ECONOMETRICA
JOURNAL OF THE ECONOMETRIC SOCIETY

A Threshold Regression Model
Author(s): Marcel G. Dagenais
Source: *Econometrica*, Vol. 37, No. 2 (Apr., 1969), pp. 193-203
Published by: The Econometric Society
Stable URL: https://www.jstor.org/stable/1913530
Accessed: 21-10-2019 13:53 UTC

Econometrica, Vol. 37, No. 2 (April, 1969)

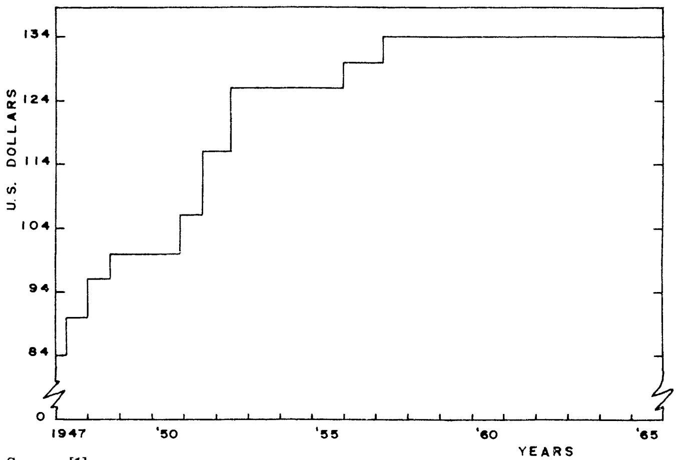
A THRESHOLD REGRESSION MODEL

BY MARCEL G. DAGENAIS[1]

A special regression model is suggested to analyze economic variables possessing steplike time paths. The dependent variable is assumed not to move until the concerted action of the independent variables and the error term induces it to overcome its reaction threshold. After the theoretical solution is presented, the model is shown to be applicable both to time series, such as newsprint prices, and to cross sections, such as household durable good purchases and plant capacity increases. Finally, computational problems are briefly discussed.

1. INTRODUCTION

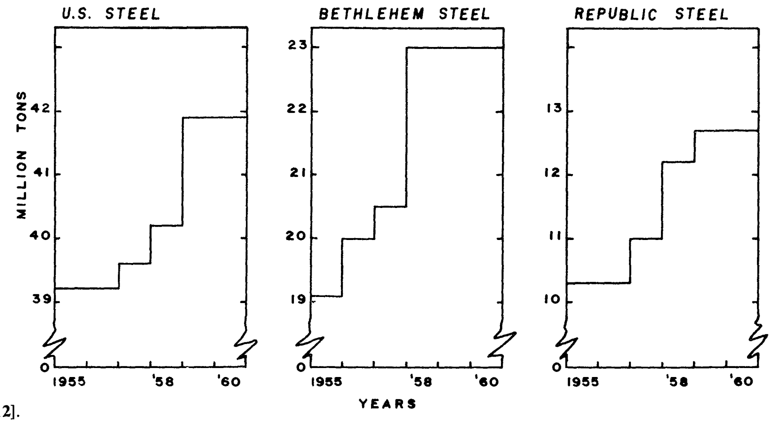
SEVERAL ECONOMIC VARIABLES possess steplike time paths. Among such variables are prices of oligopolistic industries, plant capacities, and purchases of durable goods or financial assets by households. The accompanying figures reproduce, as illustrations, the evolutions over different time periods of the New York delivered price of newsprint paper (Figure 1) and the rated ingot capacity of the three largest American steel producers (Figure 2). Discontinuities in plant capacity increases are justified by reasons of technical efficiency [**4**; **6**, p. 244], while steplike variations of oligopolistic prices are explained mainly by market control preoccupations [**2**, p. 294; **9**, pp. 99–100].



Source: [**1**].

FIGURE 1.—New York delivered newsprint prices, in U.S. dollars, 1947–1965.

[1] The author wishes to thank Professor T. I. Matuszewski for his helpful suggestions. The comments of the referees were also very useful. Messrs. C. Montmarquette, G. Lapierre, R. Depatie, and G. Gauthier assisted the author in the various phases of the project. Experiments on computational problems raised by the model were made on the computer of the "Centre de Calcul de l'Université de Montréal." The research was supported, in part, by the Canada Council of Arts.



Source [**12**].

FIGURE 2.—Rated ingot capacity of the three largest American steel producers, in million tons, 1955–1960.

The discontinuous character of durable good purchases has particularly attracted the attention of numerous econometricians in the last few years.[2] It has been generally admitted that households adjust to the desired stock of consumer durables in a stepwise manner. Most recent statistical models proposed to handle the observed discontinuities involve two separate functions. A first function with a 0, 1 regressor y^* (to buy or not to buy) is fitted linearly, by least squares, to a set of independent variables X_i ($i = 1, \ldots, k$). Then a second function relates the amounts of the purchases (y^{**}) made by the households which did buy durables, to the X_i's:

$$(1) \qquad y^* = \sum_{i=1}^{k} \beta_i X_i + \varepsilon_1,$$

$$(2) \qquad y^{**} = \sum_{i=1}^{k} \xi_i X_i + \varepsilon_2.$$

The above approach has several shortcomings. On the one hand, estimated $E(y^*)$'s might very well lie outside the 0, 1 range, for certain values of the X_i's. This anomaly, however, could be corrected by estimating the β parameters with the probit model [**15**, p. 250]. Similarly, estimated $E(y^{**})$'s could turn out negative, for certain values of the X_i's. This defect could again be avoided by using the Tobit model [**24**] in the second equation. A much more serious deficiency of the above approach is that the separate treatment of the two regressions implies that the probability distibutions of the residual errors ε_1 and ε_2 are mutually independent, while in practice this is not likely to be the case. Indeed, one might expect a positive correlation to exist between ε_1 and ε_2. The fact that households with low probabilities of purchasing decide to buy will generally indicate that extraneous causes, having little or no relationship to the independent variables,

[2] See, for example, J. Fisher [**13**], A. Goldberger and Maw Lin Lee [**16**], D. S. Huang [**17**], L. R. Klein and J. B. Lansing [**19**], G. Orcutt and A. Rivlin [**20**], De-Min Wu [**25**], and G. Chow [**5**].

have influenced their behavior. The same causes may very well induce them to spend relatively larger amounts.

In contrast with the twin linear probability function, the model proposed in the next section introduces discontinuities within the regression explaining the level of the dependent variable. Furthermore, this model does not apply exclusively to variables with only upward (or downward) reaction thresholds, such as durable good purchases, but it can also handle variables involving simultaneously both types of thresholds, such as oligopolistic prices.

On the other hand, the suggested model raises difficult computational problems, though no longer insuperable with modern computers. These problems are discussed in Section 4.

2. THE MODEL[3]

Before writing the equations of the model, let us define the symbols used as follows:

W: dependent variable;
$X_i\ (i = 1, \ldots, k)$: independent variables (fixed numbers);
$\beta_i\ (i = 0, \ldots, k)$: regression parameters;
s^*, v^*: threshold parameters;
L: mass point of the cumulative distribution function of W;
v: numerical value of the lower threshold:
s: numerical value of the upper threshold;
u, η_1, η_2: stochastic variables.

The formulation of our model is inspired from the probit regression models developed by Tobin and Rosett [**24**, **21**] a few years ago. Let

$$W = Y_1 - u \qquad (Y_1 - u < L - v), \tag{3}$$

$$W = L \qquad (Y_1 - u \geqslant L - v \quad \text{and} \quad Y_2 - u \geqslant L + s), \tag{4}$$

$$W = Y_2 - u \qquad (Y_2 - u > L + s), \tag{5}$$

where

$$Y_1 = \beta_0' + \sum_{i=1}^{k} \beta_i X_i, \tag{6}$$

$$Y_2 = \beta_0'' + \sum_{i=1}^{k} \beta_i X_i, \tag{7}$$

$$v = v^* - \eta_1 \geqslant 0, \tag{8}$$

$$s = s^* + \eta_2 \geqslant 0, \tag{9}$$

$$\eta_1 \leqslant v^*, \tag{10}$$

[3] This theoretical model has been first described in a communication given at the International Statistical Institute Meetings in Belgrade in 1965. A brief summary of this communication has appeared in the Bulletin of the International Statistical Institute [**18**].

(11) $\eta_2 \geqslant -s^*,$

(12) $\beta_0' \geqslant \beta_0''.$

The variable η_1 can never be greater than the lower threshold parameter v^*. η_2 can never be smaller than $-s^*$. The variables u, η_1, and η_2 are assumed to be mutually interdependent with correlation coefficients ρ_{u,η_1}, ρ_{u,η_2}, and ρ_{η_1,η_2}. Their joint probability density function is:

$$T(u,\eta_1,\eta_2) = f(u,\eta_1,\eta_2) \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1,\eta_2)\, d\eta_1\, d\eta_2 \tag{13}$$

where $f(u,\eta_1,\eta_2)$ is a trivariate normal distribution centered on $u = \eta_1 = \eta_2 = 0$ and $f(\eta_1,\eta_2)$ is the corresponding bivariate normal distribution, after u has been integrated out.

Figure 3 shows how W would vary with $\Sigma\, \beta_i X_i$, for an individual observation corresponding to given values of the error terms.

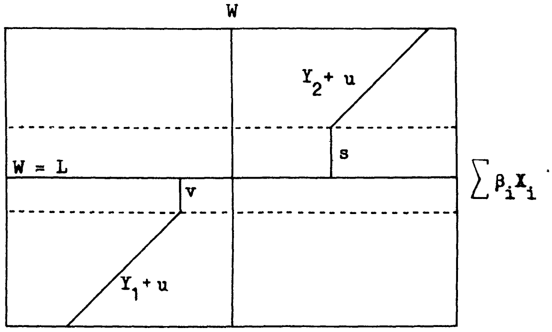


FIGURE 3

Practical applications of the model require observations on W and the X_i's; L is also assumed to be known a priori and, just as for W or the X_i's, its value may vary from observation to observation. The β's as well as s^*, v^* and the elements of the variance-covariance matrix of the three stochastic variables are the unknown parameters to be estimated.[4]

[4] A further extension of the model would consist of assuming that the values of s^* and v^* differ between observations and are also linear functions of independent variables Z_λ^* $(\lambda:1\ldots k^*)$ and Z_ω^{**} $(\omega:1\ldots k^{**})$:

(i) $s = \sum_{\lambda=1}^{k^*} \theta_\lambda^* Z_\lambda^* + \eta_2,$

(ii) $v = \sum_{\omega=1}^{k^{**}} \theta_\omega^{**} Z_\omega^{**} - \eta_1.$

The θ^*'s and θ^{**}'s would then have to be estimated instead of s^* and v^*.

Assuming that we have a random sample containing m observations with $W < L$, q observations with $W = L$, and r with $W > L$, the natural logarithm of the likelihood function becomes:[5]

$$(14)\qquad \phi^* = \sum_{j=1}^{m} \ln\left\{\int_{-s^*}^{\infty}\int_{W_j - L_j + v^*}^{v^*} f(Y_{1,j} - W_j, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\right\}$$
$$+ \sum_{p=1}^{q} \ln\left\{\int_{-\infty}^{Y_{2,p}-L_p}\int_{Y_{2,p}-u-L_p-s^*}^{+\infty}\int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du\right.$$
$$+ \int_{Y_{2,p}-L_p}^{Y_{1,p}-L_p}\int_{-s^*}^{\infty}\int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du$$
$$\left.+ \int_{Y_{1,p}-L_p}^{+\infty}\int_{-s^*}^{+\infty}\int_{-\infty}^{Y_{1,p}-u-L_p+v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du\right\}$$
$$+ \sum_{\ell=1}^{r} \ln\left\{\int_{-s^*}^{W_\ell - Ll - s^*}\int_{-\infty}^{v^*} f(Y_{2,\ell} - W_\ell, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\right\}$$
$$- (m + q + r)\ln\left\{\int_{-s^*}^{+\infty}\int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2\right\}.$$

The parameters are estimated by the values that maximize φ^*. The negative inverse of the matrix of second derivatives of φ^*, at this point, gives the asymptotic variances and covariances of the parameter estimates.

The expected value locus of W is

$$(15)\qquad E(W) = \left\{\int_{-\infty}^{L}\int_{-s^*}^{+\infty}\int_{x-L+v^*}^{v^*} xf(Y_1 - x, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, dx\right.$$
$$+ L\left[\int_{-\infty}^{Y_2-L}\int_{Y_2-u-L-s^*}^{+\infty}\int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du\right.$$
$$+ \int_{Y_2-L}^{Y_1-L}\int_{-s^*}^{+\infty}\int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du$$
$$\left.+ \int_{Y_1-L}^{+\infty}\int_{-s^*}^{+\infty}\int_{-\infty}^{Y_1-u-L+v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du\right]$$
$$\left.+ \int_{L}^{+\infty}\int_{-s^*}^{x-L-s^*}\int_{-\infty}^{v^*} xf(Y_2 - x, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, dx\right\}$$
$$\times \left\{\int_{-s^*}^{+\infty}\int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2\right\}^{-1}.$$

This locus is illustrated on Figure 4.

[5] Intermediate steps for deriving equation (14) are shown in the appendix. No evidence exists, for the moment, on the robustness of the model with respect to the hypotheses concerning the nature of the distribution function of the stochastic terms, as well as the randomness of the sample.

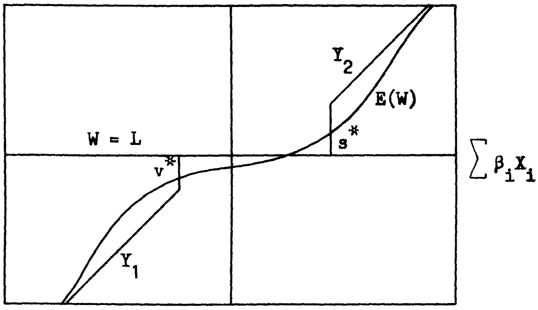


FIGURE 4

The model proposed in the preceding pages constitutes a general class which includes as special cases:

(i) the classical regression—by letting $L \to \infty$, $\rho_{u,\eta_1} \to 0$, and $\rho_{u,\eta_2} \to 0$;

(ii) the Tobit model for limited dependent variables—by postulating $\rho_{u,\eta_1} = \rho_{u,\eta_2} \to 0$ and either $\sigma^2_{\eta_1} = s^* \to 0$ and $v^* \to \infty$ or $\sigma^2_{\eta_2} = v^* \to 0$ and $s^* \to \infty$, depending upon whether the limit is a lower limit or an upper limit;

(iii) Rosett's friction model—by assuming $\sigma^2_{\eta_1} = \sigma^2_{\eta_2} = s^* = v^* \to 0$.

3. POSSIBLE APPLICATIONS OF THE MODEL

Possible applications of the model to such variables as newsprint prices, capacities of steel plants, and household durable good purchases, have been suggested earlier. Let us specify more precisely how the model could be formulated in each of these three cases.

Newsprint Price

It is well known that the New York delivered price of newsprint (W_t) is set by a leader and readily adopted by almost all North American mills.[6] Assuming that, when the industry is working below capacity, the independent variables influencing the leader's decision are the operating ratio of the industry ($X_{1,t}$) and the marginal cost of production ($X_{2,t}$) of the previous period [**9**, p. 98], the model would become:

$$W_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} - u_t \qquad (\beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} - u_t < W_{t-1} - v_t), \tag{16}$$

$$W_t = W_{t-1} \qquad (W_{t-1} - v_t \leqslant \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} - u_t \leqslant W_{t-1} + s_t), \tag{17}$$

$$W_t = \beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} - u_t \qquad (\beta_0 + \beta_1 X_{1,t} + \beta_2 X_{2,t} - u_t > W_{t-1} + s_t). \tag{18}$$

[6] See M. G. Dagenais [**9**, p. 15]. A threshold regression model similar to this one, but with threshold values assumed known a priori, was used in this reference.

In this case, the observations would be time series. The mass point L would correspond to the price of the previous period, i.e., the previous month or previous quarter. Observations of the first group (equation 16) would refer to periods of price decreases; observations in the second group would represent periods of price stability, while the third group would consist of periods of price increases. It has been assumed here that β_0' and β_0'' are equal, since there does not seem to be any reason to suppose otherwise.[7]

Steel Capacity

In the case of steel capacity, the dependent variable (W_c) could represent percentage increases of capacity of individual plants in year t, and the independent variables might be excess capacity of year $t-2$ ($X_{1,c}$) and the ratio of retained earnings to the value of physical assets in $t-1$ ($X_{2,c}$). The mass point would be located at zero. The sample would be constituted by a cross section of mills, for a given year. The equations would be the following:

$$W_c = 0 \qquad (\beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c \leqslant s_c), \tag{19}$$

$$W_c = \beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c \qquad (\beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c > s_c). \tag{20}$$

There would be no observation in group one, since only positive changes are considered; observations in group two would belong to plants reporting no additional capacity, and observations in group three would represent mills with capacity increases.

[7] The above presentation assumes that even if observations are time series, there is no serial correlation between the stochastic variables of successive periods. If such correlation existed between the u's only, it would seem possible to modify the threshold model in order to reduce it. The following transformation might be used, for example:

$$\overline{\Delta W_t} = \overline{\Delta Y_{1,t}} - u_t \qquad (\overline{\Delta Y_{1,t}} - u_t < -v_t), \tag{iii}$$

$$\overline{\Delta W_t} = 0 \qquad (\overline{\Delta Y_{1,t}} - u_t \geqslant -v_t \quad \text{and} \quad \overline{\Delta Y_{2,t}} - u_t \leqslant s_t), \tag{iv}$$

$$\overline{\Delta W_t} = \overline{\Delta Y_{2,t}} - u_t \qquad (\overline{\Delta Y_{2,t}} - u_t > s_t), \tag{v}$$

where:

$$\overline{\Delta W_t} = (W_t - W_{t-h})/\sqrt{h},$$

$$\overline{\Delta Y_{1,t}} = (Y_{1,t} - Y_{1,t-h})/\sqrt{h},$$

$$\overline{\Delta Y_{2,t}} = (Y_{2,t} - Y_{2,t-h})/\sqrt{h},$$

and h is the number of periods elapsed since the last change in W. Multiplication by $1/\sqrt{h}$ preserves the homoscedasticity of $\overline{\Delta W_t}$.

For further comments and numerical results concerning a similar approach, see M. G. Dagenais [**10**, pp. 295–299 and 320–323].

Cases involving no observation in group one (or three) are easily treated within the framework of the general model by assuming that $v^* \to \infty$ and $\sigma_{\eta_1}^2 \to 0$ (or $s^* \to \infty$ and $\sigma_{\eta_2}^2 \to 0$).

Household Durables

Following the suggestions of De-Min Wu [**25**, p. 763] and others, net expenditures on a given household durable good (W)—automobiles, for example—may be considered as depending upon the gap between desired stock and stock on hand. Neglecting the stochastic terms, we could write

$$(21) \qquad W_c = \alpha[S_c^* - (S_c - d_c)]$$

where S_c^* is the stock desired at the end of period t, S_c is stock on hand at the end of $t - 1$, and d_c is depreciation during t.

If desired stock is then assumed to be a linear function of household income (X_1),

$$(22) \qquad S_c^* = \xi_0 + \xi_1 X_{1,c},$$

the threshold model can be applied in the following way:

$$(23) \qquad W_c = 0 \qquad (\beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c \leqslant s_c),$$

$$(24) \qquad W_c = \beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c \qquad (\beta_0 + \beta_1 X_{1,c} + \beta_2 X_{2,c} - u_c > s_c),$$

where $X_{2,c} = S_c - d_c$, $\beta_0 = \alpha\xi_0$, $\beta_1 = \alpha\xi_1$, and $\beta_2 = -\alpha$.

Households reporting no net increase in their stock of the durable good in question would be in group two, while households showing an increase would constitute group three. As in the steel capacity case, there would be no observation in group one.

4. COMPUTATIONAL PROBLEMS

Empirical solutions of threshold regressions are found by iterative procedures. Current numerical experiments suggest the following comments on computational problems.

Evaluation of φ^* and $E(W)$

The likelihood function φ^* contains double and triple integrals of the normal distribution. Similarly, the expected value of W can easily be transformed so as to involve only normal integrals multiplied by constants.

The methods developed by Curnow and Dunnett [**8**] and by Steck [**22**] to evaluate double and triple integrals have proven to be very efficient.

Determination of an Initial Solution for the Parameter Estimates

Initial values of the β's and of σ_u^2 may be found by running a classical multiple regression over the total sample. First estimates of v^* and s^* may be obtained by computing the average increase and decrease of W in observations showing such variations. Estimated variances of these averages may be used as first approximations to $\sigma^2_{\eta_1}$ and $\sigma^2_{\eta_2}$. Cross products of deviations from the above averages with estimated residuals of the least squares regression yield estimates of ρ_{u,η_1} and ρ_{u,η_2}. Upper and lower bounds may then be derived for ρ_{η_1,η_2} from those of the determinant of the correlation matrix, and the middle point of the permissible interval may serve to evaluate ρ_{η_1,η_2}.

Choice of Method for Maximizing φ^*

Two methods are presently under study, namely Newton's method, which is based on the evaluation of first and second derivatives [**7**], and Davidon's method, which needs only first derivatives but yields an estimate of the matrix of second derivatives at the maximum as a by product [**11, 14, 23**].

Newton's method, when applicable, converges probably more rapidly than Davidon's technique; but it has two serious shortcomings. First, Newton's system converges, in theory, only if the path from the initial to the final solution lies on a convex hypersurface. Furthermore, this algorithm involves the computation of second derivatives of φ^*, which is a difficult task. Estimating second derivatives by numerical methods entails risks of gross inaccuracies [**3**] that might even preclude convergence, while the evaluation by analytical methods implies a great amount of programming work. Davidon's method, on the other hand, always converges in principle, and uses only first derivatives. Although it may require a greater number of iterations, it appears presently to be the most promising approach.

Ecole des Hautes Etudes Commerciales, Montréal

APPENDIX

Derivation of the Likelihood Function

In order to derive the likelihood function of our model from the premises, one must first obtain the distribution function of W. In turn, before writing this distribution function, it is useful to establish the following relationships.

For given values of the X_i's and L:

1

$$\text{(A.1)}\qquad \Pr(u' < u < u' + du) \quad \text{and} \quad Y_1 - u < L - v) = \Pr(u' < u < u + du \ \text{ and } \ \eta_1 > Y_1 - u - L + v^*)$$

$$\text{(A.2)}\qquad = \int_{-s^*}^{+\infty} \int_{Y_1 - u' - L + v^*}^{v^*} f(u', \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

2

$$\text{(A.3)}\qquad \Pr(u' < u < u' + du \quad \text{and} \quad Y_2 - u > L + s) = \Pr(u' < u < u + du \ \text{ and } \ \eta_2 < Y_2 - u - L - s^*)$$

$$\text{(A.4)}\qquad = \int_{-s^*}^{Y_2 - u' - L - s^*} \int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

3

(A.5) $\Pr(u' < u < u' + du, Y_1 - u \geqslant L - v \quad \text{and} \quad Y_2 - u \leqslant L + s)$

$= \Pr(u' < u < u' + du, \quad \eta_1 \leqslant Y_1 - u - L + v^* \quad \text{and} \quad \eta_2 \geqslant Y_2 - u - L - s^*)$.

If $u' < Y_2 - L$,

(A.6) $$\Pr(u' < u < u' + du, \quad \eta_1 \leqslant Y_1 - u - L + v^* \quad \text{and} \quad \eta_2 \geqslant Y_2 - u - L - s^*) = \int_{Y_2 - u' - L - s^*}^{+\infty} \int_{-\infty}^{v^*} f(u', \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

If $Y_1 - L \geqslant u' \geqslant Y_2 - L$,

(A.7) $$\Pr(u' < u < u' + du, \quad \eta_1 \leqslant Y_1 - u - L + v^* \quad \text{and} \quad \eta_2 \geqslant Y_2 - u - L - s^*) = \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(u', \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

If $u' > Y_1 - L$,

(A.8) $$\Pr(u' < u < u' + du, \quad \eta_1 \leqslant Y_1 - u - L + v^* \quad \text{and} \quad \eta_2 \geqslant Y_2 - u - L - s^*) = \int_{-s^*}^{+\infty} \int_{-\infty}^{Y_1 - u' - L + v^*} f(u', \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

From (A.2), (A.4), (A.6), (A.7), and (A.8), the distribution function of W is then readily inferred:

1

(A.9) $$\Pr(W' < W < W' + dW, \quad \text{when} \quad W < L) = \int_{-s^*}^{+\infty} \int_{W' - L + v^*}^{v^*} f(Y_1 - W', \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, dW \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 ,$$

where $W' = Y_1 - u'$ and therefore $u' = Y_1 - W'$.

2

(A.10) $\Pr(W = L) = \Pr(-\infty < u < +\infty, Y_1 - u \geqslant L - v \quad \text{and} \quad Y_2 - u \leqslant L + s)$

(A.11) $$= \left\{ \int_{-\infty}^{Y_2 - L} \int_{Y_2 - u - L - s^*}^{+\infty} \int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du + \int_{Y_2 - L}^{Y_1 - L} \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du + \int_{Y_1 - L}^{+\infty} \int_{-s^*}^{+\infty} \int_{-\infty}^{Y_1 - u - L + v^*} f(u, \eta_1, \eta_2)\, d\eta_1\, d\eta_2\, du \right\} \Big/ \int_{-s^*}^{\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 .$$

3

(A.12) $$\Pr(W' < W < W' + dW, \quad \text{when} \quad W > L) = \int_{-s^*}^{W' - L - s^*} \int_{-\infty}^{v^*} f(Y_2 - W', n_1, \eta_2)\, d\eta_1\, d\eta_2\, dW \Big/ \int_{-s^*}^{+\infty} \int_{-\infty}^{v^*} f(\eta_1, \eta_2)\, d\eta_1\, d\eta_2 ,$$

where $W' = Y_2 - u'$ and therefore $u' = Y_2 - W'$.

REFERENCES

[1] American Newspaper Publishers Association: *Newsprint Bulletin* (1945–1966), A.N.P.A., New York.

[2] Bain, Joe S.: *Pricing, Distribution, and Employment*, Holt, Rinehart, and Winston, New York, December, 1960.

[3] Bellman, R. E., and S. E. Dreyfus: *Applied Dynamic Programming*, Princeton University Press, 1962.

[4] CHENERY, H. B.: "Overcapacity and the Acceleration Principle," *Econometrica*, January, 1952, pp. 1–28.

[5] CHOW, GREGORY: "Statistical Demand Function for Automobiles and their Use for Forecasting," *The Demand for Durable Goods*, Arnold C. Harberger (ed.), University of Chicago Press, Chicago, 1960.

[6] CLARK, P. G.: "The Telephone Industry: A Study in Private Investment," in *Studies in the Structure of the American Economy*, by W. W. Leontief and others, Oxford University Press, New York, 1953, ch. 7, pp. 243–294.

[7] CROCKETT, J. B., AND H. CHERNOFF: "Gradient Methods of Maximization," *Pacific Journal of Mathematics*, 1955, pp. 33–50.

[8] CURNOW, R. N., AND C. N. DUNNETT: "The Numerical Evolution of Certain Multivariate Normal Integrals," *Annals of Mathematical Statistics*, 1967, pp. 571–579.

[9] DAGENAIS, MARCEL G.: *The Determination of the Output and Price Levels in the North American Newsprint Paper Industry*, Ph.D. Dissertation presented at Yale University, University Microfilm Inc., Ann Arbor, Michigan, 1964.

[10] ———: "The Short-Run Determination of Output and Shipments in the North American Newsprint Paper Industry," *Yale Economic Essays*, Fall, 1964, pp. 281–328.

[11] DAVIDON, WILLIAM C.: *Variable Metric Method for Minimization*, ANL-5990 (Rev. 2), Argonne National Laboratory, Argonne, February, 1966.

[12] *Financial Analysis of the Steel Industry*: Supplement to *Steel*, 1955–1962, Cleveland.

[13] FISHER, JANET: "An Analysis of Consumer Durable Goods Expenditures in 1957," *Review of Economics and Statistics*, XLIV, February, 1962, pp. 64–71.

[14] FLETCHER, R., AND M. J. D. POWELL: "A Rapidly Convergent Descent Method for Minimization," *Computer Journal*, July, 1963, pp. 163–168.

[15] GOLDBERGER, ARTHUR S.: *Econometric Theory*, John Wiley and Sons, Inc., New York, 1964.

[16] GOLDBERGER, ARTHUR S., AND MAN LIN LEE: "Toward a Microanalytic Model of the Household Sector," *American Economic Review*, LII, May, 1962, pp. 241–251.

[17] HUANG, DAVID S.: *The Demand for Automobiles in 1956 and 1957—A Cross-section Analysis*, Unpublished Ph.D. Dissertation, University of Washington, 1961.

[18] INTERNATIONAL STATISTICAL INSTITUTE: *Bulletin of the International Statistical Institute: Proceedings of the 35th Session*, Vol. XLI.

[19] KLEIN, LAWRENCE, AND JOHN B. LANSING: "Decisions to Purchase Consumer Durable Goods," *Journal of Marketing*, XX, October, 1955, pp. 109–132.

[20] ORCUTT, GUY H., AND ALICE RIVLIN: "An Economic and Demographic Model of the Household Sector: A Progress Report," *Demographic and Economic Change in Developed Countries*, Princeton, National Bureau of Economic Research, 1960, pp. 116–123.

[21] ROSETT, R. N.: "A Statistical Model of Friction in Economics," *Econometrica*, Vol. 27, No. 2, April, 1959, pp. 263–267.

[22] STECK, C. P.: "A Table for Computing Trivariate Normal Probabilities," *Annals of Mathematical Statistics*, September, 1958, pp. 780–800.

[23] STEWART, G. W., III: "A Modification of Davidon's Minimization Method to Accept Difference Approximations of Derivatives," *Journal of the Association for Computing Machinery*, January, 1967, pp. 72–83.

[24] TOBIN, J.: "Estimation of Relationships for Limited Dependent Variables," *Econometrica*, Vol. 26, No. 1, January, 1958, pp. 24–36.

[25] WU, DE-MIN: "An Empirical Analysis of Household Durable Goods Expenditure," *Econometrica*, Vol. 23, No. 4, October, 1965, pp. 761–780.