(1)

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Probability theory

Mathematical or classical Definition of probability

If a trial experiment results in n exhaustive events that are mutually exclusive and equally likely, and m of them are favourable to the happening of an event E, then the probability of happening of E is given by

$$p = P(E) = \frac{m}{n}.$$

The probability that E will not happen is given by

$$q = P(\bar{E}) = \frac{m-n}{n} = 1-p \Rightarrow p+q=1.$$

and also $0 \le p \le 1$ $0 \le q \le 1$.

Statistical or empirical probability : If a trial is repeated a number of times under essentially homogeneous and identical conditions then the limiting value of the ratio of the number of times the event happens to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of event symbolically if in n trials an event E happens m times then the probability of happening of E is p and given by $p = P(E) = \lim_{n\to\infty} \frac{m}{n}$.

Problems:

(1) A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue.

Sol: Total number of balls $= 3+6+7 = 16$.

Out of 16 balls 2 can be drawn in 16_{C_2} ways

i.e, n = Exhaustive number of cases $= 16_{C_2} = \frac{\lfloor 16}{\lfloor 16-2 \lfloor 2} = 120$.

Out of 6 white balls 1 ball can be drawn in $6c_1$ ways and out of 7 blue balls one ball can be drawn in $7c_1$ ways. The total number of favourable cases are $6c_1 \times 7c_1 = 6\times7 = 42 = m$

The required probability $= \frac{m}{n} = \frac{6c_1 \times 7c_1}{16_{C_2}} = \frac{42}{120} = \frac{7}{20}$

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(2) Two cards are drawn at random from a pack of 52 cards.

(a) find the probability of drawing two aces.

(b) from a pack of 52 cards three are drawn at random find the chance they are a king, a queen and a knave.

(c) Four cards are drawn at random find the probability that all are diamonds.

Sol.: (a) From a pack of 52 cards 2 can be drawn in ${}^{52}C_2$ ways, ie $n = {}^{52}C_2$.

In a pack there are 4 aces. Therefore 2 aces can be drawn in ${}^{4}C_2$ ways ie $m = {}^{4}C_2$.

Thus required prob $= \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4\times 3}{2} \times \frac{2}{52\times 51} = \frac{1}{221}$.

(b) Exhaustive events $= n = {}^{52}C_3$.

A pack of cards contain 4 kings, 4 queens and 4 knaves. A king, queen and knave can be drawn in ${}^{4}C_1$ ways. Thus the total number of favourable cases $= {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1$. Thus the required probability $= \frac{{}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_3} = \frac{4\times 4\times 4\times 6}{52\times 51\times 50} = \frac{16}{5525}$.

(C4)

(c) n = Total number of cases = ${}^{52}C_4$.

There are 13 diamonds in pack. four can be selected in

$m = {}^{13}C_4$.

the req. prob = $\frac{{}^{13}C_4}{{}^{52}C_4}$.

(8) Among the digits 1, 2, 3, 4, 5, at first one is chosen and then a second selection is made among the remaining four digits. find the prob. that an odd digit will be selected, i, first time ii, the second time iii, both times.

Sol: Total number of cases = $5 \times 4 = 20$.

i) cases in which the first digit drawn is odd

(1,2) (1,3) (1,4) (1,5) (3,1) (3,2) (3,4) (3,5) (5,1), (5,2) (5,3) (5,4).

Total number of cases = 12 (first digit odd).

$\Rightarrow$ the prob. the first digit drawn is odd = $\frac{12}{20} = 3/5$.

ii, The cases in which the second digit drawn is odd (2,1) (3,1) (4,1) (5,1) (1,3) (2,3) (4,3) (5,3), (1,5) (2,5) (3,5) (4,5). i.e 12 cases

The prob. that the second digit drawn is odd = $\frac{12}{20} = 3/5$.

(iii) The cases in which both the digits drawn are odd are (1,3), (1,5) (3,1) (3,5) (5,1) (5,3) i.e. 6 cases. Therefore the

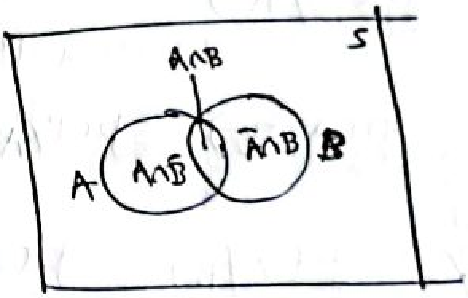
prob., that both the digits drawn are odd

$$= \frac{6}{20} = \frac{3}{10}.$$

Addition Law of probabilities: If A and B are any two events which are subsets of the sample space S and are not disjoint, then

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

Proof: we have



$A\cup B = A\cup(\bar{A}\cap B)$, since A and $(\bar{A}\cap B)$ are disjoint.

$$P(A\cup B) = P(A) + P(\bar{A}\cap B)$$

$$= P(A) + \left[P(\bar{A}\cap B) + P(A\cap B) - P(A\cap B)\right]$$

$$= P(A) + \left[P(\bar{A}\cap B) + P(A\cap B)\right] - P(A\cap B)$$

$$= P(A) + P\left[(\bar{A}\cap B)\cup(A\cap B)\right]$$ $(\bar{A}\cap B)$, $(A\cap B)$ are disjoint.

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

This is addition law of prob; can be extended to any number of events.

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Note: for any three events A, B, C we have

$$p(A\cup B\cup C) = p(A)+p(B)+p(C)-p(A\cap B)-p(B\cap C) - p(C\cap A)+p(A\cap B\cap C).$$

Conditional probability: let A and B are any two events. The prob. of happening of the event B given that A has already happened is called conditional probability and it is denoted by $p(B/A)$ and defined as

$$p(B/A) = \frac{p(A\cap B)}{p(A)}, \quad p(A) \neq 0.$$

$$\Rightarrow \quad p(A\cap B) = p(B/A) \cdot p(A), \quad p(A) \neq 0.$$

III/y $$p(A/B) = \frac{p(A\cap B)}{p(B)}, \quad p(B) \neq 0.$$

$$p(A\cap B) = p(A/B) \cdot p(B), \quad p(B) \neq 0.$$

Multiplication law of probability: The prob. of simultaneous occurence of any two events is equal to unconditional probability of one event multiplied by the conditional prob.

of the other. That is If A, B are any two events then,

$$p(A\cap B) = p(A).p(B/A), \qquad p(A)>0$$

$$p(A\cap B) = p(B).p(A/B), \qquad p(B)>0$$

It is called multiplication law of probability.

Note: let A and B are any two events such that the happening of any one event does not depend or not effect the happening of the other in this case

$$p(B/A) = p(B), \qquad p(A/B) = p(A),$$

then A and B are said to be indept events and in this case

$$p(A\cap B) = p(A).p(B).$$

problems :

(8) Three horses A, B, C are in race. A is twice likely to win as B and B is twice as likely to win as C. What are the respective probabilities of winning.

Sol: A is twice as likely to win B, $p(A)=2p(B)$

B is thrice as C $\Rightarrow$ $p(B) = 2p(C) \Rightarrow p(C) =$

we know that Total prob. is unity.

$p(A) + p(B) + p(C) = 1$

$2p(B) + p(B) + \frac{1}{2}p(B) = 1 \Rightarrow \frac{7}{2}p(B) = 1 \Rightarrow$

$p(B) = 2/7 \Rightarrow p(A) = \frac{4}{7},\ p(C) = \frac{1}{7}$.

prob. of winning for horse A is $4/7$

" " " " B $= 2/7$

" " " " C $= 1/7$

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(9) A problem is given to three students A, B, C whose chances of solving it are $1/2$, $\frac{1}{3}$, $1/4$ respty. What is the prob. that

i) the problem is solved.

ii) Exactly one of them solves the problem.

Sol: let A, B, C are the events, they solve the problem

$p(A) = 1/2 \quad p(B) = 1/3 \quad p(C) = 1/4 \Rightarrow$

not solving: $p(\bar{A}) = \frac{1}{2},\quad p(\bar{B}) = 2/3,\quad p(\bar{C}) = 3/4$. since

A, B, C solve problem independently. The prob. that the problem is solved is

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$ $=$

$p(C) =$

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p(The problem is not solved)

$= p(\bar{A}\cap\bar{B}\cap\bar{C}) = p(\bar{A}).p(\bar{B}).p(\bar{C})$

$= \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4} = \frac{1}{4}$.

p(The problem is solved) $= 1 - \frac{1}{4} = \frac{3}{4}$.

ii, p(Exactly one of them solves the problem)

$= p[(A\cap\bar{B}\cap\bar{C})\cup(\bar{A}\cap B\cap\bar{C})\cup(\bar{A}\cap\bar{B}\cap C)]$

$= p(A\cap\bar{B}\cap\bar{C}) + p(\bar{A}\cap B\cap\bar{C}) + p(\bar{A}\cap\bar{B}\cap C)$

(All are mutually exclusive events)

$= p(A).p(\bar{B}).p(\bar{C}) + p(\bar{A}).p(B).p(\bar{C}) + p(\bar{A}).p(\bar{B}).p(C)$

$= (\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}) + (\frac{1}{2}\times\frac{1}{3}\times\frac{3}{4}) + (\frac{1}{2}\times\frac{2}{3}\times\frac{1}{4})$

$= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} = \frac{11}{24}$.

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(8) A is known to hit the target in 2 out of 5 shots. B is known to hit the target in 3 out of 4 shots. Find the prob. of the target being hit when both of them try.

Sol: It is given that $p(A) = \frac{2}{5}$ $p(B) = \frac{3}{4}$.

$p(A\cup B) = p(A) + p(B) - p(A\cap B)$

since A and B are indept $p(A\cap B) = p(A).p(B)$

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$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= P(A) + P(B) - P(A) \cdot P(B)$$
$$= \frac{2}{5} + \frac{3}{4} - \frac{2}{5} \cdot \frac{3}{4}$$
$$= \frac{17}{20}.$$

The prob; that the target is being hit $= \frac{17}{20}$.

(1)

Problems on Discrete Random variable

A random variable X has the following distribution:

$X=x$:	0	1	2	3	4	5	6	7	8
$p(x)=p(X=x)$:	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$	$15K$	$17K$

Find the value of K and i) $p(x<3)$, ii) $p(x\geq 3)$
iii) $p(0<x<6)$.

Sol: Given discrete r.v is X, its values are denoted by x. and the probability at $X=x$ is denoted by $p(x)$ or $p(X=x)$. The values of random variable along with corresponding probabilities is called probability distribution, as in the given problem.

We know that the total probability is unity, i.e;

$$\sum p(X=x) = \sum_{i=0}^{8} p(X=x_i) = \sum_{i=0}^{8} p(x_i) = 1.$$

i.e $p(0)+p(1)+p(2)+---p(8)=1$

$$K+3K+5K+7K+9K+11K+13K+15K+17K=1$$

$$\Rightarrow 81K=1 \Rightarrow K=\frac{1}{81}.$$

i) $p(X<3) = p(X=0)+p(X=1)+p(X=2) = p(0)+p(1)+p(2)$
$= K+3K+5K = 9K = 9\times\frac{1}{81} = \frac{9}{81} = \frac{1}{9}.$

(2)

ii) $p(x\geq 3) = p(x=3)+p(x=4)+p(x=5)+p(x=6)+p(x=7)+p(x=8)$

$= 7k+9k+11k+13k+15k+17k = 72k$

$p(x\geq 3) = \cancel{72}^{8} \times \frac{1}{\cancel{81}_{9}} = 8/9.$

(or)

$p(x\geq 3) = 1 - p(x<3) = 1-\frac{1}{9} = 8/9.$

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iii) $p(0<x<6) = p(x=1)+p(x=2)+p(x=3)+p(x=4)+p(x=5)$

$= 3k+5k+7k+9k+11k = 35k$

$= 35\times\frac{1}{81} = \frac{35}{81}.$

$\Rightarrow p(0<x<6) = \frac{35}{81}.$

(*) Let $p(x)$ is the probability function of a discrete random variable x which assumes the values x_1, x_2, x_3, x_4 such that

$2p(x_1) = 3p(x_2) = p(x_3) = 5p(x_4)$. Find the probability distribution.

Sol: $p(x)$ is the probability at $X=x$ and it is taking the values x_1, x_2, x_3, x_4 satisfying the given condition.

(3)

$2p(x_1) = 3p(x_2) = p(x_3) = 5p(x_4)$. The the probability distribution is

$X = x : x_1 \quad x_2 \quad x_3 \quad x_4$

$p(X=x) : p(x_1) \quad p(x_2) \quad p(x_3) \quad p(x_4)$.

Let $2p(x_1) = 3p(x_2) = p(x_3) = 5p(x_4) = k$ (say)

$\Rightarrow p(x_1) = \frac{k}{2}, \; p(x_2) = \frac{k}{3}, \; p(x_3) = k, \; p(x_4) = \frac{k}{5}$.

since the total probability is unity, we have

$p(x_1) + p(x_2) + p(x_3) + p(x_4) = 1$

$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1 \Rightarrow \frac{15k + 10k + 30k + 6k}{30} = 1$

$\frac{61k}{30} = 1 \Rightarrow k = \frac{30}{61} \Rightarrow$ the required probability distribution is

$X = x : x_1 \quad x_2 \quad x_3 \quad x_4$

$p(X=x) : \frac{15}{61} \quad \frac{10}{61} \quad \frac{30}{61} \quad \frac{6}{61}$.

(4)

(2) Find the value of K and $p(x<5)$, $p(x\geq 6)$ and $p(0<x<6)$ for the following discrete distribution.

$X=x$:	0	1	2	3	4	5	6	7
$p(x=x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Sol: X is discrete random variable with given probability distribution. We know that total probability is unity. That is $\sum p(x=x)=1$

$\Rightarrow \quad p(0)+p(1)+p(2)+p(3)+p(4)+p(5)+p(6)+p(7)=1$

$\Rightarrow \quad 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow \quad 10k^2+9k=1 \Rightarrow 10k^2+9k-1=0$

$\Rightarrow \quad (10k-1)(k+1)=0 \Rightarrow k=\frac{1}{10}$ or -1

since prob., is non-negative $\therefore$ $k=-1$ is not accepted

$\Rightarrow \quad k=\frac{1}{10}$.

i) $p(x<5)=p(0)+p(1)+p(2)+p(3)+p(4)$

$p(x<5)=0+k+2k+2k+3k=8k=8\times\frac{1}{10}=\frac{8}{10}$

$p(x<5)=\frac{4}{5}$.

ii) $p(x\geq 6)=p(x=6)+p(x=7)=p(6)+p(7)$

$=2k^2+7k^2+k=9k^2+k=9\times\frac{1}{100}+\frac{1}{10}=\frac{19}{100}$.

(iii) $p(0<x<6) = p(1)+p(2)+p(3)+p(4)+p(5)$

$= 0+k+2k+2k+3k+k^2 = k^2+8k$

$= \frac{1}{100}+\frac{8}{10} = \frac{1}{100}+\frac{80}{100} = \frac{81}{100}.$

$p(0<x<6) = \frac{81}{100}.$

(8) X is a discrete r.v such that $p(x=-2)=p(x=-1)=p(x=1)=p(x=2)$ and $p(x<0)=p(x=0)=p(x>0)$. Determine the probability distribution.

Sol: Consider x as the discrete r.v with the condition

$p(x=-2)=p(x=-1)=p(x=1)=p(x=2)=k$ (let)

then we have $X: -2 \quad -1 \quad 0 \quad 1 \quad 2,$

$\Rightarrow \quad p(x<0) = p(x=-2)+p(x=-1) = k+k=2k$

$p(x>0) = p(x=1)+p(x=2) = k+k=2k$

$\Rightarrow \quad p(x<0) = p(x=0) = p(x>0) = 2k \Rightarrow p(x=0)=2k$

Thus probability distribution is

$X=x$:	-2	-1	0	1	2
$p(x=x)$:	k	k	$2k$	k	k

since the total prob; is unity $\Rightarrow$

$$\Sigma p(X=x)=1 \Rightarrow$$

$$p(-2)+p(-1)+p(0)+p(1)+p(2)=1$$

$$K+K+2K+K+K=1$$

$$6K=1 \Rightarrow K=1/6.$$

$\Rightarrow$ the required prob. distribution is

$X=x$:	-2	-1	0	1	2
$p(X=x)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

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probability mass function (pmf) of a discrete r.v 'X' is $f(x)=\frac{x}{15}$, $x=1,2,3,4,5$.

$=0$ otherwise.

Find $p(1<X<4)$ and the prob. distribution.

Sol: pmf is $f(x)=p(X=x)=\frac{x}{15}$, $x=1,2,3,4,5$.

$\Rightarrow$ the prob. distribution is

$X=x$:	1	2	3	4	5
$p(X=x)$;	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

and

$$p(1<X<4)=p(X=2)+p(X=3)=\frac{2}{15}+\frac{3}{15}=\frac{5}{15}$$

$$p(1<X<4)=\frac{1}{3}.$$

problems on continuous random variable

(A) Let X is a continuous r.v and the function $f(x)$ is called probability density function (pdf) $f(x)$ satisfy the conditions:

i) $f(x) \geq 0$ ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ $\because$ total probability is unity.

iii) $P(a < X < b) = \int_a^b f(x)\,dx$ = probability between a and b.

iv) $P(a < X \leq b) = P(a < X < b)$.

v) $P(X = a) = \int_a^a f(x)\,dx = 0$.

problems.

(A) A continuous r.v has the following pdf:

$$f(x) = k(x-1)^3, \quad 1 \leq x \leq 3.$$
$$= 0 \quad \text{otherwise}.$$

Find the value of k.

sol: we know that total probability is unity.

$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$ where $f(x)$ is the pdf.

$$\Rightarrow \int_{-\infty}^{\infty} k(x-1)^3\,dx = 1 \Rightarrow k\int_1^3 (x-1)^3\,dx = 1$$

$$\Rightarrow k.\frac{1}{4}\left[(x-1)^4\right]_1^3 = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4}$$

8) Find the value of k if the pdf

$$f(x) = kx \quad , \quad 0 \le x \le 5$$
$$= k(10-x) \quad , \quad 5 \le x \le 10$$
$$= 0 \quad , \text{ otherwise.}$$

Find $p(x \ge 5)$.

Sol: Total probability $= 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{0}^{10} f(x)\,dx = 1 \Rightarrow$$

$$\int_{0}^{5} f(x)\,dx + \int_{5}^{10} f(x)\,dx = 1 \Rightarrow \int_{0}^{5} kx\,dx + \int_{5}^{10} k(10-x)\,dx = 1$$

$$\Rightarrow k\left(\frac{x^2}{2}\right)_0^5 + k.\left(10x - \frac{x^2}{2}\right)_5^{10} = 1$$

$$\Rightarrow k \times \frac{25}{2} + k\left[\cancel{100} - \cancel{50} - 50 + \frac{25}{2}\right] = 1$$

$$\Rightarrow \frac{25k}{2} + k\left[\frac{25}{2}\right] = 1 \Rightarrow \cancel{2} \times \frac{25}{\cancel{2}} k = 1$$

$$25k = 1 \Rightarrow k = \frac{1}{25}$$

$$\text{Also } p(x \ge 5) = \int_{5}^{\infty} f(x)\,dx = \int_{5}^{10} f(x)\,dx$$

$$= \int_{5}^{10} k(10-x)\,dx = k\int_{5}^{10} (10-x)\,dx = k\left(10x - \frac{x^2}{2}\right)_5^{10}$$

$$= \frac{1}{25} \times \left[\cancel{100} - \cancel{50} - 50 + \frac{25}{2}\right] = \frac{1}{\cancel{25}} \times \frac{\cancel{25}}{2} = \frac{1}{2}$$

$$p(x \ge 5) = \frac{1}{2}.$$

Q) A continuous r.v x has the following pdf:

$$f(x) = Kx e^{-x}, \quad x>0$$
$$= 0, \quad \text{otherwise}.$$

Find the value of K.

Sol: We know that $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow$

$$\int_0^{\infty} Kx e^{-x}\,dx = 1 \Rightarrow K\left[\int_0^{\infty} x e^{-x}\,dx\right] = 1$$

$$\Rightarrow K\left[-x e^{-x} - e^{-x}\right]_0^{\infty} = 1 \Rightarrow -K\left|\left[x+1\right] e^{-x}\right|_0^{\infty} = 1$$

$$\Rightarrow -K\left[0-1\right] = 1 \Rightarrow K = 1.$$

Q) A continuous r.v X has the pdf $f(x) = Kx^2$, $0 \le x \le 1$. Find the value of K and a such that $p(x \le a) = p(x > a)$.

Sol: pdf is $f(x) = Kx^2, \quad 0 \le x \le 1$
$= 0, \quad \text{otherwise}. \Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{-\infty}^{\infty} Kx^2\,dx = 1 \Rightarrow \int_0^1 Kx^2\,dx = 1$$

$$\Rightarrow K\cdot\left(\frac{x^3}{3}\right)_0^1 = 1 \Rightarrow \frac{1}{3}K = 1 \Rightarrow K = 3.$$

$\Rightarrow$ pdf is $f(x) = 3x^2, \quad 0 \le x \le 1$
$= 0, \quad \text{otherwise}$

Now consider

$$p(x \le a) = p(x > a)$$

$$\Rightarrow \int_{-\infty}^{a} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx \Rightarrow \int_{0}^{a} 3x^2\,dx = \int_{a}^{1} 3x^2\,dx$$

$$\Rightarrow \left(\frac{x^3}{3}\right)_0^a = \left(\frac{x^3}{3}\right)_a^1 \Rightarrow a^3 = 1 - a^3 \Rightarrow 2a^3 = 1$$

$$a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}.$$

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4) pdf of a continuous r.v is $f(x) = K(1+x)$ $2 \le x \le 5$
$= 0$ otherwise.

Find $p(x<4)$.

Sol: we know that $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow K\int_{2}^{5}(1+x)\,dx = 1$

$$\Rightarrow K\left(x + \frac{x^2}{2}\right)_2^5 = 1 \Rightarrow K\left(5 + \frac{25}{2} - 2 - 2\right) = 1 \Rightarrow$$

$$K\left(1 + \frac{25}{2}\right) = 1 \Rightarrow \frac{27}{2}K = 1 \Rightarrow K = \frac{2}{27}.$$

$$p(x<4) = \int_{-\infty}^{4} f(x)\,dx = \int_{2}^{4} f(x)\,dx = \int_{2}^{4} K(1+x)\,dx$$

$$= K\int_{2}^{4}(1+x)\,dx = \frac{2}{27} \times \left(x + \frac{x^2}{2}\right)_2^4 = \frac{2}{27}(4+8-2-2)$$

$$= \frac{16}{27}$$

$$p(x<4) = \frac{16}{27}.$$

2) prove that $f(x) = 6x(1-x)$, $0 \le x \le 1$

$= 0$, otherwise

is a pdf. Also find the value of K such that $p(x<K) = p(x>K)$.

Sol: we know that $f(x)$ is pdf if $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Consider $\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 6x(1-x)\,dx$

$$= 6\int_0^1 (x - x^2)\,dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 \Rightarrow$$

$$= 6\left(\frac{1}{2} - \frac{1}{3}\right) = 6\left(\frac{3-2}{6}\right) = 6 \times \frac{1}{6} = 1$$

$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow f(x) = 6x(1-x)$, $0 \le x \le 1$

$= 0$, otherwise

is a pdf.

Consider $p(x<K) = p(x>K)$

$$\Rightarrow \int_{-\infty}^{K} f(x)\,dx = \int_K^{\infty} f(x)\,dx \Rightarrow \int_0^K f(x)\,dx = \int_K^1 f(x)\,dx$$

$$\Rightarrow \int_0^K 6x(1-x)\,dx = \int_K^1 6x(1-x)\,dx \Rightarrow$$

$$\int_0^K (x-x^2)\,dx = \int_K^1 (x-x^2)\,dx \Rightarrow \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^K = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_K^1$$

$$\Rightarrow \frac{K^2}{2} - \frac{K^3}{3} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{K^2}{2} - \frac{K^3}{3}\right)$$

$$\Rightarrow 2\left(\frac{K^2}{2} - \frac{K^3}{3}\right) = \frac{1}{6} \Rightarrow 2\left(\frac{3K^2 - 2K^3}{6}\right) = \frac{1}{6} \Rightarrow 6K^2 - 4K^3 = 1$$

$-4k^3+6k^2-1=0$ or

$4k^3-6k^2+1=0 \Rightarrow$

$k=\frac{1}{2}$ is the only real value in $(0,1)$

$\therefore \quad k=\frac{1}{2}$

$f(x)=4k(x-1)^3, \quad 1\le x\le 3$

$=0$, otherwise, is the pdf. Find $p(x<2)$

Sol: we know that $\int_{-\infty}^{\infty} f(x)\,dx=1$, if $f(x)$ is pdf.

$\Rightarrow \int_1^3 4k(x-1)^3\,dx=1 \Rightarrow 4k\int_1^3 (x-1)^3\,dx=1$

$\Rightarrow \cancel{4}k\left[\frac{(x-1)^4}{\cancel{4}}\right]_1^3=1 \Rightarrow k[16]=1 \Rightarrow$

$k=\frac{1}{16}$.

$\therefore \quad f(x)=\frac{1}{4}(x-1)^3, \quad 1\le x\le 3$

$=0$ otherwise is pdf

$p(x<2)=\int_{-\infty}^{2} f(x)\,dx=\int_1^2 f(x)\,dx=\int_1^2 \frac{1}{4}(x-1)^3\,dx$

$=\frac{1}{4}\left[\frac{(x-1)^4}{4}\right]_1^2=\frac{1}{16}\left[(x-1)^4\right]_1^2=\frac{1}{16}$.

$\Rightarrow p(x<2)=\frac{1}{16}$.

8) X is a conts. r.v with pdf as

$$f(x) = Kx \quad , 0 \le x \le 1$$
$$= K \quad , 1 \le x \le 2$$
$$= -Kx + 3K \quad , 2 \le x \le 3 .$$ Find the value K

Sol: $f(x)$ is the pdf $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \cancel{\int_{-\infty}^{0} f(x)\,dx} + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \cancel{\int_{3}^{\infty} f(x)\,dx} = 1$$

$$\Rightarrow \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} Kx\,dx + \int_{1}^{2} K\,dx + \int_{2}^{3} (-Kx+3K)\,dx = 1$$

$$\Rightarrow K\left(\frac{x^2}{2}\right)_0^1 + K(x)_1^2 + \left(-K\frac{x^2}{2} + 3Kx\right)_2^3 = 1$$

$$\Rightarrow K\left(\tfrac{1}{2} - 0\right) + K(2-1) + \left(-\frac{9K}{2} + 9K + 2K - 6K\right) = 1$$

$$\Rightarrow K + \frac{K}{2} + \frac{K}{2} = 1 \Rightarrow K + K = 1 \Rightarrow 2K = 1 \Rightarrow$$

$\underline{K = \frac{1}{2}}$ $\Rightarrow$ the pdf is

$$f(x) = \tfrac{1}{2}x \quad , 0 \le x \le 1$$
$$= \tfrac{1}{2} \quad , 1 \le x \le 2$$
$$= -\frac{x}{2} + \frac{3}{2} \quad , 2 \le x \le 3 .$$

(14)

Mataematical Expectation

Averaging procen applied a distribution with r.v, is called mathematical expectation and it is denoted by $E(x)$. For discrete distribution, let

$X = x$: $x_1 \quad x_2 \quad x_3 \quad \text{-----} \quad x_n$

$P(X=x) = p_1 \quad p_2 \quad \text{-----} \quad p_n$ where p_n is the probability at $x = x_n$. Then $E(x)$ is given by

$$E(x) = x_1 p_1 + x_2 p_2 + \cdots x_n p_n = \sum_{i=1}^{n} x_i p_i .$$

$E(x)$ is the mean of distribution. Similarly,

$$E(x^2) = x_1^2 p_1 + x_2^2 p_2 + \cdots x_n^2 p_n = \sum_{i=1}^{n} x_i^2 p_i .$$

In general

$$E(x^r) = x_1^r p_1 + x_2^r p_2 + \cdots x_n^r p_n = \sum_{i=1}^{n} x_i^r p_i .$$

$E(x^r)$ is called the rth moment about origin or it is called raw moment or Non central moment and it is denoted as μ'_r

$$\mu'_r = E(x^r) = \sum_{i=1}^{n} x_i^r p_i$$

For a continuous r.v x $E(x)$ is defined as

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx \quad \text{where } f(x) \text{ is the pdf.}$$

$$\Rightarrow E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$ — generally.

$$E(x^r) = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\, dx.$$

Variance : Let x is any random variable for the given distribution. Then the variance of x is denoted by $V(x)$ and is defined as

$$V(x) = E\left[[x - E(x)]^2\right] = E(x^2) - [E(x)]^2.$$

Properties of Mathematical Expectation :

Let a and b are any two constants and x and y are two random variables, then

i) $E(a) = a$

ii) $E(ax) = aE(x)$

(iii) $E(ax+b) = aE(x)+b$

iv) $E(x+y) = E(x)+E(y)$

v) $E(xy) = E(x).E(y)$, when x and y are indept.

i) let X is taking the values $x_1, x_2 \ldots$ with $p_1, p_2 \ldots p_n$ corresponding probabilities then the probability distribution is.

$$X = x : \quad x_1 \quad x_2 \quad x_3 \quad \ldots \quad x_n$$
$$p(X=x) : \quad p_1 \quad p_2 \quad p_3 \quad \ldots \quad p_n$$

let a and b are any two constants:

Consider $E(a) = \sum_{i=1}^{n} a p_i$ (by def. of expectation)

$$= a \sum_{i=1}^{n} p_i = a(p_1 + p_2 + \ldots p_n)$$

$$= a \times 1 = a \quad \left[\because p_1 + p_2 + \ldots p_n = 1 \text{ Total prob.} = 1 \right]$$

$\Rightarrow E(a) = a.$

ii) $E(ax+b) = \sum_{i=1}^{n} (ax_i + b) p_i = \sum_{i=1}^{n} a x_i p_i + \sum_{i=1}^{n} b p_i$

$$= a \sum_{i=1}^{n} x_i p_i + b \sum_{i=1}^{n} p_i$$

$$= a [x_1 p_1 + \ldots x_n p_n] + b [p_1 + p_2 + \ldots p_n]$$

$$= a E(x) + b$$

$\Rightarrow E(ax+b) = aE(x) + b$

Note: M

Note: Mean $= E(x) = \mu$ (meu)

$V(x) = \sigma_x^2$, where σ_x is standard deviation of x and hence

$$\sigma_x = \sqrt{V(x)}.$$

positive square root of variance is the standard deviation.

Properties of Variance: let a, b are the constants and x is r.v then

i) $V(a) = 0$

ii) $V(ax) = a^2 V(x)$

iii) $V(ax+b) = a^2 V(x)$

i) Let x is r.v (discrete) and a, b are the constants

Consider $V(a) = E(a^2) - [E(a)]^2$ by def.,

$= a^2 - [a]^2$ by $E(x)$ def.,

$= a^2 - a^2$

$= 0$

$\Rightarrow \underline{V(a) = 0}$

(18)

(ii) $V(ax) = E(a^2x^2) - [E(ax)]^2$ by def.

$= a^2E(x^2) - [aE(x)]^2$

$= a^2E(x^2) - a^2[E(x)]^2$

$= a^2[E(x^2) - [E(x)]^2]$

$V(ax) = a^2V(x)$

$\Rightarrow V(ax) = a^2V(x)$.

(iii) $V(ax+b) = E[(ax+b)^2] - [E(ax+b)]^2$

$= E[a^2x^2+b^2+2abx] - [aE(x)+b]^2$

$= E(a^2x^2)+E(b^2)+E(2abx) - [a^2[E(x)]^2+b^2 + 2abE(x)]$

$= a^2E(x^2)+\cancel{b^2}+\cancel{2abE(x)} - a^2[E(x)]^2 - \cancel{b^2} - \cancel{2abE(x)}$

$= a^2E(x^2) - a^2[E(x)]^2 = a^2[E(x^2) - [E(x)]^2]$

$= a^2 \cdot V(x)$

$\Rightarrow V(ax+b) = a^2V(x)$

Mom... the variou... like...

Moments : Moments are used to describe the various properties of a frequency distribution, like dispersion, Skewness and Kurtosis.

i) The rth moment of a variable x about mean $\bar{x}$ is defined as

$$\mu_r = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - \bar{x})^r \quad r = 0, 1, 2, 3 - - -$$

μ_r is called the central moment.

ii) The rth moment about any point A is defined as

$$\mu'_r = \frac{1}{N}\sum_{i=1}^{n} f_i (x_i - A)^r \quad r = 0, 1, 2 - - -$$

(iii) The rth moment about origin is given by

$$\mu'_r = \frac{1}{N}\sum_{i=1}^{n} f_i x_i^r .$$

where f_i is the corresponding frequency at $x = x_i$ and N is the total frequency i.e, $N = f_1 + f_2 + \cdots f_n$.

Note : $\mu_r = \frac{1}{N}\sum f_i (x_i - \bar{x})^r$, $\boxed{\sum f_i = N}$

put $r = 0$

$$\mu_0 = \frac{1}{N}\sum f_i (x_i - \bar{x})^0 = \frac{1}{N}\sum f_i = \frac{1}{N} \times f_i$$

$$= \frac{1}{N} \times N = 1 \Rightarrow \mu_0 = 1 \text{ (Always)}$$

$$\mu_1 = \frac{1}{N}\sum f_i (x_i - \bar{x})^1 = \frac{1}{N}\sum f_i x_i - \frac{1}{N}\sum f_i \bar{x}$$

$$= \frac{1}{N}\sum f_i x_i - \frac{1}{N}\cdot \bar{x}\cdot \sum f_i$$

$$\boxed{\bar{x} = \text{mean} = \frac{1}{N}\sum f_i x_i}$$

$$= \bar{x} - \frac{1}{N}\cdot \bar{x}\cdot \times \sum f_i$$

$$= \bar{x} - \bar{x} \times \frac{1}{N} \times N = \bar{x} - \bar{x} = 0.$$

when x about

$\Rightarrow \mu_1 = 0$ (Always)

put $r = 2$

$$\mu_2 = \frac{1}{N}\sum f_i (x_i - \bar{x})^2 = \sigma^2 = \text{variance}.$$

Also $\quad \mu_r' = \frac{1}{N}\sum f_i (x_i - A)^r$.

put $r = 0$

$$\mu_0' = \frac{1}{N}\sum f_i (x_i - A)^0 = \frac{1}{N}\sum f_i = \frac{1}{N}\times N = 1$$

$$\boxed{\mu_0' = 1}$$

put $r = 1$

$$\mu_1' = \frac{1}{N}\sum f_i (x_i - A)^1 = \frac{1}{N}\sum f_i x_i - \frac{A}{N}\sum f_i$$

$$\mu_1' = \bar{x} - A \times \frac{N}{N} = \bar{x} - A \Rightarrow \mu_1' = \bar{x} - A$$

$\boxed{\bar{x} = A + \mu_1'}$ when $A = 0 \Rightarrow$

$$\mu_1' = \bar{x}$$

$z\gamma$

$\mu f \cdot t$

When X is a continuous r.v the rth moment about mean is defined as

$$\mu_r = E[x - \text{mean}]^r = \int_{-\infty}^{\infty} (x - \text{mean})^r f(x)\,dx$$

where $f(x)$ is the pdf, and the raw moment μ'_r about a point $x = A$ is

$$\mu'_r = E[x - A]^r = \int_{-\infty}^{\infty} (x - A)^r f(x)\,dx$$

Also the moment about origin is

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx.$$

Relation between Central and Non-central moments

We know that the central and non-central (raw) moments are denoted by μ_r and μ'_r respectively, the relation between them is

$$\mu_1 = 0$$

$$\mu_2 = \mu'_2 - (\mu'_1)^2$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4.$$

Stewi... probabili...

Note: When x is a continuous r.v and $f(x)$ is its pdf, then

i) Arithmetic mean = A.M = mean = $\int_a^b x f(x)\,dx$

ii) Harmonic mean = H.M = $\frac{1}{H} = \int_a^b \frac{1}{x} f(x)\,dx$

where harmonic mean is H.

iii) Geometric mean = $\log G = \int_a^b [\log x] f(x)\,dx$.

where G is the geometric mean.

iv) Median : It is the point which divides the entire distribution into two equal parts. If M is the median, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = 1/2$$

(v) Mode : Mode of the distribution is the value of x for which $f(x)$ is maximum. We know for max or min $f'(x) = 0$ and for max $f''(x) < 0$

Skewness and Kurtosis: When the probability distribution is plotted in graph, it is represented in the form of a curve called probability curve. The lack of symmetry in the curve is called skewness and it is denoted by β_1 and $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$.

The peakedness or flatness of the curve is called kurtosis, and it is given by

$$\beta_2 = \frac{\mu_4}{\mu_2^2}.$$

(Q) If x is a r.v with p.m.f $f(x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5$
$= 0$, otherwise

then find $E(5x^2+2x+3)$, mean and variance.

Sol: probability mass function is $f(x) = \frac{x}{15}$, $x = 1, 2, 3, 4, 5$

The probability distribution is

$X = x$:	1	2	3	4	5
$p(X = x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$E(x) = \text{mean} = \sum x\, p(X = x)$

$$E(x) = 1\cdot\frac{1}{15} + 2\cdot\frac{2}{15} + 3\frac{3}{15} + 4\cdot\frac{4}{15} + 5\cdot\frac{5}{15}$$

$$= \frac{1}{15} + \frac{4}{15} + \frac{9}{15} + \frac{16}{15} + \frac{25}{15} = \frac{55}{15} = 3.66$$

$$E(x) = 3.66.$$

$$E(x^2) = 1^2\cdot\frac{1}{15} + 2^2\cdot\frac{2}{15} + 3^2\frac{3}{15} + 4^2\cdot\frac{4}{15} + 5^2\cdot\frac{5}{15}$$

$$= \frac{1}{15} + \frac{8}{15} + \frac{27}{15} + \frac{64}{15} + \frac{125}{15} = \frac{225}{15} = 15$$

$$E(x^2) = 15.$$

Thus mean $= E(x) = 3.66$

Variance $= V(x) = E(x^2) - [E(x)]^2 = 15 - (3.66)^2$

$$V(x) = 1.6044$$

Also $E(5x^2+2x+3) = 5E(x^2) + 2E(x) + 3$

$$= 5\times 15 + 2\times 3.66 + 3 = 85.32$$

Thus mean $= E(x) = 3.66$

Variance $= V(x) = 1.6044$

$$E(5x^2+2x+3) = 85.32$$

Also S.D = standard deviation $= \sqrt{V(x)}$

$$\sigma = S.D = \sqrt{1.6044} = 1.27.$$

3.66

(8) X is the discrete variable and it represents the number that turns up when a die is thrown. Find mean and variance.

Sol: A die has six faces represented by 1, 2, 3, 4, 5, 6, and probability of getting any number is $\frac{1}{6}$. Therefore the probability distribution is

$X = x$:	1	2	3	4	5	6
$p(X=x)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X) = \text{mean} = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6}$$

$$= \frac{1}{6}(1+2+3+4+5+6) = \frac{21}{6} = 3.5$$

Consider

$$E(x^2) = \sum x^2 p(X=x) = 1^2\times\frac{1}{6} + 2^2\times\frac{1}{6} + 3^2\times\frac{1}{6} + 4^2\times\frac{1}{6} + 5^2\times\frac{1}{6} + 6^2\times\frac{1}{6}$$

$$= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6} = 15.17$$

$$\text{variance} = V(X) = E(x^2) - [E(x)]^2$$

$$V(X) = 15.17 - (3.5)^2 = 2.92$$

$$S.D = \sqrt{V(X)} = \sqrt{2.92} = 1.71$$

(8) A coin is tossed until a head appears. What is the expectation of the number of tosses required.

Sol: Let X denotes the number of tosses required to get the first head. The first head may appear in the first toss, 2nd toss --- so on.

These events are

H, TH, TTH, TTTH, TTTTH, ----

the corresponding probabilities are

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}$ ------ that is

$X = x : 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$ ------

$p(X=x) : \frac{1}{2} \quad \frac{1}{2^2} \quad \frac{1}{2^3} \quad \frac{1}{2^4} \quad \frac{1}{2^5} \quad \frac{1}{2^6}, \frac{1}{2^7}$ ----

Thus the expected values is

$$E(X) = \sum x\, p(X=x) = \sum x_i\, p(X=x_i)$$

$$= 1\times\frac{1}{2} + 2\times\frac{1}{2^2} + 3\times\frac{1}{2^3} + 4\times\frac{1}{2^4} + 5\times\frac{1}{2^5} + ---$$

$$= \frac{1}{2}\left[1 + 2\times\left(\frac{1}{2}\right) + 3\times\left(\frac{1}{2}\right)^2 + 4\times\left(\frac{1}{2}\right)^3 + ----\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2}\right]^{-2} = \frac{1}{2}\times\left(\frac{1}{2}\right)^{-2} = \frac{1}{2}\times 2^2 = \frac{1}{2}\times 4 = 2$$

$E(X) = 2.$

$\therefore (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ---$

⑧ If $V(x)=5$ find $V(4x+3)$.

Sol: we know that $V(ax+b)=a^2V(x)$

$\Rightarrow V(4x+3)=4^2\times V(x)=16\times 5=80.$

⑧ If x and y are two independent r.v with variances 3 and 4. Find the variance of $3x+4y$.

Sol: Let $V(x)=3$ and $V(y)=4$

we know that $V(ax\pm by)=a^2V(x)\pm b^2V(y)$.

$\Rightarrow V(3x+4y)=3^2V(x)+4^2V(y)$

$=9\times 3+16\times 4=27+64=91$

$\Rightarrow V(3x+4y)=91$

⑧ Find the value of K and $p(x\geq 10)$ $p(x<5)$ and $p(5<x<10)$ when x is a continuous r.v and the pdf is

$$f(x)=Ke^{-\frac{x}{5}}, \quad x\geq 0$$
$$=0, \quad \text{otherwise.}$$

Sol: we know that $\int_{-\infty}^{\infty}f(x)\,dx=1 \Rightarrow$

$\int^{\infty}f(x)\,dx=1$.

Fir... distributio...

$$\int_0^\infty k e^{-\frac{x}{5}} dx = 1 \Rightarrow k \int_0^\infty e^{-x/5} dx = 1$$

$$= k\left[\frac{e^{-x/5}}{-1/5}\right]_0^\infty = 1 \Rightarrow -5k\left[e^{-\infty} - 1\right] = 1$$

$$= -5k[0-1] = 1 \Rightarrow 5k = 1 \Rightarrow k = \frac{1}{5}.$$

Now $p(x \geq 10) = \int_{10}^\infty f(x)\, dx = \int_{10}^\infty k e^{-\frac{x}{5}} dx$

$$= \frac{1}{5}\int_{10}^\infty e^{-x/5} dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-\frac{1}{5}}\right]_{10}^\infty = -\frac{1}{\cancel{5}}\times\frac{\cancel{5}}{1}\left[0 - e^{-2}\right]$$

$$= \frac{1}{e^2}. \Rightarrow \boxed{2<e<3}$$

$$p(x \geq 10) = \frac{1}{e^2}.$$

$$p(x<5) = \int_{-\infty}^5 f(x)\, dx = \int_0^5 f(x)\, dx = \int_0^5 k e^{-\frac{x}{5}} dx$$

$$= k\left[\frac{e^{-x/5}}{-\frac{1}{5}}\right]_0^5 = -\frac{1}{\cancel{5}}\times\frac{\cancel{5}}{1}\left[e^{-1} - 1\right] = -(e^{-1} - 1)$$

$$p(x<5) = (1 - e^{-1}) = \left(1 - \frac{1}{e}\right),$$ ~~2<e<3~~ $\boxed{2<e<3}$

$$p(5<x<10) = \int_5^{10} k e^{-x/5} dx = \frac{1}{5}\left[\frac{e^{-x/5}}{-\frac{1}{5}}\right]_5^{10}$$

$$p(5<x<10) = -\left[e^{-2} - e^{-1}\right] = e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2}$$

(8) Find mean, median, mode of a continuous distribution whose pdf is

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$
$$= 0, \quad \text{otherwise.}$$

Sol: We know that, mean $= E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$ when x is a continuous r.v and $f(x)$ is pdf.

(i) Thus, mean $= \mu_1' = E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$

$$E(x) = \int_0^1 x \cdot 6x(1-x)\,dx = \int_0^1 (6x^2 - 6x^3)\,dx$$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left[\frac{1}{3} - \frac{1}{4} - 0 + 0\right] = 6\left[\frac{4-3}{12}\right]$$

$$= \cancel{6} \times \frac{1}{\cancel{12}} = \frac{1}{2}.$$

$$\text{Mean} = \mu_1' = E(x) = \frac{1}{2}.$$

(ii) Let M is the median of the distribution. M divides the area under the curve into two equal parts, as the total area is unity, therefore we have

$$\int_{-\infty}^{M} f(x)\,dx = \int_{M}^{\infty} f(x)\,dx = \frac{1}{2} \quad \text{(or)} \quad \int_0^M f(x)\,dx = \int_M^1 f(x)\,dx = \frac{1}{2}$$

consider $\int_0^M f(x)\,dx = 1/2 \Rightarrow$

$$= \int_0^M 6x(1-x)\,dx = \frac{1}{2} \Rightarrow 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^M = 1/2$$

$$\frac{M^2}{2} - \frac{M^3}{3} = \frac{1}{12} \Rightarrow 6M^2 - 4M^3 = 1$$

(or) $4M^3 - 6M^2 + 1 = 0$ (By trail)

put $M = \frac{1}{2}$ (real root of M)

$$4\left(\frac{1}{8}\right) - 6\times\frac{1}{4} + 1 = \frac{1}{2} - \frac{3}{2} + 1 = 3/2 - 3/2 = 0$$

$\Rightarrow$ $M = \frac{1}{2}$ satisfy the equation $4M^3 - 6M^2 + 1 = 0$

$\Rightarrow$ $M = \frac{1}{2}$, that is

the median $= M = \frac{1}{2}$

(iii) <u>Mode</u>: is the value of 'x' for which $f(x)$ is maximum. By the concept of Max, Min, we have

$$f(x) = 6x(1-x) = 6x - 6x^2$$

$$f'(x) = 6 - 12x$$

$$f''(x) = -12$$

For max or min, $f'(x) = 0 \Rightarrow 6 - 12x = 0$

$x = \frac{1}{2}$

and $f''(x) = f''(\frac{1}{2}) = -12 < 0 \Rightarrow$

$f(x)$ is maximum at $x = \frac{1}{2}$ and thus

it is the mode of the distribution.

$$\text{Mode} = \frac{1}{2}$$

Note: Since mean = Median = mode = $\frac{1}{2}$,

i.e mean, median and mode are same, then the given distribution is symmetric.

(8) Find mean and variance for the distribution whose pmf is

$$f(x) = \frac{x}{15}, \quad x = 1, 2, 3, 4, 5.$$
$$= 0, \quad \text{otherwise}.$$

Sol: X is the discrete r.v with pmf

$$f(x) = \frac{x}{15}, \quad x = 1, 2, 3, 4, 5.$$

Therefore the probability distribution is

$X = x$:	1	2	3	4	5
$P(X = x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

(Repeated on page 23).

$E(2x+2)$ $V(2x+3)$

(8) pdf is $f(x) = \frac{x+2}{18}$, $-2<x<4$

$= 0$, otherwise,

Find the $E(x+2)$ and $V(2x+3)$.

Sol:

$f(x) = \frac{x+2}{18}$ $\quad -2<x<4$.

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-2}^{4} x \times \left(\frac{x+2}{18}\right) dx = \frac{1}{18}\int_{-2}^{4} (x^2+2x)\,dx$$

$$E(x) = \frac{1}{18}\left[\frac{x^3}{3} + x^2\right]_{-2}^{4} = \frac{1}{18}\left[\frac{64}{3} + 16 + \frac{8}{3} - 4\right]$$

$$= \frac{1}{18}\left[\frac{72}{3} + 12\right] = \frac{1}{18}\left[\frac{72+36}{3}\right] = \frac{\cancel{108}^{\cancel{36}\,2}}{\cancel{18}_1 \times \cancel{3}_1}$$

$E(x) = 2$

$$E(x^2) = \int_{-2}^{4} x^2 f(x)\,dx = \int_{-2}^{4} \frac{x^2(x+2)}{18}\,dx = \frac{1}{18}\int_{-2}^{4} (x^3+2x^2)\,dx$$

$$= \frac{1}{18}\left[\frac{x^4}{4} + \frac{2x^3}{3}\right]_{-2}^{4} = \frac{1}{18}\left[\frac{256}{4} + \frac{128}{3} - \frac{16}{4} + \frac{16}{3}\right]$$

$$= \frac{1}{18}\left[\frac{240}{4} + \frac{144}{3}\right] = \frac{1}{18}\left[60+48\right] = \frac{1}{\cancel{18}_{\cancel{2}}}\left[\cancel{108}^{\cancel{12}\,6}\right]$$

$E(x^2) = 6$.

$V(x) = E(x^2) - [E(x)]^2 = 6 - (2)^2 = 6 - 4 = 2$

$\therefore E(x+2) = E(x)+2 = 2+2 = 4$

$V(2x+3) = 4V(x) = 4\times 2 = 8.$

(8) Find the first four non-central moments about origin when the pdf is

$$f(x) = e^{-x}, \quad x>0$$
$$= 0, \quad \text{otherwise.}$$

Sol: The pdf of X is $f(x) = e^{-x}, \quad 0<x<\infty.$
$= 0, \quad$ otherwise

Now the rth moment about origin is

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^{\infty} x^r f(x)\,dx. \quad r = 1,2,3,4.$$

$$\mu_1' = \int_0^{\infty} x e^{-x}\,dx = \left(-xe^{-x} - e^{-x}\right)_0^{\infty} = -\left(xe^{-x} + e^{-x}\right)_0^{\infty}$$

$$\mu_1' = -\left(0+0-0-1\right) = 1$$

$$\mu_2' = \int_0^{\infty} x^2 e^{-x}\,dx = \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^{\infty}$$

$$\mu_2' = -\left[0-2\right] = 2$$

$$\mu_3' = \int_0^{\infty} x^3 e^{-x}\,dx = \left[-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$\mu_3' = 6.$$

Consider

$$\mu_4' = \int_0^\infty x^4 e^{-x}\,dx$$

$$= \left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^\infty$$

$\mu_4' = 24$ Thus, the first four noncentral moments about origin are

$\mu_1' = 1,\ \mu_2' = 2,\ \mu_3' = 6,\ \mu_4' = 24.$

(8) Find the mean and variance when the pdf is

$$f(x) = x, \quad 0<x<1$$
$$= 2-x, \quad 1<x<2$$
$$= 0, \quad \text{otherwise}.$$

Sol: The mean $= E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^2 x f(x)\,dx$

$$= \int_0^1 x f(x)\,dx + \int_1^2 x f(x)\,dx = \int_0^1 x \times x\,dx + \int_1^2 x\times(2-x)\,dx$$

$$= \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx = \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right)_1^2$$

$$= \left(\frac{1}{3} - 0\right) + \left(4 - \frac{8}{3} - 1 + \frac{1}{3}\right) = 3 + \frac{1}{3} + \frac{1}{3} - \frac{8}{3} = 3-2$$

$= 1 \implies$ Mean $= E(X) = 1$

)8.

Consider $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_{1}^{2} x^2 f(x)\,dx$

$= \int_0^1 x^2 \times x\,dx + \int_1^2 x^2(2-x)\,dx = \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3)\,dx$

$= \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_1^2 = \left(\frac{1}{4} - 0\right) + \left(\frac{16}{3} - \frac{16}{4} - \frac{2}{3} + \frac{1}{4}\right)$

$= \frac{1}{4} - \frac{16}{4} + \frac{1}{4} + \frac{16}{3} - \frac{2}{3} = \frac{-14}{4} + \frac{14}{3} = 14\left(\frac{-3+4}{12}\right)$

$= \frac{\cancel{14}^{7}}{\cancel{12}_{6}} = 7/6 \Rightarrow E(x^2) = \frac{7}{6}$.

$\Rightarrow V(x) = E(x^2) - [E(x)]^2 = \frac{7}{6} - (1)^2 = \frac{7}{6} - 1 = \frac{1}{6}$.

Thus $E(x) =$ mean $= 1$

variance $= V(x) = \frac{1}{6}$.

---

(8) The pdf of a continuous distribution is

$f(x) = Ke^{-b(x-a)}$, $a \le x < \infty$ where a, b and k are the constants. If m and σ are the mean and standard deviation of the distribution find k, a and b.

(36)

Sol: $f(x) = k e^{-b(x-a)}$

Total probability $= 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{a}^{\infty} k e^{-b(x-a)}\,dx = 1 \Rightarrow$$

$$\Rightarrow k\left[\frac{e^{-b(x-a)}}{-b}\right]_a^{\infty} = -\frac{k}{b}\left[e^{-\infty} - e^{0}\right] = -\frac{k}{b}[0-1] = 1$$

$$\frac{k}{b} = 1 \Rightarrow \boxed{k = b}$$

$$E(x) = \int_a^{\infty} x f(x)\,dx = \int_a^{\infty} x k e^{-b(x-a)}\,dx$$

$$= k\left[\frac{x e^{-b(x-a)}}{-b} - \frac{e^{-b(x-a)}}{b^2}\right]_a^{\infty}$$

$$= k\left[\frac{a}{b} + \frac{1}{b^2}\right] = b\left[\frac{a}{b} + \frac{1}{b^2}\right] = a + \frac{1}{b}.$$

$$\boxed{E(x) = \text{mean} = m = a + \frac{1}{b}}$$

$$E(x^2) = k\int_a^{\infty} x^2 e^{-b(x-a)}\,dx$$

$$= k\left[x^2\frac{e^{-b(x-a)}}{-b} - 2x\frac{e^{-b(x-a)}}{(-b)^2} + 2\frac{e^{-b(x-a)}}{(-b)^3}\right]_a^{\infty}$$

$\therefore E(x^2) = k\left[\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3}\right] = b\left[\frac{a^2}{b} + \frac{2a}{b^2} + \frac{2}{b^3}\right]$

$= a^2 + 2a/b + \frac{2}{b^2} = a^2 + \frac{2a}{b} + \frac{2}{b^2}.$

$V(x) = E(x^2) - [E(x)]^2 = a^2 + \frac{2a}{b} + \frac{2}{b^2} - (a + \frac{1}{b})^2$

$= \cancel{a^2} + \cancel{\frac{2a}{b}} + \frac{2}{b^2} - \cancel{a^2} - \frac{1}{b^2} - \cancel{\frac{2a}{b}}.$

$= \frac{2}{b^2} - \frac{1}{b^2} = \frac{1}{b^2} \Rightarrow V(x) = \frac{1}{b^2}$

$S.D = \sqrt{V(x)} = \frac{1}{b}$. but mean $= m$, $S.D = \sigma$

$\Rightarrow m = a + \frac{1}{b}$, $\sigma = \frac{1}{b}$. $\Rightarrow m = a + \sigma$

$\Rightarrow \boxed{a = m - \sigma}$; $\sigma = \frac{1}{b} \Rightarrow \boxed{b = \frac{1}{\sigma}}$

$k = b = \frac{1}{\sigma}.$

Thus, $k = \frac{1}{\sigma}$, $b = \frac{1}{\sigma}$, $a = m - \sigma$.

(8) Find the mean, median and mode of the distribution whose pdf is $f(x) = \frac{1}{2}\sin x$, $0 \le x \le \pi$.

Ans: Mean $= \pi/2$, Median $= M = \frac{\pi}{2}$, Mode $= \frac{\pi}{2}$

Distribution is symmetric.

(R) Find skewness and kurtosis of the distribution whose pdf is $f(x) = e^{-x}$, $0 < x < \infty$.

<u>Sol:</u> The pdf is $f(x) = e^{-x}$, $0 < x < \infty$, and

$$\mu'_r = r\text{th moment about origin} = E(x^r) = \int_0^\infty x^r e^{-x}\,dx$$

put $r = 1, 2, 3, 4$ we get [See pages 33 and 34]

$\mu'_1 = 1$, $\mu'_2 = 2$, $\mu'_3 = 6$, $\mu'_4 = 24$, also we know

$$\mu_1 = 0, \quad \mu_2 = \mu'_2 - (\mu'_1)^2 = 2 - (1)^2 = 2 - 1 = 1$$

$$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = 6 - 3\times2\times1 + 2\times(1)^3 = 2.$$

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$$

$$= 24 - 4\times6\times1 + 6\times2\times(1)^2 - 3(1)^4 = 24 - 24 + 12 - 3 = 12 - 3 = 9$$

Thus the central moments are

$\mu_1 = 0$, $\mu_2 = 1$, $\mu_3 = 2$, $\mu_4 = 9$, thus

$$\text{Skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(2)^2}{(1)^3} = \frac{4}{1} = 4.$$

$$\text{Kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{9}{(1)^2} = \frac{9}{1} = \frac{9}{1} = 9.$$

Skewness = 4

Kurtosis = 9

e dist —

$0<x<\infty$.

(8) The pdf of a continuous r.v is $f(x)=kx(2-x)$ in the interval $0\le x\le 2$. Find mean, variance β_1 and β_2.

Sol: $f(x)=kx(2-x)$, $0\le x\le 2$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^2 kx(2-x) = 1 \Rightarrow k\left[x^2-\frac{x^3}{3}\right]_0^2 = 1$$

$$k\left[4-\frac{8}{3}\right]=1 \Rightarrow \frac{4}{3}k=1 \Rightarrow k=\frac{3}{4}.$$

$$f(x)=\tfrac{3}{4}x(2-x),\quad 0\le x\le 2.$$

$$\mu_r' = \int_0^2 x^r f(x)\,dx = \frac{3}{4}\int_0^2 x^r . x(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^{r+1}-x^{r+2})\,dx$$

$$= \frac{3}{4}\left[\frac{2x^{r+2}}{r+2}-\frac{x^{r+3}}{r+3}\right]_0^2 = \frac{3}{4}\left[\frac{2^{r+3}}{r+2}-\frac{2^{r+3}}{r+3}\right]$$

$$= \frac{3}{4}.2^{r+3}\left[\frac{1}{r+2}-\frac{1}{r+3}\right] = \frac{3}{4}2^{r+3}.\frac{[(r+3)-(r+2)]}{(r+2)(r+3)}$$

$$= \frac{3}{4}.2^{r+3}.\frac{1}{(r+2)(r+3)} = \frac{3.2^{r+1}}{(r+2)(r+3)}.$$

$$\mu_r' = \frac{3.2^{r+1}}{(r+2)(r+3)},\quad r=1,2,3,4$$

$$\mu_1'=1,\quad \mu_2'=\frac{6}{5},\quad \mu_3'=\frac{8}{5},\quad \mu_4'=\frac{16}{17},\ \frac{16}{7}$$

Mean $= \mu'_1 = 1$.

$\mu_2 = $ Variance $= V(x) = \mu'_2 - (\mu'_1)^2 = \frac{6}{5} - 1 = \frac{1}{5}$

mean $= 1$, variance $= \frac{1}{5}$.

$\mu_1 = 0$, $\mu_2 = \frac{1}{5}$,

$\mu_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2(\mu'_1)^3 = \frac{8}{5} - 3 \times \frac{6}{5} \times 1 + 2 \times (1)^3$

$\mu_3 = \frac{8}{5} - \frac{18}{5} + 2 = \frac{8 - 18 + 10}{5} = \frac{18-18}{5} = 0$

$\mu_3 = 0$.

$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2(\mu'_1)^2 - 3(\mu'_1)^4$

$= \frac{16}{17} - 4 \times \frac{8}{5} \times 1 + 6 \times \frac{6}{5} \times (1)^2 - 3(1)^4 \cancel{= \frac{16}{17} - \frac{32}{5} + \frac{36}{5} - 3}$

$\mu_4 = \frac{16}{17} - 3 + \frac{4}{5} = \frac{80 - 15 \times 17 + 4 \times 17}{17 \times 5} = \cancel{\frac{80 - 17 \times 11}{17 \times 5}}$

$\mu_4 = \frac{3}{35}$

skewness $= \beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\mu_3^2}{\mu_2^3} = \frac{0}{(\frac{1}{5})^3} = 0$

kurtosis $= \beta_2 = \frac{\mu_4}{\mu_2^2} =$

(8) prove that the function is pdf. Find mean [Mean $= 0$].

$$f(x) = \begin{cases} \frac{(3+x)^2}{16}, & -3 \le x \le -1 \\ \frac{6-2x^2}{16}, & -1 \le x \le 1 \\ \frac{(3-x)^2}{16}, & 1 \le x \le 3 \end{cases}$$

Unit I

Relation between moments about mean interms of moments about any point A

We know that the moments about mean are the central moments and rth central moment is denoted by μ_r while moments about any point $x = A$ or $x = 0$ are called the raw or non-central moments denoted by μ'_r for, $r = 1, 2, 3, 4 \ldots$

Now we derive the relation between them.

Consider

$$\mu_r = \frac{1}{N} \sum f_i (x_i - \bar{x})^r$$ where $\bar{x}$ is the mean

and f_i is the corresponding frequency at $x = x_i$

and $N = \sum f_i$ = total frequency. Thus

$$\Rightarrow \mu_r = \frac{1}{N} \sum f_i (x_i - \bar{x})^r$$

$$= \frac{1}{N} \sum f_i (x_i - A + A - \bar{x})^r$$

$$= \frac{1}{N} \sum f_i \left[(x_i - A) - (\bar{x} - A) \right]^r$$

let $\bar{x} - A = \mu'_1$, $d_i = x_i - A$

Note

$$\mu_r = \frac{1}{N}\sum f_i\left[d_i - \mu_1'\right]^r$$

By using Binomial expansion, we have

$$\mu_r = \frac{1}{N}\sum f_i\left[{}^rc_0 d_i^r - {}^rc_1 d_i^{r-1}\mu_1' + {}^rc_2 d_i^{r-2}(\mu_1')^2 - \cdots\right]$$

$$= \frac{1}{N}\sum f_i d_i^r - {}^rc_1 \frac{1}{N}\sum f_i d_i^{r-1}\mu_1' +$$

$$+ {}^rc_2 \sum f_i d_i^{r-2}(\mu_1')^2 - \cdots$$

$$= \frac{1}{N}\sum f_i (x_i - A)^r - {}^rc_1 \frac{1}{N}\sum f_i (x_i - A)^{r-1}.\mu_1' + \cdots$$

$$\mu_r = \mu_r' - {}^rc_1 \mu_{r-1}'\mu_1' + {}^rc_2 \mu_{r-2}'(\mu_1')^2 - \cdots + (-1)^r(\mu_1')^r$$

is the relation between central Moments μ_r and non-central moments μ_r'. Taking $r = 2, 3 \ldots$ we get

$\mu_1 = 0$ (always, we know)

$$\mu_2 = \mu_2' - 2c_1\mu_1'.\mu_1' + 2c_2\mu_0'(\mu_1')^2 = \mu_2' - 2\mu_1'^2 + \mu_1'^2 =$$

$\mu_2 = \mu_2' - (\mu_1')^2$ similarly, we have

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4.$$

Note: The moments about mean (central moments) are independent of change of origin but not of scale.

$$\text{Skewness} = \beta_1 = \frac{\mu_3^2}{\mu_2^3}.$$

$$\text{Kurtosis} = \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

(8) The first four moments of a distribution about the point $x=5$ are 2, 20, 40 and 50. Find the moments about mean. Also find the moments about origin.

Sol: Given that $x=A=5$ $\mu_1'=2$, $\mu_2'=20$, $\mu_3'=40$, $\mu_4'=50$. Now the moments about mean are

$\mu_1=0$,

$\mu_2=\mu_2'-(\mu_1')^2=16$

$\mu_3=\mu_3'-3\mu_2'\mu_1'+2(\mu_1')^3=-64$,

$\mu_4=\mu_4'-4\mu_3'\mu_1'+6\mu_2'(\mu_1')^2-3(\mu_1')^4=162$.

Also we know that $\bar{x}=A+\mu_1'=5+2=7$

Therefore mean $=\mu_1'=7$ (about origin)

$\mu_2 = \mu_2' - (\mu_1')^2 \Rightarrow 16 = \mu_2' - 49 \Rightarrow \mu_2' = 65.$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 \Rightarrow -64 = \mu_3' - 3\times 65\times 7 + 2\times 343$

$\Rightarrow \mu_3' = 615.$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$162 =$ ______ do ______

$\Rightarrow \mu_4' = 5475.$

Thus, the central moments are

$\mu_1 = 0, \ \mu_2 = 16, \ \mu_3 = -64, \ \mu_4 = 162$ and

the moments about origin (Non-central moments) are

$\mu_1' = 7, \ \mu_2' = 65, \ \mu_3' = 615, \ \mu_4' = 5475.$

(*) First four moments about $x = 1$ are 5, 20, 40, 120. Find the first four central moments.

[Ans : $\mu_1 = 0, \ \mu_2 = -5, \ \mu_3 = -10, \ \mu_4 = 445$]

(*) The first four moments about origin are 1, 5, 15, 30. Find the corresponding four moments about mean.

[Ans; $\mu_1 = 0$, $\mu_2 = 4$, $\mu_3 = 2$, $\mu_4 = -3$]

Moment Generating Function: let x is any r.v and $f(x)$ is its probability function. Then, the moment generating function (MGF) is denoted and defined as

$$MGF = M_x(t) = E(e^{tx})$$ where t is the real parameter. we know that the Maclaurin's expansion of e^x is

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{------} \frac{x^n}{\lfloor n} + \text{---}$$

Consider

$$E(e^{tx}) = E\left[1 + (tx) + \frac{(tx)^2}{\lfloor 2} + \frac{(tx)^3}{\lfloor 3} + \text{---} \frac{(tx)^n}{\lfloor n} + \text{---}\right]$$

$$= E(1) + t\,E(x) + \frac{t^2}{\lfloor 2} E(x^2) + \text{---} \frac{t^n}{\lfloor n} \cdot E(x^n) + \text{---}$$

$$E(e^{tx}) = 1 + t.\mu_1' + \frac{t^2}{\lfloor 2}\mu_2' + \text{---} \frac{t^n}{\lfloor n} \cdot \mu_n' + \text{------} = \sum_{n=0}^{\infty} \frac{t^n}{\lfloor n} \mu_n'$$

and this equation generates the

moments, $\mu_1', \mu_2', \mu_3' --- \mu_n' ---$ thus $E(e^{tx})$ is the MGF for any distribution.

(I) when x is the discrete r.v, then

$$MGF = M_x(t) = E(e^{tx}) = \sum e^{tx} \cdot p(X=x)$$

where $p(X=x)$ is the pmf.

(II) when X is a continuous r.v, then

$$MGF = M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

where $f(x)$ is the pdf.

(*) Find the first four moments about origin for the distribution whose pdf is

$$f(x) = \frac{1}{2}(x+1), \quad -1<x<1$$
$$= 0 \quad , \text{ otherwise.}$$

Sol: The pdf is $f(x) = \frac{1}{2}(x+1)$, $-1<x<1$, then

$$MGF = M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-1}^{1} e^{tx} \times \frac{1}{2}(x+1)\,dx = \frac{1}{2}\int_{-1}^{1} (x+1)e^{tx}\,dx$$

$$M_X(t) = \frac{1}{2}\left[\frac{(x+1)e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{2e^{t}}{t} - \frac{e^{t}}{t^2} + \frac{e^{-t}}{t^2}\right]$$

$$= \frac{1}{2}\times 2\frac{e^{t}}{t} - \frac{1}{2t^2}\left(e^{t} - e^{-t}\right)$$

$$= \frac{e^{t}}{t} - \frac{1}{2t^2}\left[\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right)-\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}\cdots\right)\right]$$

$$= \frac{1}{t}\left[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots\right] - \frac{1}{2t^2}\left[\frac{2t}{1!}+\frac{2t^3}{3!}+\frac{2t^5}{5!}+\frac{2t^7}{7!}+\cdots\right]$$

$$= \left[\cancel{\frac{1}{t}}+1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}\cdots\right] - \left[\cancel{\frac{1}{t}}+\frac{t}{3!}+\frac{t^3}{5!}+\frac{t^5}{7!}+\cdots\right]$$

$$= 1+\left(\frac{1}{2!}-\frac{1}{3!}\right)t+\frac{t^2}{3!}+\left(\frac{1}{4!}-\frac{1}{5!}\right)t^3+\frac{t^4}{5!}+\cdots$$

$$= 1+\frac{1}{3}t+\frac{1}{6}t^2+\frac{1}{30}t^3+\frac{1}{120}t^4+\cdots$$

$$= 1+\frac{1}{3}t+\frac{1}{3}\cdot\frac{1}{2!}t^2+\frac{1}{5}\times\frac{1}{3!}t^3+\frac{1}{5}\frac{t^4}{4!}+\cdots$$

$\Rightarrow$ $\mu_1' = \frac{1}{3}$ = coeff. of t

$\mu_2' = \frac{1}{3}$ = " " $\frac{t^2}{2!}$

$\mu_3' = \frac{1}{5}$ = " " $\frac{t^3}{3!}$

$\mu_4' = \frac{1}{5}$ = " " $\frac{t^4}{4!}$.

We know that the moments about origin are

$E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \mu'_r$, that is

~~coeff of~~ coeff of $\frac{t^r}{\lfloor r} = \mu'_r = r$th raw moment.

(7) Find the mgf if $\mu'_r = (r+1)2^r$.

Sol: The rth moment is $\mu'_r = (r+1)2^r$ and

$$MGF = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{\lfloor r} \cdot \mu'_r = \sum_{r=0}^{\infty} \frac{(r+1)2^r \cdot t^r}{\lfloor r}$$

$$= \sum_{r=0}^{\infty} \frac{(r+1)(2t)^r}{\lfloor r}$$

$$= 1 + 2\frac{(2t)}{\lfloor 1} + 3 \cdot \frac{(2t)^2}{\lfloor 2} + 4 \cdot \frac{(2t)^3}{\lfloor 3} + 5 \cdot \frac{(2t)^4}{\lfloor 4} + \cdots$$

$$= 1 + \frac{2 \cdot 2^1 \times t}{\lfloor 1} + \frac{3 \cdot 2^2 \cdot t^2}{\lfloor 2} + \frac{4 \cdot 2^3 \cdot t^3}{\lfloor 3} + \frac{5 \cdot 2^4 \cdot t^4}{\lfloor 4} + \cdots$$

$$MGF = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

$$M_x(t) = \frac{1}{(1-2t)^2}$$

$\therefore \left[(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \right]$

(8) Find the mgf if pmf is

$$f(x) = \frac{1}{2^x}, \quad x = 1, 2, 3, 4 \ldots$$

Also find mean and variance.

Sol: $f(x) = \frac{1}{2^x} \Rightarrow MGF = E(e^{tx})$

$$E(e^{tx}) = M_x(t) = \sum_{x=1}^{\infty} e^{tx} \cdot f(x) = \sum e^{tx} \times \frac{1}{2^x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^t}{2}\right)^x = 1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots - 1$$

$$= \left(1 - \frac{e^t}{2}\right)^{-1} = \frac{1}{1 - \frac{e^t}{2}} - 1 = \frac{1}{\frac{2 - e^t}{2}} - 1$$

$$M_x(t) = \frac{2}{2 - e^t} - 1 = \frac{e^t}{2 - e^t}.$$

Also $\mu_r' = \left\{ \frac{d^r}{dt^r} [M_x(t)] \right\}_{t=0}$.

$$\mu_1' = \frac{d}{dt}\left[\frac{e^t}{2 - e^t}\right]_{t=0} = 2 \times -1 \times \frac{1}{(2 - e^t)^2} \times -e^t = \frac{2e^t}{2 - e^t}.$$

$$\mu_1' = \left[\frac{2\cancel{e^t}}{(2 - e^t)^2}\right]_{t=0} = \frac{2 \times e^0}{(2 - e^0)^2} = \frac{2 \times 1}{2 - 1} = \frac{2}{1} = 2$$

$\mu_1' = 2$

$$\mu_2' = \frac{d^2}{dt^2}\left[\frac{2}{(2 - e^t)}\right]_{t=0} = 2\frac{d}{dt}\left[\frac{\cancel{e^t}}{(2 - e^t)^2}\right]_{t=0} = \ldots$$

$$= \left[\frac{4e^t}{(2 - e^t)^3}\right] \ldots = 4.$$

$V(x) = \mu_2' - (\mu_1')^2 = 4 - (2)^2 = 0$.

$\mu_1' = \text{mean} = E(x) = 2$

$\mu_2 = V(x) = 0$.

(8) Find MGF and first four moments about origin for the pdf $f(x) = 2e^{-2x}$, $x \geq 0$

$= 0$, otherwise.

Sol: We know that for a continuous distribution the MGF is

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$M_x(t) = \int_0^{\infty} e^{tx} \times 2e^{-2x}\, dx = 2\int_0^{\infty} e^{(t-2)x}\, dx$$

$$= 2\int_0^{\infty} e^{-(2-t)x}\, dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$= \frac{2}{2-t} \times [0-1] = \frac{2}{2-t} = \frac{\cancel{2}}{\cancel{2}(1-\frac{t}{2})}$$

$$M_x(t) = (1-\frac{t}{2})^{-1}$$

$$M_x(t) = E(e^{tx}) = \sum_{r=0}^{\infty} \frac{t^r}{\underline{r}} \acute{\mu}_r .$$

$$= \left(1-\frac{t}{2}\right)^{-1} = \left[1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \cdots\right]$$

$$= 1 + \frac{1}{2}\cdot\frac{t}{\underline{1}} + \frac{t^2}{\underline{2}} \times \frac{1}{2} + \frac{3t^3}{\underline{3} \times 4} + \frac{3\, t^4}{\underline{4} \times 2} + \cdots$$

$$M_x(t) = 1 + \frac{t}{\underline{1}} \times \acute{\mu}_1 + \frac{t^2}{\underline{2}} \times \acute{\mu}_2 + \frac{t^3}{\underline{3}} \times \acute{\mu}_3 + \frac{t^4}{\underline{4}} \times \acute{\mu}_4 + \cdots$$

$$\Rightarrow \quad \acute{\mu}_1 = \frac{1}{2}, \quad \acute{\mu}_2 = \frac{1}{2}, \quad \acute{\mu}_3 = 3/4, \quad \acute{\mu}_4 = 3/2$$

(Q) Find the MGF whose rth non-central moment is $\acute{\mu}_r = (r+1) \times 3^r$.

$$\left[\text{Ans : MGF} = M_x(t) = \frac{1}{(1-3t)^2} = (1-3t)^{-2}\right]$$

(Q) MGF of any r.v X is $M_x(t)$, then prove that if $Y = ax+b$, where a, b are non-zero constants, then $M_y(t) = e^{bt} M_x(at)$.

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Sol: X is any r.v then

MGF $= M_x(t) = E(e^{tx})$.

Consider: $Y = aX + b$, then

$$M_Y(t) = E\left[e^{(ax+b)t}\right] = E\left[e^{axt+bt}\right]$$

$$= E\left[e^{atx} \cdot e^{bt}\right] = e^{bt} E\left[e^{atx}\right]$$

$$= e^{bt} E\left[e^{(at)x}\right]$$

$$= e^{bt} \cdot M_x(at).$$

(*) If X and Y are independent r.v and $Z = X + Y$, then $M_Z(t) = M_X(t) \cdot M_Y(t)$.

Sol: Consider that X and Y are any two ~~Indep~~ independent r.v and $Z = X + Y$ thus

$$M_X(t) = E(e^{tX}) \quad M_Y(t) = E(e^{tY}).$$

Consider

$$M_Z(t) = E(e^{tZ}) = E\left(e^{t(X+Y)}\right)$$

$$= E\left[e^{tX+tY}\right] = E\left[e^{tX} \cdot e^{tY}\right]$$

$$= E(e^{tX}) \times E(e^{tY}) = M_X(t) \cdot M_Y(t)$$

Thus $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

(*) If $M_X(t) = \frac{1}{(1-t)}$, $t<1$, then find the MGF of $Y = 2X+1$

Sol: we know that if $M_X(t)$ B the MGF of X and it is

$$M_X(t) = E(e^{tX})$$

Consider

$$M_Y(t) = E\left[e^{tY}\right] = E\left[e^{(2X+1)t}\right]$$

$$= E\left[e^{2Xt+t}\right] = E\left[e^{2tX} \cdot e^{t}\right]$$

$$= e^t E(e^{2tX}) = e^t \times M_X(2t)$$

but $M_X(t) = \frac{1}{1-t} \Rightarrow M_X(2t) = \frac{1}{1-2t} \Rightarrow$

$$M_Y(t) = \frac{e^t}{1-2t}.$$

(8) If $M_x(t) = \left(\frac{1}{5} + \frac{4}{5}e^t\right)^{15}$, find the MGF of $Y = 2x+3$.

Sol: we know that $M_x(t) = E(e^{tx})$

and $Y = ax+b$ then

$$M_Y(t) = e^{bt} M_x(at) \qquad a=2,\ b=3$$

$$M_Y(t) = e^{3t} \cdot M_x(2t)$$

$$= e^{3t} \cdot \left(\frac{1}{5} + \frac{4}{5}e^{2t}\right)^{15}$$

Baye's theorem: If $E_1, E_2, E_3 \ldots E_n$ are n-mutually exclusive events with $p(E_i) \neq 0$ for $i = 1, 2, 3 \ldots n$. for any arbitrary event A which is subset of $\bigcup_{i=1}^{n} E_i$ such that $p(A) > 0$, then

$$p(E_i/A) = \frac{p(E_i) \cdot p(A/E_i)}{\sum_{i=1}^{n} p(E_i) \cdot p(A/E_i)}.$$

proof: from conditional probability and

multiplication theorem of probability, we know that

$$p(E_i/A) = \frac{p(E_i \cap A)}{p(A)} \quad \text{--- } (1)$$

and

$$p(A \cap E_i) = p(E_i) \cdot p(A/E_i) \quad \text{--- } (2)$$

It is given that A is subset of $\bigcup_{i=1}^{n} E_i$, i.e., $A \subset \bigcup_{i=1}^{n} E_i$, then A can be written as

$$A = A \cap \left[\bigcup_{i=1}^{n} E_i\right]$$

$$A = \bigcup_{i=1}^{n} [A \cap E_i] \quad \text{--- } (3)$$

Since $E_1, E_2, E_3 \ldots E_n$ are mutually exclusive, then $A \cap E_1, A \cap E_2, \ldots A \cap E_n$ are also mutually exclusive and hence from eq (3) we have

$$p(A) = p\left[\bigcup_{i=1}^{n} (A \cap E_i)\right]$$

$$= \sum_{i=1}^{n} p(A \cap E_i)$$

By using eq (2) we have

$$P(A) = \sum_{i=1}^{n} P(E_i) \cdot P(A/E_i) \quad \text{——— (4)}$$

Equation (1) is

$$P(E_i/A) = \frac{P(E_i \cap A)}{P(A)} \quad \text{——— (1)}$$

By using equations (2) and (4) in equation (1) we have

$$P(E_i/A) = \frac{P(E_i) \cdot P(A/E_i)}{\sum_{i=1}^{n} P(E_i) \cdot P(A/E_i)}.$$

(Q) In a smoking population 70% are men and 30% are women. 10% men and 20% of women smoke a particular brand. What is the probability that a person seen smoking that brand is a man.

Sol: Let E_1 is the event that man smokes

E_2 " " " " woman smokes

then $P(E_1) = \frac{70}{100} = \frac{7}{10}$, $P(E_2) = \frac{3}{10}$.

let A represents the event that the brand is being smoken, then

$p(A/E_1) = \frac{10}{100} = \frac{1}{10}$, $p(A/E_2) = \frac{20}{100} = \frac{2}{10}$.

then

$$p(E_1/A) = \frac{p(E_1) \times p(A/E_1)}{p(E_1).p(A/E_1) + p(E_2).p(A/E_2)}$$

$$p(E_1/A) = \frac{\frac{7}{10} \cdot \frac{1}{10}}{\frac{7}{10} \cdot \frac{1}{10} + \frac{3}{10} \cdot \frac{2}{10}} = \frac{\frac{7}{100}}{\frac{7}{100} + \frac{6}{100}} = \frac{\frac{7}{100}}{\frac{13}{100}} = \frac{7}{13}.$$

the probability that the person seen moking that brand is man is

$$p(E_1/A) = \frac{7}{13}.$$

(8) the chances of x, y, z becoming manager of a certain company are 4 : 2 : 3. The probabilit that the bonus scheme will be introduced if x, y, z becomes manager are 0.3, 0.5 and 0.8 respectively. If the bonus scheme has

been introduced, what is the probability that x is appointed as manager.

Sol: Let E_1, E_2, E_3 are the events that x, y, z becomes manager, then

$$P(E_1) = \frac{4}{4+2+3} = \frac{4}{9}, \quad P(E_2) = \frac{2}{9}, \quad P(E_3) = 3/9$$

Let A is the event that bonus is included, then

$$P(A/E_1) = 0.3 = \frac{3}{10}, \quad P(A/E_2) = \frac{5}{10}, \quad P(A/E_3) = \frac{8}{10}.$$

$\Rightarrow$

$$P(E_1/A) = \frac{P(E_1) \cdot P(A/E_1)}{\sum_{i=1}^{3} P(E_i) \cdot P(A/E_i)}$$

$$= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

$$= \frac{\frac{4}{9} \times \frac{3}{10}}{\frac{4}{9} \times \frac{3}{10} + \frac{2}{9} \times \frac{5}{10} + 3/9 \times \frac{8}{10}} = \frac{6}{23}.$$

probability of x is appointed as manager and bonus scheme is included is $\frac{6}{23}$.