_{FFT}}$,
  - Analog frequency $f_n = \frac{\Omega}{2\pi T} = n\Delta f$
  - Index $n = -\frac{N_{FFT}}{2} + 1, \ldots, \frac{N_{FFT}}{2}$



OFDM Time-Frequency Grid

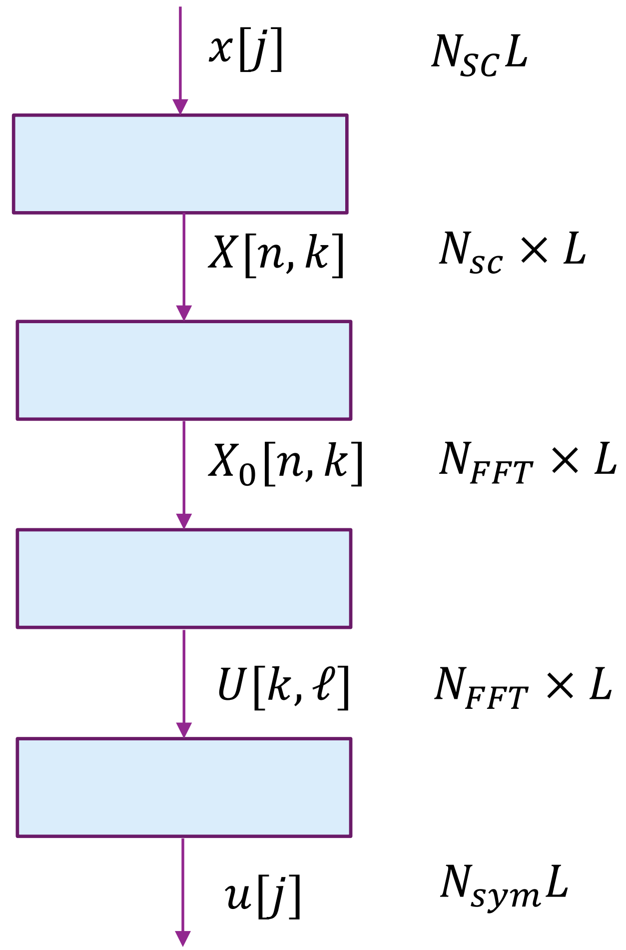


- Data to be transmitted is an array:
  $$X[n,k]$$
  - k = OFDM symbol index
  - n = subcarrier index
- Resource element:
  One time-frequency point
- TX data $X[n,k]$
  - One complex value per RE
  - Called a modulation symbol

OFDM TX Modulation

- Input: $x[j]$: Sequences of modulation symbols
- Parallelize: $X[n,k] = x[kN_{SC} + n]$
- Insert null: $X_0[n,k] = \begin{cases} X_0[k,n] & n \text{ occupied} \\ 0 & n \text{ null} \end{cases}$
- IFFT: $U[\ell,k] = \frac{1}{\sqrt{N}} \sum_n X_0[n,k]\, e^{i\Omega_n \ell}$
- Serialize with CP insertion:
  - CP: $u[kN_{sym} + \ell] = U[k, N_{FFT} - N_{CP} + \ell], \ell = 0, \ldots, N_{CP} - 1$
  - FFT: $u[kN_{sym} + N_{FFT} + \ell] = U[k, \ell], \ell = 0, \ldots, N_{FFT} - 1$

Modulation of L OFDM symbols



OFDM TX Modulation in MATLAB

Manually performing the operations

```
% Parallelize
X = reshape(x,nsc,ns);

% Zero pad and insert X around DC
nl = round(nsc/2);
X0 = zeros(nfft,ns);
X0(1:nl,:) = X(1:nl,:);
X0(nfft-nl+1:nfft,:) = X(nl+1:nsc,:);

% IFFT
U0 = ifft(X0);

% Add CP and serialize
U = [U0(nfft-ncp+1:nfft,:); U0];
u = U(:);
```

Using MATLAB's built-in function

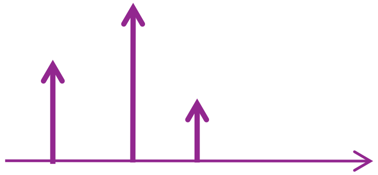
- This is preferable and easier

```
tx = comm.OFDMModulator('FFTLength', nfft, ...
    'CyclicPrefixLength', ncp,...
    'NumGuardBandCarriers', nguard, 'NumSymbols', ns );
u = tx.step(X);
```

Multipath Channel

TX OFDM symbol $u[j]$

Multipath channel



RX OFDM symbol $\mathrm{r}[j]$

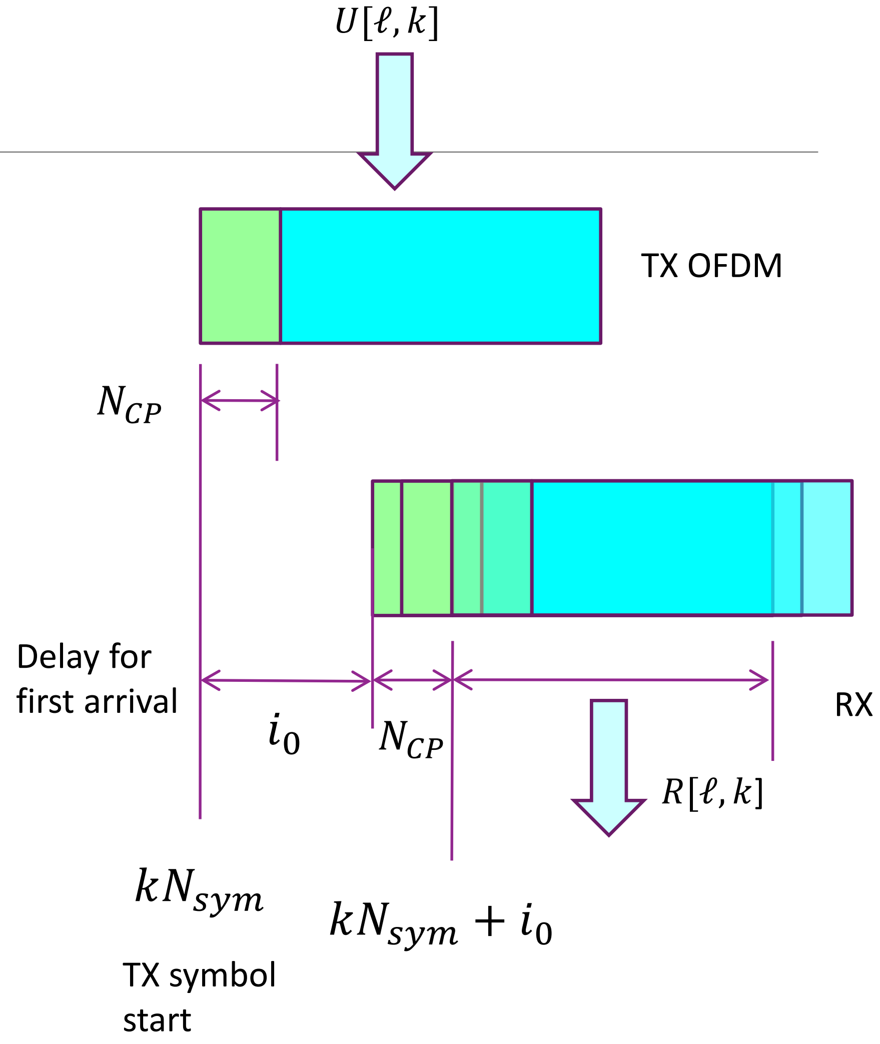
- Now consider a multi-path channel: $r[j] = u[j] * h[j + i_0]$
  - i_0 = Sample of first path
  - Assume $h[j] = 0$ for $j < 0$ and $j \geq N_{CP}$
  - That is, $h[j] \neq 0$ only for $j \in \{0,1,\dots,N_{CP}-1\}$
- Implicitly we have made two key assumptions
  - Delay spread is less than N_{CP} samples = $N_{CP}T$ total length
  - RX is aligned such that first path arrives at $j = 0$

OFDM Receiver

- Find i_0 arrival of first sample of symbol 0
  - Use synchronization signal for example
- Start FFT window after first path + CP period
  - $R[\ell,k] = r\left[kN_{sym} + i_0 + N_{CP} + \ell\right], \ell = 0,1,\ldots,N_{FFT}$
  - Start time=symbol time + first path + CP period
  - Window of length N_{FFT}
- Assume delay spread $\leq N_{CP}$
- Then, there is no ISI
  - RX window for symbol k sees only TX symbol k
- Also, channels acts as a circular convolution:

$$R[\ell,k] = \sum_j h[j]U[\ell - j,k]$$

N_{FFT}



OFDM Receiver FFT

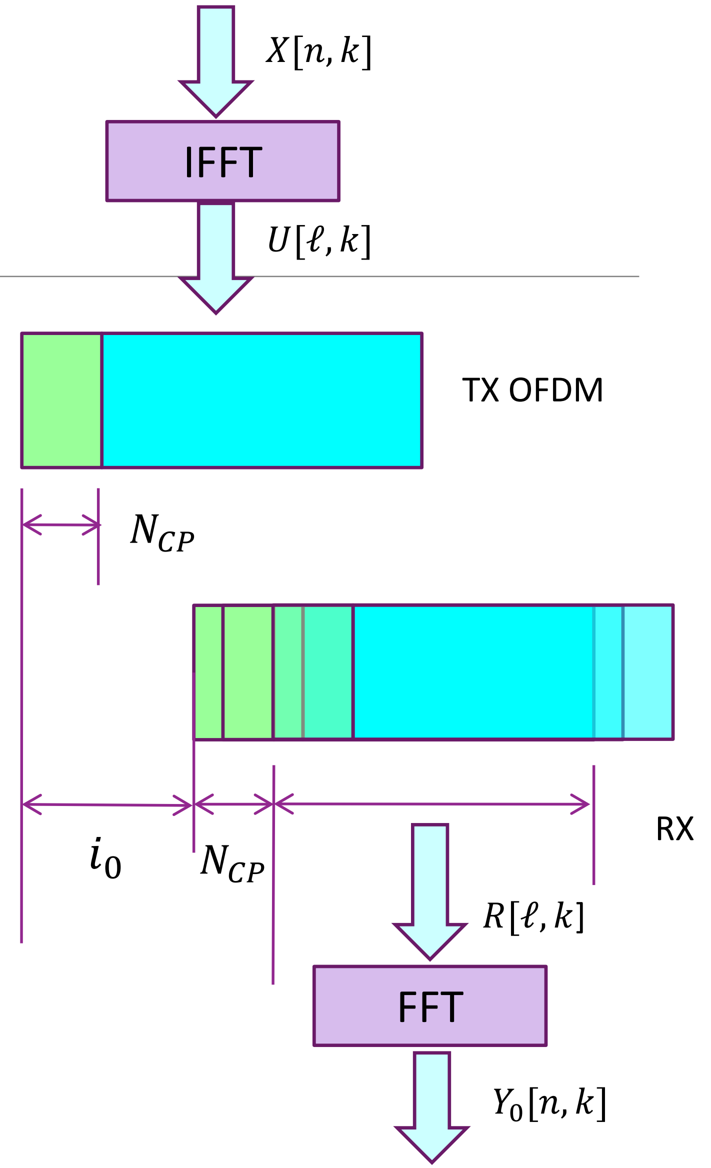
- From previous slide RX symbol is: $R[\ell,k] = \sum_j h[j]U[\ell-j,k]$
- Also, TX used an IFFT: $U[\ell,k] = \frac{1}{\sqrt{N}}\sum_n X_0[n,k]\, e^{i\Omega_n \ell}$
- At RX use an FFT:

$$Y_0[\ell,k] = \frac{1}{\sqrt{N}}\sum_n R[n,k]\, e^{-i\Omega_n \ell}$$

  - Use subscript 0 to indicate that X_0 and Y_0 contain null subcarriers
- Since channel is equivalent to a circular convolution:

$$Y_0[n,k] = H(\Omega_n)X_0[n,k]$$

  - Channel acts as scalar multiplication in each sub-carrier

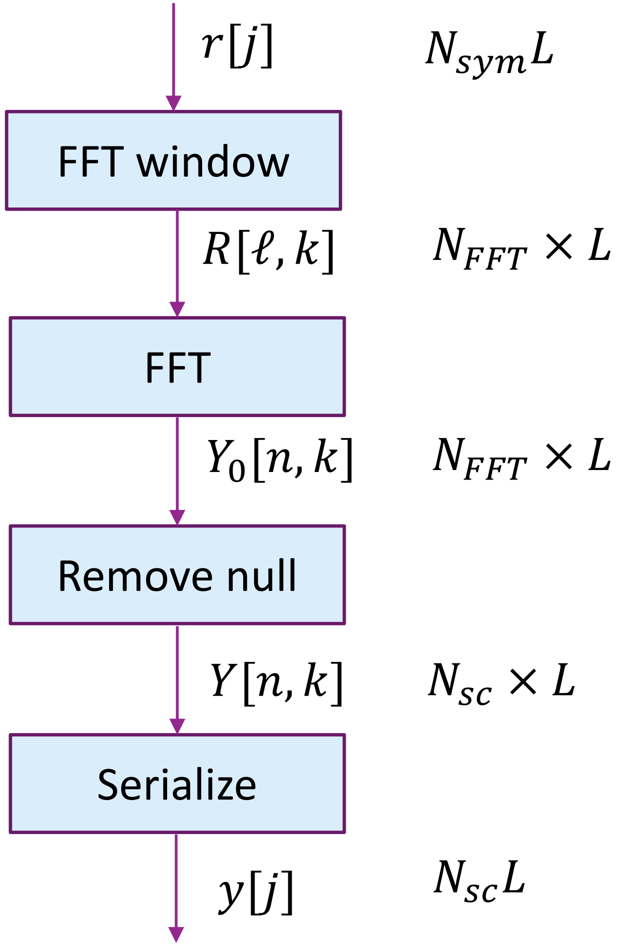


OFDM RX Summary

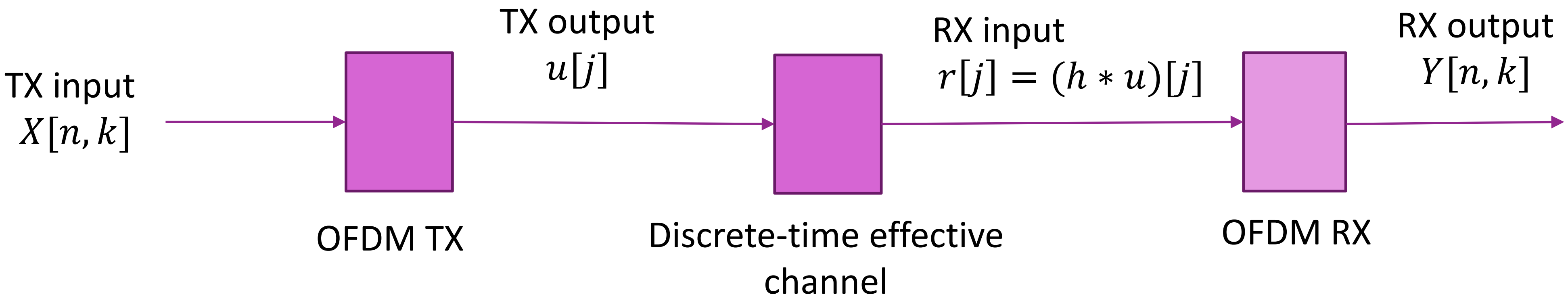
- Find i_0 arrival of first sample of symbol 0
- Take FFT window: $R[\ell, k] = r\left[i_0 + kN_{sym} + N_{CP} + \ell\right]$
  - Note the offsetting
- FFT: $Y_0[n, k] = \frac{1}{\sqrt{N}} \sum_\ell R[\ell, k]\, e^{-i\Omega_n \ell}$
- Remove null-carriers: $Y[n, k] = Y_0[n, k]$,
  - n is an occupied subcarriers
- Serialize:

$$y[kN_{sc} + n] = Y[n, k], \qquad n = 0, \ldots, N_{SC} - 1$$

Demod of L OFDM symbols



Summary



- Key relation: $Y[n,k] = H(\Omega_n)X[n,k]$
- On each subcarrier n, channel is a single complex gain
- No interference between OFDM symbols or sub-carriers
- Equalization made easy!

Degrees of Freedom, Overhead

- Each OFDM symbol has N_{SC} DoF
- Degrees of freedom per second is:

$$R = \frac{DoF\ per\ OFDM\ symbol}{Secs\ per\ OFDM\ symbol} = \frac{DoF}{sec} = \frac{N_{sc}}{N_{sym}T}$$

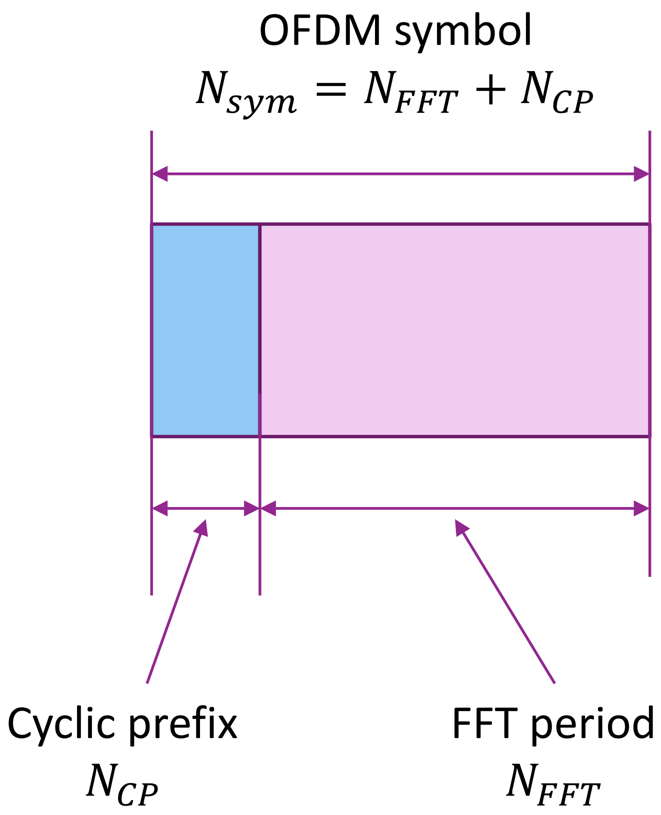
- Occupied bandwidth:

$$B = \frac{N_{sc}}{N_{FFT}T} = max\frac{DoF}{sec}$$

- Fraction Overhead:

$$\alpha = 1 - \frac{R}{B} = 1 - \frac{N_{FFT}}{N_{sym}} = \frac{N_{CP}}{N_{sym}}$$

- Conclusion: The CP is the overhead
  - The price for orthogonality between subcarriers

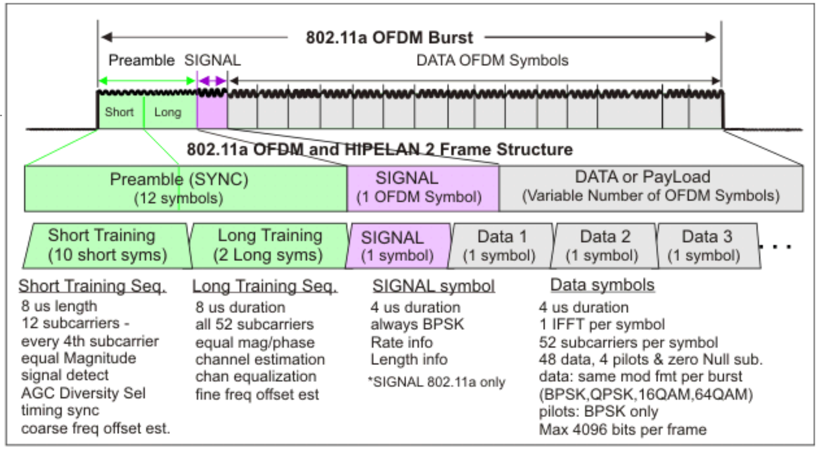


Designing the OFDM Parameters

- Select CP period, T_{CP}
  - Must be sufficiently large for maximum delay spread + timing error
  - But no larger since that results in excess overhead
- Select FFT period T_{FFT}
  - Select subcarrier spacing Δf sufficiently large to prevent inter-carrier interference from Doppler
  - Not discussed in this class (see the wireless class)
  - Then take $T_{FFT} = \frac{1}{\Delta f}$
- Select number of sub-carriers based on available bandwidth $N_{SC} = \frac{B}{\Delta f}$
- Take N_{FFT} to be smallest power of 2 $\geq N_{SC}$
  - Allows easy FFT implementation
- Sample rate = $\frac{1}{T} = N_{FFT}\Delta f$
- Number of samples in CP: $N_{CP} = \frac{T_{CP}}{T}$

Ex: 802.11a/g OFDM

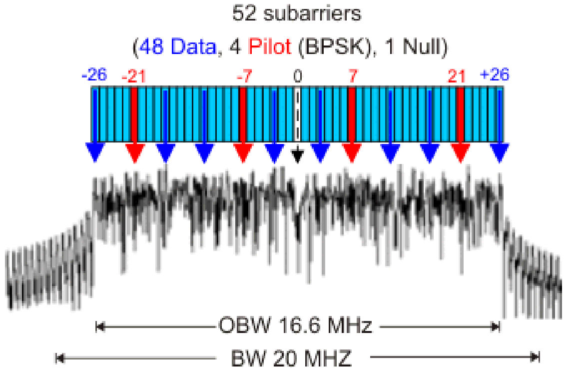
- ❑STF: For detection, AGC
- ❑2 OFDM symbols for initial channel estimate
- ❑4 pilots in remainder of symbols for tracking



Parameter	Value
Total subcarriers N_{ST}	52
Data subcarriers N_{SD}	48
Pilot subcarriers N_{SP}	4 (subcarriers -21, 7, 7, 21)
Subcarrier Frequency Spacing F_{SP}	312.5 KHz (20MHz/64)
Symbol Interval Time T_{SYM}	4 us (T_{GI} +T_{FFT})
Data Interval Time T_{DATA}	3.2 us (1/F_{SP})
Guard Interval (GI) Time T_{GI}	0.8 us (T_{FFT}/4)
IFFT/FFT Period T_{FFT}	3.2 us (1/F_{SP})
SIGNAL Symbol TIme T_{SIGNAL}	4 us (T_{GI} +T_{FFT}
Preamble $T_{PREAMBLE}$	16 us (T_{SHORT} +T_{LONG})
Short Training Sequence T_{SHORT}	8 us (10xT_{FFT}/4)
Long Training Sequence T_{LONG}	8 us (T_{GI2} + 2xT_{FFT})
Training symbol GI T_{GI2}	1.6 us (T_{FFT}/2)
FFT sample size	64 point

802.11a OFDM PHY Parameters

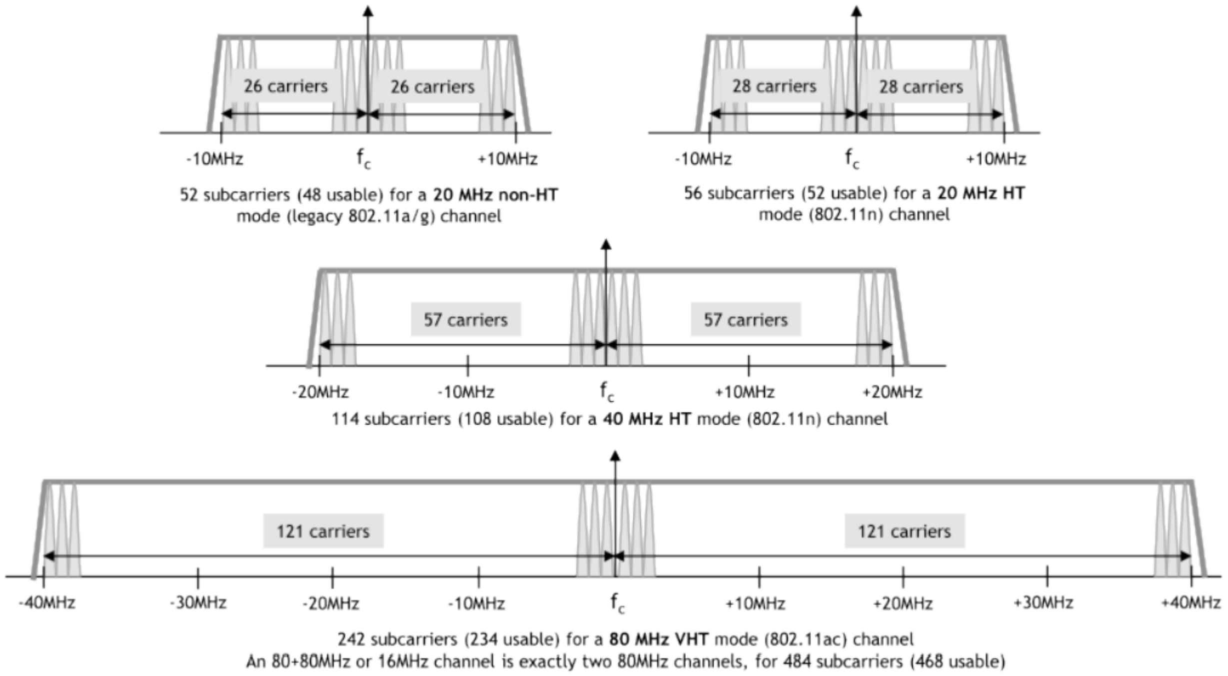
BW	20 MHZ
OBW	16.6 MHZ
Subcarrer Spacing	312.5 Khz (20MHz/64 Pt FFT)
Information Rate	**6**/9/**12**/18/**24**/36/48/54 Mbits/s
Modulation	BPSK, QPSK, 16QAM, 64QAM
Coding Rate	1/2, 2/3, 3/4
Total Subcarriers	52 (Freq Index -26 to +26)
Data Subcarriers	48
Pilot Subcarriers*	4 (-21, -7, +7, +21) *Always BPSK
DC Subcarrier	Null (0 subcarrier)



One Subcarrier = 1 constellation point
1 OFDM symbol = 52 subcarriers
1 OFDM Burst = one or more OFDM symbols

High Throughput 802.11n and 802.11ac

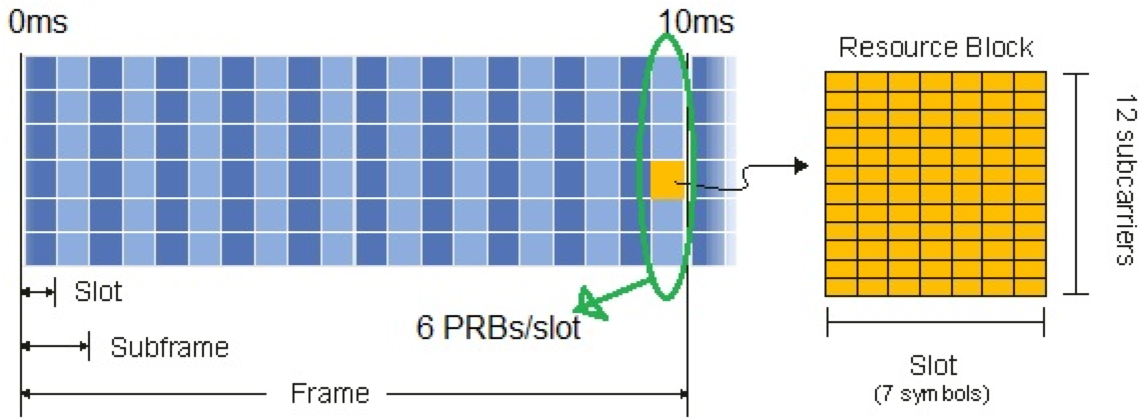
- Increase bandwidth by using more subcarriers
- Subcarrier spacing and CP length are identical



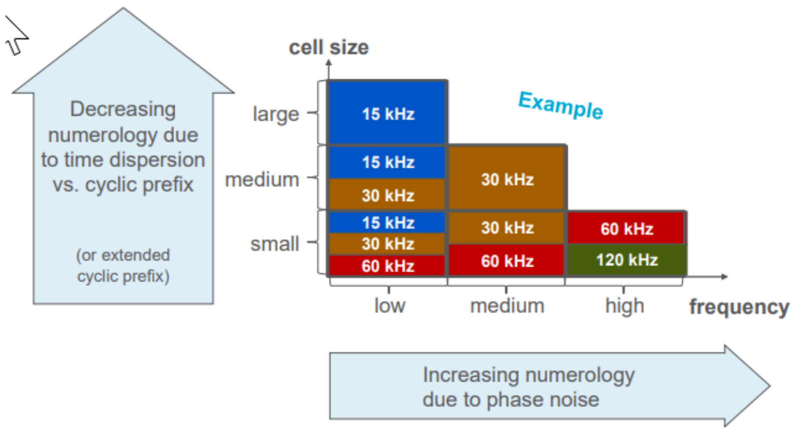
4G LTE

Channel Bandwidth (MHz)	1.4	3	5	10	15	20
Transmission Bandwidth Config. (RB)	6	15	25	50	75	100
Number of Subcarriers	72	180	300	600	900	1200
Occupied Bandwidth (MHz)	1.08	2.7	4.5	9.0	13.5	18.0

- Bandwidth allocated in resource blocks
  - 1 RB = 12 subcarriers
- Δf =15 kHz
- FFT size up to: $N_{FFT} = 2048$
- Sample rate = $N_{FFT}\Delta f$ = 30.72 MHz



5G NR OFDM



Subcarrier spacing	15kHz	30kHz (2 x 15kHz)	60kHz (4 x 15kHz)	15 x 2^nkHz, (n = 3, 4, ...)
OFDM symbol duration	66.67 μs	33.33 μs	16.67 μs	66.67/2^n μs
Cyclic prefix duration	4.69 μs	2.34 μs	1.17 μs	4.69/2^n μs
OFDM symbol including CP	71.35 μs	35.68 μs	17.84 μs	71.35/2^n μs
Number of OFDM symbols per slot	7 or 14	7 or 14	7 or 14	14
Slot duration	500 μs or 1,000 μs	250 μs or 500 μs	125 μs or 250 μs	1,000/2^n μs

- ❑Flexible numerology
- ❑Supports different cell sizes and latency / granularity

5G NR OFDM Details

- Configurable with parameter $\mu = 0,1,2,..,4$.
- Subcarrier spacing: $\Delta f = (15)2^{\mu}$ kHz
- FFT size: Typically $N_{FFT} = 1024, 2048, 4096$
- Occupied subcarriers: $12N_{RB} + 1$ DC
  - $N_{RB} =$ number of "resource blocks"
  - One RB is a group of 12 subcarriers. Basic unit of multiplexing

Parameter / Numerlogy (u)	0	1	2	3	4
Subcarrier Spacing (Khz)	15	30	60	120	240
OFDM Symbol Duration (us)	66.67	33.33	16.67	8.33	4.17
Cyclic Prefix Duration (us)	4.69	2.34	1.17	0.57	0.29
OFDM Symbol including CP (us)	71.35	35.68	17.84	8.92	4.46

Outline

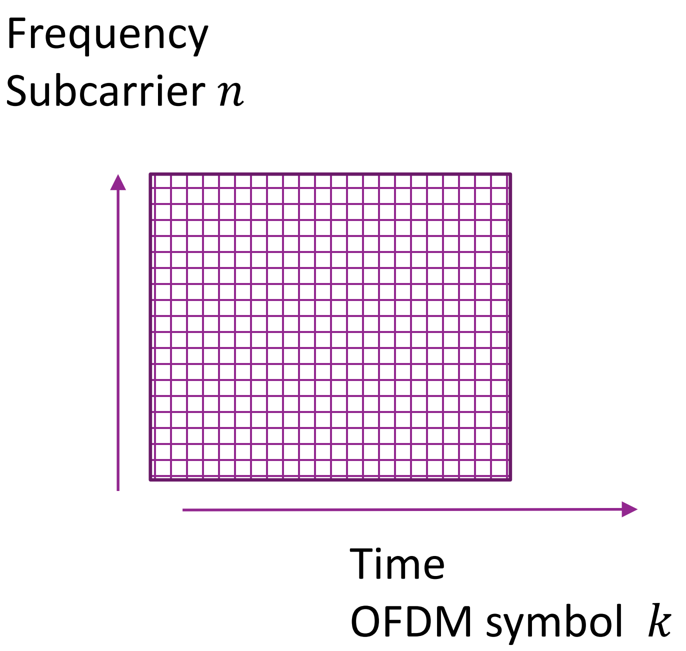
- ❑What is Equalization?
- ❑Time-Domain Equalization for Single Carrier Systems
- ❑OFDM TX and RX
- ❑OFDM Channel
- ❑OFDM Synchronization and Channel Estimation

OFDM Channel

- Key input-output equation:

$$\boxed{Y[n,k] = H[n,k]X[n,k]}$$

- $H[n,k]$ =OFDM channel gain
  - One complex gain at each time-frequency point
- The channel $H[n,k]$ varies:
  - In time: Due to multi-path fading and mobility
  - In frequency: Due to delay spread
- Topic is discussed more in the wireless class
- This class: Look at only variations in frequency



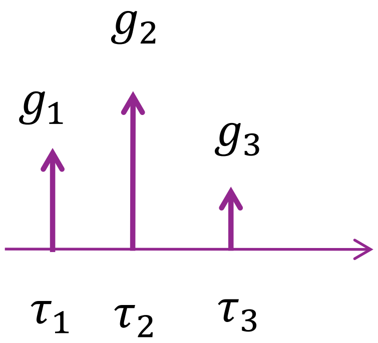
OFDM Multi-Path Channel

- Consider a time-varying multi-path channel:

$$h(t) = \sum_{\ell=1}^{L} g_\ell(t)\delta(t-\tau_\ell)$$

  - $g_\ell(t)$: Path gain. Varies to multi-path fading
  - τ_ℓ: Delay of path. Assume it is constant
  - Note: See wireless class for more about multi-path fading

Multipath channel



- Assumptions:
  - Ideal low-pass filtering
  - $g_\ell(t)$ is approximately constant in each OFDM symbol: $g_\ell[k] = g(kT_{sym})$
  - RX FFT window is aligned
- Then, discrete-time effective channel in symbol k will be approximately:

$$h[j,k] = \sum_{\ell=1}^{L} g_\ell[k]\, sinc\left(j - \frac{\tau_\ell}{T}\right)$$

Variation in Frequency

- Discrete-time time-varying impulse response is:

$$h[j,k] = \sum_{\ell=1}^{L} g_\ell[k]\, sinc\left(j - \frac{\tau_\ell}{T}\right)$$

- Frequency-response in symbol k is:

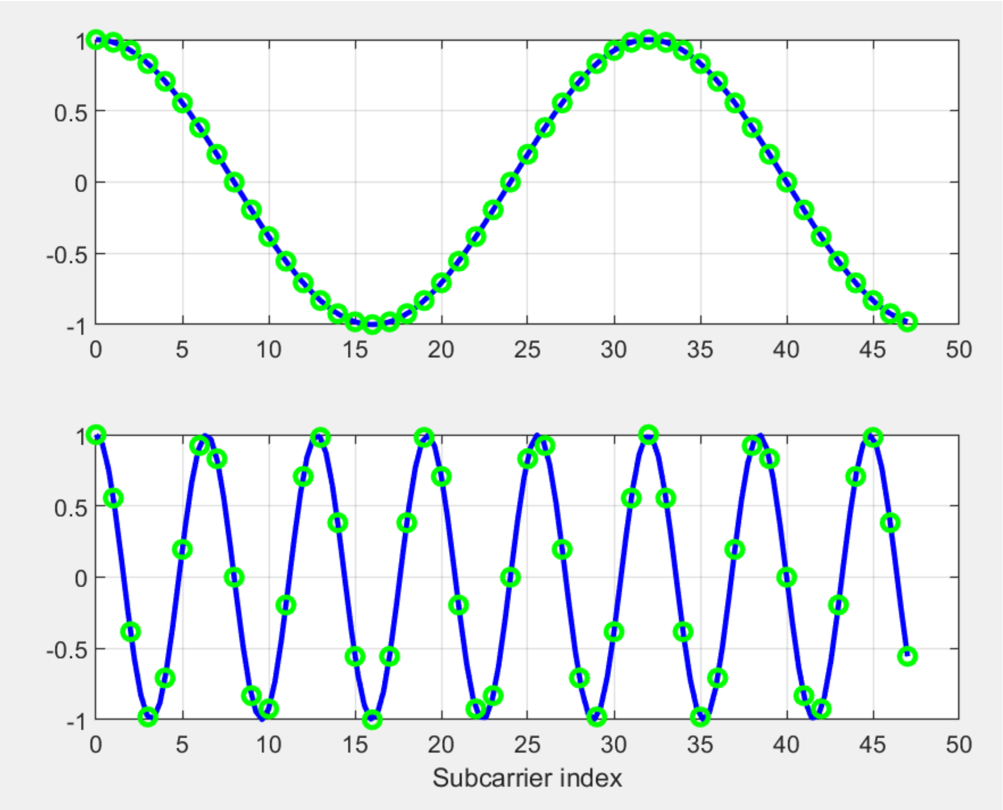
$$H_k(\Omega) = \sum_{\ell=1}^{L} g_\ell[k] e^{-\frac{i\Omega\tau_\ell}{T}}$$

- Time-varying frequency response in :

$$H[n,k] = H_k(\Omega_n) = \sum_{\ell=1}^{L} g_\ell[k] e^{-\frac{i\Omega_n\tau_\ell}{T}} = \sum_{\ell=1}^{L} g_\ell[k] e^{-\frac{i2\pi n\tau_\ell}{TN_{FFT}}}$$

- Each path causes a linear phase rotation by: $\frac{\tau_\ell}{TN_{FFT}}$ per sub-carrier

Frequency Variation: Single Path



$\tau = 0.1\mu s$
$= 2T$

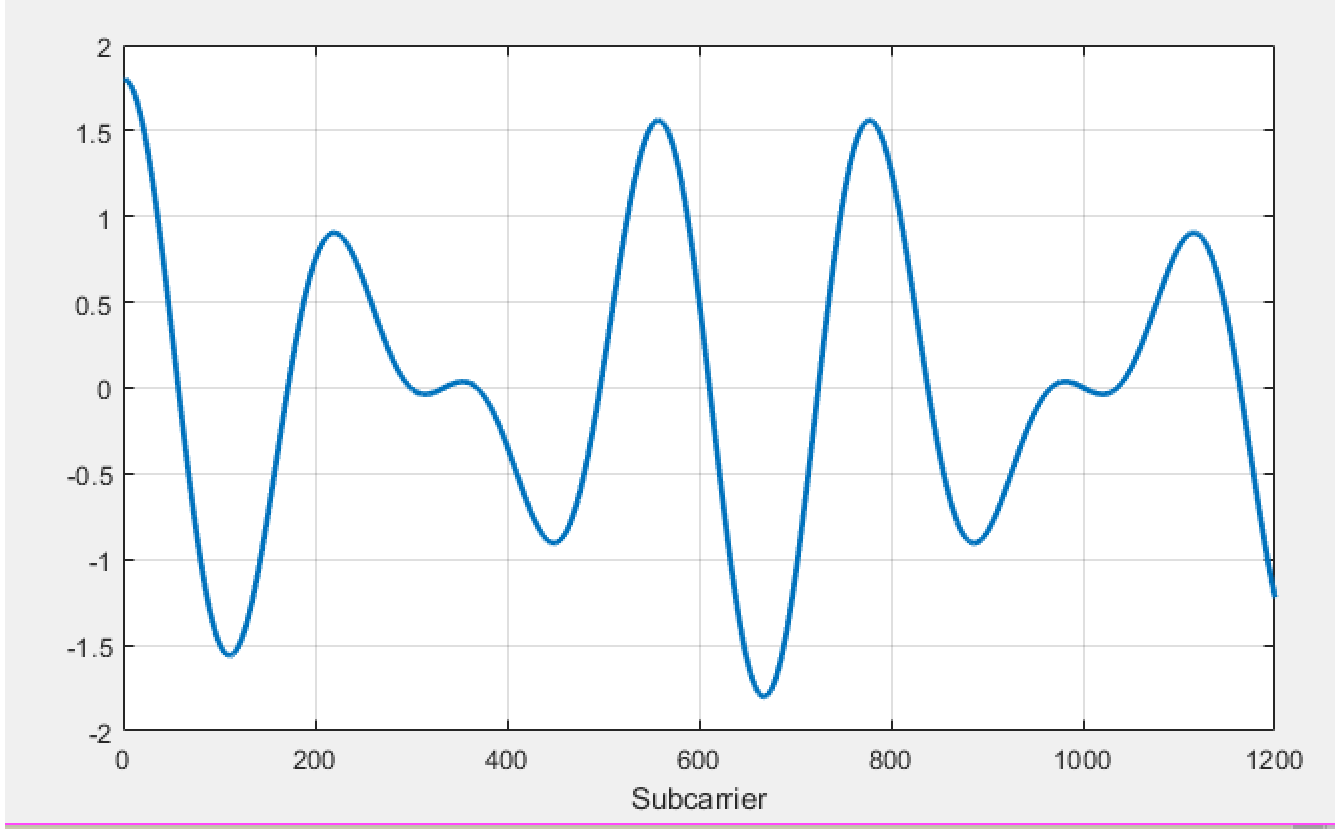
$\tau = 0.5\mu s$
$= 10T$

- Constant single path with delay τ

$$H(\Omega) = e^{-\frac{i\Omega\tau}{T}}$$

- Plotted: Real part of
  - $H(\Omega)$: Blue line
  - Sample $H[n] = H(\Omega_n)$: Green dots
- Parameters:
  - $\frac{1}{T} = 20$ MHz
  - $N_{FFT} = 64$, $N_{SC} = 48$
- Greater delays ⇒ More phase rotation

Freq Variation: Two Path Example



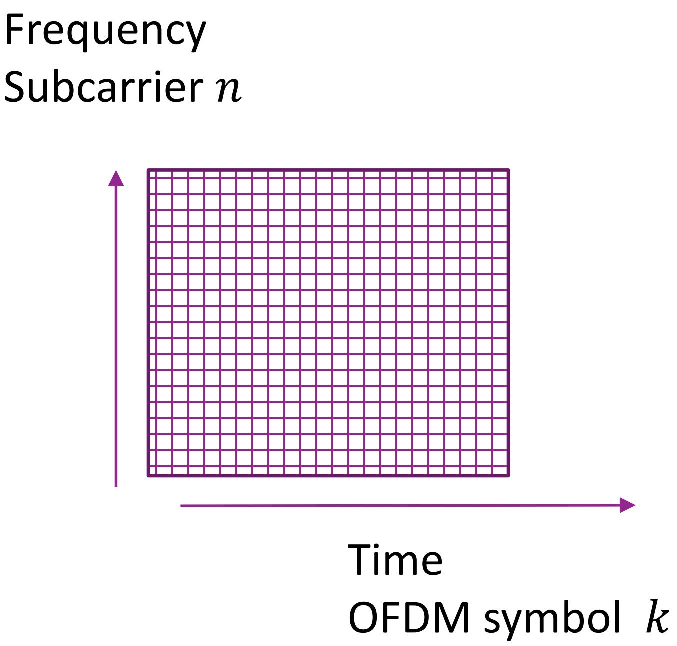
- Plotted is the real of $H[n]$
- OFDM LTE parameters
  - $\Delta f = 15$ kHz
  - N_{sc} =1200
  - $B = N_{sc}\Delta f$=18 MHz occupied BW
- Channel parameters
  - Two path channel
  - Path 1: Delay=0.25 μs, gain=1
  - Path 2: Delay=0.36 μs, gain=0.8

Outline

- ❑What is Equalization?
- ❑Time-Domain Equalization for Single Carrier Systems
- ❑OFDM TX and RX
- ❑OFDM Channel
- ❑OFDM Synchronization and Channel Estimation

OFDM Channel Estimation

- Two key tasks in OFDM
- Synchronization:
  - RX sets position for FFT window
  - Ensure there is no ISI
- Channel estimation:
  - OFDM channel is: $Y[n,k] = H[n,k]X[n,k]$
  - Must estimate channel $H[n,k]$

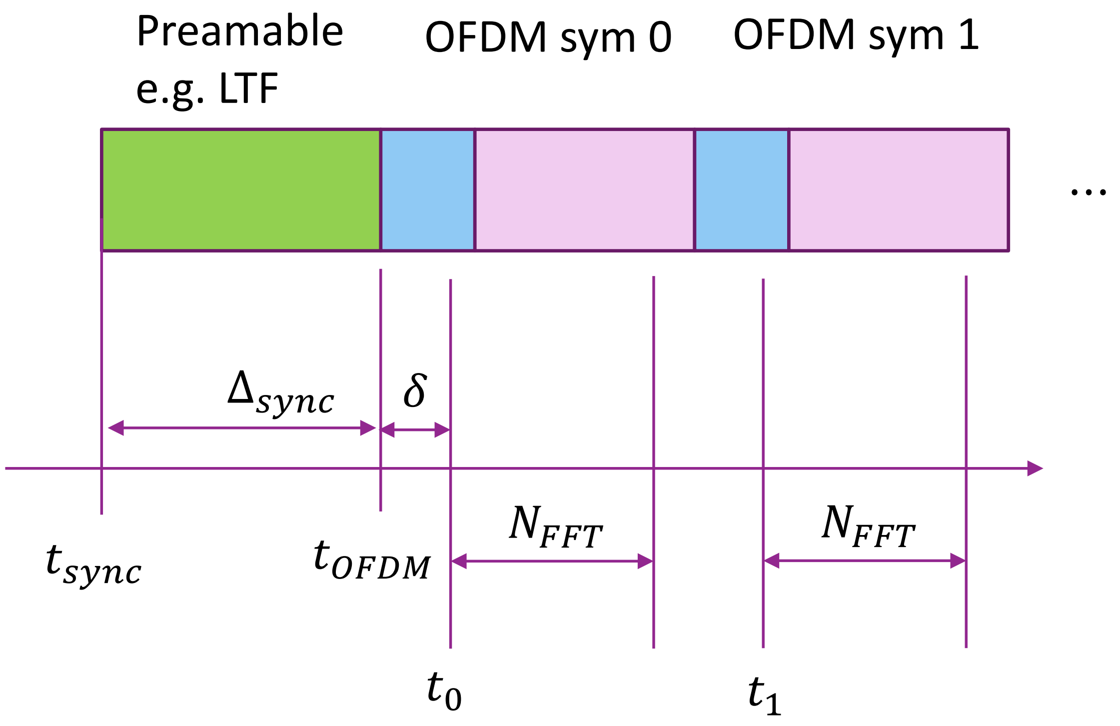


OFDM Synchronization

- Generally use two part synchronization
- Detect time of arrival of some known sync signal
  - Call this time t_{sync}
  - E.g. Long training field in 802.11g
  - Or Primary/Secondary sync signal in LTE
  - Use matched filter or technique from prev lecture
- OFDM symbol start at some known time offset

$$t_{OFDM} = t_{sync} + \Delta_{sync}$$

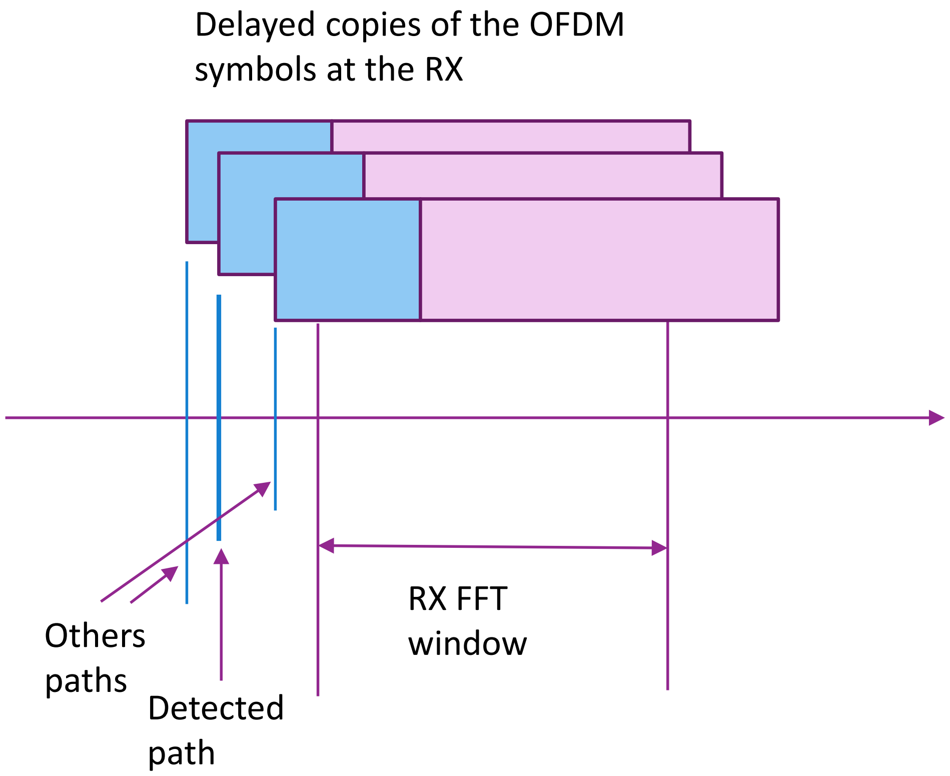
- Start FFT window shortly after t_{OFDM}
  - Some time within one CP
  - Symbol k starts at $t_k = t_{OFDM} + N_{FFT}k + \delta$
  - $\delta \in [0, N_{CP} - 1]$



OFDM Window Alignment with Multi-Path

- ❑ Suppose there is multi-path
- ❑ Pre-amble detection will generally find one path
- ❑ Typically align RX FFT window in the middle of the CP
- ❑ Then, other paths will also be in CP

- ❑ Getting this alignment correct takes some effort
- ❑ In the lab, you will:
  - Detect preamble from LTF in an 802.11g packet
  - Align the RX window for the remaining symbols

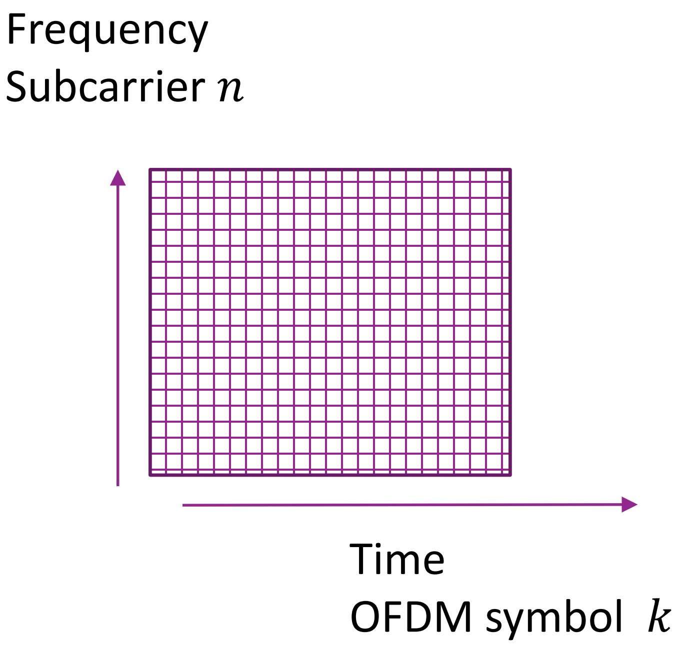


OFDM Channel Estimation

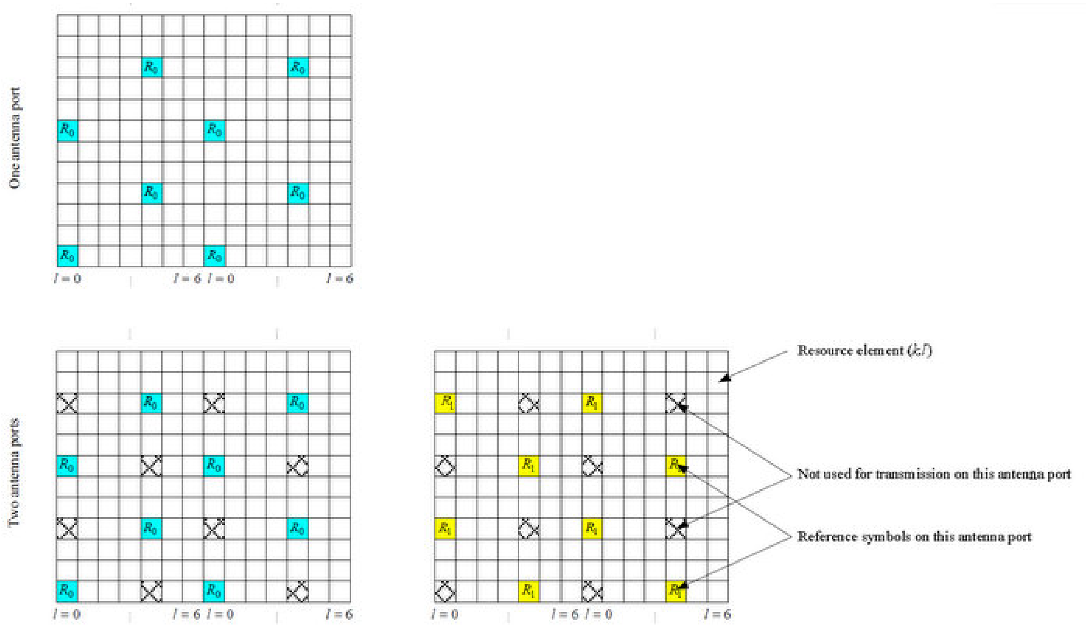
- Once RX window is properly aligned, the channel will be of the form

$$Y[n,k] = H[n,k]X[n,k]$$

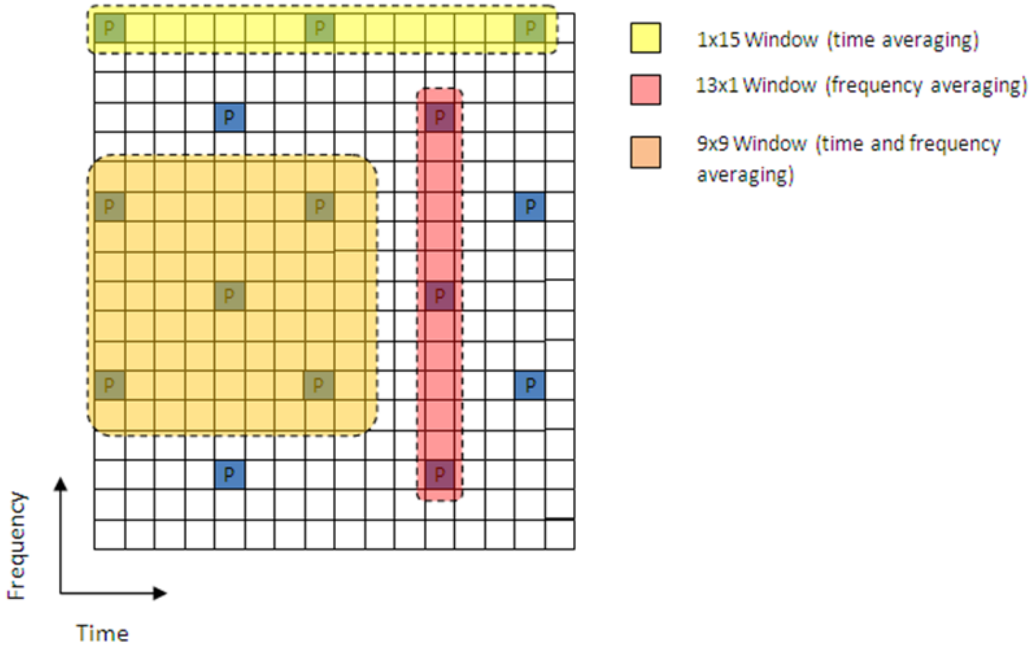
  - $H[n,k]$ =OFDM channel gain
- Channel estimation problem: Estimate $H[n,k]$
- Pilots or reference signals:
  - Transmit known $H[k,n]$ time-frequency positions (k,n)
  - Determine $H[k,n]$ at those locations
  - Interpolate to remainder of (k,n) plane
- Pilots must be spaced based on rate of variation
  - Called coherence time and bandwidth in wireless



Ex: LTE Pilots



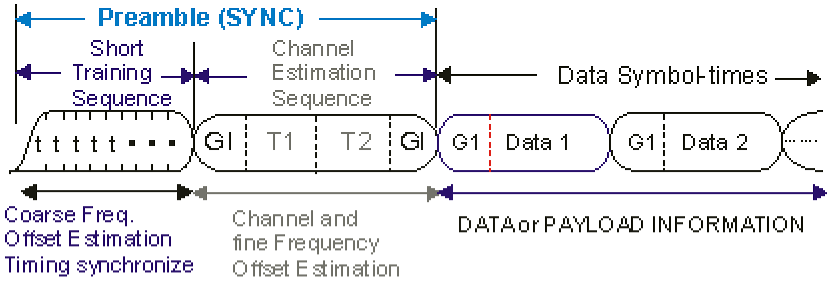
Pilots (Cell Reference signals) in one RB



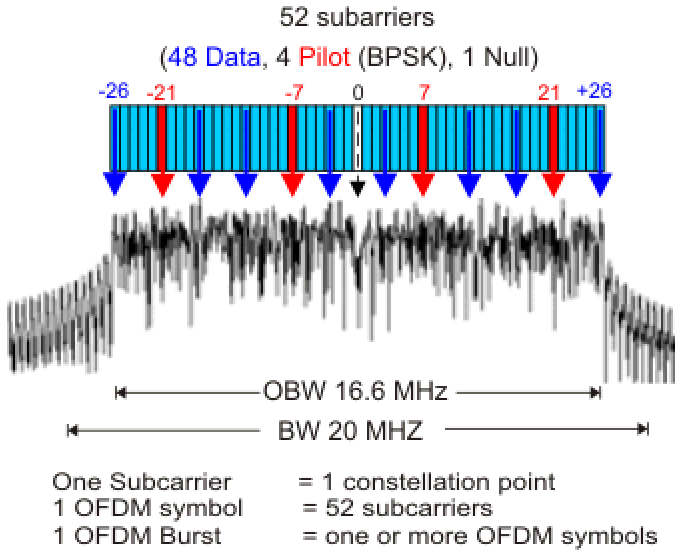
- Can average in time or frequency

Ex: 802.11a/g

- Initial estimate from channel estimation sequence
  - Reference signals on all sub-carriers
- Then tracked over time with pilots
  - Pilots on 4 of 52 sub-carriers



802.11a OFDM PHY Parameters	
BW	20 MHZ
OBW	16.6 MHZ
Subcarrer Spacing	312.5 Khz (20MHz/64 Pt FFT)
Information Rate	**6**/9/**12**/18/**24**/36/48/54 Mbits/s
Modulation	BPSK, QPSK, 16QAM, 64QAM
Coding Rate	1/2, 2/3, 3/4
Total Subcarriers	52 (Freq Index -26 to +26)
Data Subcarriers	48
Pilot Subcarriers*	4 (-21, -7, +7, +21) *Always BPSK
DC Subcarrier	Null (0 subcarrier)



802.11a OFDM Physical Parameters

Estimation Basics

- Problem: We want to estimate a parameter θ from data Y
- In OFDM channel estimation:
  - θ = true channel at some time and frequency $h[n, k]$
  - Y = observed noisy channel responses at reference signals
- An estimator is a function $\hat{\theta} = g(Y)$
  - Input is the observed data, output is the estimate of θ
- Assume Y is some random function of θ
  - There is a pdf: $p(y|\theta)$
- Mean squared error:

$$\mathrm{MSE} = E\left(\left(\hat{\theta} - \theta\right)^2 | \theta\right)$$

  - This is a function of θ, the estimator $g(\cdot)$ and the data Y
  - Can also take an average over θ if θ is random

Bias and Variance

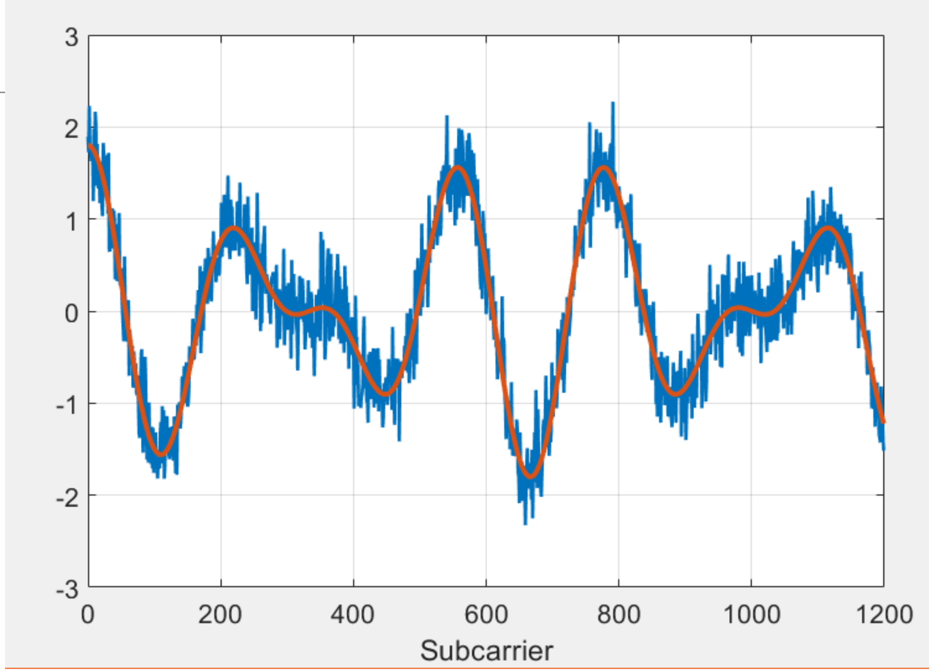
- Fix an unknown parameter θ, data distribution $Y \sim p(y|\theta)$ and an **estimator** $\hat{\theta} = g(Y)$
- Bias: $Bias(\hat{\theta}|\theta) = E[\hat{\theta}|\theta] - \theta$
  - Represents how the estimator is off, on average
- Variance: $Var(\hat{\theta}|\theta) = E\left[\left(\hat{\theta} - E[\hat{\theta}|\theta]\right)^2 \middle| \theta\right]$
  - Represents variation from average
- Both expectations are over Y
- Theorem (Bias-Variance): For any estimator,

$$MSE(\hat{\theta}|\theta) = Bias^2(\hat{\theta}|\theta) + Var(\hat{\theta}|\theta)$$

  - Proof in class
- As we will see below, there is a tradeoff in bias and variance

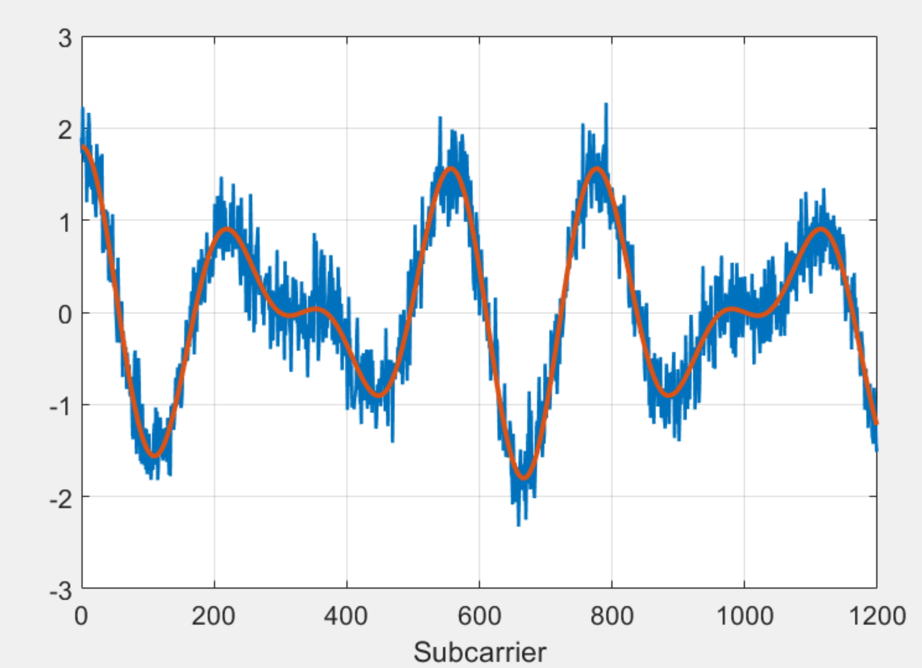
Raw Channel Estimate

- Consider estimation over frequency
  - Ignore time for now
- We have $y[n] = h[n]x[n] + w[n]$
  - $h[n]$ =unknown channel gain
  - Say we know $x[n]$ for all n (e.g. like CE field in 802.11g)
  - Assume $w[n] \sim CN(0, N_0)$, $|x[n]|^2 = E_x$ for all n
- Consider raw channel estimate: $\hat{h}_0[n] = \frac{y[n]}{x[n]}$
- Plotted to the right:
  - Two path channel from before
  - OFDM system with 1200 subcarriers (LTE 20 MHz channel)
  - SNR = 10 dB



Bias and Variance of Raw Channel Estimate

- We have $y[n] = h[n]x[n] + w[n]$
  - $w[n] \sim CN(0, N_0)$, $|x[n]|^2 = E_x$ for all n
- Raw channel estimate: $\hat{h}_0[n] = \dfrac{y[n]}{x[n]}$
- $E\left(\hat{h}_0[n]\right) = h[n] + E\left(\dfrac{w[n]}{x[n]}\right) = h[n]$
- $Bias\left(\hat{h}_0[n]\right) = E\left(\hat{h}_0[n]\right) - h[n] = 0$
  - Estimate is unbiased
- $Var\left(\hat{h}_0[n]\right) = \dfrac{1}{|x[n]|^2} var(w[n]) = \dfrac{N_0}{E_x}$
  - Inverse of the SNR
- $MSE\left(\hat{h}_0[n]\right) = Bias^2 + Var = \dfrac{N_0}{E_x}$



Adding Averaging

- Can we reduce the noise in the raw channel estimate
- Simple idea: Use averaging!

$$\hat{h}[n] = \frac{1}{2L+1} \sum_{\ell=-L}^{\ell=L} \hat{h}_0[n+\ell]$$

  - Average over window of length $2L + 1$
- Does this help?
- Compute the bias and variance
- To compute bias and variance: $\hat{h}_0[n] = h[n] + v[n], \;\; v[n] = \frac{w[n]}{x[n]}$
  - Since $\mathrm{w[n]} \sim \mathrm{CN}(0, \mathrm{N}_0)$, we have $v[n] \sim CN(0, \gamma^{-1}), \;\; \gamma = E_x/N_0$
- Bias:

$$Bias\left(\hat{h}[n]\right) = E\left(\hat{h}[n]\right) - h[n] = \frac{1}{2L+1} \sum_{\ell=-L}^{\ell=L} E\left(\hat{h}_0[n+\ell]\right) - h[n] =$$

Variance of the Averaged Estimate

- Average estimate: $\hat{h}[n] = \frac{1}{2L+1}\sum_{\ell=-L}^{\ell=L} \hat{h}_0[n+\ell]$
- Raw estimate: $\hat{h}_0[n] = h[n] + v[n], v[n] \sim CN(0, \gamma^{-1}),\ \gamma = E_x/N_0$
- Variance:
  - The noise terms $v[n]$ are independent
  - The terms $h[n]$ are considered fixed.
  - Hence,

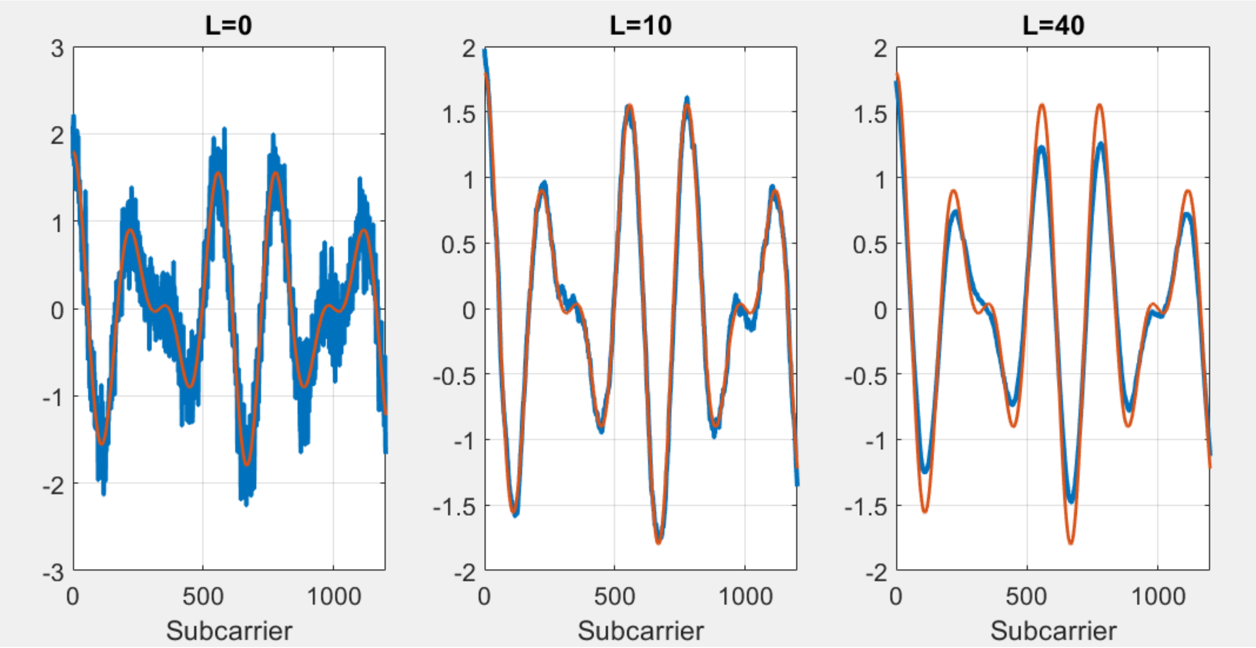
$$\begin{aligned}\mathrm{Var}\big(\hat{h}[n]\big) &= \frac{1}{(2L+1)^2}\sum_{\ell=-L}^{\ell=L} var(\hat{h}_0[n+\ell]) \\ &= \frac{1}{(2L+1)^2}\sum_{\ell=-L}^{\ell=L} var(v[n+\ell]) = \frac{(2L+1)\gamma^{-1}}{(2L+1)^2} = \frac{\gamma^{-1}}{2L+1}\end{aligned}$$

- Conclusion: Variance decreases with window length
  - Enables more averaging

Bias of the Averaged Estimate

- Average estimate: $\hat{h}[n] = \frac{1}{2L+1}\sum_{\ell=-L}^{\ell=L} \hat{h}_0[n+\ell]$
- Raw estimate: $\hat{h}_0[n] = h[n] + v[n], v[n] \sim CN(0, \gamma^{-1}),\ \gamma = E_x/N_0$
- Bias:
  - $Bias(\hat{h}[n]) = E(\hat{h}[n]) - h[n] = \frac{1}{2L+1}\sum_{\ell=-L}^{\ell=L} E(\hat{h}_0[n+\ell]) - h[n] = \frac{1}{2L+1}\sum_{\ell=-L}^{\ell=L}(h[n+\ell] - h[n])$
  - Bias depends on terms $h[n+\ell] - h[n]$
- Bias increases as window length L increases
  - On average, $h[n+\ell] - h[n]$ increases with $|\ell|$
  - Difference in two frequencies
  - Averaging over larger frequency windows averages over more distant frequencies $h[n+\ell]$
  - These may not be related to $h[n]$

Bias-Variance Illustrated



High variance

Not enough averaging

High bias

Over-smoothed

- ❑Two path channel as before
- ❑Different window lengths

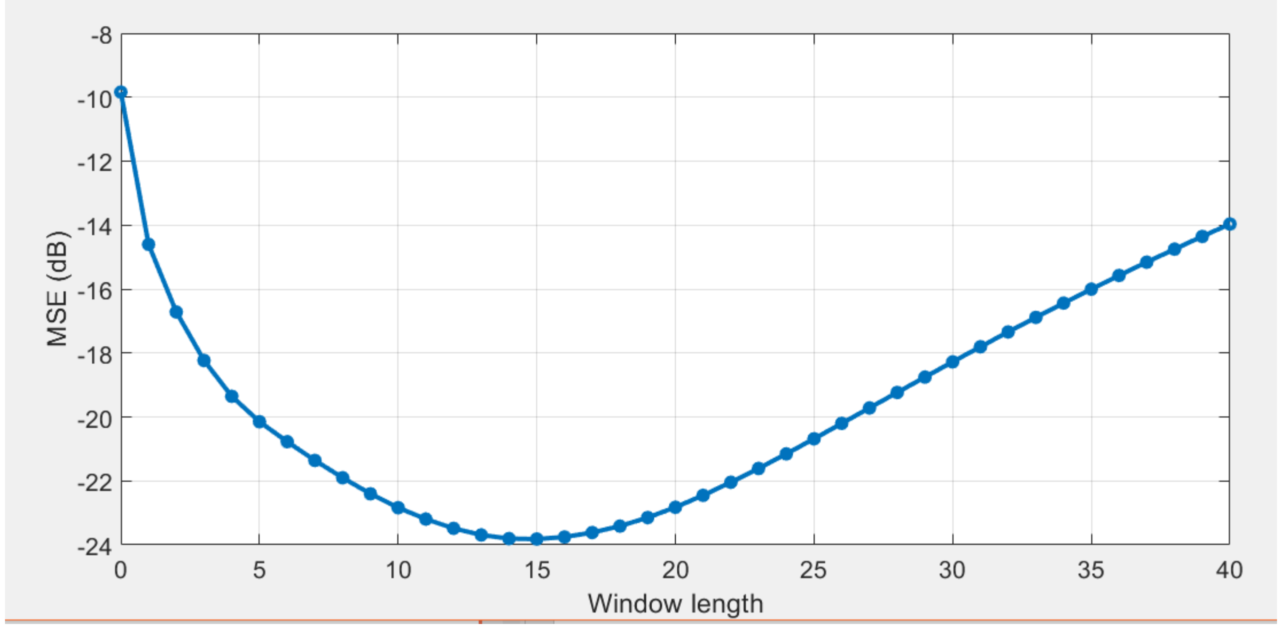
Optimizing Bias-Variance Tradeoff

High variance error

Too little averaging

High bias error

Too much averaging



- Plotted: The MSE vs. window length
- Normalized MSE:
  - $10 \log_{10} \frac{E\left|h[n]-\hat{h}[n]\right|^2}{E|h[n]|^2}$
- In reality, we would:
  - Average performance over ensemble of channels
  - Consider more complex estimators