Linear Algebra : Gilbert Strang

@) Lec-1:

#) Outline:

1) n-Linear eq.^{n}s, n-unknowns
2) Row picture
3) Column picture
4) Matrix form [AX = B]

#) n-Linear eq.^{n}s, n-unknowns:

*) When 2 eq.^{n}s & 2 unknowns are there:

Ex: $2x - y = 0$
$-x + 2y = 3$

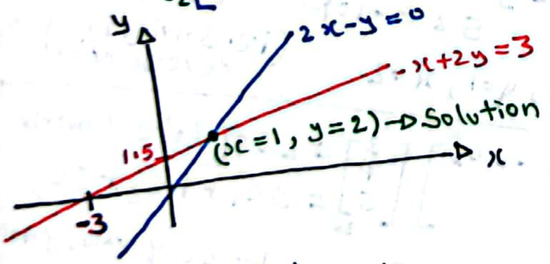
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

coefficient matrix

$AX = B$; Matrix form of Linear eq.^{n}s

*) Row picture:
Linear eq.^{n}s ≡ Linear combination of rows

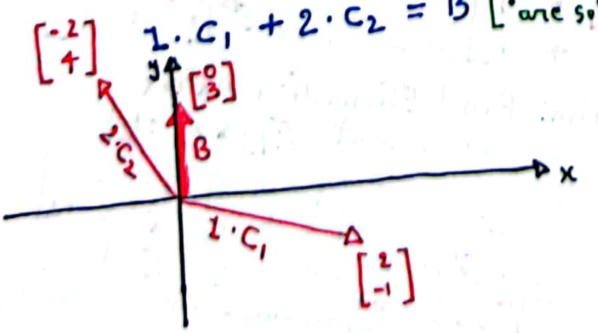
Ex: $\begin{array}{c} r_1 \\ r_2 \end{array}\begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix} = \underset{B}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$



*) Column picture:
Linear eq.^{n}s ≡ Linear comb.^{n} of columns

Ex: $x\underset{C_1}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} + y\underset{C_2}{\begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}$

$1 \cdot C_1 + 2 \cdot C_2 = B$ [∵ $x=1, y=2$ are solutions]



*) When 3-eq.^{n}s & 3-unknowns are there:-

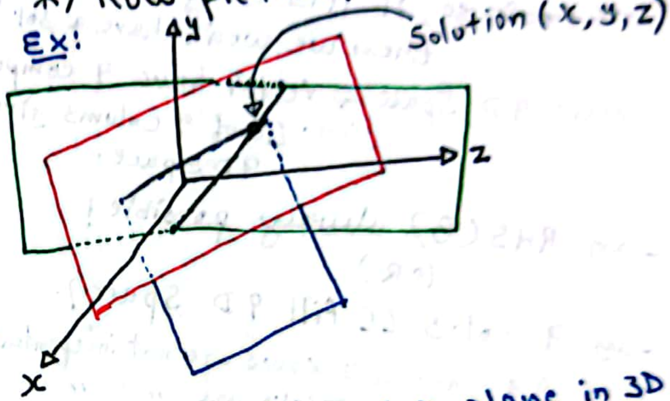
Ex: $2x - y + 0 \cdot z = 0$
$-x + 2y - z = -1$
$0 \cdot x - 3y + 4z = 4$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

$AX = B$; Matrix form of Linear eq.^{n}s

*) Row picture:

Ex:

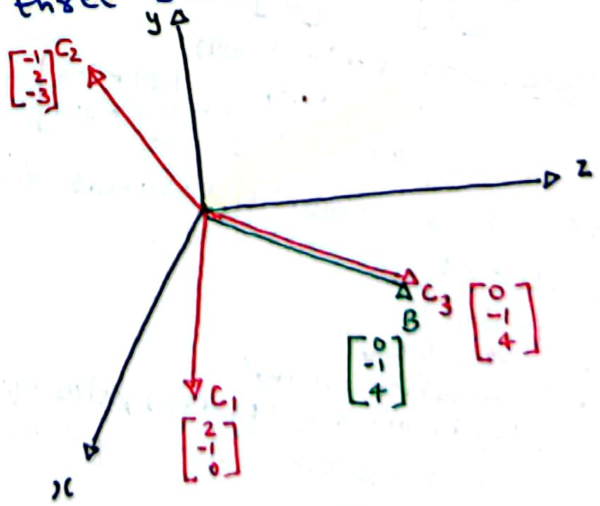


- Each row gives us a plane in 3D space
- 2 planes meet in a line
- 3 planes meet in a point (and that is our solution)
- Row picture giving us vague idea.

*) Column picture:

Ex: $x\underset{C_1}{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}} + y\underset{C_2}{\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}} + z\underset{C_3}{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}$

Our solution is a linear comb.^{n} of three 3-Dim. vectors.



$0 \cdot C_1 + 0 \cdot C_2 + 1 \cdot C_3 = B$

$\Rightarrow (x, y, z) \equiv (0, 0, 1) \equiv$ Sol.^{n} ≡ Point where 3 planes meet

If RHS, i.e. B, would've been $\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ then

$1C_1 + 1C_2 + 0C_3 = B$

$\Rightarrow (x, y, z) \equiv (1, 1, 0) \equiv$ Sol.^{n} ≡ Point where 3 planes meet

i.e. C_1, C_2, C_3 same, just the LC has changed

Q:> Can I solve $AX=B$ for every B? Is there a solution?

(OR)

Do the LC's of the columns fill up the entire 3-Dim. space?

<u>Ans</u>: $AX \equiv$ Linear comb.^{n} of col.s

For that, A has to be non-singular.

If the 3 col.s lie in the same plane, then comb^n's will also lie in the same plane. And LC will not give 3D space.

Therefore, if A is singular (or non-invertible) then we won't have a sol.^{n}

Q:> Given: 9D Space ⇒ vector have 9 components i.e. LC of 9 colums gives us 9D space.

can RHS (B) always possible?

(OR)

Can any 9 col.s LC fill 9D space?

<u>Ans</u>: If $A \equiv$ Singular ≡ rows are not independent or col's are " " then we can't find a sol.^{n}

#> Matrix form: $[AX = B]$

- $A * X = B$

⇒ matrix * vector = vector

- Ex: $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$

<u>Idea-1</u>: LC of col.s way

$1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 3 \end{bmatrix}$

<u>Idea-2</u>: Dot product

$\begin{bmatrix} \text{row 1} \cdot \text{col} \\ \text{row 2} \cdot \text{col} \end{bmatrix}$ i.e. $\begin{bmatrix} 2\cdot1 + 5\cdot2 \\ 1\cdot1 + 3\cdot2 \end{bmatrix}$

- AX is a comb.^{n} of columns of A

@> Lec-2:

#> Outline:

- System of eq.^{n}s
- Elimination method (Way in which S/W packages solves eq.^{n}s)

#> Given: A system of Linear eq.^{n}s:-

- $x + 2y + z = 2$
  $3x + 8y + z = 12$
  $0x + 4y + z = 2$

1st pivot; 2nd pivot; 3 pivots ⇒ Success

$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} = U$

2 eq.^{n}s in y & z; upper Δ matrix

- Pivots can't be 0
- Purpose of elimination: $A \rightarrow U$ (upper Δ matrix)

#> Failure of elimination: (can't get 3 pivots)

- If 1st pivot is 0 then exchange rows. (if we have a non-zero below zero pivot)
- But failure arrives when we have zero below zero pivot

<u>Ex</u>: $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & -4 \end{bmatrix} \rightarrow \; \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

only 2 pivots ⇒ Failure

#> Back Substitution:

<u>Ex</u>: $[A : B] \equiv$ Augmented matrix

row echelon

$\begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 3 & 8 & 1 & \vdots & 12 \\ 0 & 4 & 1 & \vdots & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 2 & -2 & \vdots & 6 \\ 0 & 4 & 1 & \vdots & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & \vdots & 2 \\ 0 & 2 & -2 & \vdots & 6 \\ 0 & 0 & 5 & \vdots & -10 \end{bmatrix}$

A B U C

i.e. $(A \rightarrow U, B \rightarrow C)$

⇒ $x + 2y + z = 2$
$2y - 2z = 6$
$5z = -10$

Back substituting values (i.e. solving in reverse order)

⇒ $z = -2, y = 1, x = 2$

<u>Note</u>:

- $\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}_{3\times1}$ = LC of col.s of matrix $= 3c_1 + 4c_2 + 5c_3$
- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times3}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix}$ = LC of rows of matrix $= 1\cdot r_1 + 2r_2 + 3r_3$

#> Elimination Matrices (E_{ij}):

- $A = \begin{bmatrix} 1 & 2 & 1 \\ \boxed{3} & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow[\text{Trying for rref}]{r_2 \rightarrow r_2 - 3r_1} \begin{bmatrix} 1 & 2 & 1 \\ \boxed{0} & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = A'$

Aim: Find Elimination matrix such that $E_{ij} * A = U$

E_{21} Step

$\begin{bmatrix} E_{21} \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ \boxed{0} & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

bec. trying to make zero at 2,1 position

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

Bec. taking 1 of Row1 & none of the other rows

Bec. $r_2 = r_2 - 3r_1$

Bec. taking 1 of row3 & none of the other rows

$\Rightarrow E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ & it takes my matrix

from $\begin{bmatrix} 1 & 2 & 1 \\ \boxed{3} & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$ to $\begin{bmatrix} 1 & 2 & 1 \\ \boxed{0} & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$

E_{31} Step

Now, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ \boxed{0} & 4 & 1 \end{bmatrix}$

Since already at 3,1 position 0 is there
$\Rightarrow$ No need of E_{31} as $E_{31} = I_{3\times3}$

E_{32} Step

Now, $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{4} & 1 \end{bmatrix} \xrightarrow{r_3 \to r_3 - 2r_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{0} & 5 \end{bmatrix}$

$$\begin{bmatrix} & E_{32} & \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{0} & 5 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \times \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & \boxed{0} & 5 \end{bmatrix} = U$$

- Therefore; $E_{ij} \times A = U$ Achieved

  i.e. $E_{32} \times (E_{21} \times A) = U$

  $(E_{32} \times E_{21}) \times A = U$ [$\because$ Associative Law]

  $\Rightarrow E_{ij} = (E_{32} \times E_{21}) =$ Elimination Matrix

#) Permutation Matrices (P_{ij}):

- If you want to exchange some rows or some columns, then these matrices are used.
- Ex: Exchange of rows:-

$$\begin{bmatrix} P_{ij} \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

P_{ij} on Left $\Rightarrow$ row operation $\Rightarrow 0 \cdot r_1 + 1 \cdot r_2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- Ex: Exchange of columns :-

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

P_{ij} on Right $\Rightarrow$ Col.^{m} operation $\Rightarrow 0 \cdot c_1 + 1 \cdot c_2$

#) Inverse Matrix:

- The matrix which will undo my row or column exchange operation
- Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

  operation: $r_2 = r_2 - 3r_1$

  $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$ (say)

  Inverse operation: $r_2 = r_2 + 3r_1$

  $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A^{-1}$

  i.e.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A^{-1}} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$$

- If A^{-1} exists, then $A^{-1} \cdot A = I$
  & $A \equiv$ invertible $\equiv$ Non-singular
- Commutativity:

  $A \cdot A^{-1} = I = A^{-1} \cdot A$

  For square matrices; Left inverse = Right inverse
  i.e. Yes commutative

  For Rectangular matrices; Left inverse $\neq$ Right inverse
  i.e. Not commutative

#) Multiplication of 2 matrices:

*) Regular way:-

- Dot product; row. column

$$\underbrace{\begin{bmatrix} \\ r_3 \; a_{31} \;\cdots \\ \\ \end{bmatrix}}_{A} * \underbrace{\begin{bmatrix} & c_4 & \\ & b_{14} & \\ & \vdots & \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} & & \\ & & \boxed{\;} \\ & & \end{bmatrix}}_{C}$$

$c_{34} = r_3 \cdot c_4$

$c_{34} = a_{31} \cdot b_{14} + a_{32} \cdot b_{24} + \cdots$

$c_{34} = \sum_{k=1}^{n} a_{3k} \cdot b_{k4}$

- $\underset{m\times n}{A} * \underset{n\times p}{B} = \underset{m\times p}{C}$

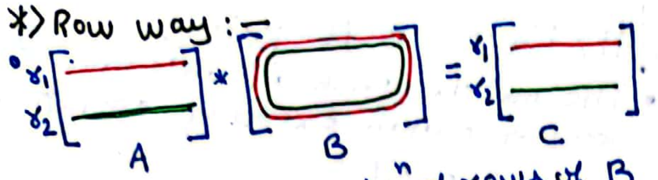
*) Column way:-

$$\underset{m\times n}{\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}} * \underset{n\times p}{\begin{bmatrix} c_1 & c_2 & c_3 \\ | & | & | \\ | & | & | \end{bmatrix}} = \underset{m\times p}{\begin{bmatrix} | & | & | \\ | & | & | \\ | & | & | \end{bmatrix}}$$

- Col.s of C are combinations of columns of A
- Multiplying A by 'p' col.s of B, one-by-one to get the 'p' col.s of C.

  As; $A \times c_1(\text{of } B) = c_1(\text{of } C)$

*) Row way :-



- Rows of C are cob.^{n} of rows of B
- $r_1(\text{of } A) * B = r_1(\text{of } C)$

Note: one col. of A $*$ one row of B = a matrix (say C)
$m\times 1 \qquad 1\times p \qquad m\times p$

$$\overset{A_1}{\begin{bmatrix}2\\3\\4\end{bmatrix}}\overset{B_1}{\begin{bmatrix}1 & 6\end{bmatrix}} = \overset{C}{\begin{bmatrix}2 & 12\\3 & 18\\4 & 24\end{bmatrix}}$$

For C; rows = multiples of B_1
col.s = multiple of A_1

*) 4^{th} way :-
- $A*B$ = Sum of (col.s of A $\times$ rows of B)
- Ex: $\begin{bmatrix}2 & 7\\3 & 8\\4 & 9\end{bmatrix}\begin{bmatrix}1 & 6\\0 & 0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1 & 6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0 & 0\end{bmatrix}$

*) Mul.^{n} by blocks way :-

$$\underset{20\times 20}{\begin{bmatrix}A_1 & A_2\\A_3 & A_4\end{bmatrix}}\underset{20\times 20}{\begin{bmatrix}B_1 & B_2\\B_3 & B_4\end{bmatrix}} = \underset{20\times 20}{\begin{bmatrix}A_1B_1+A_2B_3 & \\ & \end{bmatrix}}$$

Each block : 10×10

#> Inverse Matrix contd... :

*) Singular matrix (No inverse)

$A = \begin{bmatrix}1 & 3\\2 & 6\end{bmatrix} \rightarrow$ why this is not invertible

let, $A\cdot B = I$

Reason-1 : Column picture
since, col.s of I are comb.^{n} of col.s of A
therefore, we can't identify matrix bec. c_1 & c_2 lie on the same line.

Reason-2 : I can find a vector X with $X \neq 0$ and $AX = 0$.

i.e. $\begin{bmatrix}1 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ ($3c_1 - c_2$)

This vector X is leading my matrix A to zero, which I can't invert back.

i.e. for non-invertible (Singular) matrices :-
$AX = 0 \; ; \; X \neq 0$

*) Invertible Matrix : (non-Singular Matrix)

- $A\cdot A^{-1} = I$

$$\underset{A}{\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}}\;\begin{array}{c}j\\ \underset{A^{-1}}{\begin{bmatrix}a & c\\b & d\end{bmatrix}}\end{array} = \begin{array}{c}j\\ \underset{I}{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}\end{array}$$

$A * C_j(\text{of } A^{-1}) = C_j(\text{of } I)$

$\Rightarrow \begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ —(i)

$\begin{bmatrix}1 & 3\\2 & 7\end{bmatrix}\begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ —(ii)

Solve (i) to get a & b
solve (ii) to get c & d

- Gauss-Jorden Idea : Solve n eq.^{n}s at once.

$[A \,\vdots\, I] \longrightarrow [I \,\vdots\, A^{-1}]$

$$\left[\begin{array}{cc:cc}1 & 3 & 1 & 0\\ \boxed{2} & 7 & 0 & 1\end{array}\right]$$

$r_2 = r_2 - 2r_1$

$$\left[\begin{array}{cc:cc}1 & \boxed{3} & 1 & 0\\ 0 & 1 & -2 & 1\end{array}\right]$$

$r_1 = r_1 - 3r_2$

$$\left[\begin{array}{cc:cc}1 & 0 & 7 & -3\\ 0 & 1 & -2 & 1\end{array}\right]$$
$\qquad I \qquad\qquad A^{-1}$

Proof : why this is A^{-1}

we had; $[A \,\vdots\, I]$, then we did elimination i.e. $E * [A \,\vdots\, I]$ & we got another matrix $[I \,\vdots\, A^{-1}]$ (say)

i.e. $E * [A \,\vdots\, I] = [I \,\vdots\, A^{-1}]$

$\Rightarrow E * A = I$ && $E * I = A^{-1}$

$\Rightarrow$ E is A^{-1}

So, $E * I = A^{-1}$
$A^{-1} * I = A^{-1} \Rightarrow$ True (Holds)

#) Factorization of A into L·U :

*) $\boxed{(AB)^{-1} = B^{-1} \cdot A^{-1}}$; If A & B are invertible.

Proof: A

$A \cdot B$

$(A \cdot B)\, B^{-1}$

$(A \cdot B)\, B^{-1} \cdot A^{-1} = I$

$(A \cdot B)\, B^{-1} \cdot A^{-1} = (AB)^{-1} \cdot I$

$(A \cdot B)^{-1} (A \cdot B)\, B^{-1} \cdot A^{-1} = (A \cdot B)^{-1}$

$(I) \cdot B^{-1} \cdot A^{-1} = (A \cdot B)^{-1}$

$\Rightarrow \quad B^{-1} \cdot A^{-1} = (AB)^{-1}$

*) $\boxed{A \cdot A^{-1} = I}$

- Transpose both sides:-

$(A \cdot A^{-1})^T = I^T$

$(A^{-1})^T \cdot A^T = I$

$(A^{-1})^T \cdot \underline{A^T \cdot (A^T)^{-1}} = I \cdot (A^T)^{-1}$

$\Rightarrow \boxed{(A^{-1})^T = (A^T)^{-1}}$

*) A = LU ; (Factorization of a Matrix)

- $A = \begin{bmatrix} 2 & 1 \\ \boxed{8} & 7 \end{bmatrix}_{2\times 2}$ $\qquad A = \underbrace{E_{21}^{-1}}\, U = \underbrace{L}\, U$

Trying for UTM

$r_2 = r_2 - 4r_1$: Operation

$$\begin{bmatrix} E_{21} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = U$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}}_{E_{21}} \underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{U} = U$$

Here, $E_{21} * A = U$

$A = (E_{21})^{-1} \cdot U$

To find E_{21}^{-1}, just do the inverse operation

i.e. $r_2 = r_2 + 4r_1$: Inverse operation

$$\Rightarrow E_{21}^{-1} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$$

Therefore; $A = E_{21}^{-1} \cdot U = L\, U$

i.e. $\underbrace{\begin{bmatrix} 2 & 1 \\ 8 & 7 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}}_{U}$

- If A would've been 3x3 : Then, we would have :-

$E_{32}(E_{31}(E_{21} \cdot A)) = U$

$A = (E_{32} \cdot E_{31} \cdot E_{21})^{-1} * U$

$A = \underline{E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}} * U$

$A = \quad L \quad * U$

i.e. L is the product of inverses of the elimination matrices

Note that no row exchanges are allowed in LU decomposition, as there are no zero pivots.

Q:> Let A(3x3); If the elimination matrices are :-

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

Then find L of A = LU.

$E_{32}(E_{31}(E_{21} A)) = U$

operation: $r_2 = r_2 - 2r_1$

Inv. op.$^n \Rightarrow r_2 = r_2 + 2r_1$

$A = (E_{32} \cdot E_{31} \cdot E_{21})^{-1} \cdot U$

$= \underbrace{E_{21}^{-1} \cdot E_{31}^{-1} \cdot E_{32}^{-1}}_{L} \cdot U$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

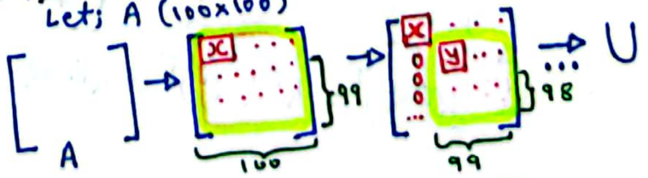
$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

- If $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{bmatrix}$; then a, b, c, d, e, f → multipliers

i.e. If no row exchanges, the multipliers go directly into L.

#) Cost of Elimination : A (n×n)

Let; A (100x100)



#operations : $\approx 99 \times 100 \approx (100)^2$ $\quad \approx 99 \times 98 \approx (99)^2 \; \ldots \approx (1)^2$

$$\Rightarrow cost = \sum_{n=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6} = O(n^3)$$

Now, if we have $[A : B]$ then :-
Let, $B(100 \times 100)$ $B(n \times n)$

$$\left[\begin{array}{c|c} O(n^3) & \underbrace{\begin{matrix} a \\ b \\ \vdots \end{matrix}}_{1} \end{array}\right]\Big\}100 \rightarrow \left[\begin{array}{c|c} \cdots & \underbrace{\begin{matrix} a \\ b \\ c \\ \vdots \end{matrix}}_{1} \end{array}\right]\Big\}99$$

i.e. #operations: 100 , 99 , ...

$$\Rightarrow \text{cost} = \sum_{n=1}^{n} n = \frac{n(n+1)}{2} = O(n^2)$$

i.e. if $[A : B]$ $O(n^3) \rightarrow O(n^2)$

#) Permutation Matrices contd... :

- P_{12} ≡ Permutation Matrix that exchanges row1 & row2

Ex:
$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_{12}} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 7 & 8 & 9 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

- For a 3x3 matrix, $\begin{matrix} r_1 \\ r_2 \\ r_3 \end{matrix}\begin{bmatrix} \text{---} \\ \text{---} \\ \text{---} \end{bmatrix}$

  # Permutation matrices exist $= 3! = 6$ P_{ij}

- These Group of matrices (6 matrices) satisfy closure, identity, inverse properties.
- P_{12} inverse is P_{21}, i.e. $\boxed{P^{-1} = P^T}$ i.e. $\boxed{P \cdot P^T = I}$
- For a 4x4 matrix, we've $4! = 24$ P's
- These are required bec. they execute row exchanges if we get a zero pivot.

#) Transpose :

- $(A_{ij})^T = A_{ji}$
- For Symmetric matrices:-

  $A^T = A$

- $R^T R$ is always symmetric; R ≡ Rectangular Matrix

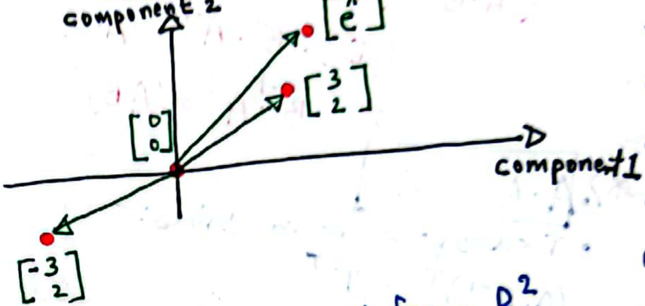
Ex:
$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 17 \end{bmatrix}$$

Proof: $(R^T R)^T = R^T (R^T)^T = R^T \cdot R$

#) Vector Spaces & Sub-Spaces:

6

- Vector Space: A space of vectors that allows me to do vector operations (adding two vectors & multiplying a vector by a scalar), and allows me to do LCs.
- R^2: A vector space of all 2-Dimensional real vectors. (its a plane)

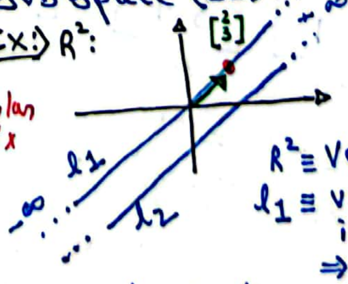


- If point (0,0) removed from R^2, then it is not a vector space. bec. $0 * \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ wouldn't be possible.
- R^3: All real vectors with 3 real components. Ex: $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$
- R^n: All column vectors with 'n' real components.

  we can take any combination, i.e. add 2 vectors, multiply by a scalar & we still will be in R^n?

- A vector space is closed under linear combinations.
- Subspace: A vector space inside another vector space is a sub-space (SS).

Ex:) R^2:



R^2 ≡ Vector space
l_1 ≡ vector space inside R^2
$\Rightarrow$ SS of R^2

All ∞ length lines that passes through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ vector are subspaces of R^2.
Like l_2 is not a subspace of R^2

- A subspace must have to go through origin.

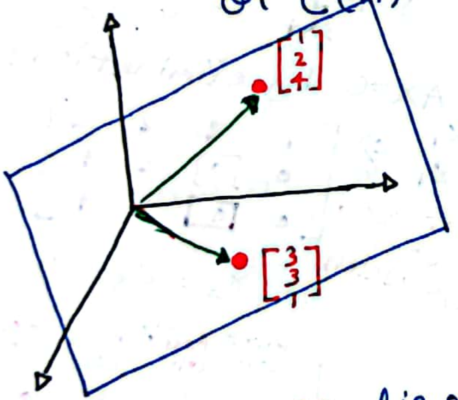
- Possible Subspaces of R^2:
  1) Entire R^2 (P)
  2) Any line through $\begin{bmatrix}0\\0\end{bmatrix}$ vector (L)
  3) Only origin $\begin{bmatrix}0\\0\end{bmatrix}$ vector (Z)
- Possible Subspaces of R^3:
  1) Entire R^3
  2) Any plane or line through origin
  3) Only origin $\begin{bmatrix}0\\0\\0\end{bmatrix}$

Q:) Create Sub-spaces out of matrix A

$$A = \begin{array}{cc} c_1 & c_2 \end{array} \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Here, all colums are 3-Dim. vectors in R^3, so, take all the LC's to form a subspace.

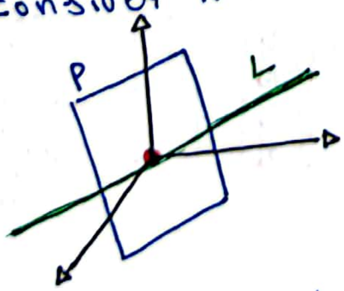
Column space of A ≡ All the LC's of colums of A
or C(A)



Take all comb^{n}'s of these 2 vectors
↓ gives
Column Space
↓ gives
a plane

Note that, if c_1 & c_2 lie on same line, then column space would've been a line. Therefore shape of column space depends on vectors.

- P and L subspaces:-
  → consider R^3:



P → All of R^2 passing throug origin

L → Line through origin

P & L are Subspaces of R^3

→ $P \cup L$ ≡ All vectors in P or L or both
≡ Not a subspace

Bec. you take a vector from P & another vector from L and add them, then you'll be out of sub-space.

→ $P \cap L$ ≡ Just the origin
≡ Yes, a subspace

Note: Conclusion: Intersection of two subspaces is always a subspace.

#) Column Space (CS) of Matrix:

*) $A = \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$ (say) ; CS of A is a subspace in R^4

- $C(A)$ ≡ All LC's of c_1, c_2, c_3
- Questions:-
  → Are those 3 columns independent?
  (or)
  → If we take all LC's of these 3 col's, do we get a 3-D space? Is there any of the column not contributing something new?
  (or)
  → Can I throw away any column to get the same column space?

Ans: $C(A)$ ≡ 2-Dim. SS of R^4
bec. c_1, c_2 contribute something new but c_3 is c_1+c_2, i.e. not contributing any thing new. i.e. c_3 is dependent so you can throw it away & still get the same column space.

- Question: Does $AX=B$ always have a solution for every B?

Ans: No:
say: $AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$

There are a lot of vector b's which are not linear comb^{n}'s of these three columns of A.
So, for those b's we can't have a solution.

Which RHSs allows me to solve this? System of eq^{n}s ($AX=B$) can only be solved when B is a vector in the column space of A ($C(A)$)
i.e. when B ≡ LC of columns of A

for example :-

If $\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ ↳ $\begin{bmatrix}1\\0\\0\end{bmatrix}$

Here, B can be $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ / $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ / $\begin{bmatrix}2\\3\\4\\5\end{bmatrix}$ / or any LC of columns

Note that: If $B=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ you can definitely have a solution.

#> Null Space of A : N(A) :

•> It contains all solutions (X's) to $AX=0$

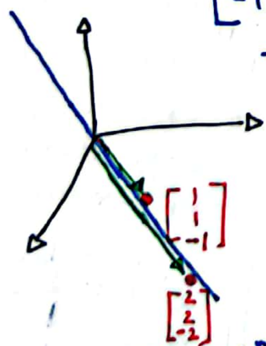
•> Example:

$$\underbrace{\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{X}=\underbrace{\begin{bmatrix}0\\0\\0\\0\end{bmatrix}}_{0}$$

Here,

$$X=\begin{bmatrix}0\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\-1\end{bmatrix},\begin{bmatrix}2\\2\\-2\end{bmatrix},K\begin{bmatrix}1\\1\\-1\end{bmatrix}=N(A)$$

i.e. $N(A)=K\begin{bmatrix}1\\1\\-1\end{bmatrix}$; $K\geq 0$



Therefore, N(A) is a line in R^3 i.e. N(A) is a SS of R^3

•> Lemma: The sol^ns to $AX=0$ (i.e. all those X's) always gives a subspace.

Proof: if $A\cdot V=0$ && $A\cdot W=0$

$\Rightarrow A(V+W)=0$

i.e. if $V\in SS$ && $W\in SS$

$\Rightarrow V+W\in SS$

if $A\cdot V=0$

$\Rightarrow A\cdot(12V)=0$

i.e. if $V\in SS \Rightarrow 12\cdot V\in SS$

Q> $\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$

$AX=B$; Do the solutions form a subspace?

Ans: Sol^ns ≡ X's ≡ $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\-1\\1\end{bmatrix}$, ... many sol^ns

≡ A plane/line that doesn't go through origin

≡ Not a SS bec. $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is not a solution.

#> Solving AX=0 (where A is rectangular)

- Pivot variables
- Special solutions

*> Let;

• $A=\begin{bmatrix}1&2&2&2\\2&4&6&8\\3&6&8&10\end{bmatrix}_{3\times4}$

C_2 is multiple of C_1 (dependent)

r_3 is dependent on r_1 & r_2

Now, solve $AX=0$ by elimination :-

$r_2=r_2-2r_1$
$r_3=r_3-3r_1$ $\Rightarrow\begin{bmatrix}1&2&2&2\\0&\boxed{0}&2&4\\0&\boxed{0}&2&4\end{bmatrix}$

zero pivot, but below is also zero, so row exchange useless

$r_3=r_3-r_2$

$$U=\begin{array}{cccc}\text{pivot col.}&\text{free col.}&\text{pivot col.}&\text{free col.}\end{array}$$
$$U=\begin{bmatrix}\boxed{1}&2&2&2\\0&\boxed{0}&\boxed{2}&4\\0&0&0&\boxed{0}\end{bmatrix}$$

[Echelon form]

(Non-Zeros in Staircase form)

• $Rank(A)=\#\,pivots=2$

• $\begin{bmatrix}1&2&2&2\\0&0&2&4\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ ~ pivot variables ~ free variables {You can put any value}

$U\cdot X=0$

$x_1+2x_2+2x_3+2x_4=0$

$2x_3+4x_4=0$

Lets put free variables $x_2=1, x_4=0$ (say)

$\Rightarrow x_1+2+2x_3=0$

$2x_3+4x_4=0$

$\Rightarrow x_1=-2,\ x_3=0$

Therefore;
Solution-1: $X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a sol.^{n} to $AX=0$ [in the null space]

also, any multiple of it is also a sol.^{n} to $Ax=0$ [in the null space]
i.e. $x = c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Now, Lets put free variables any other value, Say $x_2 = 0$, $x_4 = 1$

$\Rightarrow x_1 + 2\cdot 0 + 2x_3 + 2\times 1 = 0$

$2\times x_3 + 4\times 1 = 0$

$\Rightarrow x_3 = -2$; $x_1 = 2$

Therefore;
Solution-2: $X = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

also, $X = c\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$

- All sol.^{n} to $AX=0$
free variables can be given any values.
All sol.^{n}s $\equiv c\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c'\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \equiv$ All LC's of the special solutions

If matrix $m\times n$; rank $= r$
then # free var. $= n - r$
$= 4 - 2 = 2$

- Reduced row echelon form (R):
Getting a more cleaner U

$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 = r_1 - r_2} \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\downarrow r_2 = r_2/2$

$\begin{bmatrix} \boxed{1} & 2 & 0 & -2 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$

1) Pivot col.s $\equiv$ 1 & 3
2) Free col.s $\equiv$ 2 & 4
3) Notice: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ sitting in pivot rows & col.s

Now; $RX = 0$

$\Rightarrow x_1 + 2x_2 - 2x_4 = 0$
& $x_3 + 2x_4 = 0$

There are 2 special sol.^{n}s $\equiv \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$ And they're in the null space
[bec. we've 2 free variables]

$$\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 2 & -2 \\ 0 & 2 \end{matrix}$$
Pivot col.s (I)   Free col.s (F)

i.e. $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \equiv$ r pivot rows, r pivot col.s, n-r free col.s

$N = \underbrace{\begin{bmatrix} -F \\ I \end{bmatrix}}_{\text{col.s are the special sol.}^n\text{s}} \equiv$ bec. $RN = 0$
$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -F \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Also, w.r.t $RX = 0$
$\Rightarrow [I \;\; F]\begin{bmatrix} X_{pivot} \\ X_{free} \end{bmatrix} = 0$

$\Rightarrow \boxed{X_{pivot} = -F\cdot X_{free}}$

Note: Sol.^{n}s to $AX=0$, $UX=0$, $RX=0$ are all the same. [Just elimination]
$AX=0 \to UX=0 \to RX=0 \to RN=0$

Q:) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 2 & 8 & 10 \end{bmatrix}$

$$\begin{array}{ccc} \text{PC} & \text{PC} & \text{free col.} \end{array}\ [C_3 = C_1 + C_2]$$

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{bmatrix} \xrightarrow{\text{row exch.}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = U$

free col.

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0$
$2x_2 + 2x_3 = 0$
$\hookrightarrow x_2 = -x_3$

Lets put free variable $x_3 = 1$ (say)
$\Rightarrow x_2 = -1$, $x_1 = -1$

Therefore; $N(X) = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Now, rref:
$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{array}{cc} I & F \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{array} = R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

$\Rightarrow X = N = \begin{bmatrix} -F \\ I \end{bmatrix} = c\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

#> Solving $AX=B$: Complete Sol.^{n} :-

- **Example :-**

$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

bec. $r_3 = r_1 + r_2$

$\Rightarrow b_3 = b_1 + b_2$

Augmented coefficient matrix $= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$

Perform elimination

$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right]$

$$\begin{array}{cccc} pc & fc & pc & fc \end{array}$$
$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right]$

$\Rightarrow b_3 - b_2 - b_1 = 0$

$b_3 = b_1 + b_2$

Say; $b_1 = 1$, $b_2 = 5$, $b_3 = 6$

i.e. $b = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ (say)

$= \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

- $R(A) = R(A|B) < n \Rightarrow \infty$ Solutions
- Cond.^{n} on 'b' that makes eq.^{n} solvable :- (or) Solvability condition on RHS
  - → $AX=B$ is solvable exactly when B is in column space of A.
  - → i.e., If comb.^{n} of rows of A gives zero row, then the same comb.^{n} of the entries of B must give 0.
- The Algorithm to find complete sol.^{n} to $AX=B$:-

1.) $X_{particular}$: Set all free variables to 0 then solve $AX=B$ for pivot variables.

i.e. free variables $[x_2 = 0,\ x_4 = 0]$

$\Rightarrow \left.\begin{array}{l} x_1 + 2x_3 = 1 \\ \text{and } 2x_3 = 3 \end{array}\right\} \Rightarrow \begin{array}{l} x_3 = 3/2 \\ x_1 = -2 \end{array}$

$\Rightarrow X_{particular} = \begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}$

2.) Complete Solution $\equiv X_{particular} + X_{null\ Space}$

i.e. $A \cdot X_p = B$

$+\ A \cdot X_n = 0$

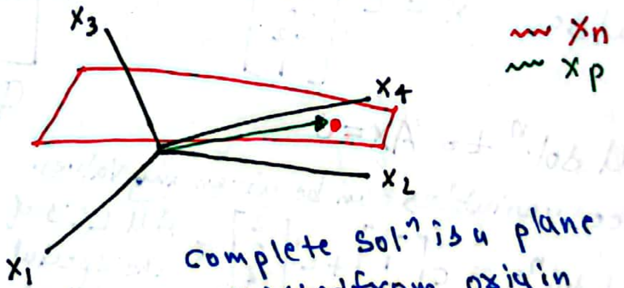
$\overline{A(X_p + X_n) = B}$ $\equiv$ Complete Sol.^{n}

i.e. $X_{complete} = \underbrace{\begin{bmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{bmatrix}}_{X_p} + \underbrace{c_1\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{X_n}$ [Solved on p.no. 9]

X_n or Special sol.^{n}s

This X_n is a Subspace, 2D SS inside R^4.

- Plot all sol.^{n} of $X_{complete}$ in R^4 :-



Complete sol.^{n} is a plane shifted from origin so, not a subspace

- **Full Rank :-**
  - → If $A_{m \times n}$ has rank r then $r \le m$ & $r \le n$
  - → 1. Full column rank (when $r = n$)
    - i.e. every col.m have 1 pivot.
    - i.e. pivot in every column
    - i.e. Free variables $\equiv 0$
    - i.e. All col.s are independent
    - i.e. $N(A) \equiv$ Only the zero vector $\equiv \{0\}$

In this case either unique sol.^{n} or no solution.

i.e. 0 or 1 sol.^{n} in $AX=B$ (X_p)

Let's say $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}$

Here, rank $= 2 = n$
[independent col.s]
[nothing in null space]

After achieving rref

$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

Now, for $AX=B$, does sol.^{n} exist?

Yes, if $B = \begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$ then $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

bec. you're taking $C_1 + C_2$

→ 2. Full Row Rank :- ($A \equiv m \times n$)

means $r = m$

I can solve $AX=B$ for which RHS?

- for every B
- Here, you have $n-r$ free variables i.e. $n-m$ free variables

For example:

$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix}$ rank $= 2 = m$

free var. $= n-r = 4-2 = 2$

↳ $R = \begin{bmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{bmatrix} \equiv \begin{bmatrix} I & F \end{bmatrix}$

In this case, there is always a sol.^{n}

i.e. Sol.^{n} exists

actually, ∞ sol.^{n}s

→ 3. Full Rank:

- means: $r = m = n$
- A is a sq. matrix
- A is invertible matrix ($|A| \neq 0$)

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$

- rref i.e. R is identity matrix

i.e. $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

- $N(A) \equiv$ zero vector
- Cond.^{n}s to solve $AX=B$:-

No cond.^{n} at all

You can solve it for every B

In this case always unique sol.^{n} (i.e. exactly 1 solution)

- Summary :-

$r=m=n$	$r=n<m$	$r=m<n$
$R=I$	$R=\begin{bmatrix} I \\ 0 \end{bmatrix}$	$R=\begin{bmatrix} I & F \end{bmatrix}$
(1 sol.^{n})	(0 or 1 sol.^{n})	I & F partly mixed. Always sol.^{n} exists bec. no zero rows (∞ sol.^{n}s)
Full Rank	Full col. Rank	Full row Rank

$r<m$ & $r<n$

$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

($\boxed{0}$ or ∞ sol.^{n}s)

↳ bec. we didn't get $0=0$ for some B's

i.e. Rank tells everything about no. of solutions.

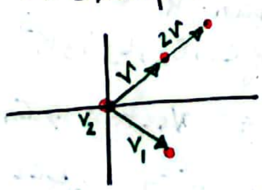
> Linear independence, Spanning of space by vectors, Basis of SS & dimension :-

- Linear independence :-

→ vectors $x_1, x_2, \ldots, x_n$ are linearly independent if no comb.^{n} gives zero vector (except the zero comb.^{n} when c's are zero)

→ $c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \neq 0$

→ Examples :- (R^2)



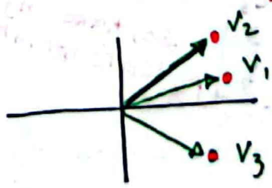
i) V & $2V$ are dependent

bec. $2(V) - 1\times(2V) = 0$

ii) V_1 & V_2 are dependent

bec. $0\times(V_1) + 6\times(V_2) = 0$

i.e. if 0 vector is present, then definitely dependent



i) V_1 & V_2 are independent

ii) V_1, V_2 & V_3 are dependent

bec. 3 vectors in a plane have to be dependent

Proof: Any one of these 3 vectors can be represented as sum of the other two vectors

i.e. $C_2 \cdot V_2 + C_3 V_3 = V_1$

Alter Proof:-

$AX = 0$

Say $\equiv \begin{array}{ccc} V_1 & V_2 & V_3 \end{array}$ $\begin{bmatrix} 2 & 1 & 2.5 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$

$m < n \Rightarrow$ free variables are there

Col's are dependent if something is there in the null space.

After putting any value of free variable you'll get some N Space ⇒ dependent

→ when $v_1, v_2, \dots v_n$ are the col's of A, they're independent if the N(A) only have the zero vector
i.e. NO free variables there ; Rank $= n$

→ They're dependent if $AC = 0$, for some non-zero C in N(A)
i.e. Yes free variables there; Rank $< n$

• **Bunch of Vectors to Span a Space:**

→ Spanning a space: $v_1, v_2, \dots v_l$ vectors span a space means the space consists of (made up of) all comb.^{n}s of those vectors.

→ The columns of a matrix span the column space.

• **Basis of SS & Dimension:**

→ Basis for a vector space is a sequence of vectors $v_1, v_2, \dots, v_d$ with 2 properties:-
i) They are independent
ii) They span the space.

→ Example: For 3D space (R^3)

One basis is: $\begin{array}{ccc} c_1 & c_2 & c_3 \\ \begin{bmatrix}1\\0\\0\end{bmatrix}, & \begin{bmatrix}0\\1\\0\end{bmatrix}, & \begin{bmatrix}0\\0\\1\end{bmatrix}\end{array}$

i) indep. ✓
ii) Span R^3 ✓
∵ comb^n of c_1, c_2, c_3 is R^3

col's

$\because \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \Rightarrow I \Rightarrow N(I) \equiv 0$ vector $\Rightarrow$ Indep.

Second basis is: $\underbrace{\begin{array}{cc} c_1 & c_2 \\ \begin{bmatrix}1\\1\\2\end{bmatrix}, & \begin{bmatrix}2\\2\\5\end{bmatrix}\end{array}}$ c_1&c_2 comb.^{n} is a plane ⇒ not 3D ⇒ span ✗

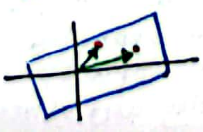
$\begin{array}{c} c_3 \\ \begin{bmatrix}3\\3\\8\end{bmatrix}\end{array}$ So, take any vector that's not in the plane

→ For R^n, n vectors give basis if $n \times n$ matrix with those col's is invertible. $|A| \neq 0$

Q:) $\underbrace{\begin{array}{cc} c_1 & c_2 \\ \begin{bmatrix}1\\1\\2\end{bmatrix} & \begin{bmatrix}2\\2\\5\end{bmatrix}\end{array}}_{\text{forms a plane inside } R^3}$; what space will they be a basis for?

i.e. 2D subspace inside R^3

They will be a basis for a plane (The one they span)



→ If matrix is invertible, their col's form a basis for R^n.

for Ex: $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\4\\8\end{bmatrix}$ form a basis for R^3

Since, col's are independent, Many basis are possible for R^3 (infinitely many) but they all have same no. of vectors.

→ Given a space, every basis for the space has the same no. of vectors, & this no. is called Dimension of the space.

Q:) Given Space is C(A);
i) Find the basis.
ii) Do they form basis of C(A)?

$$A = \begin{array}{cccc} c_1 & c_2 & c_3 & c_4 \\ \left[\begin{matrix}1\\1\\1\end{matrix}\right. & \begin{matrix}2\\1\\2\end{matrix} & \begin{matrix}3\\2\\3\end{matrix} & \left.\begin{matrix}1\\1\\1\end{matrix}\right] \\ p & p & f & f \end{array}$$

Rank(A) $= 2$ ⇓ Dim. of C(A) is 2D ⇓ 2 vectors in the Basis

i) c_1 & c_2 forms a 2D SS in R^4
The basis are: c_1, c_2 or c_1, c_3 or c_2, c_4

ii) No; c_1, c_2, c_3, c_4 are not indep.
bec. $N(A) = \begin{bmatrix}1\\1\\-1\\0\end{bmatrix}$; so $N(A) \neq$ Just the zero vector

but they span R^4 (bec. it's C(A))

→ Rank(A) = # pivot col's
= Dimension of the C(A) (NOT Dim. of A)
= # vectors in the Basis

$\boxed{\text{Note:}}$ Another basis of the C(A) is:-

$\begin{array}{cc} c_1 & c_2 \\ \begin{bmatrix}2\\2\\2\end{bmatrix}, & \begin{bmatrix}7\\5\\7\end{bmatrix}\end{array}$ → 2 & indep.

Sum of all col's

→ Dimension of Null Space:
It's letting me know that in what way the col's are dependent (i.e. add to zero)

carrying on to the previous example
dim. = 2

$$N(A) \equiv \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Putting free variables 0 & 1 and figuring out the rest.

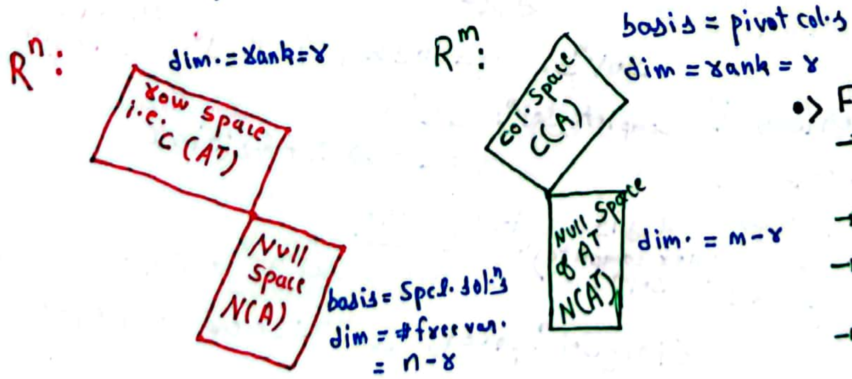
$\underbrace{\qquad}_{}$ 2 special sol^n's.
They're indep.
They span null space
⇒ They're basis

$$\dim\{N(A)\} = \#\text{free variables} = n - r$$

#> 4 fundamental Subspaces for A:

*> 4 subspaces: A is $m \times n$
1) $C(A)$ in R^m
2) $N(A)$ in R^n
3) Row space (i.e. all comb^n's of rows of A)
But I wanna work with column vectors, so...
(or) All comb^n's of A^T
i.e. Row space = $C(A^T)$ in R^n
4) $N(A^T)$ in R^m
[Left Null Space]

*> 4 Subspaces picture:



$r + (n-r) = n$ ⇒ The dimension add to n

$r + (m-r)$ ⇒ The dim. add to m

Every special sol^n comes from a free variable
i.e. # free var.'s = # Spcl. sol^n's

For each one of the 4 SS's we need to answer:-
i> what is it's Basis? ii> what is it's dimension?

*>
$$A = \begin{array}{c} \begin{array}{cccc} p & p & f & f \end{array} \\ \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \end{array}$$
← (I) ** Reference source for future

Echelon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row red.^{n} oper.^{n}s:-
$r_3 = r_3 - r_1$
$r_2 = r_2 - r_1$
$r_2 = -1 \times r_2$
$r_1 = r_1 - 2 \cdot r_2$

rref

$$\begin{array}{c} \begin{array}{cc} I & F \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} = R$$

•> $C(A) \neq C(R)$; i.e. diff. col. spaces
•> $C(A^T) = C(R^T)$; i.e. same row spaces
bec. any LC's of rows would be in row space only.
bec. row operations were done.
i.e. Basis for the row space of A or R is the first 'r' rows of R; r = rank

from R
$$\begin{array}{cc} r_1 & r_2 \\ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \end{array}$$
} Best Basis for row space

*> Left null space: $N(A^T)$:-
•> If $A^T \cdot y = 0$; then y is in the $N(A^T)$
Transpose both sides (to make y sit on left)

$$y^T \cdot A = 0^T$$

$$\begin{bmatrix} y^T \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 0^T \end{bmatrix}$$

bec. we're making y sit on left
So, Left Null Space

•> For $A^T \cdot y = 0$; what is Basis?
→ For that, I'm interested in a matrix which turned my A into R
→ i.e. Gauss Jordan Elimination
→ rref $[A_{m\times n} \; I_{m\times m}] \rightarrow [R_{m\times n} \; E_{m\times m}]$
→ Actually all these row red.^{n} op.^{n}s are as good as multiplying E
→ i.e.

$$E \cdot [A_{m\times n} \; I_{m\times m}] \rightarrow [R_{m\times n} \; E_{m\times m}]$$

$$\Rightarrow E \cdot A = R \;\;\&\;\; \underbrace{E \cdot I = E}_{}$$

This proves existence of $E \equiv r.$ red.^{n} op.^{n}s

•> Note that earlier when we used Gauss Jordan, 'R' was 'I' & 'E' was A^{-1}

i.e. $[A\ I] \longrightarrow [I\ A^{-1}]$

Now, to find E :-

Take $I_{m\times m}$ & apply same r.r.op.^{n}s (As on P.no.13)

$$\underset{I}{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}} \xrightarrow[\text{r.r.op.}^n\text{s}]{\text{same}} \underset{E}{\begin{bmatrix}-1&2&0\\1&-1&0\\-1&0&1\end{bmatrix}}$$

To verify you can do $E \cdot A = R$

•> Left Null Space ≡ Comb.^{n} of rows that gives zero row.

Null space ≡ Comb.^{n} of col.s that gives zero column.

•> $\dim\{N(A^T)\} = m - r = 3 - 2 = 1$

1 vector in Basis

$\Rightarrow [-1\ \ 0\ \ 1]$ (Last row of E)

#) A new Vector Space : $R^{n\times n}$ (i.e. matrix spaces)

*> • Lets take all 3x3 Matrices (say) to be a matrix space 'M'

• i.e. "Every 3x3 matrix is a Vector for me!"

Since, they obey all the 8 rules for vector space, so, I can call them vector space.

• we only care about $A+B$, $c \cdot A$; but not $A \cdot B$ bec. vector mul.^{n} is not defined.

• Subspaces of M :

1. All upper $\Delta_{3\times3}$ matrices (Say U)
2. All Symmetric 3x3 matrices (Say S)
3. All diagonal 3x3 matrices (Say D)

as Sym+Sym=Sym, K×Sym=Sym

Q:) $\dim\{D\} = ?$ $\quad \dim\{D\} = 3$ $\begin{bmatrix}x&&\\&x&\\&&x\end{bmatrix}$

Lets take 3 diagonal matrices

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&7\end{bmatrix}$$

They're independent matrices

⇒ They span the SS of D

⇒ They are Basis for D

(As you can form any diagonal matrix with these

• Basis for M :

$\dim\{M\} = 9$

9 matrices in Basis [Standard basis]

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}, \cdots \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$$

i.e. $\underbrace{3+3+3}_{=9} \Rightarrow \dim\{M\} = 9$

Q:) How many of these 9 matrices belong to the Symm. 3x3 subspace?

Ans: 3 ; $\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}$

• S and U subspaces of M :

S: $\dim[S] = 6$; $\begin{bmatrix}x&x&x\\&x&x\\&&x\end{bmatrix}$

U: $\dim\{U\} = 6$; $\begin{bmatrix}x&x&x\\&x&x\\&&x\end{bmatrix}$

intersection: $S \cap U$ ≡ diagonal 3x3 subspace

$\dim\{S\cap U\} = \dim\{D\} = 3$

Union: $S \cup U$ ≡ Matrices in S or U ≡ Not a subspace

bec. S & U are headed in diff. dir.^{n}s

Sum: $S + U$ ≡ any element of S + any element of U ≡ M (i.e. all 3x3 matrices)

$\dim\{S+U\} = 9$ $\quad \begin{bmatrix}x&x&x\\x&x&x\\x&x&x\end{bmatrix}$

note that: $\dim(S+U) = \dim(S) + \dim(U) - \dim(S\cap U)$

$\qquad 9 = 6 + 6 - 3$

Q:) Let; $\dfrac{d^2y}{dx^2} + y = 0$ be a diff. eq.^{n}

Sol.^{n}s: $y = \cos x$, $\sin x$]← Null Space of D. eq.^{n}

Complete sol.^{n}: LC's

$y = c_1 \cdot \cos x + c_2 \cdot \sin x$

Basis: $\underbrace{\sin x, \cos x}_{\text{Spcl. sol.}^n\text{s to } AX=0}$ (For complete sol.^{n})

$\dim\{\text{Sol.}^n\text{ space}\}$: 2 $\quad$ ∵ Second order eq.^{n}

Note that: Here, $\cos x$ & $\sin x$ are vectors

Note: $\dim\{C(A)\} = \dim\{C(A^T)\} = $ rank

#> Rank 1 Matrices:

*> • They're building blocks for all the matrices.

• Let; $A_{2\times3} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix}$

• Rank (A) = 1

• Basis for the row space is $\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}$

• dim{C(A)} = 1 = rank

• Basis for column space is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

• $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 8 & 10 \end{bmatrix} = A$

• Therefore, rank 1 matrices can be written as:-

$A = X \cdot y^T$; $X \equiv$ Column Vector, $y^T \equiv$ row vector

• A 5×17 matrix (say) of rank 4 (say) can be made by 4 ran 1 matrices.

Q:) Let, M = all 5×17 matrices.
Max.^{m} rank of M could be 5.

i) subset of set of all rank 4 matrices is a vector space?

Ans: NO; bec. if we add two such matrices of rank 4, we may get a matrix of rank 5

bec. rank(A+B) ≤ r(A) + r(B)

ii) Subset of set of all rank 1 matrices is a vector space?

Ans: NO; bec. it can add up to rank 2.

Q:) Let; R^4 be the space. Every vector inside R^4 is of the type: $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$; i.e. 4 components

If S = all vectors in R^4 with $v_1+v_2+v_3+v_4 = 0$

i) S is a subspace? Yes; • scalar mul.^{n} ✓ • v+w ✓

$\dim(S) = 3$
$= m - r$
$= 4 - 1$
$= 3$

ii) Find Basis for S:

Pivot, Free: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3 special sol.^{n}s (Spcl Basis)

iii) If S is N(A); Find A

bec AV = 0

$\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = 0$ (p f f f)

as $v_1+v_2+v_3+v_4 = 0$

$\Rightarrow A = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \Rightarrow$ rank = 1 = r

iv) Find dim{N(A)}

dim{N(A)} = n − r
= 4 − 1 = 3 ⇒ 3 vectors in Basis

v) dim{row space} or dim{C(A^T)} = 1 = r (a line in R^4)

vi) what is C(A) & dim{C(A)}

bec. all col.s are multiple of 1
⇒ C(A) = R^1

dim{C(A)} = 1 = r

vii) what is N(A^T)? = {0}

bec. no comb.^{n} of col.s gives zero row

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot X = 0$

dim{N(A^T)} = 0

#> Graphs, Networks & Incidence Matrix:

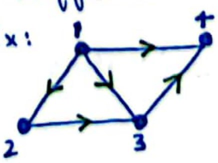
*> Graph:

• G = (V, E) = {nodes, edges}

• Here, we're generally interested that how far nodes can be seperated. (I know a guy who knows a guy.)

• Its an app.^{n} of LA, as LA uses matrices which come from different app.^{n}s in real world.

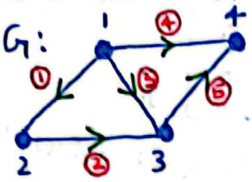
• Ex:



n = 4, e = 5

*) Incidence Matrix:

Let; G:



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1\\2\\3\\4\\5 \end{array} & A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \end{array}$$

nodes → (columns); edges ↑ (rows)

How matrix is built:- edge ① is going from vertex 1 to 2

So, $-1\ 1\ 0\ 0$ in row 1

(i.e. source $= -1$, target $= +1$)

Note that: $r_3 \equiv r_2 + r_1$

$\Rightarrow r_1, r_2, r_3$ are dependent

$\Rightarrow$ edges ①, ② & ③ form a loop

Q:) $N(A) = ?$ [How to combine col's to get 0]

If columns are indep., then $N(A) = 0$

Lets find out:-

$AX = 0$

$$\underset{A\ \ 5\times4}{\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}} \overbrace{\underset{X\ \ 4\times1}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}}}^{\text{Potentials at the nodes}} = \underset{5\times1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} x_2 - x_1 \\ x_3 - x_2 \\ x_3 - x_1 \\ x_4 - x_1 \\ x_4 - x_3 \end{bmatrix}}_{e} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Potential differences across the edges [current]

$e = AX$

$$\Rightarrow N(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; c \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{const. potential}}$$

The values of X vector that satisfy C

$$\text{Basis} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \equiv 1 \text{ dimensional}$$

$\dim(N(A)) = 1$

Q:) Suppose we ground node 4

i.e. potential $= 0$

i.e. $x_4 = 0$

$\Rightarrow \operatorname{rank}(A) = 3$; $\dim\{C(A)\} = 3$

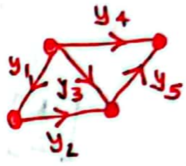
Q:) $N(A^T) = ?$

$A^T \cdot y = 0$

$\dim\{N(A^T)\} = m - r = 5 - 3 = 2$

$$\underset{4\times5}{\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}} \cdot \underset{5\times1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}} = \underset{4\times1}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{array}{c} 1\\2\\3\\4 \end{array} \underbrace{\begin{bmatrix} -y_1 - y_3 - y_4 \\ y_1 - y_2 \\ y_2 + y_3 - y_5 \\ y_4 + y_5 \end{bmatrix}}_{\text{current in = current out (for all nodes)}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



i.e. like at node 4, the total flow $= 0$

i.e. conservation of charges.

Note: The framework of Kirchoff:-

$X = x_1, x_2, x_3, x_4$ (Potentials at nodes)

$\downarrow e = AX$

$x_2 - x_1$; e.t.c (Potential differences across the edges)

$\xrightarrow[\text{Ohm's law}]{y = Ce}$ $y = y_1, y_2, y_3, y_4, y_5$ (currents on edges)

current $\propto$ Pot. diff.

$\uparrow A^T y = 0$

$A^T \cdot y = 0$ (Kirchoff's current Law, i.e. KCL)

Q:) Basis of $N(A^T) = ?$

Through observation from graph:-

2 Loops $\Rightarrow$ 2 Basis: $\underbrace{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\text{In Loop 1,}}$, $\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}}_{\text{In Loop 2,}}$

In Loop 1, Say, 1 Amp. current leaves node 1 & enters node 2.

Then 1 Amp. current leaves node 2 & enters node 3.

Then 1 Amp. current comes to node 3 from node 1 & they become zero.

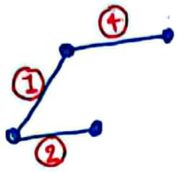
but outer loop, $\begin{bmatrix}1\\1\\0\\-1\\1\end{bmatrix}$ we don't take in the basis, bec. its sum of previous 2 vectors ⇒ not indep.

Q:> Row space (A) or $C(A^T)$?

dim. = 3 → 3 indep. rows

$$\begin{array}{ccccc} p & p & & p & \\ \end{array}$$
$$\begin{bmatrix} -1 & 0 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

↳ 3 indep. columns
⇒ 3 indep. edges in G
⇒ No loop bec. Tree



Q:> $\dim\{N(A^T)\} = m - r$

(or) # indep. loops = # indep. edges − (# nodes − 1)

i.e.

nodes − # edges + # loops = 1

Euler's formula

Ex:)

nodes − edges + loops = 1
5 − 7 + 3 = 1

Note: 3 fundamental eq.ⁿs:-

- $e = AX$
- $y = Ce$
- $A^T y = f$

e = Pot. diff.
A = Incidence matrix
X = Potentials at the nodes
C = Ohm's law const. $I \propto V$
f = Current from outside source

Basic eq.ⁿ of applied math:-

$$A^T \cdot C \cdot AX = f$$

#> Quiz :-

Q:> Let; u, v, w are non-zero vectors in R^7.
They span a subspace of R^7.
Possible dimension of subspace?

Ans: 1 or 2 or 3; can't be more bec. we've just 3 vectors

can't be 0; bec. non-zero vectors.

Q:> Let; U = 5×3 matrix ; r = 3 pivots
U is in rref
i.e. U = R

i> what is N(U)?

r = 3 ⇒ 3 indep. col.s

$$\underset{5\times3}{\begin{bmatrix} | & | & | \end{bmatrix}} \Rightarrow N(U) = \text{zero vector} = \left\{\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

ii> B is 10×3 matrix & C is 10×6 matrix

ⓐ rref of $B = \begin{bmatrix} U \\ 2U \end{bmatrix}_{10\times3}$ is?

$$\xrightarrow{\text{Row red.}^n} \begin{bmatrix} U \\ 0 \end{bmatrix} \; ; \; \text{rank}(B) = 3$$

ⓑ rref of $C = \begin{bmatrix} U & U \\ U & 0 \end{bmatrix}$

$$\xrightarrow{\text{row red.}^n} \begin{bmatrix} U & U \\ 0 & -U \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}$$

rank(C) = 6

ⓒ $\dim\{N(C^T)\}$?

C is $\underset{m}{10} \times \underset{n}{6}$

$\dim\{N(C^T)\} = m - r = 10 - 6 = 4$

Q:> $\underset{3\times3}{A} \cdot \underset{3\times1}{X} = \begin{bmatrix}2\\4\\2\end{bmatrix}$; $X = \begin{bmatrix}2\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\0\\1\end{bmatrix}$

i> what is dim{Row space (A)} ?

Since, N(A) has 2 vectors (indep.) $c\begin{bmatrix}1\\1\\0\end{bmatrix}, d\begin{bmatrix}0\\0\\1\end{bmatrix}$

⇒ $\dim\{N(A)\} = 2$

⇒ $2 = n - r$
$2 = 3 - r$
$r = 1$ ⇒ rank(A) = 1

Now, $\dim\{\text{Row space}(A)\} = \dim\{C(A^T)\} = r = 1$

ii> Find A.

Let, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Now, $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}2\\4\\2\end{bmatrix}$

⇒ $\cancel{2}a = \cancel{2}\,1$
$\cancel{2}d = \cancel{4}\,2$
$\cancel{2}g = \cancel{2}\,1$

similarly,

$$\begin{bmatrix} 1 & b & c \\ 2 & e & f \\ 1 & h & i \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{matrix} \cancel{1}+b \\ \cancel{2}+e \\ \cancel{1}+h \end{matrix} = \begin{matrix} \cancel{2}\ 1 \\ \cancel{4}\ 2 \\ \cancel{2}\ 1 \end{matrix}$$

Sim.,

$$\begin{bmatrix} 1 & 1 & c \\ 2 & 2 & f \\ 1 & 1 & i \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

$$\begin{matrix} c \\ f \\ i \end{matrix} = \begin{matrix} 2 \\ 4 \\ 2 \end{matrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

iii> $AX=b$ can be solved if?
If b is in the $C(A)$
& $C(A) = c\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Q:) If A is sq. matrix.
$N(A) = \{\text{zero vector}\}$
then $N(A^T) = ?$
$= \{\text{zero vector}\}$

Q:) Let, S = all 5×5 matrices be a subspace.
Let, S' = all invertible matrices and S' is subset of S.
Is S' a subspace? ⇒ NO
∵ Two matrices (which are invertible), their sum need not be invertible.

Q:) If $B^2=0$; then $B=0$?
Here, B is square matrix
$B=0$ need not be true always
as if $B=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$ but $B^2=0$

Q:) matrix $A_{n\times n}$ with indep. col's
$Ax=b$ always solvable? ⇒ Yes
Here, rank $= n$; 'n' pivots

Q:) Let,

$$\underset{3\times4}{B} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{X} \cdot \underbrace{\begin{array}{c} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} p & p & f & f \end{matrix} \end{array}}_{Y} = ?\ \left(\begin{matrix}\text{w/o} \\ \text{mult.}^n\end{matrix}\right)$$

we're in R^4
basis for $N(B) \subseteq R^4$
X is an invertible matrix
$\Rightarrow N(B) = N(X\cdot Y) = N(Y)$

and, Rank$(Y) = 2 \Rightarrow 2$ spcl. sol.^{n}'s in Y

$$\Rightarrow \text{basis for } N(Y) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Now, complete sol.^{n} for $BX = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X_p + X_n = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Note: There can be some other X_p but you only take one

Q:) In $A_{m\times n}$; if $m=n$
then row space = col space? ⇒ No
Ex:) $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
Here, Row space $= c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
col. space $= c\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
But, if A would've been symm.
then ans ⇒ yes

Q:) Does A & $-A$ have same col. space, Null space, row space?
⇒ yes

Q:) If A & B have same four subspaces, then A is a multiple of B? ⇒ Need not be
Ex:) $A, B \rightarrow$ invertible 6×6 matrix
$\exists (A, B) : A \neq c\cdot B$

Q:) If we exchange two rows of A, then?
Row space stays the same
Null " " "
col. space may change

Q:) $V = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ can't be in null space && row space, why?
(i.e. be a row of A)

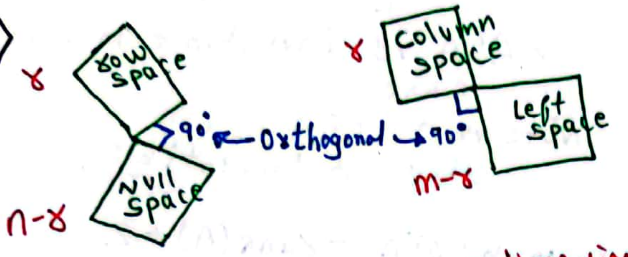
i.e. $A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ - & - & - \\ - & - & - \end{bmatrix}}_{\text{It can't be like this, bec. RHS then would be 14, not zero}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ - \\ - \end{bmatrix}$

#) Orthogonal Vectors & Orthogonal Subspaces:

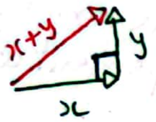
(Basis/vectors/SS's to be orthogonal)

*)



~ ≡ dimension

*) Orthogonal vectors: (perpend. vectors)



- If the dot product is zero i.e. if $x^T y = 0$ then Orthogonal vectors
- i.e. $x^T y = x_1 y_1 + x_2 y_2 + \dots = 0$

- Note that: Pythagoras theorem

$$|x|^2 + |y|^2 = |x+y|^2$$

(or) $x^T . x + y^T . y = (x+y)^T . (x+y)$

is valid only if we have orthogonal vectors.

- Proof of Pythagoras & red.^{n} of $x^T y = 0$:-

Let; $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$; Here, $|X|^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

Also, $\|X\|^2 = \underset{1\times3}{X^T} . \underset{3\times1}{X} = \underset{1\times1}{14}$

Let; $Y = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ Such that $x^T . y = 0$ (i.e. They're orthogonal)

$\|x\|^2 = 14$; $\|y\|^2 = 5$

then; $x + y = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \|x+y\|^2 = 3^2 + 1^2 + 3^2 = 19$

Therefore,

$$\|x\|^2 + \|y\|^2 = \|x+y\|^2$$

i.e. $x^T . x + y^T . y = (x+y)^T . (x+y)$

$\cancel{x^T x} + \cancel{y^T y} = \cancel{x^T x} + x^T . y + y^T . x + \cancel{y^T y}$

$2 . x^T y = 0$ $\quad [\because y^T x = x^T y]$

$\Rightarrow \boxed{x^T . y = 0}$

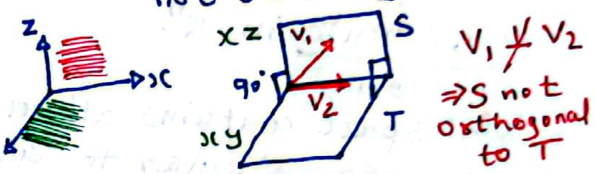
Note: Zero vector is orthogonal to all the vectors.

i.e. if $x = 0$ vector, $y =$ anything $\Rightarrow x$ & y are orthogonal

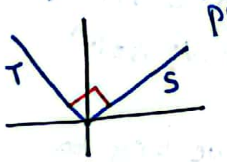
*) If; Subspace (S) is orthogonal to subspace (T)

then; it means every vector in S is orthogonal to every vector in T.

Ex 1: xz & xy planes are not orthogonal



Ex 2: Let, two subspaces in R^2 be T & S; both are a line passing through origin.



Here, S & T are orthogonal

*) Lemma: Row space is orthogonal to null space.

Proof:

For null space, when x is in the $N(A)$

then $Ax = 0$

$$\begin{bmatrix} \text{r1 of A} \\ \text{r2 of A} \\ \vdots \\ \text{rm of A} \end{bmatrix} \begin{bmatrix} X \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Row space is all LC of Rows

bec. dot product is zero

$\Rightarrow X$ is orthogonal to $\underbrace{r_1, r_2, \dots r_m}_{\text{All the rows}}$

Now, lets see if X is orthogonal to LC of rows:

$c_1 (r_1)^T x = 0$

$c_2 (r_2)^T x = 0$

(or) $c_1 (r_1^T) x + c_2 (r_2)^T x = 0$

$(c_1 r_1 + c_2 r_2 + \dots c_m r_m)^T x = 0$

$\Rightarrow$ Yes; X is orthogonal to row space.

*) Orthogonal Subspaces in 3D (R^3):

- Let; A matrix with: $n = 3$, $r = 1$

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

here, $\dim(N(A)) = n - r = 2$
$\dim(R(A)) = 1$

$N(A)$ is a plane $\perp$ to $[1\ 2\ 5]$ vector.
$[1\ 2\ 5]$ vector is normal vector (N).

It means;

Null space & Row space are orthogonal complements in R^n.
↓ means
Null space contains all vectors that are perpendicular to row space

*) Solve $AX = b$, when no sol.^{n} exists.

- It means, when b is not in $C(A)$. $\Rightarrow$ No sol.^{n}
- Here, $\underset{[eq.^n]}{m} > \underset{[unknowns]}{n}$
- i.e. Lot of information, we try to seperate noise from information.
- Lets try to find sol.^{n}:-

→ $\underset{n\times m}{A^T} \cdot \underset{m\times n}{A}$ $\equiv$ A matrix which is:-
- $n \times n$ Sq. matrix
- Symmetric "
bec. $(A^T A)^T = A^T \cdot A$

→ Now, in $AX = b$ mul. both sides by A^T

$\boxed{A^T \cdot A\hat{X} = A^T \cdot b}$ → central eq.^{n}
→ we hope this $\hat{X}$ will have solution.

→ Taking an example:-

$$\text{Let, } \overset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}} \overset{X}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}};$$

Here, In A
$m = 3$
$n = 2$
$m > n$
rank $= 2$

i.e. Here, we have 3 eq.^{n}s, 2 unknowns
$\Rightarrow$ No solution

Reason: For solvability, b needs to be in $C(A)$, then it can be solved.
$C(A)$ is a 2D plane.
but most vectors b's won't be on that plane.

Now, lets look at $A^T \cdot A$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 8 & 30 \end{bmatrix}$$

$\Rightarrow A^T \cdot A$ is invertible in this case

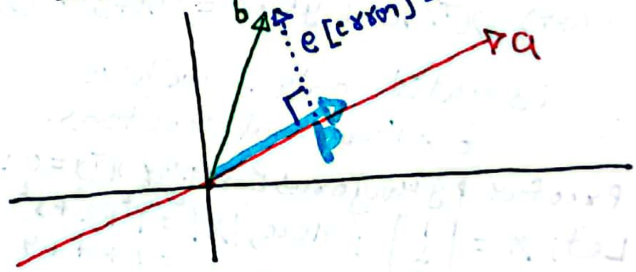
Note that: $A^T \cdot A$ is not always invertible

$\text{rank}(A^T A) = \text{rank}(A) = 2$
$|A| \neq 0$
$N(A^T A) = N(A)$

i.e.
$A^T A$ is invertible exactly if A has independent col.s.

#) Projection of vectors onto Subspaces:

*) Projecting vector 'b' onto a 1-D subspace 'a':



b: vector in R^2
a: 1-D subspace
p: Projection vector

$p = xa$; x is some multiple

Now, $a^T \cdot e = 0$ [$\because a \perp e$]
$a^T (b - p) = 0$
$a^T (b - xa) = 0$
$a^T b = x \cdot a^T a$

$$\boxed{x = \frac{a^T b}{a^T a}} \text{ — (1)}$$

$$\boxed{p = xa} \text{ — (2)}$$

$$= a \cdot \frac{a^T b}{a^T a}$$

$p = P \cdot b$ $\qquad$ $P \equiv$ Projection matrix

if $b \to 2b \Rightarrow p \to 2p$
if $a \to 2a \Rightarrow p$ stays same

$$\boxed{P = \frac{\underbrace{a \cdot a^T}_{\text{a matrix \{col x row\}}}}{\underbrace{a^T \cdot a}_{\text{a no. } \{|a|^2\}}}} \quad \text{—— (3)}$$

***) Properties of projection matrix (P):**

→ $P = \frac{a \cdot a^T}{a^T \cdot a}$; $p = P \cdot b$

→ $C(P)$ = line through 'a'

→ rank(P) = 1

→ P is symmetric

→ $P^T = P$

→ $P^2 = P$

i.e. if we project P twice, we get the same P

i.e. if project twice, you get same line.

i.e. Idempotent

***) Why projection :-**

→ Bcc. $Ax = b$ may have no solution $(m > n)$

→ My problem is that, Ax has to be in $C(A)$; but 'b' may not be in $C(A)$

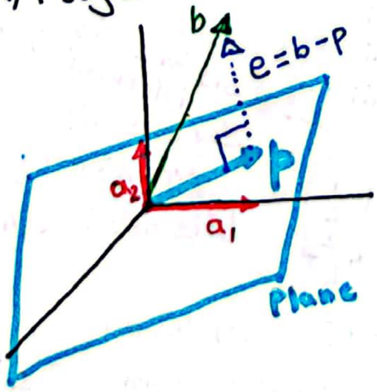
So, we change 'b' & we choose the closest vector in $C(A)$.

So, we solve $A\hat{x} = p$ instead.

$\hat{x} \equiv$ Best possible sol^n

$p \equiv$ Projection of 'b' onto $C(A)$.

***) Projection in 3-D (R^3):**



Let, a_1 & a_2 are basis for plane

i.e. Plane is plane of a_1, a_2

$\Rightarrow C(A) = \begin{bmatrix} a_1 & a_2 \\ | & | \end{bmatrix}$

Also, $e \perp$ to the plane

'b' vector is not in the plane, but we want to project it onto the plane. (i.e. 'b' is not in $C(A)$)

Now, $p = \hat{x}_1 a_1 + \hat{x}_2 a_2 = A\hat{x}$

& we want to find $\hat{x}$

i.e., Problem is to find the right comb^n of the columns so that $e \perp$ to the plane.

Now, $p = A\hat{x}$; Find $\hat{x}$

$e = b - A\hat{x}$ is $\perp$ to the plane

e is $\perp$ to a_1 & a_2 as well

bcc. a_1 & a_2 are on the Plane

Therefore,

$a_1^T \cdot (b - A\hat{x}) = 0$ —— ⟨i⟩

$a_2^T \cdot (b - A\hat{x}) = 0$ —— ⟨ii⟩

$\Rightarrow \begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (b - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow A^T \cdot (b - A\hat{x}) = 0$

$A^T \cdot e = 0$

Therefore, e is in $N(A^T)$

$\Rightarrow e$ is $\perp$ to the $C(A)$

Now, from central eq^n on p. no. 20 :-

$A^T \cdot A\hat{x} = A^T \cdot b$ or $a^T a \hat{x} = a^T b$

$\hat{x} = (A^T A)^{-1} \cdot A^T b$

Now, $p = A\hat{x} = \underbrace{A (A^T A)^{-1} A^T}_{P} b$

$p = P \cdot b$; $P = A(A^T A)^{-1} A^T$ ← Projection matrix in n-D

$= A \cdot A^{-1} \cdot (A^T)^{-1} \cdot A^T$

$= I$

(if A is sq. invertible)

but, we know that A is not sq. matrix

as $A = \begin{bmatrix} a_1 & a_2 \\ | & | \end{bmatrix} \Rightarrow$ not invertible

$\Rightarrow (A^T A)^{-1} \neq A^{-1} \cdot (A^T)^{-1}$

If 'A' would've been an invertible sq. matrix then $C(A)$ would be entire R^n

now if we project 'b' onto this whole R^n, we get $P = I$

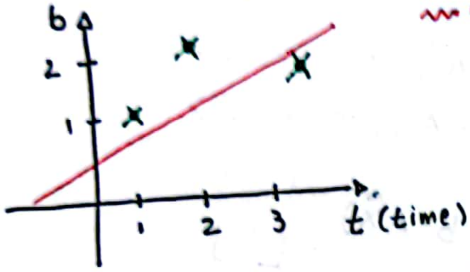
bcc. b is already in the CS

And in 3-D also,
- P is symmetric ; $P^T = P$
- $P^2 = P$; $P^2 = A(A^TA)^{-1}\cancel{A^T \cdot A(A^TA)^{-1}}A^T$

$$= A(A^TA)^{-1}A^T = P$$

*) Application of Projection of vectors:

1. Least squares fitting by a line :



✗ = Given data points
~ = best line

$b = C + Dt$

Problem: Fit the points (1,1), (2,2), (3,2) by a line [Find best line]

$b = C + Dt$ ≡ Line

From the given datapoints ✗ ;
On putting values of b & t :-

$C + D = 1$
$C + 2D = 2$
$C + 3D = 2$

2D SS

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

This is not in C(A)

There is no sol^n as 3 eq^{n}s & 2 unknowns
But to find the best solution :-
mul. both side by A^T

$$A^T \cdot A\hat{X} = A^T b$$

we try to solve this

#) Projection matrices, Least Squares, Best Straight line :-

*) Projection matrix $= P = p \cdot b$
$P = \underline{A(A^TA)^{-1}A^T} \cdot b$

case-1
$pb = b$; if b already in CS
i.e. sol^n exists

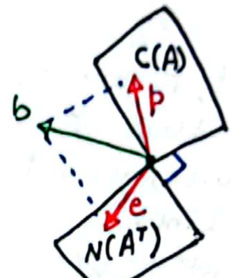
case-2
$pb = 0$; if $b \perp$ to CS
i.e. b is not in CS
i.e. No sol^n exists
i.e. b is in $N(A^T)$

Reason for case-1 :

$pb = A(A^TA)^{-1}A^T \cdot b$ [∵ b is in C(A) ⇒ $AX = b$]
$= A\cancel{(A^TA)^{-1}}\cancel{A^T \cdot A}X$
$= AX = b$

Reason for case-2 :

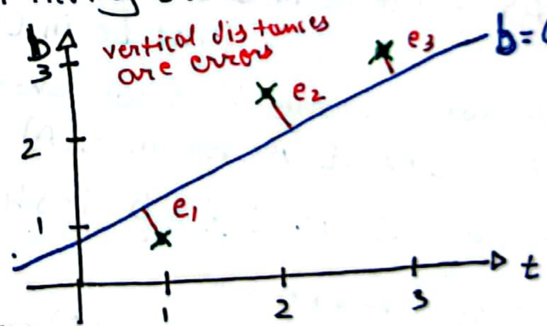
$pb = A(A^TA)^{-1}\underline{A^T \cdot b}$ [∵ $b \perp C(A)$ ⇒ b is in $N(A^T)$ ⇒ $A^T \cdot b = 0$]
$= A(A^TA)^{-1} \cdot \underline{0}$
$= 0$



$b = \underbrace{\underbrace{p}_{Pb}}_{\text{Projection onto C(A)}} + \underbrace{\underbrace{e}_{(I-P)b}}_{\text{Projection onto the perpend. space}}$

If P is sym. ⇒ $I-P$ also sym.
If $P^2 = P$ ⇒ $(I-P)^2 = (I-P)$

*) Finding the best st. line :



Fit the points : (1,1), (2,2), (3,2) by a line

- Minimize the error to find best C & D
- $C + D = 1$ — i
  $C + 2D = 2$ — ii
  $C + 3D = 2$ — iii
- $AX = b$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

Basis for CS is 2 independent col's but still No sol^n bec. b is not in C(A)

- $AX - b = e$ ≡ The error vector
  To make it a never zero quantity, sq. both sides

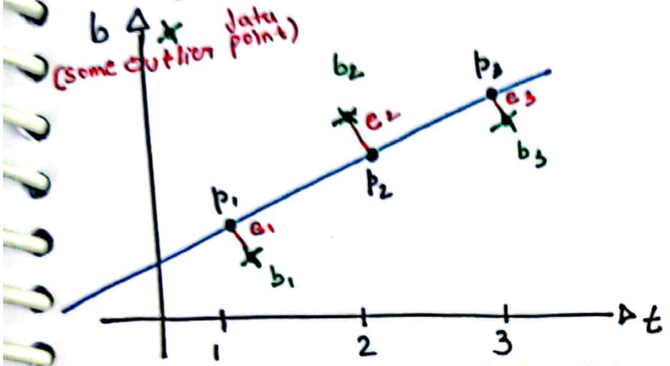
$$|AX - b|^2 = |error|^2$$

Overall error $= e_1^2 + e_2^2 + e_3^2$
e_1, e_2, e_3 are errors made in eq^{n}s i, ii & iii

- we need to minimize this sum of squares used as measure of error → $e_1^2+e_2^2+e_3^2$

It's called Linear regression

- If some data point is far away from the best line (An Outlier data point) then we won't get the best line bec. it's error would be high & e^2 would be even huge.



now, lets say if we had the data points p_1, p_2, p_3 that actually lies on the line, then after putting them on the RHSs of the eq.^{n} as $AX=p$ instead of $AX=b$, then we could have been able to get a sol.^{n}. And we would've got the Perfect line instead of the Best line.

- Finding $\hat{X} = \begin{bmatrix} \hat{C} \\ \hat{D} \end{bmatrix}$, p :-

$A^T \cdot A\hat{X} = A^T b$; most imp. eq.^{n} (estimate) to use when got error/noise

A^TA :- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 14 \end{bmatrix} \equiv$ Symm./ Idemp./ Invertible

A^Tb :- $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$

$\hat{X}$:- $\begin{bmatrix} C \\ D \end{bmatrix}$

Therefore; $3C + 6D = 5 \longrightarrow \langle i \rangle$
$6C + 14D = 11 \longrightarrow \langle ii \rangle$
Normal eq.^{n}s

Note that: You could've also used calculus:-
min. → $e_1^2+e_2^2+e_3^2$; $e = AX-b$

$f(C,D) = (C+D-1)^2 + (C+2D-2)^2 + (C+3D-2)^2$

Partial derivative w.r.t C & D

$\frac{\delta f}{\delta C} = 0 \longleftarrow \langle i \rangle$, $\frac{\delta F}{\delta D} = 0 \longleftarrow \langle ii \rangle$

Normal equations

By $\langle i \rangle$ & $\langle ii \rangle$; $D = 1/2$, $C = 2/3$

Therefore; The best fit line is : $\boxed{b = \frac{2}{3} + \frac{1}{2}t}$

- Now, after putting $t = 1, 2, 3$ we get, $p_1 = 7/6$, $p_2 = 5/3$, $p_3 = 13/6$
- Therefore; The errors $e = b - p$

$e_1 = -1/6$, $e_2 = +2/6$, $e_3 = -1/6$

verify:- $b = p + e$

$$\underset{\text{True points}}{\underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{b}} = \underset{\text{Projection points}}{\underbrace{\begin{bmatrix} 7/6 \\ 5/3 \\ 13/6 \end{bmatrix}}_{p}} + \underset{\text{Error points}}{\underbrace{\begin{bmatrix} -1/6 \\ 2/6 \\ -1/6 \end{bmatrix}}_{e}}$$

- p & e are $\perp$ to each other bec. $p \cdot e = 0$; i.e. p & e are orthogonal
- e is $\perp$ to all vectors in whole CS of A. i.e. $\perp$ to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ also

$\perp$ to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ also bec. dot = 0

- Now, Just solve $A\hat{X} = p$ to find p
- Lemma: If A has indep. cols then A^TA is invertible (like here)

Proof: Suppose $A^TAX = 0$

now we need to prove that X is zero bec. then we would say that the col.s of A^TA are indep. (as $N(A^TA) = 0$ vector)

On multiplying X^T both sides

$X^TA^TAX = 0$

$(AX)^TAX = 0$

$\Rightarrow |AX|^2 = 0$

$\Rightarrow AX = 0$

if A has indep. col.s, then $A \neq 0$

then $X = 0$

- **Lemma:** Columns are definitely indep. if they're perpendicular unit vectors (i.e. Orthonormal vectors)

Like:-

i.e. $A = I$

Note that here $A^T A = I$

Like:-