: Graph Theory :

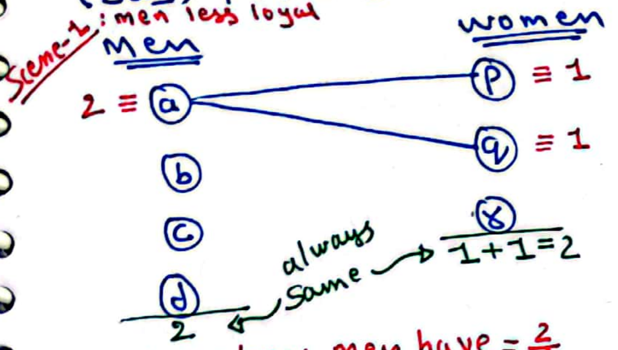
#) Graph: Some dots (nodes) and some lines (edges) connecting those dots.

#) Why study Graph (dots & lines): To model the real life situations.

Example-1: On avg., who has more opposite-gender partners, men or women?

(Say) 4 M & 3 W

Scene-1: men less loyal

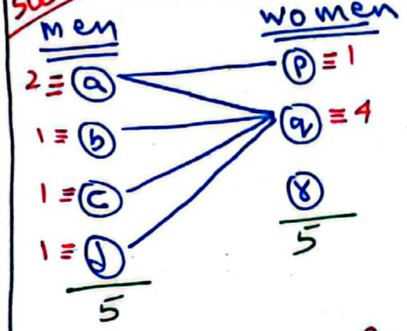


Avg. # partners men have = $\frac{2}{4}$

Avg. " " women " = $\frac{2}{3}$

⇒ On Avg., women have more opposite gender partners

Scene-2: women less loyal



$= \frac{n}{4}$

$= \frac{n}{3}$

⇒ On avg., women have more opposite gender partners

Conclusion: This has nothing to do with loyalty, only population plays the role.
In India, women population < men popul.ⁿ

$w < m$

Then, ratio of avg. # of partner for w to m:-

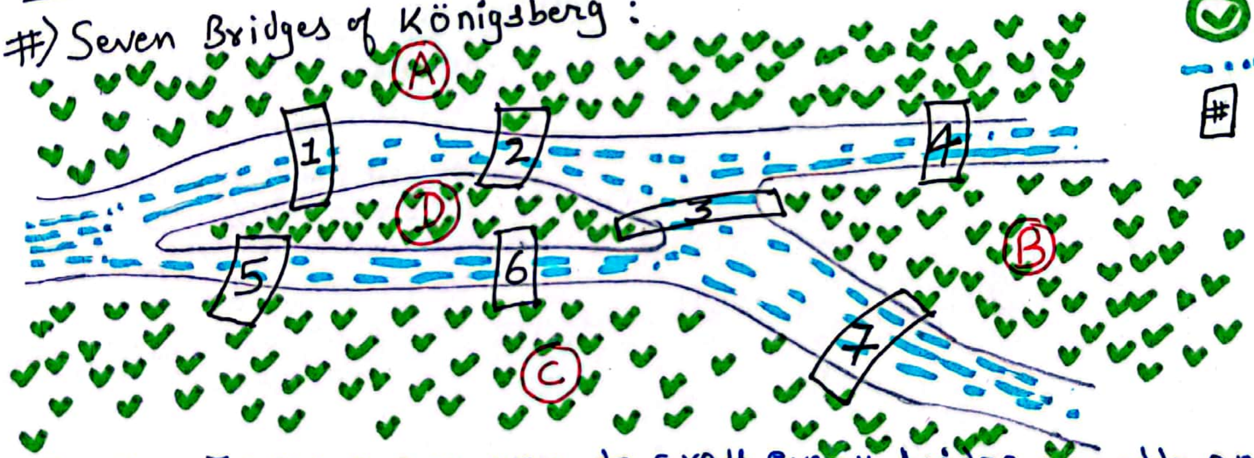
$= \frac{n/w}{n/m} = \frac{m}{w} = 1.$ ~~

Therefore, womens have more no. of opposite gender partners than men (on avg.)

Note: Do your assumptions reflect reality? ⇒ NO; Your assumptions about loyality is useless.

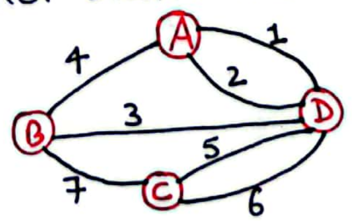
#) Seven Bridges of Königsberg:

= Land
= water
= Bridge



Problem: Is there any way, to cross every bridge exactly once?
(You can start from anywhere)

Euler created abstract structure:

G: 

vertex → abstraction for Land
edges → " " Bridge
i.e. visit every edge exactly once.

- This problem is equivalent to Euler Problem (See p.no. 27)
- Since, ∃ an odd degree vertex (in G), therefore no Euler path in G exists.
- Hence, it is proved that no solution exists for this problem.

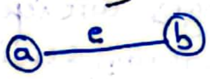
P.T.O

#) Graph theory applications:

1) Google maps
2) Internet
3) Facebook

#) Undirected vs Directed

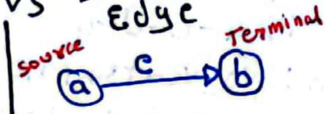
Undirected edge



$e = \{a,b\}$ → set

end points of $e = \{a,b\}$

Directed Edge



$e = (a,b)$

Set of edges but every edge is ordered pair

Note: Multi edges are parallel edges which have same endpoints.

Note: Multigraph: A graph which:-
1) Undirected graph
2) No self loops

here, multi edges are allowed

Note: Pseudograph: Any undirected graph.
here, self loops allowed.
multi edges allowed.

Note: Directed multigraph: Any directed graph.
here, self loops allowed.
multi edges allowed.

Note: Simple graph: A graph where:-
1) Undirected
2) No self loops
3) No multi edges

Note: Simple graph: A directed graph where:-
1) No self loops
2) No multi edges.

Note: In multigraph: $G(V,E)$;
$V \to$ Set of vertices
$E \to$ Multiset of edges

Ex:)

: $G(V,E)$
$V = \{a,b\}$
$E = \underbrace{\{\{a,b\},\{a,b\}\}}_{\text{multiset}}$

Note: For our convenience we can write:-

$e = \{\{a,b\}\} = ab$

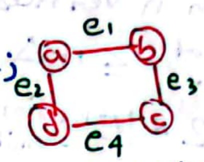
$e = \{(a,b),(b,a)\} = ab, ba$

#) Adjacent vertices

v_1 & v_2 are adjacents if $\exists$ edge b/w them

- Neighbourhood of vertex: set of vertices to which the given vertex is immediately connected.
- Adjacent edges: edges having some common endpoint.

like;



e_1 & e_2

- Degree of vertex: # of edges incident on the given vertex.

Note: But in Pseudograph,
Degree of vertex = # of times edges are incident of the vertex

Ex:)

deg(a) = 3
deg(b) = 1

- All these above definitions are for simple graphs only.
- For simple graphs, we can say that:-

$Deg(V)$ = # of neighbours of 'V'
= # of edges incident on 'V'.

Q:) Find type of Graph:

i) Facebook friendship.

- undirected edges
- No self loop
- No multiple edges

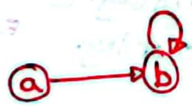
$\Rightarrow$ Simple graph

ii) Instagram following.

- Directed edges
- No self loops
- No multi edges

$\Rightarrow$ Simple Dir-Graph

iii) Instagram Likes



- Directed edges
- Self loops allowed
- No multi edges

$\Rightarrow$ Directed multigraph

- Order(G) = $|V|$
  Size(G) = $|E|$
- Degree Sequence: Degrees of vertices, written in decreasing order.

like: 6, 4, 2, 2, 0

Note: Two definitions of connected graph:-
1) If $\exists$ a path b/w every pair of vertices, then G is connected.
(or) 2) If # components = 1, then G is connected.

#) • Total degree $= \sum_{\forall v \in V} Deg(v)$

• Avg. degree $= \dfrac{\text{Total deg.}}{|V|}$

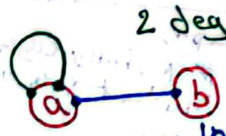
$$\delta \leq \text{Avg. deg.} \leq \Delta$$

• For graph with 'n' vertices:

$$n \cdot \delta \leq \text{Total deg.} \leq n \cdot \Delta$$

Total deg. $= 2 * |E|$; for all undirected G.

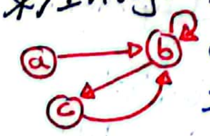
Reason: Every edge contributes 2 degree to the total deg.



i.e. # of odd deg. vertices = Even

#) Directed Graphs :-

*) Indegree/out degree:



out-deg(b) $= b^+ = 2$

In-deg(b) $= b^- = 3$

*) Adjacency: a is adjacent to b ✓

$a \to b$ (or) b is adjacent from a ✓

a, b are adjacent ✗

*) Total In-Deg. $= \sum_{v \in V} \text{In-Deg}(v)$

Total out-Deg. $= \sum_{v \in V} \text{out-Deg}(v)$

*) Each edge in directed graph, contributes 1 to Total-indeg. & 1 to Total-out deg.

So,

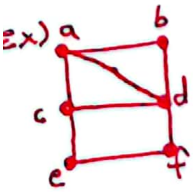
Total In-Deg. = Total out-Deg. $= |E|$

#) From now onwards we'll only study simple undirected graph.

#) Basic definitions:

*) walk: • sequence of vertices & edges where repetition of edges & vertices allowed.

Ex)



walk: a-b walk:-

i) a-b

ii) a-c-d-b

iii) a-b-d-b

iv) a-c-d-b-a-c-d-b

• Open walk ≡ Starting ending Vertex different

• closed walk ≡ " " " Same

*) Trail (footprint): • vertex repetition allowed, but edge repetition not allowed.

Ex:) a-b Trail:

i) a-b

ii) a-c-d-b

iii) a-d-c-a-b

*) Path: No vertex, no edges are repeated. here, Starting & ending vertex always different.

Ex:) a-b path:

i) a-b

ii) a-d-b

Every path is a walk.

Note: closed trail is called Circuit

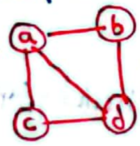
*) Cycle: • No vertex, no edges repeated, but Starting & ending vertex Same.

• cycle have atleast 3 vertices & 3 edges.

Note: Length of walk = # of edges traversed.

Note: No vertex repetition ⟹ No edge repetition

Note:



a-b-d, d-b-a } Same walks

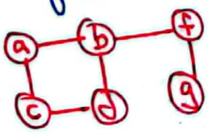
a-d-b, b-d-a, d-a-b } same cycle

here, # 3 length cycles = 2

Note: Lemma-1: In G, if there's a walk from v to w, then there's a path from v to w.

Note: To deduce a path from a walk, you just remove extra things from walk.

Ex:)



a to g walk:-

a-b-$\boxed{\text{d-c-a-b-f-b}}$-f-g ✗

a-b-f-g ≡ Path a to g

Note: Lemma-2: • There's always a walk from v to v. (Trivial walk).

• If there's a walk from v to w, then there's a walk from w to v.

• If there's a walk from a to b & a walk from b to c, then there's a walk from a to c.

#) For a directed graph;
walk, path, cycle, circuit :-

*) All these definitions are same as undirected graph, the only small difference is :-
For Directed cycle in Directed graph: You need at least 2 vertices (Not 3)

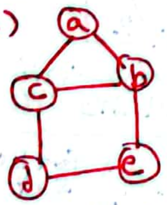
a-b-a cycle

#) For simple graph;
Distance, Diameter :-

*) Connected graph: In G, there is path b/w every 2 vertices.

*) Distance : Distance b/w V_1 & V_2 is the length of shortest path between V_1 & V_2.

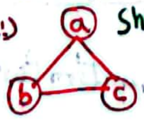
Ex:)



Dist$(a,c) = 1$
Dist$(a,e) = 2$

*) Diameter : Maximum possible shortest distance b/w two vertices V_i & V_j

Ex:)



Shortest Dist$(a,a) = 0$
" " $(a,b) = 1$
" " $(a,c) = 1$
" " $(b,b) = 0$
" " $(b,c) = 1$

} Diameter = 1

Ex:) Kashmir to KanyaKumari

#) Some Special type of Simple graphs:-

*) Regular graph: Every vertex have same degree.

*) Complete analysis of d-regular graph :-
Assume, $|V| = n$

1) Degree sequence : $d, d, d \dots d$ } n terms
2) $\Delta(G) = d$ $\quad$ Avg.deg$(G) = d$
$\delta(G) = d$ $\quad$ Total deg$(G) = n \cdot d$
3) #Edges = $n \cdot d/2$
4) Diameter
→ for disconnected regular graph = ∞
→ for connected regular graph = finite or ∞

*) Complete graph: Edge is present b/w every 2 vertices.
• Every 2 vertices are adjacent.

*) Complete analysis of K_n :-
1) # vertices = n
2) Degree of every vertex = $n-1$
3) $\Delta = \delta =$ Avg. deg. $= n-1$
4) Degree Seq. = $(n-1), (n-1), \dots (n-1)$ } n terms
5) All complete graphs (K_n) are actually $(n-1)$ Regular graphs.
6) # edges = select any 2 vertex & put edge b/w them $= {}^nC_2 = \frac{n(n-1)}{2}$
7) Diameter = 1
8) Always connected.

*) Empty/Null/Edgeless graph: vertices may be there but edges not there.
E_n

Note: Every graph must have at least one vertex.

*) Analysis of E_n :- $|V| = n$
1) Deg. of each vertex = 0
2) E_n is zero-Regular
3) $\delta = \Delta =$ Avg. deg $= 0$
4) Total deg. $= 0$
5) # edges = 0
6) E_n is connected iff $n = 1$ else disconnected
7) Diameter
→ if $n=1$ → 0
→ o/w → ∞

*) Path graph (P_n) :- A graph which looks like a path.
i.e.

$a_1 — a_2 — a_3 — \dots — a_n$

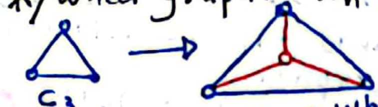
like a st. line

*) Analysis of P_n: $|V| = n$
1) Degree Sequence:
$$\equiv \begin{cases} 0 & ; P_1 \\ 1, 1 & ; P_2 \\ \underbrace{2, 2, 2 \dots 2}_{n-2 \text{ terms}}, \underbrace{1, 1}_{2 \text{ terms}} & ; P_n, n \geq 3 \end{cases}$$
2) Diameter $= \begin{cases} n-1 & ; \text{o/w} \\ 0 & ; \text{if } P_1 \text{ i.e. } n=1 \end{cases}$
3) #Edges = $n-1$
4) Total deg. $= 2(n-1)$
5) Always connected.

*) Cycle graph (C_n): $n \geq 3$;

*) Analysis of C_n ; $n \geq 3$:- $|V| = n$
1) Degree of each vertex = 2
2) Its 2-Regular
3) $\delta = \Delta =$ Avg. deg. $= 2$
Total Deg. $= 2n$
4) # Edges $= 2n/n = n$
5) Diameter $= \left\lfloor \frac{n}{2} \right\rfloor$
6) Always connected
7) Degree Seq.: $2, 2, 2, \dots 2$

*) Wheel graph: W_n ; $n \geq 3$



$W_3 \equiv$ wheel graph on 4 vertices

*) Analysis of W_n: $|V| = n+1$; $n \geq 4$

1) Deg. seq.: $n, \underbrace{3,3,3,\cdots 3}_{n \text{ terms}}$
2) W_n is not Regular graph.
3) $\#\text{Edges} = \frac{4n}{2} = 2n$
4) $\delta = 3$; $\Delta = n$; Total deg. $= n + 3n = 4n$
5) Diameter $= 2$
6) Always connected
7) Avg. degree $= \frac{4n}{n+1}$

*) Analysis of W_3: $|V| = 3+1 = 4$

- Deg. seq. = 3, 3, 3, 3
- $W_3 \equiv$ 3-Regular ($d = 3$)
- $\delta = \Delta =$ Avg. deg. $= 3$
- $\#\text{Edges} = \frac{3 \times 4}{2} = 6$
- $W_3 \equiv K_4$
- Diameter $= 1$
- connected ✓

*) Hyper cube graph:-

→ Hamming distance: # of positions where symbols are different.

Ex: $\left.\begin{matrix} ab\underline{c} \\ ab\underline{d} \end{matrix}\right] d = 1$

Ex: Google autocorrects word.

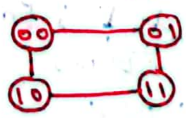
→ Hypercube graph (Q_n): or n-cube graph or n-dim. hypercube

In Q_n ; $|V| = 2^n$

and edge is there b/w u & v iff hamming dist. $(u, v) = 1$

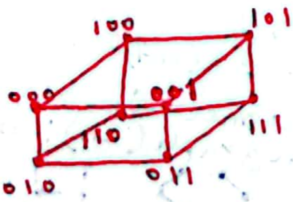
Ex-1) $Q_2 \equiv$ 2 length bit strings

$|V| = 2^2 = 4$



Ex-2) $Q_3 \equiv$ 3 length bit strings

$|V| = 2^3 = 8$



Ex-3) Q_0 | Q_1

→ Properties of Q_n:

1) order = no. of vertex $= 2^n$
2) Degree of each Vertex $= n$
3) Q_n is n-Regular ($d = n$)

$\delta = \Delta =$ Avg. deg. $= n$; Total deg. $= 2^n \times n$

4) $|E| = \frac{2^n \times n}{2}$ [∵ Total deg./2]
5) Diameter $= n$ [∵ Take any 2 vertices where all bits differ]

Note: Every Boolean Lattice $\equiv (P(A), \subseteq)$ for some set A
$\equiv$ Power set Lattice
$\equiv Q_n$; $n \geq 0$

6) Always connected

#) Interpretation of Degree Summation formula:

- $\sum_{\forall v \in V} \text{Deg}(v) = 2|E|$
- $n \times \text{Avg.} = 2|E|$

$n \times \delta \leq 2|E|$

$n \times \Delta \geq 2|E|$

Avg → Average Degree
δ → min.m degree
Δ → max.m degree

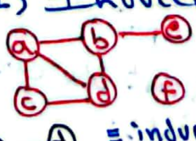
#) Subgraph:

*) Definition: H is Subgraph of G iff $V(H) \subseteq V(G)$ AND $E(H) \subseteq E(G)$

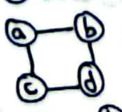
*) Subgraph types:

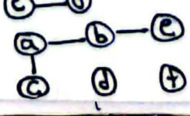
1) Spanning Subgraph: vertex deletion not allowed
2) Induced Subgraph: Extra edge deletion not allowed (vertex & it's incident edges can be deleted)

Ex1)



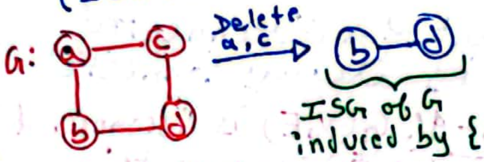
1)  $\equiv$ induced, not spanning

2)  $\equiv$ Neither

3)  $\equiv$ Spanning, not induced.

Q:) For any G, How many Spanning && induced subgraphs are there?
A:) 1 ; G itself

*) Induced Subgraph contd...
(ISG ≡ Induced subgraph)



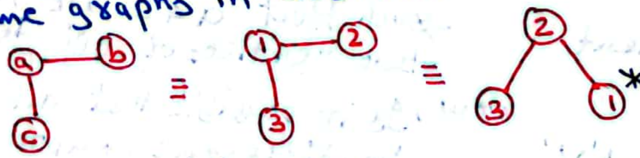
Definition of ISG:
$G(V,E)$; $S \subseteq V$
Now, Graph induced by S :
$H(S, E')$; $E' = \{ e \mid e \in E$ AND both endpoints of e are in S $\}$

Note:
- Every induced subgraph of a complete graph is also a complete graph.
- Every graph of 'n' vertex is subgraph of K_n.

#) Graph Union:
*) $G_1(V_1, E_1)$ & $G_2(V_2, E_2)$
then $G_1 \cup G_2 \equiv G(V_1 \cup V_2, E_1 \cup E_2)$

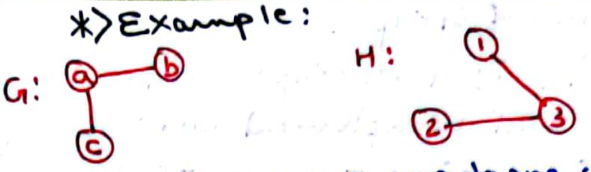
Q:) No. of different labelled undirected simple graphs on 'n' vertices?
$= 2^{^nC_2}$ or $2^{\frac{n(n-1)}{2}}$

#) Graph Isomorphism :
*) Same graphs, just drawn differently.
(or) Same graphs in abstract view.

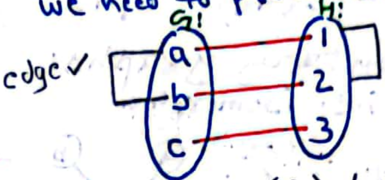
Ex:)


i.e.,
- Labelling doesn't matter
- Drawing " "
- Shape " "

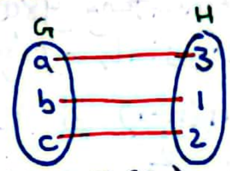
*) Formal definition: $G(V_1, E_1)$, $H(V_2, E_2)$
$G \equiv H$ iff $\exists$ bijection $f: V_1 \longrightarrow V_2$ which preserves Edges.
i.e. $(a,b) \in E_1$ iff $(f(a), f(b)) \in E_2$
i.e. • a, b have edge if their images also have edge b/w them.
• a, b don't have edge if their corresponding images also don't have edge b/w them

*) Example:


we need to prove one to one correspondense



i.e. $(a,b) \in E(G)$ but $(f(a), f(b)) \notin E(H)$
∴ so, this bijection doesn't work.
try another one,



i.e. if $(a,b) \in E(G)$ then $(f(a), f(b)) \in E(H)$
if $(b,c) \notin E(G)$ then $(f(b), f(c)) \notin E(H)$
⇒ This bijection would work.

Note: In above example we had total 6 (i.e. 3!) bijections. And for $G \equiv H$ at least one of them would work. And no bijection would work if $G \not\equiv H$.

Q:) if $|A| = 3$; $|B| = 2$; # Bijections from $A \to B$?
⇒ zero, bec. bijection exists iff $|A| = |B|$.

*) We never apply isomorphism definition to check if two graphs are isomorphic.
bec. n! bijections we need to check [in WC]

*) Proving Graph isomorphism is NP-Intermediate problem having exponential time complexity.

*) Although, proving "Not isomorphic" can be done very easily using "Graph Invarients" (properties).

*) Graph invarients:
1) Order of graph (|V|)
2) Size " " (|E|)
3) Degree sequence
4) Δ, δ
5) No. of K-length cycles
6) Connectedness
7) Diameter
8) No. of vertices with particular degree. e.t.c

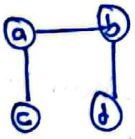
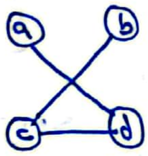
Note: If all Graph invarients are satisfied, You can't conclude that $G \equiv H$; Its used to prove $G \not\equiv H$ only.

#) Complement of Graph:

*) Every G_n is a subgraph of K_n.

∵ $G(V,E)$ का complement would be $\overline{G}(V,\overline{E})$; $\overline{E} = E(K_n) - E(G)$

*) Self complementary graphs:

- $G \equiv \overline{G}$; i.e. isomorphic.
- Ex:) G:  $\equiv \overline{G}$: 

Note: For self complementary graphs,

$G \equiv \overline{G} \Rightarrow$ edges in G = edges in $\overline{G}$ = e (say)

Now, $e + e = \frac{n(n-1)}{2} \Rightarrow e = \frac{n(n-1)}{4}$

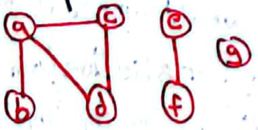
∵ e is always an integer

∴ Exactly one of 'n' or 'n-1' is divisible by 4.

i.e. $\boxed{n \bmod 4 = 0 \text{ or } 1}$

#) Components (Connected components) in a graph:

*) Component = Maximal connected Subgraph.

Ex:) 

components = 3

{a,b,c,d}, {e,f}, {g}

Note: Maximal connected means you can't add some more vertices, keeping it connected. Like in {g} you can't add more vertex.

*) In $G(V,E)$; $a,b \in V$;

Any two vertices $a, b \in$ Same component iff ∃ a path b/w a & b.

*) Defining a relation 'R' on vertex set 'V';

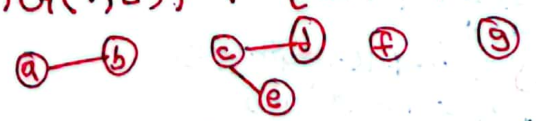
aRb iff ∃ path b/w a–b

here, R is an equivalence relation

also, # Equivalence classes = # components

(i.e. every equivalence class is a component)

Ex:) $G(V,E)$; $V = \{a,b,c,d,e,f,g\}$

G: 

V ← Base set

[a] = {a,b} [c] = {c,d,e} [g] = {g} [f] = {f} ← Equiv. classes

Note: Theorem about Self Complementary graphs:-

No. of vertices in G is 4K or 4K+1

(Converse is not true)

#) Important theorems:-

*) Complement of a disconnected graph is always connected.

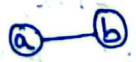
To Prove:- If $G(V,E)$ is disconnected then $\overline{G}(V,\overline{E})$ is connected.

Proof: Let, $a, b \in G$

we just need to prove that ∃ a path b/w a & b.

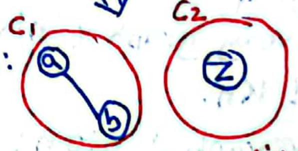
Case-1: a,b are not adjacent in G

G: 

$\overline{G}$: 

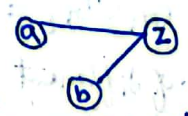
⇒ a to b ∃ path

Case-2: a,b are adjacent in G

G: 

bec. G is disconnected

∴ at least 2 components

$\overline{G}$: 

⇒ a to b ∃ path

*) If a graph has exactly two vertices of odd degrees, then they are connected by a path.

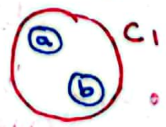
Proof: Let $G(V,E)$ & $a,b \in V$ such that a & b are the only two vertices of odd degree.

Now, Is it possible that a,b are in different component?

⇒ NO, it can't be, bec.

If a,b are in different components then this would result in a graph which have odd no. of odd degree vertices, which can't happen, i.e.

This graph has odd no. of odd degree vertices (Not possible)

Therefore, 

⇒ ∃ path b/w a & b.

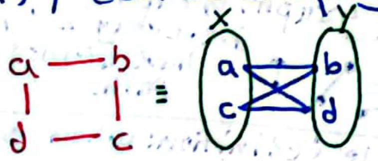
Q:) Max.^{m} no. of edges G can have if
i> G is undirected graph $\Rightarrow$ $^nC_2$
ii> G is directed graph $\Rightarrow$ $2 \times {^nC_2}$
[$\because$ you have ordered pairs (a,b), (b,a)]

#> Bipartite Graphs:

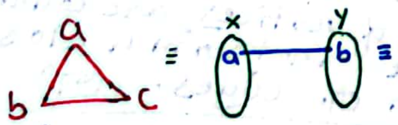
*> Definition: G(V,E) is bipartite iff $\exists$ a bipartition X, Y of V such that,

1) $X \cap Y = \phi$
2) $\forall a,b \in X$, $(a,b) \notin E(G)$ No edge b/w vertices in X
$\forall a,b \in Y$, $(a,b) \notin E(G)$ No edge b/w vertices in Y
3) $X \cup Y = V$
4) X, Y can be empty.

ex-1



ex-2



$\equiv$ Not a bipartite Graph
[$\because$ where to keep c?]

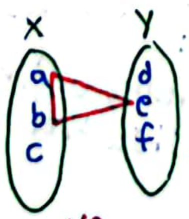
Q:) P_n is bipartite. n=?
$n \geq 1$, anything

Q:) C_n is bipartite. n=?
$C_3, C_5 \longrightarrow$ Not bipartite
$C_4, C_6 \longrightarrow$ are bipartite
$\Rightarrow n \in$ Even

Q:) K_n is bipartite. n=?
No such 'n' possible other than 1 & 2
i.e. $n = \{1, 2\}$

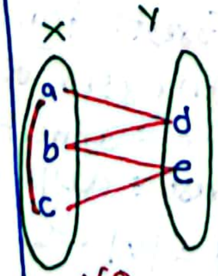
*> Theorem: A graph is bipartite iff it doesn't have an odd length cycle.

Proof: Lets try to create 3 length cycle



NO

Lets try to create 5 length cycle



NO

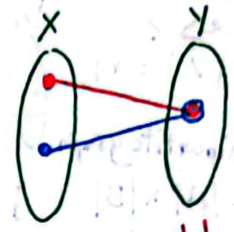
$\Rightarrow$ Not possible

i.e. No odd length cycle $\Longleftrightarrow$ Bipartite Graph

*> Theorem: If G is bipartite graph, and bipartition are X & Y, then

$$\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$$

Proof:

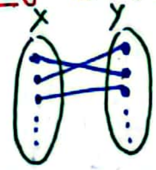


Every edge contributes a degree value 1 to X & a degree value 1 to Y

$\Rightarrow$ Total degree of X part = Total degree of Y part

*> Theorem: If a bipartite graph G is k-regular (every vertex have degree k) then $|X| = |Y|$

Proof: k-regular bipartite graph

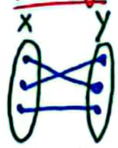


Total $\deg(X) = k|X|$
Total $\deg(Y) = k \cdot |Y|$
now, $k \cdot |X| = k \cdot |Y|$
$\Rightarrow |X| = |Y|$

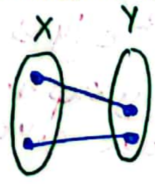
*> Theorem: Every Subgraph of a Bipartite graph is also a bipartite graph

Proof:

G:

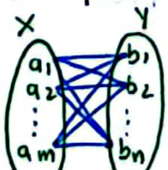


subgraph $\longrightarrow$ (Delete some vertices & edges)



: H (also a bipar.)

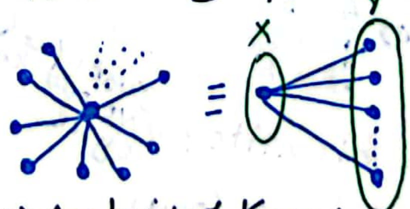
*> Complete Bipartite Graph: ($K_{m,n}$)



$|X| = m$ $|Y| = n$

$\forall a \in X$, $\forall b \in Y$
ab is an edge.

*> Star graph ($K_{1,n}$):-

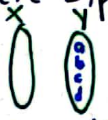


$= K_{1,n}$

*> Analysis of $K_{m,n}$; $m,n \geq 1$:-
1) $|V| = m+n$
2) $|E| = m \cdot n$
3) Degree Seq.: $\underbrace{n, n, n \ldots n}_{m \text{ times}}, \underbrace{m, m, \ldots m}_{n \text{ times}}$
4) $\Delta = \max(m,n)$
5) $\delta = \min(m,n)$
6) Diameter $= \begin{cases} 1 ; K_{1,1} \\ 2 ; K_{m,n} \text{ Except } K_{1,1} \end{cases}$

Q:) Is E_n (null graph) complete bipartite?

Yes; Ex: E_4: ⓐ ⓑ ⓒ ⓓ =



Q:) Max.^{m} no. of edges in a bipartite graph on 'n' vertices?

$G(\underbrace{A, B}, E)$; $|V| = n$

Make G as a complete bipartite graph.

max # edges $= |A| \times |B|$; $|A| = m$, $|B| = n-m$

maximize: $m \times (n-m)$

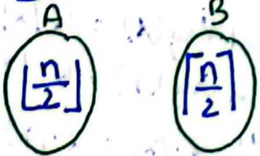
Diff. w.r.t m (bec. that's the variable)

$\frac{d}{dm}(mn - m^2) = 0$

$n - 2m = 0$

$\boxed{m = \frac{n}{2}}$

⇒ To get max.^{m} edges, partition vertices such that:



$A = \lfloor \frac{n}{2} \rfloor$, $B = \lceil \frac{n}{2} \rceil$

max.^{m} edges $= \frac{n^2}{4}$ or $\left(\frac{n}{2}\right)^2$ edges

Theorem:-

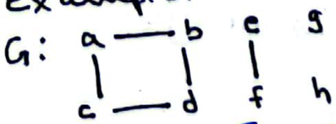
⇒ If a graph on 'n' vertices has more than $\left(\frac{n}{2}\right)^2$ edges, it can't be a bipartite graph.

Proof: Above question.

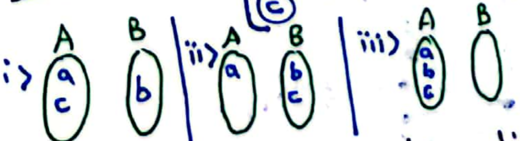
Theorem:

G is bipartite iff each of its components is bipartite.

Example:-



Note: If G: ⓐ ⓑ ⓒ ; then the following partitions would work.



Q:) which of these are bipartite? ⇒ Just check no odd length cycle

i) K_n ⇒ Yes, (only when $n = 1, 2$) ; No (o/w)

ii) P_n ⇒ Yes

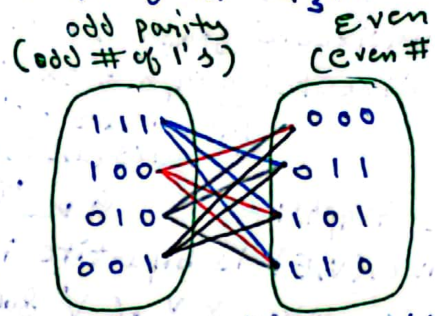
iii) C_n ⇒ Yes (only if n = even)

iv) W_n ⇒ No (∵ 3 length cycle always formed)

v) E_n ⇒ Yes (∵ No edges)

vi) Q_n ⇒ Yes (∵ You can partition into odd parity & Even parity)

Example of Q_n: Q_3, having 8 vertices



i.e. Hypercubes are bipartite. (but need not be complete bipartite)

Cyclic & Acyclic graph

*) cyclic graph: Graph containing at least one cycle.

Ex: 

*) Acyclic graph: Graph with no cycle i.e. every 2 vertices have at most one path.

Ex: 

Note:> Cycle definition: when some number of vertices are connected to one another in a closed chain of edges. (Or)

when there's some vertex where we can start off, follow a trail, and come back to the original vertex w/o repeating edges. (Or)

There are at least two vertices which have more than one path b/w them.

i.e. 

Note: There are at least 3 vertices in a cycle.

Tree

*) Tree: Connected, Acyclic, Undirected Graph.

*) Forest: Acyclic Graph (collection of Trees)

*) Analysis of Tree:

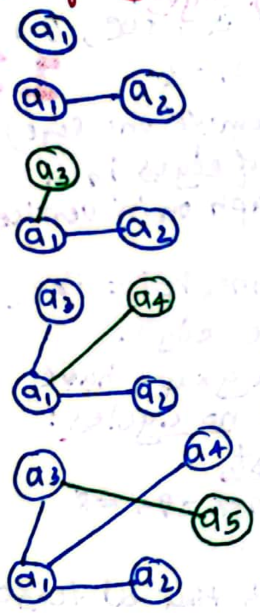
1) Tree ≡ Connected, acyclic, undirected

Forest ≡ acyclic, undirected

2) $|E| = |V| - 1$

Proof:- P.T.O

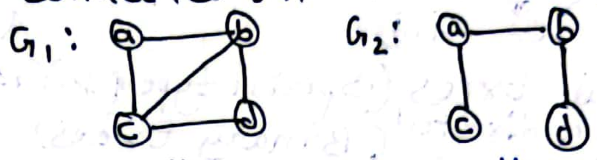
Proof by induction :-



$\lvert V\rvert = n$	$\lvert E\rvert$
1	0
2	1
3	2
4	3
5	4
...	....
n	n-1

Note: Tree + One more edge (any edge) ⇒ Cyclic graph.

Q:) Is this graph minimally connected on 4 vertices?



G_1 ⇓ No G_2 ⇓ Yes

i.e. Tree ≡ Connected with min.^{m} no. of edges

Tree ≡ Acyclic with max.^{m} # edges.

Note: Star graphs are trees.

3. Degree summation of tree:

$$\sum_{\forall v \in V} Deg(v) = 2\cdot|E| = 2(n-1)$$

$\Rightarrow$ Total deg. $= 2(n-1)$

Q:) Graph with 'n' vertices, 'n-1' edges is a tree?

⇒ NO.

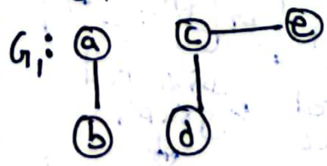
Q:) Connected graph on 'n' vertices, n-1 edges is tree?

⇒ Yes; Say you've to create a connected graph with 3 vertices :-

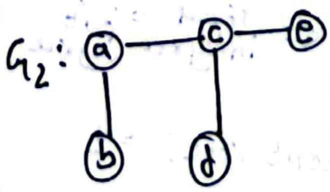
There's no other way

i.e. Tree ≡ Connected with n-1 edges

Q:) Is this "Maximally Acyclic" graph on 5 vertices?



⇒ Acyclic ✓ but not maximally acyclic, cuz you can add an edge a-c



⇒ Yes, maximally acyclic graph, cuz you can't add any more edge.

i.e. Tree ≡ Maximally acyclic graph on 'n' vertices.

≡ Forest with max.^{m} no. of edges.

*) Lemma: Every tree T with at least 2 vertices has at least two vertices of degree 1 (i.e. has at least two leafs)

Proof by contradiction :

Let, Tree T ; $n \geq 2$.

Assume we don't have at least two vertices of degree 1.

i.e. lets say we've 0 or 1 vertex with degree one.

i.e. lets say we've either 0 or 1 leaf.

i.e. # Deg(1) vertices ≤ 1

vertices with degree atleast 2 $\geq n-1$

Since, Tree is connected,

⇒ No vertex have zero degree

i.e. Every vertex degree ≥ 1

i.e. we have one vertex with degree 2

&&

we have (n-1) vertices with degree 1

(As per our assumption)

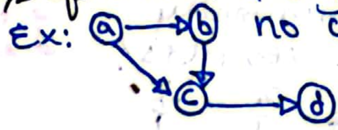
Now, Total degree $= (n-1)\times 2 + (1)\times 1 \neq 2(n-1)$

So, Contradiction

#> Directed Acyclic Graph (DAG):

*> Definition: A digraph that has no cycles.

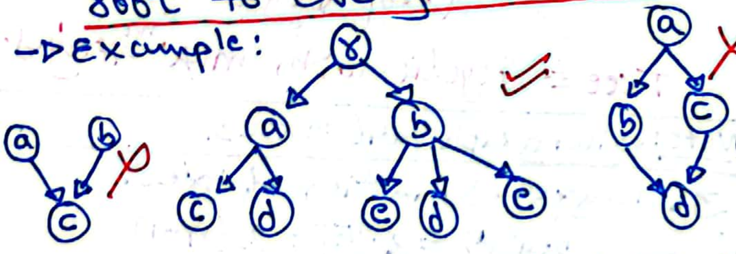
Ex:



*> Rooted trees (Special type of DAGs)

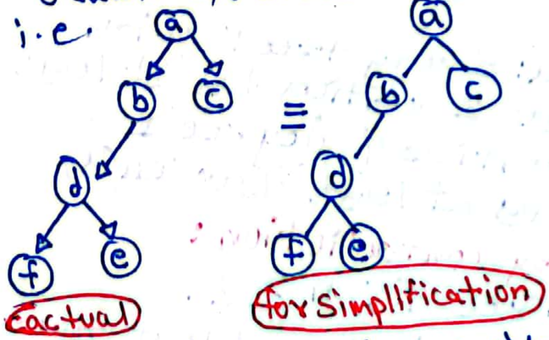
-> It is a directed graph where we have one head/root & we have a unique path from root to every other vertex.

-> Example:



-> Rooted trees, for convinience, are drawn w/o directions.

i.e.

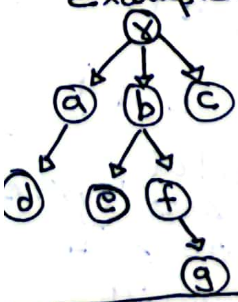


-> Binary trees are the rooted trees with atmost 2 children.

-> Height ≡ # Edges below

Depth ≡ # Edges above

Example:-



	height	depth
r	3	0
a	1	1
f	1	2
g	0	3
d	0	2

Note :> T: a here, a is root also leaf also

Note: Rooted tree is not a tree in graph theory, but we think of it as a tree sometimes

Q:) what is the max^m no. of edges in a forest which is not a tree, on n vertices? => n-2

(i.e. make a tree & remove one edge)

Q:) what is the min^m no. of edges in a connected cyclic graph on 'n' vertices? => n

bec n-1 edges + connected = Tree

Now, just add one edge.

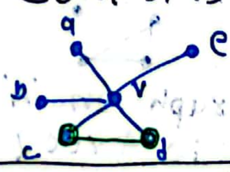
Q:) Is it possible that a graph have 9 vertices, 9 edges & no cycles?

NO, bec. in a forest, max $\#|E| = n-1 = 9-1 = 8$

Q:) True or false?

consider a graph G such that at least one vertex 'v' is connected to all other vertices & there is some other vertex of degree more than 1. Such G is not bipartite.

=> True.



odd length cycle bec. there

Note: Every tree with $n \geq 2$ has at least 2 leaves.

-> Degree (in rooted trees):

Degree of a node ≡ # of children

*> Binary Trees:

-> Definition: A binary tree is a rooted tree in which every vertex has at most two children.

Q:) L ≡ no. of leaves (Deg 0 nodes)

D'' ≡ no. of degree 2 nodes.

find relation b/w them.

=> $\boxed{L = D'' + 1}$ or $\boxed{D'' = L - 1}$

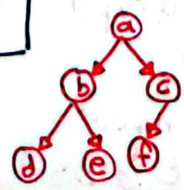
Proof-1: Binary tree -> Rooted tree -> Directed graph

$\boxed{\text{Total indeg.} = \text{Total out deg.} = |E| = n-1}$ ⇐ Total indegree = Total out degree

Now, in a binary tree,

indeg. of every node (except root) = 1

indeg. of root = 0



=> $\boxed{\text{Total indeg.} = 1(n-1) + 0(1) = n-1}$

Now, in the binary tree, Let:-

$L \equiv$ No. of Leaves (Deg. 0 nodes)

$D' \equiv$ No. of deg. 1 nodes

$D'' \equiv$ No. of deg. 2 nodes

Now, in a binary tree,

out deg. of leaves = 0

out deg. of deg. 1 nodes = 1

out deg. of deg. 2 nodes = 2

$$\Rightarrow \boxed{\text{Total out deg.} = L \times 0 + D' \times 1 + D'' \times 2 = D' + 2 \cdot D''}$$

Now, Total indeg. = Total out deg = $n-1$

$\Rightarrow D' + 2D'' = n-1$ —— (i)

also, $n = L + D' + D''$

putting in (i)

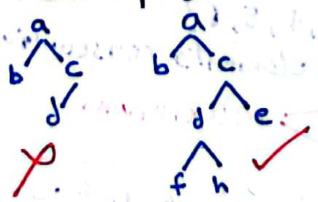
$D' + 2 \cdot D'' = L + D' + D''$

$\boxed{D'' = L-1}$ or $\boxed{L = D''+1}$

*) Full Binary Tree (FBT):

→ Definition: Every node has either zero or two children.

→ Examples:



→ In FBT, $D' = 0$

$\Rightarrow \boxed{n = L + D''}$ i.e. Internal nodes $= D''$

Q:) In FBT, if L is known then $n = ?$, $D'' = ?$

$n = L + \underline{D''} = L + \underline{(L-1)} = 2L-1$

$n = 2L-1$

Now, again; $n = L + D''$

$2L - 1 = L + D''$

$\boxed{D'' = L-1}$ or $\boxed{L = D'+1}$

Q:) In FBT, if n is known, then $D'' = ?$, $L = ?$

$n = L + D'' = L + (L-1) = 2L-1 \Rightarrow n = 2L-1$

$\Rightarrow L = \frac{n+1}{2}$

Now, $n = L + D''$

$2L - 1 = L + D''$

$D'' = L - 1 = \frac{n+1}{2} - 1 \Rightarrow D'' = \frac{n-1}{2}$

i.e. In FBT, if one of the n, L, D'' is known to you, then you can find the other two.

Q:) Min.^{m} no. of edges in a connected graph?

$\Rightarrow n-1$

Q:) How many min.^{m} edges must a graph with 'N' vertices have in order to guarantee that it is connected?

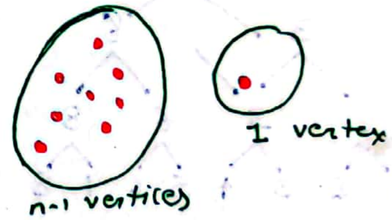
$\Rightarrow {}^{n-1}C_2 + 1$

i.e. Lemma: Every simple undirected graph with more than $(n-1)(n-2)/2$ edges is connected.

Q:) Max.^{m} no. of edges possible in an undirected graph with 'n' vertices & 'K' components?

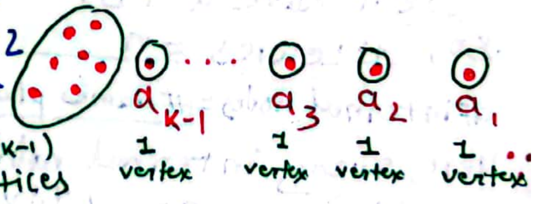
If G have 'n' vertex & 2 components:

$\max |E| = {}^{n-1}C_2$



If G have 'n' vertex & K components:

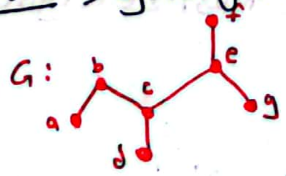
$$\boxed{\max |E| = {}^{n-K+1}C_2}$$



i.e. ${}^{n-K+1}C_2$ or $\frac{(n-K+1)(n-K)}{2}$ edges

Note: Degree of a vertex in Graph & Tree are different concepts:-

G:

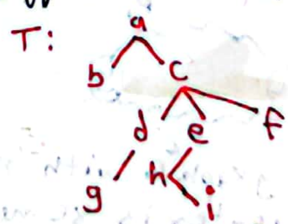


$\deg(e) = 3$

$\deg(b) = 2$

i.e. no. of edges connected

T:



$\deg(c) = 3$

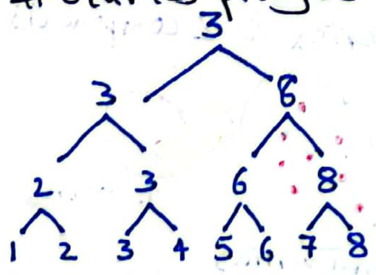
$\deg(a) = 2$

i.e. no. of child

*) Tournament Problems:
(Applⁿ of FBT)

Q:) Suppose 8 people enter a chess tournament. Use a rooted tree model of the tournament to determine how many games must be played to determine a champion, if a player is eliminated after one loss & games are played until only one entrant has not lost. (Assume there are no ties.)

#Games played = ?



#Games played
= #internal nodes
= 7

Q:) n people chess tournament:
we'll have a FBT
So, #Leaves = n
#internal nodes = $\boxed{\text{\#Games played} = n-1}$

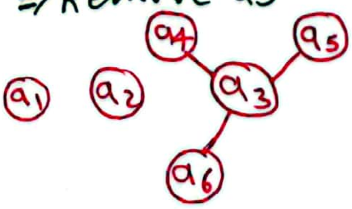
∵ Here, every internal node has 2 children or no children

Note: Lemma: If there are two different paths b/w some vertices x & y in a simple graph, then there must be a cycle in G.

Q:) A graph G with 'n' nodes & 'k' components. If a vertex is removed from G, the no. of components in the resultant graph must lie down between?

Let; G has 6 nodes

max.ᵐ no. of components
⇒ Remove a3



⇒ n-1 components

min.ᵐ no. of components
⇒ Remove an isolated vertex (Say a1)

⇒ K-1 components

#) Graph Numbers:-

*) Maximal & maximum:-

→ A set 'S' is maximal with respect to some property 'p' if nothing new can be added to S keeping the property p preserved.

→ A set 'S' is max.ᵐ w.r.t some property 'p' if for all sets 'X', who satisfies property 'p', 'S' is the largest set.

→ Example: {1,2,3,4,5}
Property p: No two elements consecutive.
Now,
{1,2} ✗
{2,4} ✓ → maximal but not maximum
{1,3} ✓
{2} ✓ → Not maximal
{1,4} ✓ → maximal but not maximum
{1,3,5} ✓ → maximal & maximum

→ In short;

Maximal ≡ No new addition possible
Maximum ≡ The largest maximal.

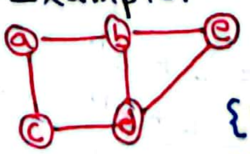
→ Maximum need not be unique, there can be multiple maximums possible
Ex: {1,2,3,4,5,6}
P: No two elements consecutive
{1,4,6} ⟩ maximum
{1,3,5}

*) Clique & Independent set:-

→ Clique: Set of vertices where every two vertex are adjacent
i.e. A subgraph which is Complete.

→ Independent set: Set of vertices where every two vertex are non-adjacent.

→ Example:



Clique	Independent set
{a,b} ✓	{a,e} ✓
{b,e,d} ✓	{a} ✓
{a,d,c} ✗	{a,d,c} ✗

→ $\omega(G)$: Denotes clique no.
$\alpha(G)$: Denotes independence no.

for above graph; $\omega(G) = 3$
$\alpha(G) = 2$

→ Example:

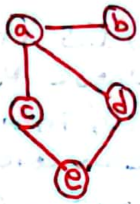
$\omega(G) = 4$

$\alpha(G) = 2$

clique	independent set
{b,d}	[e]
{b}	{a,e}
{a,b,c}	{c,e}
{a,b,c,d}	

→ Pairwise non-adjacent people sitting on table (An app.^{n} of Independent Set):

Q:) Suppose we're given a graph where vertices correspond to students in the class & edges correspond to friendships. What is the max.^{m} no. of students we can put in the exam room if no two friends should be in the room.



Ans: $\alpha(G) = 3$ (Say {c,d,b})

→ For cycle graph C_n ; $n \geq 3$:-

$$\alpha(C_n) = \lfloor \tfrac{n}{2} \rfloor$$

$$\omega(C_n) = \begin{cases} 3 & ; n = 3 \\ 2 & ; n \geq 4 \end{cases}$$

→ K_n:

$\alpha(K_n) = 1$; $\omega(K_n) = n$

→ P_n: path graph

$\alpha(P_n) = \lceil \tfrac{n}{2} \rceil$; $\omega(P_n) = \begin{cases} 2 & ; n \geq 2 \\ 1 & ; n = 1 \end{cases}$

→ Q_n: Hypercube graph

its a bipartite graph

odd parity — even parity

$\dfrac{2^n}{2}$ $\dfrac{2^n}{2}$

$\alpha(Q_n) = 2^n/2$

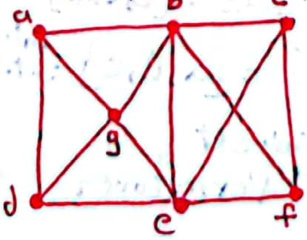
$\omega(Q_n) = 2$

→ $K_{m,n}$:

$\alpha(K_{m,n}) = \max(m,n)$

$\omega(K_{m,n}) = 2$

Q:) How many maximal cliques are there in G:



⇒ {b,e,f,c}
{a,b,g}
{b,g,e}
{d,g,e}
{a,g,d}

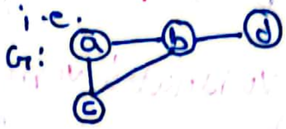
⇒ 5

Maximal Clique: A complete subgraph which is not contained in any larger complete subgraph.

→ Lemma:

- A clique 'C' in graph G is An independent set 'C' in graph $\overline{G}$.

i.e.



{a,b,c} ≡ Clique

{a,b,c} = Independent Set

- i.e. If S is max.^{m} independent set in G then S is max.^{m} clique in $\overline{G}$.
- i.e. If S is maximal independent set in G then S is maximal clique in $\overline{G}$.
- vice versa also true.

Q:) In G, we have a max.^{m} independent set 'S' and there is no larger independent set than S in graph 'G'. Now, in $\overline{G}$, S would be a clique. Can we have bigger clique (say M) than the clique S in $\overline{G}$?

⇒ Assume; $M > S$ is clique in $\overline{G}$

⇒ M is max.^{m} independent set in G

but we've assumed $M > S$

∴ S is not max.^{m} independent set in G

⇒ contradiction

Q:) what is the independence no. of a graph with at least 3 vertices & only one edge?

⇒ a_1 a_2 a_3 . . . a_n

$\alpha = n-1$

$\omega = 2$

*) Vertex Cover & Edge Cover:

→ vertex cover: Vertices which cover all edges of G.

→ Edge cover: Edges which cover all vertices of G.

→ Example:

vertex cover	Edge cover
Not minimum vertex cover ← {a,b,c,d,e}	{be, ab, cd} → mini.^{m} edge cover
mini.^{m} vertex cover ← {b,c,d}	

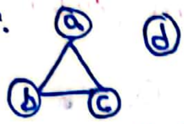
we only care about minimum vertex covering & minimum edge covering

→ One edge covers exactly 2 vertices.

One vertex can cover any no. of edges.

→ Edge cover does not exist if isolated vertex is there.

Ex:



$EC(G) = D.N.E$

i.e. EC exists iff No isolated vertex (i.e. $\delta > 0$)

→ MVC = Minimum vertex cover

Size of MVC = Vertex covering no. $= \beta(G)$

MEC = minimum edge cover

Size of MEC = Edge covering no. $= \beta'(G)$

→ Example:

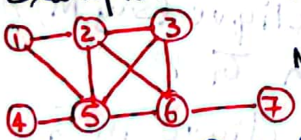
MVC = {3, 5, 1}

MEC = {37, 14, 25, 56}

$\therefore \beta = 3$

$\beta' = 4$

→ Example:



MIS = max^m independent set

MCQ = max^m clique

MVC = {2, 5, 6} ⇒ $\beta = 3$

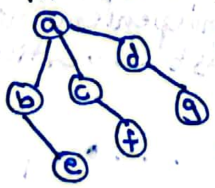
MEC = {67, 45, 12, 36} ⇒ $\beta' = 4$

MIS = {7, 3, 1, 4} ⇒ $\alpha = 4$

MCQ = ⇒ $\omega = 4$

Q:) The maximum degree vertex must always be in MVC ?

⇒ NO, need not be



MVC = {b, c, d}

→ Assume $\delta > 0$;

Size of MEC $= \beta' \geq \lceil \frac{n}{2} \rceil$ for all the graphs

Proof: Say in G, $\delta > 0$ & vertices are:

$a_1, a_2, a_3 \cdots a_n$

then; $MEC(G) = \{ \underbrace{e_1}_{\text{covers 2 vertices}}, \underbrace{e_2}_{\text{covers 2 vertices}}, \cdots \}$

= at least $\lceil \frac{n}{2} \rceil$ edges

→ For Empty graph E_n:

$MVC = \beta = 0$

$MEC = \beta' =$ Not defined

G: (say)

→ For K_n:

$MVC = \beta = n-1$

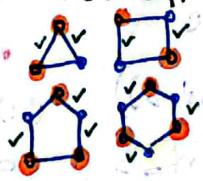
$MEC = \beta' = \lceil \frac{n}{2} \rceil$

G: (say) K_4

Q:) Does there exist a connected graph G(V, E) whose MVC is 'V' ?

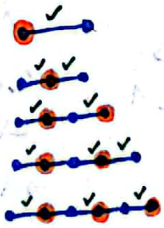
⇒ NO, in worst case (K_n), V-1 vertices can cover all edges.

→ For C_n: $\beta = MVC = \lceil \frac{n}{2} \rceil$



$\beta' = MEC = \lceil \frac{n}{2} \rceil$

→ For P_n: $n \geq 2$ {bec. for P_1 Edge cover D.N.E}



β	β'
1	1
1	2
2	2
2	3
3	3

$\beta = MVC = \lfloor \frac{n}{2} \rfloor$

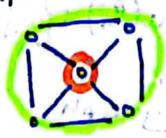
$\beta' = MEC = \lceil \frac{n}{2} \rceil$

→ For W_n: $n \geq 3$

$W_3 \equiv K_4$

$\beta = 3$

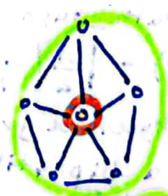
$\beta' = 2$



W_4

$\beta = 1 + \lceil \frac{n}{2} \rceil$

$\beta' = 3$



$\beta = 1 + \lceil \frac{n}{2} \rceil$

$\beta' = \lceil \frac{n}{2} \rceil$

central vertex

C_n

Here, for W_K

n is K+1

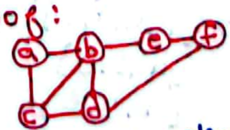
→ Note: Covering no. of a graph is by default the vertex covering no.

*) Relation b/w Vertex cover (MVC) (β) and Independent set (MIS) (α) :-

→ $\alpha + \beta = n$; n = no. of vertices

→ Proof:

G: (Say)



VC = S = {b, c, f}

$\bar{S}$ = {a, d, e} = Ind. set

i.e. VC $\xrightarrow{\text{complement}}$ Ind. set

& Ind. Set $\xrightarrow{\text{complement}}$ Vertex Cover

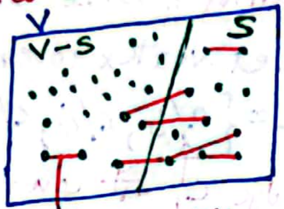
VC = vertex cover

Now, we can claim that:
In $G(V,E)$ graph

S1: let $S \subseteq V$ & S is vertex cover
then $\overline{S} \equiv (V-S) \equiv$ Ind. set

also,
let $S \subseteq V$ & S is Ind. set
S2: then $\overline{S} \equiv (V-S) \equiv$ vertex cover

Now, consider S1: i.e. S is vertex cover

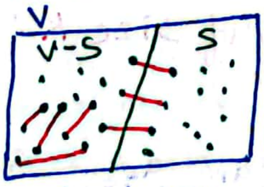


There are no edges in V-S area

All edges in S bec. its VC so covers all edges.

Not possible cuz S is vertex cover ∴ S covers all edges

Now, consider S2: i.e. S is ind. set

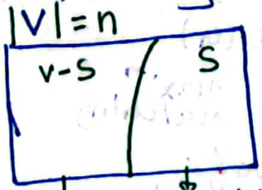


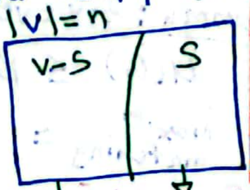
No edges in S bec. its ind. set
∴ all edges in V-S
∴ V-S is VC

Now, If you know that some S is VC
then MVC $\equiv \beta \leq |S|$

also, If you know that some S is ind. set
then MIS $\equiv \alpha \geq |S|$

Now, finding relation b/w α & β : $(|V|=n)$





$\Rightarrow \alpha \geq n-S$
$\alpha \geq n-\beta$
$\alpha+\beta \geq n$

$\Rightarrow S \leq n-\alpha$
$\beta \leq n-\alpha$
$\alpha+\beta \leq n$

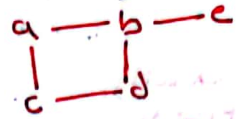
$$\alpha+\beta = n$$

Note: If $S \equiv$ MIS $\Longleftrightarrow$ $V-S \equiv \overline{S} \equiv$ MVC

*) vertex	Edges
① Min.^{m} vertex Cover $\boxed{\beta(G)}$	① Min.^{m} edge cover $\boxed{\beta'(G)}$
② max.^{m} ind. set $\boxed{\alpha(G)}$ (vertex ind. set)	② Max.^{m} Matching $\boxed{\alpha'(G) \text{ or } \mu(G)}$ (Edge ind. set)

• A matching is called Perfect Matching (PM) if it covers all vertices of the Graph.

→ Example:



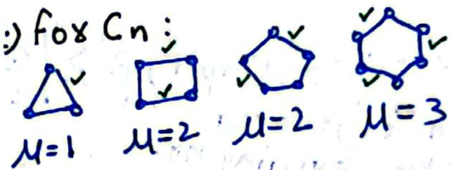
Look for non-adjacent edges

Matching :-
{ab} ✓
{ab, be} ✗
{ab, cd} ✓
{be, ac} ✓

→ If we've a matching M
$M = \{e_1 = a_1b_1, e_2 = a_2b_2, \ldots e_n = a_nb_n\}$
Then all these vertices must be different.

→ Matching No. $\equiv \alpha'(G)$ or $\mu(G)$ $\equiv$ Size of Longest matching i.e. Max.^{m} matching

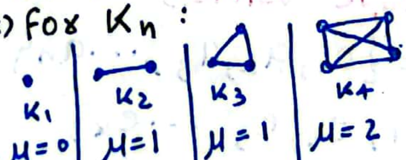
Q:) for C_n :



$\mu = \lfloor \frac{n}{2} \rfloor$

Observation: In C_n, perfect matching exists when n is even.

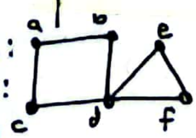
Q:) for K_n :



$K_n \rightarrow \{1, 2, 3, 4, \ldots \ldots n\}$ vertices
Longest Matching = {12, 34, 56, 78, ...}
$\Rightarrow \mu = \lfloor \frac{n}{2} \rfloor$

Observation: In K_n, perfect matching exists when n = even

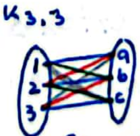
Note: G:



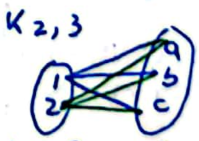
{ab, ef} → maximal but not max.^{m} matching
{ac, bd, ef} → Max.^{m} matching also, Perfect "

Note: A vertex that appears in a matching is saturated, o/w it is unsaturated.

Q:) for $K_{m,n}$



$\mu = \{1a, 2b, 3c\} = 3$
Matching is PM



$\mu = \{1a, 2b\} = 2$
Matching is not PM

$\mu = \min(m, n)$

Observation: PM exists iff $m = n$

Note: Any matching covers/saturate even no. of vertices.

Lemma: For every graph, $\mu \le \lfloor \frac{n}{2} \rfloor$

Proof: In any matching, every edge covers exactly two unique vertices.

i.e. if $G(V,E)$; $V = \{a_1, a_2, a_3, \dots a_n\}$ and matching $= \{e_1, e_2, e_3 \dots e_K\}$.

↳ covers $2K$ vertices

and $2K \le n$

$K \le \frac{n}{2}$

$\therefore \mu \le \lfloor \frac{n}{2} \rfloor$

Lemma: For every graph, $\beta' \ge \lceil \frac{n}{2} \rceil$

$\beta' \to$ min.^{m} edge cover or Edge covering no.

Proof: In any edge cover, every edge covers exactly two unique vertices.

i.e. If $G(V,E)$; $V = \{a_1, a_2, a_3, \dots a_n\}$

and min.^{m} edge cover $= \{e_1, e_2, e_3, \dots e_K\}$

↳ covers $2K$ vertices, actually covers all vertices

$\Rightarrow 2K \ge n$

$K \ge \frac{n}{2}$

$\therefore \beta' \ge \lceil \frac{n}{2} \rceil$

Lemma: If perfect matching exists, then no. of vertices should be even.

Proof: Let,

Perfect matching $= \{a_1b_1, a_2b_2, \dots a_Kb_K\}$ ↳ vertices $= 2K$

but, PM covers all the vertices $\Rightarrow |V| = 2K \Rightarrow$ even

i.e. PM exists $\Rightarrow$ $|V| =$ Even, $\not\Leftarrow$

Reverse need not true

Ex:

$|V| =$ even but PM D.N.E

$\mu = 1$

Note:

Perfect matching there $\Rightarrow$ max.^{m} matching there $\Rightarrow$ maximal matching there

Perfect matching there $\not\Leftarrow$ max.^{m} matching there $\not\Leftarrow$ maximal matching there

Lemma: For every graph;

$|\text{Any Matching}| \le |\text{Any vertex cover}|$

Proof: Let, a matching of size 3:

matching:
$a \xrightarrow{e_1} b$
$c \xrightarrow{e_2} d$
$e \xrightarrow{e_3} f$

Now, Let vertex cover

$VC = \{a/b, c/d, e/f, \dots\}$

edge e_1 को cover करने के लिए a या b लेना पड़ेगा

Sim. for e_2, e_3 also

Should cover all edges

Now, Let a matching of size K:

matching:
$a_1 \xrightarrow{e_1} b_1$
$a_2 \xrightarrow{e_2} b_2$
$\dots$
$a_n \xrightarrow{e_K} b_n$

$VC = \{a_1/b_1, a_2/b_2, \dots a_K/b_K \dots\}$

e_1 e_2 e_K

Therefore, if $\exists$ a matching of size K, then Every VC has size $\ge K$

Lemma: for every graph G;

$\mu(G) \le \beta(G) \le 2\mu(G)$

i.e. Max.^{m} matching $\le$ MVC $\le 2 \cdot$ max.^{m} matching

Proof: From above proof;

$|\text{Any matching}| \le |\text{Any vertex cover}|$

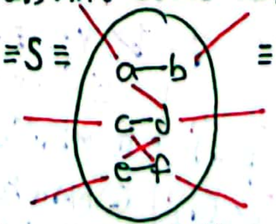
consider max.^{m} matching & min.^{m} VC

$\therefore \boxed{\mu(G) \le \beta(G)}$

Now, we just need to prove $\beta(G) \le 2 \cdot \mu(G)$

Lets assume some max.^{m} matching S

$\mu(G) \equiv S \equiv$



$\equiv \{ab, cd, ef\}$

x —— y } This edge is not possible o/w S के अंदर होती

If S is max.^{m} matching, then every edge of graph G is incident on some vertex which is there in S

Therefore, VC should be :

$VC = \{a, b, c, d, e, f\}$

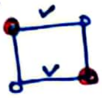
$\Rightarrow |VC| = 2 \cdot \mu(G)$

$\Rightarrow \boxed{\beta \le 2 \cdot \mu(G)}$

Therefore,

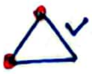
$\mu \le \beta \le 2 \cdot \mu$

Q:) Give some graph with $\mu = \beta$.



$\left.\begin{array}{l}\mu = 2 \\ \beta = 2\end{array}\right\} \Rightarrow C_n$ (n=even) $< \begin{array}{l}\mu = n/2 \\ \beta = n/2\end{array}$

Q:) Give some graph with $\beta = 2\mu$



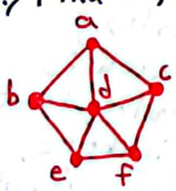
$\mu = 1$
$\beta = 2$

Lemma : Konig's Theorem

If G is a bipartite graph, then $\mu = \beta$.

(Converse need not be true)

Q:) Find α, α' or μ, β, β' :-



MIS: $\{a, e\} \Rightarrow \alpha = 2$

Max^m matching: $\{ab, cd, ef\}$

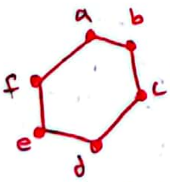
$\Rightarrow \mu$ or $\alpha' = 3$

mVC : $\{d, a, e, f\}$

$\Rightarrow \beta = 4$

mEC : $\{ab, cd, ef\}$

$\Rightarrow \beta' = 3$



$C_6 \begin{cases} \alpha = 3 \\ \beta = 3 \\ \alpha' \text{ or } \mu = 3 \\ \beta' = 3 \end{cases}$

*) Overall conclusion :

① $\alpha + \beta = n$

$\alpha' + \beta' = n$

② $\mu \le \beta \le 2\mu$

③ $\beta' \ge \lceil \frac{n}{2} \rceil$

④ $\mu \le \lfloor \frac{n}{2} \rfloor$

⑤ $\alpha \le \beta'$

⑥ Bipartite G $\underset{\not\leftarrow}{\rightarrow}$ $\mu = \beta$

Q:) For Hypercube graph Q_n :

$Q_n \rightarrow$ Bipartite graph

$\mu = \beta = 2^n/2$

$\alpha = 2^n/2$

$\beta = |V| - \alpha = 2^n - 2^{n-1} = 2^{n-1}$

Lemma: For any graph G with $\underbrace{\delta > 0}$, so that edge cover don't fall in D.N.E

$\alpha \le \beta'$

MIS $\le$ m EC

Proof: Let, $G(V, E)$

Let, Max^m Ind. set $= \alpha = \{a, b, c, d\}$ ↳ No edge among these vertices

now, Edge cover (should cover all vertices) $= \{e_1, e_2, e_3, e_4, \dots\}$

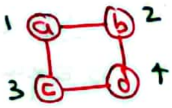
for a, for b, for c, for d

$\Rightarrow \alpha \le \beta'$.

#) Graph Coloring :

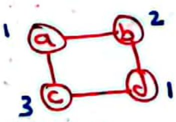
*) Vertex Coloring : Color the vertices such that no two neighbours / adjacent vertices have same colour. ≡ Proper coloring ≡ Graph coloring (By default Vertex)

*) vertex coloring examples :



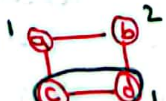
Trivial coloring : Every vertex diff. color.

$\Rightarrow$ 4 colorable ; Can we do better?



Reusing same color.

$\Rightarrow$ 3 colorable ; can we do better?



Reusing same color

$\Rightarrow$ 2 colorable $\Rightarrow$ using atmost 2 colors we can color

$\Downarrow$

$\chi \le 2$

; Can we do better?

$\Rightarrow$ No, K_2 Subgraph there

$\Downarrow$

at least 2 color needed

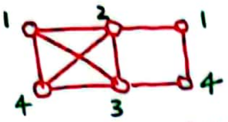
$\Rightarrow \chi \ge 2$

$\boxed{\chi = 2}$

*) K-colorable : Using at most K colors we can color, i.e. $\chi \le K$

*) Chromatic No. : Using min^m possible no. of colors, do proper coloring of the vertices.

Q:> Find $\chi(G)$:



4-colorable
$\Rightarrow \chi \le 4$

can we do better?
$\Rightarrow$ No, K_4 subgraph is there
$\Rightarrow \chi \ge 4$

$\Rightarrow \chi = 4$

*) $\chi(G)$ for some standard graphs:

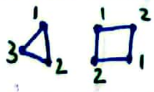
① K_n: every 2 vertices adjacent
$\Rightarrow$ No scope of reusing colors.
$\Rightarrow \chi(K_n) = n$

② E_n: Edgeless graph, no one is adjacent to any other one.
$\Rightarrow$ use same color for all
$\Rightarrow \chi(E_n) = 1$

③ P_n : Path graph

If $n \ge 2 \longrightarrow \chi(P_n) = 2$
If $n = 1 \longrightarrow \chi(P_n) = 1$

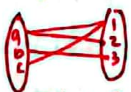
④ C_n: cycle graph



If n = even $\longrightarrow \chi = 2$
If n = odd $\longrightarrow \chi = 3$

⑤ Bipartite G : G is bipartite iff No odd cycles.

$\chi = 1$

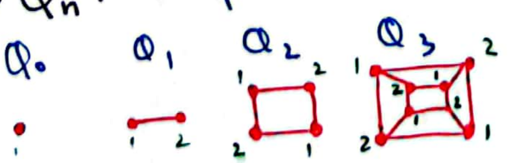


$\chi = 2$

i.e. $\chi = 1$; when No edge (E_n)
$\chi = 2$; when at least one edge there
$\Rightarrow$ Bipartite graphs are 2-colorable
$\Rightarrow \chi \le 2$

⑥ $K_{m,n}$: $\chi = 1$; when $n=0$ or $m=0$
$\chi = 2$; when $n \ge 1$ & $m \ge 1$

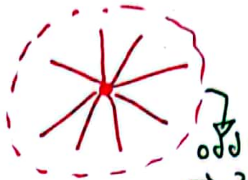
⑦ Q_n : $\rightarrow$ bipartite graph



$$\chi = \begin{cases} 2 & ; \ n \ge 1 \\ 1 & ; \ n = 0 \end{cases}$$

⑧ W_n : wheel graph on $n+1$ vertices.

n = odd



odd cycle
$\Rightarrow 3$

n = even

Even cycle
$\Rightarrow 2$

Central vertex always gets different color

$$\chi = \begin{cases} 1+3 & ; \ n = \text{odd} \\ 1+2 & ; \ n = \text{even} \end{cases}$$

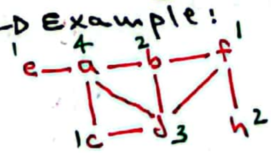
*) Algorithm for coloring:

$\rightarrow$ Greedy algorithm: being greedy for colors/Labels. i.e. using minimum colors/Labels available.
कंजूस Algo., कंजूस for colors/Labels.

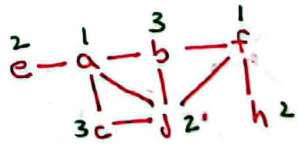
$\rightarrow$ Algorithm:
1) start at any vertex v, give it a label/color 1.
2) Go to any uncolored vertex, give minimum value which is available (i.e. min^m value that is not given to it's neighbors)

$\rightarrow$ Greedy algorithm does not guarantee to give you chromatic no. (χ).
It only tells you that G is K-colorable

$\rightarrow$ example:



Greedy Algo:
f-h-b-d-c-a-e
$\Rightarrow$ 4-colorable
$\Rightarrow \chi \le 4$



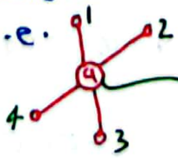
Greedy Algo:
a-f-e-d-h-c-b
$\Rightarrow$ 3-colorable
$\Rightarrow \chi \le 3$

3 length cycle there $\Rightarrow \chi \ge 3$
$\Rightarrow \chi = 3$

Q:> Let, $\Delta \equiv$ max.^{m} degree of a graph G. $\chi(G) = ?$

say $\Delta = 4$

i.e.



$\rightarrow$ In worst case 5 is available

$\Rightarrow \boxed{\chi(G) \le \Delta + 1}$

In Best case, maybe $\chi(G) = 1$

$\Rightarrow \boxed{1 \le \chi(G) \le \Delta + 1}$

*) Lemma: Relation b/w $\chi(G)$ & $\omega(G)$

chromatic no. — $\chi(G)$; clique no. — $\omega(G)$, i.e. Largest complete subgraph

$$\chi(G) \geq \omega(G)$$

Proof: Let, Graph $G(V,E)$

Assume, G contains a largest complete subgraph having 'm' vertices.

$\Rightarrow \omega(G) = m$

now, to proper color this complete subgraph, we need at least 'm' colors.

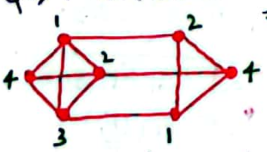
$\Rightarrow \chi(G) \geq m$

$\Rightarrow \chi(G) \geq \omega(G)$

*) Guidelines to find $\chi(G)$:

1) If standard graph (K_n; $K_{m,n}$; P_n; C_n; Q_n; W_n) then you know the answer.
2) Find largest complete subgraph and start coloring from there (Directly give different colors to all it's vertices)
3) If no complete subgraph in G, then find odd cycle $\Rightarrow \chi \geq 3$
4) $\omega(G) \leq \chi(G) \leq \Delta + 1$
5) Once you get $\chi(G) = k$, ask yourself can you do with less than k.

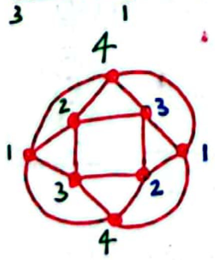
Q1) Find $\chi(G)$

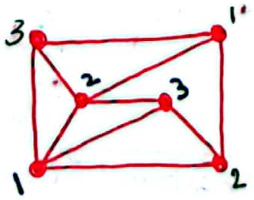


$\Rightarrow$ 4 colorable $\Rightarrow \chi \leq 4$

K_4 subgraph $\Rightarrow \chi \geq 4$

$\therefore \chi = 4$





Note: G is bipartite iff No odd length cycle. or G is bipartite iff every cycle has even length.

Note: If G is properly colored, then vertices of a particular color forms independent set.

Ex:) G —
Red = {a, d, e} → ind. set
Green = {b, c} → ind. set
Blue = {f, h} → ind. set

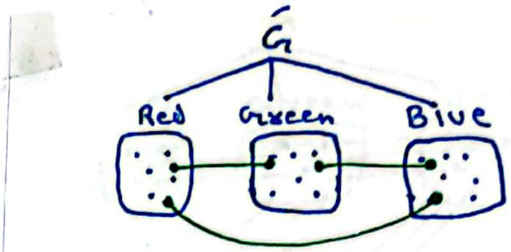
*) Lemma: If chromatic no. of G is m

i.e. if $\chi(G) = m$

then min^m # Edges in G are $^mC_2$

i.e. $|E(G)| \geq {}^mC_2$

Proof: Lets say G has chromatic no. 4 i.e. $\chi = 3$



If no edge in $G \Rightarrow \chi = 1 \neq 3$

Now, b/w every color class (in min^m coloring) at least one edge must be there (pairwise)

i.e. at least one edge b/w →
Red — Green
Red — Blue
Green — Blue

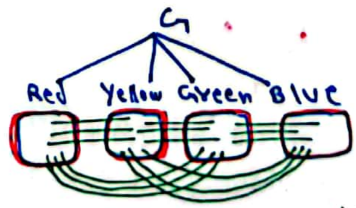
अगर ये edges ना हों तो भला merge ना कर लोगे तुम Red-Green या किसी और को, & then chromatic no. would become less than 3

$\therefore$ min^m edges in $G = {}^3C_2 = 3$

$\Rightarrow$ If $\chi = m \Rightarrow$ min^m # Edges in $G \geq {}^mC_2$

Q:) For $G(V,E)$; χ = chromatic no. Find max^m # Edges in G.

Let, G have 20 vertices $\chi(G) = 4$, such that
Red = 5 vertices
Yellow = 4 vertices
Green = 2 "
Blue = 9 "



~~~ ⇒ Independent sets

~~~ ⇒ max^m edges

→ Max^m no. of Edges possible are :—

Red—Yellow = 5×4 edges
Red—Green = 5×2 "
Red—Blue = 5×9 "
Yellow—Green = 4×2 "
Yellow—Blue = 4×9 "
Green—Blue = 2×9 "

Lemma: $|V| \le \alpha \cdot \chi$

Proof: If χ is chr. no. of G then, on an average color class have a size of: $\frac{|V|}{\chi}$

$\Rightarrow \exists$ at least one color class which contains at least $\frac{|V|}{\chi}$ vertices (on an average).

$\Rightarrow \exists$ at least one independent set of vertices which contains at least $\frac{|V|}{\chi}$ vertices (on an average).

$\Rightarrow \quad \alpha \ge \frac{|V|}{\chi} \quad \Rightarrow \quad |V| \le \alpha \cdot \chi$

*) Overall conclusions:

1) $\omega(G) \le \chi \le \Delta + 1$

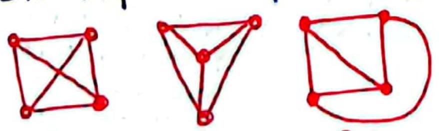
2) # Edges $\ge {}^{\chi}C_2$ (or) $|E| \ge \frac{\chi(\chi-1)}{2}$

3) $\alpha \ge \frac{|V|}{\chi}$

#) Planar Graph:

*) Planar graph: A graph which can be drawn w/o edge crossing (or)
A graph for which some planar representation is possible.

Ex: K_4 is a planar graph



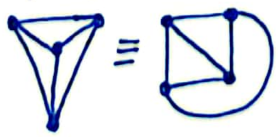
Non-Planar representation of $K_4 \equiv G_1$
Planar representation of $K_4 \equiv G_3, G_2$

Ex: $\underbrace{C_n, P_n, K_1, K_2, K_3, K_4, \text{Star graph}(K_{1,n})}_{\text{Planar graph}}$

Ex: $K_{2,n}$

$\Rightarrow$ Planar graph

Q:) Planar rep.^{n} of a planar graph is unique?
$\Rightarrow$ No;

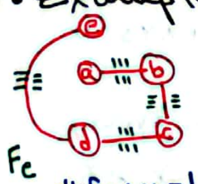


both Planar rep.^{n} of K_4

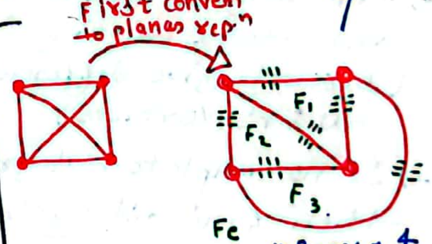
*) Face / Region in a planar rep.^{n}:
- Area surrounded by edges (or) outside of all edges.
- Ex:

Interior face $= f_1, f_2$
exterior face $= f_3$

- Concept of face is defined for planar rep.^{n} of planar graph.
- Definition of Planar Graph:
If P is planar rep.^{n} of G, then the 2-D plane is partitioned in faces/regions by P.
- Formal definition of face:
A maximal area of points in 2-D plane in which we can go from any point to any other point w/o having to cross any vertex or edge of graph.
- Degree of a face: No. of times a face is touching edges. (or)
No. of edges a face is falling/touching.
- Examples:-



#faces = 1
Deg(Fe) = 8

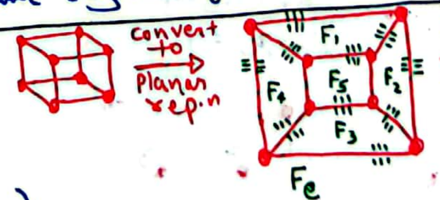


#faces = 4

same $\rightarrow$ Deg(F_1) = Deg(F_2) = Deg(F_3) = Deg(F_e) = 3

Conclusion: Two or more different planar rep.^{n}s of same graph have same #faces & same degree of respective faces.

Q_3:



#faces = 6

deg(F_1), F_2, F_3, F_4, F_5, F_e $\Big\}$ = 4

- # Exterior faces = 1 (in any finite graph)
interior faces = # Faces − 1
faces in a tree = 1
faces in a forest = 1
- while finding deg(face), consider every edge as a wall, i.e. every edge has 2 sides.

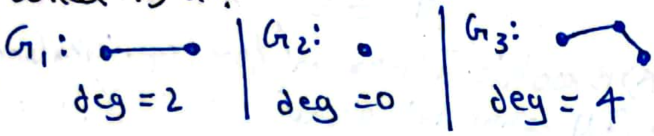
Q:) A tree has 100 vertices. Find #faces & deg(face).

$\Rightarrow$ #faces = 1 (Fe)

$Deg(Fe) = 2(n-1) = 2(99) = 198$

Q:) A connected planar graph has some face with degree 2. What is G?

$\Rightarrow$



$\Rightarrow G \equiv G_1$

Conclusion: A connected planar graph has some face with degree 2 iff the graph G is

Conclusion: A connected planar graph with #vertices ≥ 3 have deg(any face) ≥ 3.

Lemma: Total Degree of all faces $= \sum_{f \in faces} deg(f) = 2|E|$

Proof: Every edge (wall) will contribute a value of 2 in the degree summation of all faces.

$\lvert E \rvert$	0	1	2	3	...
Total degree of faces	0	2	4	6	$2\lvert E \rvert$

Q:) In G; Assume K components. Delete one vertex then the no. of components?

$\Rightarrow$ $k-1 <$ #comp. $< n-1$

Q:) In G, assume K components. Delete one edge, then the no. of components?

$\Rightarrow$ K or K+1

If edge is part of some cycle $\xrightarrow{\text{then}}$ # components unchanged.

If edge is not part of any cycle (i.e. edge is bridge) $\xrightarrow{\text{then}}$ # components increase by 1

Q:) In a planar rep.^{n}, if we delete one edge then what may happen?

$\Rightarrow$ Either (# components increase by 1 AND # faces stays the same) [when bridge edge deleted]

Or (# components stays the same AND # faces decrease by 1) [when cycle ~~at~~ edge is deleted]

*) Euler formula:

- For a finite planar graph with $|V| = v$; $|E| = e$; #comp. = c

#faces = f

$$\boxed{v + f = e + c + 1} \text{ — (i)}$$

- Proof: In terms of # of edges;

if $e = 0$; (i) becomes

$v + 1 = 0 + c + 1$

$v + 1 = v + 1$ [$\because c = v$]

$\Rightarrow$ Eq.^{n} (i) is valid

Now, lets assume eq.^{n} (i) is valid for any planar graph with e or less than e edges.

i.e. $v + f = e + c + 1$;

Now, eq.^{n} (i) should be true for planar graph with e+1 edges as well.

i.e. $v + f = (e+1) + c + 1 = e + c + 2$

Now, if delete one edge, then one of these 2 can happen

1) # components increase by 1

$v + f = (e+1-1) + (c+1) + 1$

$= e + c + 2 \Rightarrow$ Balanced

or 2) # faces decrease by 1

$v + (f-1) = (e+1-1) + c + 1$

$v + f = e + c + 2 \Rightarrow$ Balanced

$\therefore$ Eq.^{n} (i) can balance itself

$\therefore$ Eq.^{n} (i) is valid result

$\Rightarrow \boxed{v + f = e + c + 1}$

- If G is connected planar, then $c = 1$

$\Rightarrow \boxed{v + f = e + 2}$

Lemma: for a connected planar graph:

if $|V| \geq 3$

then $|E| \leq 3|V| - 6$

i.e. $\boxed{e \leq 3v - 6}$

Proof: Bcz. G is connected planar with $v \geq 3$

$\therefore$ Deg(face) ≥ 3 See conclusion on P.no.23

$\therefore$ min^m deg. of any face is 3

Now, $\sum \deg(f_i) = 2e$

$e \equiv$ # of edges

$f \equiv$ # of faces

$3 \times f \leq 2e$

$3(e + 2 - v) \leq 2e$

$\boxed{e \leq 3v - 6}$

Conclusion: If G is connected planar graph.

and $v \geq 3$

and there is no triangle (C_3) in G

then Deg$\left(\substack{\text{any} \\ \text{face}}\right) \geq 4$

i.e. Δ is the only possibility to get deg(face) = 3

Lemma: If G is connected planar graph.

and $v \geq 3$

and no C_3 (triangle) in G

then $\boxed{e \leq 2v - 4}$

Proof! here, Deg$\left(\substack{\text{any} \\ \text{face}}\right) \geq 4$

$\therefore \sum \text{Deg}(f_i) = 2e$

$4 \times f \leq 2e$

$4(e + 2 - v) \leq 2e$

$\boxed{e \leq 2v - 4}$

Q:) Prove: K_5 not planar.

$e = \frac{5 \times 4}{2} = 10$; $v = 5$ i.e. $v \geq 3$ ✓

If K_5 is planar then,

$e \leq 3v - 6$

$10 \leq 3 \times 5 - 6$

$10 \leq 9$ $\Rightarrow$ Not Satisfied

Q:) Prove $K_{3,3}$ is non-planar.

$v = 6$; $v \geq 3$ ✓ ; $e = 3 \times 3 = 9$

$K_{3,3}$ have no C_3 (triangle)

$\Rightarrow$ if $K_{3,3}$ is planar, then

$e \leq 2n - 4$

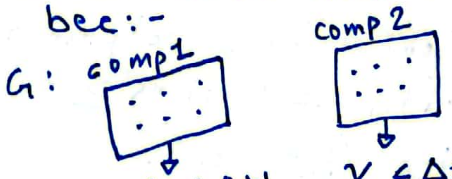
$9 \leq 2(6) - 4$

$9 \leq 8$ $\Rightarrow$ Not Satisfied

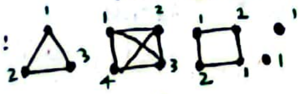
***) Brook's Theorem** (for proper Vertex Coloring)

- All connected graphs have $\chi(G) \leq \Delta(G)$ except for K_n & odd cycles.
- i.e. for K_n & C_{odd} $\rightarrow \boxed{\chi = \Delta + 1}$

  for all others $\rightarrow \boxed{\chi \leq \Delta}$
- for Disconnected graphs also the Brook's Theorem is valid bcz:-



$\therefore \chi(G) \leq \Delta + 1$

Q:) G:  ; $\chi = ?$

$\chi(G) = \max(3, 4, 2, 1, 1) = 4$

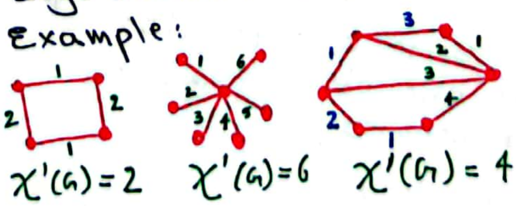
Note: Graph with K components;

then $\chi(G) = \max(\chi(C_i))$; $C_i \rightarrow$ different components

#) Graph Coloring contd...

***) Edge Coloring:**

- Give neighbour edges (edges who fall on same vertex) different colors.
- Also called proper edge coloring.
- K-edge colorable means, using atmost K colors you can proper edge color G.
- Edge chromatic no. $\equiv \chi'(G) \equiv$ Chromatic Index
- Example:

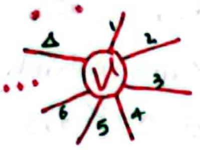


i.e. Start from edges of max^m degree vertex

Lemma: For any graph G;

$$\chi'(G) \geq \Delta(G)$$

Proof: Lets take some portion of G;



All these Δ edges are adjacent, so, have to be given different colors.
$\Rightarrow \chi'(G) \geq \Delta$

Let v be the vertex with max.^{m} degree.

*) **Vizing's Theorem** (For edge chromatic no.):

- $\chi'(G) = \Delta$ or $\Delta+1$

  i.e. $\Delta \leq \chi'(G) \leq \Delta+1$

- Graphs
  - Class 1: $\chi'(G) = \Delta$
  - Class 2: $\chi'(G) = \Delta+1$

- C_n:

  $C_{even} \equiv$ Class 1 $\qquad C_{odd} \equiv$ Class 2

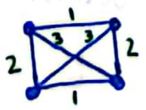
C_4	C_6	C_3	C_5
$\chi' = 2$	$\chi' = 2$	$\chi' = 3$	$\chi' = 3$
$\Delta = 2$	$\Delta = 2$	$\Delta = 2$	$\Delta = 2$

- P_n: Class 1

$\Delta = 0$, $\chi' = 0$ $\qquad$ $\Delta = 2$, $\chi' = 2$

- K_n:

  $K_{even} \equiv$ Class 1 $\qquad K_{odd} \equiv$ Class 2



$\Delta = 3$, $\chi' = 3$

$\Delta = 2$, $\chi' = 3$

- Bipartite graphs are class 1.

  i.e. $\chi' = \Delta$

  i.e. C_{even}, Q_n, Tree, P_n, $K_{m,n}$, $K_{1,n}$ (Star G) $\Big\}$ class 1

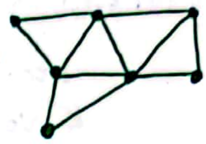
- Every class 2 Graph must have at least 3 vertices of max.^{m} degree.

  i.e. class 2 graph ($\chi' = \Delta+1$) $\Rightarrow$ At least 3 vertices with Δ (converse does not hold)

  i.e. You only check if G is not a class 2 graph

25

Ex: G:



here, $\Delta = 6$
but, we've less than 3 vertices with Δ
$\Rightarrow G$ is not class 2
$\Rightarrow G$ is class 1

Note that when you check if G is a class 2 graph, that's wrong approach

Ex:

here, $\Delta = 2$
and, we've at least 3 vertices with Δ
$\Rightarrow G$ is class 2 ✗
wrong approach
bec. G is class 1; $\chi' = \Delta = 2$

- Finding if G is class 1 or class 2 is NP hard problem.

- A regular graph with odd # vertices is class 2.

  i.e. Regular with odd $|V|$ $\Rightarrow$ class 2 (converse does not hold)

  Ex:) C_{odd}, K_{odd}

Note: $\alpha \xrightarrow{①} \beta$, $\beta \xrightarrow{②} \alpha$

① If fact α is known to us, then we can conclude fact β

② If fact β is known to us, then we can conclude fact α.

- Sometimes you've to manually edge color the graph to know $\chi'(G)$.

- Guidelines to find $\chi'(G)$:

1) $\chi'(G) = \Delta$ or $\Delta+1$
2) If less than 3 vertices of max.^{m} degree, then G is class 1 $\Rightarrow \chi' = \Delta$
3) If more than or equal to 3 vertices of max.^{m} degree, then Check if G is bipartite
   If bipartite $\longrightarrow$ then class 1 $\Rightarrow \chi' = \Delta$
4) If G is regular with odd # vertices, then class 2 $\Rightarrow \chi' = \Delta+1$
5) If nothing works $\longrightarrow$ Then do manually.

Q:) Q_n is planar?

If $n \le 3$ then Q_n Planar

If $n \ge 4$ then Q_n Not Planar

Proof: If Q_4 is planar,

then $e \le 2v - 4$

[bec. Planar; $|V| \ge 3$; No C_3]

bec. Q_n is bipartite & No odd cycle there

but, #edges = $\frac{\text{Total degree}}{2}$

$\Rightarrow e = \frac{2^n \times n}{2}$ {bec. 2^n vertices we have & each have deg. n}

$e_{(\text{in } Q_n)} = n \cdot 2^{n-1}$

$\Rightarrow 4 \times 2^{4-1} \le 2 \times 2^4 - 4$

$32 \le 32 - 4$ ⇒ Not Satisfying

⇒ Q_n is Planar iff $n \le 3$

Q:) Prove that; $\min^m$ deg. of planar graph is ≤ 5

(or) Every planar graph has some vertex with deg. ≤ 5

(or) Every planar graph has a vertex of degree at most 5.

Proof: Let, G ≡ Planar graph

assume $\delta \ge 6$ [for contradiction]

⇒ using degree summation formula

Total deg. $= 2e$

$n \times \delta \le 2e$

$v \times 6 \le 2e$

$\boxed{e \ge 3v}$

but, since G is planar ⇒ $\boxed{e \le 3v - 6}$

⇒ contradiction

$\Rightarrow \delta_{(\text{for planar graph})} \le 5$

*) Four color theorem:

Chromatic no. of any planar graph ≤ 4

Proved using a computer, bec the proof was infeasible for a human to check by hand.

*) Kuratowski's Theorem:

- Smallest non-planar graphs:-

w.r.t vertices ≡ K_5

w.r.t Edges ≡ $K_{3,3}$

- If G has a subgraph which is isomorphic to K_5 or $K_{3,3}$ then G is not planar.

(Or)

If G has clique no. ≥ 5 (i.e. G has a subgraph K_5) then G is not planar.

But the converse of these 2 statements is not true.

i.e. Subgraph K_5 or $K_{3,3}$ there → Not planar

- Original Statement of Kuratowski Theorem:

A graph is non-planar iff it contains a subgraph homeomorphic to K_5 or $K_{3,3}$

i.e. G has a subgraph which is homeomorphic to K_5 or $K_{3,3}$ ⟷ G is Not planar

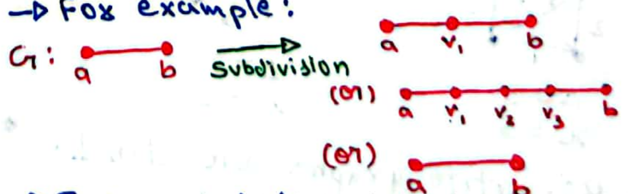
Note: Every subgraph of planar graph is planar.

*) Homeomorphic Graphs & Subdivision of a graph:-

- Subdivision of a graph:

→ By replacing an edge in G with a path, you get subdivision of G.

→ For example:



→ Every graph is subdivision of itself.

- Homeomorphic graphs:

G is homeomorphic to H iff Some subdivision of G is isomorphic to some subdivision of H

i.e. $G \overset{h}{\equiv} H$ iff

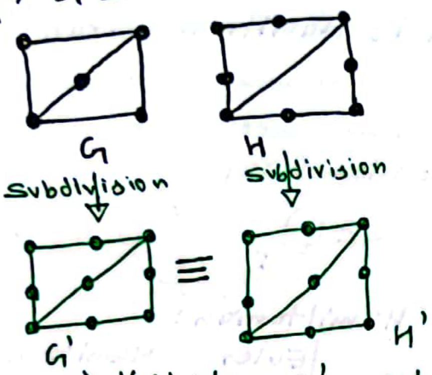
Some subdivision of G ≡ Some subdivision of H

Note: Symbols used:
G homeomorphic to H : $G \overset{h}{\equiv} H$
G Isomorphic to H : $G \equiv H$

- Example:
  $C_3 \overset{h}{\equiv} C_5$
  bec.: $C_3 \xrightarrow{\text{subdivision}} C_5$
  and $C_5 \equiv C_5$

- If $G \equiv H$ then $G \overset{h}{\equiv} H$

Q:) $G \overset{h}{\equiv} H$?



$\Rightarrow$ Yes; bec. $G' \equiv H'$

Note: Every subdivision of planar graph is planar.

Note: Every subdivision of non-planar graph is non-planar.

Note: i.e. Subdivision does not change planarity / non-planarity

- A graph is planar iff every subdivision of G is planar.

  A graph is non-planar iff some subdivision of G is non-planar.

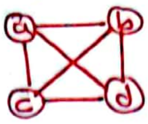
#) Euler & Hamiltonian Graphs:

*) Euler problem: Visit every edge exactly once & come back to the starting point.
Ex: 7 bridges of Königsberg.

*) Hamiltonian Problem: Visit every vertex exactly once & come back to the starting point.
Ex: Traveling salesman Problem.

*) No vertex repeats $\xrightarrow{}$ No edge repeat
$\not\leftarrow$

*) Circuit: A closed trail.

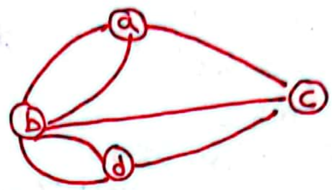


Edge repetition not allowed.

a–d–b–c–a ≡ circuit
a–d–b–c–a–b ≡ Trail

*) 7 bridges of Königsberg :
(The origin of Euler circuit)

- Abstract structure of Königsberg

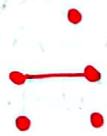


vertices ≡ Land mass
Edges ≡ Bridges

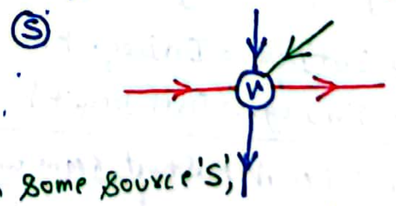
- Euler circuit: Start from any vertex, visit every edge exactly once, and come back to starting point.
- Euler Trail ≡ Euler walk ≡ Euler path: Start from any vertex, visit every edge exactly once. (You can end anywhere).
- Euler circuit ≡ Euler Tour
  ≡ Euler circle
  ≡ Euler cycle
- Euler Graph: A connected graph which has an Euler circuit.
- Conditions for a graph to be Euler graph:

1) No odd degree vertices should be there in G.

Proof: For n=1 — E.gr. ✓
For n=2 — E.gr. ✗
(or) — E.gr. ✗



For $n \geq 3$
Assume 'v' is the odd deg. vertex in G.



From some source 'S',

If you come in to 'v', then you must go out from 'v' in order to reach the 'S'. ⇒ No odd deg. vertex in Euler graph.

2) Graph should be connected.

Note: A directed graph is Euler iff every vertex have indegree = outdegree AND vertices belong to a single strongly connected component. (Discussed further)

*) Euler graph $\longleftrightarrow$ Each vertex has even degree

Q :) which is Euler graph?

1) K_{even} ✗ $K_6 \Rightarrow deg.(all) = 5$
2) K_{odd} ✓ $K_5 \Rightarrow deg.(all) = 4$
3) C_{even} ✓ Note that,
4) C_{odd} ✓ $P_1 \Rightarrow \bullet$ $\Rightarrow Deg. = 0$ $\Rightarrow$ E.Gr. ✓
5) P_{even} ✗ but, P_n ; $n \geq 2$ $\rightarrow$ E.Gr. ✗
6) P_{odd} ✗
7) Q_n In Q_n, $deg(\text{any vertex}) = n \Rightarrow$ yes; if n = even, No; if n = odd
8) W_n No ;  ← deg. = 3
9) $K_{m,n}$ Yes, iff both m, n are even
10) d-Regular Yes, iff connected AND d = even

*) Euler circuit exists iff each vertex have even degree.

Euler path exists iff at most two vertices have odd degree.

*) Euler circuit & Euler path in directed graphs:

- Euler circuit exists iff $\forall$ vertices; InDeg. = OutDeg.
- Euler path exists iff $\forall$ vertices (except Source & Final) Indeg. = OutDeg.
  AND
  For Source $\Rightarrow$ OutDeg = InDeg. + 1
  For final $\Rightarrow$ InDeg. = OutDeg. + 1

*) Hamiltonian Circuit: Start from anywhere, visit every vertex exactly once & come back to starting point.

*) Hamiltonian Graph: G is hamiltonian iff $\exists$ Hamiltonian circuit in G.

*) Hamiltonian path: Start from anywhere visit every vertex exactly once. (You can end anywhere).

*) Time complexities:
To check if G is :-

Euler graph $\Rightarrow O(V)$
[Linear Time, just find degree of each vertex]

Hamiltonian graph $\Rightarrow$ NP-Complete problem.
$\Rightarrow O(2^V)$
[Non-Polynomial time, so not feasible by computer]

*) Dirac Theorem (For Hamiltonian Graph):
If G has ($n \geq 3$ AND $\delta \geq \frac{n}{2}$)
Then G has a Hamiltonian Cycle (i.e. G is Hamiltonian Graph)

This is a one way theorem, no two way theorems for hamiltonian Graphs.

i.e. $G(n \geq 3 ; \delta \geq \frac{n}{2}) \rightarrow$ Ham. Gr. (reverse arrow crossed out)

Q :) Euler vs Hamiltonian:

Property	Euler	Hamiltonian
vertex repetition allowed?	Yes	No
Edge repetition allowed?	No	No
Omitted nodes allowed?	No	No
omitted edges allowed?	No	Yes

Q:) which graphs has an Euler circuit?

1) Any K-regular graph ; K = even
$\Rightarrow$ NO; counter example:-
G:  ; K = 2

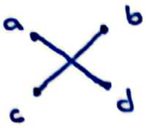
2) Complement of a cycle on 25 vertices.
If; for a graph on n vertices,
Deg(u) = m
Deg(v) = p
u, v are some particular vertices in G
Then in $\overline{G}$;
Deg(u) = (n-1) - m
Deg(v) = (n-1) - p
bec. max^m Deg = n-1

Now,
in C_{25} ; Deg(v) = 2 ; $\forall$ for all vertices
$\Rightarrow$ in $\overline{C_{25}}$; Deg(v) = (25-1) - 2
= 22 ; $\forall$ for all vertices
$\Rightarrow$ Degree is even

But, Is $\overline{C}_{25}$ connected ?

Now,

$\overline{C}_3$: a b c $\Rightarrow$ not connected

$\overline{C}_4$:  $\Rightarrow$ Not connected

$\overline{C}_5$:  $\Rightarrow$ connected

Proof that $\overline{C}_n$; $n \geq 5$ is connected:

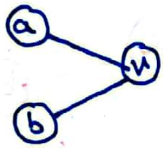
To prove that $\overline{C}_n$ is connected we need to show that $\forall a,b$ $a \underset{\text{path exist}}{\leadsto} b$

Now,

Let $a,b \in \overline{C}_5$; $a,b \in C_5$ as well

Case-1: a, b are adjacent in C_5

then, $\exists v$; which is not adjacent to any of them.

then, in $\overline{C}_5$ 

$\Rightarrow$ $a \underset{\text{path exists}}{\leadsto} b$

Case-2: a, b not adjacent in C_5

then, in $\overline{C}_5$ 

$\Rightarrow$ $a \underset{\text{path exists}}{\leadsto} b$

Hence, proved. that, $\overline{C}_n$ is connected $\forall n \geq 5$

*) **Lemma:** In $G(V,E)$; $V \geq 2$

At least two vertices will have same degree.

i.e. All vertices can't have distinct degrees.

Proof: (By Pigeon Hole Principle)

Let, $G(V,E)$; $V \geq 2$

Case-1: Connected

Possible degrees : $\boxed{1, 2, \ldots, n-1}$ Holes

vertices = $\boxed{n}$ Pigeons

$\Rightarrow$ Vertex = a_1 , a_2 , a_3 $\ldots$ a_{n-1} , a_n

Degree = 1 , 2 , 3 , $\ldots$ n-1 , $\boxed{?}$

$\Rightarrow$ At least 2 vertices will have same degree.

Case-2: Disconnected $\rightarrow$ no vertex with degree n-1

Possible degrees: 0, 1, 2, ..., n-2

vertices: n

$\Rightarrow$ vertex: a_1, a_2, a_3, ... a_{n-1}, a_n

Deg : 0 , 1 , 2 , ... n-2 $\boxed{?}$

$\Rightarrow$ At least 2 vertices will have same degree.

Therefore, Every graph ($n \geq 2$) has atleast 2 vertices of same degree.

#) Graph Realization Problem:

*) Generalized graph realization problem:

For some condition (usually degree sequence) can we create/realize a structure (Graph, Tree, etc.) of some class.

for example:

Given; Deg. Seq.: 4, 3, 2, 2, 1

i) can you realize K_m ?

$\Rightarrow$ No; bec. Graph realization cond.^{n} for K_m :-

Deg. seq. should be:

$\underbrace{(m-1), (m-1), \ldots, (m-1)}_{m \text{ times}}$

ii) can you realize Tree?

$\Rightarrow$ No; bec. Graph realization cond.^{n} for Tree :-

Total Deg. should be : $2(n-1)$

iii) can you realize simple graph?

$\Rightarrow$ Yes; bec. by using Havel-Hakimi Algo:

$\cancel{4}$, 3, 2, 2, 1

$\cancel{2}$, 1, 1, 0

0, 0 ✓

*) Note that, for all undirected graphs :-

1) Total deg. = 2|E| = Even

2) # odd deg. vertices = Even

3) Degree (v) ≥ 0 ; a non-negative integer.

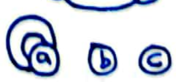
*) Graph realization problem for Pseudograph:

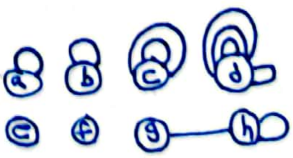
- Pseudograph (PG) → Self Loops ✓ → Multi Edges ✓ → Undirected graph

- Example:-

Degree seq.

3, 3 

4, 0, 0 

2, 2, 4, 8, 0, 0, 1, 3 

3, 2, 1, 1 No; ∵ 3 odd deg. vertices

- Graph realization cond.^{n} for pseudograph:

  Total deg. $= 2|E| =$ Even

  i.e. $\boxed{\#\text{odd degree vertices} = \text{Even}}$

  (in the Deg. seq.)

*) Graph realization for Multi-Graphs:

- Multigraph → self loop not allowed
  → Multi edges allowed
  → Undirected graph

- Example:

  4, 0 — No

  4, 2, 2

- Graph realization cond.^{n} for multigraph:
  1) No. of odd deg. vertices are even
  2) $\Delta \leq$ Total deg. of remaining vertices

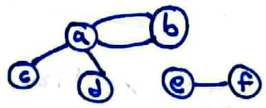
  i.e. If Deg. Seq. (non-increasing):-

  $d_1, d_2, d_3, \ldots d_n$

  then; $d_1 \leq d_2 + d_3 + \cdots d_n$

- Example:

  4, 2, 1, 1, 1, 1



  7, 6, 5, 4, 3, 1 ✓

  6, 5, 3, 3 ✗ ; 3 odd deg. vertices

  4, 2, 0, 0 ✗

  4 ✗

  10, 2, 2, 2, 1, 1 ✗

*) Graph real.^{zn} problem for Simple graph:

- Graphic (Graphic Sequence)

  Degree sequence of some simple graph.

- Havel Hakimi Theorem: (Recursive)

  $\langle d_1, d_2, d_3, \ldots, d_n \rangle$ is graphic

  iff

  $\langle \underbrace{d_2 - 1, d_3 - 1, \ldots d_k - 1}_{d_1 \text{ terms}}, \underbrace{\ldots}_{\text{Rest terms as it is}} \rangle$ is graphic

  where, $d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$

- Example:

  $\langle \cancel{3}, \underbrace{3, 2, 2}, 1, 1, 0, 0 \rangle$ → ignore cuz. isolated

  $\langle \cancel{2}, \underbrace{1, 1}, 1, 1 \rangle$

  $\langle 0, 0, 1, 1 \rangle$

  → $\langle \cancel{1}, \underline{1} \rangle$

  $\langle 0 \rangle$ is graphic: ⓐ

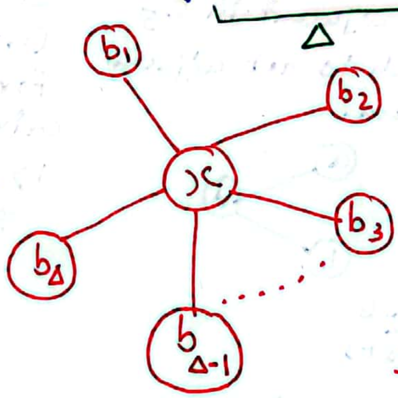
  ⇒

- Idea behind Havel Hakimi:

  Deg. Seq.: $\langle \Delta, a, b, c, \ldots \rangle$

  ⇓

  $\langle \underbrace{a-1, b-1, c-1}_{\Delta}, \underbrace{\ldots}_{\text{as it is}} \rangle$



  deg(x) ≡ Δ

  therefore, when (x) vertex removed, Δ vertices will get their degree decreased by 1.

- Guidelines to check if given deg. seq. is graphic (some simple graph exists):
  1) No. of odd deg. vertices must be even
  2) If n vertices, then; $\Delta \leq n-1$ and $\delta \geq 0$
  3) At least 2 vertices must have same degree. [$n \geq 2$]
  4) Apply Havel-Hakimi.

*) Graph realization problem for Trees:

- Graph realization cond.^{n} for trees:
  1) Total deg. $= 2(n-1)$ [$\because |E| = n-1$]
  2) $\delta \geq 1$ [$\because$ Tree is connected graph]

Note: when $n \geq 2$; Then for Tree realiz.^{n}, at least two vertices must have degree 1 (Necessary but not sufficient for tree)

- Ex: $\langle 1, 1, 3, 2, 4, 5, 6 \rangle$

  $22 \neq 2(7-1)$

  ⇒ No

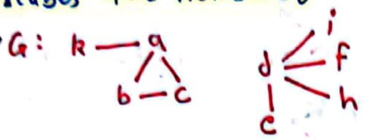
#) Cuts & Connectivity :

*) Cut vertex ≡ Articulation point — Focus on vertex

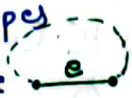
Cut Edge ≡ Bridge ≡ Non-cycle edge — Focus on edge

- cut vertex is a vertex whose removal increases the number of components.

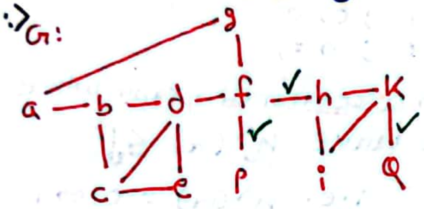
Ex:) G:



Articulation points = 2 ≡ a, d

- In any graph, there are 2 types of edges :-
  1) Cycle edge ≡ Part of some cycle 
  2) Non-cycle edge ≡ Not part of any cycle  ≡ Bridge ≡ cut edge

Ex:) G:

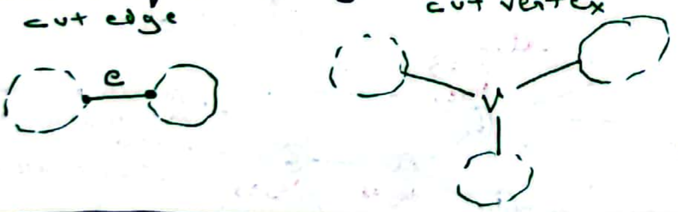


non-cycle edges ≡ # Bridges = 3

- Cut edge is an edge whose removal increases the no. of components.

Note: In questions, imagine cut vertex & cut edge as:-

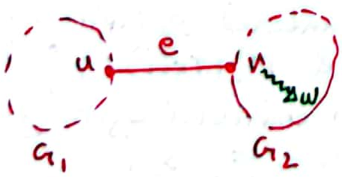
cut edge   cut vertex



- Lemma: Vertex v is a cut iff $deg(v) \geq 2$.

Proof: By definition; v is cut vertex if it's removal increases #components.

So, ⇒ v falls on one of the bridge edge.



$\because deg(v) \geq 2$

- Effect on # components when you remove one vertex :-

If initially # components = k ; $|V| = n$

then;

$$\underbrace{k-1}_{\text{when isolated vertex removed.}} \leq \#\text{components Now} \leq \underbrace{n-1}_{\text{when some central vertex removed, like in star graph.}}$$

Q:) Graph with min^m deg. 1, #compon. = k

Remove one vertex, # comp. now ?

min^m deg. 1 ⇒ No isolated vertex

$$k \leq \#\text{comp. now} \leq n-1$$

- Effect on # components when you remove one edge:-

If initially, # components = k

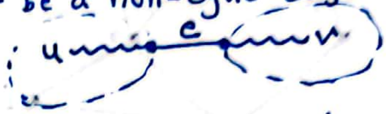
then; # comp. now = $\underbrace{k}_{\text{when cycle edge removed}}$ or $\underbrace{k+1}_{\text{when non-cycle edge removed}}$

Q:) vertex u, v of G considered.

Every path u ~~ v contains edge 'e'.

what is e?

⇒ e must be a non-cycle edge



⇒ e is a bridge (or) cut edge

Q:) If every edge of a connected graph is a cut-edge, what is G?

⇔ G is a Tree

Q:) If every edge of a graph is a cut-edge, what is G?

⇔ G is a forest

- Lemma: Let; G ; connected ; $|V| \geq 3$

If G contains a bridge

then G contains a cut vertex.

Proof: Since; G is connected; $|V| \geq 3$; G contains a bridge.

⇒

Let, a is articulation point

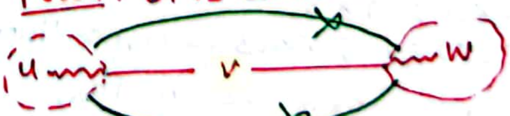
⇒ $deg(a) \geq 2$

⇒ ∃ w, such that a ~~ w

Now, if I remove a, #comp. ↑

- Lemma: A vertex 'v' of a connected graph G is a cut vertex of G iff ∃ u, w (vertices distinct from v) such that there's at least one u ~~ w path in G AND v lies on every u ~~ w path.

Proof: G is connected



~~ ⇒ These edges doesnot exist

- Cut vertex & cut edge are defined for all undirected graph (connected, Disconnected)

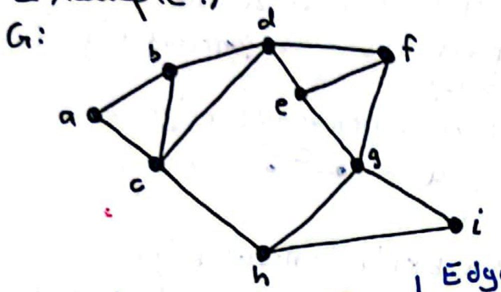
*) Vertex cut : Set of vertices / Edge Cut : Set of Edges — whose removal G disconnects OR, After removal only a single vertex remains

- Vertex cut & Edge cut are defined only for connected graphs.

- To study further concepts, now onwards we only consider connected graphs ...
  - → Vertex cut
  - → Edge cut
  - → Connectivity number ($K(G)$) Kappa

- Example :-

G:



Vertex Cut ≡ {c, d}
≡ {d, e, g, i}

Edge cut ≡
{ab, ac}
{bd, cd, ch}

$K(G) = 2$

- Connectivity number : ($K(G)$)
  - → Size of smallest vertex cut.
  - → If a cut vertex exists in a connected graph, then $K(G)=1$
  - → What does connectivity number tells you about a graph?
    - i) How difficult it is to disconnect the graph.
    - ii) Tells us how much connectedness is there.
  - → If G is disconnected or $|V| = 1$, then $K(G) = 0$; o/w $K(G) \geq 1$
  - → Connected graph G with $K(G) = 5$ means :-

    Any less than or equal to 4 vertices removal will not disconnect G.

    Minimum we need to remove 5 vertices to disconnect G.
  - → In Tree ; $n \geq 3$
    $K(T) = 1$; Trees are easy to disconnect.

Q:) Find $K(K_n)$:

K_1 ⇒ ⊙ ⇒ $K = 0$

K_2 ⇒ •—• ⇒ $K = 1$

K_3 ⇒ △ ⇒ $K = 2$

...

$\boxed{K_n \Rightarrow \quad \Rightarrow K = n-1}$

Q:) Find $K(K_{m,n})$

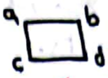
$K = \min(m, n)$

- K-connected Graph : i.e. Removal of any less than or equal to K-1 vertices to G doesn't disconnect G or leaves a single vertex in G.

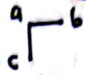
  It means, $K(G) \geq K$

  i.e. connectivity no. $\geq K$

Q:) Which of the following are 2-connected?

i)


✓ $K = 2$

ii)


✗ $K = 1$

iii) K_4 ✓ $K = 3$

iv) C_4 ✓ $K = 2$

v) P_4 ✗ $K = 1$

vi) K_6 ✓ $K = 5$

Q:) $K(W_n)$?

$K(W_n) = 3$

Q:) $K(\text{Tree})$!

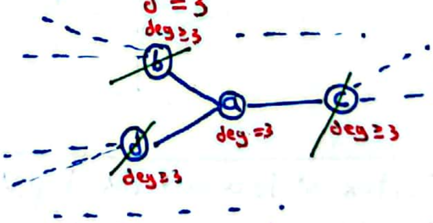
$K = \begin{cases} 1 & ; \text{o/w} \\ 0 & ; n = 1 \end{cases}$

- Lemma : In a connected graph where δ = Min^m deg. we have $K(G) \leq \delta$

Proof : Let, $\delta = 1$ ⇒ There is a pendant vertex

To leave G disconnected or leave single vertex in G
Just remove b
⇒ $K \leq 1$

Let, $\delta = 3$



To leave G disconnected or leave single vertex in G.
Just remove b, c & d

- The most connected graph ⇒ K_n & $K(K_n) = n-1$

  The least connected graph ⇒ Tree & $K(\text{Tree}) = \begin{cases} 0 & ; n=1 \\ 1 & ; \text{o/w} \end{cases}$

Q:) Min^m # edges in connected cyclic graph?

connected graph ⇒ $|E| \geq n-1$

cyclic, connected graph ⇒ $|E| \geq n$

Acyclic, connected graph ⇒ Tree ⇒ $|E| = n-1$

$\boxed{\text{Note: Deg (any cut vertex)} \geq 2}$

- Edge connectivity number: ($\lambda(G)$)
  - → Size of smallest edge cut.
  - → Lemma : $\lambda(G) \leq \delta$; δ ≡ min^m deg.

Proof: vertex 'a' has min. deg. δ

⇒ Edge cut (size is δ)

Q:) In an undirected connected graph G; n vertices; $\lambda(G) = K$;
Prove: $|E| \geq \frac{n \times K}{2}$

Ans:) w.r.t;
$\lambda(G) \leq \delta \Rightarrow K \leq \delta$
$\Rightarrow \boxed{nK \leq n\delta}$ — i

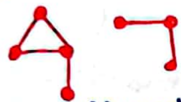
Now, $\sum \deg = 2e$
$\Rightarrow \boxed{n \cdot \delta \leq 2e}$ — ii

Therefore; $\boxed{nK \leq n\delta \leq 2e}$ ☆

$\Rightarrow e \geq \frac{nK}{2}$

#) Strongly Connected Components (SCC) in Directed graph:

*) For undirected graphs: Connectedness and components :)

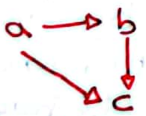
→ 

#components = #connected components = 2

→ In undirected graphs: —
i) G is connected iff #components = 1
ii) G is disconnected iff # components ≥ 2

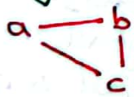
→ Def.ⁿ of component (in undirected graph): A maximal connected subgraph.

*) For directed graphs: connectedness and components :)

→ 

Not a strongly connected graph bec. no path from c to a

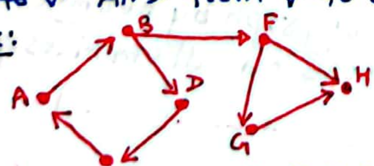
Remove dir.ⁿ ⇓



Yes a weakly connected graph, bec. after removing dir.ⁿ ∃ path b/w every pair of vertices.

→ Strongly connected vertices: Two nodes u & v are strongly connected iff ∃ a path from u to v AND from v to u.

Ex:



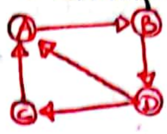
Strongly connected vertices :) ① A,B ✓ ② A,D ✓ ③ A,F ✗ ④ B,D ✓ ⑤ F,H ✗

→ Define a relation R on V. (of a Digraph)
aRb iff a,b are strongly connected.

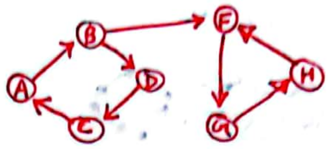
• This relation is an Equivalence relation
① Reflexive: (a,a) ≡ Trivial path exists (path of length 0 exists)
② Symmetric: if a,b are strongly connected ⇒ b,a also " "
③ Transitive: $a \leadsto b$ AND $b \leadsto c \Rightarrow a \leadsto c$

• Therefore, You can find equivalence classes.

• Example:

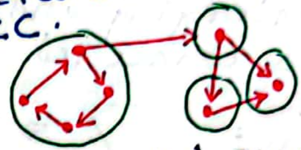


Equivalence classes:-
[A] = {A, B, C, D} = [B] = [C] = [D]
⇒ Only one equivalence class
Every two vertices are strongly connected



[A] = {A, B, D, C} = [B] = [C] = [D]
[F] = {F, G, H} = [G] = [H]

• Every equivalence class of strongly connected relation R is called a SCC.

Ex:) 

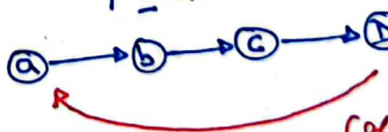
≡ 3 SCC

→ Strongly connected component (SCC): Maximal set of vertices C ⊆ V such that for every pair of vertices a & b there is a directed path from a to b AND a directed path from b to a.

Q:) Let, undirected graph G(V,E); #components = K;
Let, a, b ∈ V are not adjacent.
Now we add edge a–b in G,
#components = ?
⇒ undirected graph + one more edge
⇒ #components = K or K−1

Ex:) 

Q:) Digraph + one more edge
⇒ # SCC ? (if initially #SCC's = K)
Ans: After adding one edge in Digraph
1 ≤ #SCC ≤ K

Ex:) 

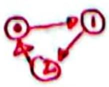
#SCC = 5
#SCC = 1 (after adding edge)

i.e. #SCC can decrease down till 1 but can't increase.

→ Strongly Connected Graph (SCG):
Def-1: If in G; #SCC = 1
Def-2: If in G; Every pair of vertices are strongly connected.

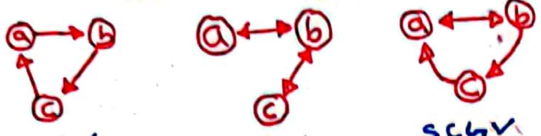
Ex:) 

#SCC = 3
Not a SCG



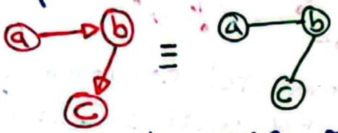
SCC = 1
Yes a SCG

Note: visualizing a SCG with 3 vertices:-



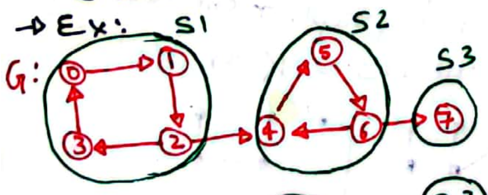
SCG ✓ SCG ✓ SCG ✓

But usually we visualize a SCG as a directed simple cycle. 

→ Weakly connected Graph (WCG):
- If the underlying undirected graph is connected.
- Ex:)  ⇒ SCG ✗, WCG ✓
- Every SCG is WCG also.

*) Associated DAG:
→ In Digraph if we represent each SCC by a single node & draw edges among these nodes if ∃ an edge from one SCC to another SCC in original G, then this modified Graph is called Associated DAG.

→ Ex:

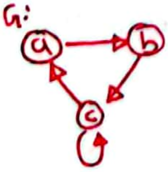


≡  ≡ G'

→ In the resulting graph there are no cycles (always) bec. if there exist some cycle, then that would mean that you can merge two different SCC's (like S_1, S_2) as a single node.
This can't happen as a SCC is maximal.

#) Adjacency Matrix & Adjacency List:

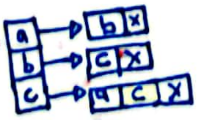
*) Graph representation in computer:-

G: 

I) Adjacency matrix:

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 0 \\ b & 0 & 0 & 1 \\ c & 1 & 0 & 1 \end{array}$$

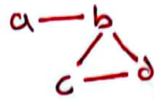
II) Adjacency List:



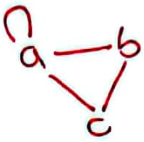
For undirected graphs adjacency matrix is symmetric i.e. $A^T = A$

*) Concepts to know before Powers of Adj. Matrix:

- Recap: walk in a graph:-



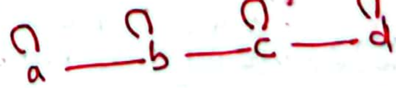
walks:-
a – b – a   2 length walk
a – b   1 length walk
a – b – c – d   3 length walk
a – b – c – b – d – c   5 length walk



No. of walks of length 3 = ?
a – a – a – a
a – a – b – a
a – b – a – a
a – a – c – a
a – c – a – a
a – b – c – a
a – c – b – a
⇒ 7 walks

- While studying Powers of adjacency matrix, we consider Graph (Directed or undirected) with no multi-edges, although loops & self loops can be there.
- You can determine connectedness in an undirected graph, by finding at least one walk from walks of length 1 to n-1, between every pair of vertices.
You need not go beyond n-1 bec. max.^m 2 connected vertices can be n-1 distance apart.
- Let; G ≡ undirected graph
G' ≡ New graph by putting self loops on all vertices of G.

Ex:) G: a — b — c — d

G': 

Point to make: If there is any walk at all, of length 1 to n-1, in G b/w a, b ~~then~~ there's a walk of length exactly 'n-1' in G' (bec. you can waste cycles/self loops)

*) Powers of Adjacency Matrix:
(For directed or undirected graph)

→ **Power 1 of Adjacency matrix:-**

- It gives us # of walks of length 1 b/w every pair of vertices.
- It's basically M^1; $M \equiv$ Adj. matrix
- Ex1)

M	a	b	c	d	e
a	1	1	0	1	0
b	1	0	1	0	0
c	0	1	1	0	0
d	$\boxed{1}$	0	0	0	0
e	0	0	0	0	0

each cell value gives us # of walk possible b/w a particular pair of vertices (d.a in this example)

→ **Power 2 of Adjacency matrix:-**

- It gives us # of walks of length 2 b/w every pair of vertices.
- It's basically $M^2 = M \times M$

(Say)

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & \cdot & \cdot & \cdot & \\ b & & & & \\ c & 0 & 1 & 1 & 1 \\ d & \cdot & \cdot & \cdot & \end{array} \times \begin{array}{c|cccc} & a & b & c & d \\ \hline a & \vdots & \vdots & & 0 \\ b & & & & 0 \\ c & & & & 1 \\ d & & & & 1 \end{array}$$

$$= \begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \boxed{2} \\ d & & & & \end{array}$$

walks of Length 2 from c to d

$c \xrightarrow{0} a \xrightarrow{0} d$

$c \xrightarrow{1} b \xrightarrow{0} d$

$c \xrightarrow{1} c \xrightarrow{1} d$

$c \xrightarrow{1} d \xrightarrow{1} d$

→ **Power K of Adjacency matrix:-**

- It gives us # of walks of length K b/w every pair of vertices.
- It's basically $M^K = M \times M \times \cdots \times M$

*) Applications of Powers of Adj. Matrix:

→ **Application-1: Finding the degree of vertices in an undirected simple graph.**

- Idea-i) $M = v\,[\ \cdots\ ]$ → # 1's is the degree of v
- Idea-ii) In M^2, diagonal entries will tell us degrees of respective vertices.

$M^2 = v\,[\ \boxed{3}\ ]$ ← Degree of v

No Self Loops

Q:) G: Undirected simple graph.
Trace $(M^2) = ?$

Ans: Trace (M^2) = Sum of main diagonal entries
$= \Sigma \text{Deg}(v)$
$= 2|E|$

Q4 G: undirected graph, which may have self loops.
How to find total deg of all vertices

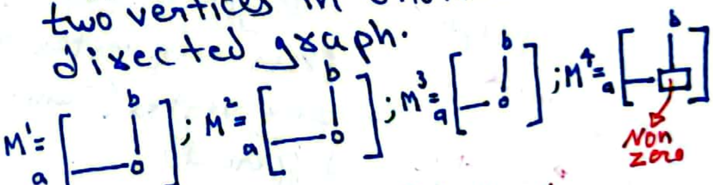
Ans: Trace $(M + M^2)$

↳ Total degree but self loop counted once

→ So, counting one more time

Note: Trace(M) = # Self loops

→ **Application-2: Finding distance b/w two vertices in undirected or directed graph.**



⇒ Distance b/w a&b is 4 (Shortest Path)

Note that, You can't directly just compute M^7 & find $M[a,b]$ = Non zero, & say that Distance b/w a&b is 7, No, that's wrong.
You go linearly from M^1, then M^2, then M^3 & so on, wherever you first get $M^k[a,b]$ = Non zero there you say that Distance is K

Note: $M^0 = I_n \equiv$ walks of length zero (Trivial walks)

→ **Application-3: To determine whether the undirected graph is connected or not**

- Idea-i) Find walks of length 1 to n-1 between every pair of vertices.

i.e. $M^1 + M^2 + M^3 + \cdots M^{n-1} = \alpha$

if all entries of α are non-zeroes ⟺ Connected

Note That: Here, $n \geq 3$ bec. at $n=2$ it may give wrong result

Ex1) $n = 2$
G: a—b : Connected
$M^1 = \begin{array}{c|cc} & a & b \\ \hline a & 0 & 1 \\ b & 1 & 0 \end{array}$ ⇒ G Not Connected

Idea-ii) Convert the undirected graph G to G' by putting self loops on each node.

i.e. Put all self loops $\equiv M + I_n$

Now,

$$(M + I_n)^{n-1} = \alpha$$

If all non-zero entries in α $\iff$ connected

bec. In G, if any walk at all there, b/w a, b then

In G', there is a walk of length exactly $n-1$.

→ **Application-4: To determine whether or not a Digraph is strongly connected.**

Strongly connected graph $\Rightarrow$ $\exists$ a walk of length at most 'n' from any vertex to any vertex (Same or different).

a to a 3 length walk

$a \to b \to c$, $c \to a$

i.e. i) $M^1 + M^2 + M^3 + \cdots M^n = \alpha$

In α, all non-zero entries $\leftrightarrow$ strongly connected

ii) $(M + I_n)^n = \alpha$

In α, all non-zero entries $\leftrightarrow$ strongly connected

→ **Application-5: Finding number of 3-length cycles (Triangles) in a simple (directed or undirected) graph.**

In G: Undirected Simple graph;

① To get walk of length 3 from b to b the only choice is a triangle.

② Because of one triangle on b, 2 walks are there from b to b of length 3.

i.e. a — b, a — c, b — c

Triangle T1

$$M^3 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 2 & \cdot & \cdot \\ \cdot & 2 & \cdot \\ \cdot & \cdot & 2 \end{bmatrix} \end{array}$$

Idea: Find M^3

Observe $\text{Trace}(M^3)$

$$\#\text{3-length cycles} = \#\text{Triangles} = \frac{\text{Trace}(M^3)}{6}$$

Because one Triangle contrib.^{n} in $\text{Trace}(M^3)$ is 6

Note that: You can't say

$$\#\text{4 length cycles} = \frac{\text{Trace}(M^4)}{8}$$

You can only say for 3-length cycle.

Bec.:-

3-length cycle possibility $\equiv$ triangle a, b, c $\equiv 1$

4-length cycle possibility $\equiv$ a — b (ये थोड़ी ना count करती मुझे), square a, b, c, d $\equiv 2$

5-length cycle possibility $\equiv$ triangle a, b, c (ये थोड़ी ना count करती मुझे)

a-b-c-a-c-a

Therefore, only 3-length cycles could be counted

In G: Directed Simple graph;

① To get walk of length 3 from a to a the only choice is a directed triangle.

② Because of one directed triangle, only 1 walk of length 3 is there from a to a.

i.e. $a \to b$, $b \to c$, $c \to a$ $\quad$ #Triangles = 1 (#3 length cycles)

$$M^3 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \end{array}$$

$a \leftrightarrow b$, $b \leftrightarrow c$, $c \leftrightarrow a$ $\quad$ #Triangles = 2 (#3 length cycles)

$$M^3 = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix} \end{array}$$

$$\#\text{3-length cycles} = \#\text{Triangles} = \frac{\text{Trace}(M^3)}{3}$$

Because one Triangle contri.^{n} in $\text{Trace}(M^3)$ is 3

Note that: In whatever app.^{n}'s you have $\alpha = M^1 + M^2 + \cdots M^{n-1}$, you can add M^n or M^{n+1} also but that's not needed, that's redundant work.

→ Application – 6: To find out transitive closure of a relation.

<u>For ex:</u> $A = \{1, 2, 3\}$

$R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$

create M:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

now,

	1	2	3
column	1, 3	2, 3	1
Row	1, 3	2	1, 2
cross product	(1, 1), (1, 3), (3, 1), (3, 3),	(2,2), (3, 2),	(1, 1), (1, 2)

Transitive closure