Game Theory

Game theory deals with situations where there are conflict of interest between persons or opposing parties.

Ex:- Launching advertisement campaigns for competing products or planning war strategies for opposing armies.

In Game theory, two opponents are called players who may have finite or infinite strategies or alternatives. Associated with each player is the payoff that one player pays to the other with respect to the each pair of strategies. These are two person zero sum games because the gain of one player is equal to the loss of the other. The game is summarized in terms of the pay-off to one player. Two players with 'm' and 'n' strategies respectively, the pay-off matrix will be of the form,

	B_1	B_2	B_n
A_1	a_{11}	a_{12}	a_{1n}
A_2	a_{21}	a_{22}	a_{2n}
$\vdots$	$\vdots$		
A_m	a_{m1}	a_{m2}	a_{mn}

if A uses strategy 'i' & B uses strategy 'j' then pay-off to the player A is a_{ij} & the pay-off to the player B is $-a_{ij}$ (gain of one is the loss of the other)

Optimal Solution

Since two person zero sum games are based on conflict of interest optimal solution selects the strategies for each player such that any change in strategies will not improve the pay-off for either of the players.

Q-

	B_1	B_2	B_3	B_4
A_1	8	-2	9	-3
A_2	6	5	6	8
A_3	-2	4	-9	5

Find the strategies that A & B should adopt and also the value of the ~~gain~~ game

Soln Pay off matrix for A.

	B_1	B_2	B_3	B_4	Row Min
A_1	8	-2	9	-3	-3
A_2	6	5	6	8	5 → max from min
A_3	-2	4	-9	5	-9
Col. Max	8	5	9	8	

↳ min from max

Value of game = 5

If company A selects strategy A_1 regardless of what B does, the worst scenario is that A losses 3% of the market share to B

company B's strategy, the payoff matrix is for A, so if it adopts strategy B_1 it loses 8% of the market share to A. Similarly for B_2 B_3 B_4

The saddle point solution is A_2 and B_2 which means that neither of the two would be willing to change the strategy.

Q-

	B_1	B_2	B_3	B_4	Row min
A_1	4	-4	-5	6	-5
A_2	-3	-4	-9	-2	-9
A_3	6	7	-8	-9	-9
A_4	7	3	-9	5	-9
Col max.	7	7	-5	6	

min-max → Best of the worst

max-min

pay off matrix for A
Find the saddle point

Sol: A_1 & B_3

value of the game is -5.

Q- pay off matrix for A

	B_1	B_2	B_3	B_4	min
A_1	1	9	6	0	0
A_2	2	3	8	4	2
A_3	-5	-2	-10	-3	-10
A_4	7	4	-2	-5	-5
max	7	9	8	4	

saddle point A_2 & B_4

$2 \le \text{value} \le 4.$

Q- Determine the values of P & Q that will make entry (2, 2) of each game a saddle point
Pay off for A:

	B_1	B_2	B_3
A_1	1	q	6
A_2	P	5	10
A_3	6	2	13

Sol^n $p \geq 5$ $q \le 5$

Mixed strategies

Q- Consider the following 2×4 game. The payoff is for player A. Find the strategies that A & B should select and also find the value of the game.

$y_1 + y_2 + y_3 + y_4 = 1$

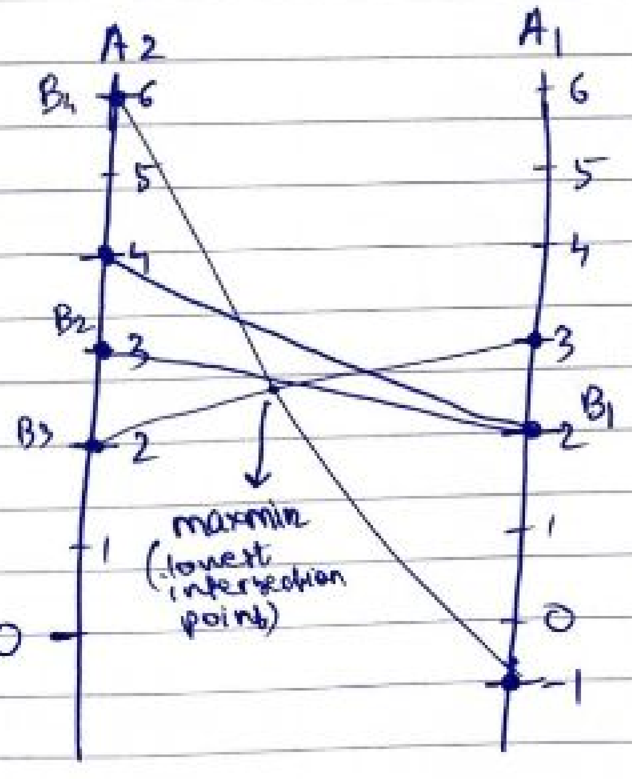
		y_1	y_2	y_3	y_4	
		B_1	B_2	B_3	B_4	
x_1:	A_1	2	2	3	-1	-1
$1-x_1$:	A_2	4	3	2	6	2
		4	3	3	6	

probabilities

A select A_1 with prob^{ty} x_1 & A_2 with the remaining prob^{ty} with $1-x_1$. B selects B_1 with prob^{ty} y_1, B_2 " " y_2, B_3 " " y_3, B_4 " " y_4

Sol^n Since there is no pure strategy or saddle point solution available, therefore it is a case of mixed strategies.

B's strategy	A's expected payoff
1	$(2-4)x_1 + 4 = -2x_1 + 4$
2	$(2-3)x_1 + 3 = -x_1 + 3$
3	$x_1 + 2$
4	$-7x_1 + 6$



Maxmin point is the intersection of B_3 & B_4 strategies, therefore we equate the two,

$$x_1 + 2 = -7x_1 + 6$$

$$8x_1 = 4$$

$$x_1 = 0.5$$

$$1 - x_1 = 0.5$$

A select A_1 with probability 0.5 & strategy A_2 with $1 - x_1 = 0.5$

Value: $x_1 + 2$

$$0.5 + 2 = \underline{\underline{2.5}}$$

$$-7x_1 + 6$$

$$-7(0.5) + 6$$

$$\underline{\underline{2.5}}$$

11/5/23

$$\begin{array}{ccc} & (y_3) & (y_4 = 1 - y_3) \\ & B_3 & B_4 \\ A_1 & \left[\begin{array}{c} 3 \\ 2 \end{array}\right. & \left.\begin{array}{c} -1 \\ 6 \end{array}\right] \\ A_2 & & \end{array}$$

A's Strategy	B's expected payoffs
1	$[3 - (-1)y_3 - 1] = 4y_3 - 1$
2	$[2 - (6)y_3 + 6 = -4y_3 + 6$

Since intersected is of B_3 & B_4.

We equate the expected payoff $\because$

$$4y_3 - 1 = -4y_3 + 6$$

$$8y_3 = 7, \quad y_3 = \frac{7}{8}, \quad y_4 = \frac{1}{8}$$

$$\therefore \text{Value} = \begin{cases} 4\left(\frac{7}{8}\right) - 1 = 2.5 \\ -4\left(\frac{7}{8}\right) + 6 = 2.5 \end{cases}$$

A adopts the strategy A_1 with prob. 0.5 and strategy A_2 with prob. $(1 - x_1 = 0.5)$.
B adopts strategy B_3 with prob. 7/8 and B_4 with prob. 1/8.
Value of game = 2.5

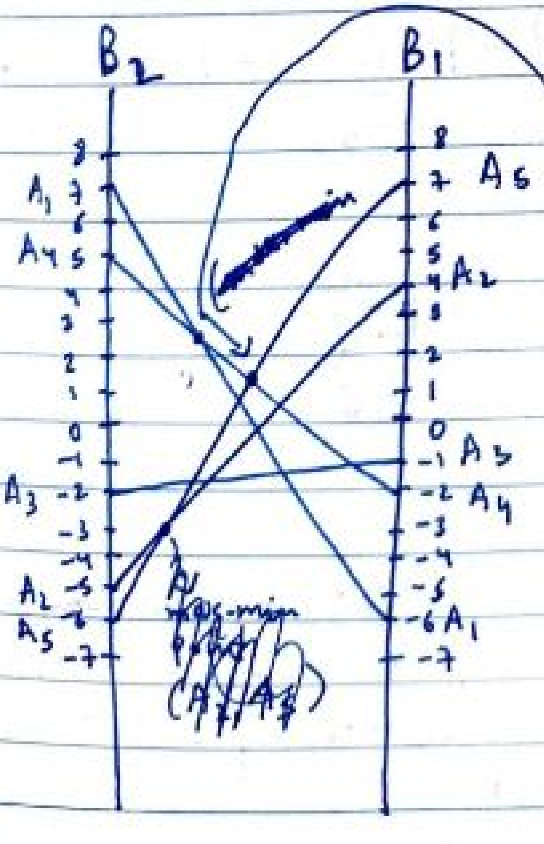
Find optimal strategies and value of game :

$$
\begin{array}{c|cc|c}
 & B_1 \; (\rightarrow y_1) & B_2 \; (\rightarrow y_2 = 1 - y_1) & \text{Lowest} \\
\hline
(x_1)\, A_1 & -6 & 7 & -6 \\
(x_2)\, A_2 & 4 & -5 & -5 \\
(x_3)\, A_3 & -1 & -2 & -2 \\
(x_4)\, A_4 & -2 & 5 & -2 \\
(x_5)\, A_5 & 7 & -6 & -6 \\
\hline
\text{Max} \rightarrow & 7 & 7 &
\end{array}
$$

(Q.)

Ans.) No pure strategy. Its a case of mixed strategies.

A's Strategy	B's expected payoff
1	$(-6-7)y_1 + 7 = -13y_1 + 7$
2	$(4-(-5))y_1 - 5 = 9y_1 - 5$
3	$(-1+2)y_1 - 2 = y_1 - 2$
4	$(-2-5)y_1 + 5 = -7y_1 + 5$
5	$(7+6)y_1 - 6 = 13y_1 - 6$



(min of max section)

min - max point (A_4, A_5)

it is the intersection of A_5 and A_4.

wrong

$$+13y_1 - 6 = -7y_1 + 5$$

$$+6y_1 = +2$$

$$y_1 = \frac{1}{3}, \quad y_2 = \frac{2}{3}$$

Value of game $= \begin{cases} -13y_1 + 7 = 8/3 \\ -7y_1 + 5 = 8/3 \end{cases}$

$$13y_1 - 6 = -7y_1 + 5$$

$$20y_1 = 11$$

$$y_1 = \frac{11}{20}, \quad y_2 = \frac{9}{20}$$

Value of game $= \begin{cases} 13\left(\frac{11}{20}\right) - 6 = 1.15 \\ -7\left(\frac{11}{20}\right) + 5 = 1.15 \end{cases}$

$$\begin{array}{cc|cc|c} & & (y_1) & (y_2) & \\ & & B_1 & B_2 & \text{Lowest} \\ \hline (x_4) & A_4 & -2 & 5 & \boxed{-2} \\ (x_5) & A_5 & 7 & -6 & -6 \\ \hline \text{Max} \rightarrow & & 7 & \boxed{5} & \end{array}$$

B's strategy

$\frac{1}{2}$

A's expected payoff

$$[-2-7]x_4 + 7 = -9x_4 + 7$$

$$[5--6]x_4 - 6 = 11x_4 - 6$$

$$-9x_4 + 7 = 11x_4 - 6$$

$$\therefore x_4 = \frac{13}{20}, \quad x_5 = 1 - \frac{13}{20} = \frac{7}{20}$$

$\therefore$ ~~A adopts~~ Value $= \begin{cases} -9\left(\frac{13}{20}\right) + 7 = 1.15 \\ 11\left(\frac{13}{20}\right) - 6 = 1.15 \end{cases}$

$\therefore$ A adopts A_4 and A_5 with probabilities $\frac{13}{20}$ and $\frac{7}{20}$ respectively.

B adopts B_1 and B_2 with prob. $\frac{11}{20}$ and $\frac{7}{20}$ respectively.

game value $= 1.15$