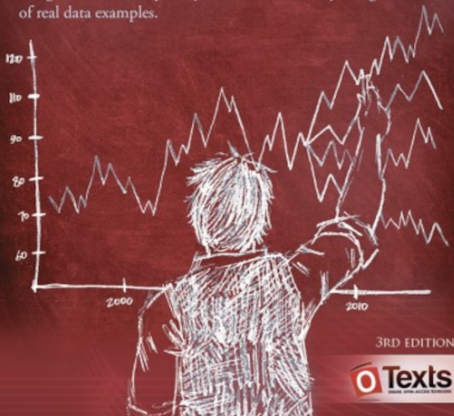


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FORECASTING

PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

 **OTexts**
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9. ARIMA models

9.8 Forecasting

OTexts.org/fpp3/

Point forecasts

- 1 Rearrange ARIMA equation so y_t is on LHS.
- 2 Rewrite equation by replacing t by $T + h$.
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \dots$

Point forecasts

ARIMA(3,1,1) forecasts: Step 1

$$(1 - \hat{\phi}_1 B - \hat{\phi}_2 B^2 - \hat{\phi}_3 B^3)(1 - B)y_t = (1 + \hat{\theta}_1 B)\varepsilon_t,$$

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$$\left[1 - (1 + \hat{\phi}_1)B + (\hat{\phi}_1 - \hat{\phi}_2)B^2 + (\hat{\phi}_2 - \hat{\phi}_3)B^3 + \hat{\phi}_3 B^4\right] y_t = (1 + \hat{\theta}_1 B)\varepsilon_t$$

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$$y_t - (1 + \hat{\phi}_1)y_{t-1} + (\hat{\phi}_1 - \hat{\phi}_2)y_{t-2} + (\hat{\phi}_2 - \hat{\phi}_3)y_{t-3} + \hat{\phi}_3 y_{t-4} = \varepsilon_t + \hat{\theta}_1 \varepsilon_{t-1}$$

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Point forecasts ($h=1$)

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$$y_{T+1} = (1 + \hat{\phi}_1)y_T - (\hat{\phi}_1 - \hat{\phi}_2)y_{T-1} - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-2} - \hat{\phi}_3y_{T-3} + \varepsilon_{T+1} + \hat{\theta}_1\varepsilon_T$$

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ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1 + \hat{\phi}_1)\hat{y}_{T+1|T} - (\hat{\phi}_1 - \hat{\phi}_2)y_T - (\hat{\phi}_2 - \hat{\phi}_3)y_{T-1} - \hat{\phi}_3y_{T-2}.$$

Prediction intervals

Assuming $\varepsilon_t \sim N(0, \sigma^2)$

95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96 \sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \hat{\theta}_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \hat{\theta}_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

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- Other models beyond scope of this book.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
 - ▶ the **uncertainty in the parameter estimates** has not been accounted for.
 - ▶ **model uncertainty** has not been accounted for.
 - ▶ the ARIMA model assumes **historical patterns will not change** during the forecast period.
 - ▶ the ARIMA model assumes **uncorrelated future errors**.