The Laplace Equation Example:

- **If we have a grid, then at each point of the grid a finite difference approximation is obtained as follows:**

$$\frac{\partial^2 T(x,y)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \quad , \quad \frac{\partial^2 T(x,y)}{\partial y^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Is approximated by :

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

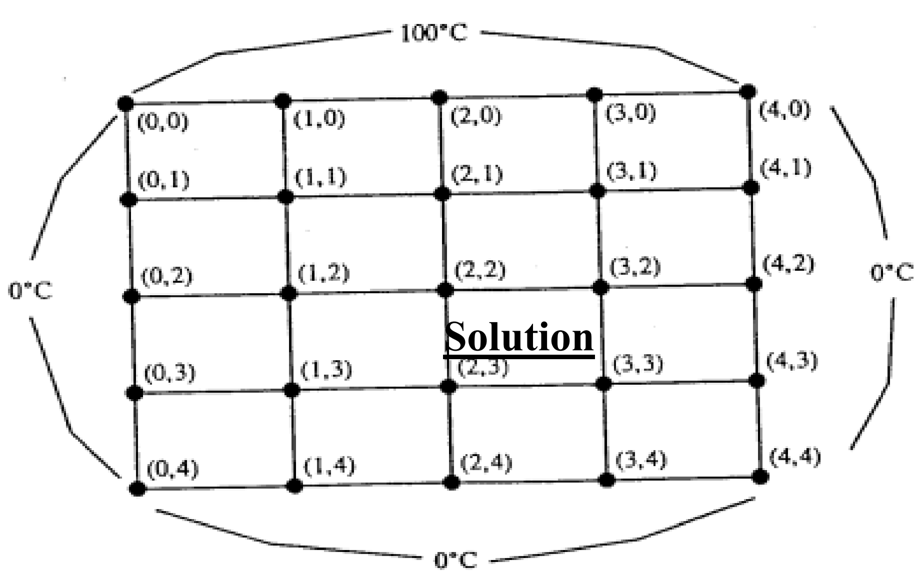
So, assume that: $\Delta x = \Delta y = h$

Then we have: $T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$

<u>Example:</u>

A two-dimensional rectangular plate ($0 \le x \le 1$, $0 \le y \le 1$) is subjected to the uniform temperature boundary conditions (with top surface maintained at 100℃ and all other surfaces at 0℃) shown in Figure 3; that is,

T(0, y)=0, T(1, y)=0, T(x, 0) = 0, and T(x, 1) = 100℃.



<u>Solution</u>

Suppose we are interested only in the values of the temperature at the nine interior nodal points (x_i, y_j), where $x_i = i\,\Delta x$ and $y_j = j\,\Delta y$, i , j = 1, 2, 3, with $\Delta x = \Delta y = \frac{1}{4}$. However, we assume symmetry for simplifying the problem. That is, we assume that $T_{3,3} = T_{1,3}$, $T_{3,2} = T_{1,2}$, and $T_{3,1} = T_{1,1}$. We thus have only six unknowns: ($T_{1,1}$, $T_{1,2}$, $T_{1,3}$) and ($T_{2,1}$, $T_{2,2}$, $T_{2,3}$). From equation (2), we then have:

$$4T_{1,1} - 0 - T_{1,2} - T_{2,1} - 100 = 0$$

$$4T_{2,1} - T_{1,1} - T_{2,2} - T_{1,1} - 100 = 0$$

$$4T_{1,2} - 0 - T_{1,3} - T_{2,2} - T_{1,1} = 0$$

$$4T_{2,2} - T_{1,2} - T_{2,3} - T_{1,2} - T_{2,1} = 0$$

$$4T_{1,3} - 0 - 0 - T_{2,3} - T_{1,2} = 0$$

$$4T_{2,3} - T_{1,3} - 0 - T_{1,3} - T_{2,2} = 0$$

After suitable rearrangement, these equations can be written in the following form:

$$\begin{bmatrix} 4 & -1 & -1 & 0 & 0 & 0 \\ -2 & 4 & 0 & -1 & 0 & 0 \\ -1 & 0 & 4 & -1 & -1 & 0 \\ 0 & -1 & -2 & 4 & 0 & -1 \\ 0 & 0 & -1 & 0 & 4 & -1 \\ 0 & 0 & 0 & -1 & -2 & 4 \end{bmatrix} \begin{bmatrix} T_{1,1} \\ T_{2,1} \\ T_{1,2} \\ T_{2,2} \\ T_{1,3} \\ T_{2,3} \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution of this system will give us temperatures at the nodal points.