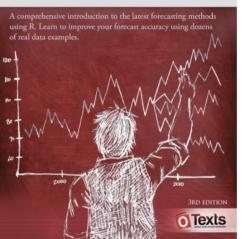
Rob J Hyndman George Athanasopoulos

FORECASTING PRINCIPLES AND PRACTICE



7. Time series regression models

7.9 Matrix formulation

OTexts.org/fpp3/

$$\mathbf{y}_t = \beta_0 + \beta_1 \mathbf{x}_{1,t} + \beta_2 \mathbf{x}_{2,t} + \cdots + \beta_k \mathbf{x}_{k,t} + \varepsilon_t.$$

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Let
$$\mathbf{y} = (y_1, \dots, y_T)', \, \boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)', \, \boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$$
 and

$$\mathbf{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \dots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \dots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,T} & x_{2,T} & \dots & x_{k,T} \end{bmatrix}.$$

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Then

$$y = X\beta + \varepsilon$$
.

Least squares estimation

Minimize: $(\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta)$

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$$\hat{\boldsymbol{\beta}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

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$$\hat{\sigma}^2 = \frac{1}{T - k - 1} (\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})$$

Note: If you fall for the dummy variable trap, (X'X) is a singular matrix.

If the errors are iid and normally distributed, then

$$\mathbf{y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}).$$

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So the likelihood is

$$L = \frac{1}{\sigma^{\mathsf{T}} (2\pi)^{\mathsf{T}/2}} \exp\left(-\frac{1}{2\sigma^2} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})' (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\right)$$

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Multiple regression forecasts

Optimal forecasts

$$\hat{y}^* = E(y^*|y, X, x^*) = x^* \hat{\beta} = x^* (X'X)^{-1} X'y$$

where \mathbf{x}^* is a row vector containing the values of the predictors for the forecasts (in the same format as \mathbf{X}).

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- This ignores any errors in \mathbf{x}^* .
- 95% prediction intervals assuming normal errors:

$$\hat{\mathbf{y}}^* \pm 1.96 \sqrt{\mathsf{Var}(\mathbf{y}^*|\mathbf{X},\mathbf{x}^*)}$$
.