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NONPARAMETRIC INSTRUMENTAL VARIABLE ESTIMATION OF STRUCTURAL QUANTILE EFFECTS

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We study the asymptotic distribution of Tikhonov regularized estimation of quantile structural effects implied by a nonseparable model. The nonparametric instrumental variable estimator is based on a minimum distance principle. We show that the minimum distance problem without regularization is locally ill-posed, and we consider penalization by the norms of the parameter and its derivatives. We derive pointwise asymptotic normality and develop a consistent estimator of the asymptotic variance. We study the small sample properties via simulation results and provide an empirical illustration of estimation of nonlinear pricing curves for telecommunications services in the United States.

KEYWORDS: Nonparametric quantile regression, instrumental variable, ill-posed inverse problems, Tikhonov regularization, nonlinear pricing curve.

1. INTRODUCTION

WE PROPOSE AND ANALYZE estimators of the nonseparable model $Y = g(X, U)$, where the error U is independent of the instrument Z and has a uniform distribution $U \sim \mathcal{U}(0, 1)$ (Chernozhukov and Hansen (2005), Chernozhukov, Imbens, and Newey (2007)). The function $g(x, u)$ is strictly monotonic increasing with respect to (w.r.t.) $u \in [0, 1]$. The variable X has compact support $\mathcal{X} = [0, 1]$ and is potentially endogenous. The variables Y and Z have compact supports $\mathcal{Y} \subset [0, 1]$ and $\mathcal{Z} = [0, 1]^{d_Z}$. The parameter of interest is the quantile structural effect $\varphi_0(x) = g(x, \tau)$ on $\mathcal{X}$ for a given $\tau \in (0, 1)$. The function φ_0 measures the structural impact of the regressor X on the τ-quantile of the dependent variable Y. Formally, it satisfies the conditional quantile restriction

$$P[Y \leq \varphi_0(X)|Z] = P[g(X, U) \leq g(X, \tau)|Z] = P[U \leq \tau|Z] = \tau, \tag{1.1}$$

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which yields the endogenous quantile regression representation (Horowitz and Lee (2007))

$$(1.2) \qquad Y = \varphi_0(X) + V, \quad P[V \leq 0|Z] = \tau.$$

The main contribution of this paper is the derivation of the large sample distribution of a Tikhonov regularized estimator of φ_0. This is the first distributional result in the literature on nonlinear problems, and it is noteworthy because of a fundamental difficulty of linearization of a nonlinear ill-posed problem such as (1.2), as pointed out in Horowitz and Lee (2007). Even though this paper focuses on a particular case (1.2) of a nonlinear ill-posed problem, the results of the paper are conceptually amenable to other problems. Indeed, the nonsmooth case (1.2) analyzed here is in some sense the hardest, so our analysis could be applied to other problems, such as nonlinear ill-posed pricing problems in finance (see, e.g., Egger and Engl (2005), Chen and Ludvigson (2009)) along similar lines.

We build on a series of fundamental papers on ill-posed endogenous mean regressions (Ai and Chen (2003), Darolles, Fan, Florens, and Renault (2011), Newey and Powell (2003), Hall and Horowitz (2005), Horowitz (2007), Blundell, Chen, and Kristensen (2007)), and the review paper by Carrasco, Florens, and Renault (CFR, 2007). The main issue in nonparametric estimation with endogeneity is overcoming ill-posedness of the associated inverse problem. It occurs since the mapping of the reduced form parameter (that is, the distribution of the data) into the structural parameter (that is, the instrumental regression function) is not continuous. We need a regularization of the estimation to recover consistency. Here we follow Gagliardini and Scaillet (GS, 2011) and study a Tikhonov regularized (TiR) estimator (Tikhonov (1963a, 1963b), Groetsch (1984), Kress (1999)). We achieve regularization by adding a compactness-inducing penalty term—the Sobolev norm—to a functional minimum distance criterion. For nonparametric instrumental variable estimation of endogenous quantile regression (NIVQR), Chernozhukov, Imbens, and Newey (2007) discussed identification and estimation via a constrained minimum distance criterion. Horowitz and Lee (2007) gave optimal consistency rates for an L^2-norm penalized estimator.

In independent work for a general setting, Chen and Pouzo (2009, 2011) studied semiparametric sieve estimation of conditional moment models based on possibly nonsmooth generalized residual functions. Specifically, Chen and Pouzo (2009) focused on semiparametric efficiency, asymptotic normality, and a weighted bootstrap procedure for the finite-dimensional parameter, and used a finite-dimensional sieve to estimate the functional parameter. They covered partially linear instrumental variable quantile regression (IVQR) as a particular example. Chen and Pouzo (2011) gave an in-depth, unifying treatment of convergence rates of penalized sieve-based estimators, and characterized when the sieve or the penalization dominates the convergence rates. Our results and those of Chen and Pouzo (2009, 2011) are complementary to each other (their results do not nest our results and vice versa, since we work in an infinite-dimensional parameter space, while Chen and Pouzo (2011) worked in finite-dimensional parameter spaces of increasing dimensions). The most important difference is the derivation of pointwise asymptotic normality, which is not available in Chen and Pouzo (2009, 2011). Our other specific contributions for NIVQR include a proof of ill-posedness and a proof of consistency under weak conditions on the penalization parameter.

We organize the rest of the paper as follows. In Section 2, we prove local ill-posedness and clarify the importance of including a derivative in the penalization. In Section 3, we prove consistency of our quantile Tikhonov regularized (Q-TiR) estimator. In Section 4, we show pointwise asymptotic normality and introduce a consistent estimator of the asymptotic variance. In Section 5, we provide computational experiments and present an empirical illustration of estimation of nonlinear pricing curves for telecommunications services in the United States. In the Appendix, we gather the technical assumptions and some proofs. We place all omitted proofs in the Supplemental Material (Gagliardini and Scaillet (2012)).

2. ILL-POSEDNESS IN NONSEPARABLE MODELS

From (1.1), the quantile structural effect φ_0 is a solution of the nonlinear functional equation $\mathcal{A}(\varphi_0) = \tau$, where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}(\varphi)(z) = \int F_{Y|X,Z}(\varphi(x)|x, z) f_{X|Z}(x|z)\, dx, \quad z \in \mathcal{Z},$$

and $F_{Y|X,Z}$ and $f_{X|Z}$ denote the cumulative distribution function (c.d.f.) of Y given X, Z and the probability density function (p.d.f.) of X given Z, respectively. Alternatively, in terms of the conditional c.d.f. $F_{U|X,Z}$ of U given X, Z, we can rewrite $\mathcal{A}(\varphi)(z) = \int F_{U|X,Z}(g^{-1}(x, \varphi(x))|x, z) f_{X|Z}(x|z)\, dx$, where $g^{-1}(x, \cdot)$ denotes the generalized inverse of function $g(x, \cdot)$ w.r.t. its second argument. The functional parameter φ_0 belongs to a subset Θ of the weighted Sobolev space $H^l[0, 1]$, $l \in \mathbb{N} \cup \{\infty\}$, that is the completion of $\{\varphi \in C^l[0, 1] \mid \|\varphi\|_H < \infty\}$ w.r.t. the weighted Sobolev norm $\|\varphi\|_H := \langle \varphi, \varphi \rangle_H^{1/2}$, where $\langle \varphi, \psi \rangle_H := \sum_{s=0}^{l} a_s \langle \nabla^s \varphi, \nabla^s \psi \rangle$, $a_s > 0$, is the weighted Sobolev scalar product and $\langle \varphi, \psi \rangle = \int \varphi(x)\psi(x)\, dx$. Below we focus on (i) $a_s = 1$, when $l < \infty$ and (ii) $a_s = 1/s!$ when $l = \infty$. The former gives the classical Sobolev space of order l (Adams and Fournier (2003)), while the latter gives the Sobolev space $H^\infty[0, 1] := \{\varphi \in C^\infty[0, 1] \mid \sum_{s=0}^{\infty} \frac{1}{s!} \langle \nabla^s \varphi, \nabla^s \varphi \rangle < \infty\}$ of infinite order (Dubinskij (1986)). These Sobolev spaces are Hilbert spaces w.r.t. $\langle \cdot, \cdot \rangle_H$, and the embeddings $H^\infty[0, 1] \subset H^l[0, 1] \subset L^2[0, 1]$, $l \in \mathbb{N}$, are compact (Adams and Fournier (2003, Theorem 6.3)). We use the L^2-norm $\|\varphi\| = \langle \varphi, \varphi \rangle^{1/2}$ as a consistency norm, and we assume that Θ is bounded and closed w.r.t. $\| \cdot \|$.

We assume that φ_0 is globally identified on Θ (see Appendix C in Chernozhukov and Hansen (2005) for a discussion) and interior.

ASSUMPTION 1: (i) $\mathcal{A}(\varphi) - \tau = 0$, $\varphi \in \Theta$, *if and only if* $\varphi = \varphi_0$. (ii) *Function* φ_0 *is an interior point of set* Θ *w.r.t. norm* $\|\cdot\|$.

We use the conditional moment restriction $m(\varphi_0, z) := \mathcal{A}(\varphi_0)(z) - \tau = 0$, $z \in \mathcal{Z}$, and consider the criterion

$$Q_\infty(\varphi) := \frac{1}{\tau(1-\tau)} E[m(\varphi, Z)^2] =: \|\mathcal{A}(\varphi) - \tau\|^2_{L^2(F_Z,\tau)},$$

where F_Z denotes the marginal distribution of Z and $L^2(F_Z, \tau)$ denotes the L^2 space w.r.t. measure $F_Z/(\tau(1-\tau))$. The true structural function φ_0 minimizes this criterion function Q_∞.

The following proposition shows that the minimum distance problem above is locally ill-posed (see, e.g., Definition 1.1 in Hofmann and Scherzer (1998)). There are sequences of increasingly oscillatory functions arbitrarily close to φ_0 that approximately minimize Q_∞ while not converging to φ_0. In other words, function φ_0 is not identified in Θ as an isolated minimum of Q_∞. Therefore, ill-posedness can lead to inconsistency of the naive analog estimators based on the empirical analog of Q_∞. To rule out these explosive solutions, we use penalization.

PROPOSITION 1: *Under Assumptions* 1(i) *and* A.3, (a) *the problem is locally ill-posed, namely for any* $r > 0$ *small enough, there exist* $\varepsilon \in (0, r)$ *and a sequence* $(\varphi_n) \subset B_r(\varphi_0) := \{\varphi \in L^2[0,1] : \|\varphi - \varphi_0\| < r\}$ *such that* $\|\varphi_n - \varphi_0\| \geq \varepsilon$ *and* $Q_\infty(\varphi_n) \to Q_\infty(\varphi_0) = 0$; (b) *any sequence* $(\varphi_n) \subset B_r(\varphi_0)$ *such that* $\|\varphi_n - \varphi_0\| \geq \varepsilon$ *for* $r > \varepsilon > 0$ *and* $Q_\infty(\varphi_n) \to 0$ *satisfies*

$$\limsup_{n\to\infty} \|\nabla\varphi_n\| = +\infty.$$

The proof of result (a) gives explicit sequences (φ_n) that generate ill-posedness. Since there is no general characterization of the ill-posedness of a nonlinear problem through conditions on its linearization, that is, on the Frechet derivative of the operator (Engl, Kunisch, and Neubauer (1989), Schock (2002)), this result does not follow from the ill-posedness of the linearized version of our problem. Under a stronger condition than Assumption 1(i), namely local injectivity of $\mathcal{A}$, the definition of local ill-posedness is equivalent to $\mathcal{A}^{-1}$ being discontinuous in a neighborhood of $\mathcal{A}(\varphi_0)$ (see Engl, Hanke, and Neubauer (2000, Chapter 10)). Part (b) provides a theoretical underpinning for including the norm $\|\nabla\varphi\|$ of the derivative in the penalty term.

3. CONSISTENCY OF THE Q-TIR ESTIMATOR

We consider a penalized criterion $L_T(\varphi) := Q_T(\varphi) + \lambda_T \|\varphi\|_H^2$, where $\lambda_T > 0$, P-almost surely (a.s.), and

$$Q_T(\varphi) := \frac{1}{T\tau(1-\tau)} \sum_{t=1}^{T} \hat{m}(\varphi, Z_t)^2.$$

The conditional moment $m(\varphi, z)$ is estimated nonparametrically by

$$\begin{aligned} \hat{m}(\varphi, z) &:= \int \hat{F}_{Y|X,Z}(\varphi(x)|x, z)\hat{f}_{X|Z}(x|z)\, dx - \tau \\ &=: \hat{\mathcal{A}}(\varphi)(z) - \tau, \quad z \in \mathcal{Z}, \end{aligned} \tag{3.1}$$

where $\hat{f}_{X|Z}$ and $\hat{F}_{Y|X,Z}$ denote kernel estimators of $f_{X|Z}$ and $F_{Y|X,Z}$ with bandwidth $h_T > 0$ and kernel K satisfying Assumption A.2.

PROPOSITION 2: *Suppose λ_T is a stochastic sequence such that $\lambda_T > 0$, $\lambda_T \to 0$, P-a.s., and $\frac{1}{\lambda_T}(\frac{\log T}{T h_T^{d_Z+1}} + h_T^{2m}) = O_p(1)$, where $m \geq 2$ is the order of differentiability of the joint density $f_{X,Y,Z}$ of (X, Y, Z). Then, under Assumptions* 1(i) *and* A.1–A.3, *the Q-TiR estimator $\hat{\varphi}$ defined by*

$$\hat{\varphi} := \arg\inf_{\varphi \in \Theta} Q_T(\varphi) + \lambda_T \|\varphi\|_H^2 \tag{3.2}$$

is consistent, namely $\|\hat{\varphi} - \varphi_0\| \xrightarrow{p} 0$, as sample size $T \to \infty$.

Term $\lambda_T \|\varphi\|_H^2$ in definition (3.2) penalizes highly oscillating components of the estimated function induced by ill-posedness and restores its consistency. In (3.2), we work with a function space-based estimator as in Horowitz and Lee (2007) (see also the suggestion in Newey and Powell (2003, p. 1573)). In Section 5, we compute the estimator based on a finite large number of polynomials. The discrepancy between the function space-based estimator and the implemented estimator is of a numerical nature, since our type of asymptotics does not rely on a sieve approach.

To show Proposition 2, we use two results. First, the Sobolev penalty implies that the sequence of estimates $\hat{\varphi}$ for $T \in \mathbb{N}$ is tight in $(L^2[0, 1], \|\cdot\|)$. This induces an effective compactification of the parameter space: there exists a compact set that contains $\hat{\varphi}$ for any large T with probability $1 - \delta$ for any arbitrarily small $\delta > 0$. Second, we obtain a suitable uniform convergence result for Q_T on an infinite-dimensional and possibly not totally bounded parameter set Θ by exploiting the specific expression of $\hat{m}(\varphi, z)$ given in (3.1). We are able to reduce the sup over Θ to a sup over a bounded subset of a finite-dimensional space. Proposition 6.2 in Chen and Pouzo (2011) states a consistency result for nonparametric additive IVQR using a series estimator for $m(\varphi, z)$ under similar conditions. In the rest of the paper, we assume a deterministic regularization parameter λ_T. The assumption in Proposition 2 becomes $h_T \asymp T^{-\eta}$ and $\lambda_T \asymp T^{-\gamma}$ for $0 < \eta < \frac{1}{d_Z+1}$, $0 < \gamma < \min\{1 - \eta(d_Z + 1), 2m\eta\}$; the relation $a_T \asymp b_T$ for positive sequences a_T and b_T means that a_T/b_T is bounded away from 0 and ∞ as $T \to \infty$.

4. ASYMPTOTIC DISTRIBUTION OF THE Q-TIR ESTIMATOR

In this section we derive a feasible asymptotic normality theorem. After deriving the first-order condition (Section 4.1), we show how to control the error induced by linearization of the problem under suitable smoothness assumptions (Sections 4.2 and 4.3). The validity of a Bahadur-type representation for the functional estimator makes it possible to show asymptotic normality (Section 4.4). We then provide a consistent estimator for the asymptotic variance (Section 4.5).

4.1. *First-Order Condition*

The asymptotic expansion of the Q-TiR estimator is derived by following the same steps as in the usual finite-dimensional setting. To cope with the functional nature of φ_0, we exploit an appropriate notion of differentiation to get the first-order condition. More precisely, we introduce the operator from $L^2[0, 1]$ to $L^2(F_Z, \tau)$ that corresponds to the Frechet derivative $A := D\mathcal{A}(\varphi_0)$ of operator $\mathcal{A}$ at φ_0,

$$A\varphi(z) = \int f_{X,Y|Z}(x, \varphi_0(x)|z)\varphi(x)\,dx,$$

and the operator from $L^2[0, 1]$ to $L^2(\hat{F}_Z, \tau)$ that corresponds to the Frechet derivative $\hat{A} := D\hat{\mathcal{A}}(\hat{\varphi})$ of operator $\hat{\mathcal{A}}$ at $\hat{\varphi}$ (see Appendix A.4),

$$\hat{A}\varphi(z) = \int \hat{f}_{X,Y|Z}(x, \hat{\varphi}(x)|z)\varphi(x)\,dx,$$

where $z \in \mathcal{Z}$ and $\varphi \in L^2[0, 1]$. The linear space $L^2(\hat{F}_Z, \tau)$ is endowed with the scalar product

$$\langle\psi_1, \psi_2\rangle_{L^2(\hat{F}_Z,\tau)} := \frac{1}{T\tau(1-\tau)} \sum_{t=1}^{T} \psi_1(Z_t)\psi_2(Z_t).$$

Under Assumption A.4(ii) and (iii), operator A is compact, which implies that the linearized version of our problem is ill-posed. Under Assumption 1(ii), the

Q-TiR estimator satisfies, with probability approaching 1 (w.p.a.1), the first-order condition

$$0 = \frac{d}{d\varepsilon} L_T(\hat{\varphi} + \varepsilon\varphi)\bigg|_{\varepsilon=0} = \frac{2}{T\tau(1-\tau)} \sum_{t=1}^{T} (\hat{\mathcal{A}}(\hat{\varphi})(Z_t) - \tau)\hat{A}\varphi(Z_t) + 2\lambda_T \langle \hat{\varphi}, \varphi \rangle_H = 2\big\langle \hat{A}^*(\hat{\mathcal{A}}(\hat{\varphi}) - \tau) + \lambda_T \hat{\varphi}, \varphi \big\rangle_H \tag{4.1}$$

for all $\varphi \in H^l[0,1]$, where the last line in (4.1) comes from the definition of the operator $\hat{A}^*$ through $\langle \hat{A}^*\psi, \varphi\rangle_H := \frac{1}{T\tau(1-\tau)} \sum_{t=1}^{T} \psi(Z_t)\hat{A}\varphi(Z_t)$ for any $\varphi \in H^l[0,1]$ and $\psi \in L^2(\hat{F}_Z, \tau)$. Operator $\hat{A}^*$ is the adjoint of $\hat{A}$ w.r.t. the scalar products $\langle \cdot, \cdot \rangle_H$ on $H^l[0,1]$ and $\langle \cdot, \cdot \rangle_{L^2(\hat{F}_Z,\tau)}$ on $L^2(\hat{F}_Z, \tau)$. It is the empirical counterpart of the adjoint operator A^* of A w.r.t. the Sobolev scalar product on $H^l[0,1]$ and the scalar product $\langle \cdot, \cdot \rangle_{L^2(F_Z,\tau)}$ on $L^2(F_Z, \tau)$. The operator A^* can be characterized in terms of the adjoint $\tilde{A}$ of A w.r.t. the $L^2[0,1]$ scalar product, defined by $\tilde{A}\psi(x) = \frac{1}{\tau(1-\tau)} \int f_{X,Y,Z}(x, \varphi_0(x), z)\psi(z)\,dz$. For $l = 1$, we have $A^* = \mathcal{D}^{-1}\tilde{A}$, where $\mathcal{D}^{-1}$ denotes the inverse of operator $\mathcal{D}: H_0^2[0,1] \to L^2[0,1]$ with $H_0^2[0,1] := \{\varphi \in H^2[0,1] : \nabla\varphi(0) = \nabla\varphi(1) = 0\}$ and $\mathcal{D}\varphi = (1 - \nabla^2)\varphi$ (see the Supplemental Material for the derivation and the characterization with $l > 1$). From (4.1) holding for all $\varphi \in H^l[0,1]$, estimate $\hat{\varphi}$ satisfies the nonlinear integro-differential equation of type II:

$$\hat{A}^*(\hat{\mathcal{A}}(\hat{\varphi}) - \tau) + \lambda_T \hat{\varphi} = 0. \tag{4.2}$$

4.2. *Highlighting the Nonlinearity Issue*

We can rewrite Equation (4.2) by using the second-order expansion

$$\hat{\mathcal{A}}(\hat{\varphi}) = \hat{\mathcal{A}}(\varphi_0) + \hat{A}_0 \Delta\hat{\varphi} + \hat{R},$$

where

$$\hat{A}_0\varphi(z) := \int \hat{f}_{X,Y|Z}(x, \varphi_0(x)|z)\varphi(x)\,dx, \quad \Delta\hat{\varphi} := \hat{\varphi} - \varphi_0,$$

and $\hat{R} := \hat{R}(\hat{\varphi}, \varphi_0)$ is the second-order residual term (see Lemma A.4). Then, after rearranging terms and performing an asymptotic expansion, we get the decomposition (Appendix A.4)

$$\Delta\hat{\varphi} = (\lambda_T + A^*A)^{-1}A^*\hat{\zeta} + \mathcal{B}_T + \mathcal{R}_T + \hat{\mathcal{K}}_T(\Delta\hat{\varphi}), \tag{4.3}$$

where

$$\hat{\zeta}(z) := \int\int \left(\tau - 1\{y \leq \varphi_0(x)\}\right) \frac{\Delta \hat{f}_{X,Y,Z}(x, y, z)}{f_Z(z)} \, dy \, dx, \quad \Delta \hat{f} = \hat{f} - f,$$

and

$$\mathcal{B}_T := [(\lambda_T + A^*A)^{-1} A^*A - 1]\varphi_0.$$

The Bahadur-type representation (4.3) of the Q-TiR estimator (see, e.g., Koenker (2005)) is key to our asymptotic normality result. The stochastic term $(\lambda_T + A^*A)^{-1}A^*\hat{\zeta}$ corresponds to the leading term, yielding asymptotic normality of the standard exogenous quantile regression. The deterministic term $\mathcal{B}_T$ is the bias function induced by regularization. The remainder term $\mathcal{R}_T$ accounts for kernel estimation of operator $\mathcal{A}$ and its expression is given in (A.6) below. The nonlinearity term $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ is defined by $\hat{\mathcal{K}}_T(\Delta\hat{\varphi}) := -(\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}\hat{A}_0^*\hat{R}$, where $\hat{A}_0^*$ is defined as $\hat{A}^*$, but with φ_0 substituted for $\hat{\varphi}$.

The major difference between our ill-posed setting and standard finite-dimensional parametric estimation problems, or well-posed functional estimation problems, concerns the behavior of the nonlinearity term $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$. Controlling $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ is a fundamental difficulty of linearization of a nonlinear ill-posed inverse problem. We prove in Lemma A.5 that $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ satisfies a quadratic bound

$$\|\hat{\mathcal{K}}_T(\Delta\hat{\varphi})\| \leq \frac{C}{\sqrt{\lambda_T}} \|\Delta\hat{\varphi}\|^2, \tag{4.4}$$

w.p.a.1, with a suitable constant C. In the right-hand side (RHS) of (4.4), the coefficient of the quadratic bound diverges as the sample size increases. Hence, the usual argument that the quadratic nonlinearity term is negligible w.r.t. the first-order term, regardless of the convergence rate of the latter, does not apply. Still, under Assumptions 2–4 discussed below and ensuring $\|\Delta\hat{\varphi}\| = o_p(\sqrt{\lambda_T})$, we can control the nonlinearity term $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ and the remainder term $\mathcal{R}_T$.

4.3. *Assumptions for Asymptotic Normality*

Let $\varphi_\lambda := \arg\inf_{\varphi\in\Theta} Q_\infty(\varphi) + \lambda\|\varphi\|_H^2$, with $\lambda > 0$, denote the nonlinear Tikhonov solution in the population. We will make the following assumptions as well as the more technical Assumptions A.4 and A.5 stated in the Appendix.

ASSUMPTION 2: *The solution φ_λ is unique and the equation $D\mathcal{A}(\varphi)^*(\mathcal{A}(\varphi) - \tau) + \lambda\varphi = 0$, $\varphi \in \Theta$, admits the unique solution $\varphi = \varphi_\lambda$ for all small $\lambda > 0$.*

This assumption involves the first-order condition for minimization of the penalized minimum distance criterion in the population. It rules out local extrema over Θ different from the global minimum φ_λ.

ASSUMPTION 3: *$A\varphi = 0$ for $\varphi \in L^2[0,1]$ if and only if $\varphi = 0$.*

This is an injectivity condition for the Frechet derivative A, that is, a local identification assumption. Since operator A is such that $A\varphi(z) = E[\{\frac{\partial g}{\partial u}(X,\tau)\}^{-1}\varphi(X)|Z=z, U=\tau]$ and $\partial g/\partial u > 0$, Assumption 3 is equivalent to completeness of X by Z conditional on $U=\tau$; see Chernozhukov, Imbens, and Newey (2007) for examples and further discussion on the relationship with completeness.

ASSUMPTION 4: (i) *The function φ_0 satisfies the source condition $\sum_{j=1}^{\infty}\frac{\langle\phi_j,\varphi_0\rangle_H^2}{\nu_j^{2\delta}} < \infty$, $\delta \in (1/2,1]$, where $\nu_j \searrow 0$ and ϕ_j are the eigenvalues and eigenfunctions of the compact, self-adjoint operator A^*A on $H^l[0,1]$, with $\|\phi_j\|_H = 1$.* (ii) *$\sum_{j=1}^{\infty}\frac{\langle\phi_j,\varphi_0\rangle_H^2}{\nu_j} < 1/c^2$, where $c := \sup_{x,y,z}|\nabla_y f_{X,Y|Z}(x,y|z)|$.* (iii) *$\Gamma(\lambda) := \inf_{\varphi\in H^l[0,1]:\|\varphi\|=1}\|A\varphi\|^2_{L^2(F_Z,\tau)} + \lambda\|\varphi\|_H^2 \geq C\lambda^a$ for $C>0$, $a \in (0,1/2)$.*

The source condition in Assumption 4(i) requires that function φ_0 can be well approximated by the elements of the eigenfunction basis $\{\phi_j : j\in\mathbb{N}\}$ associated with the larger eigenvalues of operator A^*A. The coefficients $\langle\phi_j,\varphi_0\rangle_H$ have to decrease to zero as $j\to\infty$ sufficiently fast compared to the eigenvalues ν_j. This condition controls the bias contribution as in the proof of Proposition 3.11 in CFR, and implies (see Appendix A.4)

$$\|\mathcal{B}_T\|_H = O(\lambda_T^{\delta}). \tag{4.5}$$

Hence, the larger is the parameter δ, the smaller is the regularization bias. Assumption 4(ii) is the analog of Assumption 6 in Horowitz and Lee (2007) for Sobolev penalization. It controls the distance between the nonlinear Tikhonov solution in the population and φ_0 along the lines of Proposition 10.7 in Engl, Hanke, and Neubauer (2000). Together with Assumption 4(i), it implies $\|\varphi_{\lambda_T} - \varphi_0\|_H = O(\lambda_T^{\delta})$ (see the proof of Lemma B.6 in the Supplemental Material). Finally, Assumption 4(iii) implies that

$$\inf_{\substack{\varphi\in\Theta:\\ \|\varphi-\varphi_0\|\geq d\sqrt{\lambda}}} Q_\infty(\varphi) + \lambda\|\varphi\|_H^2 - Q_\infty(\varphi_0) - \lambda\|\varphi_0\|_H^2 \geq C\lambda\Gamma(\lambda) \tag{4.6}$$

as $\lambda\to 0$, for any $C < d^2$ (see Lemma B.6 in the Supplemental Material). Inequality (4.6) replaces the usual "identifiable uniqueness" condition (White and Wooldridge (1991)) $\inf_{\varphi\in\Theta:\|\varphi-\varphi_0\|\geq\varepsilon} Q_\infty(\varphi) - Q_\infty(\varphi_0) > 0$, which does not hold with ill-posedness (see Proposition 1(a)). Function $\Gamma(\lambda)$ characterizes how well the penalized criterion distinguishes φ_0 from functions φ outside a neighborhood of φ_0 when the radius $\sqrt{\lambda}$ of the neighborhood shrinks to zero. The smaller is the parameter a, the more effective the penalization restores identifiable uniqueness. Inequality (4.6) is sharp in the sense that for a linear problem, the left-hand side (LHS) is equal to $d^2\lambda\Gamma(\lambda)+O(\lambda^{3/2})$. Assumptions 2–4 are used to control the large deviation probability $P[\|\Delta\hat{\varphi}\|\geq d\sqrt{\lambda_T}]$, $d>0$ (see Lemma B.7 in the Supplemental Material). From (4.4), this yields an upper bound for the nonlinearity term $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ (see Lemmas A.7 and A.9).

To discuss plausibility of Assumption 4(iii), let $\{\tilde{\phi}_j : j\in\mathbb{N}\}$ be an orthonormal basis in $L^2[0,1]$ of eigenfunctions of $\tilde{A}A$ to eigenvalues $\tilde{\nu}_j$. Consider the three conditions

$$(\mathrm{a})\ \sum_{j,k=1:j\neq k}^{\infty}\frac{\langle\tilde{\phi}_j,\tilde{\phi}_k\rangle_H^2}{\|\tilde{\phi}_j\|_H^2\|\tilde{\phi}_k\|_H^2}<1,\quad (\mathrm{b})\ \|\tilde{\phi}_j\|_H^2\geq C_1\omega_j,$$

$$(\mathrm{c})\ \tilde{\nu}_j\geq C_2\kappa_j,\quad C_1,C_2>0,\forall j\geq 1,$$

where $\omega_j\nearrow\infty$ and $\kappa_j\searrow 0$ are two given positive sequences. Condition (a) states that the eigenfunctions are not very correlated under $\langle\cdot,\cdot\rangle_H$. Condition (b) gives a lower bound on the speed of divergence of the Sobolev norms of the eigenfunctions in terms of sequence ω_j. Similarly, condition (c) gives a lower bound on the speed of convergence to zero of the eigenvalues in terms of sequence κ_j. The divergence of sequence ω_j must dominate the convergence to zero of sequence κ_j for Assumption 4(iii) to hold. Specifically, conditions (a)–(c) with $\omega_j=j^p$ and $\kappa_j=j^{-\tilde{\alpha}}$, $0<\tilde{\alpha}<p$, imply Assumption 4(iii) with $a=\frac{\tilde{\alpha}}{\tilde{\alpha}+p}$. The hyperbolic decay of $\tilde{\nu}_j$ corresponds to mild ill-posedness for operator $\tilde{A}A$. Moreover, conditions (a)–(c) with $\omega_j=\exp(j^2)$ and $\kappa_j=\exp(-\tilde{\alpha}j)$, $\tilde{\alpha}>0$, imply Assumption 4(iii) with any $a\in(0,1/2)$. The geometric decay of $\tilde{\nu}_j$ corresponds to severe ill-posedness for operator $\tilde{A}A$ (see CFR and Kress (1999) for the terminology).

REMARK 1: In the Supplemental Material, we provide an example where the orthonormal eigenfunctions of $\tilde{A}A$ in $L^2[0,1]$ are given by $\tilde{\phi}_j(x)=1$ and $\tilde{\phi}_j(x)=\sqrt{2}\cos(\pi(j-1)x)$, $j=2,3,\ldots$. These functions satisfy $\langle\tilde{\phi}_j,\tilde{\phi}_k\rangle_H=1\{j=k\}\sum_{s=0}^{l}a_s(\pi j)^{2s}$. Thus, for hyperbolic decay $\tilde{\nu}_j\geq Cj^{-\tilde{\alpha}}$ and finite Sobolev order $l>\tilde{\alpha}/2$, Assumption 4(iii) holds with $a=\frac{\tilde{\alpha}}{\tilde{\alpha}+2l}$. Similarly, for geometric decay $\tilde{\nu}_j\geq C\exp(-\tilde{\alpha}j)$ and Sobolev order $l=\infty$, Assumption 4(iii) holds with any $a\in(0,1/2)$.

The example in Remark 1 shows that Assumption A.4(iii) involves an adaptation condition between the speed of decay of the spectrum of operator $\tilde{A}A$

(mild, resp. severe, ill-posednesss) and the Sobolev order l (finite, resp. infinite). This is related to condition (8.28) for operator A in Engl, Hanke, and Neubauer (2000). Within our setting of assumptions for asymptotic normality, allowing for higher-order Sobolev penalties permits accommodation of various forms of ill-posedness. While the decay behaviors of the spectra of operators $\tilde{A}A$ and A^*A are expected to be tightly related, making the link explicit appears difficult in a general framework.

The regularity conditions on the eigenfunctions of A^*A are more restrictive than in Horowitz and Lee (2007) and Chen and Pouzo (2011) (see also the discussion of Assumption A.5 in Appendix A.1). They are useful to establish pointwise asymptotic normality.

4.4. *Asymptotic Normality*

Let us define the quantities

$$\sigma_T^2(x) := \sum_{j=1}^{\infty} \frac{\nu_j}{(\lambda_T + \nu_j)^2} \phi_j^2(x) \quad \text{and} \quad V_T(\lambda_T) := \frac{1}{T} \int \sigma_T^2(x)\, dx.$$

The following proposition computes the limit distribution of the Q-TiR estimator for (any sequence of) typical points. By (sequence of) typical points, we mean sets $\mathcal{X}_T \subset \mathcal{X} = [0, 1]$ such that for $x \in \mathcal{X}_T$,

$$\frac{V_T(\lambda_T)}{\sigma_T^2(x)/T} = O(1), \tag{4.7}$$

that is, the variance at a typical point x is not much smaller than the integrated variance.

PROPOSITION 3: *Suppose Assumptions* 1–4 *and* A.1–A.5 *hold. Let* h_T *and* λ_T *be such that* (a) $h_T \asymp T^{-\eta}$ *and* $\lambda_T \asymp T^{-\gamma}$ *with* $0 < \eta < \frac{1}{2(d_Z+2)}$, $0 < \gamma < \frac{1}{2}\min\{1 - \eta(d_Z + 1), m\eta, \frac{1}{1+a}\}$; (b) $\lambda_T^{2\delta} = O(V_T(\lambda_T))$, $V_T(\lambda_T) = o(\lambda_T)$; (c) $h_T^{1/4} \frac{V_T(\lambda_T;2)}{V_T(\lambda_T)} = o(1)$ *and* $h_T^{2m} \frac{V_T(\lambda_T;2m+\varepsilon_1)}{V_T(\lambda_T)} = o(1)$ *for some* $\varepsilon_1 > 1$, *where* $V_T(\lambda_T; \varepsilon) := \frac{1}{T} \sum_{j=1}^{\infty} \frac{\nu_j}{(\lambda_T+\nu_j)^2} \|\phi_j\|^2 j^{\varepsilon}$. *Then for any sequence* $x \in \mathcal{X}_T$,

$$\sqrt{T/\sigma_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x) - \mathcal{B}_T(x)) \xrightarrow{d} N(0, 1).$$

Furthermore, if $\lambda_T^{2\delta} = o(V_T(\lambda_T))$, *then*

$$\sqrt{T/\sigma_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x)) \xrightarrow{d} N(0, 1).$$

The variance function $\sigma_T^2(x)$ involves the spectrum of operator A^*A and the regularization parameter λ_T. The penalty term $\lambda_T\|\varphi\|_H^2$ in the criterion defining the Q-TiR estimator implies that the inverse eigenvalues $1/\nu_j$ of the inverse of operator A^*A are ridged with $\nu_j/(\lambda_T+\nu_j)^2$. The variance formula is reminiscent of the usual asymptotic variance of the quantile regression estimator: it involves the factor $f_{V|X,Z}(0|x,z)$ (see (1.2)) through the Frechet derivative A and the factor $\tau(1-\tau)$ through the adjoint A^*, hidden in the spectrum of A^*A. Since $\nu_j=\langle\phi_j, A^*A\phi_j\rangle_H=\|A\phi_j\|^2_{L^2(F_Z,\tau)}\leq\|A\|_{\mathcal{L}}^2\|\phi_j\|^2$, where $\|A\|_{\mathcal{L}}$ denotes the operator norm of A, we deduce that $\nu_j^{-1}\|\phi_j\|^2\geq\|A\|_{\mathcal{L}}^{-2}$, which implies $\sum_{j=1}^\infty\nu_j^{-1}\|\phi_j\|^2=\infty$. Then, by using (4.7), we get $\sigma_T^2(x)\to\infty$ as $\lambda_T\to 0$. Thus, $\sigma_T^2(x)$ summarizes the impact of ill-posedness on the nonparametric convergence rate $\sqrt{T/\sigma_T^2(x)}$. The conditions (a)–(c) on h_T and λ_T ensure that the asymptotic distribution of the nonlinear estimator $\hat{\varphi}$ is the same as the one induced by linearization. These conditions depend on the instrument dimension d_Z and the smoothness properties of the joint density of the observations via m, as well as on parameters δ and a introduced in Assumption 4. In particular, we need $\delta>1/2>a$. Since $\mathcal{B}_T(x)=O(\|\mathcal{B}_T\|_H)=O(\lambda_T^\delta)$ as shown in the proof of Proposition 3, $\lambda_T^{2\delta}=o(V_T(\lambda_T))$ is a sufficient condition for bias negligibility at a typical point. Below in Remark 3, we discuss an example of the mutual compatibility of the conditions on h_T and λ_T for linearization and bias negligibility. Finally, we show in the Supplemental Material that under a strengthening of Assumption A.5(iii), the mean integrated square error (MISE) is asymptotically like $E[\|\hat{\varphi}-\varphi_0\|^2]=M_T(\lambda_T)(1+o(1))$, where $M_T(\lambda_T):=\int(\frac{1}{T}\sigma_T^2(x)+\mathcal{B}_T(x)^2)\,dx$.

4.5. *Estimating the Asymptotic Variance*

The estimation of the asymptotic variance of $\hat{\varphi}$ requires estimation of the spectrum of operator A^*A. Let us assume the following semiparametric specification for the decay of eigenvalues ν_j and eigenfunction values $\phi_j(x)$ at $x\in\mathcal{X}$ when $j\to\infty$.

ASSUMPTION 5: (i) *The eigenvalues are such that* $\nu_j=c_{1,j}\exp(v_j'\alpha)$ *and* (ii) *the eigenfunction values are such that* $\phi_j(x)^2=c_{2,j}\exp(w_j\beta)\chi_j$, *where* $v_j'=-(j,\log j)$, $w_j=-\log j$, $\chi_j\geq 0$ *is an unknown periodic sequence with period* S, $\alpha=(\alpha_1,\alpha_2)'$ *and* β *are unknown parameters, and* $c_{1,j}, c_{2,j}>0$ *are unknown sequences converging to strictly positive constants as* $j\to\infty$.

The specification in Assumption 5(i) accommodates both geometric decay with $\alpha_2=0$ and hyperbolic decay with $\alpha_1=0$ in the eigenvalues (severe and mild ill-posedness for operator A^*A). The specification in Assumption 5(ii) accommodates both a slowly varying trend component $\exp(w_j\beta)$ and an oscillatory component χ_j in the eigenfunction values. We can relax Assumption 5 to cover more general specifications of v_j and w_j.

REMARK 2: For the geometric case of the example in Remark 1, we have $\tilde{\nu}_j = \tilde{c}_1 e^{-\alpha_1(j-1)}$, with $\tilde{c}_1, \alpha_1 > 0$. For the Sobolev order $l = 1$, we also have $\nu_j = \frac{c_1}{1+\pi^2(j-1)^2} e^{-\alpha_1(j-1)}$ and $\phi_1(x)^2 = 1$, $\phi_j(x)^2 = \frac{c_2}{1+\pi^2(j-1)^2}\cos^2(\pi(j-1)x)$, $j > 1$, with $c_1, c_2 > 0$. Such an example is compatible with Assumption 5 where $\alpha_2 = \beta = 2$ and $\chi_j = \cos^2(\pi(j-1)x)$ if x is a rational number.

Let $\bar{\nu}_j$ and $\bar{\phi}_j$ for $j \in \mathbb{N}$ be the eigenvalues and eigenfunctions of operator $\bar{A}^*\bar{A}$, normalized such that $\|\bar{\phi}_j\|_H = 1$, where $\bar{A} = D\hat{\mathcal{A}}(\bar{\varphi})$ is based on a pilot estimator $\bar{\varphi}$. The pilot estimator $\bar{\varphi}$ is a Q-TiR estimator with regularization parameter $\bar{\lambda}_T$ and bandwidth $\bar{h}_T$ satisfying the assumptions of Proposition 3. Let $n_T \le N_T$ be integers growing with T. Define $\hat{\nu}_j = \bar{\nu}_j$ for $1 \le j \le n_T$, and $\hat{\nu}_j = \bar{\nu}_{n_T}\exp((v_j - v_{n_T})'\hat{\alpha})$ for $n_T < j \le N_T$, where the estimator $\hat{\alpha}$ is computed through ordinary least squares (OLS) by regressing $\log(\bar{\nu}_j/\bar{\nu}_{n_T})$ on $v_j - v_{n_T}$ for $j = n_T/2, \dots, n_T - 1$. Similarly, define $\hat{\phi}_j(x)^2 = \bar{\phi}_j(x)^2$ for $1 \le j \le n_T$ and $\hat{\phi}_j(x)^2 = \bar{\phi}_{S,n_T}(x)^2\exp((w_j - w_{n_T})\hat{\beta})\hat{\chi}_{j \bmod S}$ for $n_T < j \le N_T$. Here $\bar{\phi}_{S,j}(x)^2 := \sum_{k=0}^{S-1}\bar{\phi}_{j-k}(x)^2$, the estimator $\hat{\beta}$ is computed through OLS by regressing $\log(\bar{\phi}_{S,j}(x)^2/\bar{\phi}_{S,n_T}(x)^2)$ on $w_j - w_{n_T}$ for $j = n_T/2, \dots, n_T - 1$, and $\hat{\chi}_j = \frac{2S}{n_T}\sum_{k:n_{T/2}\le k < n_T, k = j \bmod S}\bar{\phi}_k(x)^2/\bar{\phi}_{S,k}(x)^2$ for $j = 1, \dots, S$, where $k = j \bmod S$ if $k - j$ is an integer multiple of S. The estimator of the variance function is $\hat{\sigma}_T^2(x) = \sum_{j=1}^{N_T}\frac{\hat{\nu}_j}{(\hat{\nu}_j+\lambda_T)^2}\hat{\phi}_j(x)^2$.

We have introduced the parameter n_T, since the nonparametric estimates of the spectrum of A^*A may be unprecise in relative terms for j close to the truncation parameter N_T. The nonparametric estimates are replaced by extrapolated estimates between n_T and N_T. The extrapolation procedure exploits the supposed decay behavior of the spectrum in Assumption 5. For the eigenfunction values, we estimate and extrapolate the parametric trend component by using that the filtered spectral coefficients $\phi_{S,j}(x)^2 := \sum_{k=0}^{S-1}\phi_{j-k}(x)^2$ approaches $\exp(w_j\beta)$ for $j \to \infty$, up to a scale constant. Then we estimate the periodic component by averaging the detrended square eigenfunction values over all lags j with the same phase of the cycle.

PROPOSITION 4: *Denote* $\Delta\nu_j := \min_{k\le j}(\nu_{k-1} - \nu_k)$. *Let* N_T *be such that*

$$\sum_{j=N_T+1}^{\infty}\nu_j\|\phi_j\|^2 = o(T\lambda_T^2 V_T(\lambda_T)) \tag{4.8}$$

and let n_T *be such that* $N_T = O(n_T)$ *and*

$$\frac{1}{\exp(w_{n_T}\beta)\Delta\nu_{n_T}}\left(\frac{1}{Th_T^2} + h_T^{2m} + M_T(\bar{\lambda}_T)\right)^{1/2} = O_p(T^{-b}) \tag{4.9}$$

for $b > 0$. *Assume that* $\sigma^2_{*,T}(x)/\sigma^2_T(x) = O(1)$, *where* $\sigma^2_{*,T}(x) = \sum_{j=1}^{\infty} \frac{\nu_j}{(\nu_j+\lambda_T)^2} \times c_{2,j}\exp(w_j\beta)$. *Then, under Assumptions* 1–5 *and* A.1–A.5, $\frac{\sigma_T^2(x)}{\hat{\sigma}_T^2(x)} \xrightarrow{p} 1$.

Condition (4.8) ensures that the truncation bias is negligible. Condition (4.9) ensures that $\bar{\nu}_j$ and $\bar{\phi}_j(x)^2$ are consistent in relative terms uniformly over $1 \le j \le n_T$. It involves the asymptotic MISE $M_T(\bar{\lambda}_T)$ of the pilot estimator. The condition $N_T = O(n_T)$ and the extrapolation procedure yield relative consistency of $\hat{\nu}_j$ and $\hat{\phi}_j(x)^2$ uniformly over $1 \le j \le N_T$.

REMARK 3: Let the eigenvalues and eigenfunctions satisfy Assumption 5 with $\alpha_1, \beta > 0$, $\alpha_2 \ge 0$, and $\|\phi_j\|^2 \asymp j^{-\beta}$ (see Remarks 1 and 2). Let us verify that there exist admissible values of γ, η, $\bar{\gamma}$, and $\bar{\eta}$, so that the conditions on λ_T, h_T, $\bar{\lambda}_T$, $\bar{h}_T$, n_T, and N_T, to get Propositions 3 and 4, are all compatible. In the Supplemental Material, we prove that $V_T(\lambda_T; \varepsilon) \asymp \frac{1}{T\lambda_T[\log(1/\lambda_T)]^{\beta-\varepsilon}}$ for $\varepsilon \ge 0$, and $\sigma_T^2(x), \sigma^2_{*,T}(x) \asymp \frac{1}{\lambda_T[\log(1/\lambda_T)]^{\beta}}$. If $\frac{2}{m(1+2\delta)} < \bar{\eta} < \frac{1}{d_Z+1}\frac{2\delta-1}{2\delta+1}$ and $\frac{1}{1+2\delta} < \bar{\gamma} < \frac{1}{2}\min\{1-(d_Z+1)\bar{\eta}, m\bar{\eta}, \frac{1}{1+a}\}$, the pilot estimator satisfies the assumptions of Proposition 3. If $\frac{2}{m(1+2\delta)} < \eta < \frac{1}{d_Z+1}\frac{2\delta-1}{2\delta+1}$ and $\frac{1}{1+2\delta} < \gamma < \frac{1}{2}\min\{1-(d_Z+1)\eta, m\eta, \frac{1}{1+a}\}$, estimator $\hat{\varphi}$ is asymptotically normal with vanishing bias. The conditions on γ, η, $\bar{\gamma}$, and $\bar{\eta}$ are compatible if $m > 2(d_Z+1)$ and $\delta > \frac{1}{2} + \max\{\frac{d_Z+1}{m}, a\}$. Since $\sum_{j=n+1}^{\infty} \nu_j\|\phi_j\|^2 \le Ce^{-\alpha_1 n}$, $n \in \mathbb{N}$, and $\Delta\nu_j = \nu_{j-1} - \nu_j \asymp j^{-\alpha_2}e^{-\alpha_1 j}$, conditions (4.8) and (4.9) are satisfied for $n_T = c_1\log T$ and $N_T = c_2 \log T$ such that $c_1 < \frac{1}{\alpha_1}\min\{\frac{1-2\eta}{2}, m\eta, \frac{1-\bar{\gamma}}{2}, \delta\bar{\gamma}\}$ and $c_2 > \frac{1}{\alpha_1}\gamma$. Then $\sqrt{T/\hat{\sigma}_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x)) \xrightarrow{d} N(0,1)$, from which asymptotically valid pointwise confidence intervals can be computed.

5. MONTE CARLO RESULTS AND EMPIRICAL ILLUSTRATION

5.1. *Monte Carlo Results*

We consider an experiment, following Newey and Powell (2003), where the errors U_1^*, U_2^* and the instrument Z follow a trivariate normal distribution, with zero means, unit variances, and a correlation coefficient of .5 between U_1^* and U_2^*. We take $X^* = Z + U_2^*$, and build $X = \Phi(X^*)$, $Y = \sin(\pi X) + U_1^*$, and $U = \Phi(U_1^*)$. The median condition is $P[Y - \varphi_0(X) \le 0|Z] = .5$, where the functional parameter is $\varphi_0(x) = \sin(\pi x)$, $x \in [0,1]$. We consider sample size $T = 1000$ and a classical Sobolev penalty of order $l \in \{1, 2\}$. For $l = 1$, we consider $\bar{\lambda} = \lambda \in \{.00002, .00005, .0001\}$, while for $l = 2$, we consider $\bar{\lambda} = \lambda \in \{.0000003, .000001, .000003\}$. We use Gaussian kernels and select the bandwidth with the standard rule of thumb (Silverman (1986)). To compute $\hat{\varphi}$ and $\bar{\varphi}$, we opt for a numerical series approximation based on standardized shifted Chebyshev polynomials of the first kind, and a user-supplied analytical gradient

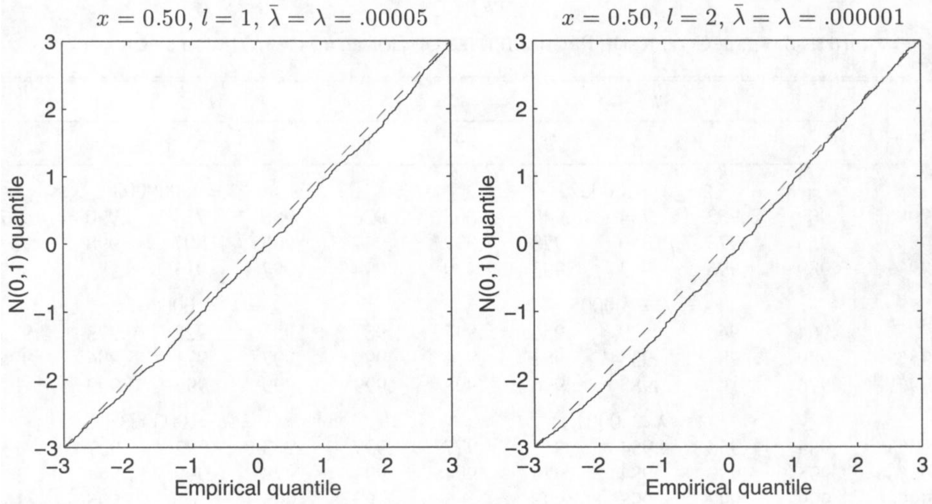

FIGURE 1.—Q-Q plot of $\sqrt{T/\hat{\sigma}_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x))$.

and Hessian optimization procedure. We report results using 16 polynomials (order 0–15); results using the first 8, or more, polynomials are nearly identical. The variance estimator $\hat{\sigma}_T^2(x)$ is computed with $N_T = n_T = 4$. Figure 1 displays the Q-Q plots of the finite sample distributions of $\sqrt{T/\hat{\sigma}_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x))$, $x = .5$, for $l = 1$, $\bar{\lambda} = \lambda = .00005$ (left), and $l = 2$, $\bar{\lambda} = \lambda = .000001$ (right), built on 1000 replications. The finite sample distributions are close to the standard normal distribution. For the selected values of $\bar{\lambda}$, λ, the regularization bias is rather small. Table I reports the finite sample coverage of pointwise confidence intervals for $\varphi_0(x)$, $x \in \{.10, .25, .50, .75, .90\}$, using the different values of l, $\bar{\lambda}$, and λ. Let us first consider the results for $l = 1$ (left panel). The finite sample coverage is close to the nominal coverage at 90%, 95%, and 99% for $x = .5$ and $\bar{\lambda} = \lambda = .00005$. At $x = .5$, we observe overrejection for $\bar{\lambda} = \lambda = .0001$ because of regularization bias and underrejection for $\bar{\lambda} = \lambda = .00002$ because of a too small number of terms in the estimated variance. For instance, with $N_T = n_T = 5$ and at $x = .5$, the finite sample coverage at 90%, 95%, and 99% is .964, .985, and .998 for $l = 1$, $\bar{\lambda} = \lambda = .00002$, and .947, .974, and .999 for $l = 2$, $\bar{\lambda} = \lambda = .0000003$. For $x = .10, .25, .75, .90$ and the considered values of $\bar{\lambda}$, λ, we typically observe some overrejection. Finally, the results for $l = 2$ are qualitatively similar to those for $l = 1$.

5.2. *Empirical Illustration*

This section presents an empirical illustration with U.S. long-distance call data extracted from the sample of Hausman and Sidak (2004). They investigated nonlinear price schedules chosen by consumers of message toll service

TABLE I

FINITE-SAMPLE COVERAGE PROBABILITIES OF CONFIDENCE INTERVALS FOR $\varphi_0(x)$

	$l=1$					$l=2$				
	$x=.1$	.25	.5	.75	.9	$x=.1$	.25	.5	.75	.9
			$\bar{\lambda}=\lambda=.00002$					$\bar{\lambda}=\lambda=.0000003$		
90%	.909	.952	.754	.947	.868	.928	.986	.718	.990	.917
95%	.956	.973	.850	.978	.937	.967	.993	.807	.998	.953
99%	.989	.996	.953	.993	.979	.994	.997	.915	1	.989
			$\bar{\lambda}=\lambda=.00005$					$\bar{\lambda}=\lambda=.000001$		
90%	.964	.968	.879	.956	.947	.989	.985	.922	.973	.986
95%	.989	.989	.935	.981	.974	.996	.997	.971	.996	.996
99%	.996	1	.983	.996	.998	.999	.999	.994	.999	1
			$\bar{\lambda}=\lambda=.0001$					$\bar{\lambda}=\lambda=.000003$		
90%	.985	.958	.954	.951	.977	.999	.943	.992	.927	1
95%	.995	.980	.984	.979	.994	.999	.978	.999	.964	1
99%	.999	.992	.998	.998	1	1	.995	1	.995	1

offered by long-distance interexchange carriers. We estimate median structural effects for nonlinear pricing curves based on the conditional quantile condition $P[Y \leq \varphi_0(X)|Z] = .5$, with $X = \Phi$ (min) and $Z = \Phi$ (Inc). Variable Y is the price per minute in dollars, min is the standardized amount of use in minutes, and Inc is the standardized logarithm of annual income. We look at clients of a leading long-distance interexchange carrier with age between 30 and 45. To study the effect of education on the chosen nonlinear price schedule, we divide the sample into people with at most 12 years of education ($T = 978$), and people with more than 12 years of education ($T = 435$).

We set $\bar{\lambda} = .0001$, and start the optimization algorithm with the NIVR estimates of GS. The specification test of Gagliardini and Scaillet (2007) does not reject the null hypothesis of the correct specification of the moment restriction used in estimating the mean pricing curve at the 5% significance level (p-value $= .32, .77$). We apply a reduction factor of .8 to the regularization parameters selected by the heuristic data-driven procedure of GS run on the linearization. For the two education categories, we get $\lambda = .154, .034$ under a Sobolev penalty with $l = 1$ and get $\lambda = .088, .001$ under a Sobolev penalty with $l = 2$. We present the empirical results with 16 polynomials. They remain virtually unchanged when gradually increasing the number of polynomials from 8 to 16. There is a stabilization of the value of the optimized objective function, of the loadings in the numerical series approximation, and of the data-driven regularization parameter. Higher order polynomials receive loadings that are closer and closer to zero. This suggests that we can limit ourselves to a small number of polynomials in this application.

Figure 2 plots the estimated nonparametric instrumental variable (NIV) median structural effect, pointwise asymptotic confidence intervals at 95%, and

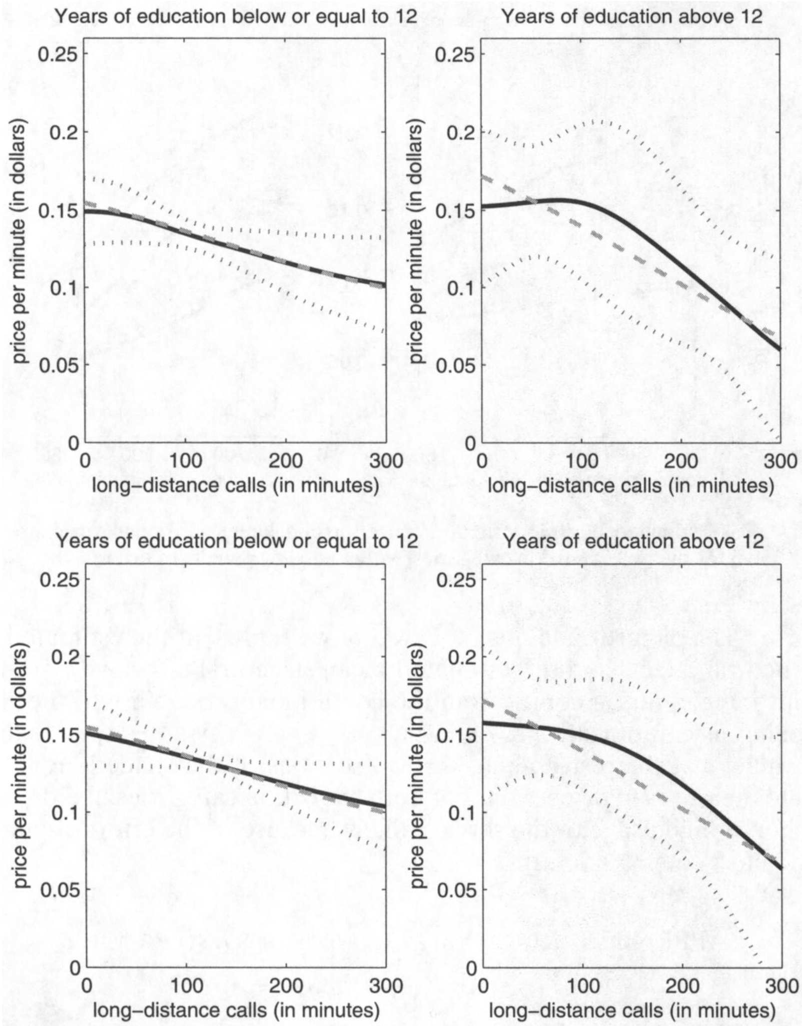

FIGURE 2.—Estimated median structural effect for the two education categories: NIVQR (solid line); IVQR (dashed line); 95% confidence intervals (dotted lines). The upper panel uses a Sobolev penalty of order $l = 1$; the lower panel uses $l = 2$.

the linear IVQR estimate for the two education categories. The upper panel for $l = 1$ and the lower panel for $l = 2$ show that estimated NIVQR and IVQR structural effects are close, and their patterns differ across the two education categories. As in Hausman and Sidak (2004) we observe that less educated customers pay more than better educated customers when the number of minutes of use increases. A possible explanation is that the latter better exploit the tariff options for long-distance calls available at those ranges.

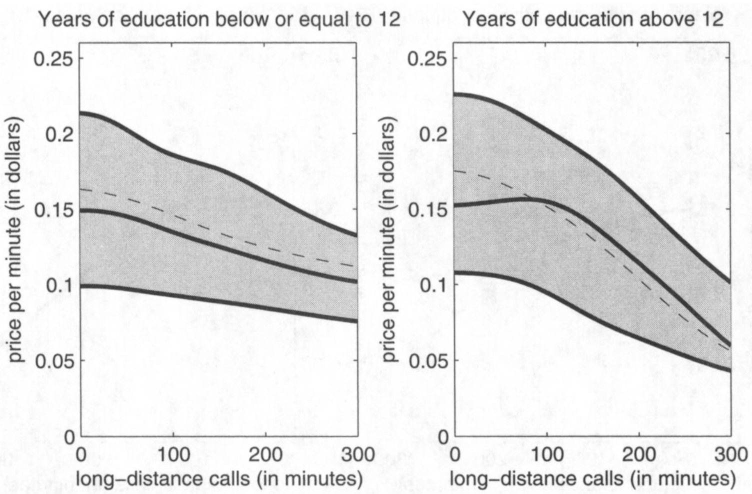

FIGURE 3.—Estimated quartile structural effects (solid lines) and mean structural effects (dashed lines) for the two education categories, with a Sobolev penalty of order $l = 1$.

Figure 3 is a picture "à la box plot" where we represent the estimated quartile structural effects and the estimated mean structural effect with $l = 1$. The box-plot interpretation comes from the conditional probability of the shaded area being asymptotically $P[g(X, .25) \leq Y \leq g(X, .75)|Z = z] = .5$ for any given value z of the instrument. Vertical sections of the shaded areas correspond to measures of dispersion. For both education categories, the dispersion is smaller at high usage of the service, likely because of the effort of people to find the most convenient tariffs.

APPENDIX: REGULARITY CONDITIONS AND PROOFS

A.1. *Assumptions*

Below we list the additional technical regularity conditions. In particular, we invoke A.1–A.3 for proofs of local ill-posedness and consistency, and invoke A.4 and A.5 for asymptotic normality. For a function f of variable s in $\mathbb{R}^{d_s}$ and a multiindex $\alpha \in \mathbb{N}^{d_s}$, we denote $\nabla^\alpha f := \nabla_{s_1}^{\alpha_1} \cdots \nabla_{s_{d_s}}^{\alpha_{d_s}} f$, $|\alpha| := \sum_{i=1}^{d_s} \alpha_i$, $\|f\|_\infty := \sup_s |f(s)|$, and $\|Df\|_\infty := \sum_{\alpha:|\alpha|=1} \|\nabla^\alpha f\|_\infty$.

ASSUMPTION A.1: (i) *$\{(X_l, Y_l, Z_l^*) : l = 1, \ldots, T^*\}$ is a sample of independent and identically distributed* (*i.i.d.*) *observations of random variable (X, Y, Z^*) that admit a density f_{X,Y,Z^*} on the support $\mathcal{X} \times \mathcal{Y} \times \mathcal{Z}^* \subset \mathbb{R}^d$, where $\mathcal{X} = [0, 1]$, $\mathcal{Y} = [0, 1]$, $\mathcal{Z}^* \subset \mathbb{R}^{d_Z}$, and $d = 2 + d_Z$.* (ii) *The density f_{X,Y,Z^*} is in class $C^m(\mathbb{R}^d)$, with $m \geq 2$, and $\nabla^\alpha f_{X,Y,Z^*}$ is uniformly continuous and bounded for any $\alpha \in \mathbb{N}^d$ with $|\alpha| = m$.* (iii) *The random variable (X, Y, Z) is such that $(X, Y, Z) =$*

(X, Y, Z^*) if $Z^* \in \mathcal{Z}$, where $\mathcal{Z} = [0,1]^{d_Z}$ is interior to $\mathcal{Z}^*$ and the density f_Z of Z is such that $\inf_{z\in\mathcal{Z}} f_Z(z) > 0$.

ASSUMPTION A.2: *The kernel K on $\mathbb{R}^d$ is such that* (i) $\int K(u)\,du = 1$ *and K is bounded,* (ii) *K has compact support,* (iii) *K is differentiable, with bounded derivatives, and* (iv) $\int u^\alpha K(u)\,du = 0$ *for any $\alpha \in \mathbb{N}^d$ with $|\alpha| < m$.*

ASSUMPTION A.3: (i) *The function $\tau \mapsto g(x,\tau)$ is strictly monotonic increasing and continuous for almost any $x \in (0,1)$ and* $\sup_{x,\tau}|g(x,\tau)| < \infty$, $\sup_{x,\tau}|\nabla_x g(x,\tau)| < \infty$. (ii) $\|Df_{X|Z}\|_\infty < \infty$. (iii) $\|DF_{U|X,Z}\|_\infty < \infty$.

ASSUMPTION A.4: (i) *There exists $h > 0$ such that function* $q(s) := \sum_{\alpha:|\alpha|\le m}\sup_{v\in B_h(s)}|\nabla^\alpha f_{X,Y,Z}(v)|$, *$s \in \mathcal{S} := \mathcal{X}\times\mathcal{Y}\times\mathcal{Z}$, is integrable and satisfies* $\int_{\mathcal{S}} \frac{q(s)^2}{f_{X,Y,Z}(s)}\,ds < \infty$, *where $B_h(s)$ denotes the ball in $\mathbb{R}^d$ of radius h centered at s.* (ii) $\|f_{X,Y|Z}\|_\infty < \infty$. (iii) $\|\nabla_y f_{X,Y|Z}\|_\infty < \infty$.

ASSUMPTION A.5: (i) $\sum_{j,k=1,j\ne k}^\infty \frac{\langle\phi_j,\phi_k\rangle^2}{\|\phi_j\|^2\|\phi_k\|^2} < \infty$. (ii) *The functions* $\psi_j(z) := \frac{1}{\sqrt{\nu_j}}(A\phi_j)(z)$, $j \in \mathbb{N}$, *satisfy* $\sup_{j\in\mathbb{N}} E[|\psi_j(Z)|^{\bar{s}}]^{1/\bar{s}} < \infty$ *for* $\bar{s} \ge 4$. (iii) *The functions ψ_j are in class $C^m(\mathbb{R}^{d_Z})$ such that* $E[|\nabla^\alpha\psi_j(Z)|^{\bar{s}}]^{1/\bar{s}} = O(j^{|\alpha|})$ *for any $\alpha \in \mathbb{N}^{d_Z}$ with $|\alpha| \le m$.*

In Assumption A.1(i), the compact support of X and Y is used for technical reasons. Assuming univariate X simplifies the exposition. Assumptions A.1(ii) and A.2 are classical conditions in kernel density estimation concerning smoothness of the density and of the kernel. In particular, when $m > 2$, K is a higher order kernel. Moreover, we assume a compact support for the kernel K to simplify the set of regularity conditions. In Assumption A.1(iii), variable Z is obtained by truncating Z^* on the compact set $\mathcal{Z}$, and the density f_Z of Z is bounded from below away from 0 on the support $\mathcal{Z}$. The corresponding observations are Z_t, $t = 1,\dots,T$, where $T \le T^*$. We get the estimator $\hat{f}_{X,Y,Z}$ of the density $f_{X,Y,Z}$ from the kernel estimator $\hat{f}_{Y,X,Z^*}(x,y,z) = \frac{1}{T^* h_T^d}\sum_{l=1}^{T^*} K((X_l - x)/h_T)K((Y_l - y)/h_T)K((Z_l^* - z)/h_T)$ of density f_{X,Y,Z^*} by normalization, namely $\hat{f}_{Y,X,Z} = \hat{f}_{Y,X,Z^*}/\int_{\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}}\hat{f}_{Y,X,Z^*} = \hat{f}_{Y,X,Z^*}/\int_{\mathcal{Z}}\hat{f}_{Z^*}$. This trick is used in the proofs to control for small values of the estimator of the density of Z that appear in denominators and to avoid edge effects. Alternative approaches to address these technical issues consist in using trimming (see, e.g., Hansen (2008)), boundary kernels (see, e.g., Hall and Horowitz (2005) for the use of such kernels in NIV regression), or density weighting (see, e.g., Horowitz and Lee (2007)). Assumption A.3(i) is a boundedness and smoothness condition on function $g(x,\tau)$ w.r.t. both its arguments. Assumption A.3(ii) and (iii) concern boundedness and smoothness of the p.d.f. of X given Z and the c.d.f. of U given X, Z, respectively.

Assumption A.4 concerns the joint density $f_{X,Y,Z}$ and the conditional density $f_{X,Y|Z}$. Specifically, Assumption A.4(i) imposes an integrability condition on a suitable measure of local variation of density $f_{X,Y,Z}$ and its derivatives. This assumption is used in the proof of Lemma A.10 to bound higher order terms in the asymptotic expansion of the estimator coming from kernel estimation bias. Assumption A.4(ii) and (iii) are used to show that $\mathcal{A}$ is Frechet differentiable, with compact Frechet derivative. These assumptions can be rewritten in terms of densities $f_{U|X,Z}$, $f_{X|Z}$ and function g. The formulation as in Assumption A.4(ii) and (iii) is closer to the use in the proofs and simplifies the exposition. Assumption A.4(iii) also implies Lipschitz behavior of the Frechet derivative operator $D\mathcal{A}(\varphi)$ w.r.t. φ in a neighborhood of the true function (assumption (ii) in Theorem 10.4 of Engl, Hanke, and Neubauer (2000)). Finally, Assumption A.5 concerns the singular system $\{\sqrt{\nu_j}, \phi_j, \psi_j; j \in \mathbb{N}\}$ of operator A (Kress (1999, p. 278)). Assumption A.5(i) requires that the $\langle \cdot, \cdot \rangle_H$ orthonormal basis of eigenfunctions of A^*A satisfy a summability condition w.r.t. $\langle \cdot, \cdot \rangle$. This assumption eases the derivation of the upper bound of the MISE in Lemma A.7. Assumption A.5(ii) and (iii) ask for the existence of bounds for moments of derivatives of functions ψ_j, $j \in \mathbb{N}$. Functions ψ_j, $j \in \mathbb{N}$, are an orthonormal system in $L^2(F_Z, \tau)$. These assumptions control for terms of the type $\int \psi_j(z)[1\{y \leq \varphi_0(x)\} - \tau]\hat{f}_{X,Y,Z}(s)\,ds$, uniformly in $j \in \mathbb{N}$, in the proof of Lemmas A.7, A.8, and A.11.

A.2. *Proof of Proposition 1*

In Step 1 we show local ill-posedness of the nonseparable setting (part (a)), and in Step 2 we prove that sequences that generate ill-posedness exhibit diverging L^2-norm of their first derivative (part (b)). We place all omitted proofs in the Supplemental Material.

Step 1—Proof of part (a). We use the next lemma, which is a local version of Proposition 10.1 in Engl, Hanke, and Neubauer (2000).

LEMMA A.1: *Suppose that*: (i) *operator $\mathcal{A}$ is compact, and* (ii) *for any $r > 0$ small enough, there exists a sequence $(\varphi_n) \subset B_r(\varphi_0)$ such that $\varphi_n \nrightarrow \varphi_0$ and $\mathcal{A}(\varphi_n) \xrightarrow{w} \mathcal{A}(\varphi_0)$, where $\xrightarrow{w}$ denotes weak convergence. Then the minimum distance problem is locally ill-posed.*

Condition (i) in Lemma A.1 follows from the next lemma, which is proved using a result in Alt (1992).

LEMMA A.2: *Under Assumption* A.3(ii) *and* (iii), *operator $\mathcal{A}$ is compact.*

Let us now verify condition (ii) in Lemma A.1. Define $\psi(x) := \sin(2\pi x)$ and $\psi_n(x) := \varepsilon\psi(nx)$, $x \in \mathcal{X}$, where $0 < \varepsilon < \min\{\tau, 1 - \tau\}$. Further, let $\varphi_n(x) := g(x, \tau + \psi_n(x))$, $x \in \mathcal{X}$. Then we deduce that $\|\varphi_n - \varphi_0\|^2 = \int_{\mathcal{X}}[g(x, \tau +$

$\varepsilon\psi(nx)) - g(x,\tau)]^2\,dx$. Split the integral w.r.t. x over the partition $((k-1)/n, k/n]$, $k = 1,\dots,n$, of $(0,1]$. It follows that

$$\begin{aligned}
&\|\varphi_n - \varphi_0\|^2 \\
&= \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \big[g(x, \tau + \varepsilon\psi(nx)) - g(x,\tau)\big]^2\,dx \\
&= \sum_{k=1}^{n} \frac{1}{n} \int_0^1 \Big[g\Big(\frac{k-1}{n} + \frac{y}{n}, \tau + \varepsilon\psi(y)\Big) - g\Big(\frac{k-1}{n} + \frac{y}{n}, \tau\Big)\Big]^2\,dy,
\end{aligned}$$

using the periodicity of ψ. Using Assumption A.3(i),

$$\begin{aligned}
\|\varphi_n - \varphi_0\|^2 = {}& \sum_{k=1}^{n} \frac{1}{n} \int_0^1 \Big[g\Big(\frac{k-1}{n}, \tau + \varepsilon\psi(y)\Big) - g\Big(\frac{k-1}{n}, \tau\Big)\Big]^2\,dy \\
&+ O(1/n).
\end{aligned}$$

The first term is a converging Riemann sum, and we get $\|\varphi_n - \varphi_0\|^2 \to I_\varepsilon := \int_{\mathcal{X}} \int_0^1 [g(x, \tau + \varepsilon\psi(y)) - g(x,\tau)]^2\,dy\,dx$ as $n \to \infty$. Then $I_\varepsilon > 0$, and $I_\varepsilon \to 0$ as $\varepsilon \to 0$ by the dominated convergence theorem. Thus, for $\varepsilon > 0$ sufficiently small, we have $(\varphi_n) \subset B_r(\varphi_0)$ and $\varphi_n \nrightarrow \varphi_0$. For $\bar{q} \in L^2(F_Z, \tau)$, we have

$$\begin{aligned}
&\langle \bar{q}, \mathcal{A}(\varphi_n)\rangle_{L^2(F_Z,\tau)} \\
&= \frac{1}{\tau(1-\tau)} \int \bar{q}(z) f_Z(z) \int_{\mathcal{X}} f_{X|Z}(x|z) F_{U|X,Z}(\tau + \psi_n(x)|x, z)\,dx\,dz.
\end{aligned}$$

Thus, we have to show

$$\begin{aligned}
\text{(A.1)}\qquad J_n := {}& \int \bar{q}(z) f_Z(z) \int_{\mathcal{X}} f_{X|Z}(x|z) F_{U|X,Z}(\tau + \psi_n(x)|x, z)\,dx\,dz \\
&\to \tau \int \bar{q}(z) f_Z(z)\,dz.
\end{aligned}$$

To this end, in J_n we split the integral w.r.t. x over the partition $((k-1)/n, k/n]$ with $k = 1,\dots,n$ and get

$$\begin{aligned}
J_n = {}& \sum_{k=1}^{n} \frac{1}{n} \int_0^1 \int \bar{q}(z) f_Z(z) f_{X|Z}\Big(\frac{k-1}{n} + \frac{1}{n}y \Big| z\Big) \\
&\times F_{U|X,Z}\Big(\tau + \varepsilon\psi(y) \Big| \frac{k-1}{n} + \frac{1}{n}y, z\Big)\,dz\,dy
\end{aligned}$$

after a change of variable and using the periodicity of function ψ. Then we have

$$J_n = \sum_{k=1}^{n} \frac{1}{n} \int \bar{q}(z) f_Z(z) f_{X|Z}\left(\frac{k-1}{n}\Big| z\right) \times \int_0^1 F_{U|X,Z}\left(\tau + \varepsilon\psi(y) \Big| \frac{k-1}{n}, z\right) dy\, dz + I_{1,n}, \tag{A.2}$$

where $|I_{1,n}| \leq \sum_{k=1}^{n} \frac{1}{n^2} \int_0^1 \int \bar{q}(z) f_Z(z) \sup_{u,x,z} |\nabla_x H(u, x|z)| y\, dz\, dy = O(1/n)$ and $H(u, x|z) := F_{U|X,Z}(u|x, z) f_{X|Z}(x|z)$, with $\sup_{u,x,z} |\nabla_x H(u, x|z)| < \infty$ from Assumption A.3(ii) and (iii). Since the Riemann sum in (A.2) converges to the corresponding integral, we get

$$J_n \to \int_{\mathcal{X}} \int \bar{q}(z) f_Z(z) f_{X|Z}(x|z) \int_0^1 F_{U|X,Z}(\tau + \varepsilon\psi(y)|x, z)\, dy\, dz\, dx =: J.$$

Using that $\int_{\mathcal{X}} f_{X|Z}(x|z) F_{U|X,Z}(u|x, z)\, dx = P[U \leq u|Z = z] = u$ by the independence of U and Z, and the uniform distribution of U, we get $J = \tau \int \bar{q}(z) f_Z(z)\, dz + \varepsilon \int \bar{q}(z) f_Z(z) \int_0^1 \psi(y)\, dy = \tau \int \bar{q}(z) f_Z(z)\, dz$, and (A.1) follows.

Step 2—Proof of part (b). The proof is by contradiction. Suppose that there exists $B < \infty$ such that $\|\nabla\varphi_n\| \leq B$ for any n large enough. Since Θ is bounded, by the compact embedding theorem (see Adams and Fournier (2003)), set $\{\varphi \in \Theta : \|\nabla\varphi\| \leq B\}$ is compact w.r.t. the norm $\|\cdot\|$. Therefore, there exists a subsequence (φ_{m_n}) that converges in norm $\|\cdot\|$ to $\varphi^* \in \Theta$, say. Since Q_∞ is continuous, we get $Q_\infty(\varphi_{m_n}) \to Q_\infty(\varphi^*)$, and thus $Q_\infty(\varphi^*) = 0$. By identification (Assumption 1(i)), we deduce $\varphi^* = \varphi_0$, and the subsequence (φ_{m_n}) converges to φ_0. But this is impossible, since $\|\varphi_{m_n} - \varphi_0\| \geq \varepsilon > 0$.

A.3. *Proof of Proposition 2*

We establish existence of the Q-TiR estimator in the Supplemental Material by using a result in Reed and Simon (1980). To prove consistency, the next lemma establishes (uniform) convergence of the minimum distance criterion $Q_T(\varphi)$ by using results in Andrews (1994), Bosq (1998), and Hansen (2008).

LEMMA A.3: *Under Assumptions* A.1, A.2, *and* A.3(ii) *and* (iii), *we have* (i) $Q_T(\varphi_0) - Q_\infty(\varphi_0) = O_p(a_T)$, *where* $a_T := \frac{\log T}{T h_T^{d_Z+1}} + h_T^{2m}$, *and* (ii) $\sup_{\varphi\in\Theta} |Q_T(\varphi) - Q_\infty(\varphi)| = O_p(\sqrt{a_T} + \frac{1}{\sqrt{T}}) = o_p(1)$ *for* $a_T = o(1)$.

By Lemma A.3(i) and the condition on λ_T, we have

$$0 \le Q_T(\hat{\varphi}) + \lambda_T \|\hat{\varphi}\|_H^2 \le Q_T(\varphi_0) + \lambda_T \|\varphi_0\|_H^2 = O_p(a_T + \lambda_T) = O_p(\lambda_T). \tag{A.3}$$

By $Q_T \ge 0$, this implies that $\lambda_T \|\hat{\varphi}\|_H^2 = O_p(\lambda_T)$, that is, $\|\hat{\varphi}\|_H^2 = O_p(1)$. Thus, by the compact embedding theorem, the sequence of minimizers $\hat{\varphi}$ is tight in $(L^2[0,1], \|\cdot\|)$. Namely, for any $\delta > 0$, there exists a compact subset K_δ of $(L^2[0,1] \cap \Theta, \|\cdot\|)$, such that $P[\hat{\varphi} \in K_\delta] \ge 1 - \delta$ for all sufficiently large sample sizes.

Next we have that for any $\varepsilon > 0$ and $\delta > 0$, and any T sufficiently large,

$$\begin{aligned}
&P[\hat{\varphi} \notin B_\varepsilon(\varphi_0)] \\
&\quad \le P\big[\{\hat{\varphi} \notin B_\varepsilon(\varphi_0)\} \cap \{\hat{\varphi} \in K_\delta\}\big] + P[\hat{\varphi} \notin K_\delta] \\
&\quad \le P\big[\{\hat{\varphi} \notin B_\varepsilon(\varphi_0)\} \cap \{\hat{\varphi} \in K_\delta\}\big] + \delta \\
&\quad \le P\Big[\inf_{\varphi \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)} Q_T(\varphi) + \lambda_T \|\varphi\|_H^2 \le Q_T(\hat{\varphi}) + \lambda_T \|\hat{\varphi}\|_H^2\Big] + \delta.
\end{aligned}$$

Using bound (A.3) and Lemma A.3(ii), we get

$$\begin{aligned}
&P[\hat{\varphi} \notin B_\varepsilon(\varphi_0)] \\
&\quad \le P\Big[\inf_{\varphi \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)} Q_\infty(\varphi) + \lambda_T \|\varphi\|_H^2 + o_p(1) \le O_p(\lambda_T)\Big] + \delta \\
&\quad \le P\Big[\inf_{\varphi \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)} Q_\infty(\varphi) + o_p(1) \le O_p(\lambda_T)\Big] + \delta \\
&\quad \le P\Big[\inf_{\varphi \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)} Q_\infty(\varphi) \le o_p(1)\Big] + \delta.
\end{aligned}$$

Now let $\kappa_{\delta,\varepsilon} := \inf_{\varphi \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)} Q_\infty(\varphi)$. By compactness of K_δ, continuity of Q_∞, and identification, we have $\kappa_{\delta,\varepsilon} = Q_\infty(\varphi^*_{\delta,\varepsilon}) > 0$ for some $\varphi^*_{\delta,\varepsilon} \in K_\delta \cap \Theta \setminus B_\varepsilon(\varphi_0)$. Thus

$$P[\hat{\varphi} \notin B_\varepsilon(\varphi_0)] \le P[\kappa_{\delta,\varepsilon} \le o_p(1)] + \delta \to \delta \quad \text{as} \quad T \to \infty.$$

Since δ can be made arbitrarily small, we conclude that $P[\hat{\varphi} \notin B_\varepsilon(\varphi_0)] \to 0$. Since $\varepsilon > 0$ is arbitrary, consistency follows.

A.4. *Proof of Proposition 3*

The steps are as follows: getting the first-order condition, deriving a Bahadur representation and an asymptotic expansion of the MISE, proving asymptotic normality, and showing bias negligibility.

Step 1—First-order condition. The following lemma provides the Frechet derivative of operator $\hat{A}$.

LEMMA A.4: *Under Assumption* A.2, *the Frechet derivative of $\hat{A}$ at $\bar{\varphi}$ is the linear operator $\bar{A} := D\hat{A}(\bar{\varphi})$ defined by $\bar{A}\varphi(z) = \int \hat{f}_{X,Y|Z}(x, \bar{\varphi}(x)|z)\varphi(x)\,dx$, $z \in \mathcal{Z}$, for $\varphi \in L^2[0,1]$. Moreover, we have $\hat{A}(\varphi) = \hat{A}(\bar{\varphi}) + \bar{A}(\varphi - \bar{\varphi}) + \hat{R}(\varphi, \bar{\varphi})$, where $\hat{R}(\varphi, \bar{\varphi})$ is such that P-a.s., $\|\hat{R}(\varphi, \bar{\varphi})\|_{L^2(\hat{F}_Z,\tau)} \leq \frac{1}{2\sqrt{\tau(1-\tau)}}\hat{c}\|\varphi - \bar{\varphi}\|^2$ and $\hat{c} := \sup_{x\in\mathcal{X}, y\in\mathbb{R}, z\in\mathcal{Z}} |\nabla_y \hat{f}_{X,Y|Z}(x, y|z)|$.*

By Assumption 1(ii), let $r > 0$ be such that $B_r(\varphi_0) \cap H^l[0,1]$ is contained in Θ. When $\|\Delta\hat{\varphi}\| < r$, we have $\forall \varphi \in H^l[0,1]$, $\exists \rho = \rho(\varphi) > 0 : \hat{\varphi} + \varepsilon\varphi \in \Theta$ for any $\varepsilon \in \mathbb{R}$ such that $|\varepsilon| < \rho$. By Lemma A.4, the estimator $\hat{\varphi}$ satisfies the first order condition (4.1). We show in the Supplemental Material that $P[\|\Delta\hat{\varphi}\| \geq r] = O(T^{-\bar{b}})$ for any $\bar{b} > 0$.

Step 2—Bahadur representation and asymptotic expansion of the MISE. We first rewrite (4.2) as

$$\Delta\hat{\varphi} = \Delta\hat{\psi} + \hat{\mathcal{K}}_T(\Delta\hat{\varphi}), \tag{A.4}$$

where $\Delta\hat{\psi} := \hat{\psi} - \varphi_0$, with $\hat{\psi} := (\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}\hat{A}_0^*\hat{r} - (\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}(\hat{A}^* - \hat{A}_0^*)(\hat{A}(\hat{\varphi}) - \tau) =: \hat{\psi}_1 + \hat{\psi}_2$ and $\hat{r} := \tau + \hat{A}_0\varphi_0 - \hat{A}(\varphi_0)$, and $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ defined in Section 4.2. The interpretation of $\hat{\psi}_1$ is as a linearized solution obtained from applying a Tikhonov regularization to the linear proxy $\hat{A}_0\varphi = \hat{r}$. The impact of nonlinearity is twofold. We face the second-order term $\hat{R}$ in $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ because of the expansion, and we face $\hat{A}^* - \hat{A}_0^*$ in $\hat{\psi}_2$ because of $\hat{\varphi}$ in $\hat{A}$. Now, we decompose $\Delta\hat{\psi}$ as

$$\Delta\hat{\psi} = (\lambda_T + A^*A)^{-1}A^*\hat{\zeta} + \mathcal{B}_T + \mathcal{R}_T, \tag{A.5}$$

$$\begin{aligned} \mathcal{R}_T = {} & [(\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1} - (\lambda_T + A^*A)^{-1}]A^*\hat{\zeta} \\ & + [(\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}\hat{A}_0^*\hat{A}_0 - (\lambda_T + A^*A)^{-1}A^*A]\varphi_0 \\ & + (\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}(\hat{A}_0^*(\hat{\zeta} - \hat{q}) - A^*\hat{\zeta}) \\ & - (\lambda_T + \hat{A}_0^*\hat{A}_0)^{-1}(\hat{A}^* - \hat{A}_0^*)(\hat{A}(\hat{\varphi}) - \tau), \end{aligned} \tag{A.6}$$

where $\hat{q} := \hat{A}(\varphi_0) - \tau + \hat{\zeta}$. The Bahadur-type representation (4.3) follows from (A.4) and (A.5).

We show by using Lemma A.5 that the nonlinearity term $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$ in Equation (A.4) satisfies a quadratic bound w.p.a.1.

LEMMA A.5: *Under Assumptions* A.1–A.3 *and* A.4(ii) *and* (iii), *and $\eta < \frac{1}{d_Z+4}$, for any $\bar{b} > 0$ and $C > \frac{1}{2\sqrt{\tau(1-\tau)}} \sup_{x,y,z} |\nabla_y f_{X,Y|Z}(x, y|z)|$, we have*

$$P\left[\|\hat{\mathcal{K}}_T(\Delta\hat{\varphi})\| > \frac{C}{\sqrt{\lambda_T}}\|\Delta\hat{\varphi}\|^2\right] = O(T^{-\bar{b}}).$$

Next, by exploiting (A.4) and the quadratic nature of $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})$, we get an asymptotic expansion of the MISE $E[\|\Delta\hat{\varphi}\|^2]$ in terms of λ_T and the expectations of powers of $\|\Delta\hat{\psi}\|$.

LEMMA A.6: *Under Assumptions* 1–4, A.1–A.3, *and* A.4(ii) *and* (iii), $\eta < \frac{1}{4+d_Z}$ *and, provided that* $\gamma < \min\{\frac{1-\eta(d_Z+1)}{1+2a}, \frac{2m\eta}{1+2a}, \frac{1}{2(1+a)}\}$, *we have that for any* $\bar{b} > 0$,

$$E[\|\Delta\hat{\varphi}\|^2] = E[\|\Delta\hat{\psi}\|^2] + O\left(\frac{1}{\sqrt{\lambda_T}}E[\|\Delta\hat{\psi}\|^3]\right) + O(T^{-\bar{b}}).$$

To compute moments of $\|\Delta\hat{\psi}\|$, we use the decomposition (A.5) and get the next upper bound for the asymptotic MISE.

LEMMA A.7: *Under Assumptions* 1–4 *and* A.1–A.5, *and* $\eta < \frac{1}{2(d_Z+2)}$, $\gamma < \frac{1}{2}\min\{1-\eta(d_Z+1), m\eta, \frac{1}{1+a}\}$, $V_T(\lambda_T) = o(\lambda_T)$, *and* $h_T^{1/4}\frac{V_T(\lambda_T;2)}{V_T(\lambda_T)} = o(1)$, *we have* $E[\|\Delta\hat{\varphi}\|^2] = O(M_T(\lambda_T))$, *where* $M_T(\lambda_T) = \int(\frac{1}{T}\sigma_T^2(x) + \mathcal{B}_T(x)^2)\,dx$.

Step 3—Asymptotic normality. From the Bahadur representation (4.3), we have

$$\begin{aligned}
&\sqrt{T/\sigma_T^2(x)}(\hat{\varphi}(x) - \varphi_0(x)) \\
&= \sqrt{T/\sigma_T^2(x)}(\lambda_T + A^*A)^{-1}A^*(\hat{\zeta} - E\hat{\zeta})(x) \\
&\quad + \sqrt{T/\sigma_T^2(x)}\mathcal{B}_T(x) + \sqrt{T/\sigma_T^2(x)}\hat{\mathcal{K}}_T(\Delta\hat{\varphi})(x) \\
&\quad + \sqrt{T/\sigma_T^2(x)}(\lambda_T + A^*A)^{-1}A^*E\hat{\zeta}(x) + \sqrt{T/\sigma_T^2(x)}\mathcal{R}_T(x) \\
&=: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) + (\mathrm{V}).
\end{aligned}$$

(a)—Asymptotic normality of I. Since $\{\phi_j : j \in \mathbb{N}\}$ is an orthonormal basis w.r.t. $\langle\cdot,\cdot\rangle_H$, we can write

$$\begin{aligned}
&(\lambda_T + A^*A)^{-1}A^*(\hat{\zeta} - E\hat{\zeta})(x) \\
&= \sum_{j=1}^{\infty}\langle\phi_j, (\lambda_T + A^*A)^{-1}A^*(\hat{\zeta} - E\hat{\zeta})\rangle_H\phi_j(x) \\
&= \sum_{j=1}^{\infty}\frac{1}{\lambda_T + \nu_j}\langle A\phi_j, \hat{\zeta} - E\hat{\zeta}\rangle_{L^2(F_{Z,\tau})}\phi_j(x)
\end{aligned}$$

for almost any $x \in [0, 1]$. Then we get

$$\sqrt{T/\sigma_T^2(x)}(\lambda_T + A^*A)^{-1}A^*(\hat{\zeta} - E\hat{\zeta})(x) = \sum_{j=1}^{\infty} w_{j,T}(x)Z_{j,T},$$

where $Z_{j,T} := \langle \psi_j, \sqrt{T}(\hat{\zeta} - E\hat{\zeta})\rangle_{L^2(F_{Z,\tau})}$ and $w_{j,T}(x) := \frac{\sqrt{\nu_j}}{\lambda_T+\nu_j}\phi_j(x)/\sigma_T(x)$, $j = 1, 2, \ldots$. Note that $\sum_{j=1}^{\infty} w_{j,T}(x)^2 = 1$. Let $g_j(r) := \frac{1}{\tau(1-\tau)}\psi_j(z)1_{\varphi_0}(w)$, where $1_{\varphi_0}(w) := 1\{y \le \varphi_0(x)\} - \tau$, $w = (y, x)$.

LEMMA A.8: *Under Assumptions* A.1, A.2, *and* A.5(iii), *and* $\gamma < m\eta$, $\frac{V_T(\lambda_T)}{\sigma_T^2(x)/T} = O(1)$, $\sqrt{h_T}\frac{V_T(\lambda_T;\varepsilon_1)}{V_T(\lambda_T)} = o(1)$, *and* $h_T^{2m}\frac{V_T(\lambda_T;2m+\varepsilon_1)}{V_T(\lambda_T)} = o(1)$, $\varepsilon_1 > 1$, *we have* $\sum_{j=1}^{\infty} w_{j,T}(x)Z_{j,T} = \frac{1}{\sqrt{T}}\sum_{t=1}^{T} Y_{tT} + o_p(1)$, *where* $Y_{tT} := \sum_{j=1}^{\infty} w_{j,T}(x)g_j(R_t)$, $R_t = (X_t, Y_t, Z_t)$.

From Lemma A.8, it is sufficient to prove that $T^{-1/2}\sum_{t=1}^{T} Y_{tT}$ is asymptotically $N(0, 1)$ distributed. Note that $E[g_j(R)] = \frac{1}{\tau(1-\tau)}\frac{1}{\sqrt{\nu_j}}E[(A\phi_j)(Z)E[1_{\varphi_0}(W)| Z]] = 0$ and $\mathrm{Cov}[g_j(R), g_k(R)] = \frac{1}{\sqrt{\nu_j}\sqrt{\nu_k}}\langle \phi_j, A^*A\phi_k\rangle_H = \delta_{j,k}$. Thus $E[Y_{tT}] = 0$ and $V[Y_{tT}] = 1$. By the Lyapunov central limit theorem (CLT), it is sufficient to show that

$$\text{(A.7)} \qquad \frac{1}{T^{1/2}}E[|Y_{tT}|^3] \to 0, \quad T \to \infty.$$

Toward this goal, using the triangular inequality and denoting $\|g_j\|_3 := E[|g_j(R)|^3]^{1/3}$, we get $E[|Y_{tT}|^3]^{1/3} \le \sum_{j=1}^{\infty} |w_{j,T}(x)|\|g_j\|_3 \le \frac{C}{\sigma_T(x)}\sum_{j=1}^{\infty}\frac{\sqrt{\nu_j}}{\lambda_T+\nu_j} \times |\phi_j(x)|$, where $C := \frac{1}{\tau(1-\tau)}\sup_{j\in\mathbb{N}} E[|\psi_j(Z)|^3]^{1/3} < \infty$ from Assumption A.5(ii). From the Cauchy–Schwarz inequality, we have $\sum_{j=1}^{\infty}\frac{\sqrt{\nu_j}}{\lambda_T+\nu_j}|\phi_j(x)| \le (\sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)^2}\phi_j(x)^2 j^{\varepsilon_2})^{1/2}(\sum_{j=1}^{\infty}\frac{1}{j^{\varepsilon_2}})^{1/2}$ and $\sum_{j=1}^{\infty}\frac{1}{j^{\varepsilon_2}} < \infty$ for any $\varepsilon_2 > 1$. From (4.7), we get $\frac{1}{T^{1/2}}E[|Y_{tT}|^3] = O([\frac{1}{T^{1/3}}\frac{\delta_T(x)}{V_T(\lambda_T)}]^{3/2})$, where $\delta_T(x) := \frac{1}{T} \times \sum_{j=1}^{\infty}\frac{\nu_j}{(\lambda_T+\nu_j)^2}\phi_j(x)^2 j^{\varepsilon_2}$. Condition (A.7) follows from $\int \frac{1}{T^{1/3}}\frac{\delta_T(x)}{V_T(\lambda_T)}\,dx = \frac{1}{T^{1/3}}\frac{V_T(\lambda_T;\varepsilon_2)}{V_T(\lambda_T)} = O(h_T^{1/4}\frac{V_T(\lambda_T;\varepsilon_2)}{V_T(\lambda_T)}) = o(1)$.

(b)—Negligibility of III. We use the following lemma.

LEMMA A.9: *Under Assumptions* A.1–A.3, *and* A.4(ii) *and* (iii), *and* $\eta < \frac{1}{d_Z+4}$, *we have* $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})(x) = O_p(\frac{1}{\sqrt{\lambda_T}}\|\Delta\hat{\varphi}\|^2)$.

By Lemmas A.7 and A.9, we get $\hat{\mathcal{K}}_T(\Delta\hat{\varphi})(x) = O_p(\frac{1}{\sqrt{\lambda_T}}M_T(\lambda_T))$. Since $\int \mathcal{B}_T(x)^2\,dx = O(\lambda_T^{2\delta})$ (see Step 4) and $\lambda_T^{2\delta} = O(V_T(\lambda_T))$, we have $M_T(\lambda_T) =$

$O(V_T(\lambda_T))$. Then we get $\sqrt{T/\sigma_T^2(x)}\hat{\mathcal{K}}_T(\Delta\hat{\varphi})(x) = O_p(\sqrt{\frac{V_T(\lambda_T)}{\lambda_T}}) = o_p(1)$ from (4.7) and $V_T(\lambda_T) = o(\lambda_T)$.

(c)—Negligibility of IV and V. Here we rely on the following lemmas.

LEMMA A.10: *Under Assumptions* A.2 *and* A.4(i), *and* $\gamma < \frac{m\eta}{2}$, $\lambda_T^{2\delta} = O(V_T(\lambda_T))$, *and* $\frac{V_T(\lambda_T)}{\sigma_T^2(x)/T} = O(1)$, *we have* $\sqrt{T/\sigma_T^2(x)}(\lambda_T + A^*A)^{-1}A^*E\hat{\zeta}(x) = o(1)$.

LEMMA A.11: *Under Assumptions* A.1–A.5, *and* $\lambda_T^{2\delta} = O(V_T(\lambda_T))$, *and* $\frac{V_T(\lambda_T)}{\sigma_T^2(x)/T} = O(1)$, $V_T(\lambda_T) = o(\lambda_T)$, $\eta < \frac{1}{2(d_Z+2)}$, *and* $\gamma < \frac{1}{2}\min\{1 - \eta(d_Z + 1), m\eta, \frac{1}{1+a}\}$, *we have* $\sqrt{T/\sigma_T^2(x)}\mathcal{R}_T(x) = o_p(1)$.

Step 4—Bias negligibility. Similarly to the proof of Proposition 3.11 in CFR, we have

$$\|\mathcal{B}_T\|_H^2 = \sum_{j=1}^{\infty} \frac{\lambda_T^2\langle\phi_j, \varphi_0\rangle_H^2}{(\lambda_T + \nu_j)^2} = \lambda_T^{2\delta}\sum_{j=1}^{\infty} \frac{\lambda_T^{2-2\delta}\nu_j^{2\delta}}{(\lambda_T + \nu_j)^2}\frac{\langle\phi_j, \varphi_0\rangle_H^2}{\nu_j^{2\delta}}$$

$$\leq \lambda_T^{2\delta}\sum_{j=1}^{\infty}\frac{\langle\phi_j, \varphi_0\rangle_H^2}{\nu_j^{2\delta}} = O(\lambda_T^{2\delta})$$

from Assumption 4(i). This proves (4.5). Moreover, from Lemma C.1 in the Supplemental Material, we have $\mathcal{B}_T(x) \leq 2\|\mathcal{B}_T\|_H$ for any $x \in [0, 1]$. Thus, from (4.7), we get $\sqrt{T/\sigma_T^2(x)}\mathcal{B}_T(x) = O(\sqrt{\lambda_T^{2\delta}/V_T(\lambda_T)}) = o(1)$ for any typical point x if $\lambda_T^{2\delta} = o(V_T(\lambda_T))$.

A.5. *Proof of Proposition 4*

Condition $\frac{\sigma_T^2(x)}{\hat{\sigma}_T^2(x)} \xrightarrow{p} 1$ is equivalent to $\frac{|\hat{\sigma}_T^2(x)-\sigma_T^2(x)|}{\sigma_T^2(x)} = o_p(1)$. Let us introduce the truncated series $\sigma_{0,T}^2(x) = \sum_{j=1}^{N_T}\frac{\nu_j}{(\nu_j+\lambda_T)^2}\phi_j(x)^2$ and the residual $\sigma_{1,T}^2(x) = \sum_{j=N_T+1}^{\infty}\frac{\nu_j}{(\nu_j+\lambda_T)^2}\phi_j(x)^2$. We have $\frac{|\hat{\sigma}_T^2(x)-\sigma_T^2(x)|}{\sigma_T^2(x)} \leq \frac{|\hat{\sigma}_T^2(x)-\sigma_{0,T}^2(x)|}{\sigma_T^2(x)} + \frac{\sigma_{1,T}^2(x)}{\sigma_T^2(x)}$. In Step 1, we show that the residual in the truncated series is negligible: $\xi_T := \frac{\sigma_{1,T}^2(x)}{\sigma_T^2(x)} = o(1)$. In Step 2, we show that the truncated estimator is consistent for the truncated series: $\delta_T := \frac{|\hat{\sigma}_T^2(x)-\sigma_{0,T}^2(x)|}{\sigma_T^2(x)} = o_p(1)$. In Step 3, we show the relative consistency of the estimated spectrum.

Step 1—Condition on the cutoff. Using $\int_0^1 \sigma_{1,T}^2(x)\,dx = \sum_{j=N_T+1}^{\infty}\frac{\nu_j}{(\nu_j+\lambda_T)^2} \times \|\phi_j\|^2 \leq \lambda_T^{-2}\sum_{j=N_T+1}^{\infty}\nu_j\|\phi_j\|^2$ and Condition (4.8), we get $\frac{\int_0^1\sigma_{1,T}^2(x)\,dx/T}{V_T(\lambda_T)} = o(1)$.

This implies $\frac{\sigma_{1,T}^2(x)/T}{V_T(\lambda_T)} = o(1)$ for almost all $x \in [0,1]$. By using $\frac{V_T(\lambda_T)}{\sigma_T^2(x)/T} = O(1)$, it follows that $\xi_T = o(1)$.

Step 2—Consistent estimation of the truncated series. We use the next lemma.

LEMMA A.12: *Let* $\epsilon_{1,T} := \sup_{1\le j\le N_T} \frac{|\hat{\nu}_j - \nu_j|}{\nu_j}$ *and* $\epsilon_{2,T} := \sup_{1\le j\le N_T} \frac{|\hat{\phi}_j(x)^2 - \phi_j(x)^2|}{\xi_j^*}$, *where* $\xi_j^* = c_{2,j}\exp(w_j\beta)$. *Then* $\delta_T \le C\epsilon_{2,T}(1+\epsilon_{1,T}) + \epsilon_{1,T} + \sqrt{1+\delta_T - \xi_T} \times \frac{\epsilon_{1,T}}{\sqrt{1-\epsilon_{1,T}}}(1+\sqrt{C\epsilon_{2,T}})$ *for a constant* $C > 0$.

Since $\xi_T = o(1)$ from Step 1, we get $\delta_T = o_p(1)$ if we show $\epsilon_{1,T} = o_p(1)$ and $\epsilon_{2,T} = o_p(1)$.

Step 3. (a)—Relative consistency of $\hat{\nu}_j$ and $\hat{\phi}_j(x)^2$ uniformly over $1 \le j \le n_T$. Let us first show that $\sup_{1\le j\le n_T} \frac{|\hat{\nu}_j - \nu_j|}{\nu_j} = O_p(T^{-b})$ and $\sup_{1\le j\le n_T} \frac{|\hat{\phi}_j(x)^2 - \phi_j(x)^2|}{\xi_j^*} = O_p(T^{-b})$ for $b > 0$. These conditions are implied by

$$\text{(A.8)} \qquad \frac{1}{\nu_{n_T}} \sup_{1\le j\le n_T} |\bar{\nu}_j - \nu_j| = o_p(T^{-b}), \qquad \frac{1}{\xi_{n_T}^*} \sup_{1\le j\le n_T} \|\bar{\phi}_j - \phi_j\| = o_p(T^{-b})$$

for $b > 0$. To prove (A.8), we use the next lemma, which follows from Lemmas 4.2 and 4.3 in Bosq (2000); see also Theorem 1 in Hall and Hosseini-Nasab (2006) for a spectral decomposition in $L^2[0,1]$.

LEMMA A.13: *For any* j, (i) $|\bar{\nu}_j - \nu_j| \le \|\hat{D}\|_H$ *and* (ii) $\|\bar{\phi}_j - \phi_j\| \le \frac{2\sqrt{2}}{\Delta\nu_j}\|\hat{D}\|_H$, *where* $\hat{D} := \bar{A}^*\bar{A} - A^*A$ *and* $\|\hat{D}\|_H := \sup_{\varphi\in H^l[0,1]:\|\varphi\|_H=1} \|\hat{D}\varphi\|_H$ *denotes the operator norm in* $H^l[0,1]$.

From Lemma A.13, an upper bound on the rate of convergence of $|\bar{\nu}_j - \nu_j|$ and $\|\bar{\phi}_j - \phi_j\|$ can be deduced from the rate of convergence of $\|\hat{D}\|_H$.

LEMMA A.14: *Under Assumptions* A.1–A.5, $\|\hat{D}\|_H = O_p(\frac{1}{\sqrt{Th_T^2}} + h_T^m + \|\bar{\varphi} - \varphi_0\|)$.

From Lemmas A.13 and A.14, and $\|\bar{\varphi} - \varphi_0\| = O_p(M_T(\bar{\lambda}_T)^{1/2})$ from Lemma A.7, we get $\sup_{1\le j\le n_T} |\bar{\nu}_j - \nu_j| = O_p(\kappa_T)$ and $\sup_{1\le j\le n_T} \|\bar{\phi}_j - \phi_j\| = O_p(\frac{\kappa_T}{\Delta\nu_{n_T}})$, where $\kappa_T := \frac{1}{\sqrt{Th_T^2}} + h_T^m + M_T(\bar{\lambda}_T)^{1/2}$. Thus, (A.8) follows from condition (4.9) and $\Delta\nu_{n_T} \le \nu_{n_T-1}$.

(b)—Relative consistency of $\hat{\nu}_j$ and $\hat{\phi}_j(x)^2$ uniformly over $1 \le j \le N_T$. Finally, we prove in the next lemma that the uniform convergence of $\hat{\nu}_j$ and $\hat{\phi}_j$ can be extended to $j \le N_T$ by the extrapolation procedure.

LEMMA A.15: *Under Assumptions* 5 *and* A.1–A.5, *and if* $N_T = O(n_T)$, *then* (i) $\sup_{n_T<j\leq N_T} \frac{|\hat{\nu}_j-\nu_j|}{\nu_j} = o_p(1)$ *and* (ii) $\sup_{n_T<j\leq N_T} \frac{|\hat{\phi}_j(x)^2-\phi_j(x)^2|}{\xi_j^*} = o_p(1)$.

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