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# FORECASTING

## PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

 **OTexts**  
OPEN TEXTS FOR PRACTICE

## 9. ARIMA models

### 9.4 Moving average models

[OTexts.org/fpp3/](http://OTexts.org/fpp3/)

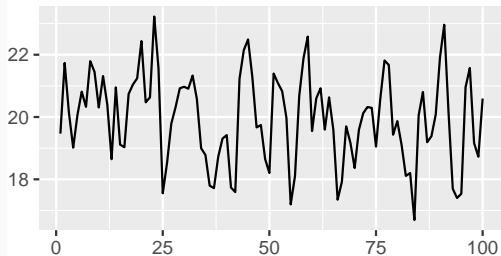
# Moving Average (MA) models

## Moving Average (MA) models:

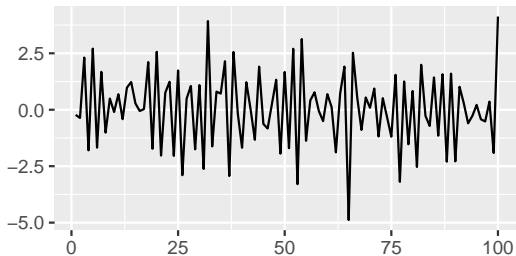
$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where  $\varepsilon_t$  is white noise. This is a multiple regression with **past errors** as predictors. *Don't confuse this with moving average smoothing!*

MA(1)



MA(2)

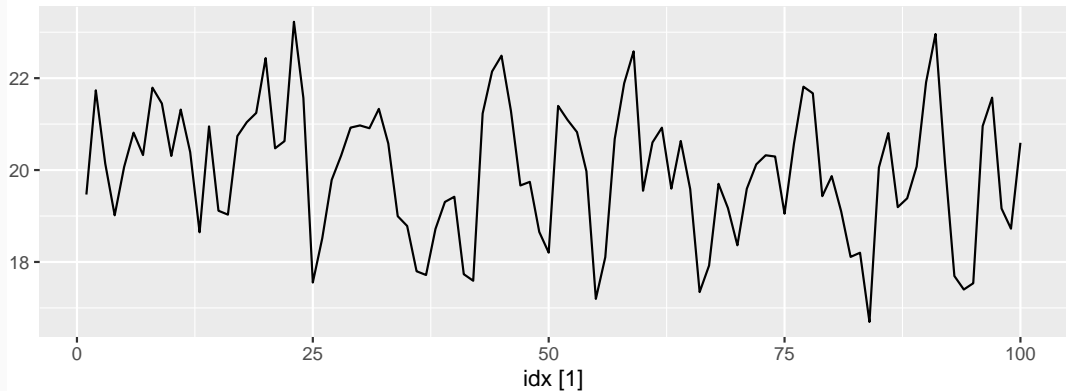


# MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$

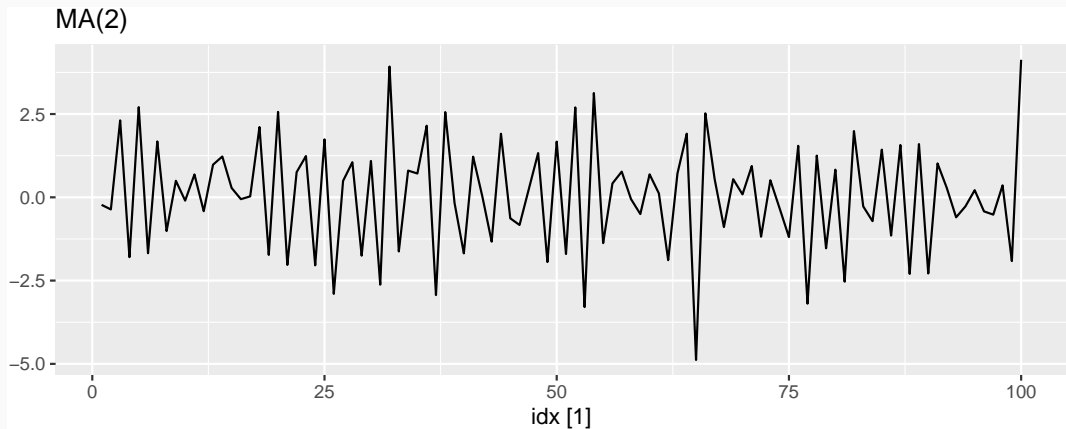
MA(1)



# MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# MA( $\infty$ ) models

It is possible to write any stationary AR( $p$ ) process as an MA( $\infty$ ) process.

**Example: AR(1)**

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \varepsilon_t \\&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&\dots\end{aligned}$$

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Provided  $-1 < \phi_1 < 1$ :

# Invertibility

- Any  $MA(q)$  process can be written as an  $AR(\infty)$  process if we impose some constraints on the MA parameters.
- Then the MA model is called “invertible”.
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

# Invertibility

## General condition for invertibility

Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.



# Invertibility

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Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

- For  $q = 1$ :  $-1 < \theta_1 < 1$ .
- For  $q = 2$ :  $-1 < \theta_2 < 1$        $\theta_2 + \theta_1 > -1$        $\theta_1 - \theta_2 < 1$ .
- More complicated conditions hold for  $q \geq 3$ .
- Estimation software takes care of this.