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# FORECASTING

## PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

 **OTexts**  
OPEN TEXTS FOR PRACTICE

## 9. ARIMA models

### 9.9 Seasonal ARIMA models

[OTexts.org/fpp3/](http://OTexts.org/fpp3/)

# Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where  $m$  = number of observations per year.

# Seasonal ARIMA models

E.g.,  $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$  model (without constant)

# Seasonal ARIMA models

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$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

# Seasonal ARIMA models

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Diagram illustrating the components of the ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model:

- $(1 - \phi_1 B)$ : Non-seasonal AR(1)
- $(1 - \Phi_1 B^4)$ : Seasonal AR(1)
- $(1 - B)$ : Non-seasonal difference
- $(1 - B^4)$ : Seasonal difference
- $(1 + \theta_1 B)$ : Non-seasonal MA(1)
- $(1 + \Theta_1 B^4)$ : Seasonal MA(1)

# Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)<sub>12</sub> will show:**

- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ....

**ARIMA(0,0,0)(1,0,0)<sub>12</sub> will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

# How does ARIMA() work for seasonal models?

## A seasonal ARIMA process

$$\Phi(B)\phi(B)(1 - B)^d(1 - B)^D y_t = c + \Theta(B)\theta(B)\varepsilon_t$$

Need to select appropriate orders:  $d, D, p, q, P, Q$ , and whether to include the intercept  $c$ .

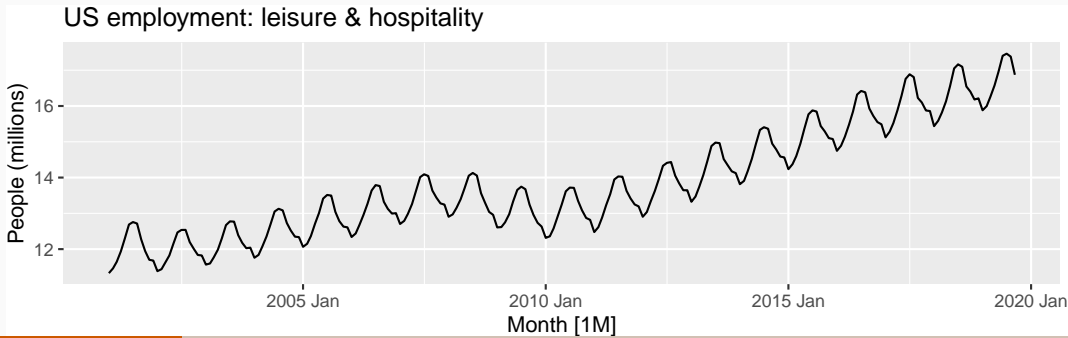
## Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences  $d$  via KPSS test and  $D$  using seasonal strength.
- Select  $p, q, P, Q$  and  $c$  by minimising AICc.
- Use stepwise search to traverse model space.



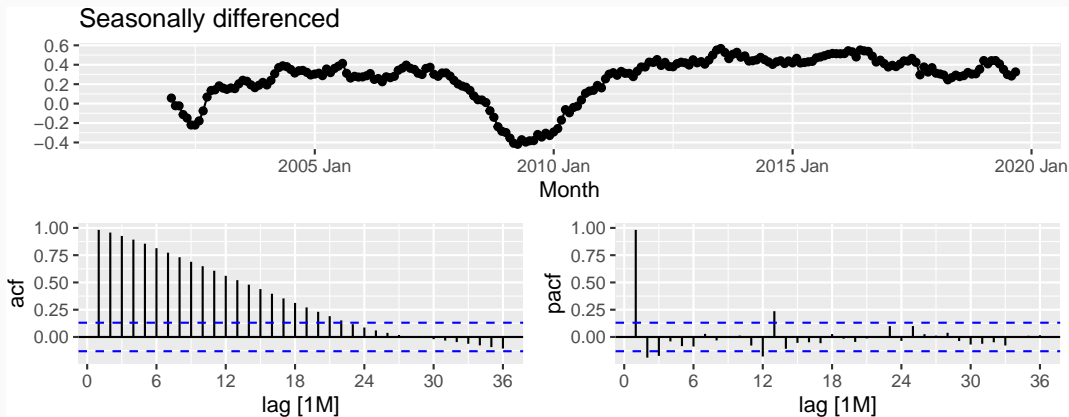
# US leisure employment

```
leisure <- us_employment |>  
  filter(Title == "Leisure and Hospitality", year(Month) > 2000) |>  
  mutate(Employed = Employed / 1000) |>  
  select(Month, Employed)  
autoplot(leisure, Employed) +  
  labs(title = "US employment: leisure & hospitality", y = "People (millions)")
```



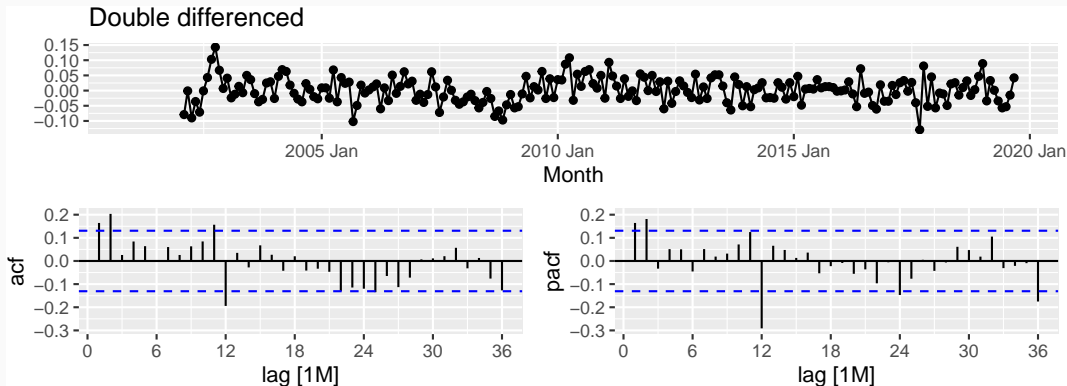
# US leisure employment

```
leisure |>  
  gg_tsdisplay(difference(Employed, 12), plot_type = "partial", lag = 36) +  
  labs(title = "Seasonally differenced", y = "")
```



# US leisure employment

```
leisure |>  
  gg_tsdisplay(difference(Employed, 12) |> difference(),  
    plot_type = "partial", lag = 36) +  
  labs(title = "Double differenced", y = "")
```



# US leisure employment

```
fit <- leisure |>
  model(arima012011 = ARIMA(Employed ~ pdq(0, 1, 2) + PDQ(0, 1, 1)),
        arima210011 = ARIMA(Employed ~ pdq(2, 1, 0) + PDQ(0, 1, 1)),
        auto = ARIMA(Employed, stepwise = FALSE, approx = FALSE))
fit |>
  pivot_longer(everything(),
               names_to = "Model name",
               values_to = "Orders")
```

```
## # A mable: 3 x 2
## # Key:      Model name [3]
##   `Model name`      Orders
##   <chr>             <model>
## 1 arima012011    <ARIMA(0,1,2)(0,1,1)[12]>
## 2 arima210011    <ARIMA(2,1,0)(0,1,1)[12]>
## 3 auto           <ARIMA(2,1,0)(1,1,1)[12]>
```

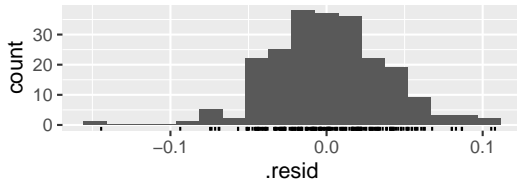
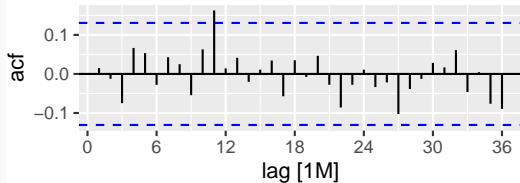
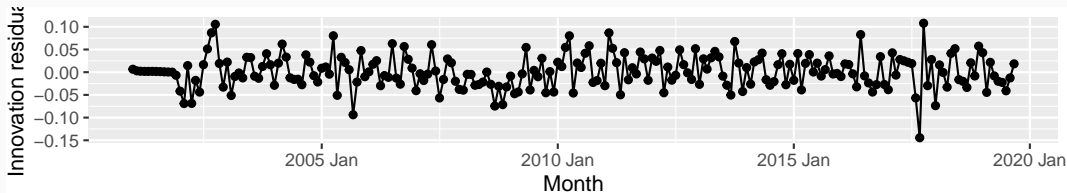
# US leisure employment

```
glance(fit) |>  
  arrange(AICc) |>  
  select(.model:BIC)
```

```
## # A tibble: 3 x 6  
##   .model      sigma2 log_lik   AIC  AICc   BIC  
##   <chr>      <dbl>   <dbl> <dbl> <dbl> <dbl>  
## 1 auto        0.00142    395. -780. -780. -763.  
## 2 arima210011 0.00145    392. -776. -776. -763.  
## 3 arima012011 0.00146    391. -775. -775. -761.
```

# US leisure employment

```
fit |>  
  select(auto) |>  
  gg_tsresiduals(lag = 36)
```



# US leisure employment

```
augment(fit) |>  
  filter(.model == "auto") |>  
  features(.innov, ljung_box, lag = 24, dof = 4)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 auto      16.6      0.680
```

# US leisure employment

```
forecast(fit, h = 36) |>  
  filter(.model == "auto") |>  
  autoplot(leisure) +  
  labs(title = "US employment: leisure & hospitality", y = "People (millions)")
```



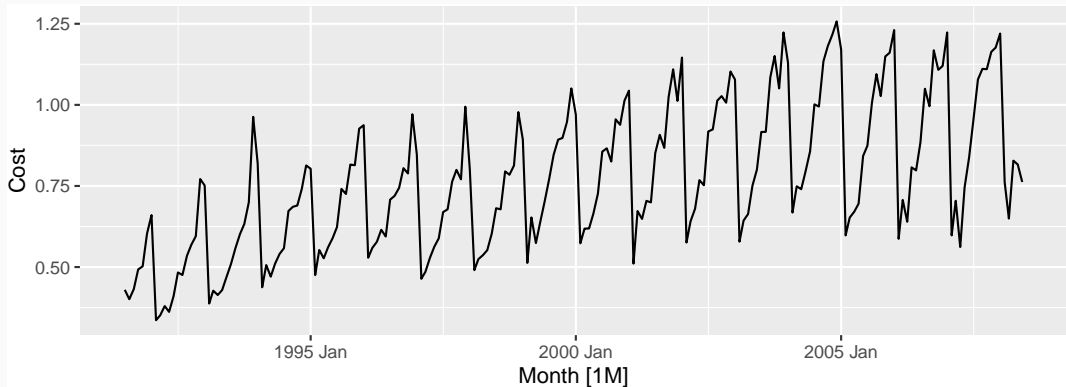


# Corticosteroid drug sales

```
h02 <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

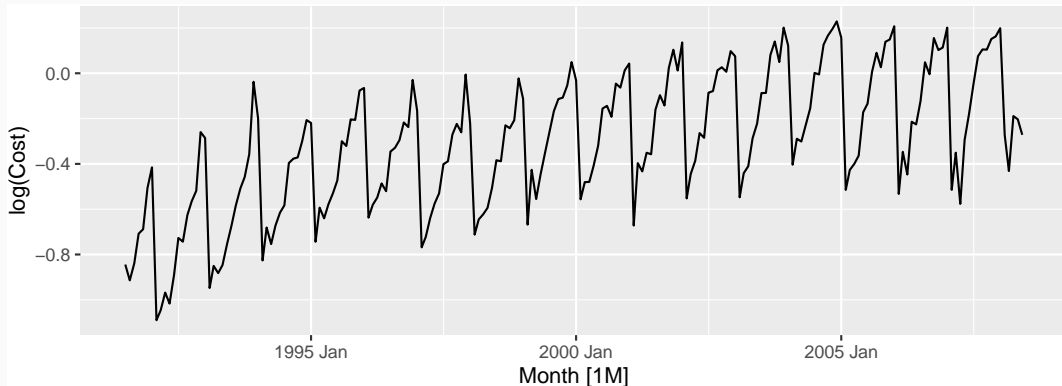
# Corticosteroid drug sales

```
h02 |> autoplot(  
  Cost  
)
```



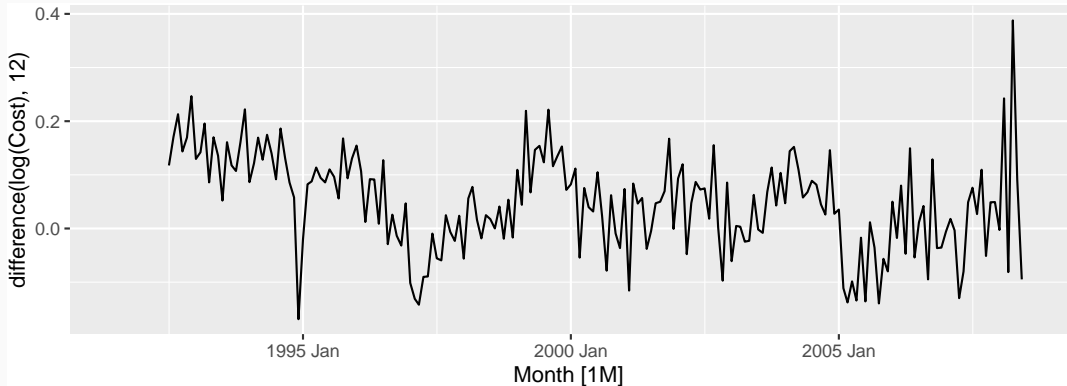
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost)  
)
```



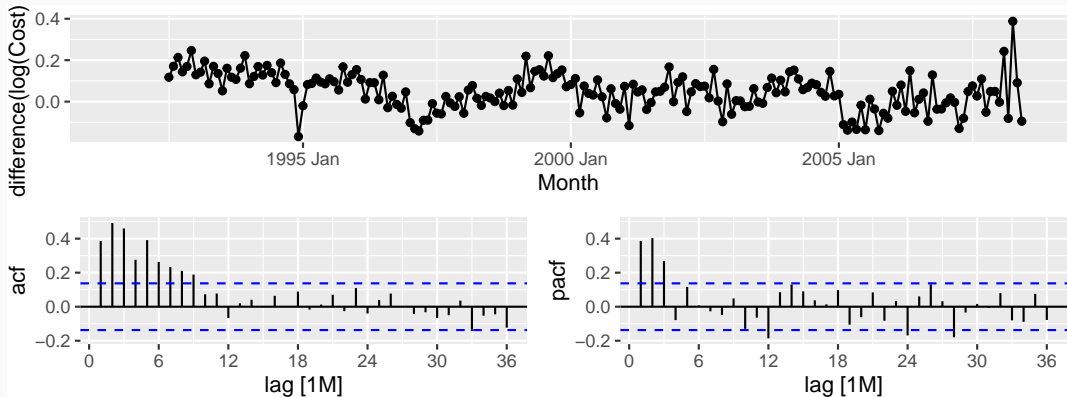
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12)  
)
```



# Corticosteroid drug sales

```
h02 |> gg_tsdisplay(difference(log(Cost), 12),  
  lag_max = 36, plot_type = "partial"  
)
```



# Corticosteroid drug sales

- Choose  $D = 1$  and  $d = 0$ .
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model:  $\text{ARIMA}(3,0,0)(2,1,0)_{12}$ .

## Corticosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485
ARIMA(3,0,1)(1,1,1)[12]	-484
ARIMA(3,0,1)(0,1,1)[12]	-484
ARIMA(3,0,1)(2,1,0)[12]	-476
ARIMA(3,0,0)(2,1,0)[12]	-475
ARIMA(3,0,2)(2,1,0)[12]	-475
ARIMA(3,0,1)(1,1,0)[12]	-463

# Corticosteroid drug sales

```
fit <- h02 |>  
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))  
report(fit)
```

```
## Series: Cost
```

```
## Model: ARIMA(3,0,1)(0,1,2)[12]
```

```
## Transformation: log(Cost)
```

```
##
```

```
## Coefficients:
```

	ar1	ar2	ar3	ma1	sma1	sma2
	-0.160	0.5481	0.5678	0.383	-0.5222	-0.1768
s.e.	0.164	0.0878	0.0942	0.190	0.0861	0.0872

```
##
```

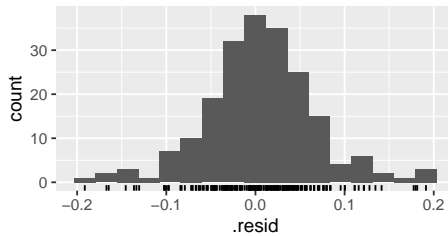
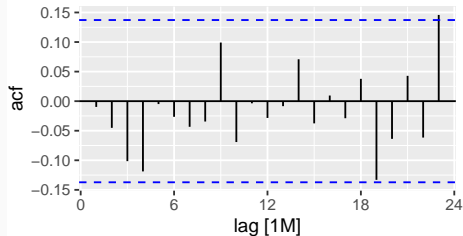
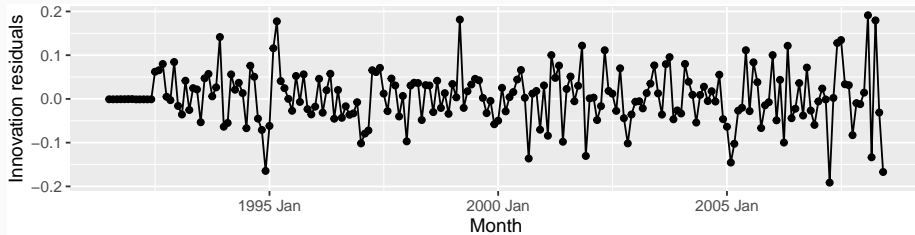
```
## sigma^2 estimated as 0.004278: log likelihood=250
```

```
## AIC=-486 AICc=-485 BIC=-463
```



# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      50.7      0.0104
```

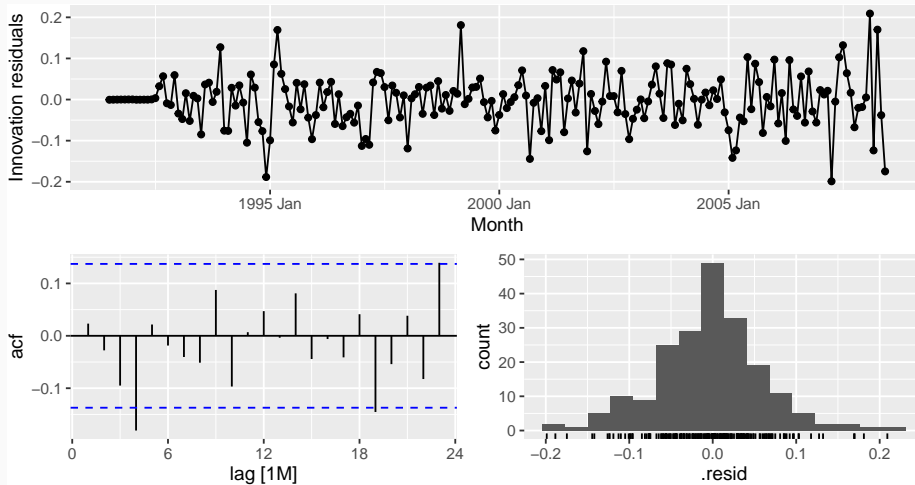
# Corticosteroid drug sales

```
fit <- h02 |> model(auto = ARIMA(log(Cost)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(2,1,0)(0,1,1)[12]  
## Transformation: log(Cost)  
##  
## Coefficients:  
##          ar1      ar2      sma1  
##      -0.8491  -0.4207  -0.6401  
## s.e.   0.0712   0.0714   0.0694  
##  
## sigma^2 estimated as 0.004387:  log likelihood=245  
## AIC=-483   AICc=-483   BIC=-470
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 auto      59.3    0.00332
```

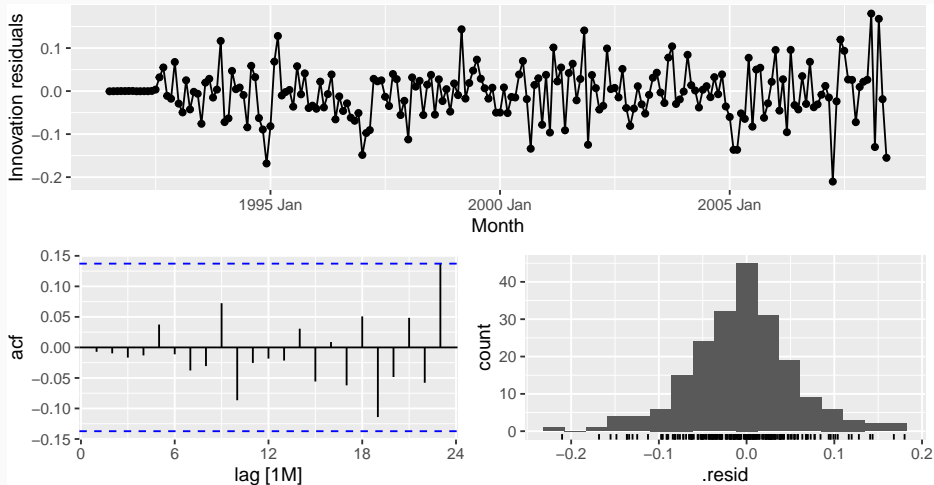
# Corticosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE, approximation = FALSE,
    order_constraint = p + q + P + Q <= 9))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(4,1,1)(2,1,2)[12]
## Transformation: log(Cost)
##
## Coefficients:
##          ar1      ar2      ar3      ar4      ma1      sar1      sar2      sma1      sma2
##      -0.0425  0.210  0.202  -0.227  -0.742  0.621  -0.383  -1.202  0.496
## s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249  0.213
##
## sigma^2 estimated as 0.004049:  log likelihood=254
## AIC=-489    AICc=-487    BIC=-456
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      36.5      0.106
```



# Corticosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 |>
  filter_index(~ "2006 Jun") |>
  model(
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit |>
  forecast(h = "2 years") |>
  accuracy(h02)
```

## Corticosteroid drug sales

.model	RMSE
ARIMA(3,0,1)(1,1,1)[12]	0.0619
ARIMA(3,0,1)(0,1,2)[12]	0.0621
ARIMA(3,0,1)(0,1,1)[12]	0.0630
ARIMA(2,1,0)(0,1,1)[12]	0.0630
ARIMA(4,1,1)(2,1,2)[12]	0.0631
ARIMA(3,0,2)(2,1,0)[12]	0.0651
ARIMA(3,0,1)(2,1,0)[12]	0.0653
ARIMA(3,0,1)(1,1,0)[12]	0.0666
ARIMA(3,0,0)(2,1,0)[12]	0.0668

# Corticosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing.  
But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

# Corticosteroid drug sales

```
fit <- h02 |>  
  model(ARIMA(Cost ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))  
fit |>  
  forecast() |>  
  autoplot(h02) + labs(y = "H02 Expenditure ($AUD)")
```

