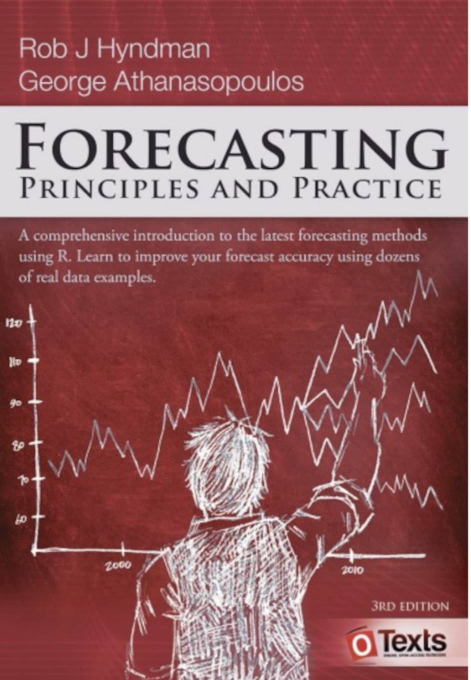

8. Exponential smoothing

8.6 Estimation and model selection

OTexts.org/fpp3/

Estimating ETS models

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, b_0, $s_0, s_{-1}, \dots, s_{-m+1}$ are estimated by maximising the "likelihood" = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

Innovations state space models

Let $\boldsymbol{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$ and $\varepsilon_t \overset{\text{iid}}{\sim} \text{N}(0, \sigma^2)$.

$$
\begin{aligned}
y_t &= \underbrace{h(\boldsymbol{x}_{t-1})}_{\mu_t} + \underbrace{k(\boldsymbol{x}_{t-1})\varepsilon_t}_{e_t} \\
\boldsymbol{x}_t &= f(\boldsymbol{x}_{t-1}) + g(\boldsymbol{x}_{t-1})\varepsilon_t
\end{aligned}
$$

Additive errors

$$k(x) = 1. \qquad y_t = \mu_t + \varepsilon_t.$$

Multiplicative errors

$$k(\boldsymbol{x}_{t-1}) = \mu_t. \qquad y_t = \mu_t(1 + \varepsilon_t).$$

$\varepsilon_t = (y_t - \mu_t)/\mu_t$ is relative error.

Innovations state space models

Estimation

$$\begin{aligned} L^*(\boldsymbol{\theta}, \boldsymbol{x}_0) &= T \log\left(\sum_{t=1}^{T} \varepsilon_t^2 \right) + 2 \sum_{t=1}^{T} \log |k(\boldsymbol{x}_{t-1})| \\ &= -2 \log(\text{Likelihood}) + \text{constant} \end{aligned}$$

- Estimate parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \phi)$ and initial states $\boldsymbol{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$ by minimizing L^*.

Parameter restrictions

Usual region

- Traditional restrictions in the methods $0 < \alpha, \beta^*, \gamma^*, \phi < 1$ (equations interpreted as weighted averages).
- In models we set $\beta = \alpha\beta^*$ and $\gamma = (1 - \alpha)\gamma^*$.
- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

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- In models we set $\beta = \alpha\beta^*$ and $\gamma = (1-\alpha)\gamma^*$.
- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts.
- Usually (but not always) less restrictive than *traditional* region.
- For example for ETS(A,N,N):
  traditional $0 < \alpha < 1$ while *admissible* $0 < \alpha < 2$.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters & initial states estimated in the model.

Model selection

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

Model selection

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k[\log(T) - 2].$$

AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.
- These are: ETS(A,N,M), ETS(A,A,M), ETS(A,A_d,M).
- Models with multiplicative errors are useful for strictly positive data, but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.

Example: National populations

```
fit <- global_economy |>
  mutate(Pop = Population / 1e6) |>
  model(ets = ETS(Pop))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country                          ets
##    <fct>                        <model>
##  1 Afghanistan             <ETS(A,A,N)>
##  2 Albania                 <ETS(M,A,N)>
##  3 Algeria                 <ETS(M,A,N)>
##  4 American Samoa          <ETS(M,A,N)>
##  5 Andorra                 <ETS(M,A,N)>
##  6 Angola                  <ETS(M,A,N)>
##  7 Antigua and Barbuda <ETS(M,A,N)>
##  8 Arab World              <ETS(M,A,N)>
##  9 Argentina               <ETS(A,A,N)>
## 10 Armenia                 <ETS(M,A,N)>
```

Example: National populations

```r
fit |>
  forecast(h = 5)
```

```
## # A fable: 1,315 x 5 [1Y]
## # Key:     Country, .model [263]
##    Country     .model  Year             Pop .mean
##    <fct>       <chr>  <dbl>          <dist> <dbl>
##  1 Afghanistan ets     2018    N(36, 0.012) 36.4
##  2 Afghanistan ets     2019    N(37, 0.059) 37.3
##  3 Afghanistan ets     2020     N(38, 0.16) 38.2
##  4 Afghanistan ets     2021     N(39, 0.35) 39.0
##  5 Afghanistan ets     2022     N(40, 0.64) 39.9
##  6 Albania     ets     2018 N(2.9, 0.00012)  2.87
##  7 Albania     ets     2019   N(2.9, 6e-04)  2.87
##  8 Albania     ets     2020  N(2.9, 0.0017)  2.87
##  9 Albania     ets     2021  N(2.9, 0.0036)  2.86
## 10 Albania     ets     2022  N(2.9, 0.0066)  2.86
## # ... with 1,305 more rows
```

Example: Australian holiday tourism

```
holidays <- tourism |>
  filter(Purpose == "Holiday")
fit <- holidays |> model(ets = ETS(Trips))
fit
```

```
## # A mable: 76 x 4
## # Key:     Region, State, Purpose [76]
##    Region                     State              Purpose          ets
##    <chr>                      <chr>              <chr>        <model>
##  1 Adelaide                   South Australia    Holiday <ETS(A,N,A)>
##  2 Adelaide Hills             South Australia    Holiday <ETS(A,A,N)>
##  3 Alice Springs              Northern Territory Holiday <ETS(M,N,A)>
##  4 Australia's Coral Coast    Western Australia  Holiday <ETS(M,N,A)>
##  5 Australia's Golden Outback Western Australia  Holiday <ETS(M,N,M)>
##  6 Australia's North West     Western Australia  Holiday <ETS(A,N,A)>
##  7 Australia's South West     Western Australia  Holiday <ETS(M,N,M)>
##  8 Ballarat                   Victoria           Holiday <ETS(M,N,A)>
##  9 Barkly                     Northern Territory Holiday <ETS(A,N,A)>
## 10 Barossa                    South Australia    Holiday <ETS(A,N,N)>
```

Example: Australian holiday tourism

```r
fit |>
  filter(Region == "Snowy Mountains") |>
  report()
```

```
## Series: Trips
## Model: ETS(M,N,A)
##   Smoothing parameters:
##     alpha = 0.157
##     gamma = 1e-04
##
##   Initial states:
##  l[0] s[0] s[-1] s[-2] s[-3]
##   142  -61   131 -42.2 -27.7
##
##   sigma^2:  0.0388
##
##  AIC AICc  BIC
##  852  854  869
```

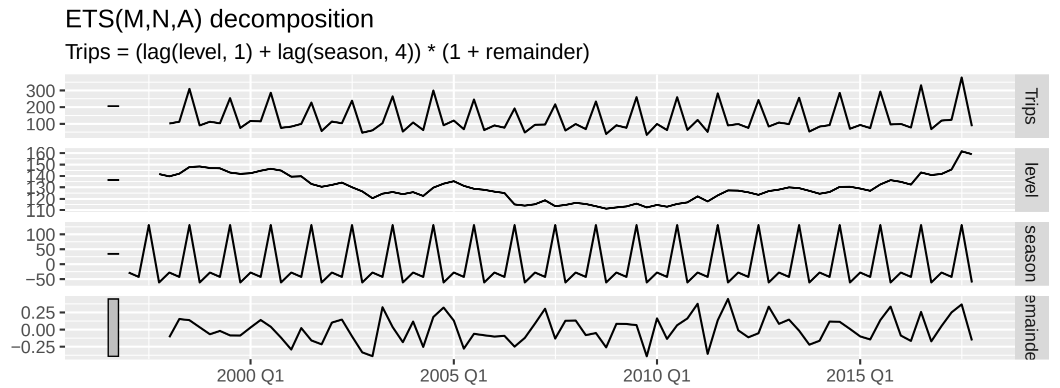
Example: Australian holiday tourism

```r
fit |>
  filter(Region == "Snowy Mountains") |>
  components(fit)
```

```
## # A dable: 84 x 9 [1Q]
## # Key:     Region, State, Purpose, .model [1]
## # :        Trips = (lag(level, 1) + lag(season, 4)) * (1 + remainder)
##    Region          State  Purpose .model Quarter Trips level season remai~1
##    <chr>           <chr>  <chr>   <chr>    <qtr> <dbl> <dbl>  <dbl>   <dbl>
##  1 Snowy Mountains New S~ Holiday ets    1997 Q1  NA      NA  -27.7 NA
##  2 Snowy Mountains New S~ Holiday ets    1997 Q2  NA      NA  -42.2 NA
##  3 Snowy Mountains New S~ Holiday ets    1997 Q3  NA      NA  131.  NA
##  4 Snowy Mountains New S~ Holiday ets    1997 Q4  NA     142. -61.0 NA
##  5 Snowy Mountains New S~ Holiday ets    1998 Q1 101.    140. -27.7 -0.113
##  6 Snowy Mountains New S~ Holiday ets    1998 Q2 112.    142. -42.2  0.154
##  7 Snowy Mountains New S~ Holiday ets    1998 Q3 310.    148. 131.   0.137
##  8 Snowy Mountains New S~ Holiday ets    1998 Q4  89.8   148. -61.0  0.0335
##  9 Snowy Mountains New S~ Holiday ets    1999 Q1 112.    147. -27.7 -0.0687
## 10 Snowy Mountains New S~ Holiday ets    1999 Q2 103.    147. -42.2 -0.0199
```

Example: Australian holiday tourism

```r
fit |>
  filter(Region == "Snowy Mountains") |>
  components(fit) |>
  autoplot()
```



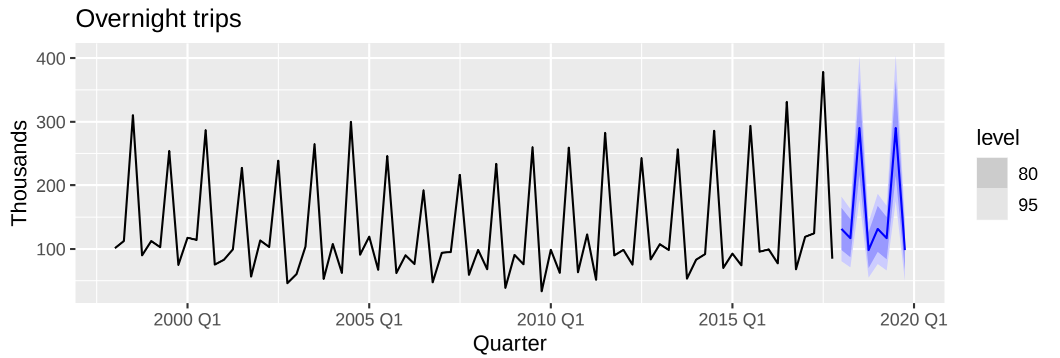
Example: Australian holiday tourism

```r
fit |> forecast()
```

```
## # A fable: 608 x 7 [1Q]
## # Key:     Region, State, Purpose, .model [76]
##    Region         State           Purpose .model Quarter        Trips .mean
##    <chr>          <chr>           <chr>   <chr>    <qtr>       <dist> <dbl>
##  1 Adelaide       South Australia Holiday ets    2018 Q1 N(210, 457) 210.
##  2 Adelaide       South Australia Holiday ets    2018 Q2 N(173, 473) 173.
##  3 Adelaide       South Australia Holiday ets    2018 Q3 N(169, 489) 169.
##  4 Adelaide       South Australia Holiday ets    2018 Q4 N(186, 505) 186.
##  5 Adelaide       South Australia Holiday ets    2019 Q1 N(210, 521) 210.
##  6 Adelaide       South Australia Holiday ets    2019 Q2 N(173, 537) 173.
##  7 Adelaide       South Australia Holiday ets    2019 Q3 N(169, 553) 169.
##  8 Adelaide       South Australia Holiday ets    2019 Q4 N(186, 569) 186.
##  9 Adelaide Hills South Australia Holiday ets    2018 Q1   N(19, 36)  19.4
## 10 Adelaide Hills South Australia Holiday ets    2018 Q2   N(20, 36)  19.6
## # ... with 598 more rows
```

Example: Australian holiday tourism

```r
fit |>
  forecast() |>
  filter(Region == "Snowy Mountains") |>
  autoplot(holidays) +
  labs(y = "Thousands", title = "Overnight trips")
```



Residuals

Response residuals

$$\hat{e}_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1}$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}}$$

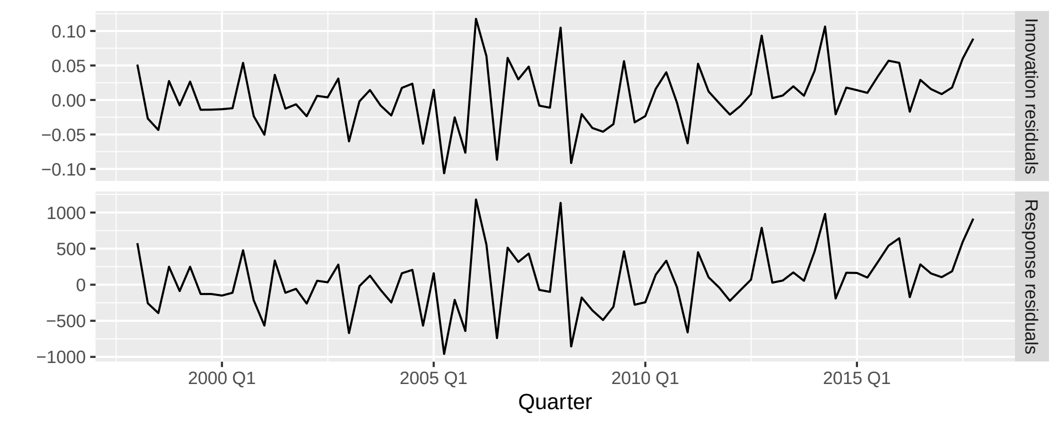
Example: Australian holiday tourism

```r
aus_holidays <- tourism |>
  filter(Purpose == "Holiday") |>
  summarise(Trips = sum(Trips))
fit <- aus_holidays |>
  model(ets = ETS(Trips)) |>
  report()
```

```
## Series: Trips
## Model: ETS(M,N,M)
##   Smoothing parameters:
##     alpha = 0.358
##     gamma = 0.000969
##
##   Initial states:
##  l[0]  s[0] s[-1] s[-2] s[-3]
##  9667 0.943 0.927 0.968  1.16
##
##   sigma^2:  0.0022
##
##  AIC AICc  BIC
## 1331 1333 1348
```

Example: Australian holiday tourism

```r
residuals(fit)
residuals(fit, type = "response")
```



Example: Australian holiday tourism

```r
fit |>
  augment()
```

```
## # A tsibble: 80 x 6 [1Q]
## # Key:       .model [1]
##    .model Quarter  Trips .fitted .resid   .innov
##    <chr>    <qtr>  <dbl>   <dbl>  <dbl>    <dbl>
##  1 ets    1998 Q1 11806.  11230.  576.   0.0513
##  2 ets    1998 Q2  9276.   9532. -257.  -0.0269
##  3 ets    1998 Q3  8642.   9036. -393.  -0.0435
##  4 ets    1998 Q4  9300.   9050.  249.   0.0275
##  5 ets    1999 Q1 11172.  11260.  -88.0 -0.00781
##  6 ets    1999 Q2  9608.   9358.  249.   0.0266
##  7 ets    1999 Q3  8914.   9042. -129.  -0.0142
##  8 ets    1999 Q4  9026.   9154. -129.  -0.0140
##  9 ets    2000 Q1 11071.  11221. -150.  -0.0134
## 10 ets    2000 Q2  9196.   9308. -111.  -0.0120
## # ... with 70 more rows
```

Example: Australian holiday tourism

```
fit |>
  augment()
```

Innovation residuals (`.innov`) are given by $\hat{\varepsilon}_t$ while regular residuals (`.resid`) are $y_t - \hat{y}_{t-1}$. They are different when the model has multiplicative errors.

```
## # A tsibble: 80 x 6 [1Q]
## # Key:       .model [1]
##    .model Quarter  Trips .fitted .resid   .innov
##    <chr>    <qtr>  <dbl>   <dbl>  <dbl>    <dbl>
##  1 ets    1998 Q1 11806.  11230.  576.   0.0513
##  2 ets    1998 Q2  9276.   9532. -257.  -0.0269
##  3 ets    1998 Q3  8642.   9036. -393.  -0.0435
##  4 ets    1998 Q4  9300.   9050.  249.   0.0275
##  5 ets    1999 Q1 11172.  11260.  -88.0 -0.00781
##  6 ets    1999 Q2  9608.   9358.  249.   0.0266
##  7 ets    1999 Q3  8914.   9042. -129.  -0.0142
##  8 ets    1999 Q4  9026.   9154. -129.  -0.0140
##  9 ets    2000 Q1 11071.  11221. -150.  -0.0134
## 10 ets    2000 Q2  9196.   9308. -111.  -0.0120
## # ... with 70 more rows
```