EM :

Ch 1 : LA

*) Types of Matrix

1) ~~Column matrix $[A_{n\times 1}]$~~
2) Row " $[A_{1\times m}]$
3) Rectangular $[A_{m\times n};\ m\neq n]$
4) Square $[A_{m\times m}]$

i) Diagonal matrix (non diagonal = 0)

$\begin{bmatrix}1&0&0\\0&2&0\\0&0&0\end{bmatrix}$ may/may not be 0

- Trace of matrix = Sum of P.D.E
- Determinant = Product " "
- ~~Inverse~~ $= \begin{bmatrix}a&0\\0&b\end{bmatrix}^{-1} = \begin{bmatrix}1/a&0\\0&1/b\end{bmatrix}$
- Product $= \begin{bmatrix}1&0\\0&2\end{bmatrix}\begin{bmatrix}10&0\\0&1\end{bmatrix} = \begin{bmatrix}10&0\\0&2\end{bmatrix}$
- Min^m no. of zero's in DM $= n^2-n = n(n-1)$

Note :) $\begin{bmatrix}0&0&0\\0&2&0\\0&0&1\end{bmatrix}^{-1} = \begin{bmatrix}1/0&0&0\\0&1/2&0\\0&0&1/1\end{bmatrix}$ Inverse Don't exist

ii) Unit/Identity matrix $(I) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$I\cdot I\cdot I = I;\ |I| = 1;\ AI = IA = A$

iii) Null/Zero matrix $= \begin{bmatrix}0&0\\0&0\end{bmatrix}$ or $\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix}$

~~(can be sq. or Rect)~~ (i.e. all zeros)

iv) Triangular matrix =

UTM $\begin{bmatrix}1&2&3\\0&0&5\\0&0&6\end{bmatrix}$ may or may not be zero

LTM $\begin{bmatrix}1&0&0\\2&3&0\\4&5&0\end{bmatrix}$

v) Scalar matrix ⇒ $A = k\cdot I$

vi) Traspose " ⇒ A^T

vii) Conjugate of matrix ⇒ $\bar{A}$

$(\overline{a+ib} = a-ib)$

(viii) Real matrix :

Symmetric $A = A^T$

$\begin{bmatrix}1&2\\2&0\end{bmatrix}$

Skew Symm. $A = -A^T$

i.e. Diagonals = 0

$a_{ij} = -a_{ji}$

$\begin{bmatrix}0&-2\\2&0\end{bmatrix}$

If A, B symm. then A·B may/may not be symm.

ix) Orthogonal matrix ⇒ $A\cdot A^T = A^T\cdot A = I$

Here • $|A| = \pm 1$; $|A| = |A^T|$; $A^{-1} = A^T$

X) Hermitian $A = A^*$

$\boxed{A^* = (A^T)^c}$ i.e. $(\overline{A^T})$

i.e.
1. Diagonals = Real
2. $a_{ij} = \bar{a}_{ji}$

$\begin{bmatrix}5&2+3i\\2-3i&2\end{bmatrix}$

Skew Hermitian $A = -A^*$

i.e.
1. Diagonals = 0 or purely imaginary
2. $a_{ij} = -\bar{a}_{ji}$

$\begin{bmatrix}0&-2-3i\\2-3i&0\end{bmatrix}$

x) Unitary Matrix : $A\cdot A^* = A^*\cdot A = I$

xi) Special matrices :

① Involutory $A^2 = I$
② Idempotent $A^2 = A$
③ Nilpotent $A^k = 0$; k = index/class

*) Singular matrix $|A| = 0$ Non singular matrix $|A| \neq 0$

*) $A^{-1} = \dfrac{\text{adj}A}{|A|}$

→ Do not exist ⇒ A is singular (Non invertible)
→ Exist ⇒ A is non- " (invertible)

*) Any Square matrix = Symm. matrix + Skew symm. matrix

$= \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

- A = Any sq. matrix, then $A\cdot A^T$ = Always Symm.
- A, B = Symmetric ⇒ A+B, A-B = Always symm.

*) Operations on matrix :

① Add^n : $A+B = B+A$; $(A+B)+C = A+(B+C)$
Commutative   Associative

② Sub^n : $A-B \neq B-A$ $(A-B)-C \neq A-(B-C)$
Commutative ✗   Associative ✗

③ Scalar Mul^n : $2\begin{bmatrix}1&2\\0&1\end{bmatrix} = \begin{bmatrix}2&4\\0&2\end{bmatrix}$; $|kA| = k^n|A|$

④ Mul^n : $\underset{(x\times y)}{A}\cdot\underset{(y\times z)}{B} = \underset{(x\times z)}{C}$; $AB \neq BA$ commutative ✗

- If A·B exists then B·A exists? ⇒ Not Always
- If AB = BA ⇒ B = I (or) B = ϕ (or) B = A^{-1}
- If AB = AC and
  - i) A is singular ⇒ |A| = 0 ⇒ A^{-1} (DNE) ⇒ B ≠ C
  - ii) A is non-singular ⇒ |A| ≠ 0 ⇒ A^{-1} exists ⇒ B = C
- AB = 0 Doesn't necessarily imply that either A or B is 0. may be 0 may not be 0
- When AB = 0 & A, B are not zero matrix then |A| = |B| = 0
- A → m×n   B → n×p   no. of mul^{n}s = m·n·p ; no. of Add^{n}s = m·(n-1)p
- $(P+Q)^2 = (P+Q)\cdot(P+Q) = P^2+Q^2+PQ+QP$

*) Determinant : $A = \begin{bmatrix}1&2&3\\0&1&1\\1&0&10\end{bmatrix}$

① Minor : $M_{32} = \begin{vmatrix}1&3\\0&1\end{vmatrix} = 1$

② Cofactor : $(-1)^{i+j}M$ $M_{32} = -\begin{vmatrix}1&3\\0&1\end{vmatrix} = -1$

•) Sarass Method : $\begin{bmatrix}1&2&0\\-1&6&1\\2&0&2\end{bmatrix}$

$(1\cdot6\cdot2)+(2\cdot1\cdot2)+(0\cdot-1\cdot0)$
$-(2\cdot6\cdot0)-(0\cdot1\cdot1)-(2\cdot-1\cdot2)$
$= 12+4+4 = 20$

Note: If Δ of $\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} = x$; then Δ of $R_2 \leftrightarrow R_3$ $\begin{vmatrix}1&2&3\\7&8&9\\4&5&6\end{vmatrix} = (-1)^1 \cdot x$

Note: $-1 * \begin{bmatrix}-6\\-6\\4\end{bmatrix} = \begin{bmatrix}+6\\+6\\-4\end{bmatrix}$

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- UTM, LTM, Diagonal Matrix :
  $|A| =$ Product of diagonal elements
  $\begin{vmatrix}1&2&3\\0&4&5\\0&0&1\end{vmatrix}, \begin{vmatrix}1&0&0\\2&4&0\\5&1&1\end{vmatrix}, \begin{vmatrix}1&0&0\\0&2&0\\0&0&3\end{vmatrix} \Rightarrow |A| =$ Product of D.E

- $|A^{-1}| = 1/|A|$, $\boxed{|X \cdot Y| = |X| \cdot |Y|}$
  $|A^T| = |A|$

- $|B| = (-1)^n |A|$; $n =$ no. of times Rows/Columns interchanged

- $|A| \xrightarrow[C_i \,:\, C_i \pm K \cdot C_j]{R_i \,:\, R_i \pm K \cdot R_j} |A|$ {Same}

- $\Delta = 0$ when {1 row or 1 column is zero}
  (or) {2 rows or 2 columns are same}
  (or) {2 rows or 2 columns are dependent}

- $\begin{vmatrix}a_1+p & b_1+q & c_1+r\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \begin{vmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix} + \begin{vmatrix}p&q&r\\a_2&b_2&c_2\\a_3&b_3&c_3\end{vmatrix}$

- Factor Theorem: If $(x-a)$ is a factor then $\Delta = 0$ when $x = a$

- $\begin{vmatrix}1+a&1&1&1\\1&1+b&1&1\\1&1&1+c&1\\1&1&1&1+d\end{vmatrix} = (abcd)\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$

- $|\text{adj} A| = |A|^{n-1}$
  $|\text{Adj}(\text{Adj } A)| = |A|^{(n-1)^2}$

- Area of Δ $(x_1,y_1)(x_2,y_2)(x_3,y_3)$
  $A = \frac{1}{2}\begin{vmatrix}x_1&y_1&1\\x_2&y_2&1\\x_3&y_3&1\end{vmatrix}$ A = 0 for collinear points

- A^{-1} Trick for 2×2
  $\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1} = \frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

- adj A Trick for 3×3 or above matrix
  $A = \begin{bmatrix}1&0&1\\-1&1&1\\0&1&0\end{bmatrix}$

  $\begin{matrix}1&0&1&1&0\\-1&1&1&-1&1\\0&1&0&0&1\\1&0&1&1&0\\-1&1&1&-1&1\end{matrix}$

  $\begin{bmatrix}-1&1&-1\\0&0&-2\\-1&-1&1\end{bmatrix} = \text{adj } A$

- A^{-1} Given, $A = ?$; $(A^{-1})^{-1} = \frac{\text{adj } A^{-1}}{|A^{-1}|}$

- $\begin{bmatrix}a&0&0\\0&b&0\\0&0&c\end{bmatrix}^{-1} = \begin{bmatrix}1/a&0&0\\0&1/b&0\\0&0&1/c\end{bmatrix}$

- Orthogonal matrix : $A^{-1} = A^T$

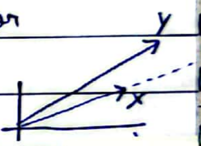
- $A \cdot A^{-1} = I$
  $\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} A^{-1} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} A^{-1} = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

- $X^2 - X + I = 0$
  $X^2 - X = -I$
  $X - I = -X^{-1}$
  $X^{-1} = I - X$

- $(7A)^{-1} = 7^{-1} \cdot A^{-1} = \frac{1}{7} \cdot A^{-1}$

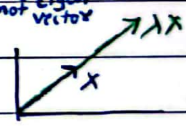
*) Eigen value, Eigen Vector :
eigen vector remain in it's span, just scaled

- $AX = \lambda X \Rightarrow X$ is Eigen vector
  $AX = \lambda Y \Rightarrow X$ is not " "



Ex:) $A = \begin{bmatrix}1&2\\2&4\end{bmatrix}, X = \begin{bmatrix}1\\1\end{bmatrix}$
$AX = \begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix} = 3\begin{bmatrix}1\\2\end{bmatrix} = \lambda Y$ Eigen vector, X is not

Ex:) $A = \begin{bmatrix}1&2\\2&4\end{bmatrix}, X = \begin{bmatrix}1\\2\end{bmatrix}$
$AX = \boxed{\begin{bmatrix}1&2\\2&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}} = \begin{bmatrix}5\\10\end{bmatrix} = \boxed{5\begin{bmatrix}1\\2\end{bmatrix}}$
AX Transposition Matrix → λX ← Eigen value, eigen vector



- $|A - \lambda I| = 0 \Rightarrow$ Characteristic eq.^{n}
  " polynomial

ex:) $A = \begin{bmatrix}-1&4\\4&-1\end{bmatrix}$
$|A - \lambda I| = \begin{vmatrix}-1-\lambda & 4\\ 4 & -1-\lambda\end{vmatrix} = 0 \Rightarrow \lambda^2 + 2\lambda - 15 = 0$
$\lambda = 3, -5$

- Characteristic eq.^{n}:
  (2×2) $\lambda^2 - S_1\lambda + |A| = 0$
  (3×3) $\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$
  - $S_1 = \Sigma$ D.E
  - $S_2 = \Sigma$ minors of D.E

Ex:) $A = \begin{bmatrix}0&1&0\\0&0&1\\-6&-11&-6\end{bmatrix}$ $S_1 = -6$
$S_2 : M_{11} + M_{22} + M_{33} = 11 + 0 + 0 = 11$

$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$
Now, $\lambda = 0$ ✗ not factor
$\lambda = 1$ ✗ " "
$\lambda = -1$ ✓ factor

Note: other names of Eigen values
- Latent roots
- Proper values
- Characteristic values

Synthetic Division:

	1	6	11	6
−1	0	−1	−5	−6
	1	5	6	$\boxed{0}$

Mul.^{n} ↗ Add ↓

$1\lambda^2 + 5\lambda + 6 = 0$
$\lambda = -2, -3$

Therefore $\lambda = -1, -2, -3$

•) $A \to \lambda_1, \lambda_2$
$A^{100} \to \lambda_1^{100}, \lambda_2^{100}$
$I + A^{100} \to 1+\lambda_1^{100}, 1+\lambda_2^{100}$

•) $A_{n\times n} \Rightarrow \lambda =$ max^m n eigen values
$A_{2n\times 2n} \Rightarrow \lambda =$ " 2n " "

•) Σ Eigen values = Trace of matrix = Σ D.E
π Eigen values = $|A|$

•) Trick: $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ → $\boxed{a+b+c = \lambda_1}$ (or)

$\boxed{\lambda_1 + \lambda_2 + \lambda_3 = \Sigma D.E}$ (Property)

$\boxed{\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|}$ (Property)

•) UTM, LTM, Diagonal matrix:
Eigen values = Diagonal elements itself

•) Eigen values → Real ⇒ 0/+/−
→ Imaginary
→ $a+ib, a-ib$
or → $a-ib, a+ib$

Ex:) $3, 3+5i, 3-5i$ ✓
$3+i, 3-i, 5+i$ ✗
$3, 1+3i, -1-3i$ ✗

I) $A = \begin{bmatrix} 8 & -4 \\ 2 & 2 \end{bmatrix}$ → $\lambda_1 = 4$, $\lambda_2 = 10-4 = 6$

$\lambda = 4, 6$
$(A-\lambda I)X = 0$

$\begin{bmatrix} 8-4 & -4 \\ 2 & 2-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 4x_1 - 4x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = 0$

$x_1 = x_2$

$X = K\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 8-6 & -4 \\ 2 & 2-6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$2x_1 - 4x_2$
$x_1 = 2x_2$

$X = K\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

II) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ $X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

method-1 $(A-\lambda I)X = 0$

$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = 0$

$(-2-\lambda)+4+3 = 0$
$\lambda = 5$

method-2 $AX = \lambda X$

$\begin{bmatrix} -2+4+3 \\ 2+2+6 \\ -1+4+0 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$5\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\lambda = 5$

•) $A \longrightarrow \lambda_1, \lambda_2$
$A^{-1} \longrightarrow \lambda_1^{-1}, \lambda_2^{-1} \to 1/\lambda_1, 1/\lambda_2$
$A^n \longrightarrow \lambda_1^n, \lambda_2^n$
$A^T \longrightarrow \lambda_1, \lambda_2$
adj $A \longrightarrow |A| \cdot A^{-1} \longrightarrow |A|\cdot\lambda_1^{-1}, |A|\cdot\lambda_2^{-1}$
$A+KI \longrightarrow \lambda_1+K(1), \lambda_2+K(1)$
λ_1+K, λ_2+K
$A^3 - 3A^2 \longrightarrow (\lambda_1)^3 - 3(\lambda_1)^2, (\lambda_2)^3 - 3(\lambda_2)^2$

•) Trace of Matrix = Σ of DE
Σ of E values
E. values

III) $A = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = -2$

$(A-\lambda I)X = 0 \Rightarrow \begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$5x_1 - 2x_2 + 2x_3 = 0$
$x_3 = 0$
$x_1 = 2/5\, x_2$ $\Rightarrow X = K\begin{bmatrix} 2/5 \\ 1 \\ 0 \end{bmatrix}$

•) Nature
1] Hermitian/Symmetric → Real
2] Skew hermitian/skew symm. → 0 or purely imag.
3] Unitary/orthogonal → $|\lambda| = 1$
4] Idempotent → $A^2 = A \Rightarrow \lambda^2 = \lambda \Rightarrow \lambda^2 - \lambda = 0$
$\Rightarrow \lambda = 0, 1$
5] Involutory → $-1, 1$

IV) $\lambda = -1, -2$; $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$AX_1 = \lambda_1 X_1$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$a - b = -1$ — ①
$c - d = 1$ — ②

$AX_2 = \lambda_2 X_2$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$a - 2b = -2$ — ③
$c - 2d = 4$ — ④

$a = 0$
$b = 1$
$c = -2$
$d = -3$
$\Rightarrow A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

•) 4 cases: [A, λ, X]
I: A ✓, λ = ?, X = ?
II: A ✓, X ✓, λ = ?
III: A ✓, λ ✓, X = ?
IV: X ✓, λ ✓, A = ?

•) Eigen values = Real ⇒
eigen vectors = Real
Eigen values = complex no. ⇒
eigen vector = Complex no.

•) $A \to \lambda_1 \to X_1$
$A^m \to \lambda_1^m \to X_1$

4

Example:

$A_{(4\times4)} = \begin{bmatrix} \cancel{2} & \cancel{3} & \cancel{4} & \cancel{5} \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ $\Delta(A)=0$, so so 3X3

$A\,(3\times3)$ $\begin{bmatrix} \cancel{4} & \cancel{5} & \cancel{6} \\ 5 & 6 & 7 \\ 10 & 11 & 12 \end{bmatrix}$ $\Delta(A)=0$, so so 2x2

$A\,(2\times2)$ $\begin{bmatrix} 6 & 7 \\ 11 & 12 \end{bmatrix}$ $\Delta(A) = -5 \neq 0$

$\boxed{\Rightarrow \text{Rank}(A) = 2}$

-) If $\lambda = 3, 3, 5$
Algebraic Multiplicity of $3 = 2$

-) If A is Symmetric Matrix, then
- Eigen values are Distinct
- $\lambda_1 \to X_1$, $\lambda_2 \to X_2$, $\lambda_3 \to X_3$ } And vectors are orthogonal vectors
  - $\to X_1 \cdot X_2 = 0$ (vectors)
  - $\to X_1^T \cdot X_2 = 0$ (Matrices)
  - $X_2^T \cdot X_3 = 0$
  - $X_1^T \cdot X_3 = 0$

-) Cayley Hamilton: Every sq. matrix satisfies its char. eq^n

A → sq. matrix

i) $A = \lambda$ ii) $\lambda = A$

$\lambda^2 - S_1\lambda + |A| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

Ex: $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $A^5 = ?$

$\lambda^2 - S_1\lambda + |A| = 0$

$\lambda^2 - 3\lambda - 10 = 0$

$A^2 - 3A - 10 = 0$

$A^2 = 3A + 10$

$A^5 = A \cdot A^2 \cdot A^2 = A(3A+10)(3A+10)$

$= A(9A^2 + 100 + 60A)$

$= \cancel{A[9(3A+10)+100+60A]}$

$= 27A^2 + 190A + 60A^2$

$= 87(3A+10) + 190A$

$= 451A + 870I$

-) Normalized Eigen Vector

x) $X = K\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$X = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

-) Rank of Matrix: Max^m no. of linearly independent Rows/Cols.
Rank = no. of non-zero rows in echelon matrix.

-) To find Rank —

M1: Reduce to Triangular matrix

M2: $|M|_{3\times3}$ → $=0 \to \rho(A) \neq 3$; $\neq 0 \to \rho(A) = 3$

i.e. $|A|_{3\times3} = 0 \longrightarrow \rho(A) \neq 3$

$|A|_{2\times2} = 0 \longrightarrow \rho(A) \neq 2$

$|A|_{1\times1} \neq 0 \longrightarrow \rho(A) = 1$

Example: —

-) $\cancel{\rho(A+B) \neq \rho(A) + \rho(B)}$

$\boxed{|A+B| \neq |A| + |B|}$

$\rho(A+B) \leq \rho(A) + \rho(B)$

$\rho(A-B) \geq \rho(A) - \rho(B)$

-) For $A_{m\times n}$; $\rho(A) \leq \min[m, n]$

-) A = sq. matrix & $\rho(A)$ = no. of rows/no. of cols
$\Rightarrow \rho(A) = n$

-) A✓, B✓ then

$\boxed{\rho(AB) \leq \min[\rho(A), \rho(B)]}$

special case -) A✓, A^T✓ then

$\rho(A \cdot A^T) = \rho(A^T A) = \rho(A) = \rho(A^T)$

-) Rank of $A \geq 0$

if A non-zero, then $\rho(A) > 0$

if A zero, then $\rho(A) = 0$

if A → col matrix (non-zero), then $\rho(AB) = 1$
B → Row ''

-) Row rank / col Rank / Rank सेम है।

BTW, col Rank of A = Row Rank of A^T
& Row Rank of A = col Rank of A = Rank of A

*) Sol^n of Linear Simultaneous Equations: —

-) $2x + 3y = 7$
$x - y = 1$

$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 1 & -1 & 1 \end{array}\right]$ A (coefficient) B (constant)

$\rho(A) = 2$
$\rho(A:B) = 2$

-) Nature:

{unique sol^n}	{IMS}	{No Sol^n}
$\rho(A) = \rho(A:B) =$ no. of variables	$\rho(A) = \rho(A:B) <$ no. of variables	$\rho(A) \neq \rho(A:B)$

-) Solution:

$a_{11}x + a_{12}y + a_{13}z = b_1$

$a_{21}x + a_{22}y + a_{23}z = b_2$

$a_{31}x + a_{32}y + a_{33}z = b_3$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}$$

$\boxed{AX = B}$

AX=B {Non zero}
<Non-Homogenous system>
- → US
- → IMS
- → NS

$|A| \neq 0 \Rightarrow$ US
$|A| = 0 \Rightarrow$ IMS (or) NoSol^n

AX=B {zero}
i.e. AX=0
<Homogeneous System>

US: $\rho(A) = \rho(A:B) = n$; $|A| \neq 0$; US/zero/Trivial sol^n

IMS: $\rho(A) = \rho(A:B) < n$; $|A| = 0$; IMS/Non-zero/Non-Trivial sol^n

• Ex: $2x+3y+z=9$
$x+2y+3z=6$
$3x+y+2z=8$

Nature: $[A:B]$

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 9 \\ 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 8 \end{array}\right]$$

$R_1 \leftrightarrow R_2$; R_2-2R_1; R_3-3R_1

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & -5 & -7 & -10 \end{array}\right]$$

R_3-5R_2

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -1 & -5 & -3 \\ 0 & 0 & 18 & 5 \end{array}\right]$$

$\rho(A) = \rho(A:B) = 3$
$\Rightarrow$ Unique sol.^{n}

Sol.^{n}: $AX=B$ <Also called Gauss elimination method>

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 5 \end{bmatrix}$$

$x = 35/18$, $z = 5/18$, $y = 29/18$

• Ex: $x+y=0$
$x-y=0$

① Homogeneous
② $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$

$|A| \neq 0$
$\Rightarrow$ Unique sol.^{n}
$(x,y) = (0,0)$
Zero sol.^{n}

• Ex: $x+y=0$
$7x+7y=0$

① Homogeneous
② $\begin{vmatrix} 1 & 1 \\ 7 & 7 \end{vmatrix} = 0$

$|A| = 0$
$\Rightarrow$ IMS
$(x,y) = (-k,k)$

• Cramer's Rule: (D = Daddu matrix, just to remember)
or
$d_1 d_2 d_3$

$a_1x+b_1y+c_1z=d_1$
$a_2x+b_2y+c_2z=d_2$
$a_3x+b_3y+c_3z=d_3$

① $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ $D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

② $x = \dfrac{D_x}{D}$

$y = \dfrac{D_y}{D}$

$z = \dfrac{D_z}{D}$

Notes)
$AX=B$
- $A \neq 0$: US
- $|A| = 0$
  - IMS: $D_x = D_y = D_z = 0$
  - NS

Note:) कुछ Sol.^n s इनसे भी निकाल सकते हो :-

$\boxed{\begin{aligned} AX &= B \\ X &= A^{-1}B \end{aligned}}$ i.e. A^{-1} निकाल के B is Mul.

PRADEEP
Page No.: But for λ_1 & λ_2 we've to check if they LI or LD.
Date:

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Note: If $A_{3\times3}$ & $\lambda_1 = 7$, $\lambda_2 = 7$, $\lambda_3 = 3$; corresponding to λ_3, we do have LI vector

No. of Linearly Independent Solutions in homogeneous System :-

Ex-I) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; No. of LI Sol.^n = ?

$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$ $\Rightarrow X = K\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2$

$(x_1, x_2, x_3) = (-2K, -2K, K)$

$\Rightarrow$ LI Sol.^n s = 1

Trick :-

$\boxed{\text{No. of LI Sol.}^n = \underset{\text{no. of vars.}}{n} - r(A)}$

$= 3 - 2 = 1$

Ex-III) $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$, LI vectors = ?

$\lambda = 1, 2, 2$

$(A - 2I)X = 0$

$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $x_3 = 0$

$-x_1 + 2x_2 + 2x_3 = 0$

$x_1 = 2x_2$

$X_1 = K\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ LI vector = 1

Ex-II) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$; Linearly Indep. Vectors = ?

$\lambda = -3, -3, 5$ -3 के लिए check करो, 5 तो है ही distinct

i) $\lambda = -3$, $X = ?$

$(A - \lambda I)X = 0$

$(A + 3I)X = 0$

$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = 3x_3 - 2x_2$

let, $x_2 = K_1$, $x_3 = K_2$

$\Rightarrow x_1 = 3K_2 - 2K_1$

$\Rightarrow X = \begin{bmatrix} 3K_2 - 2K_1 \\ 1K_1 + 0K_2 \\ 0K_1 + 1\cdot K_2 \end{bmatrix} = K_1\underset{X_1}{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} + K_2\underset{X_2}{\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$

LI vectors = 2

Trick :-

$\boxed{\text{No. of LI Vectors} = n - r(A - \lambda I)}$

$= 3 - 1 = 2$

Ex-IV) Vectors: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; LI = ?

M-1 $\boxed{\begin{aligned} \Delta = 0 &\Rightarrow \text{Dep.} \\ \Delta \neq 0 &\Rightarrow \text{Indep.} \end{aligned}}$ $\begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -2 & 0 & -2 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -2\begin{vmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = +2 \neq 0 \Rightarrow$ Indep.

M-2 Rank = n

M-3 In $K_1X_1 + K_2X_2 + K_3X_3$

If we get $K_1 = K_2 = K_3 = 0$ then LI vectors

Note:)
λ = Same $\Rightarrow$ LI / LD
λ = Distinct $\Rightarrow$ LI

*) Diagonalization of Matrix:

① $\boxed{P^{-1}AP = D}$

②(or) $\boxed{A = PDP^{-1}}$

note:) $\boxed{A^n = PD^nP^{-1}}$

$e^A = Pe^DP^{-1}$

Vectors: $X_1 \quad X_2 \dots$

Here, A = Square matrix whose Eigen Values are $(\lambda_1, \lambda_2, \dots)$ — Distinct / Same

Distinct → LI; Same → LD, LI

LD → $|P| = 0$

P = Modal Matrix $= [X_1 \; X_2 \dots]$

$P^{-1} = \dfrac{\text{adj } P}{|P|}$

$D = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \dots & \\ 0 & 0 & \lambda_3 & \\ \dots & & & \end{bmatrix}$ Digonal Matrix

Ex:) $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

① $\lambda = 5, 2$

$\lambda_1 = 2 \longrightarrow X_1 = K_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_2 = 5 \longrightarrow X_2 = K_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

② $P = [X_1 \; X_2] = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$

$P^{-1} = \dfrac{\text{adj } P}{|P|} = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$

$P^{-1}AP = \dfrac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D$

Ex-2) $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$; $A^{50} = ?$

$\lambda = 1, 3$

$\lambda_1 = 1 \longrightarrow X_1 = K_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3 \longrightarrow X_2 = K_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$A = PDP^{-1}$

$A^{50} = PD^{50}P^{-1}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^{50}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$ = Ans

Alter: $A^{50} = \alpha_1 A + \alpha_0 I$

$\lambda^{50} = \alpha_1 \lambda + \alpha_0$

$\lambda = 1 \rangle \; 1^{50} = \alpha_1 + \alpha_0$

$\lambda = 3 \rangle \; 3^{50} = 3\alpha_1 + \alpha_0$ $\rangle \; \alpha_1 = \dots, \; \alpha_0 = \dots$

Ans ⇒ $A^{50} = \alpha_1 A + \alpha_0 I$

*) Do Little approach: (LU Decomposition of Matrix)

where,

$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ $U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

Ex:) Find Sol^n of system

$x + 3y + 8z = 4$

$x + 4y + 3z = 2$

$x + 3y + 4z = 1$

Step-1: $AX = B$

$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$

Step-2: $\boxed{A = LU}$

i.e.

$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$

$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + U_{22} & l_{21}U_{13} + U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + U_{33} \end{bmatrix}$

by equating LHS = RHS

$u_{11} = 1$; $u_{12} = 3$; $u_{13} = 8$

$l_{21} = 1$; $u_{22} = 1$; $u_{23} = -5$

$l_{31} = 1$; $l_{32} = 0$; $u_{33} = -4$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Step-4: $\boxed{AX = B}$

$$(LU)X = B$$

$$\boxed{L(UX) = B}$$

$$\boxed{L\ Y = B}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\left.\begin{matrix} y_1 = 4 \\ y_2 = -2 \\ y_3 = -3 \end{matrix}\right] \Rightarrow Y = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$$\Rightarrow \boxed{UX = Y}$$

$$\begin{bmatrix} 1 & 3 & 8 \\ 0 & 1 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

$$x = \frac{-29}{4},\ y = \frac{7}{4},\ z = \frac{3}{4}$$

*) Crout's Approach:-

Step 1: $AX = B$

Step 2: $A = LU$

Step 3: $(LU)X = B$

$$L(UX) = B$$

$$L\ Y = B$$

then u calc. $Y = ✓$

Step 4: $Y = UX$

$x = ✓, y = ✓, z = ✓$

But,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

Ch 2: Prob.

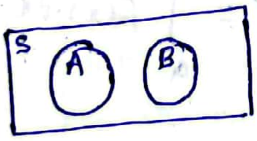
#) Basics:

1. Experiment: (Rolling a die)
2. Sample space: (All possible outcome)
   $S = \{1, 2, 3, 4, 5, 6\}$
3. Event: (Even no. on die).
   $E = \{2, 4, 6\}$
4. Probability:
   $P(\text{Even no.}) = \frac{Req.}{Total} = \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

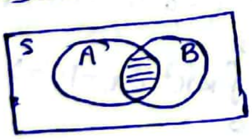
Experiment → SS → Event → P(E)

*) Different types of events:

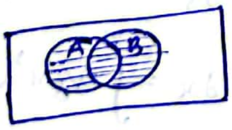
1. Complimentary events:
   $P(A') = 1 - P(A)$
2. Disjoint events:



3. Intersection:



4. Either A or B:



5. Only A:

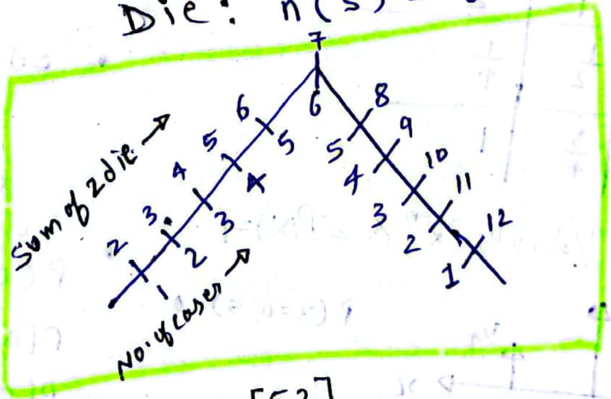
$A - (A \cap B)$

Note 1) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

6. Neither A nor B $= \overline{(A \cup B)} = S - (A \cup B)$
   $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$
7. Sure event: $P(E) = 1$
8. Null event: $P(E) = 0$
9. Equally likely event: $P(H) = P(T) = 1/2$
10. Mutually exclusive events:
    $A \cap B = \phi \Rightarrow P(A \cap B) = 0$
    $P(A \cup B) = P(A) + P(B)$
11. Independent events:
    $P(A \cap B) = P(A) \cdot P(B)$
12. Dependent events:
    $P(A \cap B) \neq P(A) \cdot P(B)$
    $= P(A) \cdot P(B/A)$
    $= P(B) \cdot P(A/B)$

#) Coin: $n(S) = 2^n$

Die: $n(S) = 6^n$



Cards: [52]

52 { Red [Heart ♡ - 13, Diamond ◇ - 13] ; Black [Spade ♤ - 13, Club ♧ - 13] }

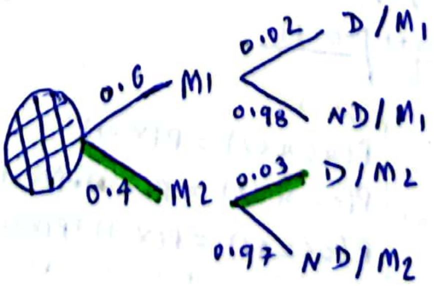
Number Cards (9) — Face cards (3)

A $\boxed{2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10}$ $\boxed{J\ Q\ K}$

Non-Face cards (10)

#) Ex:

In a factory, two machines M_1 & M_2 manufacture auto components respectively, 60% & 40% of the total production. 2% of M_1 & 3% of M_2 are defective. If a randomly drawn auto component from the combined lot is found defective, what is the probability that it was manufactured by M_2.



$P(M_2/D) = ?$

$P = \frac{Req.\ path}{Total\ path} = \frac{0.4 \times 0.03}{0.4 \times 0.03 + 0.02 \times 0.6} = 0.5$

$$P(m_2/D) = \frac{P(D/m_2)\cdot P(m_2)}{P(D/m_1)\cdot P(m_1) + P(D/m_2)\cdot P(m_2)}$$

$$= \frac{0.03\times 0.4}{0.6\times 0.02 + 0.03\times 0.4}$$

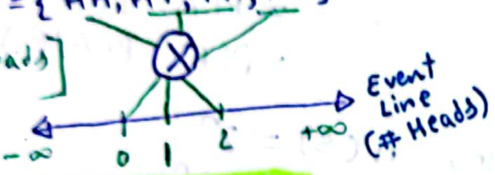
$$= 0.5$$

#) Random variables in 1D :-

Experiment: Tossing 2 coin

$S = \{HH, HT, TH, TT\}$

[X = no. of heads
X = x]

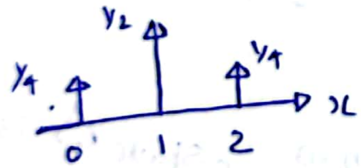


*) DRV:

X = x	0	1	2
P(X=x) pdf	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
F(x) cdf	$\frac{1}{4}$	$\frac{3}{4}$	1

P(X) = Prob. mass fxn / density fxn, & $\sum P(X) = 1$:

pdf



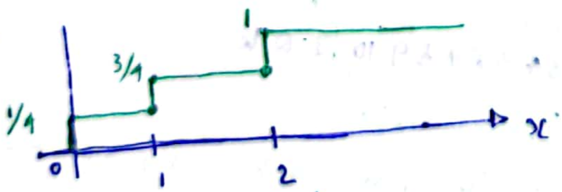
Stem plot

$P(x = 0.5) = 0$

F(x) = cumulative distrn fxn

$F(X<0) = 0$

cdf



Staircase plot

cdf < min.m value = 0
max.m " = 1

Note:) In DRV, X → countable
→ +/-/0
→ x | +1 | -0.33

Note:)

X	0	1	2
P(X)	K/4	2K/4	K/4

$\sum P = 1 \Rightarrow K/4 + 2K/4 + K/4 = 1$
$\Rightarrow K = 1$

Normalization factor / " constant

Note:)
$P(0<x<2) = P(X=1)$
$P(0\le x<2) = P(X=0)+P(X=1)$
$P(0<x\le 2) = P(X=1)+P(X=2)$

Note:) Ex: $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = K$
$X = 1,2,3,4$; pdf = ?

X	1	2	3	4
P(X)	15/61	10/61	30/61	6/61

$P(X=1) = K/2$
$\Rightarrow P(X=2) = K/3$
$P(X=3) = K$
$P(X=4) = K/5$

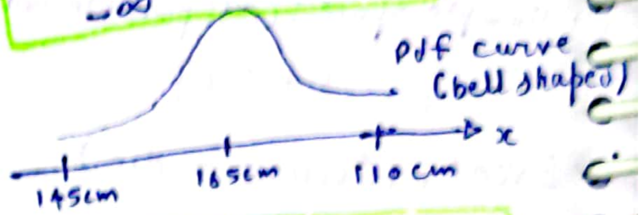
$\sum P = 1 \Rightarrow K/2 + K/3 + K + K/5 = 1$
$\Rightarrow K = 30/61$

*) CRV: In CRV, X → in Range

10

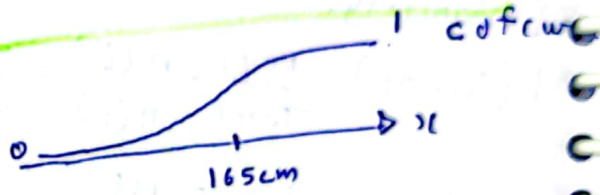
- PDF: f(x) is pdf

$$\boxed{\int_{-\infty}^{\infty} f(x)\cdot dx = 1}$$



- CDF:

$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\cdot dx}$$



- cdf ⟷ pdf

$$\boxed{\frac{d}{dx}F(x) = f(x)}$$

- $\left.\begin{array}{l} P(a<x<b) \\ P(a<x\le b) \\ P(a\le x<b) \\ P(a\le x\le b)\end{array}\right\} = \int_a^b f(x)\cdot dx$

- Ex:) $f(x) = kx^2$; $0<x<2$

i) k = ? ii) $P(0.2\le x\le 0.5)$ iii) cdf

i) $\int_{-\infty}^{\infty} f(x)\cdot dx = 1 \Rightarrow \int_0^2 kx^2\, dx = 1 \Rightarrow k = 3/8$

ii) $P(0.2\le x\le 0.5) = \int_{0.2}^{0.5} 3/8\, x^2\cdot dx = 0.014$

iii) $F(x) = \int_{-\infty}^{x} 3/8\, x^2\, dx = \int_0^x 3/8\, x^2\, dx = x^3/8$

$(0<x<2)$

- Ex:) $f(x) = 6x(1-x)$; $0\le x\le 1$

i) find b if $P(X\le b) = P(X>b)$

$$\int_{-\infty}^{b} f(x)\,dx = \int_b^{\infty} f(x)\,dx$$

$$\int_0^b 6x(1-x)\cdot dx = \int_b^1 6x(1-x)\cdot dx$$

$b = 1/2$

•EX: $f(x) = \begin{cases} k(1+x) & ; -1 < x \le 0 \\ k(1-x) & ; 0 < x \le 1 \\ 0 & ; o/w \end{cases}$

Find i) k ii) Plot f(x) iii) cdf

i) $\int_{-1}^{0} k(1+x)\,dx + \int_{0}^{1} k(1-x)\,dx = 1$

$k = 1$

ii) $f(x) = \begin{cases} 1+x & ; -1 < x \le 0 \\ 1-x & ; 0 < x \le 1 \end{cases}$

iii) $F(X < -1) = 0$

$F(-1 < X \le 0) = F(X \le 0) = \int_{-1}^{x} (1+x)\,dx$

$= x + \frac{x^2}{2} + \frac{1}{2}$

$F(0 < X \le 1) = F(X \le 1)$

$= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{x} (1-x)\,dx$

$= x - \frac{x^2}{2} + \frac{1}{2}$

$F(X > 1) = 1$

i.e.

cdf $F(x) = \begin{cases} 0 & ; x < -1 \\ x + x^2/2 + 1/2 & ; -1 < x \le 0 \\ x - x^2/2 + 1/2 & ; 0 < x \le 1 \\ 1 & ; 1 < x \end{cases}$

*) Random variables in 2D:

*) EX:) $P(X_1 = x_1, X_2 = x_2) = \frac{1}{27}(x_1 + 2x_2)$

DRV

$x_1 = 0,1,2$

$x_2 = 0,1,2$

pdf:

$P(X_1, X_2)$: $x_1 \downarrow$ / $x_2 \rightarrow$	0	1	2
0	0	2/27	4/27
1	1/27	3/27	5/27
2	2/27	4/27	6/27

Sum = 1

$P(X_1)$ =

x_1	0	1	2
$P(x_1)$	6/27	1/3	4/9

eq^n: $P(X_1) = \sum_{x_2=0}^{2} P(X_1, X_2)$

$= P(X_1, 0) + P(X_1, 1) + P(X_1, 2)$

$= \frac{x_1}{27} + \frac{x_1+2}{27} + \frac{x_1+4}{27}$

$= (x_1 + 2)/9$

$P(X_2)$ =

x_2	0	1	2
$P(x_2)$	1/9	3/9	5/9

Note:) •$P(X \le 1, Y > 2)$; $X = 0,1,2,3,4$; $Y = 0,1,2,3,4$

$= P(X = 0,1 ; Y = 3,4)$

$= P(X=0, Y=3) + P(X=0, Y=4) + P(X=1, Y=3) + P(X=1, Y=4) = 3/8$

Note:) •$P(Y=3 / X=3) = \frac{P(Y=3, X=3)}{P(X=3)}$

$X = 1,2,3$
$Y = 1,2,3$

•$P(Y \le 2) = P(Y=1) + P(Y=2)$

•$P(X+Y < 4) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1) = 1/4$

*) Ex:) $f(x,y) = \begin{cases} x^3y^3/16 & ; 0 < x < 2, 0 < y < 2 \\ 0 & ; o/w \end{cases}$

CRV

i) $f(x) = \int_{y=0}^{2} f(x,y)\,dy = \int_{0}^{2} \frac{x^3y^3}{16}\,dy = x^3/4$

ii) $f(y) = \int_{x=0}^{2} f(x,y)\,dx = \int_{0}^{2} \frac{x^3y^3}{16}\,dx = y^3/4$

iii) $f(x,y) = f(x) \cdot f(y)$ [Independent]

Ex:) $f(x,y) = \begin{cases} x+y & ; 0 < x < 1, 0 < y < 1 \\ 0 & ; o/w \end{cases}$

Is it a pdf?

$\boxed{\int_{x=0}^{1}\int_{y=0}^{1} (x+y)\,dy\,dx} = \int_{0}^{1} \left(xy + \frac{y^2}{2}\right)_0^1 dx = \int_{0}^{1} \left(x + \frac{1}{2}\right) dx = 1$

$\Rightarrow$ yes

Ex:) $f(x,y) = \begin{cases} 1/8(6-x-y) & ; 0 < x < 2, 2 < y < 4 \\ 0 & ; o/w \end{cases}$

i) $P(X < 1, Y < 3) = \int_{x=0}^{1}\int_{y=2}^{3} \frac{1}{8}(6-x-y)\,dy\,dx = 3/8$

ii) $P(X < 1 / Y < 3) = \frac{P(X<1, Y<3)}{P(Y<3)} = \frac{3/8}{5/8} = 3/5$

$P(Y < 3) = \int_{y=-\infty}^{3} f(y)\,dy = \int_{y=2}^{3} f(y)\,dy = \int_{2}^{3} \frac{5-y}{4}\,dy = \frac{5}{8}$

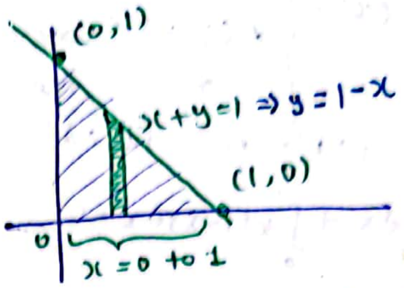
$f(y) = \int_{x=0}^{2} f(x,y)\,dx = \int_{x=0}^{2} \frac{1}{8}(6-x-y)\,dx = \frac{5-y}{4}$

Ex:) $f(x,y) = \begin{cases} e^{-(x+y)} & ; x>0,\ y>0 \\ 0 & ; o/w \end{cases}$

$P(X+Y<1) = ?$

$= \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)\,dx\,dy = \int_0^1 \left[\int_0^{1-x} e^{-y}\,dy\right] e^{-x}\,dx$

$= 1 - 2e^{-1}$



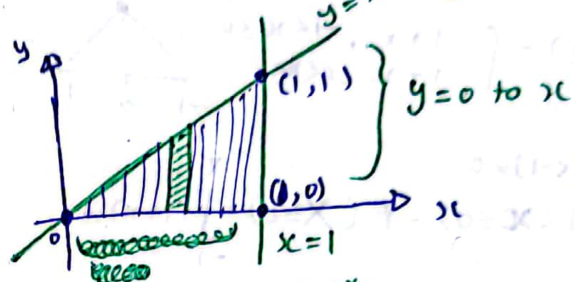
Ex:) $f(x,y) = \begin{cases} 8xy & ; 0<x<1,\ 0<y<x \\ 0 & ; o/w \end{cases}$

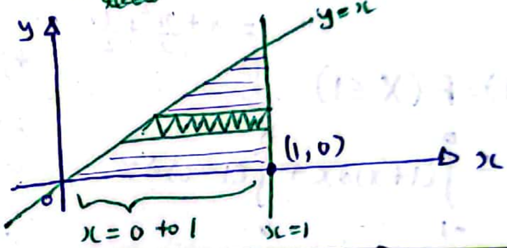
i) $f(x) = ?$

$= \int_{y=0}^{x} f(x,y)\,dy = \int_0^x 8xy\,dy = 4x^3$

ii) $f(y) = ?$

$= \int_{x=y}^{1} f(x,y)\,dx = \int_y^1 8xy\,dx = 4y(1-y^2)$





#) Expectation $[E(x)]$:-

*) Expectation = Average = mean :

• Ex:)

x_i	1	2	3	4
f_i	1	2	3	4

Avg. $= \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{30}{10} = 3$

• Ex:)

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

DRV $\boxed{E(x) = \sum_{-\infty}^{+\infty} x\cdot P(x)} = 1\times\frac{1}{10} + 2\times\frac{2}{10} + 3\times\frac{3}{10} + 4\times\frac{4}{10} = 3$

• Ex:) X = no. of heads
E(X) if 2 coins are tossed

X	0	1	2
Events	TT	HT, TH	HH
P(X)	1/4	2/4	1/4

$E(x) = \Sigma x\cdot P(x) = 1$ {one head per toss on an avg.}

• Ex:) A fair coin tossed till head appears
Expectation of no. of tosses?
x = no. of tosses

Event	H	TH	TTH	...
X	1	2	3	...
P(X)	1/2	1/2·1/2	1/2·1/2·1/2	

$E(x) = \Sigma x P(x) = 1\times\frac{1}{2} + 2\times\left(\frac{1}{2}\right)^2 + 3\times\left(\frac{1}{2}\right)^3 + \dots$

$= \frac{1}{2}\left[1 + 2\times\frac{1}{2} + 3\times\left(\frac{1}{2}\right)^2 + \dots\right]$

$= \frac{1}{2}\left[1-(1/2)\right]^{-2} = 2$

Note:) $\boxed{(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots}$

*) CRV: $\boxed{E(x) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx}$

• Ex:) $f(x) = \begin{cases} Kx(2-x) & ; 0<x<2,\ K>0 \\ 0 & ; o/w \end{cases}$

$E(x) = ?$

$\int_0^2 Kx(2-x)\,dx = 1 \Rightarrow K = 3/4$

$E(x) = \int_0^2 x\cdot\frac{3}{4}x(2-x)\,dx = 1$

Note:) In DRV, | In CRV

$E(x) = \sum_{x=-\infty}^{\infty} x\cdot P(x)$ $\quad = \int_{x=-\infty}^{\infty} x\cdot f(x)\,dx$

$E(x^2) = \sum_{x=-\infty}^{\infty} x^2\cdot P(x)$ $\quad = \int_{x=-\infty}^{\infty} x^2 f(x)\,dx$

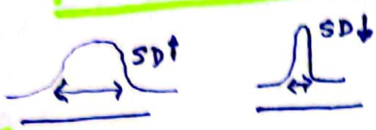
$E(g(x)) = \sum_{x=-\infty}^{\infty} g(x)\cdot P(x)$ $\quad = \int_{x=-\infty}^{\infty} g(x) f(x)\,dx$

*) Properties of Expectation :- [In DRV & CRV both]

① $E(c) = c$
② $E(aX) = a\,E(X)$
③ $E(aX+b) = a\,E(X) + b$
④ $E(X+Y) = E(X) + E(Y)$
⑤ $E(X-Y) = E(X) - E(Y)$
⑥ $E(XY) = E(X)\cdot E(Y/X)$
$= E(X)\cdot E(Y)$ [If X & Y are independent]
⑦ $E(aX^n) = a\,E(X^n)$
⑧ $E(X^2) \neq [E(X)]^2$
⑨ $E(1/X) \neq 1/E(X)$

#) Variance:

*) $\boxed{V(X) = E(X^2) - [E(X)]^2 \quad SD = \sqrt{V}}$



*) Variance properties:-

1. $V(c) = 0$
2. $V(aX) = a^2 V(X)$
3. $V(-X) = V(X)$
4. $V(X+Y) = V(X) + V(Y) + 2\,cov(X,Y)$

   if x, y independent $= V(X) + V(Y)$
5. $V(X-Y) = V(X) + V(Y) - 2\,cov(X,Y)$

   if x, y independent, $= V(X) + V(Y)$
6. $V(aX+b) = a^2 V(X)$

Ex:) $f(x,y) = \begin{cases} x+y \; ; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 \; ; & o/w \end{cases}$

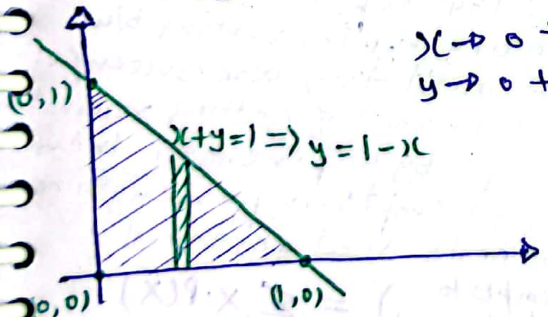
$P(x+y \le 1) = ?$

$P(X+Y \le 1) = \iint f(x,y)\,dy\,dx$

$= \iint (x+y)\,dy\,dx$

$x \to 0$ to 1

$y \to 0$ to $1-x$



$= \int_{x=0}^{1} \left[\int_{y=0}^{1-x} (x+y)\,dx \right] dy = 0.33$

#) Moment generating function:-

$E(X^1) \to 1^{st}$ moment

$E(X^2) \to 2^{nd}$ moment

...

$E(e^{tx}) = 1 + t\,E(X) + t^2 \frac{E(X^2)}{2!} + \frac{t^3 E(X^3)}{3!} + \ldots$

$E(X^n) = \left[\frac{d^n}{dt^n} E(e^{tx})\right]_{t=0} = \left[\frac{d^n}{dt^n} m_0(t)\right]_{t=0}$

Note:) $\boxed{e^x = x^0 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots}$

Ex:)

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

i) $m_0(t)$? ii) $E(X)$ iii) $E(X^2)$

i) $\boxed{E(e^{tx}) = \sum_{x=1}^{6} e^{tx} P(x)}$

$= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6)$

$= \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$

ii) $E(X)$

M1: $= \sum_{1}^{6} x P(x) = 21/6 = 7/2$

M2: $= \left[\frac{d^1}{dt^1} m_0(t)\right]_{t=0}$

$= \frac{1}{6}\left[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}\right]_{t=0}$

$= \frac{7}{2}$

iii) $E(X^2)$

M1: $= \sum_{1}^{6} x^2 P(x) = 91/6$

M2: $= \frac{d^2}{dt^2}(m_0(t))_{t=0} = \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 5^2 e^{5t} + 6^2 e^{6t}\right]_{t=0}$

$= 91/6$

#) Binomial Distribution:-

*) n = no. of trials

x = Req. for complete story (success for complete event)

p = success for 1 trial

q = failure " "

$= 1-p$

- $p + q = 1$
- p & q are independent events
- x = DRV

$\boxed{\begin{aligned} &P(x) = {}^nC_x\, p^x q^{n-x} \\ &E(X) = np \\ &V(X) = npq \end{aligned}}$

#) Bernoulli's Distr.^{n}:-

If $n=1$; Binomial $D^n \to$ Bernoulli's D^n

- $P(X) = p^x q^{1-x}$
- $E(X) = p$

  $V(X) = pq$

- $p + q = 1$
- Independent Events
- x = DRV

Ex:) The probability of a defective piece being produced is 0.01. The probability that out of 5 successive pieces, only one is defective is?

def $\begin{cases} p = 0.01 \\ q = 0.99 \end{cases}$

def $\begin{cases} x = 1 \\ n = 5 \end{cases}$

① DRV ✓ ② IE ✓ ③ p,q ✓

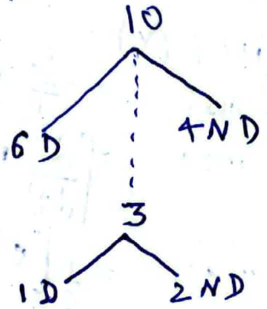
⇒ BD ✓

$P(X) = {}^5C_1 (0.01)^1 (0.99)^4$

#) Hyper Geometric Distr.^{n} :-

Ex:) 10 markers on table. 6 defective & 4 not defective. If 3 are randomly taken from ~~table~~ table P(exactly 1 marker def.)

① DRV ✓ ② IE ✗ ⇒ BD ✗



i.e. ① DRV
② 2 scenario
③ Dependent events
⇒ HGD ✓

$P = \frac{\text{Req.}}{\text{Total}}$

$= \frac{{}^6C_1 \times {}^4C_2}{{}^{10}C_3} = \frac{3}{10}$

#) Poisson distribution:-

If $n \to \infty$; BD → PD

① DRV
② IE
③ $p+q=1$
④ $n \to \infty$, $p \to 0$

$\boxed{m = \lambda = np}$ [m = mean, Exp., Avg.

$$\boxed{P(x) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-\lambda} \cdot \lambda^x}{x!}}$$

- $E(X) = np$
- $V(X) = np$

i.e. $\boxed{\text{mean} = \text{variance} = \lambda = np}$

Ex:) The no. of accidents occuring in a plant in a month follows poisson dis.^{n} with mean as 5.2. The prob. of occurence of less than 2 accidents in a month?

$X \mid \cancel{<} 2$
$\lambda \mid 5.2$

$P[X<2] = P(X=0) + P(X=1)$

$= \frac{e^{-5.2} (5.2)^0}{0!} + \frac{e^{-5.2} (5.2)^1}{1!}$

$= 0.034$

Ex:) An observer counts 240 veh/h at a specific highway location. Assume that the vehicle arrival at the locations is poisson distributed, the probability of having one vehicle arriving over a 30 sec time interval is?

$\lambda = 240 \text{ veh/h} = \frac{240 \text{ veh}}{3600 \text{ sec}}$

$= \frac{240}{120} \frac{\text{veh}}{30 \text{ sec}} = 2 \text{ veh/30 sec}$

$P(X=1) = \frac{e^{-2} \, 2^1}{1!} = 0.27$

#) Geometric distr.^{n} :-

- $X = 1 \to p$ (Prob.)
- $= 2 \to qp$
- $= 3 \to qqp$
- ...

$$\boxed{\begin{array}{l} \bullet\ P(X) = p^1 q^{x-1} \\ \bullet\ p+q = 1 \\ \bullet\ E(X) = 1/p \\ \quad V(X) = q/p^2 \end{array}}$$

Ex:) Passengers try repetedly to get a seat reservation in any train running b/w two stations until they are succesful. If there's 40% chance of getting rejeivation in any attempt by a passenger then the avg. no. of attempts a passenger needs to make to get a seat reserved?

$E(X = \text{no. of attempts to get seat reserved}) = \sum x \cdot P(X)$

Event	R	NR R	NR NR R	...
X	1	2	3	...
P(X)	0.4	0.6×0.4	$(0.6)^2(0.4)$	...

$E(X) = 1 \times 0.4 + 2 \times 0.6 \times 0.4 + 3 \times (0.6)^2 \times 0.4 + \cdots$

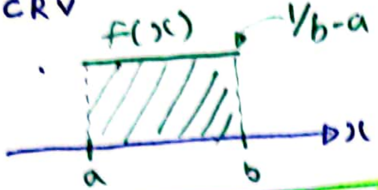
$= 0.4 [1 + 2(0.6) + 3(0.6)^2 + \cdots]$

$= 0.4 [1 - 0.6]^{-2}$

$= 2.5$

#) Uniform distribution:

• CRV

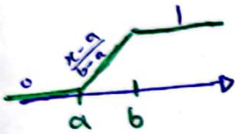


pdf

$$\boxed{f(x) = \begin{cases} \frac{1}{b-a} & ; a<x<b \\ 0 & ; o/w \end{cases}}$$

CDF

$$F(X<a) = 0$$
$$F(X>b) = 1$$
$$F(a<x<b) = \int_a^x \frac{1}{b-a}\, dx = \frac{x-a}{b-a}$$



E(X)

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$
$$= \int_a^b x \times \frac{1}{b-a}\, dx$$

$$\boxed{E(X) = \frac{a+b}{2}}$$

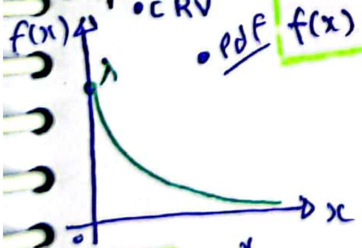
$$E(X^2) = \frac{a^2+b^2+ab}{3}$$

$$\boxed{V(X) = \frac{(b-a)^2}{12}}$$

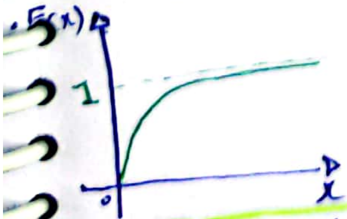
#) Exponential Distribution:—

• CRV

• pdf $\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \\ 0 & ; o/w \end{cases}}$



• cdf $\boxed{F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_0^x \lambda e^{-\lambda x} dx = 1 - e^{-\lambda x}}$



• S(x)

$$\boxed{\begin{aligned} E(X) &= 1/\lambda \\ V(X) &= 1/\lambda^2 \end{aligned}}$$
$$E(X^n) = n!/\lambda^n$$

Note:)

CRV	DRV
• UD	• BD
• Exp D	• PD
• Normal D	• GD
	• HGD

Ex:) Assume that the duration of minutes of a telephone conversation follows the exponential distribution. $f(x) = \frac{1}{5} e^{-x/5}$; $x \geq 0$. The probability that the conversation will exceed 5 minutes is?

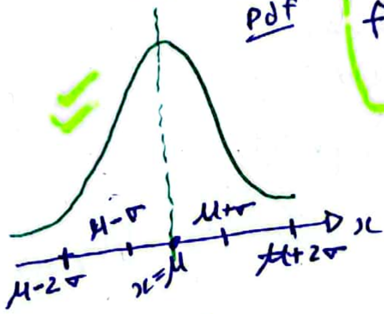
$$P(x>5) = \int_5^{\infty} \frac{1}{5} e^{-x/5} \cdot dx = 1/e$$

#) Normal Distribution:—

(Gaussian D.^{n} / Bell shaped curve)

pdf

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}$$



• x = CRV

• μ, σ $\boxed{\begin{aligned}\sigma^2 &= V \\ \sigma &= SD\end{aligned}}$

• Symmetric about $x = \mu$

i.e.

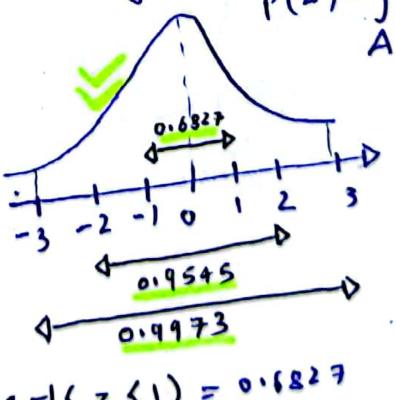
$$\int_{-\infty}^{\mu} f(x)\, dx = \int_{\mu}^{\infty} f(x) \cdot dx = \frac{1}{2}$$

⇓

(z-distr.^{n})

$$z = \frac{x-\mu}{\sigma}$$

$$P(z) = \int_A^B \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \cdot dz$$



$\begin{pmatrix} \mu = 0 \\ \sigma = 1 \end{pmatrix}$

Here, $P(Z>0) = 1/2$
$P(z<0) = 1/2$

$$P(-1<z<1) = 0.6827$$
$$P(-2<z<2) = 0.9545$$
$$P(-3<z<3) = 0.9973$$
$$P(z>0) = 1/2 \qquad P(z<0) = 1/2$$
$$P(0<z<1) = 0.6827/2$$

Note: $\{cov(X,Y)\}^2 \le var(X) \cdot var(Y)$

i.e. $\gamma = \dfrac{cov(X,Y)}{\sqrt{var(X) \cdot var(Y)}}$ $\gamma \equiv$ some constant

Note:

$cov(X,Y) = E\{(X-E(X))(Y-E(Y))\}$

$= E(XY) - E(X) \cdot E(Y)$

$\Rightarrow$ +ve / -ve / 0

#) Mean, Median, Mode:

*) Mean:

$\bar{x} = \dfrac{\Sigma x_i}{n} = \dfrac{\Sigma f_i x_i}{\Sigma f_i}$

*) Median:

→ middle no.: Arrange no.s lowest to highest

→ If no middle value, then take 2 middle no.s & take their avg.

*) mode: who occurs most frequent

→ when n is even

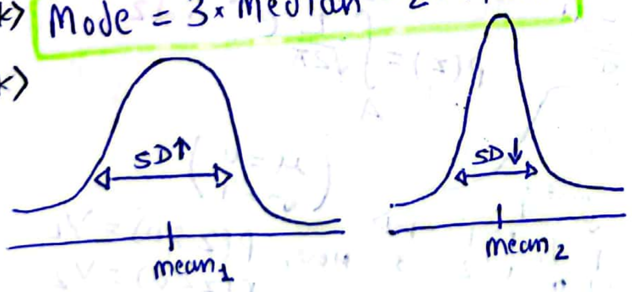
$\left(\frac{n}{2}\right)^{th}$ & $\left(\frac{n}{2}+1\right)^{th}$

→ when n is odd

$\left(\frac{n}{2}\right)^{th}$

*) $\boxed{\text{Mode} = 3 \times \text{Median} - 2 \times \text{Mean}}$

*)



*) Variance $= \Sigma(x_i^2) - (\text{mean})^2$

$SD = \sqrt{V}$

Note: Necessary condn for a fxn to be PDF

i) $f(x) \ge 0$ & ii) $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

Ex:) $f(x) = \begin{cases} -\frac{3}{8}x + 1 & ; \quad 0 \le x \le 4 \\ 0 & ; \quad o/w \end{cases}$

This f(x) can't be a PDF