DM

#) Syllabus:

1. Set Theory & Algebra
   - Set Theory
   - Relations
   - Group theory
   - Functions
2. Graph Theory
   - Graph Theory
3. Mathematical Logic
   - Propositional Logic
   - First order Logic
4. Combinatory

#) Set theory: 1

- Cardinality $|A| = n$
- Representation

i) Tabular/ Roster $\{1,2,3\}$.

ii) Set builder $\{x \mid x \in R\}$

- Numbers
  - Complex Numbers (C)
    - Real (R)
      - Rational (Q)
        - Integer (Z)
          - Whole no. (W)
            - Natural no. (N)
    - Imaginary (I)
      - Irrational (IR)

- Finite, Infinite, Countable, Uncountable
  - Finite → Exactly 'n' elements
  - Infinite → Infinite elements
  - Countable → one to one mapping b/w set & Natural numbers
  - Uncountable → can't be one to one mapping b/w set & Natural no.s
- Null ($|\phi| = 0$), Universal set
- Subset ($A \subseteq B$)
  - Proper " ($A \subset B$) ← $A \subseteq B$ & $A \neq B$
  - Equality of set ($A = B$) ← $A \subseteq B$ & $B \subseteq A$
- Power set $[P(A)][2^A]$ [set of all subsets of A]

  $|P(A)| = 2^n$

  $|A| = n$

- Operation on sets:
  - → Complement (A')
  - → Union ($A \cup B$)
  - → Intersection ($A \cap B$)
  - → Disjoint sets ($A \cap B = \phi$)
- Some Formulas:

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

- Operations on sets contd.:
  - → set difference $(A - B) = A - (A \cap B) = A \cap B'$
  - → Symmetric difference

$A \oplus B = (A \cup B) - (A \cap B)$

$= (A - B) \cup (B - A)$

1

- Laws:

* Idempotent
  - $A \cup A = A$
  - $A \cap A = A$
* Associative
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
* Commutative
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
* Distributive
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
* De-Morgan
  - $(A \cup B)' = A' \cap B'$
  - $(A \cap B)' = A' \cup B'$
* Identity
  - $A \cup \phi = A$
  - $A \cap \phi = \phi$
  - $A \cup U = U$
  - $A \cap U = A$
* Complement
  - $A \cup A' = U$
  - $A \cap A' = \phi$
  - $U' = \phi$
  - $\phi' = U$
* Involution
  - $(A')' = A$

- Cartesian product: (max.m relation possible)

if $A = \{a, b\}$ $B = \{1, 2, 3\}$

then $A \times B = \{(a,1)(a,2),(a,3),(b,1)(b,2),(b,3)\}$

$A \times B \neq B \times A$; $A \times \phi = \phi$

$|A \times B| = m \cdot n$; $|A| = m$; $|B| = n$

if $|A| = n$, then no. of Binary relations $= 2^{n^2} = |P(A \times A)|$

Note: $A \to$ set; $A \times A \to$ Binary relation (o.

#) Relations: 1

*. Subset of $A \times B$ $\{$ min $= \phi$, max $= A \times B$

*. Total relation possible $= 2^{m \cdot n}$

$|A| = m$ $|B| = n$

*. Complement of relation

$R' = (A \times B) - R$ $\left[\begin{array}{l} R \cup R' = A \times B \\ R \cap R' = \phi \end{array} \right.$

(Present in cartesian product but not in R)

*. Inverse of relation

if $R = \{(a, b)\}$, $R^{-1} = \{(b, a)\}$

*. Diagonal relation (Δ_A)

$R = \{(x, x)\}$ (All diagonal elements present nothing more)

Total diagonal rel.n possible $= 1$

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* Types of relation:

$|A| = n$ $\quad |A \times A| = n^2$

no. of Relations possible $= 2^{n^2}$

I) Reflexive relation: (All diagonals must present)

- $(x, x) \in R$ $\quad$ Extra elements may present
- Possible $= \boxed{2^{n(n-1)}}$

II) Irreflexive relation: (No diagonal must present)

- $(x, x) \notin R$
- Possible $= \boxed{2^{n(n-1)}}$

III) Symmetric relation:

- if $(a,b) \in R$ then $(b,a) \in R$
- Possible $= \boxed{2^{n(n+1)/2} = 2^{\frac{n(n-1)}{2}} \times 2^n}$

IV) Anti-Symmetric relation: (Diagonal Allowed but Symmetry not allowed)

- diagonal elements may present
- No symmetric pair allowed
- Possible $= \boxed{2^n \cdot 3^{\frac{n(n-1)}{2}}}$
- Largest anti symmetric rel.^{n} $= \frac{n(n+1)}{2}$ elements

V) Asymmetric relation:

- No diagonal pairs (x, x)
- No symmetric pairs $(a,b), (b,a)$
- Possible $= \boxed{3^{\frac{n(n-1)}{2}}}$
- Largest Asym. rel.^{n} $= \frac{n(n-1)}{2}$ elements
- Every asymmetric is also anti-symm.

VI) Transitive relation:

- if $(a,b); (b,c) \in R$ then $(a,c) \in R$
- Possible =

$\lvert A \rvert$	No.
0	1
1	2
2	13
3	171
4	3994

$TC = O(n^3)$ [∵ matrix mul.^{n}]

* Transitive closure of relation:

→ Warshall's Algorithm:

It gives expected transitive pairs a relation should have.

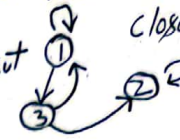
Example –

Note: Poset also satisfies (RAT)

$A = \{1, 2, 3\}$ $\quad R = \{(1,1), (1,3), (2,2), (3,1), (3,2)\}$

	1	2	3
1	1	0	1
2	0	1	0
3	1	1	0

or Shortcut



Closure = (1,1), (1,3), (1,2), (3,3), (3,1), (3,2), (2,2)

2

	1	2	3
Column	1, 3 ×	2, 3 ×	1 ×
Row	1, 3	2	1, 2
cross product	1,1 1,3 3,1 3,3	2,2 3,2	1,1 1,2

Note: Bell's No.

$B_n = 1, 1, 2, 5, 15, 52$ (i = 0, 1, 2, 3, 4, 5) = No. of eq. rel.^{n} on set with i elements

* Equivalence relation

- Reflexive + Symmetric + Transitive $\quad$ ReST

Largest = n^2, smallest = n

* Equivalence class: $[x]$

if Relation = R, Set = A

$[x] = \{ y \mid y \in A \ \& \ (x,y) \in R \}$

- Partition of a set: The set we get from equivalence class
- Example:

$A = \{1,2,3,4,5\}$; $R = \{(1,1), (2,2), (3,3), (1,4), (5,5), (1,4), (4,1), (2,5), (5,2)\}$

Partition of A: –

(Equivalence classes)

$[1] = \{1, 4\}$

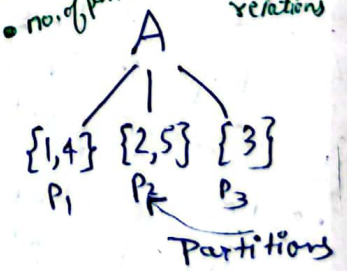
$[2] = \{2, 5\}$

$[3] = \{3\}$

$[4] = \{1, 4\}$

$[5] = \{2, 5\}$

- no. of partitions = no. of equiv. relations



$P_1 \cap P_2 \cap P_3 = \phi$

$P_1 \cup P_2 \cup P_3 = A$

Note: Equivalence relation $= \{P_1 \times P_1\} \cup \{P_2 \times P_2\} \cup \{P_3 \times P_3\}$

* Partial order relation: (POR)

→ Reflexive

→ Anti-Symmetric (RAnT)

→ Transitive

* Partial ordering set: (POSet)

(set A with POR 'R')

ex: $[A, /]$, $[A, \leq]$, $[P(S), \subseteq]$

ex: $A = \{1,2,3,4\}$ Poset $= [A, \leq]$

$\forall a \in A$; $(a,b) \in R$ iff $a \leq b$; $R = \{(1,1), (1,2) \ldots\}$

*. Total order relation: (TOR)

- Poset [A, R] with every pair of element comparable i.e. (a,b) or (b,a) ∈ R
- Ex: A = {1, 2, 3, 6}; [A, /] is a poset but not total order relation.

	1	2	3	6
1	11	12	13	16
2		22		26
3			33	36
6				66

Poset ✓
TOR ✗

Ex: A = {1, 2, 4, 8}; [A, /] is a poset & a total order rel.^n

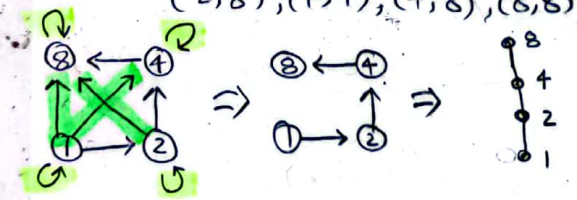
	1	2	4	8
1	11	12	14	18
2		22	24	28
4			44	48
8				88

Poset ✓
TOR ✓

*. POSET to Hasse diagram Conversion:

vertex Draw → edge draw → Remove reflexive/transitive edges → Remove direction & arrange in height

Ex: R = {(1,1), (1,2), (1,4), (1,8), (2,2), (2,4), (2,8), (4,4), (4,8), (8,8)}



Not Allowed in Hasse dia.: → Horizontal edge, → Reflexive edge, → Transitive edge
(Hasse dia. is invalid अगर ये सब हुआ)

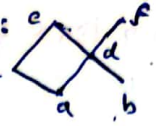
*. Elements of a poset:

Maximal मैं किसी से बात नहीं करता
Minimal मुझसे कोई बात नहीं करता
Maximum (Greatest) मैं किसी से बात नहीं करता, सब मुझसे बात करते हैं
Minimum (Least) मैं सबसे बात करता हूँ, मुझसे कोई बात नहीं करता

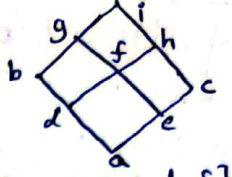
Every Hasse dia. have: → At least 1 minimal
" " 1 maximal
At most 1 Greatest
" " 1 Least

Upper bound: Given set के सारे मुझसे बात करते हैं
Lower bound: मैं Given set के सारे से बात करता हूँ

Ex:



minimal: a, b
minimum: ϕ
maximal: e, f
maximum: ϕ



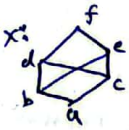
Given set = {d, f}
UB: f, g, h, i
LB: a, d

Least Upper Bound (∨) (Join): पहली दफा ऊपर मिले

Greatest Lower Bound (∧) (Meet): पहली दफा नीचे मिले

Join, meet में एक ही element आएगा, अगर ज्यादा आए तो ϕ

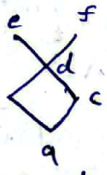
Ex:



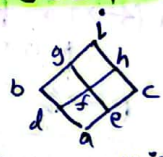
B = {d, e}
Join: f
meet: ϕ

Join Semi lattice: Join exist for every element
Meet semi lattice: Meet " " " "
Lattice: Both JSL & MSL

Ex:



e ∨ f = ϕ
⇒ Meet S.L



⇒ Lattice

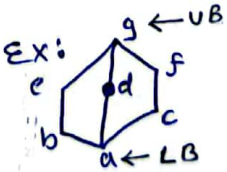
Complement: for a & a^c in lattice:-
i) $a \vee a^c$ = UB of lattice
ii) $a \wedge a^c$ = LB of Lattice

Distributive Lattice: 0 or 1 complement (for every element)

Complemented lattice: 1 or more complement (for every element)

Boolean Algebra: Exactly 1 compliment (for every element)

Ex:



e^c = {f, d, c}
C.L ✓ D.L ✗

Note: [A, /]; (a,b) ∈ R iff a/b
i.e. a divides b
i.e. $\frac{b}{a}$ = some constant no.

#) Proposit.^n & First order logic : 3

•) Proposition: जिसका Ans T/F में दे सके

•) Negation ($\neg$):

p	$\neg p$
F	T
T	F

Conjunction ($\wedge$):

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction ($\vee$):

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Implication ($\rightarrow$): If p then q

If p then q
p implies q
p only if q
p is sufficient for q
sufficient cond.^n for q is p
q if p
q whenever p
q when p

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

q is necessary for p
necessary cond.^n for p is q
q follows from p
q unless $\neg p$
If p, q

Biconditional ($\leftrightarrow$): p iff q
p is necessary & sufficient for q

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

$p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$

XOR: $\oplus$: P or q but not both

p	q	$p \oplus q$
T	T	F
F	F	F
T	F	T
F	T	T

•) Some terms:

$p \rightarrow q$	implication
$q \rightarrow p$	converse
$\neg p \rightarrow \neg q$	inverse
$\neg q \rightarrow \neg p$	contrapositive

$p \rightarrow q = \neg q \rightarrow \neg p$

$p \rightarrow q = \neg p \vee q$

Modus ponens

$$\frac{p \rightarrow q \quad p}{q}$$

Modus tollens

$$\frac{p \rightarrow q \quad \neg q}{\neg p}$$

•) Some cases:

1) Tautology (valid): Always True
2) Contradic.^n (unsatisfiable): Always False
3) Contingency: neither Taut. nor Contr.
4) Satisfiable: At least 1 True
5) Fx.^nally complete set: $\{\wedge, \neg\}$
$\{\vee, \neg\}$

•) First order logic: 4

a) Universal quantifier ($\forall_x$)

$\forall_x P(x)$ For all value of x, P(x) is true

b) Existential quantifier ($\exists_x$)

$\exists_x P(x)$ For at least one value of x P(x) is true

$\forall_x [Indian(x) \rightarrow Cricket(x)]$
Every indian likes cricket

$\exists_x [Indian(x) \wedge samosa(x)]$
Some indians like samosa

Note: $\exists_x$ is not used with $\rightarrow$

$\neg[\forall_x P(x)] = \exists_x (\neg P(x))$

$\neg[\exists_x P(x)] = \forall_x (\neg P(x))$

x likes y or y is liked by x : $L(x,y)$

$\forall_x \forall_y L(x,y) = \forall_y \forall_x L(x,y)$

$\exists_x \exists_y L(x,y) = \exists_y \exists_x L(x,y)$

$\forall_x \exists_y L(x,y) \neq \exists_y \forall_x L(x,y)$

$\forall_y \exists_x L(x,y) \neq \exists_x \forall_y L(x,y)$

$\exists_x (P(x) \vee Q(x)) = \exists_x P(x) \vee \exists_x Q(x)$

$\exists_x (P(x) \wedge Q(x)) \neq \exists_x P(x) \wedge \exists_x Q(x)$

$\forall_x (P(x) \vee Q(x)) \neq \forall_x P(x) \vee \forall_x Q(x)$

$\forall_x (P(x) \wedge Q(x)) = \forall_x P(x) \wedge \forall_x Q(x)$

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#) Group Theory : 1

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•) Properties –

1) Closure: $a*b \in A$; $\forall a,b \in A$

2) Associative: $(a*b)*c = a*(b*c)$; $\forall a,b,c \in A$

3) Identity: $a*e = e*a = a$; $\forall a \in A$

e is exactly 1 element, also, $e \in A$

4) Inverse: $a*a^{-1} = a^{-1}*a = e$

$\forall a \in A$ & $a^{-1} \in A$

5) Comutative: $a*b = b*a$; $\forall a,b \in A$

•) Group Types –

1) Algebraic Structure = Closure Property

2) Semi Group = Closure & Associative

3) Monoid = Closure, Associative, Identity

4) Group = closure, Ass., Iden., Inverse

5) Abelian group = closure
Ass.
Iden.
Inv.
commutative

All Semesters, Many Groups Abandoned

•) Finite Gp.: gp. with finite elements
Ex: $(\{1\}, *)$

•) Order of Gp.: no. of elements in Gp.

•) Addition modulo ($+_m$)

$a +_m b = a+b$; $a+b < m$

$a+b-m$; $a+b \geq m$

Ex:

$A = \{0,1,2,3,4\}$

$(A, +_5)$

0 → self inv.

1 A → inv.

2, 3 → "

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

•) Multiplication modulo ($*_m$)

$a *_m b = a*b$; $a*b < m$

$(a*b)\%m$; $a*b \geq m$

Ex: $A = \{1,2,3,4\}$

$(A, *_5)$

$*_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

•) Order of Element:

For gp. $(A,*)$ & $a \in A$

Now, $a^n = e$ Order$(a) = n$

Note: $a^n = a*a*a \cdots$

Note: order (identity) = 1

order (a) = order(a^{-1})

Ex: $\{0,1,2,3\}, +_4$ identity = 0

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

i) Order(0): $0' = 0$ ⇒ order = 1

ii) Order(1): $1' = 1$
$1^2 = 1+_4 1 = 2$
$1^3 = 2+_4 1 = 3$
$1^4 = 3+_4 1 = 0$
⇒ order = 4

iii) Order(2):
$2^1 = 2$
$2^2 = 2+_4 2 = 0$
⇒ order = 2

iv) Order(3):
$3^1 = 3$
$3^2 = 3+_4 3 = 2$
$3^3 = 2+_4 3 = 1$
$3^4 = 1+_4 3 = 0$
⇒ order = 4

Generators = 3 & 1

Note: Generator Element = Element of gp. able to express every element of set by raising power

•cyclic gp. = At least 1 generator present in gp.

•Order of generator = no. of elements in gp.

•If a & b are inv. of one-another, their order is same & both are generators

•Any group of prime order is cyclic.

•) Subgroup: when subset of gp. is also gp.

Ex: $G = \{1,3,5,7\}, *_8$

$g = \{1,3\}, *_8$

$*_8$	1	3
1	1	3
3	3	1

Lagrange's Theorem for Subgp

⇒ Order(gp.) % Order(Sub gp.) = 0

Lagrange's Theorem for no. of generators:

⇒ A is cyclic gp., no. of generators in A = $\phi(n)$

n = order of gp.
p = prime factor of n

$$\phi(n) = \frac{n(p_1-1)(p_2-1)\cdots(p_k-1)}{p_1 \cdot p_2 \cdots p_k}$$

Ex: $O(G) = 12$, $\phi(12) = \frac{12(2-1)(3-1)}{2 \times 3}$

= 4 generators

2	12
2	6
3	3
	1

= 2, 3

#) Graph Theory: (2)

•) Graph types:

	Self Loop	Parallel edge
Simple Graph	No	No
Multi "	No	Yes
Pseudo "	Yes	No
Graph	Yes	Yes

- Finite graph = finite vertices
- Null " = zero edges
- Trivial " = 1 vertex, without edge
- Complete graph: Every vertex connected

$e \to$ edges, $n \to$ vertices $\quad e = \frac{n(n-1)}{2}$

- $\underbrace{n}_{\text{vertices}} \longrightarrow \underbrace{\frac{n(n-1)}{2}}_{\text{edges}}$ $\nearrow 2^{\frac{n(n-1)}{2}}$ } no. of simple possible graph with 'n' vertices

$\searrow {}^{\frac{n(n-1)}{2}}C_e$ } 'n' vertices & 'e' edges

•) Degree: No. of edges connected to a vertex

- Pendent ⇒ one degree
- Isolated ⇒ zero "
- $\delta(G) \to$ min.^{m} degree
  $\Delta(G) \to$ max.^{m} "

ex V✓ A ex V✗ A—B A—B

- Hand Shaking Theorem:

Sum of degree of all vertex $= 2 \times$ no. of edges.

$E \to$ Total edges, $V(G) \to$ Total vertex

$$\sum d(v) = 2 \times E$$

no. of odd degree vertex = even

$$\delta(G) \times V(G) \le 2 \cdot E \le \Delta(G) \times V(G)$$

- Havel-Hakimi Algorithm: Predicts whether given degree sequence represent simple non-directed graph or not.
  +ve result ⇒ Graphic

Ex: Degree seq. = {6, 5, 5, 4, 3, 3, 2, 2, 2}

~~6~~ 5 5 4 3 3 2 2 2

4 4 3 2 2 1 2 2

~~4~~ 4 3 2 2 2 2 1

3 2 1 1 2 2 1

3 2 2 2 1 1 1

. . .

0 1 1 0

~~0~~ 1 0 0

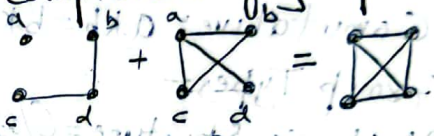
$\boxed{0\ 0\ 0}$

⇒ Valid degree seq. to construct a graph

•) Some popular graphs:

1) Bi-partite graph = Vertex set be partitioned into two non-empty disjoint subsets. 6
2) Complete Bi-partite = Every vertex in Ist partition is connected to every vertex in IInd partition. ($K_{m,n}$) ($m \times n$ edges)
3) Cycle graph (C_n): C_5
4) Wheel graph (W_n): W_5
5) Regular graph: All vertex have same degree

•) Complement of graph:



G G' K_4

$$E(G) + E(G') = E(K_n) = \frac{n(n-1)}{2}$$

•) Graph Traversal:

Walk: No edge repeat, Vertex may repeat
Closed walk: Starting/ending vertex same.
Path: Open walk with no vertex repeti.^{n}
Open walk: Starting/ending vertex not same

•) Connected graph: At least one path b/w every pair of vertex.

$$\text{edges} > \frac{(n-1)(n-2)}{2}$$ } for a graph to be connected

•) Euler graph: i) Graph is connected
ii) All vertex → even degree

•) Eulerian graph: Edge set is decomposable into cycles.

•) Unicursal line: Open walk which covers all edges without any edge repeated.

•) Unicursal graph: Connected graph having unicursal line.

•) Euler line: Add edge b/w initial & final vertex of unicursal line.

•) Eulerian circuit: use every edge exactly once & starting/ending vertex are same.
(Eulerian circuit = Euler circuit)

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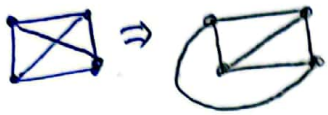
•> Hamiltonian graph:
Hamiltonian circuit होता

Ex:

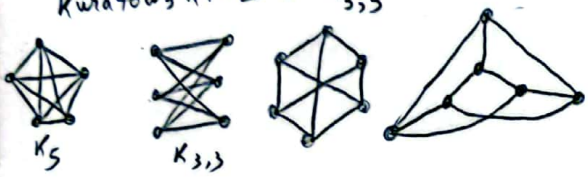
✓ Ham. (circuit देखो)
× Euler (odd degree देखो)

Graph always hamiltonian if deg. of every vertex $\geq n/2$

•> Planer Graph: Draw on plane, no edge cross each other.

ex:  ✓

•> Simplest non-Planer graphs:
Kuratowski-I : K_5
Kuratowski-II : $K_{3,3}$



•> Euler's Formulae -
For Finite, planar, connected graph

$\boxed{r = e - v + 2}$

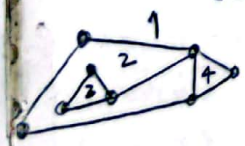
r = regions
K = connected components

For disconnected graphs
$\boxed{r = e - v + (K+1)}$

No. of bounded faces = $r - 1$

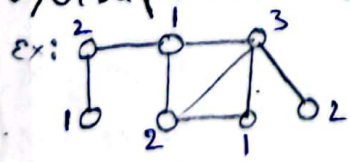
Some other Formula
$\boxed{2e \geq 3r}$ } For connected planar graph

Sum of degree of regions $= 2 \times |E|$



Ex: deg(1) = 5
deg(4) = 3

•> Graph coloring: → vertex coloring
→ Region "

Ex: 

$\chi(G) = 3$ = Chromatic no.
⇒ 3 colorable

$\chi(G) \leq 1 + \Delta(G)$

5 color Theorem: Any planar graph is atmost 5 chromatic

4 color Theorem: Any planar graph is 4 chromatic

•> Tree / Forest: nodes of degree 2 = leafs − 1.

$e = n - 1$

$\boxed{e = n - K}$ (For forest)

*Eccentricity = Farthest vertex distance
Diameter = max.m eccentricity
Radius = min.m ecc.
Center = Vertex having min.m eccentricity

•> Spanning tree / Forest:

G: T:

Branches = Rank(γ) = $n - 1$ or $n - K$
Chords = Nullity(μ) = $e - n + 1$ or $e - n + K$

Rank + Nullity = no. of edges

Branch: Edge of G that is present in T
Chord : " " " " " not " " "

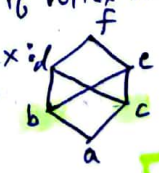
No. of Spanning Trees $= N^{N-2}$
(For complete graph with 'N' vertex)

•> Cutset (edges): set of edges whose removal leaves 'G' disconnected.

$\lambda(G)$ connectivity (edges): edges in smallest cutset.
if edge connectivity = 1, ⇒ bridge

•> cutset (vertex): set of vertex whose removal leaves 'G' disconnected.

$k(G)$ Connectivity (vertex): vertex in smallest cutset
if vertex connectivity = 1, ⇒ Articulat.n pt.

Ex: 

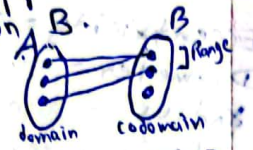
$k(G) = 2$
$\lambda(G) = 2$
$\delta(G) = 2$

$\boxed{k(G) \leq \lambda(G) \leq \delta(G)}$

•> To check isomorphism:
|vertices|,
|edges|,
vertices with given degree
Degree of vertex neighbour
min.m / max.m cycle length ; no. of cycles
Connected / disconnected
Chromatic no.
Identify similar vertices & delete both, keep seperate

#> Function : 1

•) $f: A \to B$ is a fx^n if each element of A is mapped with a one ~~unique~~ element on B.



A is domain
B is co-domain

Range $= y \mid y \in B \wedge (x,y) \in f$

Range $\subseteq B$

if $|A| = m, |B| = n$; no. of fx^ns $= n^m$

•) Fx^n Composition – $fog(x) = f(g(x))$
$gof(x) = g(f(x))$

•) One to one (Injection)
If every element of A has distinct image in B

no. of fx^ns $= \boxed{{}^nP_m}$

But if $|A| = |B| = n$; no of fx^ns $= n!$

•) Onto (Surjection)
If every element of B is mapped by at least one element of A

Range of f = B = codomain of f

no. of fx^ns $= \boxed{n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \dots}$

•) Bijection
if f is one-one & onto
possible if $|A| = |B|$

no of fx^ns $= \boxed{n!}$

•) Inverse:
$f^{-1} : B \to A$
inverse exists if f is bijection

Q:) $f(x,y) = (x+y, x-y)$
f is bijective
$f^{-1}(x,y)$ is}

Note:
$\{0,1\}^4 \Rightarrow \{000, 001, \dots\}$ 16 elements

a) $f^{-1}(x,y) = \left(\frac{1}{x+y}, \frac{1}{x-y}\right)$

b) $f^{-1}(x,y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$

Sol:) $f(2,3) = (2+3, 2-3) = (5,-1)$
$\Rightarrow f^{-1}(5,-1) = (2,3)$
putting 5, -1 in options we get (b) ✓

#> Graph theory contd... 2

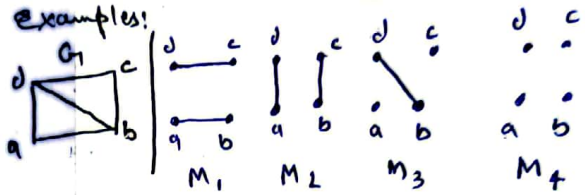
Matching: Every vertex is associated with either 0 or 1 edge 8
i.e. $\deg(v) \le 1$; $\forall v$

Maximal matching: If no other edge can be added without violating the degree condition

Maximum matching: Matching with max.^{m} no. of edges = matching no.

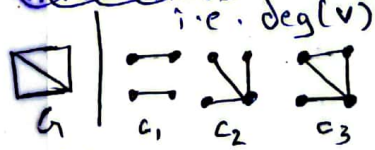
Perfect matching: Every vertex connected to exactly one edge.

examples:



Note:)
• If G has perfect match $\Rightarrow |V|$ = Even
• For K_n, no. of perfect match $= \frac{(2n)!}{n! \cdot 2^n}$

Covering:
i) Line covering: G के सारे vertex cover करने है
i.e. $\deg(v) \ge 1$ $\forall v$

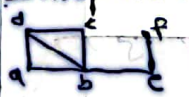


Minimal line covering: If no edge can be deleted from C without losing a vertex Ex: C_1, C_2
Minimum line covering: C with min.^{m} edges Ex: C_1
= Line covering no.

Line covering no. $\ge \left\lceil \frac{n}{2} \right\rceil$

No minimal covering contain a cycle

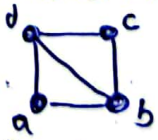
Independent line set: No two edges are adjacent.



$L_1 = \{(b,d)\}$
bd, ef
ad, bc, ef = Max.^{m} independent line set
Independent line no. is

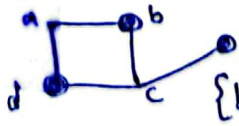
Note:)
Line covering no. + Independent line no. $= |V|$
$\alpha_1 + \beta_1 = |V|$

vertex covering: Every edge is incident with a vertex
Ex:
$K_1 = b, d$
$K_2 = a, b, c$
$K_3 = b, c, d$



min.^{m} vertex covering is vertex covering no.

Independent vertex set: No two vertex are adjacent



{b, d, e} = max.^{m} independent vertex set

Note:)
vertex covering no. + Independent vertex no. $= |V|$
$\alpha_2 + \beta_2 = |V|$

Note:

- If $\pi_1 = \{\{a,b,c\}, \{d\}\}$ then eq. rel.^{n} is $\pi_1 \times \pi_1 = [\{a,b,c\} \times \{a,b,c\}, \{d\} \times \{d\}]$
- If a set S is countable, then $P(S)$ i.e. 2^S is uncountable.
- Every subset of a countable set is either countable or finite.
- Set of real no.s → Uncountable
  Set of rational no.s → countable
- $b \Leftrightarrow c$ का मतलब है कि wff में b को c से replace कर सकते हो या c को b से replace कर सकते हो
- G is bipartite iff it does not contain any cycle of odd length.
- A connected graph has a Euler circuit $\Leftrightarrow$ All of it's vertices have even degree
- A connected graph has Euler path $\Leftrightarrow$ exactly 2 of it's vertices have odd degree.

Note:

- In proposition logic; S_1, S_2, S_3 are consistent if $S_1 \wedge S_2 \wedge S_3 = T$
- Property of power set
  $P(A \cap B) = P(A) \cap P(B)$
- Property of complete graph
  → For K_n (Directed / Undirected), there always exists hamiltonian path.
  → # Hamiltonian cycles $= \frac{(n-1)!}{2}$
- Recurrence for no. of edges in K_n

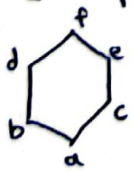
$a_3 = 3$

$a_4 = a_3 + 3$

$a_5 = a_4 + 4$ ...

$\Rightarrow a_n = a_{n-1} + (n-1)$

- In a poset



here, $(a,a) \in R$

$\boxed{\begin{matrix}(a,b) \in R \\ (b,a) \notin R\end{matrix}}$ ⇒ a-b pair is comparable

$\boxed{\begin{matrix}(c,b) \notin R \\ (b,c) \notin R\end{matrix}}$ ⇒ b-c pair is not comparable

- Property of abelian group
  In an abelian group, every element is self inverse.
  i.e. $a = a^{-1}$ or $a * a = e$

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$: DM: Combinatorics

#) Basic counting principles:-

*) Combinatorics: count without counting (i.e. without actually enumerating all cases).

*) Sum rule: (union of disjoint sets)

- Set A, B;
  If A & B are disjoint then $|A \cup B| = |A| + |B|$
  i.e. $n(A \cup B) = n(A) + n(B)$

- Task T $\Rightarrow$ $n_1 + n_2 + n_3$ ways

  Task T_1 (1) OR Task T_2 (1) OR Task T_3
  $T_1 \to n_1$ ways, $T_2 \to n_2$ ways, $T_3 \to n_3$ ways

  (2) mutually Disjoint

- Ex: 37 faculty, 83 students, we have to choose one person. (No one is both faculty & student) $\to$ Disjoint

  Task T: choose 1 person $\Rightarrow$ $37 + 83$ ways

  choose one faculty (1) OR choose one student
  $\to$ 37 ways, $\to$ 83 ways

  Disjoint

*) Product rule: (cross product of two sets)

- Task T_1 AND Task T_2 $\Rightarrow$ $m \times n$ ways

  m:
  way 1 ——— n ways
  way 2 ——— n ways
  ...
  way m ——— n ways

  Note: For every way of T_1 there are n ways for T_2

- Task T_1 AND Task T_2 $\Rightarrow$ $3 \times n + 1 \times k$ ways

  way 1 ——— n ways
  way 2 ——— n ways
  way 3 ——— n ways
  way 4 ——— k ways

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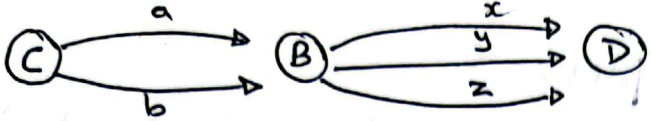
• Ex:



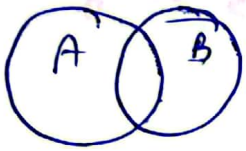
Task T : C to D $\Longrightarrow$ 2×3 ways

C to B AND B to D

2 3

*) Subtraction rule : (Inclusion Exclusion of two sets)

• Set A, B ; A & B are not disjoint



$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

• Ex: How many 8 length bit strings which either start with 1 or end with 00.

T : 8 length $\Longrightarrow 2^7 + 2^6 - 2^5$

1 - - - - - - - ① (08) - - - - - - 00

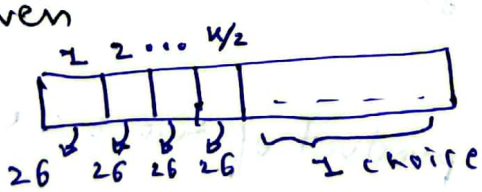
2^7 2^6

② Not disjoint

$A \cap B$: 1 - - - - - 00 2^5

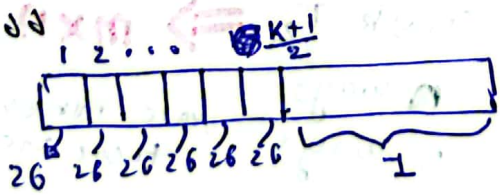
• Ex: k length english pallindrome words, such that :-

i) k = even



$\Rightarrow (26)^{k/2}$ ways

ii) k = odd

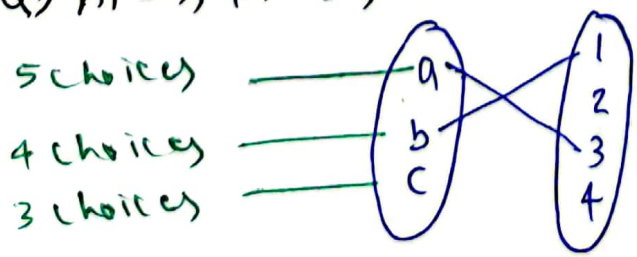


$\Rightarrow (26)^{\frac{k+1}{2}}$ ways

iii) k = anything

$\Rightarrow (26)^{\lceil \frac{k}{2} \rceil}$

Q:) $|A| = 3$, $|B| = 5$; How many 1 to 1 fx^ns from A to B possible ?



Ans : $5 \times 4 \times 3$

*) Counting by cases [Mutually exclusive, Exhaustive cases]:

- Mutually exclusive cases ⇒ Disjoint cases ⇒ you can apply sum rule
- Exhaustive cases ⇒ All desired possibilities covered
- Ex:)

→ Target: 8 length bit strings

case-1: Starting with 101] ME ✓
case-2: " " 001] EX ✗

→ Target: 8 length bit strings

case-1: Starting with 0] ME ✓
case-2: " " 1] EX ✓

- Ex: ID is made up of 3 letters followed by 2 digits. How many ID's possible with even no. of A's?

case-1: #A's = 0

L	L	L	D	D
25	25	25	10	10

$= (25)^3 \times (10)^2$

case-2: #A's = 2

Mutually exclusive

→ case-i)

A	A			
		25	10	10

→ case-ii)

	A	A		
25			10	10

→ case-iii)

A		A		
	25		10	10

$= 3 \times (25) \times (10)^2$

+

*) Complement rule: (Desired = Total − undesired)

- Ex:) $|A| = 3$, $|B| = 5$, How many fx.ns from A to B are not injective?

$\#$ non 1-1 fx.n $= \#$ Total fx.n $- \#$ 1-1 fx.n

$= 5^3 - (5 \times 4 \times 3)$

- Ex:) ID is made of 3 letters followed by 2 digits. How many ID's have some repetition of letters

= Total − No repetition of letters

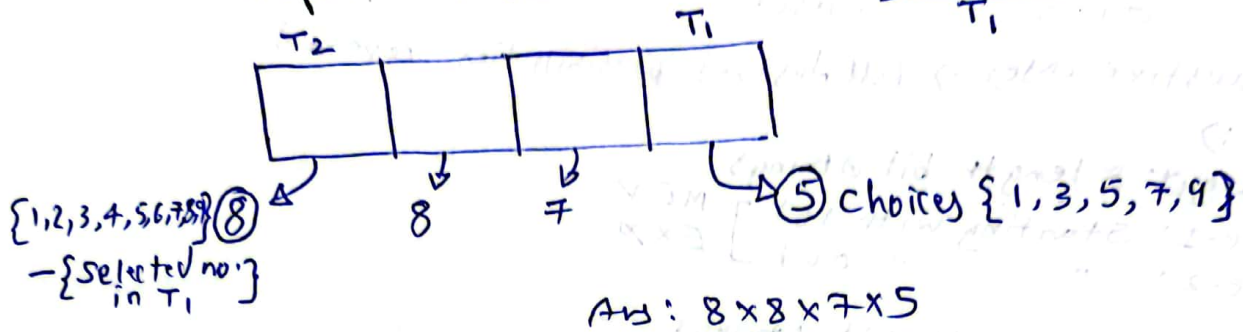
$= (26)^3 \times (10)^2 - 26 \times 25 \times 24 \times (10)^2$

Note:) while applying product rule, start with the most restricted place.

Q:) How many 4 digit no.s are there, not starting with 0, without repetition, also must be an odd no.

T_2 14

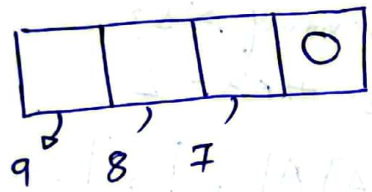
T_1



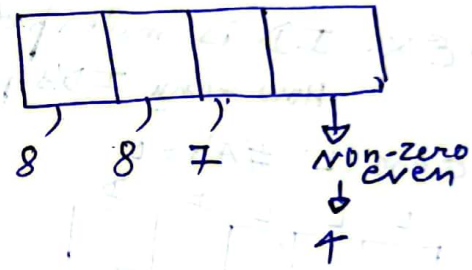
Ans: $8 \times 8 \times 7 \times 5$

Q:) ... same ...
, also must be an even no.

case-1:

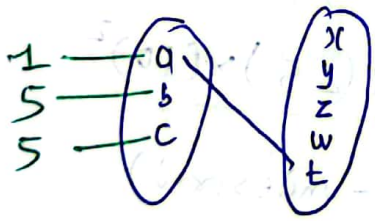


case-2:



$9 \times 8 \times 7 + 8 \times 8 \times 4 \times 7$

Q:) $|A| = 3$, $|B| = 5$; $A = \{a, b, c\}$, $B = \{x, y, z, w, t\}$. How many fx.ns from A to B are there such that 'a' maps to 't'



Ans: 5×5

Q:) There are 6 jobs with distinct difficulty levels, and 3 computers with distinct processing speeds. Each job is assigned to a computer such that Fastest computer gets toughest job, Slowest computer gets easiest job, every computer gets at least one job. No. of ways to do?

$J_1 < J_2 < J_3 < J_4 < J_5 < J_6$

1 choice → $C_1 < C_2 < C_3$ ← 1 choice

At least one job to C_2 = Total − (No job to C_2)

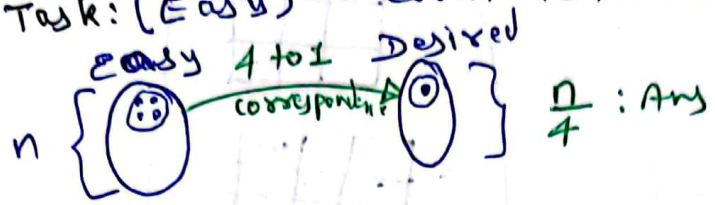
→ J_2 to J_5 2 choices for each

$= 3^4 - 2^4$

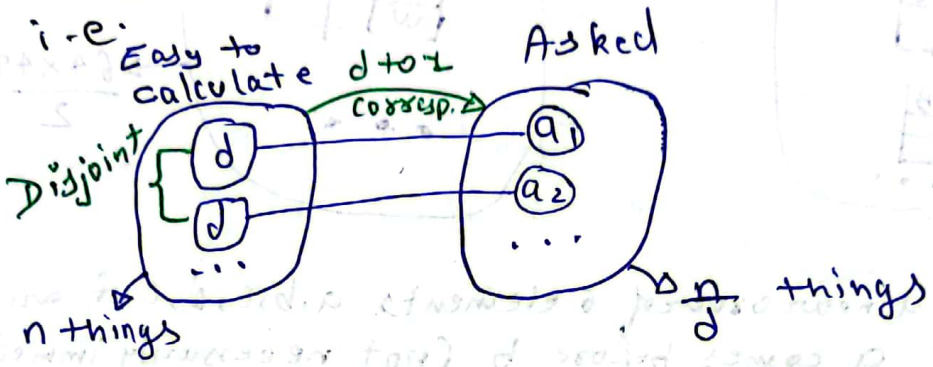
*) Division rule: (Life is so much better if we ignore small differences)

• Idea:) Task: (Desired) → Count no. of cows in field

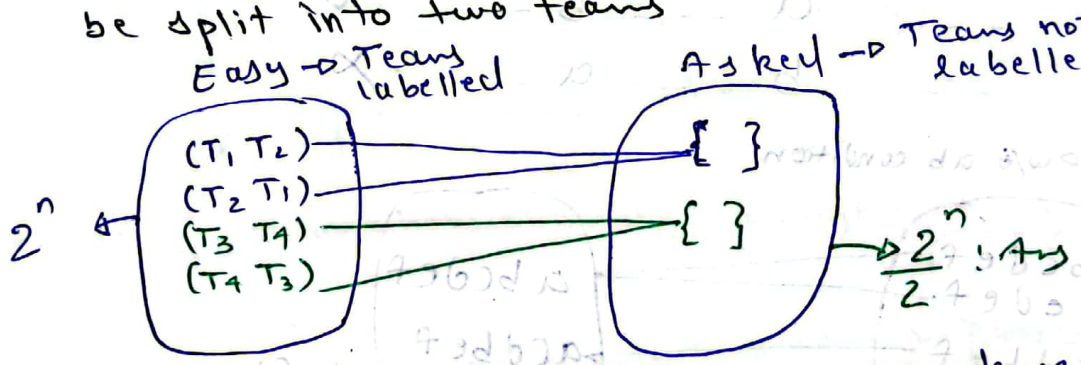
Task: (Easy) → Count total no. of legs in the field



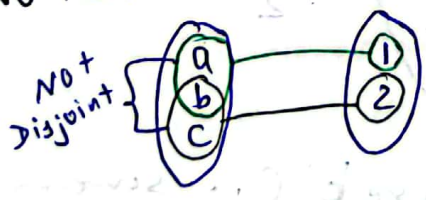
i.e.



• Ex:) How many ways can the full committee of n people be split into two teams

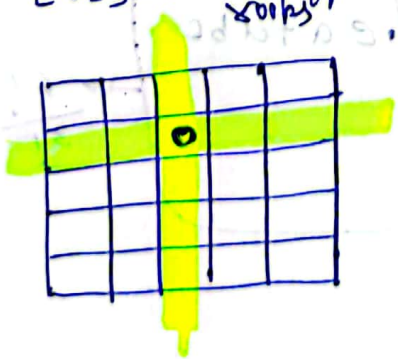


• Note:) The below is not a 2 to 1 correspondence cuz 'b' having two images $b < \begin{matrix} 1 \\ 2 \end{matrix}$



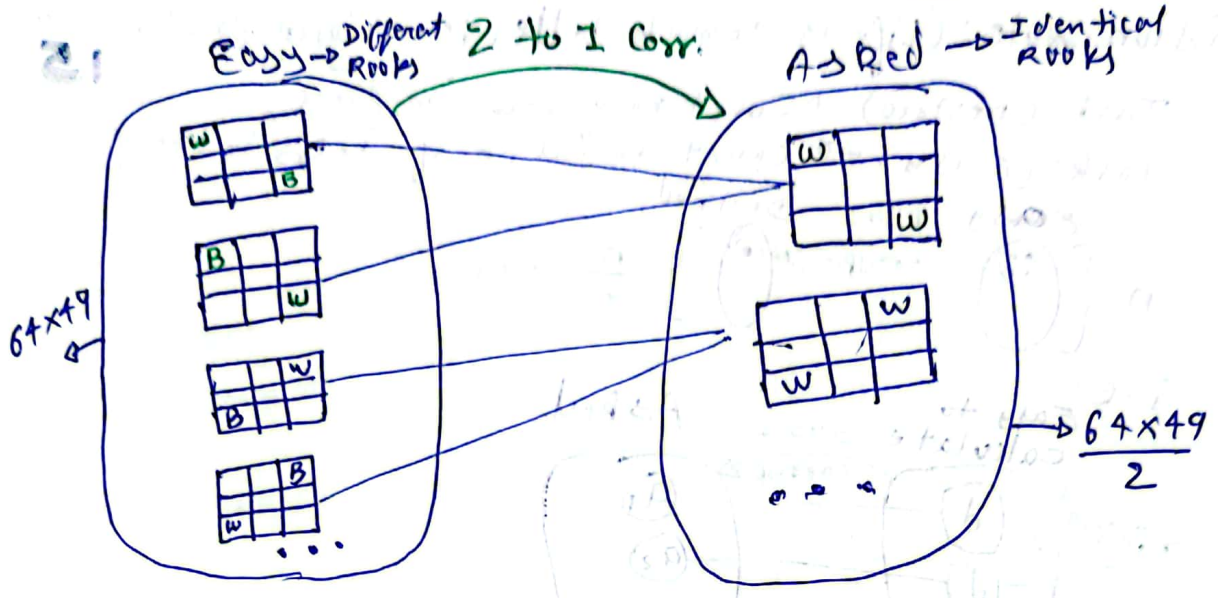
Q:) How many ways can we place two identical rooks (♖) on an 8X8 chessboard so that they occupy different rows & diff. cols.

Easy → Two different rooks



Black Rook 64 choices AND white Rook (64 − 15) choices

Ans: 64 × 49

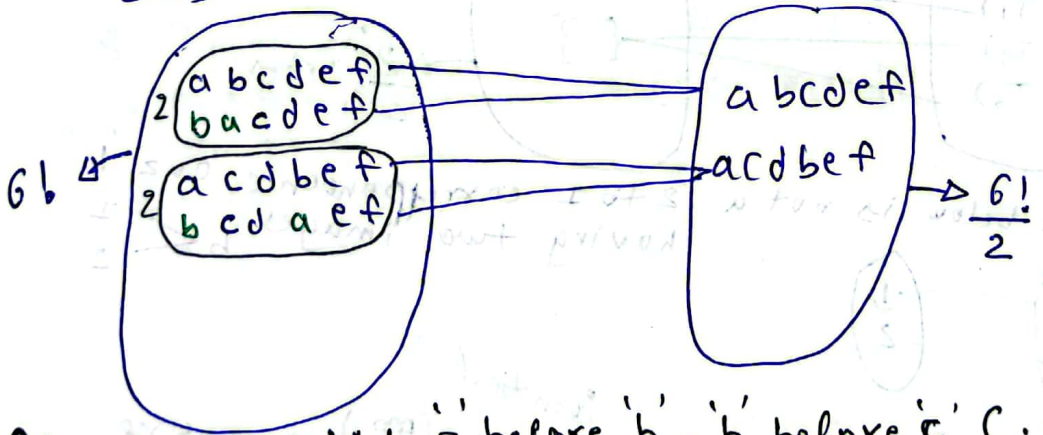


Q:> How many linear order of 6 elements a, b, c, d, e, f are there such that a comes before b (not necessarily immediately)?

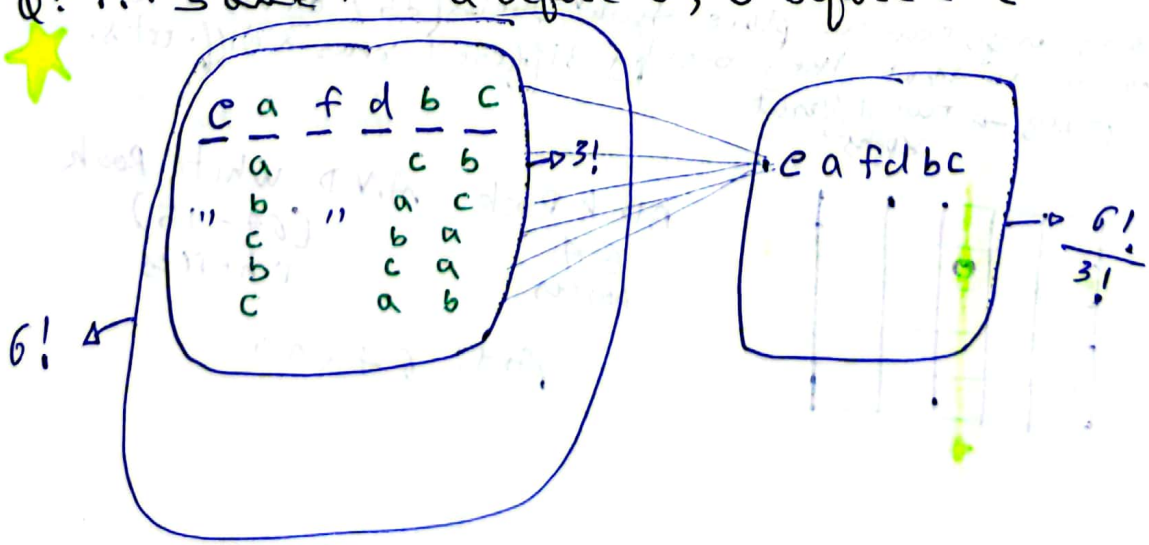
_ _ _ _ a _ _ _ b _ _ _ ✓

_ _ _ b _ _ _ a _ _ _ ✗

Easy → w/o ab condition



Q: ... same ... 'a' before 'b', 'b' before 'c' (... same ...)



*) Factorial:

- Definition: $n! = \begin{cases} n(n-1)! \\ 0! = 1 \end{cases}$; $n \in$ Whole no.

- Shortcuts:
  $^{10}C_2 \rightarrow \frac{10 \times 9}{2!} = 45$ | $^{10}P_2 = 10 \times 9 = 90$

- Remember:
  $^nC_1 = n$ | $^nP_1 = n$ | $^nC_2 = \frac{n(n-1)}{2}$ | $^nC_n = 1$ | $^nP_n = n!$

  $^nP_r = r! \times {}^nC_r$ | $^nC_{n+1} = 0$

*) Permutation & combination:

Permutation (ordered Arrgmt)	Combination (selection)
Arranging people	Selecting people
Picking $1^{st}, 2^{nd}$ & 3rd winner	Picking 3 winners
Shuffling	Choosing
It's An app'n of product rule	It's an app'n of ~~[illegible]~~ Division rule

- Ex:) No. of permutations/ordering/arrangements of 5 distinct elements.

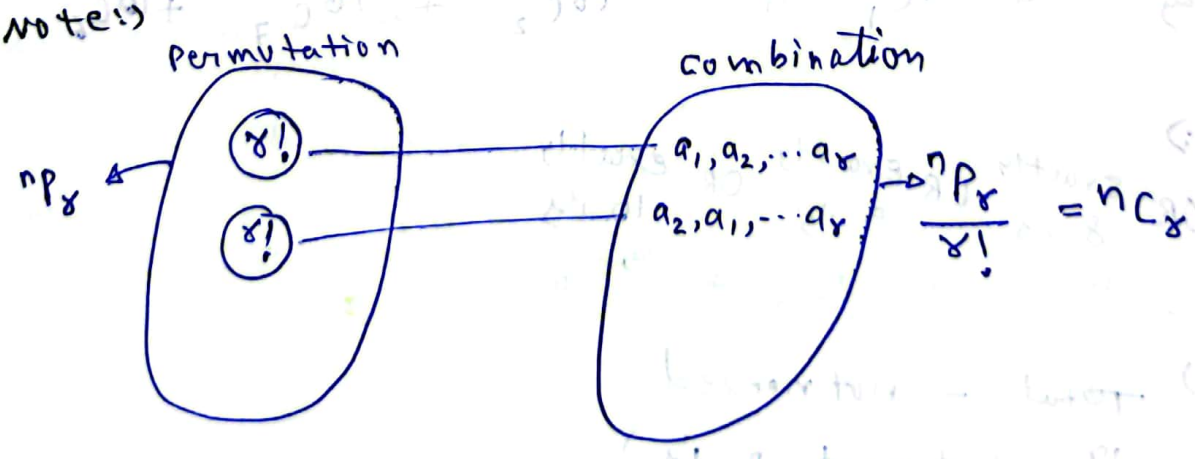
  $\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5!$

- Ex:) In how many ways can we select 3 students from a group of 5 students to stand in line for a picture.

  m1: abc, bac } Different photos
  $^5P_3$

  m2: $^5C_3 \times 3!$

- Note:)



Q: Hand in poker = choose 5 random cards

Full house = 3 card same value, other 2 cards another same value

$(5\clubsuit, 5\spadesuit, 5\heartsuit, 4\heartsuit, 4\diamondsuit)$

How many full houses are possible

full house → Select a value which has 3 cards → For this value we need 3 suits

→ Select a value which has 2 cards → for this value we need 2 suits

$$\underbrace{{}^{13}C_1 \times {}^{4}C_3}_{\text{3 cards of a value}} * \underbrace{{}^{12}C_1 \times {}^{4}C_2}_{\text{2 cards of another value}}$$

Q:) How many bit strings of length 10 have :-

1) Exactly 3 0's
2) more 0's than 1's
3) at least 7 1's
4) at least 3 1's

Ans:) 1:) ${}^{10}C_3$

2:)

(10 0's, 0 1's) OR (9 0's, 1 1's) OR (8 0's, 2 1's) OR (7, 3) OR (6, 4)

1 way + ${}^{10}C_1$ + ${}^{10}C_2$ + ${}^{10}C_7$ + ${}^{10}C_6$

3:)

Exactly 7 1's OR Exactly 8 1's OR Exactly 9 1's OR exactly 10 1's

${}^{10}C_7$ + ${}^{10}C_8$ + ${}^{10}C_9$ + ${}^{10}C_{10}$

4:) Total − not needed

2^{10} − (at most 2 1's)

$2^{10} - ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2)$

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Q:) How many permutations of ABCDEFGH contain the string ABC

$\boxed{ABC}$ DEFGH $\Rightarrow 6!$

Q:) How many permutations of ABCDEFG contain string:-

a) BCD $\rightarrow 5!$
b) CFGA $\rightarrow 4!$
c) BA and GF $\rightarrow 5!$
d) ABC & DE $\rightarrow 4!$
e) ABC & CDE $\rightarrow 3!$
f) CBA & BED $\rightarrow 0$

Q:) How many ways are there for 8 m & 5 w to stand in a line so that no two women stand next to each other?

x m x m x m x m x m x m x m x m x

$^9C_5 \times 5! \times 8!$

Q:) Suppose 12 stocks have been traded & 7 increased, 3 decreased, 2 stayed the same. In how many ways could this happen?

inc. Dec. same.

$^{12}C_7 * {}^5C_3 * {}^2C_2$

Q:) How many 3 digit no.s can be made using only two different digits

choose 2 time occuring digit * choose 1 time occuring digit * 3

$^{10}C_1 \times {}^9C_1 \times 3$

$\times 3$ bec. (a,b)

aab
baa
aba

#) Combinatorial identities :- make a story, think of it in two ways

1) $^nC_r = {}^nC_{n-r}$

Story: keep r $\equiv$ rejecting n-r

2) Pascal's identity:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Story:) side-1:

Class $\rightarrow$ n+1 students $\rightarrow$ select k students

$\Rightarrow {}^{n+1}C_k$

side-2:

a particular student $\Rightarrow x$

take x OR don't take x

$^nC_{k-1} + {}^nC_k$

Note: variation

$^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$

Note: variation

$^nC_k = {}^{n-2}C_{k-2} + {}^{n-2}C_k + 2\,{}^{n-2}C_{k-1}$

Story:) side 1:) n students select k of them

side 2:) Take two particular students x, y

$^{n-2}C_{k-2} + {}^{n-2}C_k + 2\,{}^{n-2}C_{k-1}$

Both x,y selected — neither selected — one of them selected

3) Recursive def.^{n} of binomial coefficient:

$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

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4) Vandermonde's Identity

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots \binom{n}{r}\binom{m}{0} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

Story:) n women, m men; Select r people

side-1 ⇒ ${}^{n+m}C_r$

side-2 ⇒ (0 w, r m) OR (1 w, r-1 m) OR ...

Note: variations

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{n-k}\binom{m}{r-k} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{m-r+k}$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \overset{\text{men}}{\binom{n}{k}}\overset{\text{women}}{\binom{n}{n-k}} = \sum_{k=0}^{n} \binom{n}{k}^2$$

5) ${}^{2n}C_2 = 2 \cdot {}^{n}C_2 + n^2$

Story:) n boys, n girls; select 2

Side-1:) ${}^{2n}C_2$

Side-2:) $\underset{\text{both boys}}{{}^{n}C_2} + \underset{\text{both girls}}{{}^{n}C_2} + \underset{\text{1 boy 1 girl}}{{}^{n}C_1 \times {}^{n}C_1}$

6) $\sum_{r=0}^{n} {}^{n}C_r = 2^n$

Story:) Create subset of a set $[n] = \{1, 2, 3 \ldots n\}$

side1:) #subsets $= 2^n$

side2:) #subset of size 0 + #subset of size 1 + ... + # subsets of size n

${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_n$

7) $\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$

Story:) n people; Select subset from n people AND make one of them president

Side-1) first make a subset from n people, then from that subset make one president

$$\sum_{k=0}^{n} {}^{n}C_k \times k$$

Side-2) first select one president, then select a subset of remaining people

$$\binom{n}{1} * 2^{n-1}$$

8) ${}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^{r}C_r = {}^{n+1}C_{r+1}$

Story:) from n+1 numbers, select r+1 numbers

side-1) ${}^{n+1}C_{r+1}$

Side-2) Select r+1, but first you select which no. is the max.^{m} no. in your subset of r+1 numbers.

1 2 3 ... r r+1 r+2 ... n-2 n-1 n n+1

Suppose maximum no. that I selected is :- $(n+1)^{th}$ OR $(n)^{th}$ OR $(n-1)^{th}$ ... $(r+1)^{th}$

$\Downarrow$ $\Downarrow$ $\Downarrow$ $\Downarrow$

${}^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r \cdots {}^{r}C_r$

Therefore $${}^{n+1}C_{r+1} = \sum_{k=r}^{n} {}^{k}C_r$$

जब $(n+1)^{th}$ no. max.^{m} no. है तो मुझे उससे छोटे n no.s में से ही r no.s choose करने पड़ेंगे; $(n+1)^{th}$ तो मैंने choose कर ही लिया

9) Hockey stick identity (similar to 8)

$$\sum_{i=r}^{n} {}^{i}C_{r} = {}^{n+1}C_{r+1}$$

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \dots + \binom{r+1}{r} + \binom{r}{r}$$

11) $\binom{n}{r} = \frac{n}{r} \times \binom{n-1}{r-1}$

12) $\binom{k}{2} + \binom{k}{k-2} + k^2 = \binom{2k}{2}$

n boys, n girls, select 2

$$\binom{2n}{2} = \underbrace{\binom{n}{2}}_{\text{both boy}} + \underbrace{\binom{n}{n-2}}_{\text{both girl}} + \underbrace{\binom{n}{1}\binom{n}{1}}_{\text{one boy one girl}}$$

10) $k\binom{n}{k} = n\binom{n-1}{k-1}$

story:) n people, make a team of k, with one of them as captain

side 1) Select k people then select one captain $\binom{n}{k} \times k$

side 2) Select a captain first, then select k-1 people from remaining n-1 people $n \times \binom{n-1}{k-1}$

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#) Binomial Theorem :-

*) $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r} = \sum_{r=0}^{n} \binom{n}{n-r} a^r b^{n-r}$ ⇒ n+1 terms

Ex:) $(a+b)^4 = \sum_{r=0}^{4} \binom{4}{r} a^r b^{4-r}$ ⇒ 5 terms

*) Questions-

1) coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$

$^{25}C_{13}\, x^{12} y^{13}$

⇒ $^{25}C_{13}$

2) coefficient of $x^{12}y^{14}$ in $(x+y)^{25}$

$0 \cdot x^{12} y^{14}$

⇒ 0

3) coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$

$\left.\begin{matrix} a = 2x \\ b = -3y \end{matrix}\right\} (a+b)^{25} \Rightarrow {}^{25}C_{12}\, a^{12} b^{13}$

$^{25}C_{12} (2x)^{12} (-3y)^{13}$

⇒ $^{25}C_{12} \times 2^{12} \times (-3)^{13}$

4) Coefficient of $2x^{12}y^{14}$ in $(x^3+2y^2)^{10}$

$(x^3)^4 (2y^2)^{10-6}$ ⇒ $x^{12} (2y^2)^4$

⇒ it's coefficient is 0

5) coefficient of $2x^{12}y^{12}$ in $(x^3+2y^2)^{10}$

$^{10}C_4 (x^3)^4 (2y^2)^{10-4}$

$= \boxed{^{10}C_4 \times 2^5} \times 2\, x^{12} \cdot y^{12}$

Note: $(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + {}^nC_2 x^2 + \dots {}^nC_n x^n$

$= 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots$

Note: In above eq.^{n}, put $x = -1$

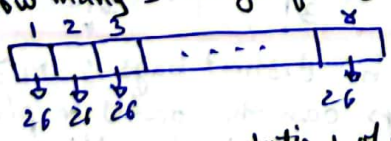
$0 = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots$

⇒ $\boxed{{}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots}$

⇒ $\boxed{\sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0}$

#) Permutation with repetation

Q:) How many strings of length 'r'?

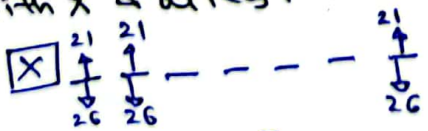


⇒ $(26)^r$

Q:) How many permutations of Success?

$= \frac{7!}{3! \cdot 2!}$

Q:) 8 letter english word starting with X & atleast one vowel



$(26)^7 - (21)^7$

All possibilities − No vowel

Q:) Arrangements of word 'Permutations' if there are always 4 letters b/w P & S.

P _ _ _ _ S
S _ _ _ _ P } 2 ways

12 positions

1 2 3 4 5 6 7 8 9 10 11 12

for P, S → { 1→06, 2→7, ⋮, 7→12 } ⇒ 7 positions

Ans: $7 \times 2 \times \frac{10!}{2!}$

Q:) Permute word ABCDEF, containing ABC substring

$\boxed{ABC}$ DEF ⇒ 4!

Q:) Permute 6 people, such that AB stand next to each other

$\boxed{AB}$ C DEF ⇒ 5! × 2!

Q:) Arrange 'Wondering' if all 3 vowels consecutively appearing.

Oei W n d r n g

$$\frac{7! \times 3!}{2!}$$

Q:) ABCDEF ; contains AB subsequence

	A			B	

$\underbrace{{}^6C_2 \times 1}_{\text{for A, B}} \times \underbrace{4!}_{\text{for remaining}}$

$= \frac{6!}{2!}$

Q:) same ↑ -- ABC subsequence

	A		B	C	

${}^6C_3 \times 1! \times 3! = \frac{6!}{3!}$

Q:) 10 speakers; S1 wants to speak before S2 & S2 " " " " S3.

$\underbrace{{}^{10}C_3 \times 1!}_{\text{for } S_1, S_2, S_3} \times 7! = \frac{10!}{3!}$

Q:) 1 2 3 4 5 6 7 8 9; 4 comes before 1, 2, 9, 8

$\underbrace{{}^9C_5 \times 4!}_{\text{for } 4, 1, 2, 9, 8} \times 4!$

Q:) Rainbow; A is always before I & I " " " O

$\underbrace{{}^7C_3 \times 1!}_{\text{for } A, O, I} \times 4! = \frac{7!}{3!}$

Q:) ABCDEFG ; A & B are never next to each other.

22

⊗C⊗D⊗E⊗F⊗G⊗

${}^6C_2 \times 2! \times 5!$

Q:) same ; exactly two consecutive vowels

$\underbrace{{}^3C_2 \times 2!}_{\text{which two vowels together}} \times \underbrace{\frac{6!}{2!}}_{\text{Remainin}} \times \underbrace{{}^7P_2}_{I,\ O}$ 7 gaps

Q:) ABCDEF; AB, CD subsequences

	A	C	B	D		

$\underbrace{{}^6C_2 \times 1!}_{\text{for A, B}} \times \underbrace{{}^4C_2 \times 1!}_{\text{for C, D}} \times 2! = \frac{6!}{2! \times 2!}$

Q:) ABCD ; AB, CD substring

$\boxed{AB}\boxed{CD} \Rightarrow 2!$

Q:) Permute 5, 2, 1, 8, 9, 7 ; if
- 5 must come first
- 8 must come before 9 & 7
- 2 must come before 1

5 | | | | | |

$\left.\begin{matrix} 8\ 9\ 7 \\ 8\ 7\ 9 \end{matrix}\right\} 2 \quad \underbrace{{}^5C_3 \times 2!}_{\text{for } 8, 9, 7} \times \underbrace{{}^2C_2 \times 1!}_{\text{for } 1, 2}$

Q:) ABCDEFGH contain subsequence CAB or BED

CAB OR BED — subsequence CABED

$\frac{8!}{3!} + \frac{8!}{3!} - \frac{8!}{5!}$

Q:) IODB: 'm' identical balls are to be placed in 'n' distinct bags; $m \geq kn$, k is a natural no. ≥ 1. In how many ways can the balls be placed in the bags if each bag must contain at least k balls.?

$B_1 \quad B_2 \quad \cdots \quad B_n$

$k \quad k \quad k$

Remaining stars $= m - kn$

Bars $= n - 1$

$\Rightarrow {}^{m-kn+n-1}C_{n-1}$

#) Distributing objects into Boxes:-

*) Overview of templates:-

1) DODB - Diff. obj. Diff. boxes

Objects: a, b, c, d
Boxes: B_1, B_2 } $\rightarrow 2^4 = 16$ ways

B_1	B_2
ab	cd
cd	ab
...	...

] Different

2) IODB - Identical obj. diff. Boxes

Objects: a, a, a, a } → chocolates
Boxes: B_1, B_2 } → kids (Boys)

B_1	B_2
0	4
1	3
3	1
2	2
4	0

these counts matters

3) DOIB - Diff. obj identical boxes

Objects: a, b, c, d } → friends
Boxes: R, R } → Rooms in hotel

R	R
a	bcd
b	acd
c	abd
cd	ab
ab	cd

] Same

4) IOIB: identical object identical boxes

Objects - a, a, a, a
Boxes - R, R

R	R
a	aaa
aaa	a
aa	aaa
	aa

] Same

*) DODB template:
(Simple product rule with combination)

Q:) How many ways to distribute hands of 5 cards to each of 4 players from the deck of 52 cards.
players → boxes
cards → objects

Template: DODB

A	B	C	D	Trash
5	5	5	5	32

→ sum of all = 52

$$^{52}C_5 \times {}^{47}C_5 \times {}^{42}C_5 \times {}^{37}C_5$$

$$= \frac{52!}{5! \times 5! \times 5! \times 5! \times 32!}$$

Ans: 3

Q:) 8 distinct objects; 3 boys; Everyone gets atleast 2 object

A	B	C
2+2	2	2
2	2+1	2+1

→ $^3C_1$ (a boy gets 4 obj)
→ $^3C_1$ (a boy gets 2 obj)

$$^3C_1 \times {}^8C_4 \times {}^4C_2 \times {}^2C_2 + {}^3C_1 \times {}^8C_2 \times {}^6C_3 \times {}^3C_3$$

*) IODB template: (Star bars problem) (Santa give chocos to children)

→ Star bars problem:-
r stars; n bars

$\underbrace{* * * \ldots *}_{r \text{ stars}}$ $\underbrace{| | | \ldots |}_{n \text{ bars}}$

#Permutations = $^{n+r}C_n = {}^{n+r}C_r$

→ Q:) n+r length bit string containing exactly r zeros

#Permutations = $^{n+r}C_r = {}^{n+r}C_n$

→ Variation:
#Bars ⇒ n-1
#Stars ⇒ r

#permutations = $^{n+r-1}C_r$
= $^{n+r-1}C_{n-1}$

→ IODB problems == Star bars problem

Q:) 5 identical chocos; 3 children

A	B	C
0	0	5
0	5	0
2	3	0
* *	* * *	–

stars ⇒ 5
Bars ⇒ 2

#Per. = $^{5+2}C_2 = {}^{5+2}C_5$

Q:) 10 identical balls, 8 different bins

A_1	A_2	...	A_8
		...	

Stars ⇒ 10
Bars ⇒ 7 ⇒ $^{17}C_7$ or $^{17}C_{10}$

Q:)

Q:) 30 identical objects, 3 different boxes. Each box must have atleast 5 items.

A:)

A	B	C
5	5	5

Remaining stars $= 30 - 3\times5 = 15$

Bars $= 2$

$\Rightarrow {}^{17}C_2$ or ${}^{17}C_{15}$

Q:) ways to split 15 identical marbles to 5 different non-empty boxes

24

A_1	A_2	A_3	A_4	A_5
1	1	1	1	1

Remaining stars $= 15 - 5 = 10$

Bars $= 4$

$\Rightarrow {}^{14}C_4$ or ${}^{14}C_{10}$

→ Combination with repetition == Star bars problem:

Q:) Apples, Mangoes, Oranges are there in market. You want 4 fruits. # ways?

A	M	O
0	2	2

Stars $= 4$

Bars $= 2$

$\Rightarrow {}^{6}C_2$

Q:) 4 different kind of cookies on sale. How many ways to choose 6 cookies

C_1	C_2	C_3	C_4
0	2	2	2

Stars $= 6$

Bars $= 3$

$\Rightarrow {}^{9}C_3$ or ${}^{9}C_6$

Q:) 6 different types of beads. ways to choose 24 beads.

b_1	b_2	---	b_6
2	4	---	6

Stars $= 24$

Bars $= 5$

$\Rightarrow {}^{29}C_5$ or ${}^{29}C_{24}$

→ (Non-ve) Integer sol.^{n} of an eq.^{n} == Star bars problem:

Q:) Eq.^{n}: $x+y+z = 10$; $x, y, z \geq 0$

How many non-ve integer sol.^{n}?

x	y	z
5	5	0
5	0	5

Stars $= 10$

Bars $= 2$

$\Rightarrow {}^{12}C_2$

Q:) Eq.^{n}: $x_1+x_2+x_3 = 11$

where x_1, x_2, x_3 are +ve integers

$\Rightarrow x_i \geq 1$

$x_1 + x_2 + x_3 = 11$; $x_i \geq 1$

1 1 1 (Santa-chocolate logic)

$y_1+y_2+y_3 = 8$; $y_i \geq 0$

→ Stars $= 8$

Bars $= 2$

$\Rightarrow {}^{10}C_2$

Q:) $a+b+c = 10$; $a \geq 1$ $b \geq 2$ $c \geq 3$

a b c

1 2 3

$x+y+z = 4$; $x, y, z \geq 0$

$\Rightarrow {}^{6}C_2$

Q:) $a+b+c \leq 8$

a	b	c	p
2	3	1	2

→ pocket of Santa

i.e. $a+b+c+p = 8 \Rightarrow {}^{11}C_3$

Q:) $a+b+c = 10$; $a \geq 2$ $b \geq -2$ $c \geq 0$

a b c

2 2

$x+y+z = 10$; $x, y, z \geq 0$

$\Rightarrow {}^{12}C_2$

CamScanner

Q:) $x_1+x_2+x_3+x_4+x_5=20$
All are +ve integers, & $x_3 \le 3$

case-1: $x_3=1$
$x_i \leftarrow 1$
Remaining stars $=15$

x_1	x_2	x_4	x_5

Bars $=3$
$\Rightarrow {}^{18}C_3$

case-2: $x_3=2$
$x_i \leftarrow 1$
Remaining stars $=14$
Bars $=3$
$\Rightarrow {}^{17}C_3$

case-3: $x_3=3$
$x_i \leftarrow 1$
Rem. stars $=13$
Bars $=3$
$\Rightarrow {}^{16}C_3$

$\Rightarrow {}^{18}C_3+{}^{17}C_3+{}^{16}C_3$

Q:) 3 digit no. sum of digits is 9
$a+b+c=9$; $a\ge 1$, $b,c\ge 0$
$\Rightarrow {}^{10}C_2$

counting

→ Multiset problem == Star bars problem:
• Multiset: its just like a set, but repetition allowed. symbol is $\langle \ldots \rangle$. Although its unordered just like sets.
Ex: $\langle 1,2\rangle$
$\langle 1,1,2\rangle$
note that $\langle 1,2\rangle \ne \langle 1,1,2\rangle$

Note:) List ⇒ sequence or string or ordered tuples
Here order matters

Q:) Using 1 & 2 how many multiset of size 3 possible?

1	2
3	0
2	1
1	2
0	3

stars $=3$
Bars $=1$
$\Rightarrow {}^4C_1$

Q:) $\{a,b,c,d\}$; multiset of size 2

a	b	c	d
2	0	0	0
1	0	0	1

stars $=2$
Bars $=3$
$\Rightarrow {}^5C_2$

Note:) Following problems are equivalent:-
1. Combination with repetition
2. IODB
3. Non-ve integer sol^n of the eq^n
4. Counting no. of multisets on a given set
5. Counting no. of +ve integer sequences $a_1, a_2, a_3 \ldots a_k$ where $1 \le a_1 \le a_2 \le a_3 \ldots \le a_k \le n$ (ie ~~integer composition~~ or non-decreasing integer sequence)
6. Positive Integer Composition (ordered summation of +ve integers)

→ Non decreasing integer sequence == Star bars problem:

Q:) The number of 4 digit no.s having their digits in non-dec. order (L to R) constructed by using digits belonging to set $\{1,2,3\}$

from $\{1,2,3\}$ → 4 digit no. with digits in non-Dec. order

Ex: 1233

①	②	③	
2	2	0	→ 1122
1	1	2	→ 1233
4	0	0	→ 1111
0	2	2	→ 2233

⇒ stars $=4$
Bars $=2$
$\Rightarrow {}^6C_2$ or ${}^6C_4$

Q:) Find the no. of non-decreasing sequences of length n using the digits from $\{1, 2, \ldots m\}$

①	②	③	...	ⓜ
			...	

Stars = n
Bars = m-1 $\Rightarrow {}^{m+n-1}C_n$

Q:) what's the no. of n-digit natural numbers in which digits are in non-decreasing order? You can use digits from $\{1, 2, 3, \ldots 9\}$

$\Rightarrow$

①	②	...	⑨

$\Rightarrow$ n stars
8 bars $\Rightarrow {}^{n+8}C_8$

Q:) same ...
non-increasing order?

$\{0, 1, 2, \ldots 9\}$

⓪	①	②	...	⑨
2	0	1	0 ...	1
n	0	0	0	0

like
$\rightarrow$ 9200 ✓
$\rightarrow$ 0000 ✗

$\Rightarrow$ n stars
9 bars
$\Rightarrow {}^{n+9}C_9 - 1$ ← for all 0's

Q:) 7 digit decreasing numbers ~~(and n E)~~?

$\{0, 1, 2, \ldots 9\}$

⓪	①	...	⑨
2	1	0	4

Stars = 7
Bars = 9 $\Rightarrow {}^{16}C_7 - 1$ ← for all 0's

Note!
In non-Decreasing
staying same or inc.
11123 ⇒ zero can't be considered

In non-increasing
staying same or dec.
11110 ⇒ zero can be considered

→ Some remaining examples of DODB :-

Q:) How many ways are there to deal hands of 5 cards to each of six players from 52 card deck.

DODB !

B_1	B_2	B_3	...	B_6	Trash
5	5	5		5	22

${}^{52}C_5 \; {}^{47}C_5 \; {}^{42}C_5 \; {}^{37}C_5 \; {}^{32}C_5 \; {}^{27}C_5 \; {}^{22}C_{22}$

i.e. $\dfrac{52!}{(5!)^6 \times 22!}$

Q:) How many ways can you deal the 52 card deck to 4 people so that everyone gets 13 cards & the oldest player gets queen of spades

A (oldest)	B	C	D
Q♠			

${}^{51}C_{12} \times {}^{39}C_{13} \times {}^{26}C_{13} \times {}^{13}C_{13}$

i.e. $\dfrac{51!}{12! \times (13!)^3}$

Q:) How many ways are there to distribute 15 different objects to 5 different boxes, such that boxes have 1, 2, 3, 4, 5 objects in them?

Boxes:	A	B	C	D	E
x	1	2	3	4	5
x	1	3	2	4	5
x	2	1	3	5	4
x	-	-	-	-	

$\Rightarrow 5!$ ways

To find x :)

A	B	C	D	E
1	2	3	4	5

$x = \dfrac{15!}{2! \times 3! \times 4! \times 5!}$

final answer $= x \times 5!$

Q:) same
2, 2, 3, 1, 7 objects in them.

Boxes:	A	B	C	D	E
x	1	2	2	3	7
x	2	1	2	3	7
	-	-	-		

$\Rightarrow \dfrac{5!}{2!}$ ways

for x:

A	B	C	D	E
7	2	2	1	3

$x = \dfrac{15!}{7! \times 2! \times 2! \times 3!}$

final answer $= x \times \dfrac{5!}{2!}$

Q:) what is the total no. of ways to distribute 'n' different balls into 'm' different boxes, such that total no. of balls in each box can vary.

n Balls: B_1, B_2 .... B_n

$B_1 \to m$ choices, $B_2 \to m$ choices, $B_n \to m$ choices

$\Rightarrow m^n$

Note:) To find out which are boxes which are objects?

How many total you want

Examples:)

1) Size k multisets from n objects.
   obj, boxes

2) choosing (k) elements from (n) kind of elements
   k → obj, n → boxes

27

→ mixed Question:

Q:) use digits from $D = \{1, 2, 3 \ldots 9\}$

1) How many 4 digit no.s are there $\Rightarrow 9^4$
2) How many 4 digit no.s have 4 different digits $\Rightarrow 9 \times 8 \times 7 \times 6$
3) How many 4 element subsets does D have $\Rightarrow {}^9C_4$
4) How many 4 digit non-decreasing no.s are there?

$\Rightarrow$ IODB

1	2	3	----	9
* *	*	-		*

$\Rightarrow 1129$

${}^{12}C_4 \Leftarrow$ {stars = 4, bars = 8

5) How many 4 digit strictly increasing no.s are there?

→ not IODB

$\Rightarrow {}^9C_4 \times 1$

2	5	7	9

6) How many 4 digit non-increasing no.s are there $\Rightarrow {}^{12}C_4$ or ${}^{12}C_8$

→ IODB contd. : Integer composition (ordered summation of +ve integers)

• composition of 4:-

4	1+1+2
1+3	1+2+1
3+1	2+1+1
2+2	1+1+1+1

$\Rightarrow$ 8 total

one part → (4)
two parts → (1+3), (3+1), (2+2)
Three parts → (1+1+2), (1+2+1), (2+1+1)
Four parts → (1+1+1+1)

• composition of 6 into 3 parts:-

$\Rightarrow a + b + c = 6$; $a, b, c \geq 1$

$1 \quad 1 \quad 1$

Remaining stars = 3
Bars = 2

$\Rightarrow {}^5C_2$

Q:) The set of compositions of n into k parts

$B_1 + B_2 + \ldots B_k = n$; $B_i \geq 1$

$1 \quad 1 \quad 1$

Remaining stars = n − k
Bars = k − 1

$\Rightarrow {}^{n-1}C_{k-1}$

Q:) Positive integer n; How many compositions.

$a \quad b \quad c \quad \ldots \quad r$; $a, b, c, r \geq 1$

r part compositions $= {}^{n-1}C_{r-1}$

Total no. of compositions $= \sum_{r=1}^{n} \binom{n-1}{r-1}$

let, $r - 1 = x$
at $r = 1$, $x = 0$
at $r = n$, $x = n - 1$

$\Rightarrow \sum_{x=0}^{n-1} \binom{n-1}{x} = 2^{n-1}$

Note:) Integer composition of n == IODB with no. of boxes varying from 1 to n & every box contains at least one object.

→ DOIB template : Friends trip to haunted hotel with same rooms == No. of partition of a set

Q:) 5 friends - Creating two groups of size 3 & size 2. How many ways?

a, b, c, d, e $\quad \underbrace{{}^5C_3}_{\text{who 3 are together}} \times \underbrace{{}^2C_2}_{\text{who 2 are together}}$

Q:) 4 friends. Creating two groups of size 2 & size 2 How many ways?

a, b, c, d $\quad \dfrac{{}^4C_2 \times {}^2C_2}{2!}$

Q:) 6 friends creating two groups of size 3 & size 3. How many ways?

a, b, c, d, e, f $\quad \dfrac{{}^6C_3 \times {}^3C_3}{2!}$

abc def
def abc } same

Q:) 8 friends. creating three groups of size 2, size 2, size 4

$\dfrac{{}^8C_4 \times {}^4C_2 \times {}^2C_2}{2!}$

Q:) 9 friends. creating four grps of size 2, size 2, size 2, & size 3. ways?

$\dfrac{{}^9C_2 \times {}^7C_2 \times {}^5C_2}{3!} \times {}^3C_3$

Q:) 3 friends. How many ways partitions can be created?

1 part → abc ⇒ 1 way
2 part → (2, 1) ⇒ ${}^3C_2 \times {}^1C_1$ ⇒ 3 ways
3 part → (1, 1, 1) ⇒ 1 way
⇒ 5 ways

Note:) DOIB ≡ Partitions of a set
≡ # eq. rel^n on a set

Q:) How many partitions of 5 element set?

1 part → (abcde) → 1 way
2 part → (1, 4) → ${}^5C_1 \times {}^4C_4$
(2, 3) → ${}^5C_2 \times {}^3C_3$
3 part → (1, 1, 3) → ${}^5C_3 \times \dfrac{{}^2C_1 \times {}^1C_1}{2!}$
4 part → (1, 1, 1, 2) → ${}^5C_2 \times \dfrac{{}^3C_1 \times {}^2C_1 \times {}^1C_1}{3!}$
5 part → (1, 1, 1, 1, 1) → 1 way

Q:) 6 identical books, 4 identical boxes. Ways?

$\underbrace{a, a, a, a, a, a}_{\text{Books}}$ $\quad \underbrace{B, B, B, B}_{\text{Boxes}}$

1 part → (6) → 1 way
2 part → (1 5) → 1 way
(2 4) → 1 way
(3 3) → 1 way
3 part → (1, 1, 4) → 1 way
(1, 2, 3) → 1 way
(2, 2, 2) → 1 way
4 part → (1, 1, 1, 3) → 1 way
(1, 1, 2, 2) → 1 way
⇒ Total 9 ways

→ IOIB template == Positive Integer Partition

• Positive integer partition is actually unordered summation of +ve integers. (Partition Number)

• Composition of 4

4
1 + 3
3 + 1
2 + 2
1 + 1 + 2
1 + 2 + 1
2 + 1 + 1
1 + 1 + 1 + 1

Partition of 4

4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

Q:) Partition of a set of 3 elements

(3) → 1 way
(2, 1) → ${}^3C_2$ way
(1, 1, 1) → 1 way
⇒ 5 partitions

Partition of 3

P(3) = 3
1 + 2
1 + 1 + 1
⇒ 3

Note:) DOIB = Partition of a set of n elements
IOIB = Partition of an natural no. n
IODB = Composition of natural no.

Q:) Partition of 6 i.e. $P(6) = ?$

1 part → 6 → 1 way

2 part → 1+5 → 1 way
2+4 → 1 way
3+3 → 1 way

3 part → 1+1+4 → 1 way
1+2+3 → 1 way
2+2+2 → 1 way

4 part → 1+1+1+3 → 1 way
1+1+2+2 → 1 way

5 part → 1+1+1+1+2 → 1 way

6 part → 1+1+1+1+1+1 → 1 way

$\Rightarrow P(6) = 11$

Q:) 1 to 100 ; neither multiple of 2 nor of 9.

= Total − (by 2 OR by 9)

$= 100 - \left(\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{9}\right\rfloor - \left\lfloor\frac{100}{18}\right\rfloor\right)$

Q:) How many +ve integers less than 1000 (i.e. 1 to 999)

a) are divisible by 7
b) _____ but not by 11
c) are divisible by both 7 & 11
d) are divisible by either 7 or 11
e) are divisible by exactly one of 7 & 11
f) are divisible by neither 7 nor 11
g) have distinct digits & are even no.s
h) have " " & are even no.s

a) $\left\lfloor\frac{999}{7}\right\rfloor$

b) by 7 − by 7 & 11

$\left\lfloor\frac{999}{7}\right\rfloor - \left\lfloor\frac{999}{77}\right\rfloor$

c) by 7 & 11 $\Rightarrow \left\lfloor\frac{999}{77}\right\rfloor$

d) by 7 or by 11

$= \left\lfloor\frac{999}{7}\right\rfloor + \left\lfloor\frac{999}{11}\right\rfloor - \left\lfloor\frac{999}{77}\right\rfloor$

e) by 7 ⊕ by 11

$= \left\lfloor\frac{999}{7}\right\rfloor + \left\lfloor\frac{999}{11}\right\rfloor - 2\left\lfloor\frac{999}{77}\right\rfloor$

f) $\overline{\text{by 7}} \cap \overline{\text{by 11}} \Rightarrow \overline{\text{by 7} \cup \text{by 11}}$

$\Rightarrow$ Total − (by 7 ∪ by 11)

$= 999 - \left(\left\lfloor\frac{999}{7}\right\rfloor + \left\lfloor\frac{999}{11}\right\rfloor - \left\lfloor\frac{999}{77}\right\rfloor\right)$

g) case-1: 1 digit ⇒ $\underline{9}$ ways

case-2: 2 " ⇒ $\underline{9}\,\underline{9}$ ⇒ 81 ways
(1 to 9) (0 to 9) − previous

case-3: 3 digit ⇒ $\underline{9}\,\underline{9}\,\underline{8}$ ⇒ 9×9×8

h) case-1: $\underline{4}$

case-2: $\underset{(1\text{ to }9)}{\underline{9}}\ \underline{1} + \underline{8}\ \overset{\text{non zero}}{\underline{4}}$ ⇒ 41

case-3: $\underline{9}\,\underline{8}\,\underline{1} + \underline{8}\,\underline{8}\ \overset{\text{non zero}}{\underline{4}}$

#) Inclusion-Exclusion Principle:

*) • $|A \cup B| = |A| + |B| - |A \cap B|$

• $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

• $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |B \cap C| - |C \cap D| - |A \cap D| - |A \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |B \cap C \cap D| + |C \cap D \cap A| + |A \cap B \cap D| - |A \cap B \cap C \cap D|$

*) Priority:)
complement > ∩ > ∪

Q:) How many integers from 1 to 40 are multiples of 6?

$\left\lfloor\frac{40}{6}\right\rfloor \Rightarrow 6$

Q:) from a to n ; How many integers divisible by b?

$\left\lfloor\frac{n}{b}\right\rfloor + 1$

Q:) How many integers from a to b are multiples of n?

1 ——— a ⇒ $\left\lfloor\frac{a}{n}\right\rfloor$

1 ——— b ⇒ $\left\lfloor\frac{b}{n}\right\rfloor$

1 — a — b ⇒ $\underbrace{\left\lfloor\frac{b}{n}\right\rfloor}_{1\text{ to }b} - \underbrace{\left\lfloor\frac{a-1}{n}\right\rfloor}_{1\text{ to }a-1}$

a−1

Q:) from 1 to 100, how many multiples of 2 or 3?

$\left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor - \left\lfloor\frac{100}{6}\right\rfloor$

*) Applications of inclusion-exclusion principle :-

- Derangement (no one in their own place) (D_n)
- Euler's totient fx.^{n} (# of coprimes of 'n', from 1 to n-1) (ϕ_n)
- # of onto fx.^{n}s (# onto fx.^{n}s = Total − not onto)
- The Sieve of Eratosthenes (# Prime no.s $\le n$)

*) Derangement :- (No one in their own place)

Ex: word: abc
↳ a is 1st, b is 2nd, c is 3rd

abc ✗
acb ✗ $\Rightarrow D_3 = 2$
bac ✗
bca ✓
cab ✓
cba ✗

Ex: D_2
word: ab
ab ✗ $\Rightarrow D_2 = 1$
ba ✓

Ex: D_4
word: abcd

'a' not in it's own place

derangements $= \bar{a} \cap \bar{b} \cap \bar{c} \cap \bar{d}$

$= \text{Total} - (\text{at least one person in it's own place})$

a in it's own place

$= 4! - (a \text{ OR } b \text{ OR } c \text{ OR } d)$

$= 4! - (a \cup b \cup c \cup d)$

Now,

$a \cup b \cup c \cup d = (a+b+c+d)$ → same → a: [a | | |] $\Rightarrow 3!$

$-ab - ac - ad - bc - bd - cd$ → same → bc: [| b | c |] $\Rightarrow 2!$

$+abc + abd + acd + bcd$ → same → acd: [a | | c | d] $\Rightarrow 1!$

$-abcd$ → same → abcd $\Rightarrow 1$

$= {}^4C_1 \times 3! - {}^4C_2 \times 2! + {}^4C_3 \times 1! - {}^4C_4 \times 1!$

Conclusion:-

$D_n = n! - ({}^nC_1 \times (n-1)! - {}^nC_2 \times (n-2)! + {}^nC_3 \times (n-3)! - \dots)$

= Total − at least one person in it's own place

Now, After taking out n! common

$$D_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \frac{1}{n!}\right]$$

Note:) D_0 is 1 bec. of recurrence relation of D_n:-

$$D_n = (n-1)\left[D_{n-1} + D_{n-2}\right]$$

Q: LIMIT; # Derangements

case-1: I _ _ I _ _ _ _ _ T

I _ _ _ _ _ _ I

$= \underbrace{2}_{\text{for T}} \times 2! = 4$

case-2: I _ _ _ _ I

m

$= \underbrace{2}_{\text{for m}} \times 2! = 4$

case-3: _ _ _ I I

L

$= \underbrace{2}_{\text{for L}} \times 2! = 4$

total = 12

Q:) i) $a_1, a_2, a_3, \ldots a_{10}$

Those permutations where atleast one not in their home.

= Total − Everyone in home

$= 10! - 1$

ii) Same... Everyone in their home

$= 1$

iii) Same... At least one in their home

= Total − no one in their home

$= 10! - D_{10}$

iv) First 5 people in first 5 positions but not in their own home

$= D_5 \times 5!$

v) first 5 in first 5 positions, but not in their own home. Last 5 not in their own home

$= D_5 \times D_5$

*) Euler's totient fx^n:-

- For any $n \in N$, it computes how many co-prime no.s are there from 1 to n-1.
- Note:- → 1 is neither prime nor composite
  - → Prime no. ⇒ Exactly 2 divisors
  - → Composite no. ⇒ at least 3 divisors
  - → Every composite no. is divisible by some prime no.
  - → If P is prime and P divides a·b then either P divides a or P divides b
  - → Prime factorization:- $n = P_1^{n_1} \cdot P_2^{n_2} \cdot P_3^{n_3} \cdots P_k^{n_k}$
  - → To check if a no. 'n' is prime

Ex: n=101

Step-1: $\lfloor \sqrt{n} \rfloor = \lfloor \sqrt{101} \rfloor = 10$

Step-2: write primes ≤ 10

$= 2, 3, 5, 7$

Step-3: Try dividing 101 with these

$\frac{101}{2}$ ✗ $\frac{101}{3}$ ✗ $\frac{101}{5}$ ✗ $\frac{101}{7}$ ✗

⇒ 101 is prime

→ Two no.s n_1 & n_2 are coprime if $GCD(n_1, n_2) = 1$

Ex:) (14, 25)

1 is coprime with every I+

- Questions:-

Q-1) for 20, how many co-prime? (i.e. 1 to 20 how many co-prime)

$n = 20 = 2^2 \times 5$

co-primes = 20 − { Divisible by 2 OR Divisible by 5 }

$= 20 - \left\{\left\lfloor\frac{20}{2}\right\rfloor + \left\lfloor\frac{20}{5}\right\rfloor - \left\lfloor\frac{20}{10}\right\rfloor\right\}$

$= 8$

- Euler's totient fx^n: (no. of coprimes of n i.e. no. of coprimes from 1 to n-1)

If $n = P_1^{n_1} \cdot P_2^{n_2} \cdot P_3^{n_3} \cdots P_k^{n_k}$

then,

$$\phi(n) = \frac{n(P_1-1)(P_2-1)\cdots(P_k-1)}{P_1 \cdot P_2 \cdots P_k}$$

- Note:

coprimes from 1 to n = # coprimes from 1 to n-1

*) Sieve of Eratosthenes:

(How many primes are there $\le n$)

→ We can find the no. of primes not exceeding a specified positive integer.

→ Q:) How many primes ≤ 40?

$= \underset{40}{\text{Total}} - \text{composites} - 1$ → for 1

$\lfloor\sqrt{40}\rfloor = 6 \Rightarrow$ any composite ≤ 40 should be div. by 2 or 3 or 5

But except 2, 3, 5 themselves

$\Rightarrow \left\lfloor\frac{40}{2}\right\rfloor + \left\lfloor\frac{40}{3}\right\rfloor + \left\lfloor\frac{40}{5}\right\rfloor - \left\lfloor\frac{40}{6}\right\rfloor - \left\lfloor\frac{40}{10}\right\rfloor - \left\lfloor\frac{40}{15}\right\rfloor + \left\lfloor\frac{40}{30}\right\rfloor \Rightarrow 30$

$\Rightarrow 30 - 3$

$= 40 - (30-3) - 1 = 12$

Q:) How many primes ≤ 100

$\#\text{primes} = \underset{100}{\text{Total}} - \underset{x}{\#\text{composites}} - 1$

$x: \lfloor\sqrt{100}\rfloor \Rightarrow 10 \Rightarrow$ div. by 2, 3, 5, 7

Except 2, 3, 5, 7 themselves

$x = \left\lfloor\frac{100}{2}\right\rfloor + \left\lfloor\frac{100}{3}\right\rfloor + \left\lfloor\frac{100}{5}\right\rfloor + \left\lfloor\frac{100}{7}\right\rfloor - \cdots$

$)(=$ something

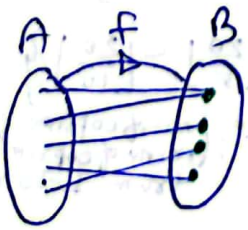
$\#\text{primes} = \text{Total} - (\text{something} - 4) - 1$

*) The number of onto functions:-

→ Onto fx^n:-

$f: A \to B$; every element of B is mapped by atleast one element of A

then f is onto (Surjective)



i.e.

$\forall y \in B, \exists x \in A;\ f(x) = y$

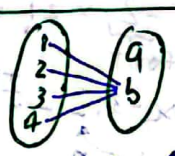
Q ⇒ The no. of onto functions from $X = \{1,2,3,4\}$ to $Y = \{a,b\}$?

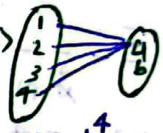
$\#\text{onto fx}^n\text{s} = \text{Total} - (\text{Not onto})$

$= \underset{2^4}{\text{Total}} - (\bar{a} \cup \bar{b})$

$\bar{a}$ means, 'a' does not have preimage

$\bar{a} \cup \bar{b} = \underset{1^4}{\bar{a}} + \underset{1^4}{\bar{b}} - \underset{0}{\bar{a}\cdot\bar{b}}$

$\bar{a} \Rightarrow$ 

$\bar{b} \Rightarrow$ 

No. of such fx^n $= 1^4$ $\Rightarrow 1^4$

$\bar{a}\bar{b} \Rightarrow$

$\Rightarrow 0$

ये तो fx^n की Definition से ही बाहर हो गया

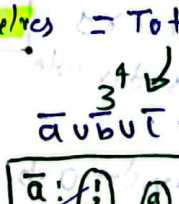
$\#\text{onto fx}^n\text{s} = 2^4 - 2 - 0 = 14$

Q:) The number of onto fx^{n}s (Surjective) from $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$?

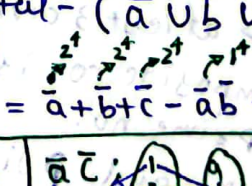
$\#\text{Onto fx}^n\text{s} = \text{Total} - (\text{Not onto})$

$= \text{Total} - (\bar{a} \cup \bar{b} \cup \bar{c})$

$\bar{a} \cup \bar{b} \cup \bar{c} = \underset{3^4}{\bar{a}} + \underset{2^4}{\bar{b}} + \underset{2^4}{\bar{c}} - \underset{2^4}{\bar{a}\bar{b}} - \underset{1^4}{\bar{b}\bar{c}} - \underset{1^4}{\bar{c}\bar{a}} + \underset{0}{\bar{a}\bar{b}\bar{c}}$

$\bar{a}$: 

i.e. b or c $\Rightarrow 2^4$

$\bar{a}\bar{c}$: 

$\Rightarrow 1^4$

$\bar{a}\bar{b}\bar{c}$:

$\Rightarrow 0$

$\#\text{onto fx}^n\text{s} = 3^4 - (3\cdot 2^4 - 3\cdot 1^4 + 0)$

$= 3^4 - 45 = 81 - 45$

Q:) How many ways are there to assign 6 different jobs to 4 different employees? at least one job is given to every employee

You can apply DODB, but its less efficient here.

Method-2 : By IEP:-

$6 \to 3$ onto fx^{n}s

$$\#\text{onto} = 3^6 - (\underset{\bar{a}\,\bar{b}\,\bar{c}}{{}^3C_1 \times 2^6} - \underset{\bar{a}\bar{b}\,\bar{b}\bar{c}\,\bar{a}\bar{c}}{{}^3C_2 \times 1^6} + \underset{\bar{a}\bar{b}\bar{c}}{0})$$

Generating functions:-

→ Aim: sequence से fx^n generate करना

→ Definition: (Ordinary generating fx^n)

If, Sequence $= a_0, a_1, a_2, a_3, \dots a_k$

AND sequence is infinite

then,

$$G(x) = a_0 + a_1 x + a_2 x^2 + \dots a_k x^k$$

power series form (open form)

$$= \sum_{k=0}^{\infty} a_k x^k$$

closed form

Note:) Sequence is like a string

→ Examples:-

1) $S = 1, 1, 1, 1, \dots$

$G(x) = 1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + \dots$

$= \sum_{i=0}^{\infty} x^i$

2) $S = 1, a, a^2, a^3, \dots$

$G(x) = 1\cdot x^0 + a\cdot x^1 + a^2\cdot x^2 + \dots$

$= \sum_{i=0}^{\infty} a^i \cdot x^i$

3) $G(x) = x^2 - x^4 + x^5 + x^6 + x^7 + \dots$

$= 0\cdot x^0 + 0 x^1 + 1\cdot x^2 + (-1)\cdot x^3 + (-1) x^4 + 1\cdot x^5 + 1\cdot x^6 \dots$

$\Rightarrow S = 0, 0, 1, 0, -1, 1, 1, 1, \dots$

Note:) sum of infinite GP $= \frac{a}{1-r}$

• In $G(x)$, coefficient of x^n is a_n.

4) $S = 1, 1, 1, 1, \dots$

$G(x) = 1 + x + x^2 + x^3 + \dots$

$= \frac{1}{1-x}$

5) $S = 1, 2, 3$

$\Rightarrow S = 1, 2, 3, 0, 0, 0, \dots$

$G(x) = 1 + 2x + 3x^2 + 0x^3 + \dots$

$= 1 + 2x + 3x^2$

6) $G(x) = x^6 - 1$

$\Rightarrow S = \overset{0}{-1}, \overset{1}{0}, \overset{2}{0}, \overset{3}{0}, \overset{4}{0}, \overset{5}{0}, \overset{6}{1}, \overset{7}{0}, \overset{8}{0}, \overset{9}{0}, 0, 0, 0 \dots$

$= -1, 0, 0, 0, 0, 0, 1$

Note: AP, GP, AGP knowledge

• AP:

$S = \underset{1^{st}}{a}, \underset{2^{nd}}{a+d}, \underset{3^{rd}}{a+2d}, \dots, \underset{n^{th}}{a+(n-1)d}$

$\Sigma S = \frac{n}{2}(a+l)$

• GP:

$S = \underset{1^{st}}{a}, \underset{2^{nd}}{ar}, \underset{3^{rd}}{ar^2}, \dots, \underset{n^{th}}{ar^{n-1}}$

$\Sigma S = \frac{a(r^n - 1)}{r-1}$

If GP ∞ then $\Sigma_\infty = \frac{a}{1-r}$

• AGP:

$S = \underset{1^{st}}{a}, \underset{2^{nd}}{(a+d)r}, (a+2d)r^2, \dots \underset{n^{th}}{[a+(n-1)d]r^{n-1}}$

$\Sigma S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$ ← ∞ AGP का सम

7) $S = \{a_k\}; k \geq 0; a_k = 3$

$\sum_{k=0}^{\infty} a_k x^k = ?$

$\sum_{k=0}^{\infty} a_k x^k = 3x^0 + 3x^1 + 3x^2 + \dots = G(x)$

$\Rightarrow G(x) = 3 \sum_{i=0}^{\infty} x^i$

$= 3 \times \frac{1}{1-x}$

8) $S = \{a_k\}; k \geq 0; a_k = k+1$

$S = a_0, a_1, a_2, \dots$

$= 1, 2, 3, \dots$

$G(s) = \underset{a}{1} + \underset{(a+d)r}{2x} + 3x^2 + 4x^3 + \dots$ ∞ AGP

correct form

$= \frac{1}{1-x} + \frac{1\cdot x}{(1-x)^2} = \frac{1}{(1-x)^2}$

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→ Differentiation of $G(x)$:-

$S = a_0, a_1, a_2, \ldots$

$G(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$\Rightarrow \boxed{G(0) = a_0}$

$G'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$

$\Rightarrow \boxed{G'(0) = a_1}$ but $G''(0) \neq a_2$

$G''(0) = 2!\,a_2$

$G'''(0) = 3!\,a_3$

Therefore,

$\boxed{G^n(0) = n!\,a_n}$

Note:) Alter to solve ∞ AGP

Suppose,

$G(x) = 1x + 2x^2 + 3x^3 + \ldots$

a (a+d)r ✗ not in correct format

So, either you can do $x \times G(x)$

or :-

$= 0 + 1x + 2x^2 + 3x^3 + \ldots \infty$

a (a+d)r ✓ correct format

$= \frac{0}{(1-x)} + \frac{1 \cdot x}{(1-x)^2} = \frac{x}{(1-x)^2}$

Note:) Don't treat $G(x)$ as a formula to find any coefficient values

Don't put values of x

→ Examples contd...

9) $G(x) = (2x - 3)^3$; $S = ?$

$= (2x)^3 + (-3)^3 + 3(2x)(-3)^2 + 3(2x)^2(-3)$

$= -27 + 3^3 \times 2x - 3^2 2^2 x^2 + 2^3 x^3 + 0x^4 + \ldots$

$\Rightarrow S = -27, 3^3 2, -3^2 2^2, 2^3, 0, 0, 0, \ldots$

10) $G(x) = (1+x)^2$; $S = ?$

$= 1 + x^2 + 2x$

$\Rightarrow S = 1, 2, 1, 0, 0, 0 \ldots$

11) $S = \{a_k\}$; $k \geq 0$; $a_k = 2^k$; $G = ?$

$S = a_0, a_1, a_2, a_3, \ldots$

$= 2^0, 2^1, 2^2, 2^3 \ldots$

$\Rightarrow G(x) = 1 + 2^1 x + 2^2 x^2 + 2^3 x^3 + \ldots \infty$ GP $= \frac{1}{1-2x}$

12) Find sum of series

$S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + 100 \cdot 2^{100}$

→ Series in AGP, but it is finite so, ∞ AGP formula will not work so, Apply procedure

$2S = \quad 1 \cdot 2^2 + 2 \cdot 2^3 + \ldots + 100 \times 2^{101}$

$S - 2S = \underbrace{1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + \ldots + 1 \cdot 2^{100}}_{\text{GP } \#\text{terms} = 100} - 100 \times 2^{101}$

$-S = 2\frac{(2^{100} - 1)}{2 - 1} - 100 \cdot 2^{101}$

$S = 99 \cdot 2^{101} + 2$

13) Find sum of series

$S = \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \ldots \infty$ GP

a (a+d)r ✗ Not in desired format

$= \frac{1}{10}\left\{1 + \frac{2}{10} + \frac{3}{10^2} + \ldots \infty\right\}$

a (a+d)r ✓

$= \frac{1}{10}\left\{\frac{1}{1 - \frac{1}{10}} + \frac{\frac{1}{10} \times 1}{\left(1 - \frac{1}{10}\right)^2}\right\}$

14) $a_k = C(m, k)$; $k = 0, 1, 2, \ldots m$

m = +ve integer

find $G(x)$

$S: \binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots \binom{m}{m}$

$G(x) = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{m}x^m$

$= (1+x)^m$

15) Which one of the following is the closed form for the generating fx^n of the sequence $\{a_n\}$; $n \geq 0$ defined as :-

$a_n = \begin{cases} n+1 & ; n = \text{odd} \\ 1 & ; n = \text{even} \end{cases}$

✓ A) $\frac{x(1+x^2)}{(1-x^2)^2} + \frac{1}{1-x}$

B) $\frac{x(3-x^2)}{(1-x^2)^2} + \frac{1}{1-x}$

C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$

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$S: \overset{a_0}{1}, \overset{a_1}{2}, \overset{a_2}{1}, \overset{a_3}{4}, \overset{a_4}{1}, \overset{a_5}{6}, \overset{a_6}{1}, \overset{a_7}{8}, \ldots$

$G(x) = 1 + 2x + 1x^2 + 4x^3 + 1x^4 + \cdots$

$= (A_{\text{even terms}} + B_{\text{odd terms}})$

$A: 1 + x^2 + x^4 + \cdots \infty_{GP} = \frac{1}{1-x^2}$

$B: 2x + 4x^3 + 6x^5 + \cdots \infty_{AGP}$

$2x\{1 + 2x^2 + 3x^4 + \cdots \infty_{AGP}\}$

$= 2x\left\{\frac{1}{1-x^2} + \frac{1 \cdot x^2}{(1-x^2)^2}\right\}$

$\Rightarrow G(x) = A + B$

$= \frac{2x}{(1-x^2)} + \frac{2x^3}{(1-x^2)^2} + \frac{1}{1-x^2}$

$= \frac{2x^3}{(1-x^2)^2} + \frac{2x}{(1-x^2)} - \frac{x}{1-x^2} + \frac{1}{1-x}$

$= \frac{x^3+x}{(1-x^2)^2} + \frac{1}{1-x}$

*) Extended binomial coefficient:-

→ Binomial coefficient:-

${}^nC_r$; $n \in N$; $r \in$ Whole no.

when $r > n$ then ${}^nC_r = 0$

→ Extended binomial coefficient:-

$${}^nC_r = \frac{\underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ terms}}}{r!}$$

→ examples:-

1) ${}^{1/2}C_2 = \frac{(\frac{1}{2})(\frac{1}{2}-1)}{2!} = -\frac{1}{8}$

2) ${}^{-2}C_4 = \frac{(-2)(-2-1)(-2-2)(-2-3)}{4!} = 5$

→ i.e. Normal Binomial coefficient

$$n \in N,\ r \in W \quad {}^nC_r = \begin{cases} 0 & ; r > n \\ 1 & ; r = 0 \\ \frac{n!}{r!(n-r)!} & ; 0 \le r \le n \end{cases}$$

Extended Binomial coefficient

$$n \in \text{Real},\ r \in W \quad {}^nC_r = \begin{cases} 1 & ; r = 0 \\ \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} & ; r \neq 0 \end{cases}$$

→ Binomial theorem:-

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{n}x^n$

$\boxed{n \in N}$ $\underbrace{+ \binom{n}{n+1}x^{n+1} + \binom{n}{n+2}x^{n+2} + \cdots}_{\to 0}$

$\frac{x+1-x}{(1+x)(1-x)} = \frac{1}{1-x} - \frac{x}{1-x^2}$

→ Extended binomial theorem:-

$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{n}x^n$

$\underbrace{\binom{n}{n+1}x^{n+1} + \binom{n}{n+2}x^{n+2} + \cdots}_{\to \text{not } 0}$ $\boxed{n \in R}$

Q:) $(1+x)^{-2}$; coefficient x^4?

${}^{-2}C_4 = \frac{-2 \times -3 \times -4 \times -5}{4!} = 5$

bec. ${}^nC_4 x^4$

Q:) $(2+x)^{-2}$; coefficient of x^4?

$\binom{-2}{2}(1 + \frac{x}{2})^{-2}$

${}^nC_4(\frac{x}{2})^4 \Rightarrow \binom{-2}{2} \times {}^{-2}C_4 \times \frac{x^4}{16}$

$\frac{1}{2^2} \times \frac{(-2)(-3)(-4)(-5)}{4!} \times \frac{x^4}{16} = \frac{5}{2^6}x^4$

Q:) Coefficient of x^{12} in $(x^3+x^4+x^5+x^6+\cdots)^3$

$\Rightarrow x^9\underbrace{(1+x+x^2+x^3+\cdots)^3}$

$\Rightarrow x^3$ in

$= x^9\left(\frac{1}{1-x}\right)^3 = x^9(1-x)^{-3}$

$x^9 \times {}^{-3}C_3(-x)^3$

$= x^{12} \times \frac{(-3)(-4)(-5)}{3!} \times (-1)^3$

$= 10x^{12}$

Q:) GF of sequence $\{a_n\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$; $a_3 - a_0 = ?$

$G(0) = \frac{1+0}{(1-0)^3} = 1 = a(0)$

$G(z) = \frac{1+z}{(1-z)^3} = \frac{1}{(1-z)^3} + \frac{z}{(1-z)^3}$

$a(3)$ = coefficient of z^3 in $G(z)$ — coeff. of z^3 — coeff. of z^2

$\frac{1}{(1-z)^3} = (1-z)^{-3} \Rightarrow {}^{-3}C_3 \times (-z)^3$

$= \frac{-3 \times -4 \times -5}{3!}(-z)^3$

$= +10 \times z^3$

$z(1-z)^{-3} \Rightarrow z \times {}^{-3}C_2 \times (-z)^2 = 6z^3$

now,

$a(3) = (10+6) = 16$

$a_3 - a_0 = 16 - 1 = 15$

Q:) Find closed form expression of sequence $\{a_n\}$; $a_n = 2n+3$, $n = 0, 1, 2, \ldots$

A) $\frac{3}{(1-x)^2}$

B) $\frac{3x}{(1-x)^2}$

C) $\frac{2-x}{(1-x)^2}$

D) $\frac{3-x}{(1-x)^2}$

$G(0) = a_0 = 3 \Rightarrow$ B ✗ C ✗

$S: 3, 5, 7, 9, \ldots$

$G(x): 3 + 5x + 7x^2 + \ldots$

$= \frac{3}{1-x} + \frac{2x}{(1-x)^2}$

$= \frac{3-x}{(1-x)^2}$

Q:) $G(x) = \frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} g(i)x^i$

what is $g(i)$? $\rightarrow$ coeff of x^i

A) i

B) $i+1$

C) $2i$

D) 2^i

$G(x) = (1-x)^{-2}$

$\Rightarrow {}^{-2}C_i(-x)^i$

$\Rightarrow {}^{-2}C_i \times (-1)^i \times x^i$

$\Rightarrow \frac{-2 \times -3 \times -4 \times -5 \times \cdots (i+1)}{i!} \times (-1)^i$

$\Rightarrow (-1)^i \times \frac{(i+1)!}{i!} \times (-1)^i$

$\Rightarrow (i+1)$

Q:) Coefficient of x^3 in $(1+x)^3(2+x^2)^{10}$ is?

$G(x) = (1 + x^3 + 3x^2 + 3x)(2+x^2)^{10}$

$(2+x^2)^{10} + 3x^2(2+x^2)^{10} + 3x(2+x^2)^{10} + x^3(2+x^2)$

coeff. of x^3 $= 0$ — coeff. of x^1 $= 0$ — coeff. of x^2 $= A$ — coeff. of constant term $= B$

A: ${}^{10}C_9(2)^9(x^2)^1 \times 3x$

B: ${}^{10}C_{10}(2)^{10}(x^2)^0 \times x^3$

$\Rightarrow A+B \Rightarrow 30 \times 2^9 + 2^{10}$

$(5+1) \times 2^{10} = 2^{14}$

Pigeon hole principle:-

Q:) a, b, c, d can I guarentee that at least 2 friends will have to share a room

Hotel | a | b | cd |
R_1 R_2 R_3

$\Rightarrow$ Yes

Definition:-

If $k+1$ or more pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons.

i.e. If there are 'k' pigeon holes (room) and more than 'k' pigeons (friends) then at least two pigeons (friends) will go to same room (share a room)

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Q:) You are meeting your friend 8 times. Can I guarantee that you're meeting at least two times on same day of week?

⇒ yes

M	T	W		S
1	1	1	...	1

$\underbrace{\quad\quad\quad}_{7}$ +1

Q:) How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

⇒ Denial is the best way

$a_1 \quad a_2 \quad \cdots \quad a_{100} \quad a_{101} + a_{102}$

$\downarrow \quad \downarrow \quad \cdots \quad \downarrow \quad \downarrow$

$0 \quad 1 \quad \cdots \quad 99 \quad 100$

⇒ 102

Q:) we have exam, graded from 0 to 999; max^m how many students we can have such that no two students get same grade.

$0 \quad 1 \quad \cdots \quad 999$

$\downarrow \quad \downarrow \quad \cdots \quad \downarrow$

$a_0 \quad a_1 \quad \cdots \quad a_{999}$

⇒ 1000

Q:) $\{a, b, c, d, e, f, g, h\}$; 3 rooms

Can I guarantee at least 3 will go to same room?

ab	cd	ef
2	2	2
R_1	R_2	R_3

2 still remaining

⇒ yes

Q:) Given a group of 100 people, at minimum, how many people were born in the same month?

⇒ $\left\lceil \frac{100}{12} \right\rceil = 9$

Q:) 100 people; max^m how many people you can guarantee that they were born in same month

Jan	Feb	Mar	...	Dec
8	8	8		8

+1

$\left\lceil \frac{100}{12} \right\rceil = 9$

*) Generalized pigeon hole principle:-

If N objects are placed into K boxes then there is at least one box containing at least $\left\lceil \frac{N}{K} \right\rceil$ objects

i.e. N (friends)

K (rooms)

At least how many friends together?

R_1	R_2	R_3	...	R_K

$\left\lceil \frac{N}{K} \right\rceil$

Q:) Among any set of 21 decimal digits there must be 3 decimal digits that are the same. True?

Decimal digits ⇒ 0 to 9

⇒ 10

$\left\lceil \frac{21}{10} \right\rceil = 3$ ⇒ True ✓

Q:) what is the min^m no. of students required in a course to be sure that at least 6 will receive the same grade if there are 5 possible grades (A, B, C, D, F)

A	B	C	D	F
5	5	5	5	5

+1

⇒ 26

Q:) 'n' rooms; Make sure that at least one room with at least 'r' people in it. Then how many min^m people we want?

worst case:-

R_1	R_2	R_3	...	R_n
r-1	r-1	r-1		r-1

⇒ $n(r-1)+1$

Q:) A drawer contains 20 red & 20 white socks. what is the least no. of socks that need to be pulled out of the drawer to ensure that at least one matching pair is obtained?

=> 10 pairs of red & 10 pairs of white

R_{1L} — R_{1R}
R_{2L} — R_{2R}
W_{1L} — W_{1R}
W_{2L} — W_{2R}

Drawer.

In worst case
↳ All left socks

=> 20

Now,
20+1 will ensure

=> Ans: 21

Q:) Let, $B = \{1, 2, 3, \ldots 100\}$ 'n' distinct integers will be chosen from B. How big does 'n' have to be to ensure that the sum of two of the chosen integers is 101?

Pairs who can give me 101

(1, 100)
(2, 98)
⋮
(50, 51)

=> 50 +1

Q:) Min^m no. of cards to be dealt from 52 cards to guarentee that three cards are from same suit?

2 2 2 2

=> 8+1

Q:) How many natural no.s we have to pick to make sure that at least two have same remainder when divided by 5?

worst case

Rem 0	Rem 1	Rem 2	Rem 3	Rem 4
↓	↓	↓	↓	↓
1	1	1	1	1

no. c

=> 5+1

Q:) what is the min^m no. of ordered pairs of non-negative no.s that should be chosen to ensure that there are two pairs (a,b) & (c,d) in the chosen set such that $a \equiv c \bmod 3$ && $b \equiv d \bmod 2$

i.e. there are two pairs (a,b) & (c,d) such that,
a & c have same remainder when divided by 3
b & d have same remainder when divided by 2

Ex:) (1, 6) & (4, 8)

Now, creating worst cases:-
mod 3 => rem = {0, 1, 2}
mod 2 => rem = {0, 1}

- (rem 0, rem 0)
- (rem 1, rem 0)
- (rem 2, rem 0)
- (rem 0, rem 1)
- (rem 1, rem 1)
- (rem 2, rem 1)

+1 (say (rem 1, rem 0) (100, 122))

Ans: 7

Q:) A = {1, 2, 3, 4, 5, 6, 7, 8}; n distinct integers will be chosen from A. How big does 'n' have to be to ensure that the sum of two of the chosen integers is 9?

=> Denial is the best way
=> create worst case

Pairs who can give me 9

(1, 8)
(2, 7)
(3, 6)
(4, 5)

=> 4+1

Q:) what is the min^m no. of ordered pairs of non-negative numbers that should be chosen to ensure that there are two pairs (a,b) & (c,d) in the chosen set such that, $a \equiv c \bmod 3$ and $b \equiv d \bmod 5$

Ans: $3 \times 5 + 1$

Q:) How many ordered pairs of integers (a,b) are needed to guarentee that there are two ordered pairs (a_1,b_1) and (a_2,b_2) such that
$a_1 \bmod 5 = a_2 \bmod 5$ AND
$b_1 \bmod 5 = b_2 \bmod 5$

Ans: $5 \times 5 + 1$

Q:) Triplets (a_1,b_1,c_1); (a_2,b_2,c_2) such that
$a_1 \equiv a_2 \bmod k$
$b_1 \equiv b_2 \bmod m$
$c_1 \equiv c_2 \bmod n$

Ans: $kmn+1$

Q:) The rules for the university of Bombay five-a-side cricket competition specify that the members of each team must have birthdays in the same month. What is the min^m no. of mathematics students needed to be enrolled in the department to guarentee that they can raise a team?

4	4	4	. . . . .	4
J	F	M		D

$\Rightarrow 12 \times 4 + 1$

Q:) what is the min^m no. of students needed to guarentee that there are at least 6 students whos birthdays fall in same month?

5	5	5	. . . . .	5
J	F	M		D

$= 12 \times 5 + 1 = 61$

#) Recurrence Relations:

*) Sequence: ordered list of elements

- like: 1, 2, 3, 5, 8
  1, 3, 9, 27, 81, . . .
- Here, $1,2,3 \neq 2,1,3$
- $\langle a_0, a_1, a_2, a_3, \ldots \rangle$
  a_0 → zeroth element
- 0, 0, 1, 2, 3, 4, . . .
  0, 1, 2, 3, 4, . . .
  } different sequence

*) Examples:

→ Arithmetic Sequence
$\langle 3, 8, 13, 18, 23, \ldots \rangle$

$\boxed{a_n = a_{n-1} + 5}$, $n \geq 1$ $\quad a_0 = 3$

→ Fibonacci sequence
$\langle \underset{a_0}{0}, \underset{a_1}{1}, \underset{a_2}{1}, \underset{a_3}{2}, \underset{a_4}{3}, 5, 8, \ldots \rangle$

initial conditions

$\boxed{a_0 = 0, \; a_1 = 1 \quad a_n = a_{n-1} + a_{n-2}; \; n \geq 2}$

→ Natural no.'s sequence
$\langle 1, 2, 3, 4, 5, 6, \ldots \rangle$

$a_n = a_{n-1} + 1; \; n \geq 1$

$a_0 = 1$

Note: An eq.^{n} → more than 1 sequence may be possible.

An eq.^{n} + Initial cond.^{n}s → Unique Sequence.

Q:) Find recurrence relation

Seq: $\langle 3, 7, 11, 15, 19, 23, \ldots \rangle$

$a_n = a_{n-1} + 4$
$a_0 = 3$

Seq.: $\langle 2, 6, 18, 54, 162, \ldots \rangle$

$a_n = 3(a_{n-1})$
$a_0 = 2$

Seq.: $\langle 6, 13, 27, 55, 111, \ldots \rangle$

$a_n = 2(a_{n-1}) + 1$
$a_0 = 6$

CamScanner

Q:) Find recurrence relation for these fx.ns :-

→ Factorial(n) : n!

$f(n) = n \cdot f(n-1)$

→ Multiplication (a, b) : a*b

$f(a,b) = f(a, b-1) + a$

→ Power (a, b) : a^b

$f(a,b) = f(a, b-1) * a$

→ fibonacci :

$f(n) = f(n-1) + f(n-2)$

→ Sum of list of no.s

$\langle \underbrace{a_0, a_1, \ldots, a_{n-1}}, a_n \rangle$

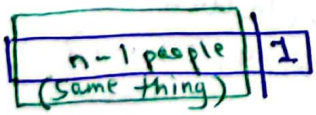
$S(n) = S(n-1) + a_n$

Q:) Find a recurrence relation & give initial conditions for the no. of bit strings of length 'n' that do not have two consecutive 0's. How many such bit strings are there of length 5?

initial condition $\begin{cases} f(1) = 2 & [\because 0, 1] \\ f(2) = 3 & [\because 01, 10, 11] \end{cases}$

now,

f(n) = no. of bit strings of length n which do not have two consecutive 0's



$\Rightarrow f(n-1)$

$\Rightarrow f(n-2)$

People
Same thing

This bit has to be 1 cuz. we're looking for NO Two consecutive 0's

Recursive $\{ f(n) = f(n-1) + f(n-2) \; ; \; n \geq 3$

Q:) Find a recurrence relation for the no. of permutations of a set with 'n' elements.

$S = \{a_1, a_2, a_3 \ldots a_n\}$

P(n) = no. of permutations of 'n' elements

$\Rightarrow P_{n-1}$

$\Rightarrow P_{n-1}$

$\Rightarrow P_{n-1}$

$P(n) = n \cdot P(n-1) \; ; \; n \geq 2$

$P(1) = 1$

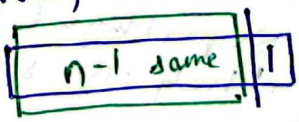
Q:) Find a recurrence relation for the no. of bit strings of length 'n' that contain a pair of consecutive 0's

T(n) = The no. of bit strings of length 'n' that contain a pair of consecutive 0's

$T(1) = 0 \quad [\because \cancel{0}, \cancel{1}]$

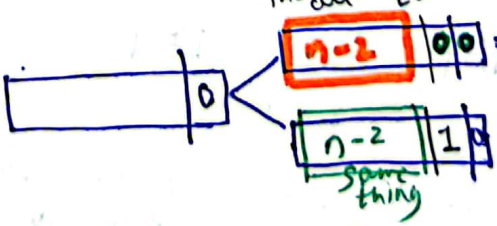
$T(2) = 1 \quad [\because 00, 01, 10, 11]$

now,



$\Rightarrow T(n-1)$

These n-2 people all combinations taken / cuz I got 00



$\Rightarrow 2^{n-2}$

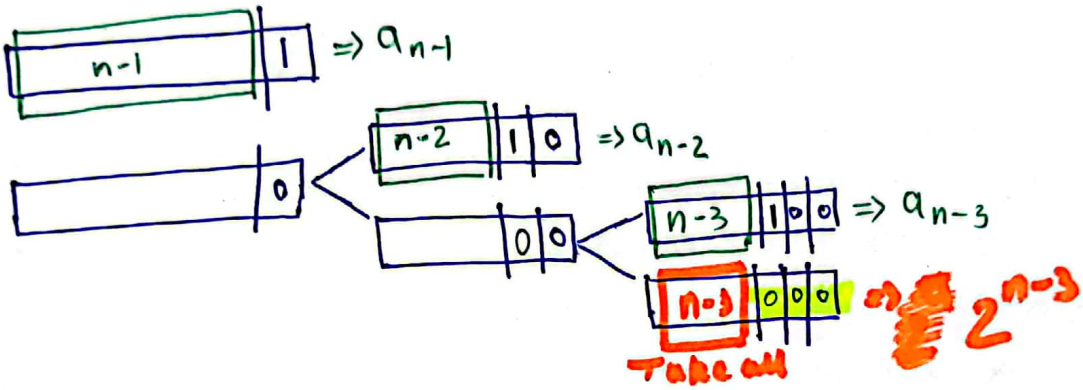
$\Rightarrow T(n-2)$

$T(n) = T(n-1) + 2^{n-2} + T(n-2) \; ; n \geq 3$

Q:) Find a recurrence relation for the no. of bit strings of length 'n' that contain three consecutive 0's.

a_n = No. of n-length bit strings which contain 3 consecutive 0's

$$\left.\begin{aligned} a_1 &= 0 \\ a_2 &= 0 \\ a_3 &= 1 \end{aligned}\right\} \text{Initial conditions}$$



$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$$

Q:) How many ways are there to pay a bill of 17 pesos using :-

- a currency with coins of values 1 peso, 2 pesos, 5 pesos, 10 pesos
- bills with values 5 pesos and 10 pesos

Here, order matters

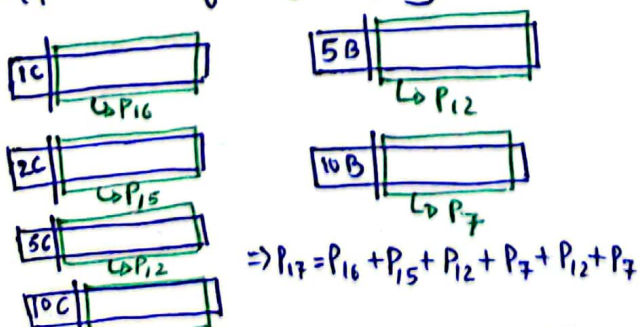
so, 5 bill, 1 coin ≠ 1 coin, 5 bill

Given;

1C	5B
2C	10B
5C	
10C	

Pay: 17 pesos

P_{17} = No. of ways to pay 17 pesos



$\Rightarrow P_{17} = P_{16} + P_{15} + P_{12} + P_7 + P_{12} + P_7$

Q:) Let a_n be the no. of n-bit strings that do not contain two consecutive 1's. which of the following is the recurrence relation for a_n.?

A) $a_n = a_{n-1} + 2a_{n-2}$ B) $a_n = a_{n-1} + a_{n-2}$ C) $a_n = 2a_{n-1} + a_{n-2}$ D) $a_n = 2a_{n-1} + 2a_{n-2}$

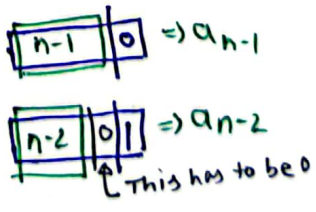
Method-1: Go by elimination

$a_1 = 2$ $a_3 = 5$

$a_2 = 3$ $\Rightarrow$ A, C, D wrong

Method-2: Recurrence formation

a_n = no. of 'n' bit strings which do not contain two consecutive 1's



$$a_n = a_{n-1} + a_{n-2}$$