

TSKS14

# Multiple Antenna Communications

Lecture 7, 2020

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# Outline of this lecture

- Uplink Massive MIMO
  - Pilot transmission
  - Channel estimation
- Capacity lower bound
  - Computation using channel estimates
  - Maximization

# Recall: Uplink Massive MIMO system model

- Received signal:

$$\mathbf{y} = \sqrt{\rho_{ul}} \mathbf{G} \mathbf{x} + \mathbf{w}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \quad \mathbf{G} = \begin{pmatrix} g_1^1 & \cdots & g_K^1 \\ \vdots & \ddots & \vdots \\ g_1^M & \cdots & g_K^M \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_M \end{pmatrix}$$

- Parameters are normalized: Maximum power is  $\rho_{ul}$   
 $x_1, \dots, x_K$  has power  $\leq 1$
- Channel of user  $k$ :  $g_k^1, \dots, g_k^M \sim \mathcal{CN}(0, \beta_k)$
- Normalized noise:  $\mathbf{w} \sim \mathcal{CN}(\mathbf{0}, \mathbf{I}_M)$

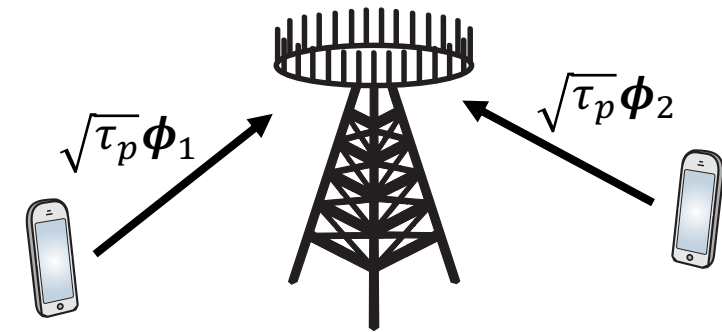
Large-scale fading coefficient

# Sending pilot sequences

- Send pilot matrix  $\sqrt{\tau_p} \mathbf{\Phi} = \sqrt{\tau_p} [\boldsymbol{\phi}_1 \dots \boldsymbol{\phi}_K]$  over  $\tau_p$  uses of the channel:

$$\mathbf{Y}_p = \sqrt{\tau_p \rho_{ul}} \mathbf{G} \mathbf{\Phi}^H + \mathbf{W}_p$$

- Stacking received signals in  $M \times \tau_p$  matrix  $\mathbf{Y}_p$
- Noise  $\mathbf{W}_p$  with i.i.d.  $CN(0,1)$  elements



- Despreading of pilot signal:

$$\mathbf{Y}'_p = \mathbf{Y}_p \mathbf{\Phi} = \sqrt{\tau_p \rho_{ul}} \underbrace{\mathbf{G} \mathbf{\Phi}^H \mathbf{\Phi}}_{= \mathbf{I}_K} + \underbrace{\mathbf{W}_p \mathbf{\Phi}}_{\text{Still i.i.d. } CN(0,1) \text{ elements}}$$

# Estimating Gaussian variable in noise

- Consider  $y = \sqrt{p}g + w$  where
  - $p$  is a constant,  $g \sim \mathcal{CN}(0, \beta)$ ,  $w \sim \mathcal{CN}(0, 1)$

Mean squared error:  
 $E\{|\hat{g} - g|^2\}$

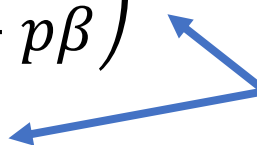
Minimum mean squared error (MMSE) estimator:

$$\hat{g} = E\{g|y\} = \frac{\sqrt{p}\beta}{1 + p\beta} y$$

Estimation error:  $\tilde{g} = \hat{g} - g \sim \mathcal{CN}\left(0, \beta - \frac{p\beta^2}{1 + p\beta}\right)$

Estimate:  $\hat{g} \sim \mathcal{CN}\left(0, \frac{p\beta^2}{1 + p\beta}\right)$

Independent random variables



# Estimates of the channels

$$[\mathbf{Y}'_p]_{mk} = \sqrt{\tau_p \rho_{ul}} g_k^m + [\mathbf{W}_p \mathbf{\Phi}]_{mk}$$

- MMSE estimate of  $g_k^m$  from user  $k$  to antenna  $m$

- Estimate:

$$\hat{g}_k^m = E\{g_k^m | \mathbf{Y}'_p\} = \frac{\sqrt{\tau_p \rho_{ul}} \beta_k}{1 + \tau_p \rho_{ul} \beta_k} [\mathbf{Y}'_p]_{mk} \sim CN(0, \gamma_k)$$

- Estimation error:

$$\tilde{g}_k^m = \hat{g}_k^m - g_k^m \sim CN(0, \beta_k - \gamma_k)$$

where

$$\gamma_k = \frac{\tau_p \rho_{ul} \beta_k^2}{1 + \tau_p \rho_{ul} \beta_k}$$

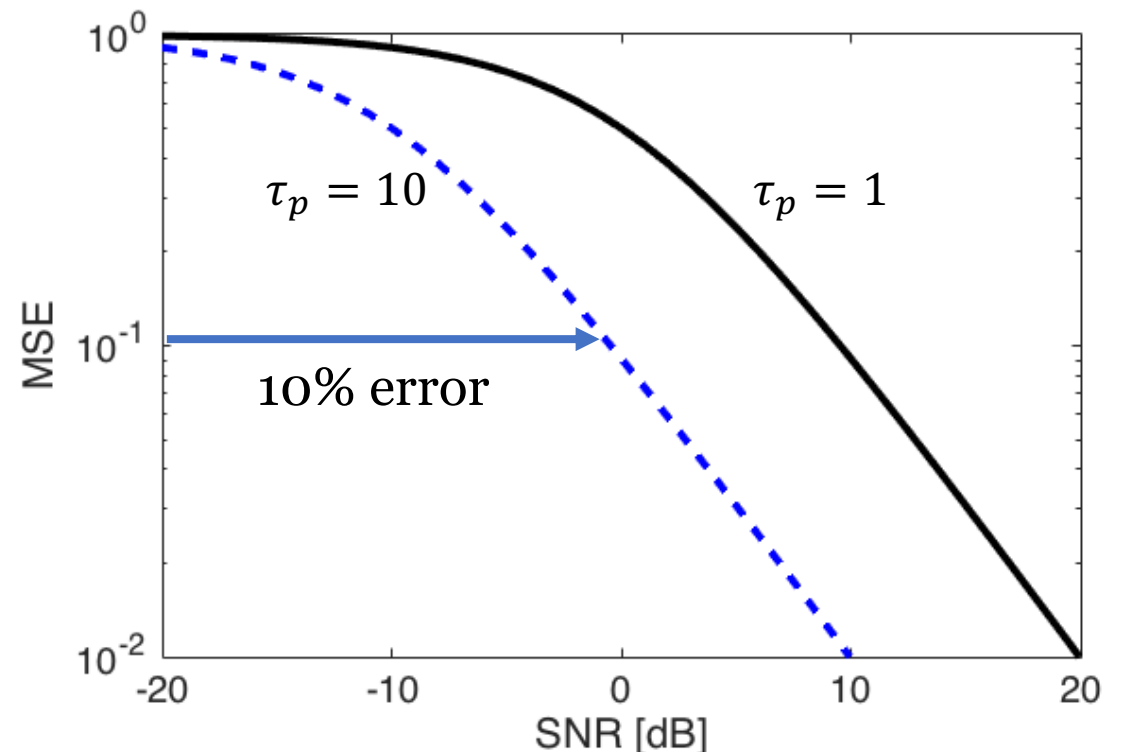
# How good is the channel estimate?

- Mean squared error (MSE):

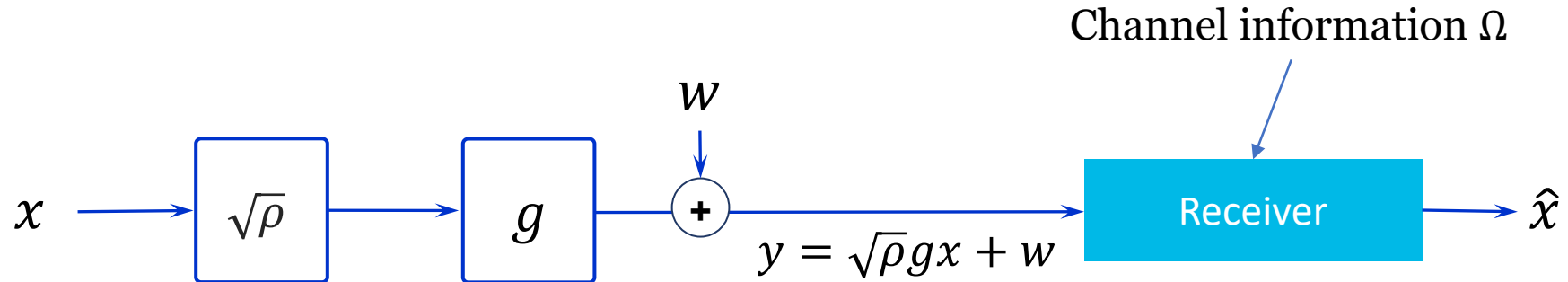
$$E\{|\hat{g}_k^m - g_k^m|^2\} = E\{|\tilde{g}_k^m|^2\} = \beta_k - \gamma_k = \beta_k - \frac{\tau_p \rho_{ul} \beta_k^2}{1 + \tau_p \rho_{ul} \beta_k}$$

- Goes to zero as  $\rho_{ul} \rightarrow \infty$  or  $\tau_p \rightarrow \infty$  (perfect estimate)

Simulation:  
 $\beta_k = 1$   
 $\text{SNR} = \rho_{ul} \beta_k = \rho_{ul}$



# A capacity lower bound



- Desired signal  $x$ , power  $\rho$ , and  $g$  and  $w$  uncorrelated
- Channel coefficient  $g$ , known channel information  $\Omega$

Capacity lower bound:

$$C \geq E \left\{ \log_2 \left( 1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho \text{Var}\{g|\Omega\} + \text{Var}\{w|\Omega\}} \right) \right\}$$



# Uplink data transmission

- Received signal:

$$\mathbf{y} = \sqrt{\rho_{ul}} \mathbf{G} \mathbf{x} + \mathbf{w}$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_M \end{pmatrix} \quad \mathbf{G} = \begin{pmatrix} g_1^1 & \cdots & g_K^1 \\ \vdots & \ddots & \vdots \\ g_1^M & \cdots & g_K^M \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_K \end{pmatrix} \quad \mathbf{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_M \end{pmatrix}$$

- Signals:  $x_k = \sqrt{\eta_k} q_k$  where  $q_k \sim CN(0,1)$  data symbol  
 $0 \leq \eta_k \leq 1$  controls the power
- Channel of user  $k$ :  $g_k^1, \dots, g_k^M \sim CN(0, \beta_k)$
- Normalized noise:  $\mathbf{w} \sim CN(\mathbf{0}, \mathbf{I}_M)$

Large-scale fading coefficient

# Linear receiver processing

- Received signal:

$$\mathbf{y} = \sqrt{\rho_{ul}} \mathbf{G} \mathbf{D}_{\eta}^{1/2} \mathbf{q} + \mathbf{w}$$

where  $\mathbf{D}_{\eta} = \begin{pmatrix} \eta_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \eta_K \end{pmatrix}$ ,  $\mathbf{q} = \begin{pmatrix} q_1 \\ \vdots \\ q_K \end{pmatrix}$

- Assign receiver filter  $\mathbf{a}_i$  for user  $i$ 
  - Select it to make

$$\begin{aligned} \mathbf{a}_i^H \mathbf{y} &= \sqrt{\rho_{ul}} \mathbf{a}_i^H \mathbf{G} \mathbf{D}_{\eta}^{1/2} \mathbf{q} + \mathbf{a}_i^H \mathbf{w} \\ &= \sum_{k=1}^K \mathbf{a}_i^H \mathbf{g}_k \sqrt{\rho_{ul} \eta_k} q_k + \mathbf{a}_i^H \mathbf{w} \approx q_i \end{aligned}$$

No successive  
interference cancelation

In some “good” sense

# Capacity bound for User $i$ using Channel Estimates

- General capacity lower bound:

$$C \geq E \left\{ \log_2 \left( 1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho \text{Var}\{g|\Omega\} + \text{Var}\{w'|\Omega\}} \right) \right\}$$

In our case:

- $\rho = \rho_{ul}\eta_i$
- $g = \mathbf{a}_i^H \mathbf{g}_i$
- $x = q_i$
- $\Omega = \{\hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\}$
- $w' = \sum_{k=1, k \neq i}^K \mathbf{a}_i^H \mathbf{g}_k \sqrt{\rho_{ul}\eta_k} q_k + \mathbf{a}_i^H \mathbf{w}$



We need to compute each term!

# Computing the expectation in the numerator

$$\begin{aligned} E\{g|\Omega\} &= E\{\mathbf{a}_i^H \mathbf{g}_i | \hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\} \\ &= \mathbf{a}_i^H E\{\hat{\mathbf{g}}_i | \hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\} - \mathbf{a}_i^H E\{\tilde{\mathbf{g}}_i | \hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\} \\ &= \mathbf{a}_i^H \hat{\mathbf{g}}_i \end{aligned}$$

We have used

- $g = \mathbf{a}_i^H \mathbf{g}_i$
- $\mathbf{a}_i$  selected based on  $\Omega = \{\hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\}$
- $\mathbf{g}_i = \hat{\mathbf{g}}_i - \tilde{\mathbf{g}}_i$  where  $\hat{\mathbf{g}}_i$  and  $\tilde{\mathbf{g}}_i$  are independent
- $E\{\tilde{\mathbf{g}}_i | \hat{\mathbf{g}}_1, \dots, \hat{\mathbf{g}}_K\} = E\{\tilde{\mathbf{g}}_i\} = \mathbf{0}$

$$C \geq E \left\{ \log_2 \left( 1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho \text{Var}\{g|\Omega\} + \text{Var}\{w|\Omega\}} \right) \right\}$$

# Computing the first term in the denominator

$$\text{Var}\{g|\Omega\} = E\{|g|^2|\Omega\} - |E\{g|\Omega\}|^2$$

- We have already computed  $E\{g|\Omega\}$
- Note that

$$\begin{aligned} E\{|g|^2|\Omega\} &= E\left\{|\mathbf{a}_i^H \mathbf{g}_i|^2|\Omega\right\} \\ &= \mathbf{a}_i^H E\left\{\hat{\mathbf{g}}_i \hat{\mathbf{g}}_i^H + \tilde{\mathbf{g}}_i \tilde{\mathbf{g}}_i^H - \hat{\mathbf{g}}_i \tilde{\mathbf{g}}_i^H - \tilde{\mathbf{g}}_i \hat{\mathbf{g}}_i^H|\Omega\right\} \mathbf{a}_i \\ &= \mathbf{a}_i^H (\hat{\mathbf{g}}_i \hat{\mathbf{g}}_i^H + (\beta_i - \gamma_i) \mathbf{I}_M - \mathbf{0} - \mathbf{0}) \mathbf{a}_i = \mathbf{a}_i^H (\hat{\mathbf{g}}_i \hat{\mathbf{g}}_i^H + (\beta_i - \gamma_i) \mathbf{I}_M) \mathbf{a}_i \end{aligned}$$

We used the same properties as before.

$$C \geq E \left\{ \log_2 \left( 1 + \frac{\rho |E\{g|\Omega\}|^2}{\rho \text{Var}\{g|\Omega\} + \text{Var}\{w|\Omega\}} \right) \right\}$$

# Computing the second term in the denominator

$$\text{Var}\{w'|\Omega\} = \text{Var}\left\{\sum_{k=1, k \neq i}^K \mathbf{a}_i^H \mathbf{g}_k \sqrt{\rho_{ul}\eta_k} q_k + \mathbf{a}_i^H \mathbf{w} | \Omega\right\}$$

- $E\{w'|\Omega\} = 0$  since  $E\{q_k\} = 0, E\{\mathbf{w}\} = \mathbf{0}$

$$\begin{aligned} \text{Var}\{w'|\Omega\} &= E\{|w'|^2|\Omega\} \\ &= \sum_{k=1, k \neq i}^K E\left\{|\mathbf{a}_i^H \mathbf{g}_k|^2 | \Omega\right\} \rho_{ul}\eta_k E\{|q_k|^2 | \Omega\} + E\left\{|\mathbf{a}_i^H \mathbf{w}|^2 | \Omega\right\} \\ &= \sum_{k=1, k \neq i}^K \mathbf{a}_i^H (\hat{\mathbf{g}}_k \hat{\mathbf{g}}_k^H + (\beta_k - \gamma_k) \mathbf{I}_M) \mathbf{a}_i \rho_{ul}\eta_k + \mathbf{a}_i^H \mathbf{I}_M \mathbf{a}_i \end{aligned}$$

## Result: Capacity lower bound of User $i$

$$C \geq E \left\{ \log_2 \left( 1 + \frac{\rho_{ul} \eta_i |\mathbf{a}_i^H \hat{\mathbf{g}}_i|^2}{\mathbf{a}_i^H \mathbf{B}_i \mathbf{a}_i} \right) \right\}$$

where

$$\mathbf{B}_i = \sum_{k=1, k \neq i}^K \rho_{ul} \eta_k \hat{\mathbf{g}}_k \hat{\mathbf{g}}_k^H + \sum_{k=1}^K \rho_{ul} \eta_k (\beta_k - \gamma_k) \mathbf{I}_M + \mathbf{I}_M$$

How to pick  $\mathbf{a}_i$ ?

# Generalized Rayleigh Quotient

The ratio

$$\frac{|\mathbf{a}^H \mathbf{b}|^2}{\mathbf{a}^H \mathbf{B} \mathbf{a}}$$

for given  $\mathbf{b}$  and invertible  $\mathbf{B}$  is maximized by  $\mathbf{a} = \mathbf{B}^{-1} \mathbf{b}$

- Intuition:

- The ratio  $\frac{|\mathbf{a}^H \mathbf{b}|^2}{\mathbf{a}^H \mathbf{a}}$  is maximized by  $\mathbf{a} = \mathbf{b}$
- The extra term  $\mathbf{B}^{-1}$  is “whitening”

Extension of maximum  
ratio combining



# Maximizing the capacity lower bound

$$C \geq E \left\{ \log_2 \left( 1 + \frac{|\mathbf{a}_i^H \mathbf{b}_i|^2}{\mathbf{a}_i^H \mathbf{B}_i \mathbf{a}_i} \right) \right\}$$

where  $\mathbf{b}_i = \sqrt{\rho_{ul}\eta_i} \hat{\mathbf{g}}_i$  and

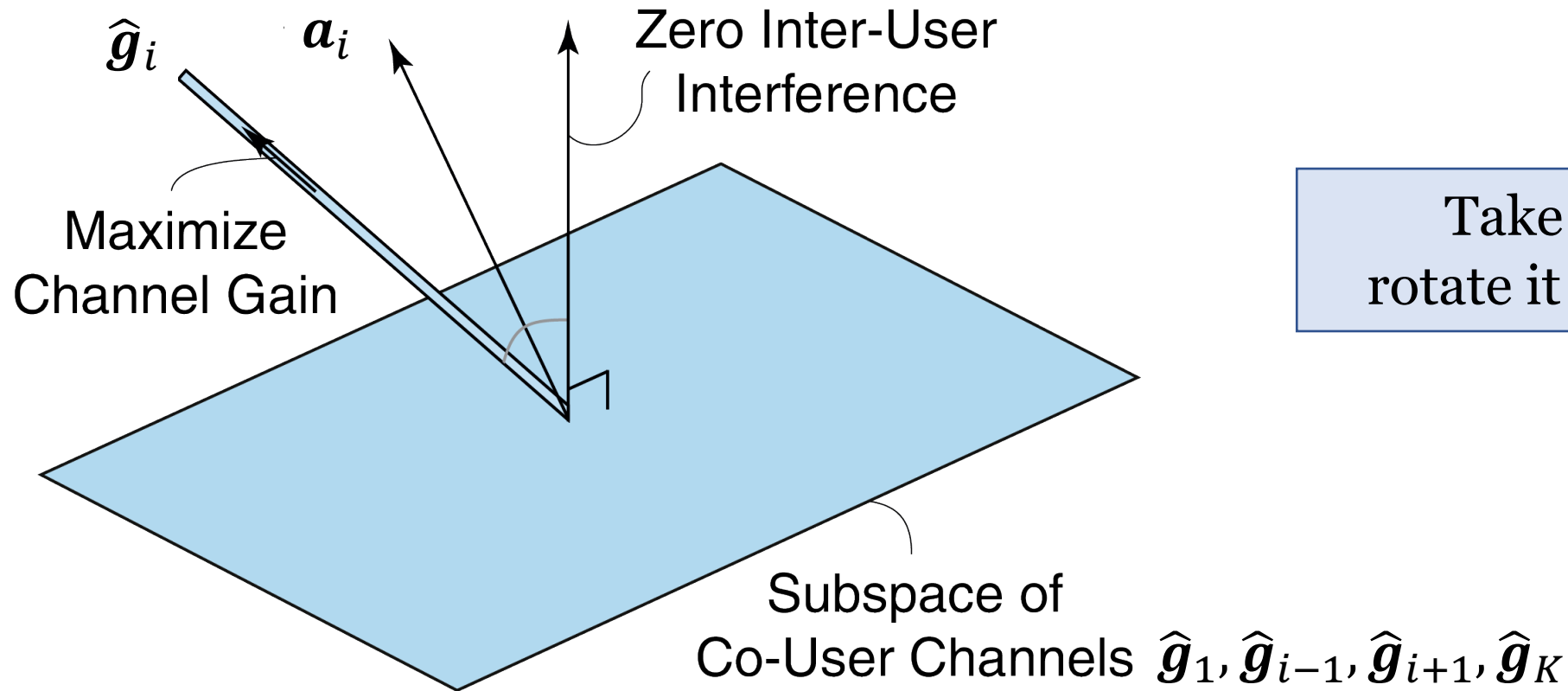
$$\mathbf{B}_i = \sum_{k=1, k \neq i}^K \rho_{ul}\eta_k \hat{\mathbf{g}}_k \hat{\mathbf{g}}_k^H + \sum_{k=1}^K \rho_{ul}\eta_k (\beta_k - \gamma_k) \mathbf{I}_M + \mathbf{I}_M$$

Maximized by selecting

$$\mathbf{a}_i = \mathbf{B}_i^{-1} \mathbf{b}_i = \sqrt{\rho_{ul}\eta_i} \mathbf{B}_i^{-1} \hat{\mathbf{g}}_i$$

Called MMSE combining

# Interpretation of $\mathbf{a}_i = \sqrt{\rho_{ul}\eta_i} \mathbf{B}_i^{-1} \hat{\mathbf{g}}_i$ (whitening)



Take  $\hat{\mathbf{g}}_i$  and  
rotate it using  $\mathbf{B}_i^{-1}$

# Summary

- Channel estimation
  - Send pilot sequences of length  $\tau_p \geq K$  in the uplink
  - Estimate channels using MMSE estimation
- Data transmission
  - General capacity lower bound
  - Can be computed when MMSE channel estimates are known
  - The linear receiver was optimized

End of Lecture 7

# TSKS14 Multiple Antenna Communications