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NOTES AND COMMENTS

INFERENCE IN DYNAMIC DISCRETE CHOICE MODELS WITH SERIALLY CORRELATED UNOBSERVED STATE VARIABLES

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This paper develops a method for inference in dynamic discrete choice models with serially correlated unobserved state variables. Estimation of these models involves computing high-dimensional integrals that are present in the solution to the dynamic program and in the likelihood function. First, the paper proposes a Bayesian Markov chain Monte Carlo estimation procedure that can handle the problem of multidimensional integration in the likelihood function. Second, the paper presents an efficient algorithm for solving the dynamic program suitable for use in conjunction with the proposed estimation procedure.

KEYWORDS: Dynamic discrete choice models, Bayesian estimation, MCMC, nearest neighbors, random grids.

1. INTRODUCTION

DYNAMIC DISCRETE CHOICE MODELS (DDCMs) describe the behavior of a forward-looking economic agent who chooses between several alternatives repeatedly over time. Estimation of the deep structural parameters of these models is a theoretically appealing and promising area in empirical economics. One important feature of DDCMs that was often assumed away in the literature due to computational difficulties is serial correlation in unobserved state variables. Ability, productivity, health status, taste idiosyncrasies, and many other unobservables are, however, likely to be persistent over time. This paper develops a computationally attractive method for inference in DDCMs with serially correlated unobservables.

Advances in simulation methods and computing speed over the last two decades made the Bayesian approach to statistical inference practical. Bayesian methods are now applied to many problems in statistics and econometrics that are difficult to tackle by the classical approach. Static discrete choice models and, more generally, models with latent variables, are one of those areas where the Bayesian approach was particularly fruitful; see for example Albert and Chib (1993), McCulloch and Rossi (1994), and Geweke, Keane, and Runkle (1994). Similarly to the static case, the likelihood function for a DDCM can be thought of as an integral over latent variables (the unobserved state

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variables). If the unobservables are serially correlated, computing this integral is very hard. A Markov chain Monte Carlo (MCMC) algorithm is employed in this paper to handle this issue.

An important obstacle for Bayesian estimation of DDCMs is the computational burden of solving the dynamic program (DP) at each iteration of the estimation procedure. Imai, Jain, and Ching (2005), from now on IJC, were the first to attack this problem and consider application of Bayesian methods for estimation of DDCMs. Their method uses an MCMC algorithm that solves the DP and estimates the parameters at the same time. The Bellman equation is iterated only once for each draw of the parameters. To obtain the approximations of the expected value functions for the current MCMC draw of the parameters, the authors used kernel smoothing over the approximations of the value functions from the previous MCMC iterations.

This paper extends the work of IJC in several dimensions. IJC employd MCMC to "solve the DP problem and estimate the parameters simultaneously" rather than handle more flexible specifications for unobservables. Their theory does not apply to a Gibbs sampler that includes blocks for simulating unobservables. In contrast, I develop an algorithm that applies MCMC to handle serially correlated unobservables and possibly other interesting forms of heterogeneity that would lead to hard integration problems in computing the likelihood function. Second, the algorithm developed in this paper can be applied to more general DDCMs: models with infinite state space and random state transitions (IJC's algorithm works for finite state space and deterministic transitions for all state variables except independent and identically distributed (i.i.d.) errors). I achieve this more general applicability of the algorithm in part by using nearest neighbors instead of the kernel smoothing used by IJC. Also, in addition to approximating the value function in the parameter space, an algorithm for solving the DP has to deal with an integration problem for computing the expectations of the value functions. My prescriptions for handling this integration problem differ from IJC's. Finally, this paper develops theory that justifies statistical inference made on the basis of the algorithm's output. In the Bayesian framework, most inference exercises involve computing posterior expectations of some functions. IJC showed that the last draw from their algorithm will converge in distribution to the posterior. I show that sample averages from my algorithm can be used to approximate posterior expectations, and this is exactly how MCMC output is used in practice.

The proposed method was experimentally evaluated on two different DDCMs: a binary choice model of optimal bus engine replacement (Rust (1987)) and a model of medical care use and work absence (Gilleskie (1998)). Experiments are excluded from this paper for brevity. They can be found in Norets (2007, 2008). In summary, experiments demonstrate that ignoring serial correlation in unobservables of DDCMs can lead to serious misspecification errors and that the proposed method for handling serially correlated unobservables is feasible, accurate, and reliable.

The paper is organized as follows. Section 2 describes setup and estimation of a general DDCM. The algorithm for solving the DP and corresponding convergence results are presented in Sections 3 and 4. Proofs of the theoretical results can be found in the Supplemental Material (Norets (2009b)).

2. SETUP AND ESTIMATION OF DDCMS

Eckstein and Wolpin (1989), Rust (1994), and Aguirregabiria and Mira (2007) surveyed the literature on estimation of DDCMs. Below, I introduce a general model setup and emphasize possible advantages of the Bayesian approach to the estimation of these models, especially in treating the time dependence in unobservables. I also briefly discuss most relevant previous research.

Under weak regularity conditions (see, e.g., Rust (1994)), a DDCM can be described by the Bellman equation

$$V(s_t;\theta) = \max_{d_t\in D} \mathcal{V}(s_t, d_t;\theta), \tag{1}$$

where $\mathcal{V}(s_t, d_t;\theta) = u(s_t, d_t;\theta) + \beta E\{V(s_{t+1};\theta)|s_t, d_t;\theta\}$ is an alternative-specific value function, $u(s_t, d_t;\theta)$ is a per-period utility function, $s_t \in S$ is a vector of state variables, d_t is a control from a finite set D, $\theta \in \Theta$ is a vector of parameters, β is a time discount factor, and $V(s_t;\theta)$ is a value function or lifetime utility of the agent. The state variables are assumed to evolve according to a controlled first order Markov process with a transition law denoted by $f(s_{t+1}|s_t, d_t;\theta)$ for $t \geq 1$; the distribution of the initial state is denoted by $f(s_1|\theta)$. This formulation embraces a finite horizon case if time t is included in the vector of the state variables.

In estimable DDCMs, some state variables, denoted here by y_t, are assumed to be unobserved by econometricians. The observed states are denoted by x_t. All the state variables $s_t = (x_t, y_t)$ are known to the agent at time t. Examples of the unobserved state variables include taste idiosyncrasy, health status, ability, and returns to patents. The unobservables play an important role in the estimation. The likelihood function is a product of integrals over the unobservables

$$p(x, d|\theta) = \prod_{i=1}^{I} \int p\big(y_{T_i,i}, x_{T_i,i}, d_{T_i,i}, \dots, y_{1,i}, x_{1,i}, d_{1,i}|\theta\big)\, d\big(y_{T_i,i}\cdots y_{1,i}\big), \tag{2}$$

where $(x, y, d) = \{x_{t,i}, y_{t,i}, d_{t,i}\}_{t=1}^{T_i}$, $i \in \{1, \dots, I\}$, I is the number of the observed individuals, T_i is the number of time periods individual i is observed,

$$\begin{aligned} &p\big(y_{T_i,i}, x_{t,i}, d_{t,i}, \dots, y_{1,i}, x_{1,i}, d_{1,i}|\theta\big) \\ &\quad = \prod_{t=1}^{T_i} p(d_{t,i}|y_{t,i}, x_{t,i};\theta) f(x_{t,i}, y_{t,i}|x_{t-1,i}, y_{t-1,i}, d_{t-1,i};\theta), \end{aligned}$$

$f(\cdot|\cdot;\theta)$ is the state transition density, $\{x_{0,i}, y_{0,i}, d_{0,i}\} = \emptyset$, and $p(d_{t,i}|y_{t,i}, x_{t,i};\theta)$ is the choice probability conditional on all state variables.

In general, evaluation of the likelihood function in (2) involves computing multidimensional integrals of an order equal to T_i times the number of components in y_t, which becomes very difficult for large T_i and/or multidimensional unobservables y_t. That is why in previous literature the unobservables were often assumed to be i.i.d. In a series of papers, Rust developed a dynamic multinomial logit model, where he assumed that the utility function of the agents is additively separable in the unobservables and that the unobservables are extreme value i.i.d. In this case, the integration in (2) can be performed analytically. Pakes (1986) used Monte Carlo simulations to approximate the likelihood function in a model of binary choice with a serially correlated one-dimensional unobservable. More recently, several authors estimated models with particular forms of serial correlation in unobservables by adopting the method of Keane and Wolpin (1994), which uses Monte Carlo simulations to compute the likelihood and interpolating regressions to speed up the solution to the DP.[2] Even for DDCMs with special forms of serial correlation that reduce the dimension of integration in (2), estimation is still very hard. In this paper, I propose a computationally attractive Bayesian approach to estimation of DDCMs with serial correlation in unobservables.

In the Bayesian framework, the high-dimensional integration over y_t for each parameter value can be circumvented by employing Gibbs sampling and data augmentation. In models with latent variables, the Gibbs sampler typically has two types of blocks: (a) parameters conditional on other parameters, latent variables, and the data; (b) latent variables conditional on other latent variables, parameters, and the data (this step is called data augmentation). Draws from this Gibbs sampler form a Markov chain with the stationary distribution equal to the joint distribution of the parameters and the latent variables conditional on the data. The densities for both types of blocks are proportional to the joint density of the data, the latent variables, and the parameters. Therefore, to construct the Gibbs sampler, we need to be able to evaluate the joint density of the data, the latent variables, and the parameters. For a textbook treatment of these ideas, see Chapter 6 in Geweke (2005).

It is straightforward to obtain an analytical expression for the joint density of the data, the latent variables, and the parameters under the parameterization

[2]For example, Erdem and Keane (1996) estimated a model in which consumer perceptions of products are modelled by a sum of a parameter and an i.i.d. component, and thus are serially correlated. Consumer product usage requirements are modelled similarly in Erdem, Imai, and Keane (2003). In Sullivan (2006), a job match-specific wage draw persists for the duration of a match. In Keane and Wolpin (2006), women draw from husbands earnings distribution and the draw stays fixed for the duration of the match. It is also common to allow for serial correlation in unobservables induced by latent types (see, for example, Keane and Wolpin (1997)). I thank an anonymous referee for bringing these references to my attention.

of the Gibbs sampler in which the unobserved state variables are directly used as the latent variables in the sampler

$$(3)\qquad p(\theta, x, d, y) = p(\theta)\prod_{i=1}^{I}\prod_{t=1}^{T_i} p(d_{t,i}|x_{t,i}, y_{t,i}; \theta) \times f(x_{t,i}, y_{t,i}|x_{t-1,i}, y_{t-1,i}, d_{t-1,i}; \theta),$$

where $p(d_{t,i}|x_{t,i}, y_{t,i}; \theta) = 1_{\{\mathcal{V}(y_{t,i},x_{t,i},d_{t,i};\theta)\geq\mathcal{V}(y_{t,i},x_{t,i},d;\theta),\forall d\in D\}}(y_{t,i}, x_{t,i}, d_{t,i}; \theta)$ is an indicator function and $p(\theta)$ is a prior density for the parameters. In this Gibbs sampler, the conditional density of a parameter given the data, the rest of the parameters, and the latent variables will be proportional to (3). Since (3) includes a product of indicator functions $p(d_{t,i}|y_{t,i}, x_{t,i}; \theta)$, in this Gibbs sampler, the distributions for parameter blocks will be truncated to a region defined by inequality constraints that are nonlinear in θ:

$$(4)\qquad \mathcal{V}(y_{t,i}, x_{t,i}, d_{t,i}; \theta) \geq \mathcal{V}(y_{t,i}, x_{t,i}, d; \theta) \quad \forall d \in D, \forall t \in \{1, \dots, T_i\}, \forall i \in \{1, \dots, I\}.$$

For realistic sample sizes, the number of these constraints is very large and the algorithm is impractical; for example, parameter draws from an acceptance sampling algorithm never got accepted in experiments with a sample size of more than 100 observations. The same situation occurs under the parameterization in which $u_{t,d,i} = u(y_{t,i}, x_{t,i}, d_{t,i}; \theta)$ are used as the latent variables in the sampler instead of some or all of the components of $y_{t,i}$.

The complicated truncation region (4) in drawing the parameter blocks could be avoided if we use $\mathcal{V}_{t,i} = \{\mathcal{V}_{t,d,i} = \mathcal{V}(s_{t,i}, d; \theta), d \in D\}$ as latent variables in the sampler. Under this parameterization, the joint density of the data, the latent variables, and the parameters (needed for construction of the Gibbs sampler) does not have a convenient analytical form because $\mathcal{V}_{t,d,i}$ depends on other unobservables through the expected value function, which can only be approximated numerically. In general, even evaluation of a kernel of this distribution is not easy. However, under some reasonable assumptions on the unobservables, a feasible Gibbs sampler can be constructed. In particular, let us assume that the unobserved part of the state vector includes some components that do not affect the distribution of the future state. Let us denote them by ν_t and denote the other (possibly serially correlated) components by ϵ_t; so, $y_t = (\nu_t, \epsilon_t)$. This assumption means that the transition law $f(x_{t+1}, \nu_{t+1}, \epsilon_{t+1}|x_t, \epsilon_t, d; \theta)$ and thus the expected value function $E\{V(s_{t+1}; \theta)|s_t, d; \theta\}$ do not depend on ν_t.

The presence of ν_t is well justified in an estimable model. If the support of these unobservables is sufficiently large and if they enter the utility function in a particular way, then the econometric model will be consistent with any possible sequence of observed choices (specification for unobservables is then called saturated (Rust (1994, p. 3102))). If, in contrast, all the unobservables do affect the expected value function $E\{V(s_{t+1};\theta)|s_t, d;\theta\}$, then the desirable saturation property might not hold or be very difficult to establish.

Since the expected value function $E\{V(s_{t+1};\theta)|s_t, d;\theta\}$ does not depend on ν_t, the alternative specific value functions $\mathcal{V}_{t,i} = \{u(\nu_{t,i}, \epsilon_{t,i}, x_{t,i}, d;\theta) + \beta E[V(s_{t+1};\theta)|\epsilon_{t,i}, x_{t,i}, d;\theta], d \in D\}$ will depend on $\nu_{t,i}$ only through $u(\nu_{t,i}, \epsilon_{t,i}, x_{t,i}, d;\theta)$. The per-period utility $u(\cdot)$ and the distribution for $\nu_{t,i}$ can be specified in such a way that $p(\mathcal{V}_{t,i}|\theta, x_{t,i}, \epsilon_{t,i})$ has a convenient analytical expression (or at least a quickly computable density kernel). In this case, a marginal conditional decomposition of the joint distribution of the data, the parameters, and the latent variables will consist of parts with analytical or easily computable expressions. Construction of the Gibbs sampler in this case is illustrated by the following example.

EXAMPLE 1—A Model of Optimal Bus Engine Replacement (Rust (1987)): In this model, a maintenance superintendent of a transportation company decides every time period whether to replace an engine for each bus in the company's fleet. The observed state variable is bus mileage x_t since the last engine replacement. The per-period utility is the negative of per-period costs. If the engine is not replaced at time t, then $u(x_t, \epsilon_t, \nu_t, d_t = 1;\alpha) = \alpha_1 x_t + \epsilon_t$; otherwise, $u(x_t, \epsilon_t, \nu_t, d_t = 2;\alpha) = \alpha_2 + \nu_t$, where ϵ_t and ν_t are the unobserved state variables, α_1 is the negative of per-period maintenance costs per unit of mileage, and α_2 is the negative of the costs of engine replacement. The bus mileage is discretized into $M = 90$ intervals $X = \{1, \dots, M\}$. The change in the mileage $(x_{t+1} - x_t)$ evolves according to a multinomial distribution on $\{0, 1, 2\}$ with parameters $\eta = (\eta_1, \eta_2, \eta_3)$.

Rust assumed that ϵ_t and ν_t are extreme value i.i.d. Under this assumption, the integrals over $y_t = (\epsilon_t, \nu_t)$ in the Bellman equation (1) and in the likelihood function (2) can be computed analytically. Rust used the maximum likelihood method to estimate the model. Since the expression for the likelihood function involves the expected value functions, Rust's algorithm solves the DP numerically on each iteration of the estimation procedure. Rust's assumptions on unobservables considerably reduce computational burden. However, it is reasonable to expect that engine-specific maintenance costs represented by ϵ_t are serially correlated. Thus, one could assume ν_t is i.i.d. $N(0, h_\nu^{-1})$ truncated to an interval $[-\overline{\nu}, \overline{\nu}]$, ϵ_t is $N(\rho\epsilon_{t-1}, h_\epsilon^{-1})$ truncated to $E = [-\overline{\epsilon}, \overline{\epsilon}]$, and $\epsilon_0 = 0$. When ϵ_t is serially correlated, the dimension of integration in the likelihood function can exceed 200 for Rust's data. It would be very hard to compute these integrals on each iteration of an estimation procedure. The Gibbs sampler with data augmentation described below can handle this problem.

Each bus/engine i is observed for T_i time periods: $\{x_{t,i}, d_{t,i}\}_{t=1}^{T_i}$ for $i = 1, \dots, I$. When the engine is replaced, the state is reinitialized: $x_{t-1} = 1$, $\epsilon_{t-1} = 0$. Therefore, a bus with a replaced engine can be treated as a separate observation. The parameters are $\theta = (\alpha, \eta, \rho, h_\epsilon)$; h_ν is fixed for normalization. The latent variables are $\{\mathcal{V}_{t,i}, \epsilon_{t,i}\}_{t=1}^{T_i}$, $i = 1, \dots, I$, where $\mathcal{V}_{t,i} = x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} - \nu_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})$ and $F_{t,i}(\theta, \epsilon) = \beta(E[V(x', \epsilon', \nu'; \theta)|\epsilon, x_{t,i}, d_{t,i} = 1; \theta] - E[V(x', \epsilon', \nu'; \theta)|\epsilon, x_{t,i}, d_{t,i} = 2; \theta])$. A compact space for parameters (required by the theory in the following sections) Θ is defined as $\alpha_i \in [-\overline{\alpha}, \overline{\alpha}]$, $\rho \in [-\overline{\rho}, \overline{\rho}]$, $h_\epsilon \in [\overline{h_\epsilon^l}, \overline{h_\epsilon^r}]$, and η belongs to a two-dimensional simplex.

The joint distribution of the data, the parameters, and the latent variables is

$$
\begin{aligned}
(5)\qquad & p(\theta; \{x_{t,i}, d_{t,i}; \mathcal{V}_{t,i}, \epsilon_{t,i}\}_{t=1}^{T_i}; i = 1, \dots, I) \\
&= p(\theta) \prod_{i=1}^{I} \prod_{t=1}^{T_i} \big[p(d_{t,i}|\mathcal{V}_{t,i}) p(\mathcal{V}_{t,i}|x_{t,i}, \epsilon_{t,i}; \theta) \\
&\qquad \times p(x_{t,i}|x_{t-1,i}; d_{t-1,i}; \eta) p(\epsilon_{t,i}|\epsilon_{t-1,i}, \rho, h_\epsilon) \big],
\end{aligned}
$$

where $p(\theta)$ is a prior,

$$
\begin{aligned}
& p(x_{t,i}|x_{t-1,i}; d_{t-1,i}; \eta) = \eta_{x_{t,i} - x_{t-1,i} + 1}, \\
& p(d_{t,i}|\mathcal{V}_{t,i}) = 1_{\{d_{t,i}=1, \mathcal{V}_{t,i} \geq 0 \text{ or } d_{t,i}=2, \mathcal{V}_{t,i} < 0\}}, \\
& p(\epsilon_{t,i}|\epsilon_{t-1,i}, \theta) \\
&\quad = \frac{h_\epsilon^{1/2} \exp\{-0.5 h_\epsilon (\epsilon_{t,i} - \rho\epsilon_{t-1,i})^2\}}{\sqrt{2\pi}[\Phi([\overline{\epsilon} - \rho\epsilon_{t-1,i}] h_\epsilon^{0.5}) - \Phi([-\overline{\epsilon} - \rho\epsilon_{t-1,i}] h_\epsilon^{0.5})]} 1_E(\epsilon_{t,i}),
\end{aligned}
$$

and

$$
\begin{aligned}
& p(\mathcal{V}_{t,i}|x_{t,i}, \epsilon_{t,i}; \theta) \\
(6)\qquad &\quad = \exp\big\{-0.5 h_\nu \big(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})]\big)^2\big\} \\
(7)\qquad &\qquad \cdot 1_{[-\overline{\nu}, \overline{\nu}]}\big(\mathcal{V}_{t,i} - [x_{t,i}\alpha_1 - \alpha_2 + \epsilon_{t,i} + F_{t,i}(\theta, \epsilon_{t,i})]\big) \\
&\qquad \cdot \frac{h_\nu^{0.5}}{\sqrt{2\pi}[\Phi(\overline{\nu} h_\nu^{0.5}) - \Phi(-\overline{\nu} h_\nu^{0.5})]}.
\end{aligned}
$$

Densities for Gibbs sampler blocks will be proportional to the joint distribution in (5). In this Gibbs sampler the observed choice optimality constraints do not involve parameters and affect only blocks for simulating $\mathcal{V}_{t,i}|\cdots$, which will have a normal truncated distribution proportional to (6) and (7), and also truncated to R^+ if $d_{t,i} = 1$ or to R^- otherwise. Efficient algorithms for simulating from truncated normal distributions are readily available; see, for example, Geweke (1991).

The density for $\epsilon_{t,i}|\cdots$ is

$$
\begin{aligned}
&p(\epsilon_{t,i}|\cdots) \\
&\quad\propto \frac{\exp\{-0.5h_\nu(\mathcal{V}_{t,i}-[x_{t,i}\alpha_1-\alpha_2+\epsilon_{t,i}+F_{t,i}(\theta,\epsilon_{t,i})])^2\}}{\Phi([\overline{\epsilon}-\rho\epsilon_{t-1,i}]h_\epsilon^{0.5})-\Phi([-\overline{\epsilon}-\rho\epsilon_{t-1,i}]h_\epsilon^{0.5})} \\
&\qquad\cdot 1_{[-\overline{\nu},\overline{\nu}]}\big(\mathcal{V}_{t,i}-[x_{t,i}\alpha_1-\alpha_2+\epsilon_{t,i}+F_{t,i}(\theta,\epsilon_{t,i})]\big) \\
&\qquad\cdot \exp\{-0.5h_\epsilon(\epsilon_{t+1,i}-\rho\epsilon_{t,i})^2-0.5h_\epsilon(\epsilon_{t,i}-\rho\epsilon_{t-1,i})^2\}\cdot 1_E(\epsilon_{t,i}).
\end{aligned}
\tag{8}
$$

Direct simulation from $\epsilon_{t,i}|\cdots$ could be difficult. However, the kernel of this density can be evaluated numerically (approximations to $F_{t,i}(\theta,\epsilon_{t,i})$ are discussed in the next section). Therefore, a Metropolis-within-Gibbs[3] algorithm can be used for this Gibbs sampler block. A convenient transition density for this Metropolis-within-Gibbs step is a truncated normal density proportional to (8).

Assuming a normal prior $N(\underline{\rho},\underline{h}_\rho^{-1})$ truncated to $[-\overline{\rho},\overline{\rho}]$,

$$
\begin{aligned}
p(\rho|\cdots) \propto &\frac{\exp\left\{-0.5h_\nu\sum_{i,t}(\mathcal{V}_{t,i}-[x_{t,i}\alpha_1-\alpha_2+\epsilon_{t,i}+F_{t,i}(\theta,\epsilon_{t,i})])^2\right\}}{\prod_{i,t}\Phi([\overline{\epsilon}-\rho\epsilon_{t-1,i}]h_\epsilon^{0.5})-\Phi([-\overline{\epsilon}-\rho\epsilon_{t-1,i}]h_\epsilon^{0.5})} \\
&\cdot\prod_{i,t}1_{[-\overline{\nu},\overline{\nu}]}\big(\mathcal{V}_{t,i}-[x_{t,i}\alpha_1-\alpha_2+\epsilon_{t,i}+F_{t,i}(\theta,\epsilon_{t,i})]\big) \\
&\cdot\exp\{-0.5\overline{h}_\rho(\rho-\overline{\rho})^2\}\cdot 1_{[-\overline{\rho},\overline{\rho}]}(\rho),
\end{aligned}
\tag{9}
$$

where $\overline{h}_\rho=\underline{h}_\rho+\sum_i\sum_{t=2}^{T_i}\epsilon_{t-1,i}^2$ and $\overline{\rho}=\overline{h}_\rho^{-1}(\underline{h}_\rho\underline{\rho}+h_\epsilon\sum_i\sum_{t=2}^{T_i}\epsilon_{t,i}\epsilon_{t-1,i})$. A Metropolis-within-Gibbs algorithm with truncated normal transition density proportional to (9) can be used for this Gibbs sampler block. Blocks for other parameters can be constructed in a similar way; see Norets (2007).

The Gibbs sampler presented in this example can be generalized and applied to different models with other interesting forms of heterogeneity such as individual-specific parameters. Also, components of ν_t do not have to enter the utility function linearly. The essential requirement is the ability to evaluate a kernel of $p(\mathcal{V}_{t,i}|\theta,x_{t,i},\epsilon_{t,i})$ quickly. The Gibbs sampler outlined above requires computing the expected value functions for each new parameter draw θ^m from

[3]To produce draws from some target distribution, the Metropolis or Metropolis–Hastings MCMC algorithm only needs values of a kernel of the target density. The draws are simulated from a transition density and they are accepted with probability that depends on the values of the target density kernel and the transition density. For more details, see, for example, Chib and Greenberg (1995).

the MCMC iteration m and each observation in the sample. The following section describes how the approximations of the expected value functions can be efficiently obtained.

3. ALGORITHM FOR SOLVING THE DP

For a discussion of methods for solving the DP for a given parameter vector θ, see Rust (1996). Below, I introduce a method of solving the DP suitable for use in conjunction with the Bayesian estimation of a general DDCM. This method uses an idea from Imai, Jain, and Ching (2005): to iterate the Bellman equation only once at each step of the estimation procedure and use information from previous steps to approximate the expectations in the Bellman equation. However, the way the previous information is used differs for the two methods. A detailed comparison is given in Section 3.2.

3.1. *Algorithm Description*

In contrast to conventional value function iteration, this algorithm iterates the Bellman equation only once for each parameter draw. First, I will describe how the DP solving algorithm works and then how the output of the DP solving algorithm is used to approximate the expected value functions in the Gibbs sampler.

The DP solving algorithm takes a sequence of parameter draws θ^m, $m = 1, 2, \dots,$ as an input from the Gibbs sampler, where m denotes the Gibbs sampler iteration. For each θ^m, the algorithm generates random states $s^{m,j} \in S$, $j = 1, \dots, \hat{N}(m)$. At each random state, the approximations of the value functions $V^m(s^{m,j}; \theta^m)$ are computed by iterating the Bellman equation once. At this one iteration of the Bellman equation, the future expected value functions are computed by importance sampling over value functions $V^k(s^{k,j}; \theta^k)$ from previous iterations $k < m$.

The random states $s^{m,j}$ are generated from a density $g(\cdot) > 0$ on S. This density $g(\cdot)$ is used as an importance sampling source density in approximating the expected value functions. The collection of the random states $\{s^{m,j}\}_{j=1}^{\hat{N}(m)}$ will be referred to below as the random grid. (Rust (1997) showed that value function iteration on random grids from a uniform distribution breaks the curse of dimensionality for DDCMs.) The number of points in the random grid at iteration m is denoted by $\hat{N}(m)$ and will be referred to below as the size of the random grid (at iteration m).

For each point in the current random grid $s^{m,j}$, $j = 1, \dots, \hat{N}(m)$, the approximation of the value function $V^m(s^{m,j}; \theta^m)$ is computed according to

$$V^m(s; \theta) = \max_{d \in D}\{u(s, d; \theta) + \beta \hat{E}^{(m)}[V(s'; \theta) | s, d; \theta]\}. \tag{10}$$

Not all of the previously computed value functions $V^k(s^{k,j};\theta^k)$, $k<m$, are used in importance sampling for computing $\hat{E}^{(m)}[V(s';\theta)|s,d;\theta]$ in (10). To converge, the algorithm has to forget the remote past. Thus, at each iteration m, I keep track only of the history of length $N(m)$: $\{\theta^k; s^{k,j}, V^k(s^{k,j};\theta^k), j=1,\dots,\hat{N}(k)\}_{k=m-N(m)}^{m-1}$. In this history, I find $\tilde{N}(m)$ closest to θ parameter draws. Only the value functions corresponding to these nearest neighbors are used in importance sampling. Formally, let $\{k_1,\dots,k_{\tilde{N}(m)}\}$ be the iteration numbers of the nearest neighbors of θ in the current history: $k_1=\arg\min_{i\in\{m-N(m),\dots,m-1\}}\|\theta-\theta^i\|$ and

$$k_j=\underset{i\in\{m-N(m),\dots,m-1\}\setminus\{k_1,\dots,k_{j-1}\}}{\arg\min}\|\theta-\theta^i\|,\quad j=2,\dots,\tilde{N}(m). \tag{11}$$

If the arg min returns a multivalued result, I use the lexicographic order for $(\theta^i-\theta)$ to decide which θ^i is chosen first. If the result of the lexicographic selection is also multivalued, $\theta^i=\theta^j$, then I choose θ^i over θ^j if $i>j$. This particular way to resolve the multivaluedness of the arg min might seem irrelevant for implementing the method in practice; however, it is used in the proof of the measurability of the supremum of the approximation error, which is necessary for the uniform convergence results. A reasonable choice for the norm in (11) would be $\|\theta\|=\sqrt{\theta^T\underline{H}_\theta\theta}$, where $\underline{H}_\theta$ is the prior precision for the parameters. Importance sampling is performed as

$$\hat{E}^{(m)}[V(s';\theta)|s,d;\theta] \tag{12}$$
$$=\sum_{i=1}^{\tilde{N}(m)}\sum_{j=1}^{\hat{N}(k_i)}\frac{V^{k_i}(s^{k_i,j};\theta^{k_i})\cdot f(s^{k_i,j}\mid s,d;\theta)/g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)}\sum_{q=1}^{\hat{N}(k_r)}f(s^{k_r,q}\mid s,d;\theta)/g(s^{k_r,q})}$$

$$=\sum_{i=1}^{\tilde{N}(m)}\sum_{j=1}^{\hat{N}(k_i)}V^{k_i}(s^{k_i,j};\theta^{k_i})W_{k_i,j,m}(s,d,\theta). \tag{13}$$

The target density for importance sampling is the state transition density $f(\cdot|s,d;\theta)$. The source density is the density $g(\cdot)$ from which the random grid on the state space is generated. In general, $g(\cdot)$ should give reasonably high probabilities to all parts of the state space that are likely under $f(\cdot|s,d;\theta)$ with reasonable values of the parameter θ. To reduce the variance of the approximation of expectations produced by importance sampling, one should make $g(\cdot)$ relatively high for the states that result in large absolute values for value functions ($g(s')$ that minimizes the variance of the importance sampling approximation to the expectation is proportional to $|V(s';\theta)f(s'|s,d;\theta)|$).

Section 3.3 formally presents the assumptions on model primitives and restrictions on $g(\cdot)$, $\hat{N}(m)$, $N(m)$, and $\tilde{N}(m)$ that are sufficient for algorithm convergence.

After $V^m(s^{m,j};\theta^m)$ are computed from (10) and (12), they can be used in a formula similar to (12) to obtain the approximations of the expectations $E[V(s_{t+1};\theta^m)|x_{t,i},\epsilon^m_{t,i},d;\theta^m]$ on iteration m of the Gibbs sampler.

3.2. *Comparison With Imai, Jain, and Ching (2005)*

An algorithm for solving the DP has to deal with an integration problem for computing the expectations of the value functions in addition to approximating the value function in the parameter space. My prescriptions for handling this integration problem differ from IJC's. IJC used kernel smoothing over all $N(m)$ previously computed value functions to approximate the expected value functions. They also generated only one new state at each iteration, $\hat{N}(m)=1\ \forall m$. For a finite observed state space, deterministic transitions for the observed states, and i.i.d. unobservables IJC proved convergence of their DP solution approximations. To handle compact state space and random state transitions I introduce growing random grids: $\hat{N}(m)$ increases with m. A fixed random grid size that works for IJC's i.i.d. errors does not seem to be enough for general random transitions. When the size of the random grid grows, the nearest neighbor (NN) algorithm that I use to approximate value functions in the parameter space is computationally much more efficient than the kernel smoothing used by IJC. The computational advantage of using the NN algorithm in this case stems from the fact that importance sampling over the random grids has to be performed only for a few nearest neighbors and not for the whole tracked history of length $N(m)$. The convergence results I obtain are also stronger. IJC proved uniform convergence in probability for their DP solution approximations. For the NN algorithm, I establish complete uniform convergence, which implies uniform a.s. convergence. Furthermore, the NN algorithm easily accommodates more than one iteration of the Bellman equation for each parameter draw to improve the approximation precision in practice. Overall, the nearest neighbors method is not just a substitute for kernel smoothing that might work better in higher dimensions (see, e.g., Scott (1992, pp. 189–190)), but an essential part of the algorithm that, in conjunction with random grids, makes it computationally efficient and applicable to more general model specifications.

3.3. *Theoretical Results*

The following assumptions on the model primitives and the algorithm parameters are made:

ASSUMPTION 1: *$\Theta\subset R^{J_\Theta}$ and $S\subset R^{J_S}$ are compact, and $\beta\in(0,1)$ is known.*

The assumption of compactness of the parameter space is standard in econometrics. Fixing β is also a usual practice in the literature on estimation of DDCMs.

ASSUMPTION 2: *$u(s, d; \theta)$ is continuous in (θ, s) (and thus bounded on compacts).*

ASSUMPTION 3: *$f(s'|s, d; \theta)$ is continuous in (θ, s, s') and $g(s)$ is continuous in s. Discrete state variables can be accommodated by defining densities with respect to the counting measure.*

Assumptions 1–3 imply continuity of $V(s; \theta)$ in (θ, s) (see Proposition 4 in the Supplemental Material (Norets (2009b)) or Norets (2009a) for more general results).

ASSUMPTION 4: *The density of the state transition $f(\cdot|\cdot)$ and the importance sampling density $g(\cdot)$ are bounded above and away from zero, which gives*

$$\inf_{\theta,s',s,d} f(s'|s, d; \theta)/g(s') \geq \underline{f} > 0 \quad \textit{and}$$

$$\sup_{\theta,s',s,d} f(s'|s, d; \theta)/g(s') \leq \overline{f} < \infty.$$

Assumption 4 can be relaxed. The support of the transition density can be allowed to depend on the decision d and the discrete state variables if they take a finite number of values. Deterministic transitions for discrete state variables and, in some cases, for continuous state variables (e.g., setting $\epsilon_t = 0$ when $d_t = 2$ in Rust's engine replacement model) can be accommodated. Corollaries 1 and 2 below describe changes in the DP solving algorithm required to relax Assumption 4.

ASSUMPTION 5: *$\exists \hat{\delta} > 0$ such that $P(\theta^{m+1} \in A|\omega^m) \geq \hat{\delta}\lambda(A)$ for any Borel measurable $A \subset \Theta$, any m, and any feasible history $\omega^m = \{\omega_1, \ldots, \omega_m\}$, where λ is the Lebesgue measure. The history includes all the parameter and latent variable draws from the Gibbs sampler and all the random grids from the DP solving algorithm: $\omega_t = \{\theta^t, \mathcal{V}^t, \epsilon^t; s^{t,j}, j = 1, \ldots, \hat{N}(t)\}$.*

Assumption 5 means that at each iteration of the algorithm, the parameter draw can get into any part of Θ. This assumption should be verified for each specific DDCM and the corresponding parameterization of the Gibbs sampler. The assumption is only a little stronger than standard conditions for convergence of the Gibbs sampler; see Corollary 4.5.1 in Geweke (2005). Since a careful practitioner of MCMC would have to establish convergence of the Gibbs sampler, a verification of Assumption 5 should not require much extra effort.

ASSUMPTION 6: *Let* $1 > \gamma_0 > \gamma_1 > \gamma_2 \geq 0$ *and* $N(t) = [t^{\gamma_1}]$, $\tilde{N}(t) = [t^{\gamma_2}]$, $\hat{N}(t) = [t^{\gamma_1 - \gamma_2}]$, *and* $\hat{N}(0) = 1$, *where* $[x]$ *is the integer part of* x.

Multiplying the functions of t in Assumption 6 by positive constants will not affect any of the theoretical results below.

THEOREM 1: *Under Assumptions 1–6, the approximation to the expected value function in (12) converges uniformly and completely to the exact value: that is, the following statements hold:*

(i) $\sup_{s,\theta,d} |\hat{E}^{(t)}[V(s';\theta) \mid s,d;\theta] - E[V(s';\theta) \mid s,d;\theta]|$ *is measurable.*

(ii) *For any* $\tilde{\epsilon} > 0$ *there exists a sequence* $\{z_t\}$ *such that* $\sum_{t=0}^{\infty} z_t < \infty$ *and*

$$P\Big(\sup_{s,\theta,d} \big|\hat{E}^{(t)}[V(s';\theta) \mid s,d;\theta] - E[V(s';\theta) \mid s,d;\theta]\big| > \tilde{\epsilon}\Big) \leq z_t.$$

COROLLARY 1: *Let the state space be a product of a finite set* S_f *and a bounded rectangle* $S_c \in R^{J_{S_c}}$, $S = S_f \times S_c$. *Let* $f(s'_f, s'_c | s_f, s_c; \theta)$ *be the state transition density with respect to the product of the counting measure on* S_f *and the Lebesgue measure on* S_c. *Assume for any* $s_f \in S_f$ *and* $d \in D$, *we can define* $S(s_f, d) \subset S$ *such that* $f(s'_f, s'_c | s_f, s_c, d; \theta) > 0$ *for any* $(s'_f, s'_c) \in S(s_f, d)$ *and any* $s_c \in S_c$ *and* $f(s'_f, s'_c | s_f, s_c, d; \theta) = 0$ *for any* $(s'_f, s'_c) \notin S(s_f, d)$ *and any* $s_c \in S_c$. *For each* $s_f \in S_f$ *and* $d \in D$, *let density* $g_{s_f,d}(\cdot)$ *be such that* $\inf_{\theta\in\Theta,(s'_f,s'_c)\in S(s_f,d),s_c\in S_c} f(s'_f, s'_c | s_f, s_c, d; \theta)/g_{s_f,d}(s'_f, s'_c) \geq \underline{f} > 0$ *and* $\sup_{\theta\in\Theta,(s'_f,s'_c)\in S(s_f,d),s_c\in S_c} f(s'_f, s'_c | s_f, s_c, d; \theta)/g_{s_f,d}(s'_f, s'_c) \leq \overline{f} < \infty$. *In the DP solving algorithm, generate the random grid over the state space for each discrete state* $s_f \in S_f$ *and decision* $d \in D$: $s^{m,j}_{s_f,d} \sim g_{s_f,d}(\cdot)$, *and use these grids to compute the approximations of the expectations* $E(V(s';\theta)|s_f, s_c, d;\theta)$. *Then the conclusions of Theorem 1 hold.*

COROLLARY 2: *If the transition for the discrete states is independent from the other states, then a more efficient alternative would also work. Let us denote the transition probability for the discrete states by* $f(s'_f | s_f, d; \theta)$. *Suppose that for* $f(s'_c | s_c, d; \theta)$ *and some* $g(\cdot)$ *defined on* S_c, *Assumption 4 holds and the random grid* $s^{m,j}_c$ *is generated only on* S_c *from* $g(\cdot)$. *Consider the following approximation of the expectations,* $\hat{E}^{(m)}[V(s';\theta)|s_f, s_c, d;\theta]$, *in the DP solving algorithm:*

$$\sum_{s'_f \in S_f(s_f,d)} f(s'_f | s_f, d; \theta) \times \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} \frac{V^{k_i}(s'_f, s^{k_i,j}; \theta^{k_i}) f(s^{k_i,j} \mid s, d; \theta)/g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\hat{N}(k_r)} f(s^{k_r,q} \mid s, d; \theta)/g(s^{k_r,q})}, \tag{14}$$

where $S_f(s_f, d)$ denotes the set of possible future discrete states given the current state s_f and decision d. Then the conclusions of Theorem 1 hold.

4. CONVERGENCE OF POSTERIOR EXPECTATIONS

In Bayesian analysis, most inference exercises involve computing posterior expectations of some functions. For example, the posterior mean and the posterior standard deviation of a parameter and the posterior probability that a parameter belongs to a set can all be expressed in terms of posterior expectations. More importantly, the answers to the policy questions that DDCMs address also take this form. Using the uniform complete convergence of the approximations of the expected value functions, I prove the complete convergence of the approximated posterior expectations under mild assumptions on a kernel of the posterior distribution.

ASSUMPTION 7: *Assume that $\epsilon_{t,i}$, θ, and $\nu_{t,k,i}$ have compact supports E, Θ, and $[-\overline{\nu}, \overline{\nu}]$ correspondingly, where $\nu_{t,k,i}$ denotes the kth component of $\nu_{t,i}$. Let the joint posterior distribution of the parameters and the latent variables be proportional to a product of a continuous function and indicator functions,*

$$
\begin{aligned}
(15)\qquad p(\theta, \mathcal{V}, \epsilon; F|d, x) \propto{}& r(\theta, \mathcal{V}, \epsilon; F(\theta, \epsilon)) \cdot 1_{\Theta}(\theta) \\
&\cdot \left(\prod_{i,t} 1_E(\epsilon_{t,i}) p(d_{t,i}|\mathcal{V}_{t,i})\right) \\
&\cdot \left(\prod_{i,t,k} 1_{[-\overline{\nu},\overline{\nu}]}\big(q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}(\theta, \epsilon_{t,i}))\big)\right),
\end{aligned}
$$

where $r(\theta, \mathcal{V}, \epsilon; F)$ and $q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i})$ are continuous in $(\theta, \mathcal{V}, \epsilon, F)$, $F = \{F_{t,d,i}, \forall i, t, d\}$ stands for a vector of the expected value functions, and $F_{t,i}$ are the corresponding subvectors. Also assume that the level curves of $q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i})$ corresponding to $\overline{\nu}$ and $-\overline{\nu}$ have zero Lebesgue measure,

$$
\begin{aligned}
(16)\qquad &\lambda[(\theta, \mathcal{V}, \epsilon) : q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = \overline{\nu}] \\
&\quad = \lambda[(\theta, \mathcal{V}, \epsilon) : q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = -\overline{\nu}] = 0.
\end{aligned}
$$

This assumption is likely to be satisfied for most models formulated on a bounded state space, in which distributions are truncated to bounded regions required by the theory. The kernel of the joint distribution for the engine replacement example from Section 2 has the form in (15). Condition (16) is also easy to verify. In Rust's model, $q_d(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) = \Delta u(x_{t,i}, d) + \epsilon_{t,d,i} + F_{t,d,i}(\theta, \epsilon_{t,i}) - \mathcal{V}_{t,d,i} = \overline{\nu}$ defines a continuous function $\mathcal{V}_{t,d,i} = \Delta u(x_{t,i}, d) + \epsilon_{t,d,i} + F_{t,d,i}(\theta, \epsilon_{t,i}) - \overline{\nu}$. Since the Lebesgue measure of the graph of a continuous function is zero, (16) will be satisfied.

THEOREM 2: *Let $h(\theta, \mathcal{V}, \epsilon)$ be a bounded function. Under Assumptions 1–7, the expectation of $h(\theta, \mathcal{V}, \epsilon)$ with respect to the approximated posterior that uses the DP solution approximations $\hat{F}^n$ from step n of the DP solving algorithm converges completely (and thus a.s.) to the true posterior expectation of $h(\theta, \mathcal{V}, \epsilon)$ as $n \to \infty$. In particular, for any $\varepsilon > 0$, there exists a sequence $\{z_n\}$ such that $\sum_{n=0}^{\infty} z_n < \infty$ and the probability*

$$P\left(\left|\int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F|d, x)\, d(\theta, \mathcal{V}, \epsilon) - \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; \hat{F}^n|d, x)\, d(\theta, \mathcal{V}, \epsilon)\right| > \varepsilon\right)$$

is bounded above by z_n.

One way to apply Theorem 2 is to stop the DP solving algorithm at an iteration m and run the Gibbs sampler for extra n iterations using the DP solution $\hat{E}^{(m)}[V(s'; \theta)|s, d; \theta]$ from iteration m. If the Gibbs sampler is uniformly ergodic (see Tierney (1994)) for any fixed approximation $\hat{E}^{(m)}[V(s'; \theta)|s, d; \theta]$, then for any $\delta > 0$ and $\varepsilon > 0$ there exist m and N such that for all $n \geq N$,

$$P\left(\left|\frac{1}{n}\sum_{i=m+1}^{m+n} h(\theta^i, \mathcal{V}^i, \epsilon^i) - \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F|d, x)\, d(\theta, \mathcal{V}, \epsilon)\right| > \varepsilon\right) \leq \delta.$$

If we do not stop the DP solving algorithm and run it together with the Gibbs sampler, then the stochastic process for $(\theta^i, \mathcal{V}^i, \epsilon^i)$ will not be a Markov chain. In this case, results from the adaptive MCMC literature (e.g., Roberts and Rosenthal (2006)) can be adapted to prove laws of large numbers and convergence in distribution for $(\theta^i, \mathcal{V}^i, \epsilon^i)$.

THEOREM 3: *Let us define the following two conditions.* (a) *The MCMC algorithm that uses the exact DP solutions is uniformly ergodic: for any $\varepsilon > 0$ there is N such that*

$$\left\|P^N((\theta, \mathcal{V}, \epsilon), \cdot; F) - P(\cdot; F|d, x)\right\| \leq \varepsilon$$

for any $(\theta, \mathcal{V}, \epsilon)$, where $P^N(\cdot, \cdot)$ is the Markov transition kernel implied by N iterations of the MCMC algorithm, $P(\cdot; F|d, x)$ is the posterior probability measure, and $\|\cdot\|$ is the bounded variation norm.

(b) *The transition kernel that uses the approximate DP solutions converges uniformly in probability to the transition kernel that uses the exact solutions*

$$\sup_{\theta,\mathcal{V},\epsilon} \left\| P((\theta,\mathcal{V},\epsilon),\cdot;F) - P((\theta,\mathcal{V},\epsilon),\cdot;F^n) \right\| \xrightarrow{P} 0, \quad \text{as } n \to \infty.$$

Conditions (*a*) *and* (*b*) *imply the following two results*:

(i) *The MCMC algorithm that uses the approximate DP solutions is ergodic*: *for any* $(\theta^0, \mathcal{V}^0, \epsilon^0)$ *and any* $\varepsilon > 0$ *there exists* M *such that for any* $i \geq M$,

$$\sup_A \left| P((\theta^i, \mathcal{V}^i, \epsilon^i) \in A | \theta^0, \mathcal{V}^0, \epsilon^0) - P(A; F|d, x) \right| \leq \varepsilon.$$

(ii) *A weak law of large numbers* (*WLLN*) *holds*: *for any* $(\theta^0, \mathcal{V}^0, \epsilon^0)$ *and a bounded function* $h(\cdot)$,

$$\sum_{i=1}^n h(\theta^i, \mathcal{V}^i, \epsilon^i) \Big/ n \xrightarrow{P} \int h(\theta, \mathcal{V}, \epsilon) p(\theta, \mathcal{V}, \epsilon; F|d, x)\, d(\theta, \mathcal{V}, \epsilon).$$

Norets (2007) showed how to establish condition (a) for the MCMC algorithm used for inference in the engine replacement example. A verification of condition (b) involves arguments and assumptions similar to those employed in the statement and proof of Theorem 2. (Theorem 2 implies strong convergence for the approximated posterior probability, while here we need a similar result for the approximated Markov transition probability.)

5. CONCLUSION

This paper presents a feasible method for Bayesian inference in dynamic discrete choice models with serially correlated unobserved state variables. I construct the Gibbs sampler, employing data augmentation and Metropolis steps, that can successfully handle multidimensional integration in the likelihood function of these models. The computational burden of solving the DP at each iteration of the estimation algorithm can be reduced by efficient use of the information obtained on previous iterations. Serially correlated unobservables are not the only possible source of intractable integrals in the likelihood function of DDCMs. The Gibbs sampler algorithm can be extended to allow for other interesting features in DDCMs such as individual-specific coefficients, missing data, macroshocks, and cohort effects. The proposed theoretical framework is flexible and leaves room for experimentation. For details on implementation and experiments, the interested reader is referred to Norets (2007, 2008). Overall, combined with efficient DP solution strategies, standard computational tools of Bayesian analysis seem to be very promising in making more elaborate DDCMs estimable.

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SUPPLEMENT TO "INFERENCE IN DYNAMIC DISCRETE CHOICE MODELS WITH SERIALLY CORRELATED UNOBSERVED STATE VARIABLES"

(*Econometrica*, Vol. 77, No. 5, September 2009, 1665–1682)

BY ANDRIY NORETS

THIS SUPPLEMENT is organized as follows. In Section S1, nonuniform convergence results for the DP solution algorithm are presented. In Section S2, these results are extended to the uniform convergence. In Section S3, convergence of posterior expectations is proved. The final section of the supplement presents auxiliary results referred to in proofs and the main paper.

S1. NONUNIFORM CONVERGENCE

THEOREM 4: *Under Assumptions* 1–6, *the approximation to the expected value function in* (12) *converges completely* (*and thus a.s.*) *to the true value with probability bounds that are uniform over parameter and state spaces*; *that is*, *for any* $\tilde{\epsilon} > 0$, *there exists a sequence* $\{z_t\}$ *such that* $\sum_{t=0}^{\infty} z_t < \infty$, *and for any* $\theta \in \Theta$, $s \in S$, *and* $d \in D$,

$$P\Big(\Big|\hat{E}^{(t)}[V(s';\theta) \mid s,d;\theta] - E[V(s';\theta) \mid s,d;\theta]\Big| > \tilde{\epsilon}\Big) \le z_t.$$

PROOF: Let us decompose the error of approximation into three parts:

$$\begin{aligned}
&\Big|\hat{E}^{(t)}[V(s';\theta) \mid s,d;\theta] - E[V(s';\theta) \mid s,d;\theta]\Big| \\
&\quad = \left|\sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} V^{k_i}(s^{k_i,j};\theta^{k_i}) W_{k_i,j,t}(s,d,\theta) - E[V(s';\theta) \mid s,d;\theta]\right| \\
&\quad \le \left|\sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} V(s^{k_i,j};\theta) W_{k_i,j,t}(s,d,\theta) - E[V(s';\theta) \mid s,d;\theta]\right| \\
&\qquad + \left|\sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} (V(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta)) W_{k_i,j,t}(s,d,\theta)\right| \\
&\qquad + \left|\sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} (V^{k_i}(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta^{k_i})) W_{k_i,j,t}(s,d,\theta)\right| \\
&\quad = A_1^t(\theta,s,d) + A_2^t(\theta,s,d) + A_3^t(\theta,s,d) \\
&\quad \le \max_d A_1^t(\theta,s,d) + \max_d A_2^t(\theta,s,d) + \max_d A_3^t(\theta,s,d)
\end{aligned}$$

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$$= A_1^t(\theta, s) + A_2^t(\theta, s) + A_3^t(\theta, s).$$

In Lemma 1, I show that $A_1^t(\theta, s)$ converges to zero completely with bounds on probabilities that are independent of θ and s. The proof uses Hoeffding's inequality, implying a strong law of large numbers (SLLN) for bounded random variables. However, some additional work is required since $s^{k_i,j}$ do not constitute a random sample. Using the continuity of the value function $V(\cdot)$, the compactness of the parameter space Θ, and the assumption that each parameter draw can get into any point in Θ (Assumption 5), I show analogous result for $A_2^t(\theta, s)$ in Lemma 2. In Lemma 3, I bound $A_3^t(\theta, s)$ by a weighted sum of $A_1^t(\theta, s)$ and $A_2^t(\theta, s)$ from previous iterations. Due to very fast convergence of $A_1^t(\theta, s)$ and $A_2^t(\theta, s)$, $A_3^t(\theta, s)$ also converges to zero completely. Thus, from the three lemmas the result follows. Formally, according to Lemmas 1, 2, and 3, there exist $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$, and T such that $\forall \theta \in \Theta$, $\forall s \in S$, and $\forall t > T$,

$$P\big(|A_1^t(\theta, s)| > \tilde{\epsilon}/3\big) \le e^{-0.5\delta_1 t^{\gamma_1}},$$
$$P\big(|A_2^t(\theta, s)| > \tilde{\epsilon}/3\big) \le e^{-0.5\delta_2 t^{\gamma_1}},$$
$$P\big(|A_3^t(\theta, s)| > \tilde{\epsilon}/3\big) \le e^{-\delta_3 t^{\gamma_0\gamma_1}}.$$

Combining the above equations gives

$$\begin{aligned}
&P\big(\big|\hat{E}^{(t)}[V(s';\theta) \mid s, d; \theta] - E[V(s';\theta) \mid s, d; \theta]\big| > \tilde{\epsilon}\big) \\
&\quad \le P(A_1^t(\theta, s) + A_2^t(\theta, s) + A_3^t(\theta, s) > \tilde{\epsilon}) \\
&\quad \le P\big(|A_1^t(\theta, s)| > \tilde{\epsilon}/3\big) + P\big(|A_2^t(\theta, s)| > \tilde{\epsilon}/3\big) + P\big(|A_3^t(\theta, s)| > \tilde{\epsilon}/3\big) \\
&\quad \le e^{-0.5\delta_1 t^{\gamma_1}} + e^{-0.5\delta_2 t^{\gamma_1}} + e^{-\delta_3 t^{\gamma_0\gamma_1}} \quad (\forall t > T) \\
&\quad = z_t \quad (\forall t > T).
\end{aligned}$$

For $t \le T$, set $z_t = 1$. Proposition 9 shows that $\sum_{t=0}^{\infty} z_t < \infty$. The lemmas are stated and proved below. *Q.E.D.*

LEMMA 1: *Given $\tilde{\epsilon} > 0$, there exist $\delta > 0$ and T such that for any $\theta \in \Theta$, $s \in S$, and $t > T$,*

$$P\big(|A_1^t(\theta, s)| > \tilde{\epsilon}\big) \le e^{-\delta \tilde{N}(t)\hat{N}(t-N(t))} \le e^{-0.5\delta t^{\gamma_1}}. \tag{17}$$

PROOF: Fix a combination $m = \{m_1, \dots, m_{\tilde{N}(t)}\}$ from $\{t - N(t), \dots, t-1\}$. Let

$$\begin{aligned}
&X(\omega^{t-1}, \theta, s, d, m) \\
&\quad = \left| \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(m_i)} \Big(\big(V(s^{m_i,j};\theta) - E[V(s';\theta) \mid s, d; \theta]\big) \right.
\end{aligned} \tag{18}$$

$$\times f(s^{m_i,j} \mid s, d; \theta)/g(s^{m_i,j}) \Bigg/ \left(\sum_{r=1}^{\tilde{N}(t)} \sum_{q=1}^{\hat{N}(m_r)} f(s^{m_r,q} \mid s, d; \theta)/g(s^{m_r,q}) \right) \Bigg) \Bigg|.$$

Since the importance sampling weights are bounded away from zero by $\underline{f} > 0$ (see Assumption 4),

$$\begin{aligned}
(19) \qquad & [X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}] \\
& \subset \Bigg[\Bigg| \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(m_i)} \Bigg(\big(V(s^{m_i,j}; \theta) - E[V(s'; \theta) \mid s, d; \theta] \big) \\
& \quad \times f(s^{m_i,j} \mid s, d; \theta)/g(s^{m_i,j}) \\
& \quad \Bigg/ \left(\sum_{r=1}^{\tilde{N}(t)} \hat{N}(m_r) \inf_{\theta, s, s', d} f(s' \mid s, d; \theta)/g(s') \right) \Bigg) \Bigg| > \tilde{\epsilon} \Bigg] \\
& = \Bigg[\Bigg| \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(m_i)} \big(V(s^{m_i,j}; \theta) - E[V(s'; \theta) \mid s, d; \theta] \big) \frac{f(s^{m_i,j} \mid s, d; \theta)}{g(s^{m_i,j})} \Bigg| \\
& \quad > \tilde{\epsilon} \underline{f} \sum_{i=1}^{\tilde{N}(t)} \hat{N}(m_i) \Bigg].
\end{aligned}$$

Using (19) and then applying Hoeffding (1963)'s inequality, we get

$$\begin{aligned}
(20) \qquad & P(X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}) \\
& \leq P \Bigg[\Bigg| \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(m_i)} \big(V(s^{m_i,j}; \theta) - E[V(s'; \theta) \mid s, d; \theta] \big) \frac{f(s^{m_i,j} \mid s, d; \theta)}{g(s^{m_i,j})} \Bigg| \\
& \qquad > \tilde{\epsilon} \underline{f} \sum_{i=1}^{\tilde{N}(t)} \hat{N}(m_i) \Bigg] \\
& \leq 2 \exp\left\{ \frac{-2 \underline{f}^2 \tilde{\epsilon}^2}{(b-a)^2} \sum_{r=1}^{\tilde{N}(t)} \hat{N}(m_r) \right\},
\end{aligned}$$

where a and b are correspondingly the lower and upper bounds on $(V(s^{m_i,j};\theta) - E[V(s';\theta) \mid s;\theta])f(s^{m_i,j} \mid s,d;\theta)/g(s^{m_i,j})$. Hoeffding's inequality applies since $s^{m_i,j}$ are independent, the summands have expectations equal to zero,

$$\int \frac{(V(s^{m_i,j};\theta) - E[V(s';\theta) \mid s;\theta])f(s^{m_i,j} \mid s,d;\theta)}{g(s^{m_i,j})} g(s^{m_i,j})\, ds^{m_i,j} = 0, \tag{21}$$

and a and b are finite by Assumptions 1 and 2.

Since $\hat{N}(\cdot)$ is nondecreasing, (20) implies

$$\begin{aligned} P(X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}) &\leq 2\exp\left\{\frac{-2\underline{f}^2\tilde{\epsilon}^2}{(b-a)^2}\tilde{N}(t)\hat{N}(t-N(t))\right\} \\ &= 2\exp\{-4\delta\tilde{N}(t)\hat{N}(t-N(t))\}, \end{aligned} \tag{22}$$

where the last equality defines $\delta > 0$.

Since $|A_1^t(\theta,s,d)| < \max_m X(\omega^{t-1},\theta,s,d,m)$,

$$\begin{aligned} &P(|A_1^t(\theta,s,d)| > \tilde{\epsilon}) \\ &\quad\leq P\Big[\max_m X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}\Big] \\ &\quad= P\Big(\bigcup_m [X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}]\Big) \\ &\quad\leq \sum_m P[X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}] \\ &\quad\leq 2\exp\{-4\delta\tilde{N}(t)\hat{N}(t-N(t))\}\frac{N(t)!}{(N(t)-\tilde{N}(t))!\tilde{N}(t)!}, \end{aligned} \tag{23}$$

where the summation, the maximization, and the union are taken over all possible combinations m, and $N(t)!/((N(t)-\tilde{N}(t))!\tilde{N}(t)!)$ is the number of the possible combinations.

Assumption 6 and Proposition 7 show that $\exists T_1$ such that $\forall t > T_1$,

$$\begin{aligned} &\exp\{-4\delta\tilde{N}(t)\hat{N}(t-N(t))\}\frac{N(t)!}{(N(t)-\tilde{N}(t))!\tilde{N}(t)!} \\ &\quad\leq \exp\{-2\delta\tilde{N}(t)\hat{N}(t-N(t))\}. \end{aligned} \tag{24}$$

Finally,

$$\begin{aligned} &P(|A_1^t(\theta,s)| > \tilde{\epsilon}) \\ &\quad= P\Big(\max_{d\in D}|A_1^t(\theta,s,d)| > \tilde{\epsilon}\Big) \end{aligned} \tag{25}$$

$$= P\Big(\bigcup_{d\in D}\big[|A_1^t(\theta, s, d)| > \tilde{\epsilon}\big]\Big)$$
$$\le \operatorname{card}(D)2\exp\big\{-2\delta\tilde{N}(t)\hat{N}(t-N(t))\big\} \quad (\forall t > T_1)$$
$$\le \exp\big\{-\delta\tilde{N}(t)\hat{N}(t-N(t))\big\} \quad (\forall t > T_2 \ge T_1),$$

where such T_2 exists since $\operatorname{card}(D)2\exp\{-\delta\tilde{N}(t)\hat{N}(t-N(t))\} \to 0$. The last inequality in (17) follows since $\tilde{N}(t)\hat{N}(t-N(t)) \ge t^{\gamma_1} - t^{\gamma_1-\gamma_2} \ge 0.5t^{\gamma_1}$ for any t larger than some $T \ge T_2$. *Q.E.D.*

LEMMA 2: *Given $\tilde{\epsilon} > 0$, there exist $\delta > 0$ and T such that for any $\theta \in \Theta$, $s \in S$, and $t > T$,*

$$P\big(|A_2^t(\theta, s)| > \tilde{\epsilon}\big) \le e^{-\delta(N(t)-\tilde{N}(t))} \le e^{-0.5\delta t^{\gamma_1}}. \tag{26}$$

PROOF: Let us find an event encompassing $[|A_2^t(\theta, s)| > \tilde{\epsilon}]$, for which the probability can be easily bounded

$$\begin{aligned}
&\big[|A_2^t(\theta, s, d)| > \tilde{\epsilon}\big] \\
&\quad = \left[\left|\sum_{i=1}^{\tilde{N}(t)}\sum_{j=1}^{\hat{N}(t)}(V(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta))W_{k_i,j,t}(s, d, \theta)\right| > \tilde{\epsilon}\right] \\
&\quad \subset \left[\sum_{i=1}^{\tilde{N}(t)}\sum_{j=1}^{\hat{N}(t)}|V(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta)|W_{k_i,j,t}(s, d, \theta) > \tilde{\epsilon}\right] \\
&\quad \subset \big[\exists k_i, j : |V(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta)| > \tilde{\epsilon}\big].
\end{aligned} \tag{27}$$

Since $V(s;\theta)$ is continuous and $\Theta \times S$ is a compact, $\exists\tilde{\delta}_{\tilde{\epsilon}} > 0$ such that $\|(s_1, \theta_1) - (s_2, \theta_2)\| \le \tilde{\delta}_{\tilde{\epsilon}}$ implies $|V(s_1;\theta_1) - V(s_2;\theta_2)| \le \tilde{\epsilon}$. Therefore,

$$\begin{aligned}
&\big[\exists k_i, j : |V(s^{k_i,j};\theta^{k_i}) - V(s^{k_i,j};\theta)| > \tilde{\epsilon}\big] \\
&\quad \subset \big[\exists k_i, j : \|(s^{k_i,j}, \theta^{k_i}) - (s^{k_i,j}, \theta)\| > \tilde{\delta}_{\tilde{\epsilon}}\big] = [\exists k_i : \|\theta^{k_i} - \theta\| > \tilde{\delta}_{\tilde{\epsilon}}].
\end{aligned} \tag{28}$$

Because k_i are the indices of the parameters from the previous iterations that are the closest to θ,

$$\begin{aligned}
&[\exists k_i : \|\theta^{k_i} - \theta\| > \tilde{\delta}_{\tilde{\epsilon}}] \\
&\quad \subset \big[\forall j \in \{t - N(t), \dots, t-1\} \setminus \big\{k_1, \dots, k_{\tilde{N}(t)}\big\} : \|\theta^j - \theta\| > \tilde{\delta}_{\tilde{\epsilon}}\big]
\end{aligned} \tag{29}$$

$$\subset \bigcup_{\substack{(j_1,\ldots,j_{N(t)-\tilde{N}(t)}) \\ j_m\in\{t-N(t),\ldots,t-1\}, \\ m\neq l\Rightarrow j_m\neq j_l}} \bigcap_{m=1}^{N(t)-\tilde{N}(t)} [\|\theta^{j_m}-\theta\| > \tilde{\delta}_{\tilde{\epsilon}}].$$

Fix some $(j_1,\ldots,j_{N(t)-\tilde{N}(t)})$. Then by Assumption 5,

$$\begin{aligned} P\big([\|\theta^{j_m}-\theta\| > \tilde{\delta}_{\tilde{\epsilon}}] \mid \omega^{j_m-1}\big) &= 1 - P\big([\|\theta^{j_m}-\theta\| < \tilde{\delta}_{\tilde{\epsilon}}] \mid \omega^{j_m-1}\big) \\ &\le 1 - \hat{\delta}\lambda\big[B_{\tilde{\delta}_{\tilde{\epsilon}}}(\theta)\cap\Theta\big] \\ &\le 1 - \hat{\delta}[\tilde{\delta}_{\tilde{\epsilon}}/J_{\Theta}^{0.5}]^J \\ &= \exp\big\{-4\big(-0.25\log(1-\hat{\delta}[\tilde{\delta}_{\tilde{\epsilon}}/J_{\Theta}^{0.5}]^J)\big)\big\} \\ &= e^{-4\delta}, \end{aligned} \tag{30}$$

where the last equality defines $\delta > 0$, J_Θ is the dimensionality of rectangle Θ, and $B_\cdot(\cdot)$ is a ball in R^{J_Θ}. It holds for any history ω^{j_m-1}. Thus for fixed $(j_1,\ldots,j_{N(t)-\tilde{N}(t)})$,

$$P\left(\bigcap_{m=1}^{N(t)-\tilde{N}(t)} [\|\theta^{j_m}-\theta\| > \tilde{\delta}_{\tilde{\epsilon}}]\right) \le e^{-4\delta(N(t)-\tilde{N}(t))}. \tag{31}$$

Since the union in (29) is taken over $N(t)!/(\tilde{N}(t)!(N(t)-\tilde{N}(t))!)$ events,

$$\begin{aligned} P\big[|A_2^t(x,\theta,\epsilon)| > \tilde{\epsilon}\big] &\le e^{-4\delta(N(t)-\tilde{N}(t))}\frac{N(t)!}{\tilde{N}(t)!(N(t)-\tilde{N}(t))!} \\ &\le e^{-2\delta(N(t)-\tilde{N}(t))} \quad \forall t > T_2, \end{aligned} \tag{32}$$

where the second inequality and existence of T_2 follows from Assumption 6 and Proposition 7. Finally,

$$\begin{aligned} P\big(|A_2^t(\theta,s)| > \tilde{\epsilon}\big) &= P\Big(\max_{d\in D}|A_2^t(\theta,s,d)| > \tilde{\epsilon}\Big) \\ &= P\Big(\bigcup_{d\in D}\big[|A_2^t(\theta,s,d)| > \tilde{\epsilon}\big]\Big) \\ &\le \mathrm{card}(D)e^{-2\delta(N(t)-\tilde{N}(t))} \quad (\forall t > T_2) \\ &\le e^{-\delta(N(t)-\tilde{N}(t))} \quad (\forall t > T_3 \ge T_2), \end{aligned} \tag{33}$$

where such T_3 exists since $\mathrm{card}(D)e^{-\delta(N(t)-\tilde{N}(t))} \to 0$. The last inequality in (26) follows since $N(t)-\tilde{N}(t) \ge [t^{\gamma_1}] - [t^{\gamma_2}] \ge t^{\gamma_1} - 1 - t^{\gamma_2} \ge 0.5t^{\gamma_1}$ for any t larger than some $T \ge T_3$. *Q.E.D.*

LEMMA 3: *Given* $\tilde{\epsilon} > 0$, *there exist* $\delta > 0$ *and* T *such that* $\forall \theta \in \Theta$, $\forall s \in S$, *and* $\forall t > T$,

$$P\big(|A_3^t(\theta, s)| > \tilde{\epsilon}\big) \leq e^{-\delta t^{\gamma_0 \gamma_1}}. \tag{34}$$

PROOF: First let us show that for any positive integer m, $\forall \theta \in \Theta$, and $\forall s \in S$,

$$\begin{aligned} A_3^t(\theta, s) \leq \frac{\beta}{1-\beta}\Big[&\max_{i=t-mN(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j})\Big) \\ &+ \max_{i=t-mN(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j})\Big)\Big] \\ &+ \beta^m \max_{i=t-mN(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j})\Big). \end{aligned} \tag{35}$$

By definition,

$$A_3^t(\theta, s, d) = \left| \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} \big(V^{k_i}(s^{k_i,j}; \theta^{k_i}) - V(s^{k_i,j}; \theta^{k_i})\big) W_{k_i,j,t}(s, d, \theta) \right|. \tag{36}$$

Since $\max_d a(d) - \max_d b(d) \leq \max_d\{a(d) - b(d)\}$,

$$\begin{aligned} &|V^{k_i}(s^{k_i,j}; \theta^{k_i}) - V(s^{k_i,j}; \theta^{k_i})| \\ &\quad = \Big| \max_{d \in D}\big\{u(s^{k_i,j}, d) + \beta \hat{E}^{(k_i)}[V(s'; \theta^{k_i}) \mid s^{k_i,j}, d; \theta^{k_i}]\big\} \\ &\qquad - \max_{d \in D}\big\{u(s^{k_i,j}, d) + \beta E[V(s'; \theta^{k_i}) \mid s^{k_i,j}, d; \theta^{k_i}]\big\}\Big| \\ &\quad \leq \Big| \max_{d \in D}\big\{\beta \hat{E}^{(k_i)}[V(s'; \theta^{k_i}) \mid s^{k_i,j}, d; \theta^{k_i}] \\ &\qquad - \beta E[V(s'; \theta^{k_i}) \mid s^{k_i,j}, d; \theta^{k_i}]\big\}\Big|. \end{aligned} \tag{37}$$

From (37) and definition of $A_l^t(\cdot)$ given in Theorem 4,

$$\begin{aligned} &|V^{k_i}(s^{k_i,j}; \theta^{k_i}) - V(s^{k_i,j}; \theta^{k_i})| \\ &\quad \leq \beta \max_{d \in D}\big(A_1^{k_i}(\theta^{k_i}, s^{k_i,j}, d) + A_2^{k_i}(\theta^{k_i}, s^{k_i,j}, d) + A_3^{k_i}(\theta^{k_i}, s^{k_i,j}, d)\big) \\ &\quad \leq \beta\big(A_1^{k_i}(\theta^{k_i}, s^{k_i,j}) + A_2^{k_i}(\theta^{k_i}, s^{k_i,j}) + A_3^{k_i}(\theta^{k_i}, s^{k_i,j})\big). \end{aligned} \tag{38}$$

Combining (36) and (38) gives

$$A_3^t(\theta, s, d) \leq \beta \sum_{i=1}^{\tilde{N}(t)} \sum_{j=1}^{\hat{N}(k_i)} \big(A_1^{k_i}(\theta^{k_i}, s^{k_i,j}) + A_2^{k_i}(\theta^{k_i}, s^{k_i,j}) \tag{39}$$

$$
\begin{aligned}
&\quad + A_3^{k_i}(\theta^{k_i}, s^{k_i,j})\big) W_{k_i,j,t}(s,d,\theta) \\
&\le \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j}) \Big),
\end{aligned}
$$

where the second inequality follows from the fact that $\forall i \in \{1, \dots, \tilde{N}(t)\}$, $k_i \in \{t-N(t), \dots, t-1\}$ and the weights sum to 1. Since the right-hand side (r.h.s.) of (39) does not depend on d,

$$
\begin{aligned}
A_3^t(\theta, s) &\le \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j}) \Big).
\end{aligned} \tag{40}
$$

To facilitate the description of the iterative process on (40) that will lead to (35), let $M(t,0) = t$ and $M(t,i) = M(t,i-1) - N(M(t,i-1))$. Then

$$
\begin{aligned}
A_3^t(\theta, s) &\le \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta \max_{i=t-N(t),t-1} \Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta^2 \max_{i=t-N(t)-N[t-N(t)],t-2} \Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta^2 \max_{i=t-N(t)-N[t-N(t)],t-2} \Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \beta^2 \max_{i=t-N(t)-N[t-N(t)],t-2} \Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j}) \Big) \\
&\le \sum_{k=1}^{m} \beta^k \Big[\max_{i=M(t,k),t-k} \Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j}) \Big) \\
&\quad + \max_{i=M(t,k),t-k} \Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j}) \Big) \Big] \\
&\quad + \beta^m \max_{i=M(t,m),t-m} \Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j}) \Big),
\end{aligned} \tag{41}
$$

from which (35) follows since $\sum_{k=1}^{m}\beta^k < \beta/(1-\beta)$ and $M(t,m) \geq t - mN(t)\ \forall m$.

Inequality in (35) is shown to hold for any m. Let $m(t) = [(t - t^{\gamma_0})/N(t)]$ ($[x]$ is the integer part of x) and notice that $M(t, m(t)) \geq t - m(t)N(t) \geq t^{\gamma_0}$. Since $A_3^i(\theta^i, s^{i,j})$ is bounded above by some $\bar{A}_3 < \infty$ (utility function, and state and parameter spaces are bounded),

$$\begin{aligned}
&P\big[|A_3^t(\theta, s)| > \tilde{\epsilon}\big] \\
&\quad \leq P\bigg[\frac{\beta}{1-\beta}\bigg\{\max_{i=t-m(t)N(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j})\Big) \\
&\qquad + \max_{i=t-m(t)N(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j})\Big)\bigg\} \\
&\qquad + \beta^{m(t)} \max_{i=t-m(t)N(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_3^i(\theta^i, s^{i,j})\Big) > \tilde{\epsilon}\bigg] \\
&\quad \leq P\bigg[\max_{i=t-m(t)N(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_1^i(\theta^i, s^{i,j})\Big) > \frac{\tilde{\epsilon}(1-\beta)}{3\beta}\bigg] \\
&\qquad + P\bigg[\max_{i=t-m(t)N(t),t-1}\Big(\max_{j=1,\hat{N}(i)} A_2^i(\theta^i, s^{i,j})\Big) > \frac{\tilde{\epsilon}(1-\beta)}{3\beta}\bigg] \\
&\qquad + P\bigg[\beta^{m(t)}\bar{A}_3 > \frac{\tilde{\epsilon}}{3}\bigg] \\
&\quad \leq \sum_{i=t-m(t)N(t)}^{t-1}\sum_{j=1}^{\hat{N}(i)}\bigg\{P\bigg[A_1^i(\theta^i, s^{i,j}) > \frac{\tilde{\epsilon}(1-\beta)}{3\beta}\bigg] \\
&\qquad + P\bigg[A_2^i(\theta^i, s^{i,j}) > \frac{\tilde{\epsilon}(1-\beta)}{3\beta}\bigg]\bigg\}.
\end{aligned}$$

The last inequality holds for $t > T_3$, where T_3 satisfies $P(\beta^{m(t)}\bar{A}_3 > \tilde{\epsilon}/3) = 0\ \forall t > T_3$. Such T_3 exists since $m(t) \to \infty$.

Since $t - m(t)N(t) \to \infty$, by Lemma 1 and Lemma 2, there exist $\delta_1 > 0$, $\delta_2 > 0$, T_1, and T_2 such that $\forall t > \max(T_1, T_2, T_3)$,

$$\begin{aligned}
(42)\qquad &P(|A_3^t(\theta, s)| > \tilde{\epsilon}) \\
&\quad \leq \sum_{i=t-m(t)N(t)}^{t-1} \hat{N}(i)\big[e^{-\delta_1 \tilde{N}(i)\hat{N}(i-N(i))} + e^{-\delta_2(N(i)-\tilde{N}(i))}\big].
\end{aligned}$$

Proposition 8 shows that there exist $\delta > 0$ and T_4 such that the r.h.s. of (42) is no larger than $\exp(-\delta t^{\gamma_0\gamma_1})\ \forall t > T_4$. Thus, setting $T = \max(T_1, T_2, T_3, T_4)$ completes the proof. *Q.E.D.*

S2. EXTENSION TO THE UNIFORM CONVERGENCE

First, note that the approximation error is not a continuous function of (θ, s). Thus, we cannot apply the standard results to show the measurability of the supremum of the approximation error over the state and parameter spaces. Proposition 1 below and Proposition 3 establish the measurability in this case. Next, Lemma 4 shows that a uniform version of Lemma 1 holds; Lemma 5 shows that a uniform version of Lemma 2 also holds; a uniform version of Lemma 3 holds trivially since the right-hand side of the key inequality (35) does not depend on (θ, s). Theorem 1 follows from the uniform versions of the lemmas in the same way as Theorem 4 follows from Lemmas 1–3.

PROPOSITION 1: *Let $f(\omega, \theta)$ be a measurable function on $(\Omega \times \Theta, \sigma(\mathcal{A} \times \mathcal{B}))$ with values in R. Assume that Θ has a countable subset $\tilde{\Theta}$ and that for any $\omega \in \Omega$ and any $\theta \in \Theta$ there exists a sequence in $\tilde{\Theta}$, $\{\tilde{\theta}_n\}$ such that $f(\omega, \tilde{\theta}_n) \to f(\omega, \theta)$. Then $\sup_{\theta\in\Theta} f(\omega, \theta)$ is measurable with respect to (w.r.t.) $(\Omega, \mathcal{A})$ (the proposition can be used to show that the supremum of a random function with some simple discontinuities, e.g., jumps, on a separable space is measurable).*

PROOF: First, let us show that for an arbitrary t,

$$\bigcup_{\theta\in\Theta}[f(\omega, \theta) > t] = \bigcup_{\theta\in\tilde{\Theta}}[f(\omega, \theta) > t]. \tag{43}$$

Assume $\omega_1 \in \bigcup_{\theta\in\Theta}[f(\omega, \theta) > t]$. This means there exists $\theta_1 \in \Theta$ such that $f(\omega_1, \theta_1) > t$. By the theorem's assumption, $\exists\{\tilde{\theta}_n\}$ is such that $f(\omega_1, \tilde{\theta}_n) \to f(\omega_1, \theta_1)$. Then $\exists n$, $f(\omega_1, \tilde{\theta}_n) > t$. Thus, $\omega_1 \in \bigcup_{\theta\in\tilde{\Theta}}[f(\omega, \theta) > t]$ and (43) is proved.

Note that $[\sup_{\theta\in\Theta} f(\omega, \theta) > t] = \bigcup_{\theta\in\Theta}[f(\omega, \theta) > t] = \bigcup_{\theta\in\tilde{\Theta}}[f(\omega, \theta) > t]$ is a countable union of sets from $\mathcal{A}$ and thus also belongs to $\mathcal{A}$. *Q.E.D.*

To apply the proposition for establishing the measurability of the supremum of the approximation errors, let the set of rational numbers contained in $\Theta \times S$ play the role of the countable subset $\tilde{\Theta}$. Proposition 3 shows that for any given history ω^{t-1} and any (θ, s), it is always possible to find a sequence with rational coordinates $(\tilde{\theta}_n) \to \theta$ such that for all n, $(\tilde{\theta}_n)$ and θ have the same iteration indices for the nearest neighbors. For a given history ω^{t-1}, the approximation error is continuous in (θ, s) on the subsets of $\Theta \times S$ that give the same iteration indices of the nearest neighbors. Using any rational sequence $s^n \to s$ gives $f(\omega, (\widetilde{\theta, s})_n) \to f(\omega, (\theta, s))$ as required in the proposition. Thus, the supremum of the approximation error is measurable.

LEMMA 4: *Given $\tilde{\epsilon} > 0$, there exist $\delta > 0$ and T such that $\forall t > T$,*

$$P\Big(\sup_{\theta\in\Theta, s\in S} |A_1^t(\theta, s)| > \tilde{\epsilon}\Big) \le e^{-\delta\tilde{N}(t)\hat{N}(t-N(t))} \le e^{-0.5\delta t^{\gamma_1}}. \tag{44}$$

PROOF: Fix a combination $m = \{m_1, \dots, m_{\tilde{N}(t)}\}$ from $\{t - N(t), \dots, t-1\}$. Lemma 1 defines $X(\omega^{t-1}, \theta, s, d, m)$ in (18). By Proposition 5, $\{X(\omega^{t-1}, \theta, s, d, m)\}_{\omega^{t-1}}$ are equicontinuous on $\Theta \times S$: there exists $\tilde{\delta}(\tilde{\epsilon}) > 0$ such that $\|(\theta_1, s_1) - ((\theta_2, s_2))\| < \tilde{\delta}(\tilde{\epsilon})$ implies $|X(\omega^{t-1}, \theta_1, s_1, d, m) - X(\omega^{t-1}, \theta_2, s_2, d, m)| < \tilde{\epsilon}/2$. Since $\Theta \times S$ is a compact set, it can be covered by M balls: $\Theta \times S \subset \bigcup_{i=1}^M B_i$ with radius $\tilde{\delta}(\tilde{\epsilon})$ and centers at (θ_i, s_i), where $M < \infty$ depends only on $\tilde{\epsilon}$. It follows that

$$\text{(45)} \qquad \Big[\sup_{\theta,s} X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}\Big] = \bigcup_{\theta,s} [X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}] = \bigcup_{i=1}^{M} \bigcup_{(\theta,s)\in B_i} [X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}].$$

Let us show that

$$\text{(46)} \qquad \bigcup_{(\theta,s)\in B_i} [X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}] \subset \Big[X(\omega^{t-1}, \theta_i, s_i, d, m) > \frac{\tilde{\epsilon}}{2}\Big].$$

If $\omega_*^{t-1} \in \bigcup_{(\theta,s)\in B_i} [X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}]$, then $\exists (\theta^*, s^*) \in B_i(\theta_i, s_i)$ such that $X(\omega_*^{t-1}, \theta^*, s^*, d, m) > \tilde{\epsilon}$. Since $\|(\theta^*, s^*) - (\theta_i, s_i)\| \leq \tilde{\delta}(\tilde{\epsilon})$, $X(\omega_*^{t-1}, \theta_i, s_i, d, m) \geq X(\omega_*^{t-1}, \theta^*, s^*, d, m) - \tilde{\epsilon}/2$. This implies $\omega_*^{t-1} \in [X(\omega^{t-1}, \theta_i, s_i, d, m) > \frac{\tilde{\epsilon}}{2}]$.

Since $\sup_{\theta,s} |A_1^t(\theta, s, d)| < \max_m \sup_{\theta,s} X(\omega^{t-1}, \theta, s, d, m)$,

$$\text{(47)} \qquad \begin{aligned} &P\Big(\sup_{\theta,s} |A_1^t(\theta, s, d)| > \tilde{\epsilon}\Big) \\ &\quad \leq P\Big[\max_m \sup_{\theta,s} X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}\Big] \\ &\qquad \text{(max is over all possible combinations } m\text{)} \\ &\quad \leq P\Big(\bigcup_m \Big[\sup_{\theta,s} X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}\Big]\Big) \\ &\quad \leq \sum_m P\Big[\sup_{\theta,s} X(\omega^{t-1}, \theta, s, d, m) > \tilde{\epsilon}\Big] \\ &\quad \leq \sum_m P\Big(\bigcup_{i=1}^M \Big[X(\omega^{t-1}, \theta_i, s_i, d, m) > \frac{\tilde{\epsilon}}{2}\Big]\Big) \quad \text{(by (45) and (46))} \\ &\quad \leq M \frac{N(t)!}{(N(t) - \tilde{N}(t))!\tilde{N}(t)!} 2\exp\big\{-4\delta \tilde{N}(t)\hat{N}(t - N(t))\big\}, \end{aligned}$$

where $N(t)!/((N(t)-\tilde{N}(t))!\tilde{N}(t)!)$ is the number of different combinations m and $2\exp\{-4\delta\tilde{N}(t)\hat{N}(t-N(t))\}$ is the bound from (22) in Lemma 1. From the last inequality, the proof follows steps of the argument starting after (23) in the proof of Lemma 1. *Q.E.D.*

LEMMA 5: *Given* $\tilde{\epsilon}>0$, *there exist* $\delta>0$ *and* T *such that* $\forall t>T$,

$$P\Big(\sup_{\theta,s}|A_2^t(\theta,s)|>\tilde{\epsilon}\Big)\le e^{-\delta(N(t)-\tilde{N}(t))}\le e^{-0.5\delta t^{\gamma_1}}. \tag{48}$$

PROOF: From Lemma 2,

$$\begin{aligned}&\big[|A_2^t(\theta,s,d)|>\tilde{\epsilon}\big]\\ &\quad\subset\bigcup_{\substack{(j_1,\dots,j_{N(t)-\tilde{N}(t)})\\ j_m\in\{t-N(t),\dots,t-1\},\\ m\neq l\Rightarrow j_m\neq j_l}}\ \bigcap_{m=1}^{N(t)-\tilde{N}(t)}[\|\theta^{j_m}-\theta\|>\tilde{\delta}_{\tilde{\epsilon}}].\end{aligned} \tag{49}$$

This implies that

$$\begin{aligned}&\Big[\sup_{\theta,s}|A_2^t(\theta,s,d)|>\tilde{\epsilon}\Big]\\ &\quad=\bigcup_{\theta,s}\big[|A_2^t(\theta,s,d)|>\tilde{\epsilon}\big]\\ &\quad\subset\bigcup_{\theta\in\Theta}\Bigg\{\bigcup_{\substack{(j_1,\dots,j_{N(t)-\tilde{N}(t)})\\ j_m\in\{t-N(t),\dots,t-1\},\\ m\neq l\Rightarrow j_m\neq j_l}}\ \bigcap_{m=1}^{N(t)-\tilde{N}(t)}[\|\theta^{j_m}-\theta\|>\tilde{\delta}_{\tilde{\epsilon}}]\Bigg\}.\end{aligned} \tag{50}$$

Since Θ is a rectangle in R^{J_Θ}, it can be covered by a finite number of balls with radius $\tilde{\delta}_{\tilde{\epsilon}}/2$:

$$\Theta\subset\bigcup_{i=1}^{M}B(\theta_i),\qquad M=\text{const}\cdot(\tilde{\delta}_{\tilde{\epsilon}}/2)^{-J_\Theta}. \tag{51}$$

Let us prove the fact

$$\bigcup_{\theta\in B(\theta_i)}\ \bigcap_{m=1}^{N(t)-\tilde{N}(t)}[\|\theta^{j_m}-\theta\|>\tilde{\delta}_{\tilde{\epsilon}}]\subset\bigcap_{m=1}^{N(t)-\tilde{N}(t)}[\theta^{j_m}\notin B(\theta_i)]. \tag{52}$$

Assume $\omega^{t-1} \in (\bigcap_{m=1}^{N(t)-\tilde{N}(t)}[\theta^{j_m} \notin B(\theta_i)])^c$. There exists m such that $\theta^{j_m} \in B(\theta_i)$. It follows that $\forall \theta \in B(\theta_i), \exists \theta^{j_m}, \|\theta^{j_m} - \theta\| \leq \tilde{\delta}_{\tilde{\epsilon}}$. Thus, ω^{t-1} belongs to the set

$$\bigcap_{\theta \in B(\theta_i)} \bigcup_{m=1}^{N(t)-\tilde{N}(t)} [\|\theta^{j_m} - \theta\| \leq \tilde{\delta}_{\tilde{\epsilon}}] = \left(\bigcup_{\theta \in B(\theta_i)} \bigcap_{m=1}^{N(t)-\tilde{N}(t)} [\|\theta^{j_m} - \theta\| > \tilde{\delta}_{\tilde{\epsilon}}] \right)^c. \tag{53}$$

Therefore, the claim in (52) is proved.

By the same argument as for (31) from Lemma 2, we can establish that

$$P\left(\bigcap_{m=1}^{N(t)-\tilde{N}(t)} [\theta^{j_m} \notin B(\theta_i)] \right) \leq e^{-4\delta(N(t)-\tilde{N}(t))} \tag{54}$$

for some positive δ.

From (50), (51), and (52)

$$\begin{aligned} &\left[\sup_{\theta,s} |A_2^t(\theta, s, d)| > \tilde{\epsilon} \right] \\ &\quad \subset \bigcup_{\substack{(j_1,\ldots,j_{N(t)-\tilde{N}(t)}) \\ j_m \in \{t-N(t),\ldots,t-1\}, \\ m \neq l \Rightarrow j_m \neq j_l}} \bigcup_{i=1}^{M} \left(\bigcap_{m=1}^{N(t)-\tilde{N}(t)} [\theta^{j_m} \notin B(\theta_i)] \right). \end{aligned} \tag{55}$$

Using (54) and (55) gives

$$P\left[\sup_{\theta,s} |A_2^t(\theta, s, d)| > \tilde{\epsilon} \right] \leq \frac{N(t)!}{\tilde{N}(t)!(N(t) - \tilde{N}(t))!} M e^{-4\delta(N(t)-\tilde{N}(t))}. \tag{56}$$

The rest of the proof follows the corresponding steps in Lemma 2. *Q.E.D.*

S3. PROOF OF CONVERGENCE OF POSTERIOR EXPECTATIONS

PROOF OF THEOREM 2: First, let us introduce some notation shortcuts:

$$\begin{aligned} r &= r(\theta, \mathcal{V}, \epsilon; F(\theta, \epsilon)), \\ \hat{r} &= r(\theta, \mathcal{V}, \epsilon; \hat{F}^n(\theta, \epsilon)), \\ 1_{\{\cdot\}} &= 1_\Theta(\theta) \cdot \left(\prod_{i,t} 1_E(\epsilon_{t,i}) p(d_{t,i} | \mathcal{V}_{t,i}) \right) \\ &\quad \cdot \left(\prod_{i,t,k} 1_{[-\overline{\nu},\overline{\nu}]}\big(q(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}(\theta, \epsilon_{t,i}))\big) \right), \end{aligned}$$

$$\hat{1}_{\{\cdot\}} = 1_{\Theta}(\theta) \cdot \left(\prod_{i,t} 1_E(\epsilon_{t,i}) p(d_{t,i}|\mathcal{V}_{t,i}) \right)$$
$$\cdot \left(\prod_{i,t,k} 1_{[-\overline{\nu},\overline{\nu}]}\big(q(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, \hat{F}^n_{t,i}(\theta, \epsilon_{t,i}))\big) \right),$$
$$\int h(\theta, \mathcal{V}, \epsilon)\, d(\theta, \mathcal{V}, \epsilon) = \int h,$$
$$p = p(\theta, \mathcal{V}, \epsilon; F|d, x) = \frac{r \cdot 1_{\{\cdot\}}}{\int r \cdot 1_{\{\cdot\}}},$$
$$\hat{p} = p(\theta, \mathcal{V}, \epsilon; \hat{F}^n|d, x) = \frac{\hat{r} \cdot \hat{1}_{\{\cdot\}}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}}.$$

The probability that the approximation error exceeds $\varepsilon > 0$ can be bounded by the sum of two terms

$$P\left[\left| \int h \cdot p - \int h \cdot \hat{p} \right| > \varepsilon \right] \tag{57}$$
$$\leq P(\|F - \hat{F}\| > \delta_F)$$
$$+ P\left(\left[\left| \int h \cdot p - \int h \cdot \hat{p} \right| > \varepsilon \right] \cap [\|F - \hat{F}\| \leq \delta_F] \right), \tag{58}$$

where $\|F - \hat{F}\| = \sup_{s,\theta,d} |F(s, \theta, d) - \hat{F}(s, \theta, d)|$, $F(s, \theta, d)$ is the expected value function (or the difference of expected value functions, depending on the parameterization of the Gibbs sampler), and $\hat{F}$ is the approximation to F from the DP solving algorithm on its iteration n (fixed in this proof). I will show that for a sufficiently small $\delta_F > 0$, the set in (58) is empty. Then, by Theorem 1, the term in (57) can be bounded by z_n corresponding to δ_F:

$$\left[\left| \int h \cdot p - \int h \cdot \hat{p} \right| > \varepsilon \right] \cap [\|F - \hat{F}\| \leq \delta_F]$$
$$\subset \left[\int |p - \hat{p}| > \varepsilon / \|h\| \right] \cap [\|F - \hat{F}\| \leq \delta_F]$$
$$\subset \left(\left[\int_{\hat{1}_{\{\cdot\}} = 1_{\{\cdot\}}} |p - \hat{p}| > \varepsilon / (2\|h\|) \right] \cap [\|F - \hat{F}\| \leq \delta_F] \right) \tag{59}$$
$$\cup \left(\left[\int_{\hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}} |p - \hat{p}| > \varepsilon / (2\|h\|) \right] \cap [\|F - \hat{F}\| \leq \delta_F] \right). \tag{60}$$

Let us start with (59):

$$\left(\left[\int_{\hat{1}_{\{\cdot\}}=1_{\{\cdot\}}} |p-\hat{p}| > \frac{\varepsilon}{(2\|h\|)}\right] \cap [\|F-\hat{F}\| \leq \delta_F]\right) \tag{61}$$

$$= \left[\int_{\hat{1}_{\{\cdot\}}=1_{\{\cdot\}}} \left| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right| > \frac{\varepsilon}{2\|h\|}\right] \cap [\|F-\hat{F}\| \leq \delta_F]$$

$$\subset \left[\left\| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right\| > \frac{\varepsilon}{2\|h\|\overline{\lambda}}\right] \cap [\|F-\hat{F}\| \leq \delta_F],$$

where $\overline{\lambda} < \infty$ is the Lebesgue measure of the space for the parameters and the latent variables. For $\delta_{\text{Sp}} \in (0, \int r \cdot 1_{\{\cdot\}})$:

$$\left[\left\| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right\| > \frac{\varepsilon}{2\|h\|\overline{\lambda}}\right] \cap [\|F-\hat{F}\| \leq \delta_F]$$

$$= \left(\left[\left\| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right\| > \frac{\varepsilon}{2\|h\|\overline{\lambda}}\right] \cap [\|F-\hat{F}\| \leq \delta_F]\right.$$

$$\left.\cap \left[\left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right| > \delta_{\text{Sp}}\right]\right) \tag{62}$$

$$\cup \left(\left[\left\| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right\| > \frac{\varepsilon}{2\|h\|\overline{\lambda}}\right] \cap [\|F-\hat{F}\| \leq \delta_F]\right.$$

$$\left.\cap \left[\left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right| \leq \delta_{\text{Sp}}\right]\right). \tag{63}$$

By Proposition 2 for δ_{Sp} there exists $\delta_F^1 > 0$ such that $[|\int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}}| > \delta_{\text{Sp}}] = \emptyset$. Thus, (62) (the whole two-line expression in parentheses) is the empty set for any $\delta_F < \delta_F^1$. Now, let us work with (63) (again, both lines in parentheses):

$$\left\| \frac{r}{\int r \cdot 1_{\{\cdot\}}} - \frac{\hat{r}}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \right\| \leq \frac{\|r\| \cdot \left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right|}{\int r \cdot 1_{\{\cdot\}} \cdot \int \hat{r} \cdot \hat{1}_{\{\cdot\}}} + \frac{\|\hat{r} - r\|}{\int \hat{r} \cdot \hat{1}_{\{\cdot\}}} \tag{64}$$

$$
\leq \frac{\|r\| \cdot \left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right|}{\int r \cdot 1_{\{\cdot\}} \cdot \left(\int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}} \right)}
+ \frac{\|\hat{r} - r\|}{\int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}}}.
$$

This inequality shows that (63) is a subset of the union of the two sets

$$
\left[\frac{\|r\| \cdot \left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right|}{\int r \cdot 1_{\{\cdot\}} \cdot \left(\int \int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}} \right)} > \frac{\varepsilon}{4\|h\|\overline{\lambda}} \right] \cap [\|F - \hat{F}\| \leq \delta_F]
\cap \left[\left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right| \leq \delta_{\mathrm{Sp}} \right] \tag{65}
$$

and

$$
\left[\frac{\|\hat{r} - r\|}{\int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}}} > \frac{\varepsilon}{4\|h\|\overline{\lambda}} \right] \cap [\|F - \hat{F}\| \leq \delta_F]
\cap \left[\left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right| \leq \delta_{\mathrm{Sp}} \right]. \tag{66}
$$

I will show that both of them are empty for sufficiently small δ_F. By Proposition 2, there exists $\delta_F^2 > 0$ such that

$$
\left[\left| \int r \cdot 1_{\{\cdot\}} - \int \hat{r} \cdot \hat{1}_{\{\cdot\}} \right| > \frac{\varepsilon \cdot \int r \cdot 1_{\{\cdot\}} \cdot \left(\int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}} \right)}{4\|h\|\overline{\lambda}\|r\|} \right] = \emptyset
$$

whenever $\|F - \hat{F}\| \leq \delta_F^2$. Therefore, (65) is equal to the empty set for $\delta_F \leq \delta_F^2$. Since r is continuous in components of F, there exists $\delta_F^3 > 0$ such that

$$
\|\hat{r} - r\| < \frac{\varepsilon \cdot \left(\int r \cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}} \right)}{4\|h\|\overline{\lambda}\|r\|}
$$

whenever $\|F - \hat{F}\| \leq \delta_F^3$. Therefore, for $\delta_F \leq \delta_F^3$, (66) is equal to the empty set and so is (63). Thus far we showed that (59) is equal to the empty set for $\delta_F \leq \min_{i=1,2,3}(\delta_F^i)$.

Now, let us work with (60). Note that

$$\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} |p-\hat{p}| \leq \left(\frac{\|r\|}{\int r\cdot 1_{\{\cdot\}}} + \frac{\|\hat{r}\|}{\int \hat{r}\cdot \hat{1}_{\{\cdot\}}}\right)\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} 1$$
$$\leq \left(\frac{\|r\|}{\int r\cdot 1_{\{\cdot\}}} + \frac{\|\hat{r}\|}{\int r\cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}}}\right)\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} 1.$$

Thus, (60) is a subset of the set

$$\left(\left[\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} |p-\hat{p}| > \frac{\varepsilon}{(2\|h\|)}\right] \cap [\|F-\hat{F}\| \leq \delta_F]\right) \tag{67}$$
$$\subset \left(\left[\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} 1 > \varepsilon \Big/ \left(2\|h\|\left(\frac{\|r\|}{\int r\cdot 1_{\{\cdot\}}} + \frac{\|\hat{r}\|}{\int r\cdot 1_{\{\cdot\}} - \delta_{\mathrm{Sp}}}\right)\right)\right]\right.$$
$$\left.\cap [\|F-\hat{F}\| \leq \delta_F]\right).$$

Using the same argument as the one starting from (71) in Proposition 2, I can show that there exists $\delta_F^4 > 0$ such that $\forall \delta_F < \delta_F^4$, (60) will be the empty set. Setting $\delta_F = \min_{i=1,2,3,4}\{\delta_F^i\}$ completes the proof of the theorem. *Q.E.D.*

PROPOSITION 2: *For any* $\varepsilon > 0$, *there exists* $\delta_F > 0$ *such that*

$$[\|F-\hat{F}\| < \delta_F] \cap \left[\left|\int \hat{r}\cdot \hat{1}_{\{\cdot\}} - \int r\cdot 1_{\{\cdot\}}\right| > \varepsilon\right] = \emptyset. \tag{68}$$

PROOF:

$$\left[\left|\int \hat{r}\cdot \hat{1}_{\{\cdot\}} - \int r\cdot 1_{\{\cdot\}}\right| > \varepsilon\right] \subset \left[\int \left|\hat{r}\cdot \hat{1}_{\{\cdot\}} - r\cdot 1_{\{\cdot\}}\right| > \varepsilon\right]$$
$$\subset \left[\int_{\hat{1}_{\{\cdot\}}= 1_{\{\cdot\}}} \left|\hat{r}\cdot \hat{1}_{\{\cdot\}} - r\cdot 1_{\{\cdot\}}\right| > \varepsilon/2\right] \tag{69}$$
$$\cup \left[\int_{\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}} \left|\hat{r}\cdot \hat{1}_{\{\cdot\}} - r\cdot 1_{\{\cdot\}}\right| > \varepsilon/2\right]. \tag{70}$$

Let us show that the intersection of (69) and $[\|F-\hat{F}\| < \delta_F]$ is the empty set for a sufficiently small δ_F:

$$\left[\int_{\hat{1}_{\{\cdot\}}= 1_{\{\cdot\}}} \left|\hat{r}\cdot \hat{1}_{\{\cdot\}} - r\cdot 1_{\{\cdot\}}\right| > \varepsilon/2\right] \subset \left[\int_{\hat{1}_{\{\cdot\}}= 1_{\{\cdot\}}} \left|\hat{r} - r\right| > \varepsilon/2\right]$$

$$\subset [\|\hat{r} - r\| > \varepsilon/(2\overline{\lambda})],$$

where $\overline{\lambda} < \infty$ is the Lebesgue measure of the bounded space for the parameters and the latent variables on which the integration is performed: $\Theta \times E \times \cdots \times E \times \mathbf{V} \times \cdots \times \mathbf{V}$, where $\mathbf{V} \subset R$ is the space for the alternative specific value functions $\mathcal{V}_{t,d,i}$. By Assumption 7, r is continuous in components of F. Thus, $\exists \delta_F^1 > 0$ such that $\|F - \hat{F}\| < \delta_F^1$ implies $\|\hat{r} - r\| < \varepsilon/(2\overline{\lambda})$, which means that the intersection of (69) and $[\|F - \hat{F}\| < \delta_F]$ is the empty set for $\forall \delta_F < \delta_F^1$.

Let us show that the intersection of (70) and $[\|F - \hat{F}\| < \delta_F]$ is the empty set for a sufficiently small δ_F. First, note that

$$\int_{\hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}} \left|\hat{r} \cdot \hat{1}_{\{\cdot\}} - r \cdot 1_{\{\cdot\}}\right| \leq (\|r\| + \|\hat{r}\|) \int_{\hat{1}_{\{\cdot\}} \neq \cdot 1_{\{\cdot\}}} 1, \tag{71}$$

where $\|r\| < \infty$ and $\|\hat{r}\| < \overline{r} < \infty$ for any $\hat{F}$ (everything is bounded in the model). Thus,

$$\begin{aligned} &[\|F - \hat{F}\| < \delta_F] \cap \left[\int_{\hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}} \left|\hat{r} \cdot \hat{1}_{\{\cdot\}} - r \cdot 1_{\{\cdot\}}\right| > \varepsilon/2\right] \\ &\quad \subset [\|F - \hat{F}\| < \delta_F] \cap \left[\int_{\hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}} 1 > \varepsilon/\big(2(\|r\| + \|\hat{r}\|)\big)\right] \\ &\quad = [\|F - \hat{F}\| < \delta_F] \cap \big[\lambda\big[\hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}\big] > \varepsilon/\big(2(\|r\| + \|\hat{r}\|)\big)\big], \end{aligned} \tag{72}$$

where $\lambda(\cdot)$ is the Lebesgue measure on the space of the parameters and the latent variables.

By Assumption 7, q_k is continuous in components of F. Thus, for any $\delta_q > 0$, there exists $\delta_F(\delta_q) > 0$ such that $\|F - \hat{F}\| < \delta_F(\delta_q)$ implies $\max_k \|\hat{q}_k - q_k\| < \delta_q$. On the space of the parameters and the latent variables (these are not subsets of the underlying probability space),

$$\begin{aligned} &\big[(\theta, \mathcal{V}, \epsilon) : \hat{1}_{\{\cdot\}} \neq 1_{\{\cdot\}}\big] \\ &\quad \subset \bigcup_{i,t,k} \big[(\theta, \mathcal{V}, \epsilon) : q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) \in B_{\delta_q}(\overline{\nu}) \cup B_{\delta_q}(-\overline{\nu})\big] \end{aligned} \tag{73}$$

if $\|F - \hat{F}\| < \delta_F(\delta_q)$. To prove this claim, assume $\forall i, t, k$ $q_k(\theta, \mathcal{V}_{t,i}, \epsilon_{t,i}, F_{t,i}) \notin B_{\delta_q}(\overline{\nu}) \cup B_{\delta_q}(-\overline{\nu})$. So the distance between q_k and the truncation region edges $-\overline{\nu}$ and $\overline{\nu}$ is larger than δ_q for all i, t, k. But then, since $\|\hat{q}_k - q_k\| < \delta_q$, $\hat{1}_{\{\cdot\}} = 1_{\{\cdot\}}$ and the claim (73) is proved.

Note that

$$\lim_{\delta_q\to 0}\lambda\Big(\bigcup_{i,t,k}\big[(\theta,\mathcal{V},\epsilon):q_k(\theta,\mathcal{V}_{t,i},\epsilon_{t,i},F_{t,i})\in B_{\delta_q}(\overline{\nu})\cup B_{\delta_q}(-\overline{\nu})\big]\Big) \tag{74}$$

$$\leq \sum_{i,t,k}\lim_{\delta_q\to 0}\lambda\big[(\theta,\mathcal{V},\epsilon):q_k(\theta,\mathcal{V}_{t,i},\epsilon_{t,i},F_{t,i})\in B_{\delta_q}(\overline{\nu})\cup B_{\delta_q}(-\overline{\nu})\big]$$

$$= \sum_{i,t,k}\lambda\big[(\theta,\mathcal{V},\epsilon):q_k(\theta,\mathcal{V}_{t,i},\epsilon_{t,i},F_{t,i})\in\{\overline{\nu},-\overline{\nu}\}\big],$$

where the last equality holds by the monotone property of measures (the Lebesgue measure in this case) and by the fact that $\bigcap_{\delta_q>0}[q_k\in B_{\delta_q}(\overline{\nu})] = [q_k=\overline{\nu}]$.

By Assumption 7, $\lambda[(\theta,\mathcal{V},\epsilon):q_k=\overline{\nu}]=\lambda[(\theta,\mathcal{V},\epsilon):q_k=-\overline{\nu}]=0$. Therefore, the limit in (74) is equal to zero and there exists $\delta_q^*>0$ such that if $\|F-\hat{F}\|<\delta_F(\delta_q^*)$, then

$$\lambda\big[\hat{1}_{\{\cdot\}}\neq 1_{\{\cdot\}}\big]<\varepsilon/\big(2(\|r\|+\|\hat{r}\|)\big).$$

So $\forall\delta_F\in(0,\delta_F(\delta_q^*)]$, the intersection of (70) and $[\|F-\hat{F}\|<\delta_F]$ is the empty set. Setting $\delta_F=\min\{\delta_F(\delta_q^*),\delta_F^1\}$ completes the proof of the proposition. *Q.E.D.*

PROOF OF THEOREM 3: The proof is the same as the proofs of analogous results in Roberts and Rosenthal (2006). Conditions (a) and (b) here can be used instead of their assumptions (a) and (b) on page 8. Result (i) follows from their proof of Theorem 5 and result (ii) follows from their proof of Theorem 23 (the only required change in the proofs is to use the exact transition kernel from my setup instead of $P_{\Gamma_{K-N}}$ in their formula (3)). *Q.E.D.*

S4. AUXILIARY RESULTS

PROPOSITION 3: *For any $\{\theta^1,\dots,\theta^N\}$ and θ in R^n, and any $\tilde{N}\leq N$, there exists a sequence of rational numbers $q_m\to\theta$ such that for any m, q_m and θ have the same set of indices for the nearest neighbors*: $\{k_1,\dots,k_{\tilde{N}}\}$ *defined by* (11).

PROOF: The outcomes of selecting the nearest neighbors can be classified into two cases. The trivial case occurs when there exists a ball around θ with radius r such that $\|\theta^{k_i}-\theta\|<r$ and $\|\theta^j-\theta\|>r+d$ for $d>0$ and $j\neq k_i$. Then, applying the triangle inequality twice, we get $\forall q\in B_{d/4}(\theta)$, $\|\theta^{k_i}-q\|<r+d/2<\|\theta^j-q\|$ $\forall j\neq k_i$. For this case the proposition holds trivially.

The other case occurs when there exists a ball at θ with radius r_1 such that the closure of the ball includes all the nearest neighbors and the boundary of the ball includes one or more θ^j that are not included in the set of the nearest neighbors. For this case, I will construct a ball in the vicinity of θ such that it can be made as close to θ as needed and such that for any point inside this ball, the set of the nearest neighbors is the same as for θ.

As described in the main paper (see (11)), the selection of the nearest neighbors on the boundary of $B_{r_1}(\theta)$ is conducted by the lexicographic comparison of $(\theta^j - \theta)$. Let us denote vectors $(\theta^j - \theta)$ such that θ^j is on the boundary of $B_{r_1}(\theta)$: $\|\theta^j - \theta\| = r_1$ by $x^{0,i}$, $i = 1, \dots, M_x^0$. The results of the lexicographic selection process can be represented as

$$
\begin{aligned}
(75)\qquad & z^{k,i} = (r_1 - a_1, \dots, r_{k-1} - a_{k-1}, z_k^{k,i}, \dots, z_n^{k,i}), \\
& x^{k,i} = (r_1 - a_1, \dots, r_{k-1} - a_{k-1}, r_k - a_k, x_{k+1}^{k,i}, \dots, x_n^{k,i}), \\
& y^{k,i} = (r_1 - a_1, \dots, r_{k-1} - a_{k-1}, y_k^{k,i}, \dots, y_n^{k,i}),
\end{aligned}
$$

where a geometric interpretation of variables r_k and a_k is given in Figure 1,

$$
(76)\qquad z_k^{k,i} > r_k - a_k > y_k^{k,i},
$$

and $k = 1, \dots, K$ for some $K \leq n$. Vectors $z^{k,i}$, $i = 1, \dots, M_z^k$, are those vectors included in the set of nearest neighbors for which the decision of inclusion was obtained from the lexicographic comparison for the coordinate k. Vectors $x^{k,i}$, $i = 1, \dots, M_x^k$, are the vectors for which the decision has not yet been made after comparing coordinates k. Vectors $y^{k,i}$, $i = 1, \dots, M_y^k$, are the vectors for which the decision of not including them in the set of the nearest neighbors was obtained from comparing coordinate k. Vectors $x^{k+1,i}$, $y^{k+1,i}$, and $z^{k+1,i}$ are all selected from $x^{k,i}$. The lexicographic selection will end at some coordinate K with unique x^K. This vector is denoted by x, not by z, to emphasize the fact that if there are multiple repetitions of $\theta + x^K = \theta^i = \theta^j$, $i \neq j$, in the history, then not all the repetitions have to be selected for the set of nearest neighbors

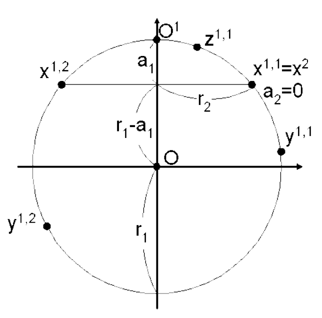

FIGURE 1.—Nearest neighbors.

(the ones with larger iteration number will be selected first). Of course, this is true only for the last selected nearest neighbor; for all the previous ones, all the repetitions are included. Note that vectors $z^{k,i}$, $x^{k,i}$, and $y^{k,i}$ are constructed in the system of coordinates with the origin at θ, so we should add θ to all of them to get back to the original coordinate system.

A graphical illustration might be helpful for understanding the idea of the proof (the proof was actually constructed from similar graphical examples in R^2 and R^3). The figure shows an example in which two nearest neighbors have to be chosen for point O. Since the required number of nearest neighbors is smaller than the number of points on the circle, we can always find a_1 such that all the points with the first coordinate strictly above $r_1 - a_1$ will be included in the set of the nearest neighbors and all the points with the first coordinate strictly below $r_1 - a_1$ will not be. For the points with the coordinate equal to $r_1 - a_1$, the selection process continues to the next dimension.

If we did not use the lexicographic comparison and just resolved the multi-valuedness of arg min by choosing vectors with larger iteration numbers first, then the proposition would not hold (a counterexample could be easily found in R^2).

If the following conditions hold, then the same nearest neighbors from the surface of $B_{r_1}(\theta)$ will be chosen for $(\theta + b)$ and θ:

$$\|b - y^{k,i}\| > \|b - x^K\| > \|b - z^{k,i}\| \quad \forall k, i. \tag{77}$$

The condition says that $(x^K + \theta)$, which is the last nearest neighbor selected for θ, also has to be selected last for $(\theta + b)$ and that vectors on the boundary of $B_{r_1}(\theta)$ that are not the selected nearest neighbors for θ ($y^{k,i}, \forall k, i$) should not be the selected nearest neighbors for $(\theta + b)$. Since $\|y^{k,i}\| = \|x^K\| = \|z^{k,i}\| = r_1$, these conditions are equivalent to

$$b^T(x^K - y^{k,i}) > 0 \quad \text{and} \quad b^T(z^{k,i} - x^K) > 0. \tag{78}$$

Define

$$\begin{aligned} d = \min_{k=1,K} \min \Big\{ &\min_i [z_k^{k,i} - (r_k - a_k)], \\ &\min_i [(r_k - a_k) - y_k^{k,i}] \Big\}, \qquad d > 0 \text{ by construction.} \end{aligned}$$

For given $\epsilon_1 > 0$, let

$$\begin{aligned} &\epsilon_{k+1} = \min\{\epsilon_k, \epsilon_k d/(4nr_1)\}, \\ &\epsilon(\epsilon_1) = (\epsilon_1, \ldots, \epsilon_n), \\ &\delta(\epsilon_1) = \epsilon_n d/(8nr_1). \end{aligned} \tag{79}$$

Let $b \in B_{\delta(\epsilon_1)}(\epsilon(\epsilon_1))$ and $l = b - \epsilon(\epsilon_1)$. Let us show that (78) holds for any such b:

$$b^T(x^K - y^{k,i}) = (r_k - a_k - y_k^{k,i})\epsilon_k + \sum_{m=k+1}^{n}(x_m^K - y_m^{k,i})\epsilon_m + \sum_{m=k}^{n}(x_m^K - y_m^{k,i})l_m. \tag{80}$$

Note that $|l_k| \leq \delta(\epsilon_1)$ and $|x_m^K - y_m^{k,i}| \leq 2r_1$:

$$\begin{aligned} b^T(x^K - y^{k,i}) &\geq (r_k - a_k - y_k^{k,i})\epsilon_k - n2r_1 \max_{m=k+1,n} \epsilon_m - n2r_1\delta(\epsilon_1) \\ &\geq d\epsilon_k - n2r_1\frac{\epsilon_k d}{4nr_1} - n2r_1\frac{\epsilon_k d}{8nr_1} \\ &= \frac{d\epsilon_k}{4} > 0. \end{aligned} \tag{81}$$

Analogously,

$$\begin{aligned} b^T(z^{k,i} - x^K) &\geq [z_k^{k,i} - (r_k - a_k)]\epsilon_k + \sum_{m=k+1}^{n}(z_m^{k,i} - x_m^K)\epsilon_m \\ &\quad + \sum_{m=k}^{n}(z_m^{k,i} - x_m^K)l_m \\ &\geq d\epsilon_k - n2r_1 \max_{m=k+1,n} \epsilon_m - n2r_1\delta(\epsilon_1) \\ &\geq \frac{d\epsilon_k}{4} > 0. \end{aligned} \tag{82}$$

Thus, the order of selecting the nearest neighbors on the surface of $B_{r_1}(\theta)$ is the same for θ and any $\theta + b$ if $b \in B_{\delta(\epsilon_1)}(\epsilon(\epsilon_1))$ for any $\epsilon_1 > 0$. Making ϵ_1 sufficiently small, we can guarantee that all θ^j satisfying $\|\theta^j - \theta\| < r_1$ will be chosen as the nearest neighbors for $\theta + b$ before the vectors on the surface of $B_{r_1}(\theta)$ and that θ^j satisfying $\|\theta^j - \theta\| > r_1$ will not be chosen at all. For any $\epsilon_1 > 0$, $B_{\delta(\epsilon_1)}(\theta + \epsilon(\epsilon_1))$ will contain rational numbers. Letting a positive sequence $\{\epsilon_1^m\}$ go to zero and choosing $q_m \in B_\delta(\theta + \epsilon^m) \cap Q$ will give the sought sequence $\{q_m\}$. *Q.E.D.*

PROPOSITION 4: *If Θ and S are compact, $u(s, d; \theta)$ is continuous in (s, θ), and $f(s' \mid s, d; \theta)$ is continuous in (θ, s, s'), then $V(s; \theta)$ and $E\{V(s'; \theta) \mid s, d; \theta\}$ are continuous in (θ, s).*

PROOF: The proof of the proposition follows closely the standard proof of the continuity of value functions with respect to the state variables (see, for example, Chapters 3 and 4 of Stokey and Lucas (1989)). Let us consider the Bellman operator Γ on the Banach space of bounded functions B with sup norm: $V : \Theta \times S \to X$, where X is a bounded subset of R:

$$\Gamma(V)(s;\theta) = \max_d \left\{ u(s,d;\theta) + \beta \int V(s';\theta) f(s' \mid s,d;\theta)\, ds' \right\}.$$

Blackwell's sufficient conditions for contraction are satisfied for this operator, so Γ is a contraction mapping on B. The set of continuous functions C is a closed subset in B. Thus, it suffices show that $\Gamma(C) \subset C$ (this trivially implies that the fixed point of Γ is a continuous function).

Let $V(s;\theta)$ be a continuous function in B ($V \in C$). Let us show that $\Gamma(V)$ is also continuous:

$$\begin{aligned}
(83)\qquad & |\Gamma(V)(s_1;\theta_1) - \Gamma(V)(s_2;\theta_2)| \\
&\le \max_d \bigg| u(s_1,d;\theta_1) - u(s_2,d;\theta_2) \\
&\quad + \beta \int V(s';\theta_1) f(s' \mid s_1,d;\theta_1)\, ds' \\
&\quad - \beta \int V(s';\theta_2) f(s' \mid s_2,d;\theta_2)\, ds' \bigg| \\
&\le \max_d |u(s_1,d;\theta_1) - u(s_2,d;\theta_2)| \\
&\quad + \beta \max_d \bigg| \int [V(s';\theta_1) f(s' \mid s_1,d;\theta_1) \\
&\quad - V(s';\theta_2) f(s' \mid s_2,d;\theta_2)]\, ds' \bigg|.
\end{aligned}$$

Given $\epsilon > 0$, there exists $\delta_1 > 0$ such that $\|(s_1;\theta_1) - (s_2;\theta_2)\| < \delta_1$ implies $\max_d |u(s_1,d;\theta_1) - u(s_2,d;\theta_2)| < \epsilon/2$:

$$\begin{aligned}
(84)\qquad & \left| \int [V(s';\theta_1) f(s' \mid s_1,d;\theta_1) - V(s';\theta_2) f(s' \mid s_2,d;\theta_2)]\, ds' \right| \\
&\le \max_d \sup_{s'} \left| V(s';\theta_1) f(s' \mid s_1,d;\theta_1) - V(s';\theta_2) f(s' \mid s_2,d;\theta_2) \right| \cdot \lambda(S).
\end{aligned}$$

Since $V(s';\theta)f(s'\mid s,d;\theta)$ is continuous on compact $\Theta\times S\times S$, for $\epsilon>0$ there exists $\delta_2^d>0$ such that $\|(s_1,s';\theta_1)-(s_2,s';\theta_2)\|=\|(s_1;\theta_1)-(s_2;\theta_2)\|<\delta_2^d$ implies

$$\sup_{s'}|V(s';\theta_1)f(s'\mid s_1,d;\theta_1)-V(s';\theta_2)f(s'\mid s_2,d;\theta_2)|<\frac{\epsilon}{2\lambda(S)}.$$

Thus, for $\delta=\min\{\delta_1,\min_d\delta_2^d\}$, $\|(s_1;\theta_1)-(s_2;\theta_2)\|<\delta$ implies $|\Gamma(V)(s_1;\theta_1)-\Gamma(V)(s_2;\theta_2)|<\epsilon$. So $\Gamma(V)$ is a continuous function. The continuity of $E\{V(s';\theta)\mid s,d;\theta\}$ follows from the continuity of $V(s';\theta)$ by an analogous argument. *Q.E.D.*

PROPOSITION 5: *A family of functions $X(\omega^{t-1},\theta,s,d,m)$ defined in* (18) *is equicontinuous on $\Theta\times S$ if Θ and S are compacts, $V(s;\theta)$ and $E[V(s';\theta)\mid s,d;\theta]$ are continuous in (θ,s), and $f(s'\mid s,d;\theta)/g(s')$ is continuous in (θ,s,s') and satisfies Assumption* 4.

PROOF: Let us introduce the following notation shortcuts: T will denote the number of terms in the sum defining $X(\omega^{t-1},\theta,s,d,m)$. Consider two arbitrary points (θ_1,s_1) and (θ_2,s_2), let $V_j^i=V(s^j;\theta_i)-EV(\theta_i,s_i)$ and

$$W_j^i=\frac{f_j^i/g_j^i}{\sum f_k^i/g_k^i}=\frac{f(s^j\mid s_i,d;\theta_i)/g(s^j)}{\sum f(s^k\mid s_i,d;\theta_i)/g(s^k)}.$$

Then

$$\begin{aligned}
&|X(\omega^{t-1},\theta_1,s_1,d,m)-X(\omega^{t-1},\theta_2,s_2,d,m)|\\
&\quad=\left|\sum_{j=1}^T V_j^1W_j^1-\sum_{j=1}^T V_j^2W_j^2\pm\sum_{j=1}^T V_j^2W_j^1\right|\\
&\quad\le\left|\sum_{j=1}^T (V_j^1-V_j^2)W_j^1\right| &&(85)\\
&\qquad+\left|\sum_{j=1}^T V_j^2(W_j^1-W_j^2)\right|. &&(86)
\end{aligned}$$

By the proposition's hypothesis, $V(s;\theta)$ and $E[V(s';\theta)\mid s,d;\theta]$ are continuous in (θ,s) on a compact set. Thus, given $\epsilon>0$ $\exists\delta_1>0$ such that

$$\|(\theta_1,s_1,s^j)-(\theta_2,s_2,s^j)\|=\|(\theta_1,s_1)-(\theta_2,s_2)\|<\delta_1$$

implies $|V(s^j;\theta_1) - EV(\theta_1, s_1) - (V(s^j;\theta_2) - EV(\theta_2, s_2))| < \epsilon/2$. Since the weights sum to 1, (85) is bounded above by $\epsilon/2$. Let us similarly bound (86):

$$
\begin{aligned}
(87)\qquad & \left|\sum_{j=1}^{T} V_j^2 (W_j^1 - W_j^2)\right| \\
&= \left|\sum_{j=1}^{T} V_j^2 \left(\frac{f_j^1/g_j^1}{\sum f_k^1/g_k^1} - \frac{f_j^2/g_j^2}{\sum f_k^2/g_k^2}\right)\right| \\
&= \Big|\Big(\Big(\sum f_k^2/g_k^2\Big)\Big[\sum V_j^2 (f_j^1/g_j^1 - f_j^2/g_j^2)\Big] \\
&\quad + \Big[\sum f_k^2/g_k^2 - \sum f_k^1/g_k^1\Big]\Big(\sum V_j^2 f_j^2/g_j^2\Big)\Big) \\
&\quad \Big/ \Big(\sum f_k^1/g_k^1 \cdot \sum f_k^2/g_k^2\Big)\Big| \\
&\le \frac{\overline{V} \cdot \max_j |f_j^1/g_j^1 - f_j^2/g_j^2| \cdot T}{\underline{f} T} + \frac{T \cdot \max_j |f_j^1/g_j^1 - f_j^2/g_j^2| \cdot \overline{V} \cdot \overline{f} \cdot T}{\underline{f}^2 T^2} \\
&\le \max_j \left|\frac{f_j^1}{g_j^1} - \frac{f_j^2}{g_j^2}\right| \cdot \overline{V}\left(\frac{1}{\underline{f}} + \frac{\overline{f}}{\underline{f}^2}\right),
\end{aligned}
$$

where $\overline{f}$ and $\underline{f}$ are the upper and lower bounds on f/g introduced in Assumption 4, and $\overline{V} < \infty$ is an upper bound on V_j^i. Since $f(s' \mid s, d; \theta)/g(s')$ is continuous in (θ, s, s') on compact $\Theta \times S \times S$, for any $\epsilon > 0$, there exists $\delta_2 > 0$ such that $\|(\theta_1, s_1, s^j) - (\theta_2, s_2, s^j)\| = \|(\theta_1, s_1) - (\theta_2, s_2)\| < \delta_2$ implies

$$
\left|\frac{f(s^j \mid s_1, d; \theta_1)}{g(s^j)} - \frac{f(s^j \mid s_2, d; \theta_2)}{g(s^j)}\right| < \frac{\epsilon/2}{\|V\|\left(\dfrac{1}{\underline{f}} + \dfrac{\overline{f}}{\underline{f}^2}\right)} \qquad \forall j.
$$

Thus, (86) is also bounded above by $\epsilon/2$. For given $\epsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$. Then $\|(\theta_1, s_1) - (\theta_2, s_2)\| < \delta$ implies $|X(\omega^{t-1}, \theta_1, s_1, d, m) - X(\omega^{t-1}, \theta_1, s_1, d, m)| < \epsilon/2 + \epsilon/2 = \epsilon$. *Q.E.D.*

PROPOSITION 6: *Assume that in the DP solving algorithm, the same random grid over the state space is used at each iteration: $s^{m_1,j} = s^{m_2,j} = s^j$ for any m_1, m_2, and j, where $s^j \overset{\text{i.i.d.}}{\sim} g(\cdot)$. If the number of the nearest neighbors is constant, γ_2 in Assumption 6 is equal to zero and $\tilde{N}(t) = \tilde{N}$, then all the theoretical results proved in the paper will hold.*

PROOF: Only the proof of Lemma 1 is affected by the change, since in the other parts, I use only one fact about the weights in the importance sampling: the weights are in [0, 1]. Thus let us show that Lemma 1 holds.

In Lemma 1, the terms in the sum (19) corresponding to the same s^j should be grouped into one term multiplied by the number of such terms:

$$
\begin{aligned}
(88)\qquad & P(X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}) \\
&= P\Bigg[\Bigg|\sum_{j=1}^{\hat{N}(\max\{m_i\})} \big(M_j(t,m)(V(s^j;\theta) - E[V(s';\theta)\mid s,d;\theta]) \\
&\qquad \times f(s^j\mid s,d;\theta)/g(s^j)\big) \\
&\qquad \Big/\Bigg(\sum_{r=1}^{\hat{N}(\max\{m_r\})} M_r(t,m) f(s^r\mid s,d;\theta)/g(s^r)\Bigg)\Bigg| > \tilde{\epsilon}\Bigg] \\
&\le P\Bigg[\Bigg|\sum_{j=1}^{\hat{N}(\max\{m_i\})} M_j(t,m)(V(s^j;\theta) - E[V(s';\theta)\mid s,d;\theta]) \\
&\qquad \times \frac{f(s^j\mid s,d;\theta)}{g(s^j)}\Bigg| > \tilde{\epsilon}\underline{f}\hat{N}(\max\{m_i\})\Bigg],
\end{aligned}
$$

where $M_j(t,m) \in \{1,\dots,\tilde{N}(t)\}$ denotes the number of the terms corresponding to s^j and $\hat{N}(\max\{m_r\})$ is the largest grid size. The inequality above follows since

$$
\sum_{j=1}^{\hat{N}(\max\{m_i\})} M_j(t,m) f(s^j\mid s,d;\theta)/g(s^j) \ge \underline{f}\hat{N}(\max\{m_i\}).
$$

The summands in (88) are bounded by $(\tilde{N}a, \tilde{N}b)$, where a and b were defined in Lemma 1. Application of Hoeffding's inequality to (88) gives

$$
\begin{aligned}
(89)\qquad P(X(\omega^{t-1},\theta,s,d,m) > \tilde{\epsilon}) &\le 2\exp\{-4\delta\tilde{N}\hat{N}(\max\{m_i\})\} \\
&\le 2\exp\{-4\delta\tilde{N}\hat{N}(t-N(t))\},
\end{aligned}
$$

where $0 < \delta = \tilde{\epsilon}^2\underline{f}^2/(2(b-a)^2\tilde{N}^3)$. The rest of the argument follows the steps in Lemma 1 starting after (22). *Q.E.D.*

PROOF OF COROLLARY 1: Given the assumptions made in the first part of this proposition, the proofs of Lemma 1 and its uniform extension Lemma 4 apply without any changes. The rest of the results are not affected at all. *Q.E.D.*

PROOF OF COROLLARY 2: If (14) is used for approximating the expectations, then in the proofs of Lemmas 1 and 4 let us separate the expression for $X(\cdot)$ into $K = \text{card}(S_f(s_f, d))$ terms corresponding to each possible future discrete state:

$$
\begin{aligned}
(90)\qquad & X(\omega^{t-1}, \theta, s, d, m) \\
&= f(s'_{f,1} \mid s_f, d; \theta) \\
&\quad \times \left\{ \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} \frac{V^{k_i}(s'_{f,1}, s^{k_i,j}; \theta^{k_i}) f(s^{k_i,j} \mid s_c, d; \theta)/g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\hat{N}(k_r)} f(s^{k_r,q} \mid s_c; \theta)/g(s^{k_r,q})} \right. \\
&\quad \left. - E[V(s'; \theta) \mid s'_f = s'_{f,1}, s_c, d; \theta] \right\} + \cdots \\
&\quad + f(s'_{f,K} \mid s_f, d; \theta) \\
&\quad \times \left\{ \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} \frac{V^{k_i}(s'_{f,K}, s^{k_i,j}; \theta^{k_i}) f(s^{k_i,j} \mid s_c, d; \theta)/g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\hat{N}(k_r)} f(s^{k_r,q} \mid s_c; \theta)/g(s^{k_r,q})} \right. \\
&\quad \left. - E[V(s'; \theta) \mid s'_f = s'_{f,K}, s_c, d; \theta] \right\}.
\end{aligned}
$$

Then, applying the argument from Lemmas 1 and 4, we can bound the probabilities for $k = 1, \dots, K$,

$$
(91)\qquad P\left[\left| f(s'_{f,k} \mid s_f, d; \theta) \sum_{i=1}^{\tilde{N}(m)} \sum_{j=1}^{\hat{N}(k_i)} \frac{V^{k_i}(s'_{f,k}, s^{k_i,j}; \theta^{k_i}) f(s^{k_i,j} \mid s_c, d; \theta)/g(s^{k_i,j})}{\sum_{r=1}^{\tilde{N}(m)} \sum_{q=1}^{\hat{N}(k_r)} f(s^{k_r,q} \mid s_c; \theta)/g(s^{k_r,q})} \right. \right.
$$

$$
(92)\qquad \left. \left. - E[V(s'; \theta) \mid s'_f = s'_{f,k}, s_c, d; \theta] \right| > \frac{\epsilon}{K} \right],
$$

and Lemmas 1 and 4 will hold. The proofs of the other lemmas are not affected at all, since the weights on the value functions in expectation approximations are still nonnegative and sum to 1. *Q.E.D.*

PROPOSITION 7: *If x_t, z_t, and y_t are integer sequences with* $\lim_{t\to\infty} y_t/z_t = 0$, $\lim_{t\to\infty} z_t = \infty$, *and* $\limsup_{t\to\infty} z_t/x_t < \infty$, *then* $\forall\delta > 0$ $\exists T$ *such that* $\forall t > T$,

$$e^{-\delta x_t}\frac{z_t!}{(z_t - y_t)!y_t!} \le e^{-0.5\delta x_t}.$$

PROOF: To prove the inequality let us work with the logarithm of the left hand side

$$\begin{aligned}
(93)\qquad & \log\left[e^{-\delta x_t}\frac{z_t!}{(z_t - y_t)!y_t!}\right] \\
&= -\delta x_t + \sum_{i=z_t-y_t+1}^{z_t}\log(i) - \sum_{i=1}^{y_t}\log(i) \\
&\le -\delta x_t + \int_{z_t-y_t+1}^{z_t+1}\log(i)\,di - \int_1^{y_t}\log(i)\,di \\
&= -\delta x_t + (z_t+1)\log(z_t+1) - (z_t - y_t + 1)\log(z_t - y_t + 1) \\
&\quad - [(z_t+1) - (z_t - y_t + 1)] - \{y_t\log(y_t) - 1\log(1) - [y_t - 1]\} \\
&= -\delta x_t + z_t[\log(z_t+1) - \log(z_t - y_t + 1)] \\
&\quad + y_t[\log(z_t - y_t + 1) - \log(y_t)] \\
&\quad + \log(z_t+1) - \log(z_t - y_t + 1) - y_t\log(y_t) - 1 \\
&\le -\delta x_t + z_t\log\frac{z_t+1}{z_t - y_t + 1} + y_t\log\frac{z_t - y_t + 1}{y_t} + \log\frac{z_t+1}{z_t - y_t + 1} \\
&= x_t\left[-\delta + \frac{z_t}{x_t}\log\frac{z_t+1}{z_t - y_t + 1}\right. \\
&\quad \left. + \frac{(z_t - y_t + 1)y_t}{x_t(z_t - y_t + 1)}\log\frac{z_t - y_t + 1}{y_t} + \frac{1}{x_t}\log\frac{z_t+1}{z_t - y_t + 1}\right] \\
&\le -0.5\delta x_t \quad \forall t > T.
\end{aligned}$$

There exists such T that the last inequality holds, since all the terms in (93) converge to zero. Exponentiating the obtained inequality completes the proof. *Q.E.D.*

PROPOSITION 8: *For any* $\delta_1 > 0$ *and* $\delta_2 > 0$, *there exist* $\delta > 0$ *and* T *such that* $\forall t > T$,

$$(94)\qquad \sum_{i=t-m(t)N(t)}^{t-1}\hat{N}(i)\left[e^{-\delta_1\tilde{N}(i)\hat{N}(i-N(i))} + e^{-\delta_2(N(i)-\tilde{N}(i))}\right] \le \exp\{-\delta t^{\gamma_0\gamma_1}\}.$$

PROOF: The following inequalities will be used below:

$$t - m(t)N(t) \geq t - \frac{t - t^{\gamma_0}}{N(t)} N(t) = t^{\gamma_0}, \tag{95}$$

$$t - m(t)N(t) \leq t - \left(\frac{t - t^{\gamma_0}}{N(t)} - 1\right) N(t) \leq t^{\gamma_0} + t^{\gamma_1} < 2t^{\gamma_0}, \tag{96}$$

$$\begin{aligned} \tilde{N}(t - m(t)N(t)) &= \big[(t - m(t)N(t))^{\gamma_2}\big] \\ &\geq [t^{\gamma_0\gamma_2}] \geq t^{\gamma_0\gamma_2} - 1 \geq 0.5t^{\gamma_0\gamma_2} \quad \forall t > T_1 = 2^{1/(\gamma_0\gamma_2)}, \end{aligned} \tag{97}$$

$$N(t - m(t)N(t)) = \big[(t - m(t)N(t))^{\gamma_1}\big] \leq (2t^{\gamma_0})^{\gamma_1} \leq 2^{\gamma_1} t^{\gamma_0\gamma_1}, \tag{98}$$

$$\begin{aligned} &\hat{N}\big(t - m(t)N(t) - N(t - m(t)N(t))\big) \\ &\quad = \big[\big(t - m(t)N(t) - N(t - m(t)N(t))\big)^{\gamma_1 - \gamma_2}\big] \\ &\quad \geq (t^{\gamma_0} - 2^{\gamma_1} t^{\gamma_0\gamma_1})^{\gamma_1 - \gamma_2} - 1 \quad \text{(by (95) and (98))} \\ &\quad \geq \frac{t^{\gamma_0(\gamma_1 - \gamma_2)}}{2^{\gamma_1 - \gamma_2}} - 1 \quad (\forall t > 2^{(1+\gamma_1)/(\gamma_0(1-\gamma_1))}) \\ &\quad \geq \frac{t^{\gamma_0(\gamma_1 - \gamma_2)}}{2^{1+\gamma_1 - \gamma_2}} \quad (\forall t > T_2), \end{aligned} \tag{99}$$

where $T_2 = \max\{2^{(1+\gamma_1-\gamma_2)/(\gamma_0(\gamma_1-\gamma_2))}, 2^{(1+\gamma_1)/(\gamma_0(1-\gamma_1))}\}$.

Combining (97) and (99) gives

$$\begin{aligned} &\exp\big\{-\delta_1 \tilde{N}(t - m(t)N(t))\hat{N}\big(t - m(t)N(t) - N(t - m(t)N(t))\big)\big\} \\ &\quad \leq \exp\left\{-\frac{\delta_1 t^{\gamma_0\gamma_1}}{2^{2+\gamma_1-\gamma_2}}\right\} = \exp\{-\tilde{\delta}_1 t^{\gamma_0\gamma_1}\}, \end{aligned} \tag{100}$$

where the last equality defines $\tilde{\delta}_1 > 0$, and

$$\begin{aligned} &N(t - m(t)N(t)) - \tilde{N}(t - m(t)N(t)) \\ &\quad = \big[(t - m(t)N(t))^{\gamma_1}\big] - \big[(t - m(t)N(t))^{\gamma_2}\big] \\ &\quad \geq [t^{\gamma_0\gamma_1}] - [2^{\gamma_2} t^{\gamma_0\gamma_2}] \quad \text{(by (95) and (96))} \\ &\quad \geq t^{\gamma_0\gamma_1} - 1 - 2^{\gamma_2} t^{\gamma_0\gamma_2} \\ &\quad \geq 0.5t^{\gamma_0\gamma_1} \quad \text{for } t \text{ larger than some } T_3, \end{aligned} \tag{101}$$

where such T_3 exists since $(0.5t^{\gamma_0\gamma_1} - 1 - 2^{\gamma_2} t^{\gamma_0\gamma_2}) \to \infty$.

Taking an upper bound on summands in (94) and multiplying it by the number of terms in the sum gives the upper bound on the sum:

$$\begin{aligned}(102)\qquad &\sum_{i=t-m(t)N(t)}^{t-1} \hat{N}(i)\big[e^{-\delta_1 \tilde{N}(i)\hat{N}(i-N(i))} + e^{-\delta_2(N(i)-\tilde{N}(i))}\big] \\ &\leq \big((t-1)-(t-m(t)N(t))+1\big) \times \hat{N}(t-1) \\ &\quad \times \big[e^{-\delta_1 \tilde{N}(t-m(t)N(t))\hat{N}(t-m(t)N(t)-N(t-m(t)N(t)))} \\ &\quad + e^{-\delta_2(N(t-m(t)N(t))-\tilde{N}(t-m(t)N(t)))}\big].\end{aligned}$$

Inequalities in (100), (101), and (102) imply

$$\begin{aligned}(103)\qquad &\sum_{i=t-m(t)N(t)}^{t-1} \hat{N}(i)\big[e^{-\delta_1 \tilde{N}(i)\hat{N}(i-N(i))} + e^{-\delta_2(N(i)-\tilde{N}(i))}\big] \\ &\leq t^{1+\gamma_1-\gamma_2}(\exp\{-\tilde{\delta}_1 t^{\gamma_0\gamma_1}\} + \exp\{-0.5\delta_2 t^{\gamma_0\gamma_1}\}) \\ &\leq 2t^{1+\gamma_1-\gamma_2}\exp\{-\min(\tilde{\delta}_1, 0.5\delta_2)t^{\gamma_0\gamma_1}\},\end{aligned}$$

where $\tilde{\delta}_1$ was defined in (100).

Note that $(2t^{1+\gamma_1-\gamma_2}\exp\{-0.5\min(\tilde{\delta}_1, 0.5\delta_2)t^{\gamma_0\gamma_1}\}) \to \infty$ and therefore $\exists T \geq \max(T_1, T_2, T_3)$ such that $\forall t > T$,

$$(104)\qquad 2t^{1+\gamma_1-\gamma_2}\exp\{-\min(\tilde{\delta}_1, 0.5\delta_2)t^{\gamma_0\gamma_1}\} \leq \exp\{-\delta t^{\gamma_0\gamma_1}\},$$

where $\delta = 0.5\min(\tilde{\delta}_1, 0.5\delta_2)$. This completes the proof. *Q.E.D.*

PROPOSITION 9: *For any $a > 0$ and $\delta > 0$, $\sum_{t=1}^{\infty}\exp\{-\delta t^a\} < \infty$.*

PROOF—Sketch: The sum above is a lower sum for the improper integral $\int_0^\infty \exp\{-\delta t^a\}\,dt$. One way to show that it is finite is to do a transformation of variables $y = t^a$ and then bound the obtained integral by an integral of the form $\int_0^\infty y^n \exp\{-\delta y\}\,dy$, where n is an integer. It follows by induction and integration by parts that this integral is finite. *Q.E.D.*

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