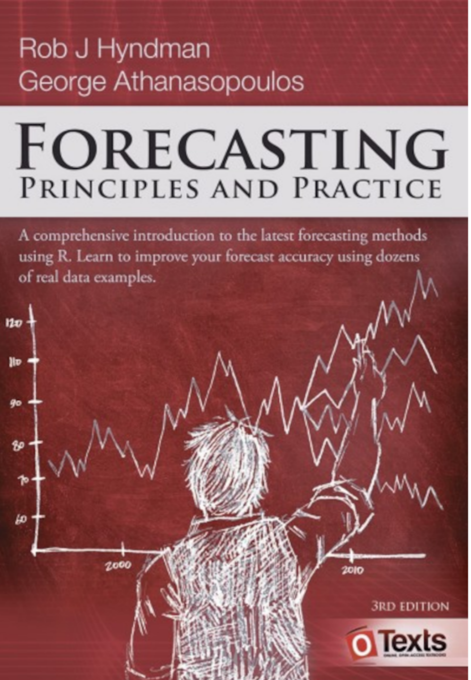

8. Exponential smoothing

8.6 Estimation and model selection

OTexts.org/fpp3/

Estimating ETS models

- Smoothing parameters α, β, γ and ϕ, and the initial states ℓ_0, b_0, $s_0, s_{-1}, \dots, s_{-m+1}$ are estimated by maximising the "likelihood" = the probability of the data arising from the specified model.
- For models with additive errors equivalent to minimising SSE.
- For models with multiplicative errors, **not** equivalent to minimising SSE.

Innovations state space models

Let $\boldsymbol{x}_t = (\ell_t, b_t, s_t, s_{t-1}, \dots, s_{t-m+1})$ and $\varepsilon_t \overset{\text{iid}}{\sim} \text{N}(0, \sigma^2)$.

$$\begin{aligned} y_t &= \underbrace{h(\boldsymbol{x}_{t-1})}_{\mu_t} + \underbrace{k(\boldsymbol{x}_{t-1})\varepsilon_t}_{e_t} \\ \boldsymbol{x}_t &= f(\boldsymbol{x}_{t-1}) + g(\boldsymbol{x}_{t-1})\varepsilon_t \end{aligned}$$

Additive errors

$$k(\boldsymbol{x}_{t-1}) = 1. \qquad y_t = \mu_t + \varepsilon_t.$$

Multiplicative errors

$$k(\boldsymbol{x}_{t-1}) = \mu_t. \qquad y_t = \mu_t(1 + \varepsilon_t).$$

$\varepsilon_t = (y_t - \mu_t)/\mu_t$ is relative error.

Innovations state space models

Estimation

$$L^*(\boldsymbol{\theta}, \boldsymbol{x}_0) = T \log\left(\sum_{t=1}^{T} \varepsilon_t^2\right) + 2\sum_{t=1}^{T} \log\left|k(\boldsymbol{x}_{t-1})\right|$$
$$= -2\log(\text{Likelihood}) + \text{constant}$$

- Estimate parameters $\boldsymbol{\theta} = (\alpha, \beta, \gamma, \phi)$ and initial states $\boldsymbol{x}_0 = (\ell_0, b_0, s_0, s_{-1}, \dots, s_{-m+1})$ by minimizing L^*.

Parameter restrictions

Usual region

- Traditional restrictions in the methods $0 < \alpha, \beta^*, \gamma^*, \phi < 1$ (equations interpreted as weighted averages).
- In models we set $\beta = \alpha\beta^*$ and $\gamma = (1 - \alpha)\gamma^*$.
- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

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- Therefore $0 < \alpha < 1, \quad 0 < \beta < \alpha \quad$ and $0 < \gamma < 1 - \alpha$.
- $0.8 < \phi < 0.98$ — to prevent numerical difficulties.

Admissible region

- To prevent observations in the distant past having a continuing effect on current forecasts.
- Usually (but not always) less restrictive than *traditional* region.
- For example for ETS(A,N,N):
  traditional $0 < \alpha < 1$ while *admissible* $0 < \alpha < 2$.

Model selection

Akaike's Information Criterion

$$\text{AIC} = -2\log(\text{L}) + 2k$$

where L is the likelihood and k is the number of parameters & initial states estimated in the model.

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Corrected AIC

$$\text{AIC}_{\text{c}} = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

Model selection

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Corrected AIC

$$\text{AIC}_\text{c} = \text{AIC} + \frac{2k(k+1)}{T-k-1}$$

which is the AIC corrected (for small sample bias).

Bayesian Information Criterion

$$\text{BIC} = \text{AIC} + k[\log(T) - 2].$$

AIC and cross-validation

Minimizing the AIC assuming Gaussian residuals is asymptotically equivalent to minimizing one-step time series cross validation MSE.

Automatic forecasting

From Hyndman et al. (IJF, 2002):

- Apply each model that is appropriate to the data. Optimize parameters and initial values using MLE (or some other criterion).
- Select best method using AICc:
- Produce forecasts using best method.
- Obtain forecast intervals using underlying state space model.

Method performed very well in M3 competition.

Some unstable models

- Some of the combinations of (Error, Trend, Seasonal) can lead to numerical difficulties; see equations with division by a state.
- These are: ETS(A,N,M), ETS(A,A,M), ETS(A,A_d,M).
- Models with multiplicative errors are useful for strictly positive data, but are not numerically stable with data containing zeros or negative values. In that case only the six fully additive models will be applied.

Example: National populations

```
fit <- global_economy |>
  mutate(Pop = Population / 1e6) |>
  model(ets = ETS(Pop))
fit
```

```
## # A mable: 263 x 2
## # Key:     Country [263]
##    Country                       ets
##    <fct>                     <model>
##  1 Afghanistan          <ETS(A,A,N)>
##  2 Albania              <ETS(M,A,N)>
##  3 Algeria              <ETS(M,A,N)>
##  4 American Samoa       <ETS(M,A,N)>
##  5 Andorra              <ETS(M,A,N)>
##  6 Angola               <ETS(M,A,N)>
##  7 Antigua and Barbuda <ETS(M,A,N)>
##  8 Arab World           <ETS(M,A,N)>
##  9 Argentina            <ETS(A,A,N)>
## 10 Armenia              <ETS(M,A,N)>
```

Example: National populations

```
fit |>
  forecast(h = 5)
```

```
## # A fable: 1,315 x 5 [1Y]
## # Key:     Country, .model [263]
##    Country     .model  Year             Pop .mean
##    <fct>       <chr>  <dbl>          <dist> <dbl>
##  1 Afghanistan ets     2018    N(36, 0.012) 36.4
##  2 Afghanistan ets     2019    N(37, 0.059) 37.3
##  3 Afghanistan ets     2020     N(38, 0.16) 38.2
##  4 Afghanistan ets     2021     N(39, 0.35) 39.0
##  5 Afghanistan ets     2022     N(40, 0.64) 39.9
##  6 Albania     ets     2018 N(2.9, 0.00012)  2.87
##  7 Albania     ets     2019   N(2.9, 6e-04)  2.87
##  8 Albania     ets     2020  N(2.9, 0.0017)  2.87
##  9 Albania     ets     2021  N(2.9, 0.0036)  2.86
## 10 Albania     ets     2022  N(2.9, 0.0066)  2.86
## # ... with 1,305 more rows
```

Residuals

Response residuals

$$e_t = y_t - \hat{y}_{t|t-1}$$

Innovation residuals

Additive error model:

$$\hat{\varepsilon}_t = y_t - \hat{y}_{t|t-1} = e_t$$

Multiplicative error model:

$$\hat{\varepsilon}_t = \frac{y_t - \hat{y}_{t|t-1}}{\hat{y}_{t|t-1}} \neq e_t$$

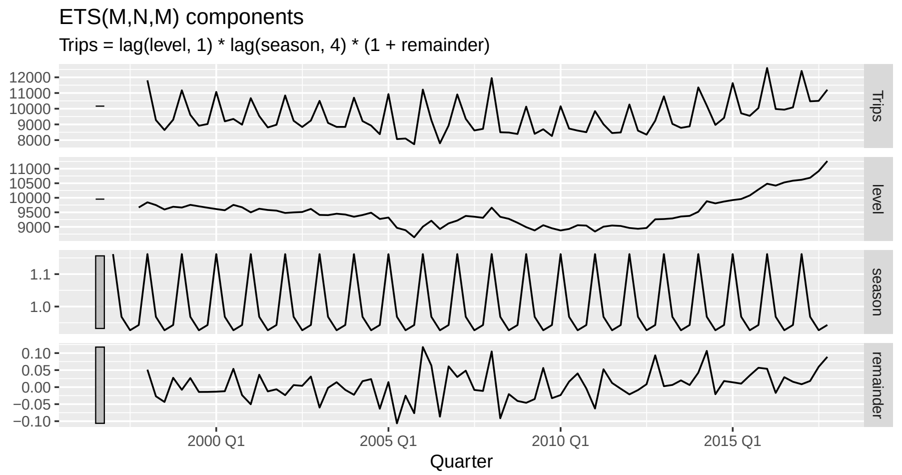
Example: Australian holiday tourism

```r
aus_holidays <- tourism |>
  filter(Purpose == "Holiday") |>
  summarise(Trips = sum(Trips))
fit <- aus_holidays |>
  model(ets = ETS(Trips)) |>
  report()
```

```
## Series: Trips
## Model: ETS(M,N,M)
##   Smoothing parameters:
##     alpha = 0.358
##     gamma = 0.000969
##
##   Initial states:
##   l[0]  s[0] s[-1] s[-2] s[-3]
##   9667 0.943 0.927 0.968  1.16
##
##   sigma^2:  0.0022
##
##   AIC AICc  BIC
## 1331 1333 1348
```

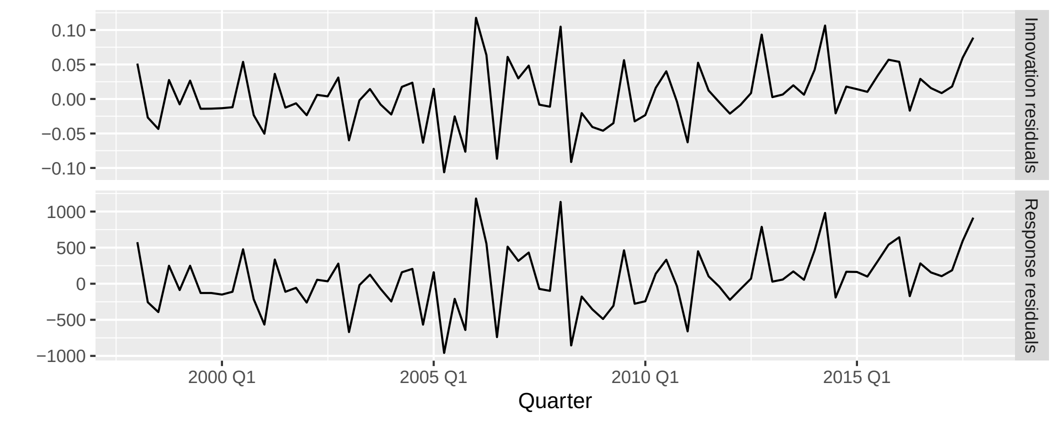
Example: Australian holiday tourism

```r
components(fit) |> autoplot() + labs(title = "ETS(M,N,M) components")
```



Example: Australian holiday tourism

```
residuals(fit)
residuals(fit, type = "response")
```



Example: Australian holiday tourism

```r
fit |>
  augment()
```

```
## # A tsibble: 80 x 6 [1Q]
## # Key:       .model [1]
##    .model Quarter  Trips .fitted .resid   .innov
##    <chr>    <qtr>  <dbl>   <dbl>  <dbl>    <dbl>
##  1 ets    1998 Q1 11806.  11230.  576.   0.0513
##  2 ets    1998 Q2  9276.   9532. -257.  -0.0269
##  3 ets    1998 Q3  8642.   9036. -393.  -0.0435
##  4 ets    1998 Q4  9300.   9050.  249.   0.0275
##  5 ets    1999 Q1 11172.  11260.  -88.0 -0.00781
##  6 ets    1999 Q2  9608.   9358.  249.   0.0266
##  7 ets    1999 Q3  8914.   9042. -129.  -0.0142
##  8 ets    1999 Q4  9026.   9154. -129.  -0.0140
##  9 ets    2000 Q1 11071.  11221. -150.  -0.0134
## 10 ets    2000 Q2  9196.   9308. -111.  -0.0120
## # ... with 70 more rows
```

Example: Australian holiday tourism

```r
fit |>
  augment()
```

Innovation residuals (`.innov`) are given by $\hat{\varepsilon}_t$ while regular residuals (`.resid`) are $y_t - \hat{y}_{t/t-1}$. They are different when the model has multiplicative errors.

```
## # A tsibble: 80 x 6 [1Q]
## # Key:       .model [1]
##    .model Quarter  Trips .fitted .resid   .innov
##    <chr>    <qtr>  <dbl>   <dbl>  <dbl>    <dbl>
##  1 ets    1998 Q1 11806.  11230.  576.   0.0513
##  2 ets    1998 Q2  9276.   9532. -257.  -0.0269
##  3 ets    1998 Q3  8642.   9036. -393.  -0.0435
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## # ... with 70 more rows
```