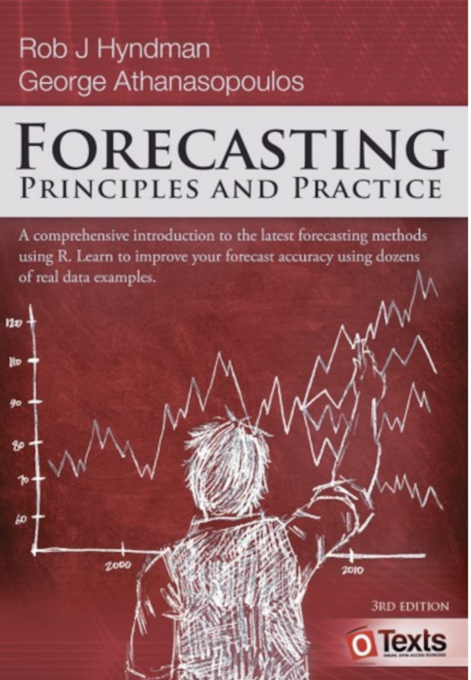


8. Exponential smoothing

8.2 Methods with trend

OTexts.org/fpp3/

Holt's linear trend

Component form

$$
\begin{aligned}
\text{Forecast} \qquad & \hat{y}_{t+h|t} = \ell_t + hb_t \\
\text{Level} \qquad & \ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
\text{Trend} \qquad & b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},
\end{aligned}
$$

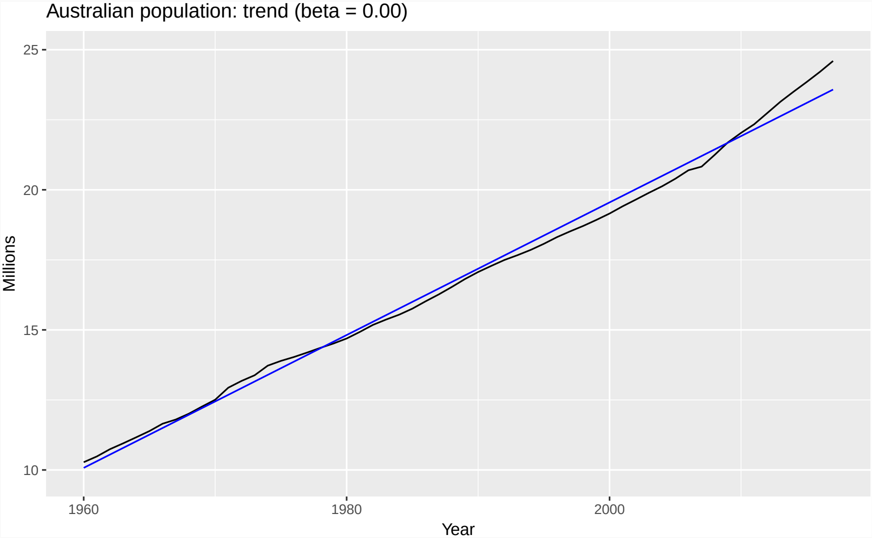
Holt's linear trend

Component form

Forecast	$\hat{y}_{t+h\|t} = \ell_t + hb_t$
Level	$\ell_t = \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1})$
Trend	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1},$

- Two smoothing parameters α and β^* $(0 \le \alpha, \beta^* \le 1)$.
- ℓ_t level: weighted average between y_t and one-step ahead forecast for time t, $(\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1})$
- b_t slope: weighted average of $(\ell_t - \ell_{t-1})$ and b_{t-1}, current and previous estimate of slope.
- Choose $\alpha, \beta^*, \ell_0, b_0$ to minimise SSE.

Exponential smoothing: trend/slope



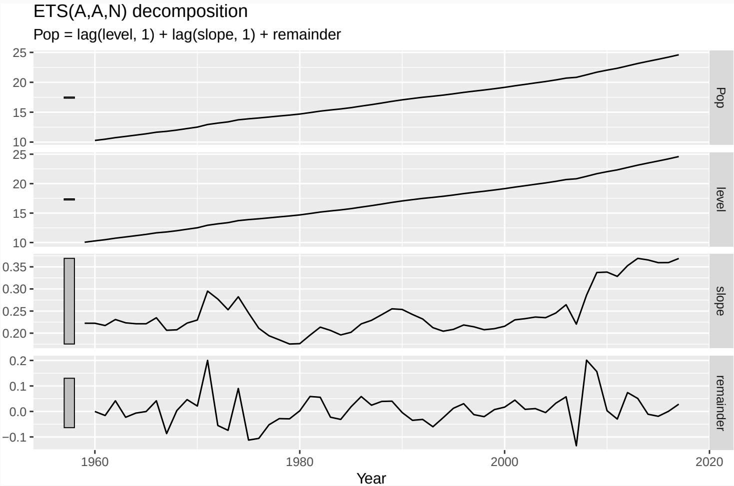
Example: Australian population

```r
aus_economy <- global_economy |>
  filter(Code == "AUS") |>
  mutate(Pop = Population / 1e6)
fit <- aus_economy |>
  model(AAN = ETS(Pop ~ error("A") + trend("A") + season("N")))
report(fit)
```

```
## Series: Pop
## Model: ETS(A,A,N)
##   Smoothing parameters:
##     alpha = 1
##     beta  = 0.327
##
##   Initial states:
##  l[0]  b[0]
##  10.1 0.222
##
##   sigma^2:  0.0041
##
##   AIC  AICc   BIC
## -77.0 -75.8 -66.7
```

Example: Australian population

```r
components(fit) |> autoplot()
```



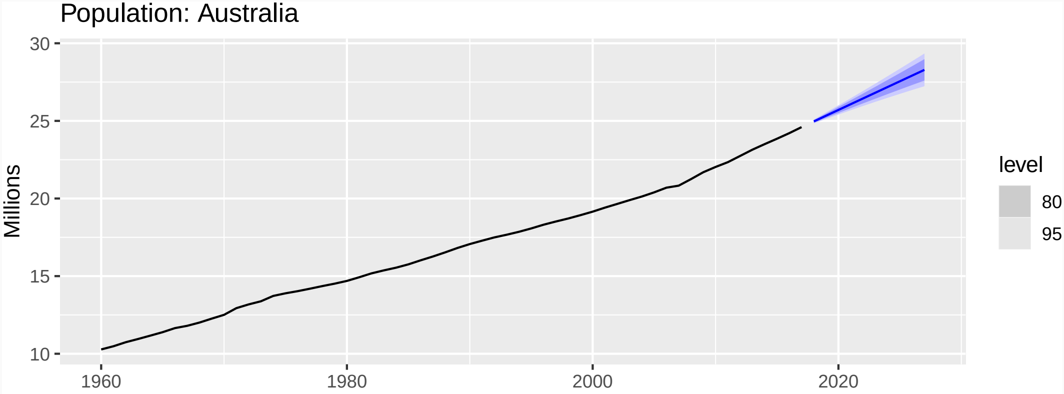
Example: Australian population

```r
components(fit) |>
  left_join(fitted(fit), by = c("Country", ".model", "Year"))
```

```
## # A dable: 59 x 8 [1Y]
## # Key:     Country, .model [1]
## # :        Pop = lag(level, 1) + lag(slope, 1) + remainder
##    Country   .model  Year   Pop level slope remainder .fitted
##    <fct>     <chr>  <dbl> <dbl> <dbl> <dbl>     <dbl>   <dbl>
##  1 Australia AAN     1959  NA    10.1 0.222 NA           NA
##  2 Australia AAN     1960  10.3  10.3 0.222 -0.000145    10.3
##  3 Australia AAN     1961  10.5  10.5 0.217 -0.0159      10.5
##  4 Australia AAN     1962  10.7  10.7 0.231  0.0418      10.7
##  5 Australia AAN     1963  11.0  11.0 0.223 -0.0229      11.0
##  6 Australia AAN     1964  11.2  11.2 0.221 -0.00641     11.2
##  7 Australia AAN     1965  11.4  11.4 0.221 -0.000314    11.4
##  8 Australia AAN     1966  11.7  11.7 0.235  0.0418      11.6
##  9 Australia AAN     1967  11.8  11.8 0.206 -0.0869      11.9
## 10 Australia AAN     1968  12.0  12.0 0.208  0.00350     12.0
## # ... with 49 more rows
```

Example: Australian population

```
fit |>
  forecast(h = 10) |>
  autoplot(aus_economy) +
  labs(y = "Millions", title = "Population: Australia")
```



Damped trend method

Component form

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t \\
\ell_t &= \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.
\end{aligned}$$

Damped trend method

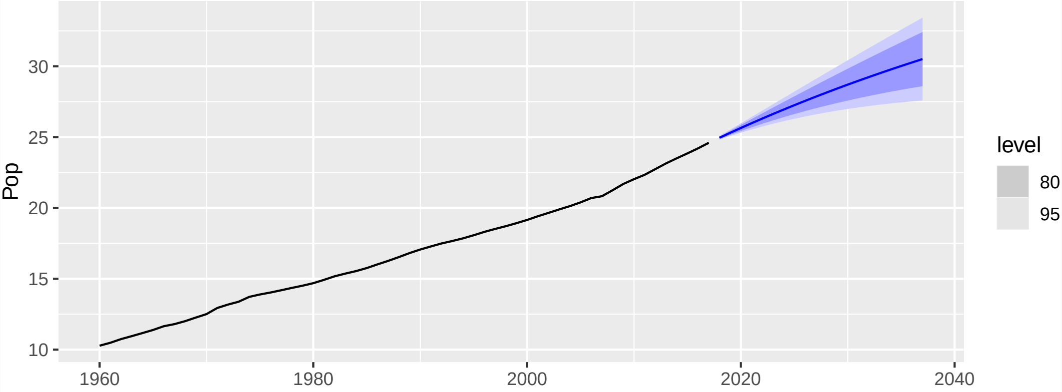
Component form

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + (\phi + \phi^2 + \cdots + \phi^h) b_t \\
\ell_t &= \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.
\end{aligned}$$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \to \infty$, $\hat{y}_{T+h|T} \to \ell_T + \phi b_T/(1 - \phi)$.
- Short-run forecasts trended, long-run forecasts constant.

Example: Australian population

```
aus_economy |>
  model(holt = ETS(Pop ~ error("A") + trend("Ad") + season("N"))) |>
  forecast(h = 20) |>
  autoplot(aus_economy)
```



Example: Australian population

```r
fit <- aus_economy |>
  filter(Year <= 2010) |>
  model(
    ses = ETS(Pop ~ error("A") + trend("N") + season("N")),
    holt = ETS(Pop ~ error("A") + trend("A") + season("N")),
    damped = ETS(Pop ~ error("A") + trend("Ad") + season("N"))
  )
```

```r
tidy(fit)
accuracy(fit)
```

Example: Australian population

term	SES	Linear trend	Damped trend
α	1.00	1.00	1.00
β^*		0.30	0.40
ϕ			0.98
ℓ_0	10.28	10.05	10.04
b_0		0.22	0.25
Training RMSE	0.24	0.06	0.07
Test RMSE	1.63	0.15	0.21
Test MASE	6.18	0.55	0.75
Test MAPE	6.09	0.55	0.74
Test MAE	1.45	0.13	0.18