Debiased machine learning of conditional average treatment effects and other causal functions

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Summary: This paper provides estimation and inference methods for the best linear predictor (approximation) of a structural function, such as conditional average structural and treatment effects, and structural derivatives, based on modern machine learning tools. We represent this structural function as a conditional expectation of an unbiased signal that depends on a nuisance parameter, which we estimate by modern machine learning techniques. We first adjust the signal to make it insensitive (Neyman-orthogonal) with respect to the first-stage regularisation bias. We then project the signal onto a set of basis functions, which grow with sample size, to get the best linear predictor of the structural function. We derive a complete set of results for estimation and simultaneous inference on all parameters of the best linear predictor, conducting inference by Gaussian bootstrap. When the structural function is smooth and the basis is sufficiently rich, our estimation and inference results automatically target this function. When basis functions are group indicators, the best linear predictor reduces to the group average treatment/structural effect, and our inference automatically targets these parameters. We demonstrate our method by estimating uniform confidence bands for the average price elasticity of gasoline demand conditional on income.

Keywords: *High-dimensional statistics*, *heterogeneous treatment effect*, *conditional average treatment effect*, *group average effects*, *debiased/orthogonal estimation*, *machine learning*, *double robustness*, *continuous treatment effects*, *dose–response functions*.

JEL codes: *C14*, *C55*.

1. INTRODUCTION AND MOTIVATION

This paper gives a method for estimating and conducting inference on a nonparametric function $g(x)$ that summarises heterogeneous treatment/causal/structural effects conditional on a small set of covariates X. We assume that this function can be represented as a conditional expectation function

$$g(x) = \mathbb{E}[Y(\eta_0)|X = x], \tag{1.1}$$

where the random variable $Y(\eta_0)$, which we refer to as a signal, depends on a nuisance function $\eta_0(z)$ of a (potentially very) high-dimensional control vector Z. Examples of the nonparametric

target function include the conditional average treatment effect (CATE), continuous treatment effects (CTEs), as well as many others discussed below. Examples of the nuisance functions

$$\eta_0 = \eta_0(z)$$

include the propensity score, the conditional density, and the regression function, among others. In summary,

$$\dim(Z) \text{ is high}; \quad \dim(X) \text{ is low}.$$

Although there are many possible choices of signals $Y(\eta)$, we focus on signals that have the orthogonality property (Neyman, 1959). Formally, we require the pathwise derivative of the conditional expectation to be zero conditional on X:

$$\partial_r \mathbb{E}[Y(\eta_0 + r(\eta - \eta_0))|X = x]|_{r=0} = 0, \quad \text{for all } x \text{ and } \eta. \tag{1.2}$$

If the signal $Y(\eta)$ is orthogonal, its plug-in estimate $Y(\widehat{\eta})$ is insensitive to bias in the estimation of $\widehat{\eta}$, which results from applying modern adaptive learning methods in high dimensions. Under mild conditions, $Y(\widehat{\eta})$ delivers a high-quality estimator of the target function $g(x)$.

We demonstrate the importance of the orthogonality property for CTEs, studied in Imbens (2000), Gill and Robins (2001), and Kennedy et al. (2017). Let $X \in \mathbb{R}$ be a one-dimensional continuous treatment, and let Y^x be the potential outcome corresponding to the subject's response after receiving x units of treatment. The observed data vector $V = (X, Z, Y)$ consists of the treatment X, the control vector Z, and the observed outcome $Y = Y^X$. If potential outcomes $\{Y^x, x \in \mathbb{R}\}$ are independent of treatment X conditional on controls Z, the average potential outcome is identified as

$$\mathbb{E}[Y^x] = \mathbb{E}\mu_0(x, Z) = \int \mu_0(x, z) d P_Z(z), \tag{1.3}$$

where $\mu_0(x, z) = \mathbb{E}[Y|X = x, Z = z]$ is the regression function of the observed outcome. Because the control vector Z is high dimensional, it is necessary to estimate the regression function $\mu_0(x, z)$ with some regularised technique to achieve convergence.

A naive approach to estimate $\mathbb{E}[Y^x]$ is to consider a sample average

$$\widetilde{g}(x) = \int \widehat{\mu}(x, z) d\widehat{P}_Z(z),$$

where $\widehat{\mu}(x, z)$ is a regularised estimate of $\mu_0(x, z)$, and $\widehat{P}_Z$ is the empirical analog of P_Z. This approach results in a biased estimate, and the bias of estimation error, $\widehat{\mu}(x, Z) - \mu_0(x, Z)$, does not vanish faster than $N^{-1/2}$. The plug-in estimator inherits this first-order bias, because the moment equation (1.3) is not orthogonal to perturbations of μ:

$$\partial_r \mathbb{E}[(\mu_0 + r(\mu - \mu_0)) \circ (x, Z)]|_{r=0} = \mathbb{E}[\mu(x, Z) - \mu_0(x, Z)] \neq 0.$$

This bias implies that the plug-in estimator $\widetilde{g}$ will not converge at the optimal rate.

To deliver a high-quality estimate of $\mathbb{E}[Y^x]$, we represent

$$g(x) = \mathbb{E}[Y^x]$$

as a special case of the signal framework (1.1). We choose the signal $Y(\eta)$ to be the doubly robust signal from Kennedy et al. (2017):

$$Y(\eta) := \frac{Y - \mu(X, Z)}{s(X|Z)} w(X) + \int \mu(X, z) d P_Z(z), \tag{1.4}$$

where the nuisance parameter

$$\eta_0(x, z) = \{s_0(x|z), \mu_0(x, z), w_0(x)\}$$

consists of the regression function $\mu_0(x, z)$, the conditional density $s_0(x|z)$ of X given Z, and the marginal treatment density $w_0(x)$. This procedure is more costly because the nuisance parameter now includes two more functions, $s_0(x|z)$ and $w_0(x)$, in addition to $\mu_0(x, z)$. However, the signal in (1.4) has the benefit of being conditionally orthogonal with respect to each nuisance function in $\eta_0(x, z)$:

$$\mathbb{E}\left[\begin{array}{c} -\int_{z\in\mathcal{Z}}(\mu(X, z) - \mu_0(X, z))dP_Z(z) + \int_{z\in\mathcal{Z}}(\mu(x, z) - \mu_0(x, z))dP_Z(z) \\[2mm] \dfrac{\mu_0(X, Z) - Y}{s_0^2(X|Z)}(s(X|Z) - s_0(X|Z)) \\[2mm] \dfrac{Y - \mu_0(X, Z)}{s_0(X|Z)}(w(X) - w_0(X)) \end{array}\middle| X = x\right] = 0.$$

Because this signal is conditionally orthogonal to the nuisance function, the bias of the estimation error, $\widehat{\eta}(X, Z) - \eta_0(X, Z)$, does not create first-order bias in the estimated signal $Y(\widehat{\eta})$ and affects only its higher-order bias. As a result, the estimate of the target function based on $Y(\widehat{\eta})$ is high quality under plausible conditions.

In the second stage, we consider a linear projection of an orthogonal signal $Y(\eta)$ onto a vector of basis functions $p(X)$,

$$\beta := \arg\min_{b\in\mathbb{R}^d} \mathbb{E}(Y(\eta) - p(X)'b)^2.$$

The choice of basis functions depends on the desired interpretation of the linear approximation. For example, consider partitioning the support of X into d mutually exclusive groups $\{G_k\}_{k=1}^d$. Setting

$$p_k(x) = \mathbf{1}\{x \in G_k\}, \quad k \in \{1, 2, \ldots, d\}$$

implies that $p(x)'\beta_0$ is a group average treatment effect for group k such that $x \in G_k$. Our inference will target this parameter, allowing the number of groups to increase at some rate.

For another example, let $p(x) \in \mathbb{R}^d$ be a d-dimensional dictionary of series/sieve basis functions, e.g., polynomials, splines, or wavelets. Then, $p(x)'\beta_0$ corresponds to the best linear approximation to the target function $g(x)$ in the given dictionary. Under some smoothness conditions, as the dimension of the dictionary becomes large, $p(x)'\beta_0$ will approximate $g(x)$, and our inference will target this function. We derive a complete set of results for estimation and simultaneous inference on all parameters of the best linear predictor, conducting inference by Gaussian bootstrap. When the structural function is smooth and the basis is sufficiently rich, our estimation and inference results automatically target this function. When basis functions are group indicators, the best linear predictor reduces to the group average treatment/structural effect, and our inference automatically targets these parameters.

1.1. Literature review

This paper builds on three bodies of research within the semiparametric literature: orthogonal (debiased) machine learning, least-squares series estimation, and treatment effects / missing data problems. Orthogonal machine learning (Chernozhukov et al., 2016; Chernozhukov et al.,

2018) proposes inference on a fixed-dimensional target parameter β_0 in the presence of a high-dimensional nuisance function η in a semiparametric moment problem. If the moment condition is orthogonal to perturbations of η, estimating η by machine learning methods has no first-order effect on the asymptotic distribution of the target parameter β_0. In particular, plugging in an estimate of η obtained on a separate sample results in a $\sqrt{N}$-consistent asymptotically normal estimate whose asymptotic variance is the same as if the econometrician knew $\eta = \eta_0$. This result makes it possible to use highly complex machine learning methods to estimate the nuisance function η, such as ℓ_1 penalised methods in sparse models (Bühlmann and van der Geer, 2011; Belloni et al., 2016; ℓ_2 boosting in sparse linear models Luo and Spindler, 2016), and other methods for classes of neural nets, regression trees, and random forests. The present paper extends the orthogonal machine learning literature by allowing the target parameter to be a function—that is, an infinite-dimensional parameter. Next, our paper contributes to a large body of work on "debiased" inference for parameters after regularisation or model selection (Belloni et al., 2017; Belloni et al., 2014; Belloni et al., 2016; van der Geer et al., 2014; Javanmard and Montanari, 2014; Zhang and Zhang, 2014), with the crucial difference being that our (ultimate) target parameter $g(x)$ is infinite dimensional, whereas in those papers the target parameter is finite dimensional.

The second building block of our paper is the literature on least-squares series estimation (Newey, 2007; Newey, 2009; Belloni et al., 2015; Chen and Christensen, 2015), which establishes pointwise and uniform limit theory for least-squares series estimation. We extend this theory by allowing the dependent variable of the series projection to depend on an unknown nuisance parameter η. We show that series properties continue to hold without any additional strong assumptions on the problem design.

Finally, we also contribute to the literature on estimating the CATEs and group average treatment effects with missing data (Robins and Rotnitzky, 1995; Hahn, 1998; Graham, 2011; Graham et al., 2012; Hirano et al., 2003; Abrevaya et al., 2015; Athey and Imbens, 2016; Grimmer et al., 2017; Oprescu et al., 2018, among others). After we released the working paper version of the present article (Semenova and Chernozhukov, 2018), many methods (Jacob, 2019; Fan et al., 2019; Zimmert and Lechner, 2019; Colangelo and Lee, 2020) have been proposed for estimating group, incremental, or heterogeneous treatment effects in the presence of high-dimensional controls. Our framework covers many more examples than just CATEs or CTEs and uses series estimators, as opposed to kernels, to localise the structural function.

In a related paper, Chernozhukov et al. (2017) studied CATEs in randomised control trials with a known propensity score. Recognising a widespread interest in estimating CATE by modern machine learning techniques, Chernozhukov et al. (2017) studies the best linear projection of the true CATE function onto an arbitrary machine learning estimator of CATE under consideration, constructed on an auxiliary sample. The analysis of Chernozhukov et al. (2017) does not require any assumptions on the machine learning estimator; however, that paper targets a specific feature of CATE—the best linear projection of CATE—rather than CATE itself. In contrast, our work operates in a classic observational setting, with many potential controls, and targets the true CATE function. In our setting, modern regularised methods are used to estimate the propensity score, but they are required to approximate this parameter sufficiently well. To sum up, the present paper delivers a sharper characterisation of CATE in a more challenging setting, in exchange for stronger assumptions about the first-stage machine learning estimate.

2. SETUP

2.1. Examples

In this section, we describe our main examples. For each example, we provide an orthogonal signal $Y(\eta)$ obeying (1.1)–(1.2).

EXAMPLE 2.1 (CONTINUOUS TREATMENT EFFECTS). Let $X \in \mathbb{R}$ be a continuous treatment variable, Z be a vector of the controls, Y^x stand for the potential outcomes corresponding to the subject's response after receiving x units of treatment, and $Y = Y^X$ be the observed outcome. The observed data V is $V = (X, Z, Y)$. For a given value x, the target function is the average potential outcome

$$g(x) = \mathbb{E}[Y^x].$$

A standard way to identify the function $g(x)$ is to assume unconfoundedness. Suppose all of the potential outcomes $\{Y^x, x \in \mathbb{R}\}$ are independent of X conditional on Z,

$$\{Y^x, x \in \mathbb{R}\} \perp X|Z.$$

Then, $g(x)$ is identified as

$$g(x) = \mathbb{E}\mu_0(x, Z),$$

where $\mu_0(x, z) = \mathbb{E}[Y|X = x, Z = z]$ is the regression function. Lemma 4.3 shows that the doubly robust signal from Kennedy et al. (2017),

$$Y(\eta) := \frac{Y - \mu(X, Z)}{s(X|Z)} w(X) + \int \mu(X, z) dP_Z(z), \qquad (2.5)$$

is conditionally orthogonal with respect to the nuisance parameter

$$\eta_0(x, z) := \{s_0(x|z), \mu_0(x, z), w_0(x)\},$$

consisting of the conditional treatment density (a.k.a. generalised propensity score)

$$s_0(x|z) = \left. \frac{dP(X \le t|Z = z)}{dt} \right|_{t=x},$$

regression function $\mu_0(x, z) = \mathbb{E}[Y|X = x, Z = z]$, and the marginal treatment density

$$w_0(x) = \left. \frac{d\mathrm{P}(X \le t)}{dt} \right|_{t=x} = \mathbb{E}_Z s_0(x|Z).$$

Theorems 4.6 and 4.7 establish pointwise and uniform asymptotic normality for the orthogonal estimator of CTEs.

In the examples below, the vector X represents a low-dimensional subset of potential controls Z.

EXAMPLE 2.2 (CONDITIONAL AVERAGE TREATMENT EFFECT). Let Y_1 and Y_0 be the potential outcomes corresponding to a subject's response with and without receiving a binary treatment, respectively. Let $D = 1$ be a dummy for whether a subject is treated. The object of interest is the CATE

$$g(x) := \mathbb{E}[Y_1 - Y_0|X = x].$$

Because an individual cannot be treated and untreated at the same time, the econometrician observes only the actual outcome $Y = DY_1 + (1 - D)Y_0$ but not the treatment effect $Y_1 - Y_0$.

A standard way to make progress in this problem is to assume unconfoundedness (Rosenbaum and Rubin, 1983). Suppose there exists an observable control vector Z such that treatment status D is independent of the potential outcomes Y_1, Y_0 conditional on Z,

$$Y_1, Y_0 \perp D|Z.$$

Define the conditional probability of treatment receipt as $s_0(z) = \mathrm{P}(D = 1|Z = z)$. Consider a Robins and Rotnitzky (1995)–type orthogonal signal $Y(\eta)$,

$$Y(\eta) := \mu(1, Z) - \mu(0, Z) + \frac{D[Y - \mu(1, Z)]}{s(Z)} - \frac{(1 - D)[Y - \mu(0, Z)]}{1 - s(Z)}, \qquad (2.6)$$

where $\mu_0(d, z) = \mathbb{E}[Y|D = d, Z = z]$ is the conditional expectation function of Y. Corollary 4.1 shows that (6) is orthogonal with respect to the nuisance parameter $\eta_0(z) := \{s_0(z), \mu_0(1, z), \mu_0(0, z)\}$ and establishes pointwise and uniform asymptotic theory for the orthogonal estimator of CATE.

EXAMPLE 2.3 (REGRESSION FUNCTION WITH PARTIALLY MISSING OUTCOME). Suppose a researcher is interested in the conditional expectation of a variable Y^* given X:

$$g(x) := \mathbb{E}[Y^*|X = x],$$

where Y^* is partially missing. Let $D = 1$ be a dummy for whether the outcome Y^* is observed, Z be a control vector, $Y = DY^*$ be the observed outcome, and $V = (D, Z, Y)$ be the data vector. Because the researcher does not control D, a standard way to make progress is to assume there exists an observable control vector Z such that Y^* is independent of D given Z,

$$Y^* \perp D|Z.$$

Corollary 4.2 shows that the signal $Y(\eta)$, defined as

$$Y(\eta) := \mu(Z) + \frac{D[Y - \mu(Z)]}{s(Z)}, \qquad (2.7)$$

is orthogonal with respect to the nuisance parameter

$$\eta_0(z) := \{s_0(z), \mu_0(z)\},$$

where $\mu_0(z) = \mathbb{E}[Y|Z = z, D = 1]$ is the conditional expectation function of the observed outcome Y.

EXAMPLE 2.4 (CONDITIONAL AVERAGE PARTIAL DERIVATIVE). Let $D \in \mathbb{R}$ be a continuous treatment variable, Z be a vector of the controls, Y^d stand for the potential outcomes corresponding to the subject's response after receiving d units of treatment, $Y = Y^D$ be the observed outcome, and $V = (D, Z, Y)$ be the data vector. Let X be a subvector of controls Z. The target function is the average partial derivative conditional on a covariate vector X,

$$g(x) = \partial_d \mathbb{E}[Y^D|X = x].$$

A standard way to identify the function $g(x)$ is to assume unconfoundedness. Suppose the potential outcome Y^d is independent of D conditional on Z,

$$\{Y^d, d \in \mathbb{R}\} \perp D|Z.$$

Then, $g(x)$ is identified as

$$g(x) = \mathbb{E}[\partial_d \mu_0(D, Z)|X = x],$$

where $\mu_0(d, z) = \mathbb{E}[Y|D = d, Z = z]$ is the regression function. Corollary 4.3 shows that the signal

$$Y(\eta) := -\partial_d \log s(D|Z)[Y - \mu(D, Z)] + \partial_d \mu(D, Z) \tag{2.8}$$

is orthogonal with respect to the nuisance parameter

$$\eta_0(d, z) = \{\mu_0(d, z), s_0(d|z)\},$$

where $s_0(d|z)$ is the conditional density of D given Z.

2.2. Overview of main results

The first main contribution of this paper is to provide sufficient conditions for pointwise and uniform asymptotic Gaussian approximation of the target function. We approximate the target function $g(x)$ by a linear form $p(x)'\beta_0$:

$$g(x) = p(x)'\beta_0 + r_g(x),$$

where $p(x)$ is a d-vector of basis functions of x, $r_g(x)$ is the linear approximation error, and β_0 is the best linear predictor / approximation parameter, defined by the normal equation

$$\mathbb{E}p(X)[g(X) - p(X)'\beta_0] = \mathbb{E}p(X)r_g(X) = 0.$$

We construct the *orthogonal estimator* $\widehat{\beta}$, the two-stage estimator of β_0, as follows. In the first stage, we construct an estimate $\widehat{\eta}$ of the nuisance parameter η_0, using a high-quality machine learning estimator capable of dealing with the high-dimensional covariate vector Z. In the second stage we construct an estimate $\widehat{Y}_i$ of the signal Y_i as $\widehat{Y}_i := Y_i(\widehat{\eta})$ and run ordinary least squares of $\widehat{Y}_i$ on $p(X_i)$. We use different samples to estimate η in the first stage and β_0 in the second stage in a form of cross-fitting.

DEFINITION 2.1 (CROSS-FITTING).

(1) *For a random sample of size N, denote a K-fold random partition of the sample indices $[N] = \{1, 2, \ldots, N\}$ by $(J_k)_{k=1}^K$, where K is the number of partitions, and the sample size of each fold is $n = N/K$. For each $k \in [K] = \{1, 2, \ldots, K\}$ define $J_k^c = \{1, 2, \ldots, N\}\backslash J_k$.*
(2) *For each $k \in [K]$, construct an estimator $\widehat{\eta}_k = \widehat{\eta}(V_{i \in J_k^c})$ of the nuisance parameter η_0 by using only the data $\{V_j : j \in J_k^c\}$. For any observation $i \in J_k$, define $\widehat{Y}_i := Y_i(\widehat{\eta}_k)$.*

DEFINITION 2.2 (ORTHOGONAL ESTIMATOR). *Given $(\widehat{Y}_i)_{i=1}^N$, define*

$$\widehat{\beta} := \left(\frac{1}{N}\sum_{i=1}^N p(X_i)p(X_i)'\right)^{-1} \frac{1}{N}\sum_{i=1}^N p(X_i)\widehat{Y}_i.$$

Figure 1 gives a schematic description of the orthogonal estimator. Under mild conditions on η, the orthogonal estimator delivers a high-quality estimate $p(x)'\widehat{\beta}$ of the pseudo-target function $p(x)'\beta_0$ with the following properties:

(1) With probability (w.p.) $\to 1$, the mean squared error of $p(x)'\widehat{\beta}$ is bounded by

$$\left(\frac{1}{N} \sum_{i=1}^{N} (p(X_i)'(\widehat{\beta} - \beta_0))^2 \right)^{1/2} = O_P\left(\sqrt{\frac{d}{N}} \right).$$

(2) The estimator $p(x)'\widehat{\beta}$ of the pseudo-target function $p(x)'\beta_0$ is asymptotically linear:

$$\sqrt{N} \frac{p(x)'(\widehat{\beta} - \beta_0)}{\sqrt{p(x)'\Omega p(x)}} = G_N(x) + o_P(1/\sqrt{\log N}),$$

where the empirical process $G_N(x)$ is approximated by a Gaussian process

$$G(x) = \frac{p(x)'}{\sqrt{p(x)'\Omega p(x)}} N(0, \Omega)$$

uniformly over $x \in \mathcal{X}$, and the covariance matrix Ω can be consistently estimated by a sample analog $\widehat{\Omega}$.

(3) If the misspecification error $r_g(x)$ is small, the pseudo-target function $p(x)'\beta_0$ can be replaced by the target function $g(x)$:

$$\sqrt{N} \frac{p(x)'\widehat{\beta} - g(x)}{\sqrt{p(x)'\Omega p(x)}} = G_N(x) + o_P(1/\sqrt{\log N}).$$

(4) Simultaneous inference is facilitated by Gaussian bootstrap, which relies on simulating the empirical Gaussian process:

$$G^\star(x) = \frac{p(x)'}{\sqrt{p(x)'\Omega p(x)}} N(0, \widehat{\Omega}).$$

The quantiles of the suprema of $x \mapsto G(x)$ can be consistently approximated by simulation, which makes it possible to construct uniform confidence bands for $x \mapsto p(x)'\beta$ and $x \mapsto g(x)$.

Our results accommodate high-dimensional / highly complex modern machine learning methods to estimate η, such as random forests, neural networks, and ℓ_1-shrinkage estimators, as well as procedures that estimate $\widehat{\eta}$ by classic nonparametric methods. The only requirement we impose on the estimation of $\widehat{\eta}$ is that it converges to the true nuisance parameter η_0 at a fast enough rate $o_P(N^{-1/4-\delta})$ for some $\delta \geq 0$. This requirement is satisfied under structural assumptions on η_0, such as approximate sparsity of η_0 with respect to some dictionary, or if η_0 is well approximated by trees or by sparse neural and deep neural nets. For the CATE, it is straightforward to apply our method by using the best_linear_predictor command in the *R* grf package, available from Tibshirani et al. (2017).
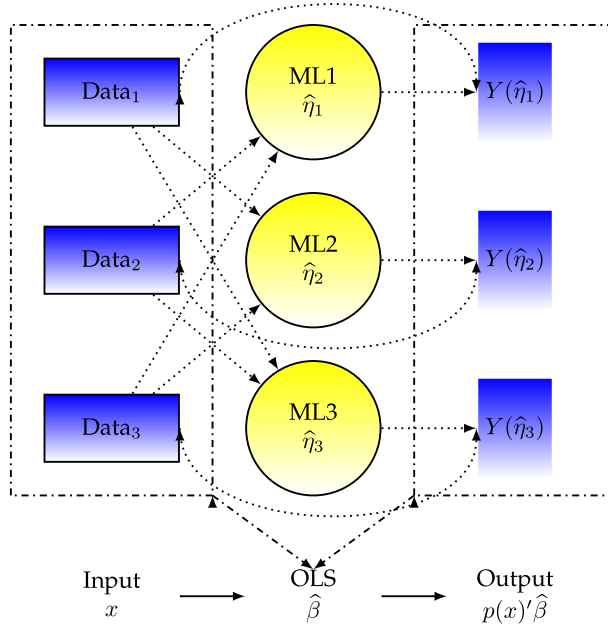
Define the covariance matrix of the basis functions as

$$Q = \mathbb{E}p(X)p(X)',$$

and define its empirical analog as

$$\widehat{Q} = \frac{1}{N} \sum_{i=1}^{N} p(X_i)p(X_i)'.$$

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Figure 1. Graphical representation of the orthogonal estimator (OE) with cross-fitting. *First stage.* The rectangles represent the partition of the data into $K = 3$ subsets. The circles represent $K = 3$ instances of the machine learning algorithm (ML), whose training sets are indicated by straight arrows. For each partition $k \in \{1, 2, 3\}$, the signal $Y(\widehat{\eta}_k)$ is estimated using the Data$_k$ and ML instance $\widehat{\eta}_k$. *Second stage.* The ordinary least squares (OLS) estimator $\widehat{\beta} = (\sum_{i=1}^{N} p(X_i) p(X_i)')^{-1} \sum_{i=1}^{N} p(X_i) Y_i(\widehat{\eta})$ is estimated on the full data set with covariate vector $p(X)$ and outcome variable $Y(\widehat{\eta})$, where $p(X)$ is a vector of series terms. For a given point of interest x (input), the OE estimator of function $g(x)$ is $p(x)'\widehat{\beta}$ (output).

In Examples 2.2 through 2.4, the asymptotic covariance matrix of the orthogonal estimator is

$$\Omega = Q^{-1} \mathbb{E} p(X) p(X)' (U + r_g(X))^2 Q^{-1},$$

and its empirical analog is

$$\widehat{\Omega} := \widehat{Q}^{-1} \mathbb{E}_N p(X_i) p(X_i)' (Y_i(\widehat{\eta}) - p(X_i)'\widehat{\beta})^2 \widehat{Q}^{-1}. \tag{2.9}$$

In the case of CTEs (Example 2.1), the asymptotic variance contains an additional component that we describe in Section 4.

DEFINITION 2.3 (POINTWISE AND UNIFORM CONFIDENCE BANDS). *Let* $\widehat{g}(x) = p(x)'\widehat{\beta}$. *Denote the t-statistic as*

$$t_N(x) := \frac{\widehat{g}(x) - g(x)}{\widehat{\sigma}_N(x)}, \tag{2.10}$$

where $\widehat{\sigma}_N(x) = \sqrt{p(x)'\widehat{\Omega}p(x)/N}$, and denote the bootstrapped t-statistic as

$$\widehat{t}_N^b(x) := \frac{p(x)'\widehat{\Omega}^{1/2}/\sqrt{N}}{\widehat{\sigma}_N(x)}\mathcal{N}_d^b,$$

where $\mathcal{N}_d^b$ is a bootstrap draw from $N(0, I_d)$. Define the confidence bands for $g(x)$ as

$$[\underline{i}(x), \overline{i}(x)] := [\widehat{g}(x) - c_N(1-\alpha)\widehat{\sigma}_N(x), \quad \widehat{g}(x) + c_N(1-\alpha)\widehat{\sigma}_N(x)], \quad x \in \mathcal{X}, \quad (2.11)$$

where the critical value $c_N(1-\alpha)$ is the $(1-\alpha)$-quantile of $N(0,1)$ for the pointwise bands, and $c_N(1-\alpha)$ is the $(1-\alpha)$-quantile of $\sup_{x \in \mathcal{X}} |\widehat{t}_N^b(x)|$ for the uniform bands.

In the case of CTEs (Example 2.1), the orthogonal signal (2.5) involves an auxiliary nuisance parameter—the expectation of $\mu(x, Z)$ with respect to Z for each value of x. We estimate this parameter by a leave-one-out sample average,

$$Y_i^{\dagger}(\widehat{\eta}) = \frac{Y_i^o - \widehat{\mu}(X_i, Z_i)}{\widehat{s}(X_i|Z_i)}\widehat{w}(X_i) + \frac{1}{n-1}\sum_{j \in J_k, j \neq i}\widehat{\mu}(X_i, Z_j), \quad i \in J_k, \quad (2.12)$$

where $\widehat{\eta}(x, z) = \widehat{\eta}_k(x, z)$ is estimated on J_k^c for each k and the sample average in the second summand is taken over data $(V_j)_{j \in J_k}$ excluding observation i. Having replaced $Y_i(\widehat{\eta})$ by $Y_i^{\dagger}(\widehat{\eta})$ in Definition 2.2, we obtain an asymptotically linear estimator $p(x)'\widehat{\beta}^{\dagger}$ of the pseudo-target function:

$$\frac{\sqrt{N}p(x)'(\widehat{\beta}^{\dagger} - \beta_0)}{\sqrt{p(x)'\Omega^{\dagger}p(x)}} = G_N(x) + o_P(1/\sqrt{\log N}),$$

where $\Omega^{\dagger}$ is the asymptotic variance of $\widehat{\beta}^{\dagger}$, and the empirical process $G_N(x)$ is approximated by a Gaussian process

$$G(x) = \frac{p(x)'}{\sqrt{p(x)'\Omega^{\dagger}p(x)}}N(0, \Omega^{\dagger})$$

uniformly over $x \in \mathcal{X}$.

3. MAIN THEORETICAL RESULTS

We use the empirical process notation. For a generic function f and a generic sample $(V_i)_{i=1}^N$, denote the empirical sample average by $\mathbb{E}_N f(V_i) := \frac{1}{N}\sum_{i=1}^N f(V_i)$ and the scaled, demeaned sample average by

$$\mathbb{G}_N f(V_i) := 1/\sqrt{N}\sum_{i=1}^N [f(V_i) - \int f(v)dP(v)].$$

The following assumptions impose regularity conditions on the covariate distribution, error terms, and the estimator of the nuisance parameter.

ASSUMPTION 3.1 (IDENTIFICATION). *Let $Q := \mathbb{E}p(X)p(X)' = Q_d$ denote the population covariance matrix of $p(X)$. Assume that there exist $0 < C_{\min} < C_{\max} < \infty$ that do not depend on d so that $C_{\min} \leq \min \mathrm{eig}(Q) \leq \max \mathrm{eig}(Q) \leq C_{\max}$ for all d.*

ASSUMPTION 3.2 (GROWTH CONDITION). *We assume that the* sup-*norm of the basis functions* $\xi_d := \sup_{x \in \mathcal{X}} \| p(x) \| = \sup_{x \in \mathcal{X}} (\sum_{j=1}^{d} p_j(x)^2)^{1/2}$ *grows sufficiently slowly:*

$$\sqrt{\frac{\xi_d^2 \log N}{N}} = o(1).$$

ASSUMPTION 3.3 (MISSPECIFICATION ERROR). *There exists a sequence of finite constants* l_d, r_d *such that the norms of the misspecification error are controlled as follows:*

$$\| r_g \|_{P,2} := \sqrt{\int r_g(x)^2 dP(x)} \lesssim r_d \text{ and } \| r_g \|_{P,\infty} := \sup_{x \in \mathcal{X}} |r_g(x)| \lesssim l_d r_d.$$

Assumption 3.3 introduces the rate of decay of the misspecification error. Specifically, the sequence of constants r_d bounds the mean squared misspecification error. In addition, the sequence $l_d r_d$ bounds the worst-case misspecification error uniformly over the domain $\mathcal{X}$, where l_d is the modulus of continuity of the worst-case error with respect to mean squared error.

Define the stochastic error U as

$$U := Y - g(X)$$

and the lower and upper bounds on its second moment conditional on X as

$$\underline{\sigma}^2 := \inf_{x \in \mathcal{X}} \mathbb{E}[U^2 | X = x], \quad \overline{\sigma}^2 := \sup_{x \in \mathcal{X}} \mathbb{E}[U^2 | X = x].$$

ASSUMPTION 3.4 (ERROR ASSUMPTION). *The second moment of the sampling error U conditional on X is bounded from above:* $\overline{\sigma}^2 \lesssim 1$.

To describe the first-stage rate requirement, Assumption 3.5 introduces a sequence of nuisance realisation sets $\mathcal{T}_N$ for the nuisance parameter η_0. As sample size N increases, the sets $\mathcal{T}_N$ shrink around the true value η_0. The shrinkage speed is described in terms of the statistical rates B_N and Λ_N.

ASSUMPTION 3.5 (SMALL BIAS CONDITION). *There exists a sequence $\epsilon_N = o(1)$, such that with probability at least $1 - \epsilon_N$, for all $k \in [K]$, the first-stage estimate $\widehat{\eta}_k$, obtained by cross-fitting (Definition 2.1), belongs to a shrinking neighborhood of η_0, denoted by $\mathcal{T}_N$. Uniformly over $\mathcal{T}_N$, the following mean square convergence holds:*

$$B_N := \sqrt{N} \sup_{\eta \in \mathcal{T}_N} \| \mathbb{E} p(X)[Y(\eta) - Y(\eta_0)] \| = o(1),$$

$$\Lambda_N := \sup_{\eta \in \mathcal{T}_N} (\mathbb{E} \| p(X)[Y(\eta) - Y(\eta_0)] \|^2)^{1/2} = o(1).$$

In particular, Λ_N can be bounded as $\Lambda_N \lesssim \xi_d \sup_{\eta \in \mathcal{T}_N} (\mathbb{E}(Y(\eta) - Y(\eta_0))^2)^{1/2}$.

REMARK 3.1 (SUFFICIENT CONDITIONS FOR ASSUMPTION 3.5). Assumption 3.5 is stated in a high-level form in order to accommodate various machine learning estimators. We demonstrate the plausibility of Assumption 3.5 for a high-dimensional sparse model for Example 2.3 in Appendix B, adapting the work of Belloni et al. (2017). Furthermore, one can also use deep neural networks (Schmidt-Hieber, 2017; Farrell et al., 2018) and random forest in small (Wager and Walther, 2015) dimensions and high (Syrganis and Zampetakis, 2020) dimensions with sparsity structure.

3.1. Pointwise limit theory

In this section, we establish pointwise asymptotic properties for the orthogonal estimator. Our first result is concerned with mean square convergence rate and pointwise linearisation.

LEMMA 3.1 (CONVERGENCE RATE AND POINTWISE LINEARIZATION). *Let Assumptions 3.1 through 3.5 hold. Then, the following statements hold:*

(a) *The ℓ_2-norm of the estimation error is bounded as:*

$$\|\widehat{\beta} - \beta_0\|_2 \lesssim_P \sqrt{\frac{d}{N}} + \left[\sqrt{\frac{d}{N}} l_d r_d \wedge \xi_d r_d / \sqrt{N}\right],$$

which implies a bound on the mean squared error of the estimate $p(x)'\widehat{\beta}$ of the pseudo-target function $p(x)'\beta_0$:

$$(\mathbb{E}_N(p(X_i)'(\widehat{\beta} - \beta_0))^2)^{1/2} \lesssim_P \sqrt{\frac{d}{N}} + \left[\sqrt{\frac{d}{N}} l_d r_d \wedge \xi_d r_d / \sqrt{N}\right].$$

(b) *For any $\alpha \in \mathcal{S}^{d-1} := \{\alpha \in \mathbb{R}^d : \|\alpha\| = 1\}$, the estimator $\widehat{\beta}$ is approximately linear:*

$$\sqrt{N}\alpha'(\widehat{\beta} - \beta) = \alpha' Q^{-1} \mathbb{G}_N p(X_i)(U_i + r_g(X_i)) + R_{1,N}(\alpha),$$

where the remainder term $R_{1,N}(\alpha)$ is bounded as

$$R_{1,N}(\alpha) \lesssim_P B_N + \Lambda_N + \sqrt{\frac{\xi_d^2 \log N}{N}} \left(1 + \min\left\{l_d r_d \sqrt{d}, \xi_d r_d\right\}\right).$$

Under a small bias condition, Lemma 3.1 states that the orthogonal estimator converges at the oracle rate and achieves oracle asymptotic linearity representation, where the oracle knows the true value of the nuisance parameter η_0.

THEOREM 3.1 (POINTWISE NORMALITY OF THE ORTHOGONAL ESTIMATOR). *Suppose Assumptions 3.1 through 3.5 hold. In addition, suppose $(\xi_d^2 \log N/N)^{1/2} \cdot (1 + l_d r_d \sqrt{d}) = o(1)$, $1 \lesssim \sigma^2$, and the Lindeberg condition holds:* $\sup_{x \in \mathcal{X}} \mathbb{E}[U^2 1_{|U|>M}|X = x] \to 0$, $M \to \infty$. *Then, for any $\alpha \in \mathcal{S}^{d-1}$, the orthogonal estimator is asymptotically normal:*

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| P\left(\frac{\sqrt{N}\alpha'(\widehat{\beta} - \beta_0)}{\sqrt{\alpha'\Omega\alpha}} < t\right) - \Phi(t)\right| = 0.$$

Moreover, for any $x_0 = x_{0,N} \in \mathcal{X}$, the estimator $p(x_0)'\widehat{\beta}$ of the pseudo-target value $p(x_0)'\beta_0$ is asymptotically normal:

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| P\left(\frac{\sqrt{N}p(x_0)'(\widehat{\beta} - \beta_0)}{\sqrt{p(x_0)'\Omega p(x_0)}} < t\right) - \Phi(t)\right| = 0,$$

and if the approximation error is negligible relative to the estimation error, namely $\sqrt{N}r_g(x_0) = o(\|\Omega^{1/2}p(x_0)\|)$, then $\widehat{g}(x) = p(x_0)'\widehat{\beta}$ is asymptotically normal:

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| P\left(\frac{\sqrt{N}(\widehat{g}(x_0) - g(x_0))}{\sqrt{p(x_0)'\Omega p(x_0)}} < t\right) - \Phi(t)\right| = 0.$$

Theorem 3.1 delivers the pointwise convergence in distribution of the orthogonal estimator for any point x_0 that can depend on N.

3.2. Uniform limit theory

In this section, we establish uniform asymptotic properties for the orthogonal estimator. Not surprisingly, stronger conditions are required for our results to hold when compared with the pointwise case. Let $m > 2$. The following assumption controls the tails of the regression errors.

ASSUMPTION 3.6 (TAIL BOUNDS). *There exists a constant $m > 2$ such that the upper bound of the mth moment of $|U|$ is bounded conditional on X:*

$$\sup_{x \in \mathcal{X}} \mathbb{E}[|U|^m | X = x] \lesssim 1.$$

Denote by $\alpha(x) := p(x)/\|p(x)\|$ the normalised value of basis functions vector $p(x)$. Define the Lipschitz constant as:

$$\xi_d^L = \sup_{x,x' \in \mathcal{X}, x \neq x'} \frac{\|\alpha(x) - \alpha(x')\|}{\|x - x'\|}.$$

ASSUMPTION 3.7 (BASIS). *Basis functions are well behaved, namely* $(i)(\xi_d^L)^{2m/(m-2)} \log N / N \lesssim 1$ *and* (ii) $\log \xi_d^L \lesssim \log d$ *for the same m as in Assumption 3.6.*

ASSUMPTION 3.8 (CONDITION FOR MATRIX ESTIMATION). *Let $\mathcal{T}_N$ be as in Assumption 3.5. Uniformly over $\mathcal{T}_N$, the following convergence holds:*

$$\kappa_N^1 := \sup_{\eta \in \mathcal{T}_N} \mathbb{E}[\max_{1 \leq i \leq N} |Y_i(\eta) - Y_i(\eta_0)|] = o(1),$$

$$\kappa_N := \sup_{\eta \in \mathcal{T}_N} (\mathbb{E} \max_{1 \leq i \leq N} (Y_i(\eta) - Y_i(\eta_0))^2)^{1/2} = o(1).$$

Lemma 3.2 establishes asymptotic linearity representation uniformly over the domain $\mathcal{X}$ and uniform convergence rate.

LEMMA 3.2 (UNIFORM RATE AND UNIFORM LINEARIZATION). *Suppose Assumptions 3.1 through 3.7 hold.*

(a) *The orthogonal estimator is approximately linear uniformly over $\mathcal{X}$:*

$$|\sqrt{N}\alpha(x)'(\widehat{\beta} - \beta_0) - \alpha'(x)Q^{-1}\mathbb{G}_N p(X_i)[U_i + r_g(X_i)]| \leq R_{1,N}(\alpha(x)),$$

where $R_{1,N}(\alpha(x))$, summarising the impact of unknown design and the first-stage error, obeys

$$\sup_{x \in \mathcal{X}} R_{1,N}(\alpha(x)) \lesssim_P B_N + \Lambda_N + \sqrt{\frac{\xi_d^2 \log N}{N}}(N^{1/m}\sqrt{\log N} + \sqrt{d}l_d r_d) =: \bar{R}_{1N}$$

uniformly over $x \in \mathcal{X}$. Moreover,

$$|\sqrt{N}\alpha(x)'(\widehat{\beta} - \beta_0) - \alpha'(x)Q^{-1}\mathbb{G}_N p(X_i)U_i| \leq R_{1,N}(\alpha(x)) + R_{2,N}(\alpha(x)),$$

where $R_{2,N}(\alpha(x))$, summarising the impact of misspecification error, obeys

$$R_{2,N}(\alpha(x)) \lesssim_P \sqrt{\log N}l_d r_d =: \bar{R}_{2N}$$

uniformly over $x \in \mathcal{X}$.

(b) The estimator $p(x)'\widehat{\beta}$ of the pseudo-target $p(x)'\beta_0$ converges uniformly over $\mathcal{X}$:

$$\sup_{x \in \mathcal{X}} |p(x)'(\widehat{\beta} - \beta_0)| \lesssim_P \frac{\xi_d}{\sqrt{N}}[\sqrt{\log N} + \bar{R}_{1N} + \bar{R}_{2N}].$$

REMARK 3.2 (OPTIMAL UNIFORM RATE IN HOLDER CLASS). Suppose the true function $g(x)$ belongs to the Holder smoothness class of order k, denoted by $\Sigma_k(\mathcal{X})$. Suppose $l_d r_d \lesssim d^{-k/\dim(X)}, \xi_d \lesssim \sqrt{d}, \bar{R}_{1N} + \bar{R}_{2N} \lesssim (\log N)^{1/2}$. Then, the optimal number d of technical regressors that comprise a vector $p(x)$ obeys

$$d \asymp (\log N/N)^{-\dim(X)/(2k+\dim(X))}.$$

This choice of d yields the optimal uniform rate:

$$\sup_{x \in \mathcal{X}} |\widehat{g}(x) - g(x)| \lesssim_P \left(\frac{\log N}{N}\right)^{k/(2k+\dim(X))}.$$

Theorem 3.2 establishes a strong approximation of the orthogonal estimator's series process by a sequence of zero-mean Gaussian processes.

THEOREM 3.2 (STRONG APPROXIMATION BY A GAUSSIAN PROCESS). *Suppose Assumptions 3.1 through 3.7 hold with $m \geq 3$. Let $\bar{a}_N$ be a sequence of positive numbers s.t. $\bar{a}_N^{-1} = o(1)$. Suppose (i) $\bar{R}_{1N} = o(\bar{a}_N^{-1})$, (ii) $1 \lesssim \underline{\sigma}^2$, and (iii) $d^4 \bar{a}_N^6 \xi_d^2 (1 + l_d^3 r_d^3)^2 \log^2 N/N = o(1)$. Then, for some $\mathcal{N}_d \sim N(0, I_d)$, the following statement holds for $e(x) = \Omega^{1/2} p(x)$*

$$\sqrt{N} \frac{p(x)'(\widehat{\beta} - \beta_0)}{\|e(x)\|} =_d \frac{e(x)}{\|e(x)\|} \mathcal{N}_d + o_P(\bar{a}_N^{-1}) \text{ in } \ell^\infty(\mathcal{X}).$$

In addition, if $\sup_{x \in \mathcal{X}} \sqrt{N} |r(x)|/\|e(x)\| = o(\bar{a}_N^{-1})$, then, for $\widehat{g}(x) = p(x)'\widehat{\beta}$,

$$\sqrt{N} \frac{\widehat{g}(x) - g(x)}{\|e(x)\|} =_d \frac{e(x)}{\|e(x)\|} \mathcal{N}_d + o_P(\bar{a}_N^{-1}) \text{ in } \ell^\infty(\mathcal{X}).$$

Theorem 3.3 establishes the convergence rate for the covariance matrix estimator $\widehat{\Omega}$. It extends Theorem 4.6 of Belloni et al. (2015), allowing the signal $Y(\eta_0)$ to depend on an unknown nuisance parameter η_0.

THEOREM 3.3 (MATRICES ESTIMATION). *Suppose Assumptions 3.1 through 3.8 hold. In addition, suppose (i) $\bar{R}_{1N} + \bar{R}_{2N} \lesssim \sqrt{\log N}$ and (ii) $(N^{1/m} + l_d r_d)(\sqrt{\frac{\xi_d^2 \log N}{N}} + \kappa_N^1) = o(1)$. Then, the estimator $\widehat{\Omega}$, defined in (9), converges in the matrix operator norm with the following rate:*

$$\|\widehat{\Omega} - \Omega\| \lesssim_P \left(N^{1/m} + l_d r_d\right) \cdot \left(\sqrt{\frac{\xi_d^2 \log N}{N}} + \kappa_N^1\right) + \kappa_N^2 =: a_N.$$

Moreover, for $\sigma_N(x) = \sqrt{p(x)'\Omega p(x)/N}$ and $\widehat{\sigma}_N(x) = \sqrt{p(x)'\widehat{\Omega} p(x)/N}$, the following bound holds:

$$\sup_{x \in \mathcal{X}} \left|\frac{\widehat{\sigma}_N(x)}{\sigma_N(x)} - 1\right| \lesssim_P \|\widehat{\Omega} - \Omega\| \lesssim_P a_N. \tag{3.13}$$

Theorem 3.4 establishes validity of empirical (Gaussian) bootstrap.

THEOREM 3.4 (VALIDITY OF GAUSSIAN BOOTSTRAP). *Suppose the assumptions of Theorem 3.2 hold with $\bar{a}_N = \log N$ and the assumptions of Theorem 3.3 hold with $a_N = O(N^{-b})$ for some $b > 0$. In addition, suppose (i) $1 \lesssim \underline{\sigma}^2$ and (ii) there exists a sequence ξ'_N obeying $1 \lesssim \xi'_N \lesssim \|p(x)\|$ uniformly for all $x \in \mathcal{X}$ so that $\|p(x) - p(x')\|/\xi'_N \le L_N \|x - x'\|$, where $\log L_N \lesssim \log N$. Let $\mathcal{N}_d^b$ be a bootstrap draw from $N(0, I_d)$ and P^* be a probability conditional on data $(V_i)_{i=1}^N$. Then, the following approximation holds uniformly in $\ell^\infty(\mathcal{X})$:*

$$\frac{p(x)'\widehat{\Omega}^{1/2}}{\|\widehat{\Omega}^{1/2}p(x)\|}\mathcal{N}_d^b =^d \frac{p(x)'\Omega^{1/2}}{\|\Omega^{1/2}p(x)\|}\mathcal{N}_d^b + o_{P^*}(\log^{-1} N). \tag{3.14}$$

Theorem 3.5 establishes the asymptotic validity of uniform confidence bands. It also shows that the uniform width of the bands is of the same order as the uniform rate of convergence.

THEOREM 3.5 (VALIDITY OF UNIFORM CONFIDENCE BANDS). *Let Assumptions 3.1 through 3.8 hold with $m \ge 4$. In addition, suppose (i) $\bar{R}_{1N} + \bar{R}_{2N} \lesssim \log^{-1/2} N$, (ii) $\xi_d \log^2 N/N^{1/2-1/m} = o(1)$, (iii) $1 \lesssim \underline{\sigma}^2$, (iv) $\sup_{x\in\mathcal{X}} \sqrt{N}|r_g(x)|/\|p(x)\| = o(\log^{-1/2} N)$, and (v) $d^4\xi_d^2(1 + l_d^3 r_d^3)^2 \log^5 N/N = o(1)$. Then,*

$$P\Big(\sup_{x\in\mathcal{X}} |t_N(x)| \le c_N(1-\alpha)\Big) = 1 - \alpha + o(1)$$

for t_N defined in (10). As a consequence, the confidence bands defined in (11) satisfy

$$P\big(g(x) \in [\underline{i}(x), \bar{i}(x)] \quad \forall x \in \mathcal{X}\big) = 1 - \alpha + o(1).$$

The width of the confidence bands $2c_N(1-\alpha)\widehat{\sigma}_N(x)$ obeys

$$2c_N(1-\alpha)\widehat{\sigma}_N(x) \lesssim_P \sigma_N(x)\sqrt{\log N} \lesssim \sqrt{\frac{\xi_d^2 \log N}{N}}$$

uniformly over $x \in \mathcal{X}$.

4. APPLICATIONS

In this section, we apply the results of Section 3 for empirically relevant settings, described in Examples 2.1 through 2.4.

4.1. Continuous Treatment Effects

Consider the setup of Example 2.1. We provide sufficient low-level conditions on the first-stage nuisance parameter $\eta_0(x, z)$ such that the pointwise and uniform Gaussian approximations established in Section 3 hold.

Assume that there exists a sequence of numbers $\epsilon_N = o(1)$ and sequences of neighborhoods S_N of $s_0(\cdot|\cdot)$, M_N of $\mu_0(\cdot, \cdot)$, and W_N of $w_0(\cdot)$ such that the first-stage estimate

$$\{\widehat{s}(\cdot|\cdot), \widehat{\mu}(\cdot, \cdot), \widehat{w}(\cdot)\}$$

belongs to the set $\{S_N \times M_N \times W_N\}$ with probability at least $1 - \epsilon_N$. The shrinkage speed of this set is measured by the following statistical rates:

$$\mathbf{s}_{N,q} := \sup_{s \in S_N} (\mathbb{E}(s(X|Z) - s_0(X|Z))^q)^{1/q},$$

$$\mathbf{m}_{N,q} := \sup_{\mu \in M_N} (\mathbb{E}(\mu(X, Z) - \mu_0(X, Z))^q)^{1/q},$$

$$\mathbf{w}_{N,q} := \sup_{w \in W_N} (\mathbb{E}(w(X) - w_0(X))^q)^{1/q},$$

where q is either a positive number $q \geq 2$ or $q = \infty$, which corresponds to ℓ_∞-norm (sup-norm). For $q = 2$, we will refer to $\mathbf{s}_N := \mathbf{s}_{N,2}$ as the conditional density mean square rate, $\mathbf{m}_N := \mathbf{m}_{N,2}$ as the regression function mean square rate, and $\mathbf{w}_N = \mathbf{w}_{N,2}$ as the marginal density mean square rate.

ASSUMPTION 4.9 (FIRST-STAGE RATE OF CTE). *Assume that mean square rates* $\mathbf{s}_N, \mathbf{m}_N$, *and* $\mathbf{w}_N$ *decay sufficiently fast,*

$$\xi_d(\mathbf{s}_N \vee \mathbf{m}_N \vee \mathbf{w}_N) = o(1),$$

and one of two alternative conditions holds. (a) Bounded basis. There exists $\bar{B} < \infty$ *so that* $\sup_{x \in \mathcal{X}} \|p(x)\|_\infty \leq \bar{B}$ *and* $\sqrt{N}\sqrt{d}(\mathbf{m}_N \mathbf{s}_N \vee \mathbf{m}_N \mathbf{w}_N) = o(1)$. *(b) Unbounded basis. There exist* $\kappa, \gamma \in [1, \infty]$, $1/\kappa + 1/\gamma = 1$ *so that* $\sqrt{N}\sqrt{d}(\mathbf{m}_{N,2\gamma} \mathbf{s}_{N,2\kappa} \vee \mathbf{m}_{N,2\gamma} \mathbf{w}_{N,2\kappa}) = o(1)$. *Furthermore, there exists a constant* $\bar{C} < \infty$ *such that* $\sup_{\mu \in M_N} \sup_{(x,z) \in \mathcal{X} \times \mathcal{Z}} |\mu(x, z)| < \bar{C}$, $\sup_{s \in S_N} \sup_{(x,z) \in \mathcal{X} \times \mathcal{Z}} s^{-1}(x|z) < \bar{C}$, *and* $\sup_{s \in S_N} \sup_{(x,z) \in \mathcal{X} \times \mathcal{Z}} s(x|z) < \bar{C}$,

$$\sup_{w \in W_N} \sup_{x \in \mathcal{X}} w^{-1}(x) < \bar{C}.$$

LEMMA 4.3 (ORTHOGONAL SIGNAL FOR CTE). *Suppose Assumption 4.9 holds. Then, the orthogonal signal* $Y(\eta)$, *defined in* (5), *satisfies Assumption 3.5.*

As discussed in the Introduction and Motivation, the estimator $\widehat{\beta}^\dagger$ takes the form

$$\widehat{\beta}^\dagger = \widehat{Q}^{-1} \frac{1}{N} \sum_{i=1}^{N} p(X_i) Y_i^\dagger(\widehat{\eta}),$$

where $Y_i^\dagger(\widehat{\eta})$ is as in (12). Because μ enters linearly in (5), the error term $Y_i^\dagger(\widehat{\eta}) - Y_i(\widehat{\eta})$ does not introduce bias in $\widehat{\beta}^\dagger$ but introduces an extra term in asymptotic variance, which we characterise below. For a function $\mu(x, z)$, define its demeaned analog $\mu^0(x, z)$ as

$$\mu^0(x, z) := \mu(x, z) - \mathbb{E}\mu(x, Z)$$

and the kernel function as

$$\tau(v_1, v_2; \mu) = \frac{1}{2}(p(x_1)\mu^0(x_1, z_2) + p(x_2)\mu^0(x_2, z_1)),$$

where $v_1 = (x_1, z_1)$ and $v_2 = (x_2, z_2)$. Finally, define the Hajek projection of $\tau(v_1, v_2; \mu)$ as

$$\tau_1(v; \mu) := \mathbb{E}\tau(v, V; \mu) = \mathbb{E}p(X)\mu^0(X, z) = \tau_1(z; \mu). \tag{4.15}$$

Decompose the orthogonal estimator $\widehat{\beta}^\dagger$ into the sum of its infeasible analog $\widehat{\beta}$ and a mean-zero U-statistic with the kernel function $\tau(v_1, v_2; \mu)$

$$\widehat{\beta}^\dagger =: \widehat{\beta} + \widehat{Q}^{-1} \frac{1}{K} \sum_{k=1}^{K} \frac{1}{n(n-1)} \sum_{i,j \in J_k, i \neq j} \tau(V_i, V_j; \widehat{\mu}).$$

Theorem 4.6 establishes pointwise asymptotic normality of the orthogonal estimator. Its asymptotic variance is

$$\Omega^\dagger = Q^{-1} \Sigma^\dagger Q^{-1},$$

where $\Sigma^\dagger$ is

$$\Sigma^\dagger = \mathbb{E}[p(X)(U + r_g(X)) + \tau_1(Z; \mu_0)][p(X)(U + r_g(X)) + \tau_1(Z; \mu_0)]'.$$

THEOREM 4.6 (POINTWISE ASYMPTOTIC THEORY FOR CTES) *Suppose Assumptions 3.1 through 3.4 and 4.9 hold. Let $C_{\min}^\dagger > 0$ be an absolute constant. In addition, suppose (i) $(\xi_d^2 \log N / N)^{1/2} \cdot (1 + l_d r_d \sqrt{d}) = o(1)$, (ii) $\sqrt{d\xi_d^4 \log^3 N / N^2} + \sqrt{d}\mathbf{m}_N = o(1)$, $\xi_d l_d r_d = o(N^{1/2})$, (iii) $\min \operatorname{eig} \Omega^\dagger \geq C_{\min}^\dagger$, and (iv) the Lindeberg condition holds: $\sup_{x \in \mathcal{X}} \mathbb{E}[U^2 1_{|U| > M} | X = x] \to 0$, $M \to \infty$. Then, for any $x_0 = x_{0,N} \in \mathcal{X}$ the estimator $p(x_0)'\widehat{\beta}^\dagger$ of the pseudo-target $p(x_0)'\beta_0$ is asymptotically normal:*

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \mathrm{P} \left(\frac{\sqrt{N} p(x_0)'(\widehat{\beta}^\dagger - \beta_0)}{\sqrt{p(x_0)'\Omega^\dagger p(x_0)}} < t \right) - \Phi(t) \right| = 0,$$

and if the approximation error is negligible relative to the estimation error, namely $\sqrt{N} r_g(x_0) = o(\|(\Omega^\dagger)^{1/2} p(x_0)\|)$, then $\widehat{g}^\dagger(x_0) = p(x_0)'\widehat{\beta}^\dagger$ is asymptotically normal:

$$\lim_{N \to \infty} \sup_{t \in \mathbb{R}} \left| \mathrm{P} \left(\frac{\sqrt{N}(\widehat{g}^\dagger(x_0) - g(x_0))}{\sqrt{p(x_0)'\Omega^\dagger p(x_0)}} < t \right) - \Phi(t) \right| = 0.$$

THEOREM 4.7 (UNIFORM ASYMPTOTIC THEORY FOR CTES). *Suppose Assumptions 3.1 through 3.7 hold with $\xi_d^L / C_{min} \geq e^2/16 \vee e$. In addition, suppose $\sqrt{d\xi_d^2 \log N / N} = o(1)$ and $\xi_d \log^2 N / N = o(1)$. Then, the following statements hold.*

(a) *The orthogonal estimator converges uniformly over $\mathcal{X}$:*

$$\sup_{x \in \mathcal{X}} |p(x)'(\widehat{\beta}^\dagger - \beta_0)| \lesssim_P \frac{\xi_d}{\sqrt{N}} \left(\sqrt{\log N} + \bar{R}_{1N} + \bar{R}_{2N} + \sqrt{d}\mathbf{m}_N \right),$$

   where $\bar{R}_{1N}$ and $\bar{R}_{2N}$ are as in Lemma 3.2.

(b) *Suppose Assumptions 3.6 and 3.7 hold with $m \geq 3$. Then, the statement of Theorem 3.2 holds for $\widehat{\beta}^\dagger$ in place of $\widehat{\beta}$, $\Omega^\dagger$ in place of Ω, and $e(x) := (\Omega^\dagger)^{1/2} p(x)$.*

Theorem 4.7 establishes a uniform convergence rate and strong Gaussian approximation for CTEs. We compare our Theorems 4.6 and 4.7 with Kennedy et al. (2017), who introduced the doubly robust score for the average potential outcome. First, by virtue of sample splitting, we do not impose any complexity requirements on the estimator of the first-stage nuisance parameters. In particular, unlike Theorem 2 of Kennedy et al. (2017), we do not require the function class containing $\widehat{\mu}(\cdot, \cdot), \widehat{s}(\cdot|\cdot), \widehat{w}(\cdot)$ to have bounded uniform entropy integrals. As a result, our method accommodates a wide class of modern regularised methods to be used to

estimate first-stage parameters, as discussed in Remark 3.1. Second, Theorem 4.7 offers uniform asymptotic statements, whereas the work by Kennedy et al. (2017) offers only pointwise results. Finally, Kennedy et al. (2017) uses local linear regression in the second stage and delivers convergence at a $\sqrt{N}h^{\dim(X)}$ rate, where $h = h(N)$ is the kernel bandwidth. In contrast, our method delivers $\sqrt{N}$-approximation for the normalised projection $p(x)'\beta_0/\|p(x)\|$.

4.2. Conditional Average Treatment Effect

Consider the setup of Example 2.2. We provide sufficient low-level conditions on the regression functions $\mu_0(1, \cdot)$, $\mu_0(0, \cdot)$ and the propensity score $s_0(\cdot)$ such that the pointwise and uniform Gaussian approximations of Section 3 hold.

ASSUMPTION 4.10 (STRONG OVERLAP).

(a) *The propensity score is bounded above and below. Specifically, there exists $\bar{\pi}_0 > 0$ such that*
$$0 < \bar{\pi}_0 < s_0(z) < 1 - \bar{\pi}_0 < 1 \text{ for any } z \in \mathcal{Z}.$$
(b) *The propensity score is bounded below. Specifically, there exists $\bar{\pi}_0 > 0$ such that $0 < \bar{\pi}_0 < s_0(z) < 1$ for any $z \in \mathcal{Z}$.*

In the context of Example 2.2, Assumption 4.10(a) ensures that the probability of assignment to the treatment and control group is bounded away from zero. In the context of Example 2.3, Assumption 4.10(b) ensures that the probability of observing the response Y^* is bounded away from zero.

Given the true functions $s_0(\cdot)$, $\mu_0(1, \cdot)$, $\mu_0(0, \cdot)$ and sequences of shrinking neighborhoods S_N of $s_0(\cdot)$ and M_N of $\mu_0(1, \cdot)$ and of $\mu_0(0, \cdot)$, define the following rates:

$$\mathbf{s}_{N,q} := \sup_{s \in \mathcal{S}_N} (\mathbb{E}(s(Z) - s_0(Z))^q)^{1/q},$$

$$\mathbf{m}_{N,q} := \sup_{\mu \in M_N} (\mathbb{E}(\mu(1, Z) - \mu_0(1, Z))^q)^{1/q} \vee \sup_{\mu \in M_N} (\mathbb{E}(\mu(0, Z) - \mu_0(0, Z))^q)^{1/q},$$

where $q \geq 2$ is either a positive number or $q = \infty$. We will refer to $\mathbf{s}_N := \mathbf{s}_{N,2}$ as the propensity score mean square rate and $\mathbf{m}_N := \mathbf{m}_{N,2}$ as the regression function mean square rate.

ASSUMPTION 4.11 (FIRST-STAGE RATE FOR CATE). *Assume that there exists a sequence of numbers $\epsilon_N = o(1)$ and sequences of neighborhoods S_N of $s_0(\cdot)$ and M_N of $\mu_0(1, \cdot)$ and $\mu_0(0, \cdot)$ such that the first-stage estimate $\{\widehat{s}(\cdot), \widehat{\mu}(1, \cdot), \widehat{\mu}(0, \cdot)\}$ belongs to the set $\{S_N \times M_N \times M_N\}$ w.p. at least $1 - \epsilon_N$. Assume that mean square rates $\mathbf{s}_N$, $\mathbf{m}_N$ decay sufficiently fast:*

$$\xi_d(\mathbf{s}_N \vee \mathbf{m}_N) = o(1),$$

and one of two alternative conditions holds. (a) Bounded basis. There exists $\bar{B} < \infty$ so that $\sup_{x \in \mathcal{X}} \|p(x)\|_\infty \leq \bar{B}$ and $\sqrt{N}\sqrt{d}\mathbf{m}_N\mathbf{s}_N = o(1)$. (b) Unbounded basis. There exist $\kappa, \gamma \in [1, \infty]$, $1/\kappa + 1/\gamma = 1$ so that $\sqrt{N}\sqrt{d}\mathbf{m}_{N,2\gamma}\mathbf{s}_{N,2\kappa} = o(1)$. Finally, the functions in S_N and M_N are bounded uniformly over their domain:

$$\sup_{\mu \in M_N} \sup_{z \in \mathcal{Z}} \sup_{d \in \{1,0\}} |\mu(d, z)| \vee \sup_{s \in S_N} \sup_{z \in \mathcal{Z}} s^{-1}(z) < \bar{C} < \infty.$$

COROLLARY 4.1 (ASYMPTOTIC THEORY FOR CATE). *Under Assumptions 4.10(a) and 4.11, the orthogonal signal $Y(\eta)$, given by Equation (6), satisfies Assumption 3.5. As a result, the statements of Theorems 3.1 through 3.5 hold for CATE.*

4.3. Regression function with partially missing outcome

Consider the setup of Example 2.3. Define the regression function rate $\mathbf{m}_{N,q}$ as

$$\mathbf{m}_{N,q} := \sup_{\mu \in M_N} (\mathbb{E}(\mu(Z) - \mu_0(Z))^q)^{1/q},$$

where $q \geq 2$ is either a positive number or $q = \infty$, and let $\mathbf{s}_{N,q}$ be as defined in Section 4.2. We show that pointwise and uniform Gaussian approximations of Section 3 hold for regression function with partially missing outcome.

COROLLARY 4.2 (ASYMPTOTIC THEORY FOR REGRESSION FUNCTION WITH PARTIALLY MISSING OUTCOME). *Suppose Assumptions 4.10(b) and 4.11 hold for $\mathbf{s}_{N,q}$, defined in Example 4.2, and $\mathbf{m}_{N,q}$, redefined above. Then, the orthogonal signal $Y(\eta)$, given by Equation (7), satisfies Assumption 3.5. Then, the statements of Theorems 3.1 through 3.5 hold for the regression function with partially missing outcome.*

We give an example of low-level sparse conditions for the propensity score $s(\cdot)$ and regression $\mu(\cdot)$ in Appendix B.

4.4. Conditional average partial derivative

Consider the setup of Example 2.4. We provide sufficient low-level conditions on the regression functions $s(d|z)$, $\mu(d, z)$ such that the pointwise and uniform Gaussian approximations of Section 3 hold.

Given a true function $s_0(\cdot|\cdot)$, $\mu_0(\cdot, \cdot)$, let S_N, M_N be a sequence of shrinking neighborhoods of $s_0(\cdot|\cdot)$ and $\mu_0(\cdot, \cdot)$, constrained as follows:

$$\mathbf{s}_{N,q} := \sup_{s \in \mathcal{S}_N} (\mathbb{E}(s(D|Z) - s_0(D|Z))^q)^{1/q} \vee \sup_{s \in \mathcal{S}_N} (\mathbb{E}(\partial_d s(D|Z)) - \partial_d s_0(D|Z))^q)^{1/q}$$

$$\mathbf{m}_N := \sup_{\mu \in M_N} (\mathbb{E}(\mu(D, Z) - \mu_0(D, Z))^q)^{1/q} \vee \sup_{\mu \in M_N} (\mathbb{E}(\partial_d \mu(D, Z) - \partial_d \mu_0(D, Z))^q)^{1/q},$$

where $q \geq 2$ is either a positive number or $q = \infty$. We will refer to $\mathbf{s}_N := \mathbf{s}_{N,2}$ as the mean square conditional density rate and $\mathbf{m}_N := \mathbf{m}_{N,2}$ as the mean square regression function rate.

ASSUMPTION 4.12 (FIRST-STAGE RATE FOR CONDITIONAL AVERAGE PARTIAL DERIVATIVE). *Assume that there exists a sequence of numbers $\epsilon_N = o(1)$ and a sequence of neighborhoods S_N, M_N of functions $s_0(\cdot|\cdot)$, $\mu_0(\cdot, \cdot)$ such that the first-stage estimate $\{\widehat{s}(\cdot|\cdot), \widehat{\mu}(\cdot)\}$ belongs to the set $\{S_N \times M_N\}$ w.p. at least $1 - \epsilon_N$. Assume that mean square rates $\mathbf{s}_N$, $\mathbf{m}_N$ decay sufficiently fast $\xi_d(\mathbf{s}_N \vee \mathbf{m}_N) = o(1)$, and one of two alternative conditions hold. (a) Bounded basis. There exists $\bar{B} < \infty$ so that $\sup_{x \in \mathcal{X}} \|p(x)\|_\infty \leq \bar{B}$ and $\sqrt{N}\sqrt{d}\mathbf{m}_N \mathbf{s}_N = o(1)$. (b) Unbounded basis. There exist $\kappa, \gamma \in [1, \infty]$, $1/\kappa + 1/\gamma = 1$ so that $\sqrt{N}\sqrt{d}\mathbf{m}_{N,2\gamma}\mathbf{s}_{N,2\kappa} = o(1)$. The functions in M_N and S_N are bounded uniformly over their domain:*

$$\sup_{\mu \in M_N} \sup_{(d,z) \in \mathrm{R} \times \mathcal{Z}} |\mu(d, z)| \vee \sup_{s \in \mathcal{S}_N} \sup_{(d,z) \in \mathrm{R} \times \mathcal{Z}} \max\{s(d|z), s^{-1}(d|z)\} < \bar{\mathcal{C}}.$$

COROLLARY 4.3 (ASYMPTOTIC THEORY FOR CONDITIONAL AVERAGE PARTIAL DERIVATIVE). *Suppose Assumption 4.12 holds. Then, the orthogonal signal $Y(\eta)$, given by Equation (7), satisfies Assumption 3.5. Then, the statements of Theorems 3.1 through 3.5 hold for conditional average partial derivative.*

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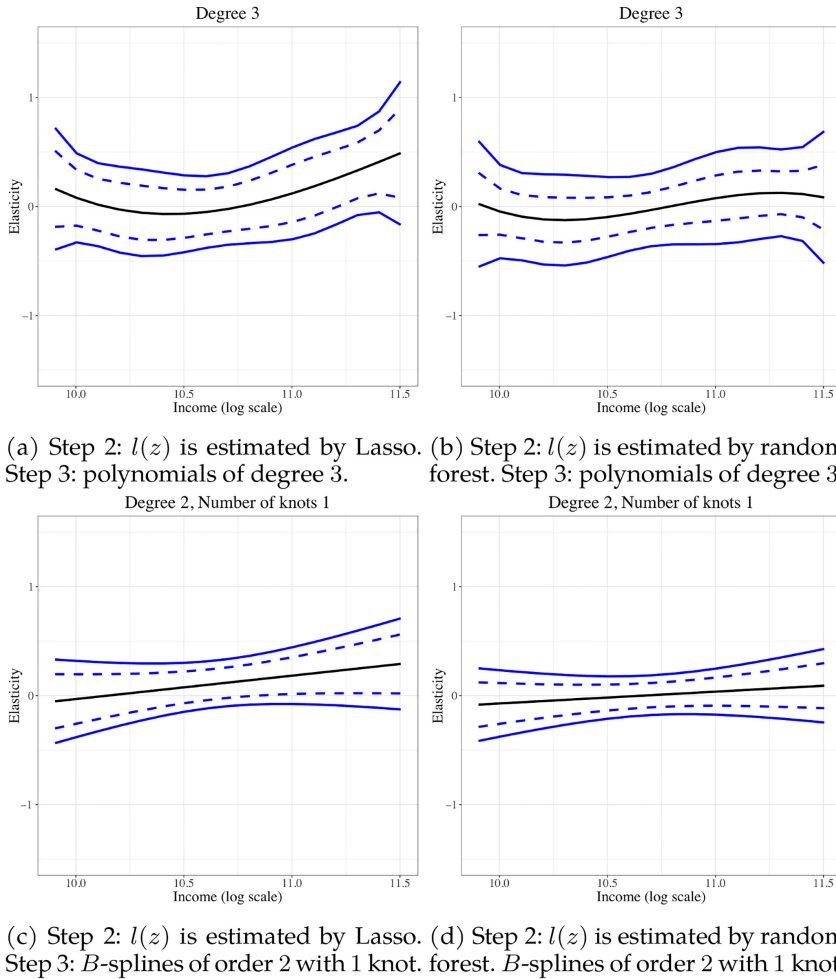(a) Polynomial degree $d = 1$ (b) Polynomial degree $d = 2$
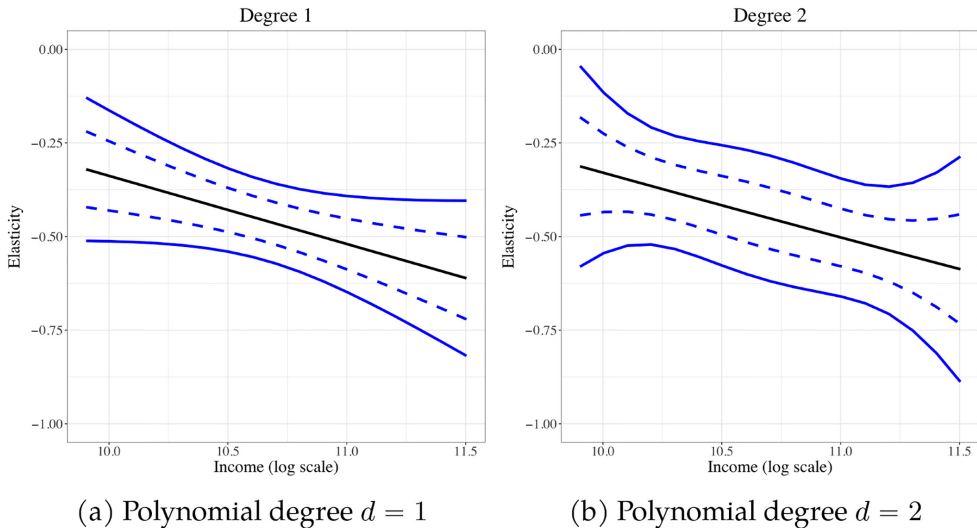
Figure 2. 95% confidence bands for the best linear approximation of the average price elasticity conditional on income without accounting for the demographic controls in the first stage. The black line is the estimated function, and the dashed blue lines and the solid blue lines are the pointwise and the uniform confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function $\mu(d, x) = \mathbb{E}[Y | D = d, X = x]$, (2) second-stage estimation of the conditional density $s(d|x)$, and (3) third-stage estimation of the target function $g(x)$ by least-squares series. Step 1 is performed by using least-squares series regression using polynomial functions $\{1, x, \ldots, x^q\}, q = 3$ whose power q is chosen by cross-validation out of $\{1, 2, 3\}$. Step 2 is performed by kernel density estimator with the Silverman choice of bandwidth. Step 3 is performed by using polynomial functions $\{1, x, \ldots, x^d\}$ and is shown for $d = 1$ and $d = 2$. Uniform confidence bands are based on $B = 200$ repetitions of weighted (Bayes) bootstrap algorithm, described in Belloni et al. (2015).

5. EMPIRICAL APPLICATION: INFERENCE ON CONDITIONAL AVERAGE ELASTICITY

To show the immediate usefulness of the method, we consider an important problem of inference on structural derivatives. We apply our methods to study the household demand for gasoline, a question studied in Hausman and Newey (1995), Schmalensee and Stoker (1999), Yatchew and No (2001), and Blundell et al. (2012). These papers estimated the demand function and the average price elasticity for various demographic groups. The dependence of the price elasticity on the household income was highlighted in Blundell et al. (2012), who have estimated the elasticity by low, middle, and high-income groups and found its relationship with income to be non-monotonic. To gain more insight into this question, we estimate the average price elasticity as a function of income and provide simultaneous confidence bands for it.
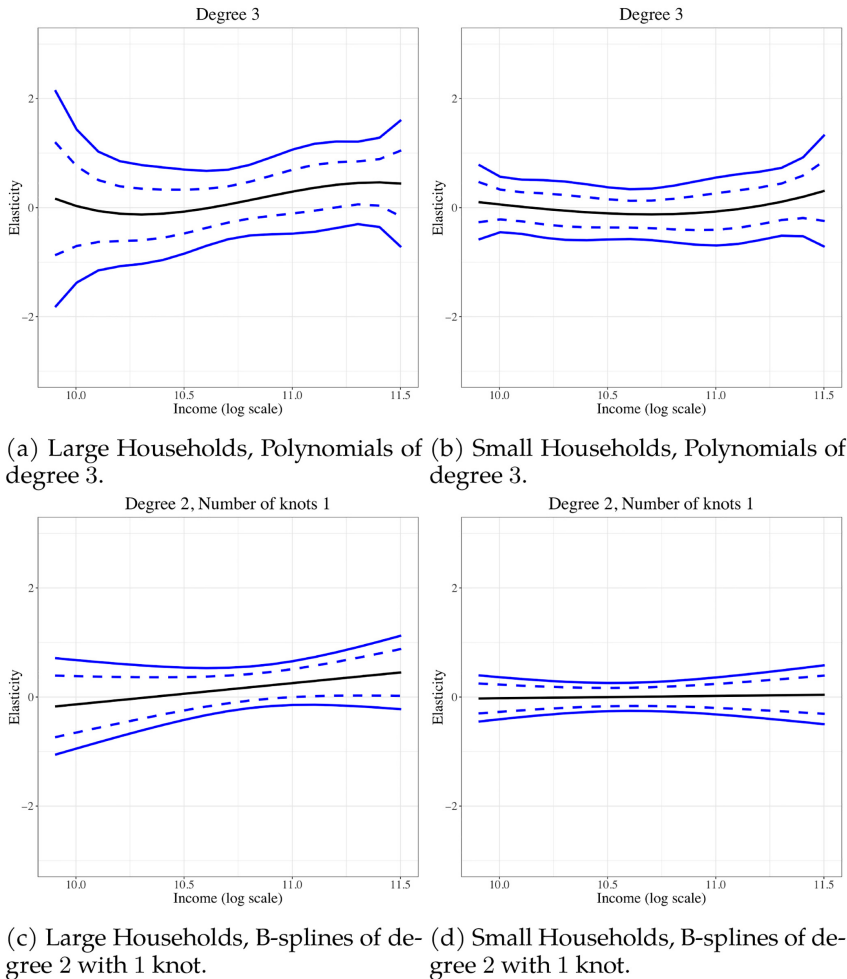
The data for our analysis are the same as in Yatchew and No (2001), coming from the National Private Vehicle Use Survey, conducted by Statistics Canada between October 1994 and September

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(a) Step 2: $l(z)$ is estimated by Lasso. (b) Step 2: $l(z)$ is estimated by random
Step 3: polynomials of degree 3. forest. Step 3: polynomials of degree 3.

(c) Step 2: $l(z)$ is estimated by Lasso. (d) Step 2: $l(z)$ is estimated by random
Step 3: B-splines of order 2 with 1 knot. forest. B-splines of order 2 with 1 knot.

Figure 3. 95% confidence bands for the best linear approximation of the average price elasticity conditional on income with accounting for the demographic controls in the first stage. The black line is the estimated function, and the dashed (solid) blue lines are the pointwise (uniform) confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function $\mu(d, z)$, (2) second-stage estimation of the conditional density $s(d|z)$, and (3) third-stage estimation of the target function $g(x)$ by least-squares series. Step 1 is performed by using Lasso with standardised covariates and the penalty choice $\lambda = 2.2\sqrt{n}\widehat{\sigma}\,\Phi^{-1}(1 - \gamma/2p)$, where $\gamma = 0.1/\log n$ and $\widehat{\sigma}$ is the estimate of the residual variance. Step 2 is performed by estimating the regression function of $l(z) = \mathbb{E}[D|Z = z]$ and estimating the density $\phi(d - l(z))$ of the residual $d - l(z)$ by the adaptive kernel density estimator of Portnoy and Koenker (1989) with the Silverman choice of bandwidth. The regression function $l(z)$ is estimated Lasso (a, c) and random forest (b, d). Step 3 is performed using B-splines of order 2 with the number of knots equal to one (c, d) and polynomial functions of order 3 (a, b). Uniform confidence bands are based on $B = 200$ repetitions of weighted (Bayes) bootstrap algorithm, described in Belloni et al. (2015).

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(a) Large Households, Polynomials of degree 3.

(b) Small Households, Polynomials of degree 3.

(c) Large Households, B-splines of degree 2 with 1 knot.

(d) Small Households, B-splines of degree 2 with 1 knot.

Figure 4. 95% confidence bands for the best linear approximation of the average price elasticity conditional on income with accounting for the demographic controls in the first stage by household size. The black line is the estimated function, and the dashed (solid) blue lines are the pointwise (uniform) confidence bands. The estimation algorithm has three steps: (1) first-stage estimation of the conditional expectation function $\mu(d, z)$, (2) second-stage estimation of the conditional density $s(d|z)$, and (3) third-stage estimation of the target function $g(x)$ by least-squares series. Step 1 is performed by using Lasso with standardised covariates and the penalty choice $\lambda = 2.2\sqrt{n}\widehat{\sigma}\Phi^{-1}(1 - \gamma/2p)$, where $\gamma = 0.1/\log n$ and $\widehat{\sigma}$ is the estimate of the residual variance. Step 2 is performed by estimating the regression function of $l(z) = \mathbb{E}[D|Z = z]$ and estimating the density $\phi(d - l(z))$ of the residual $d - l(z)$ by adaptive kernel density estimator of Portnoy and Koenker (1989) with the Silverman choice of bandwidth. The regression function $l(z)$ is estimated Lasso. Step 3 is performed by using B-splines of order 2 with the number of knots equal to one (c, d) and using non-orthogonal polynomial functions of degree 3 (a, b). Uniform confidence bands are based on $B = 200$ repetitions of weighted (Bayes) bootstrap algorithm, described in Belloni et al. (2015).

1996. The data set is based on fuel purchase diaries and contains detailed information about fuel prices, fuel consumption patterns, vehicles, and demographic characteristics. We use the same selection procedure as in Yatchew and No (2001) and Belloni et al. (2019), focusing on a sample of the households with non-zero licensed drivers, vehicles, and distances driven, which leaves us with 5,001 observations.

The object of interest is the average predicted percentage change in the demand due to a unit percentage change in the price, holding the observed demographic characteristics fixed, conditional on income. In the context of Example 2.4, this corresponds to the conditional average derivative

$$g(x) = \mathbb{E}[\partial_d \mu(D, Z) | X = x],$$

$$\mu(d, z) = \mathbb{E}[Y | D = d, Z = z],$$

where Y is the logarithm of gas consumption, D is the logarithm of price per liter, X is log income, and Z are the observed subject characteristics, such as household size and composition, distance driven, the type of fuel used, and income. We use the orthogonal signal $Y(\eta)$ of (8).

The choice of the estimators in the first and the second stages is as follows. To estimate the conditional expectation function $\mu(d, z)$ and its partial derivative $\partial_d \mu(d, z)$, we consider a linear model

$$\mu(d, z) = b(d, z)' \omega,$$

where the basis function $b(d, z)$ includes price, price squared, income, income squared, and their interactions with 28 time, geographical, and household composition dummies. All in all, we have 91 explanatory variables. We estimate the coefficient vector ω using Lasso with the penalty level chosen as in Belloni et al. (2014) and plug the estimate $\widehat{\omega}$ into the expression for the derivative,

$$\partial_d \mu(d, z) = \partial_d b(d, z)' \omega,$$

to estimate $\partial_d \mu(d, z)$.

To estimate the conditional density $s(d|z)$, we consider a model:

$$D = l(Z) + U, \quad U \perp Z,$$

where $l(z) = \mathbb{E}[D | Z = z]$ is the conditional expectation of price variable D given covariates Z, and U is an independent continuously distributed shock with univariate density $\phi(\cdot)$. Under this assumption, the log density $\partial_d \log s(d|z)$ equals to

$$\partial_d \log s(d|z) = \frac{\phi'(d - l(z))}{\phi(d - l(z))}.$$

We estimate $\phi(u) : \mathbb{R} \to \mathbb{R}^+$ by an adaptive kernel density estimator of Portnoy and Koenker (1989) with Silverman choice of bandwidth. Finally, we plug in the estimates of $\mu(d, z)$, $\partial_d \mu(d, z)$, and $s(d|z)$ into Equation (1.4) to get an estimate of $\widehat{Y}$ and estimate $g(x)$ by least-squares series regression of $\widehat{Y}$ on X. We try both polynomial basis function and B-splines to construct technical regressors.

Figures 3 and 4 report the estimate of the target function (the black line) and the pointwise (the dashed blue lines) and the uniform confidence (the solid blue lines) bands for the average price elasticity conditional on income, where the significance level $\alpha = 0.05$. The panels of Figure 3 correspond to different choices of the first-stage estimates of the nuisance functions $\mu(d, z)$ and $s(d|z)$ and dictionaries of technical regressors. The panels of Figure 4 correspond to the subsamples of large and small households and to different choices of the dictionaries.

The summary of our empirical findings based on Figures 3 and 4 is as follows. We find the elasticity to be in the range $(-1, 0)$ and significant for the majority of income levels. The estimates based on B-splines (Figure 3,c and d) are monotonically increasing in income, which is intuitive. The estimates based on polynomial functions are non-monotonic in income. For every algorithm in Figure 3 we cannot reject the null hypothesis of constant price elasticity for all income levels; for each estimation procedure, the uniform confidence bands contain the constant function. Figure 4 shows the average price elasticity conditional on income for small and large households. For the majority of income levels, we find large households to be more price elastic than the small ones, but the difference is not significant at any income level.

To demonstrate the relevance of demographic data Z in the first-stage estimation, we also show the average predicted effect of the price change on the gasoline consumption (in logs), without accounting for the covariates in the first stage. In particular, this effect equals to $\mathbb{E}[\partial_d \mu(D, X)|X = x]$, where $\mu(d, x) = \mathbb{E}[Y|D = d, X = x]$ is the conditional expectation of gas consumption given income and price. Figure 2 shows this predictive effect, approximated by the polynomials of degree $d \in \{1, 2\}$, conditional on income. By contrast to the results in Figure 3, the slope of the polynomial of degree $d = 1$ has a negative relationship between income and price elasticity, which presents evidence that the demographics strongly confound the relationship between income and price elasticity.

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SUPPORTING INFORMATION

Additional Supporting Information may be found in the online version of this article at the publisher's website:

Online Supplement

Co-editor Jaap Abbring handled this manuscript.

ONLINE SUPPLEMENT

This document contains supplementary materials for the article 'Debiased Machine Learning of Conditional Average Treatment Effects and Other Causal Functions' in the *Econometrics Journal*. Appendix A contains proofs. Appendix B contains verification of high-level conditions.

APPENDIX A: PROOFS

Glossary. For two sequences of random variables $a_N, b_N, N \geq 1 : a_N \lesssim_P b_n$ means $a_N = O_P(b_N)$. For two sequences of numbers $a_N, b_N, N \geq 1$, $a_N \lesssim b_N$ means $a_N = O(b_N)$. Let $a \wedge b = \min\{a, b\}, a \vee b = \max\{a, b\}$. The ℓ_2 norm of a vector is denoted by $\|\cdot\|$, the ℓ_1 norm is denoted by $\|\cdot\|_1$, the ℓ_∞ is denoted by $\|\cdot\|_\infty$, and ℓ_0 is denoted by $\|\cdot\|_0$. Given a vector $\delta \in \mathbb{R}^p$ and a set of indices $T \subset \{1, ..., p\}$, we denote by δ_T the vector in $\mathbb{R}^p$ for which $\delta_{Tj} = \delta_j, j \in T$ and $\delta_{Tj} = 0, j \notin T$. Let $\xi_d := \sup_{x \in \mathcal{X}} \|p(x)\| = \sup_{x \in \mathcal{X}} (\sum_{j=1}^d p_j(x)^2)^{1/2}$. For a matrix Q, let $\|Q\|$ be the maximal eigenvalue of Q. For a random variable V, let $\|V\|_{P,q} := (\int |V|^q dP)^{1/q}$. The random sample $(V_i)_{i=1}^N$ is a sequence of independent copies of a random element V taking values in a measurable space $(\mathcal{V}, \mathcal{A}_\mathcal{V})$ according to a probability law P. Let $p_i := p(X_i)$ and $r_i := r_g(X_i)$. Let $w_X(x)$ be the marginal density of x, $w_{X,Z}(x, z)$ be the joint density of x, z, $w_Z(z)$ be the marginal density of z, and $s(x|z)$ be the conditional density of x given z.

Define an event $\mathcal{E}_N := \{\widehat{\eta}_k \in T_N \quad \forall k \in [K]\}$, such that the nuisance parameter estimate $\widehat{\eta}_k$ belongs to the realisation set T_N for each fold $k \in [K]$. By union bound, this event holds with high probability.

$$P(\mathcal{E}_N) \geq 1 - K\epsilon_N = 1 - o(1).$$

For a given partition k in $\{1, 2, \ldots, K\}$, define the partition-specific averages

$$\mathbb{E}_{n,k} f(V_i) := \frac{1}{n} \sum_{i \in J_k} f(V_i), \quad \mathbb{G}_{n,k} f(V_i) := \frac{1}{\sqrt{n}} \sum_{i \in J_k} [f(V_i) - \int f(v) dP(v)].$$

A.1. Useful technical lemmas

LEMMA A.1 (LLN FOR MATRICES). *Let $Q_i, i = 1, 2, \ldots, N$ be independent not necessarily identically distributed. symmetric non-negative $d \times d$-matrices such that $d \geq e^2$ and $\|Q_i\| \leq M$ a.s. Let $Q = \frac{1}{N} \sum_{i=1}^N \mathbb{E} Q_i$ denote average value of population covariance matrices. Then, the following statement holds:*

$$\mathbb{E} \|\widehat{Q} - Q\| \leq \sqrt{\frac{M(1 + \|Q\|) \log N}{N}}.$$

In particular, if $Q_i = p_i p_i'$ with $\|p_i\| \leq \xi_d$, then

$$\mathbb{E} \|\widehat{Q} - Q\| \leq \sqrt{\frac{\xi_d^2 (1 + \|Q\|) \log N}{N}}.$$

Proof can be found in Rudelson (1999). The statement is taken from Belloni et al. (2015).

LEMMA A.2 (CONDITIONAL CONVERGENCE IMPLIES UNCONDITIONAL). *Let $\{X_m\}_{m \geq 1}$ and $\{Y_m\}_{m \geq 1}$ be sequences of random vectors. (i) If for $\epsilon_m \to 0$, $P(\|X_m\| > \epsilon_m | Y_m) \to_P 0$, then $P(\|X_m\| > \epsilon_m) \to 0$. In particular, this occurs if $\mathbb{E}[\|X_m\|^q / \epsilon_m^q | Y_m] \to_P 0$ for some $q \geq 1$, by Markov inequality. (ii) Let $\{A_m\}_{m \geq 1}$ be a sequence of positive constants. If $\|X_m\| = O_P(A_m)$ conditional on Y_m, namely, that for any $\ell_m \to \infty$, $P(\|X_m\| > \ell_m A_m | Y_m) \to_P 0$, then $X_m = O_P(A_m)$ unconditionally, namely, that for any $\ell_m \to \infty$, $P(\|X_m\| > \ell_m A_m) \to 0$.*

Lemma A.2 is a restatement of Lemma 6.1 of Chernozhukov et al. (2018).

A.2. Proofs of lemmas from Section 3

LEMMA A.3 (NO EFFECT OF FIRST-STAGE ERROR). *Suppose Assumption 3.5 holds. Then,*

$$\sqrt{N}\|\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\| = O_P(B_N + \Lambda_N) = o(1).$$

Proof of Lemma A.3 The total sum is the sum of K partial sums, defined for each partition $(J_k)_{k=1}^K$:

$$\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)] = \frac{1}{K} \sum_{k=1}^K \mathbb{E}_{n,k} p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)] - \mathbb{E}\big[p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]|(V_i)_{i\in J_k^c}\big]$$

$$+ \mathbb{E}\big[p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]|(V_i)_{i\in J_k^c}\big]$$

$$=: \frac{1}{K} \sum_{k=1}^K [I_{1,k} + I_{2,k}].$$

Conditional on $(V_i)_{i\in J_k}$, the estimator $\widehat{\eta} = \widehat{\eta}_k$ is non-stochastic. On the event $\mathcal{E}_N$,

$$\mathbb{E}[\|\sqrt{n} I_{1,k}\|^2 | \mathcal{E}_N, (V_i)_{i\in J_k^c}] \leq \mathbb{E}[\|p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\|^2 | \mathcal{E}_N, (V_i)_{i\in J_k^c}]$$

$$\leq \sup_{\eta \in T_N} \mathbb{E}\|p_i[Y_i(\eta) - Y_i(\eta_0)]\|^2 = \Lambda_N^2 \qquad \text{(Assumption 3.5)}$$

Hence, $\sqrt{n} I_{1,k} = O_P(\Lambda_N)$ by Lemma A.2. To bound $I_{2,k}$, recognise that on the event $\mathcal{E}_N$

$$\mathbb{E}[\|\sqrt{n} I_{2,k}\| | \mathcal{E}_N, (V_i)_{i\in J_k^c}] \leq \sup_{\eta \in T_N} \sqrt{n}\|\mathbb{E} p_i(Y_i(\eta) - Y_i(\eta_0))|(V_i)_{i\in J_k}\|$$

$$\leq \sup_{\eta \in T_N} \sqrt{n}\|\mathbb{E} p_i(Y_i(\eta) - Y_i(\eta_0))\| \leq B_N.$$

Therefore, $\sqrt{n} I_{2,k} = O_P(B_N)$. $\qquad\qquad\square$

Proof of Lemma 3.1 (a)

$$\|\widehat{\beta} - \beta_0\| = \|\widehat{Q}^{-1}\mathbb{E}_N p_i Y_i(\widehat{\eta}) - \beta_0\| \leq \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\|$$

$$+ \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[Y_i(\eta_0) - p_i'\beta_0]\|$$

$$=: S_1 + \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[Y_i(\eta_0) - g(X_i)]\| + \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[g(X_i) - p_i'\beta_0]\|$$

$$=: S_1 + \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i U_i\| + \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i r_i\|.$$

The effect of sampling error (second term) is bounded as

$$\|\mathbb{E}_N p_i U_i\| \lesssim_P (\mathbb{E}[\|\mathbb{E}_N p_i U_i\|^2])^{1/2} \leq (\mathbb{E} U_i^2 p_i' p_i / N)^{1/2} \lesssim \overline{\sigma}\sqrt{d/N}.$$

The effect of approximation error (third term) is bounded as

$$\|\mathbb{E}_N p_i r_i\| \lesssim_P (\mathbb{E}[\|\mathbb{E}_N p_i r_i\|^2])^{1/2} \lesssim l_d r_d \sqrt{\frac{\mathbb{E}\|p_i\|^2}{N}} = l_d r_d \sqrt{\frac{d}{N}}.$$

Alternatively, the third term can be also bounded as

$$\|\mathbb{E}_N p_i r_i\| \lesssim_P (\mathbb{E}[\|\mathbb{E}_N p_i r_i\|^2])^{1/2} \leq \xi_d \sqrt{\frac{\mathbb{E} r_i^2}{N}} = \xi_d r_d / \sqrt{N}.$$

With high probability, $\|\widehat{Q}^{-1}\| \leq 2\|Q^{-1}\| \leq 2/C_{\min}$. Lemma A.3 implies

$$\|S_1\| \leq \|\widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\| \leq \|\widehat{Q}^{-1}\|[B_N/\sqrt{N} + \Lambda_N/\sqrt{N}] = o_P(N^{-1/2}). \qquad\square$$

Proof of Lemma 3.1 (b) Decomposing

$$\sqrt{N}[\mathbb{E}_N p_i Y_i(\widehat{\eta}) - \widehat{Q}\beta_0] = \sqrt{N}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)] + \sqrt{N}\mathbb{E}_N p_i[Y_i(\eta_0) - p_i'\beta_0]$$
$$= \sqrt{N}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)] + \mathbb{G}_N p_i[Y_i(\eta_0) - p_i'\beta_0]$$
$$= \sqrt{N}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)] + \mathbb{G}_N p_i U_i + \mathbb{G}_N p_i r_i$$

we obtain:

$$\sqrt{N}\alpha'\,[\widehat{\beta} - \beta_0] = \sqrt{N}\alpha'\widehat{Q}^{-1}[\mathbb{E}_N p_i Y_i(\widehat{\eta}) - \widehat{Q}\beta_0]$$
$$= \sqrt{N}\alpha'\widehat{Q}^{-1}[\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]] + \alpha'\widehat{Q}^{-1}\mathbb{G}_N[p_i[r_i + U_i]]$$
$$= \alpha'Q^{-1}\mathbb{G}_N[p_i[r_i + U_i]]$$
$$+ \alpha^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{G}_N[p_i(U_i + r_i)] + \sqrt{N}\alpha^\top Q^{-1}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]$$
$$+ \sqrt{N}\alpha^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]$$
$$=: \alpha'Q^{-1}\mathbb{G}_N[p_i[r_i + U_i]] + R_{1,N}(\alpha),$$

where the remainder term $R_{1,N}$ is

$$R_{1,N}(\alpha) = I_1 + I_2 + I_3.$$

Decomposing I_1 into sampling and approximation parts gives

$$I_1 = \sqrt{N}\alpha^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{E}_N p_i U_i + \alpha^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{G}_N p_i r_i =: I_{1,a} + I_{1,b}.$$

Observe that $\mathbb{E}[I_{1,a}|(X_i)_{i=1}^N] = 0$. As shown in the proof of Lemma 4.1 in Belloni et al. (2015), its second moment is bounded as

$$\mathbb{E}[I_{1,a}^2|(X_i)_{i=1}^N] \le \alpha^\top[\widehat{Q}^{-1} - Q^{-1}]Q[\widehat{Q}^{-1} - Q^{-1}]\alpha\overline{\sigma}^2 \lesssim_P \frac{\xi_d^2 \log N}{N}\overline{\sigma}^2,$$

which implies $I_{1,a} = O_P\left(\sqrt{\frac{\xi_d^2 \log N}{N}}\right)$. Likewise, by the proof of Lemma 4.1 in Belloni et al. (2015),

$$|I_{1,b}| \lesssim_P \sqrt{\frac{\xi_d^2 \log N}{N}}[l_d r_d \sqrt{d} \wedge \xi_d r_d].$$

By Lemma A.3,

$$|I_2| \lesssim_P \|Q\|^{-1}\|\sqrt{N}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\| \lesssim_P 1/C_{\min}[\Lambda_N + B_N] = o(1),$$

$$|I_3| \lesssim_P \|\alpha\|\|[\widehat{Q}^{-1} - Q^{-1}]\|\|\sqrt{N}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\| \lesssim_P \sqrt{\frac{\xi_d^2 \log N}{N}}[\Lambda_N + B_N] = o(1).$$

Therefore, with probability approaching one,

$$\sup_{\eta \in T_N} \|R_{1,N}(\alpha)\| \lesssim_P \Lambda_N + B_N + \sqrt{\frac{\xi_d^2 \log N}{N}}\left(1 + \min\left\{l_d r_d \sqrt{d}, \xi_d r_d\right\}\right).$$

$\square$

Proof of Lemma 3.2 (a) Define

$$I_1(x) = \alpha(x)^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{G}_N[p_i(U_i + r_i)],$$
$$I_2(x) = \sqrt{N}\alpha(x)^\top Q^{-1}\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)],$$
$$I_3(x) = \sqrt{N}\alpha(x)^\top[\widehat{Q}^{-1} - Q^{-1}]\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)].$$

Decompose

$$\sqrt{N}\alpha(x)^{\top}(\widehat{\beta} - \beta_0) = \sqrt{N}\alpha(x)^{\top}Q^{-1}\mathbb{G}_N[p_i[r_i + U_i]] + I_1(x) + I_2(x) + I_3(x)$$

Step 1. According to Step 1 of the proof of Lemma 4.2 in Belloni et al. (2015), the following bound holds

$$\sup_{x\in\mathcal{X}}|\alpha(x)^{\top}[\widehat{Q}^{-1} - Q^{-1}]\mathbb{G}_N p_i U_i| \lesssim_P N^{1/m}\sqrt{\frac{\xi_d^2 \log N}{N}}$$

by Assumptions 3.2, 3.6, 3.7.

Step 2. According to Step 2 of the proof of Lemma 4.2 in Belloni et al. (2015), the following bound holds

$$\sup_{x\in\mathcal{X}}|\alpha(x)^{\top}[\widehat{Q}^{-1} - Q^{-1}]\mathbb{G}_N p_i r_i| \lesssim_P \sqrt{\frac{\xi_d^2 \log N}{N}}l_d r_d\sqrt{d}$$

by Assumptions 3.2 and 3.3. Steps 1 and 2 give the bound on $\sup_{x\in\mathcal{X}} I_1(x)$.

Step 3. The bound on $I_2(x) + I_3(x)$ follows from Lemma A.3

$$\sup_{x\in\mathcal{X}}|I_2(x)| \leq \sup_{x\in\mathcal{X}}\|\alpha(x)\|C_{\min}^{-1}\|\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\|$$

$$\lesssim_P B_N + \Lambda_N = o(1)$$

$$\sup_{x\in\mathcal{X}}|I_3(x)| \leq \sup_{x\in\mathcal{X}}\|\alpha(x)\|\|Q^{-1} - \widehat{Q}^{-1}\|\|\mathbb{E}_N p_i[Y_i(\widehat{\eta}) - Y_i(\eta_0)]\|$$

$$\lesssim_P \sqrt{\frac{\xi_d^2 \log N}{N}}(B_N + \Lambda_N) = o(1).$$

Lemma 3.2(b) follows from Theorem 4.3 in Belloni et al. (2015). □

A.3. Proofs of main results from Section 3

Proof of Theorem 3.1 Proof of Theorem 3.1 follows from Lemma 3.1 and the proof of Theorem 4.2 in Belloni et al. (2015). □

Proof of Theorem 3.2 Proof of Theorem 3.2 follows from Lemma 3.2 and the proof of Theorem 4.4 in Belloni et al. (2015). □

Proof of Theorem 3.3 Define the estimator of the matrix $\widehat{\Sigma}$ as

$$\widehat{\Sigma} := \mathbb{E}_N p_i p_i^{\top}\widehat{U}_i^2 := \mathbb{E}_N p_i p_i^{\top}(Y_i(\widehat{\eta}) - p(X_i)'\widehat{\beta})^2,$$

and $\widehat{\Omega} := \widehat{Q}^{-1}\widehat{\Sigma}\widehat{Q}^{-1}$. Decompose the difference $\widehat{\Sigma} - \Sigma$ as

$$\widehat{\Sigma} - \Sigma = \mathbb{E}_N p_i p_i'[\widehat{U}_i^2 - \{U_i + r_i\}^2] + \mathbb{E}_N p_i p_i'\{U_i + r_i\}^2 - \Sigma =: I_1 + I_2.$$

In Belloni et al. (2015), it was shown that $I_2 \lesssim_P (N^{1/m} + l_d r_d)\sqrt{\frac{\xi_d^2 \log N}{N}}$. Therefore, it suffices to prove a bound on I_1. Recognise that

$$\widehat{U}_i = Y_i(\eta_0) - p_i'\beta_0 + p_i'\beta_0 - p_i'\widehat{\beta} + Y_i(\widehat{\eta}) - Y_i(\eta_0)$$

$$= (U_i + r_i) + (p_i'(\beta_0 - \widehat{\beta})) + (Y_i(\widehat{\eta}) - Y_i(\eta_0)) = a + b + c.$$

Plugging $|(a+b+c)^2 - a^2| = |2a(b+c) + (b+c)^2| \le 2(c^2 + b^2 + |ac| + |ab|)$ into I_1 gives:

$$
\begin{aligned}
I_1 := \|\mathbb{E}_N p_i p_i'[\widehat{U}_i^2 - \{U_i + r_i\}^2]\| &\le 2\|\mathbb{E}_N p_i p_i'(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2\| \\
&\quad + 2\|\mathbb{E}_N p_i p_i'(U_i + r_i)(Y_i(\widehat{\eta}) - Y_i(\eta_0))\| \\
&\quad + 2\|\mathbb{E}_N p_i p_i'(p_i'(\widehat{\beta} - \beta_0))^2\| \\
&\quad + 2\|\mathbb{E}_N p_i p_i'(U_i + r_i)p_i'(\widehat{\beta} - \beta_0)\| \\
&:= 2(C^2 + AC + B^2 + AB).
\end{aligned}
$$

For each $k \in \{1, 2, \ldots, K\}$, conditional on $(V_j)_{j \in J_k^c}$ and the event $\mathcal{E}_N$,

$$
\max_{i \in J_k}(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2 \lesssim_P \mathbb{E}[\max_{i \in J_k}(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2 | (V_j)_{j \in J_k^c}, \mathcal{E}_N] + o_P(1)
$$

$$
\lesssim \sup_{\eta \in \mathcal{T}_N} \mathbb{E} \max_{i \in J_k}(Y_i(\eta) - Y_i(\eta_0))^2 = \kappa_n^2.
$$

Therefore,

$$
\max_{i \in \{1, 2, \ldots, N\}}(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2 \lesssim_P K\kappa_n^2 + o_P(1).
$$

$$
C^2 \lesssim_P \max_{1 \le i \le N}(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2 \|\mathbb{E}_N p_i p_i'\|
$$

$$
\lesssim_P \max_{1 \le i \le N}(Y_i(\widehat{\eta}) - Y_i(\eta_0))^2 O_P(1) \lesssim_P \kappa_n^2 \asymp \kappa_N^2
$$

The bound on AC can be seen as

$$
AC \lesssim_P \max_{1 \le i \le N}|Y_i(\widehat{\eta}) - Y_i(\eta_0)| \max_{1 \le i \le N}\{|U_i| + |r_i|\}\|\mathbb{E}_N p_i p_i'\|
$$

$$
\lesssim_P \max_{1 \le i \le N}|Y_i(\widehat{\eta}) - Y_i(\eta_0)| \max_{1 \le i \le N}\{|U_i| + |r_i|\}O_P(1)
$$

$$
\lesssim_P \kappa_N^1(N^{1/m} + l_d r_d).
$$

The bound on $B^2 + AB$ is established in the proof of Theorem 4.6 in Belloni et al. (2015). It is

$$
B^2 + AB \lesssim_P (N^{1/m} + l_d r_d)\sqrt{\frac{\xi_d^2 \log N}{N}}.
$$

Collecting the bounds on $C^2 + AC + B^2 + BC$ yields:

$$
I_1 \le 2(C^2 + AC + B^2 + AB) \lesssim_P (N^{1/m} + l_d r_d)\left(\sqrt{\frac{\xi_d^2 \log N}{N}} + \kappa_N^1\right) + \kappa_N^2. \tag{A.1}
$$

Thus, the statement of Theorem follows from the proof of Theorem 4.6 of Belloni et al. (2015) and (1). Finally, (13) follows from the proof of Lemma 5.1 in Belloni et al. (2015). $\qquad \square$

Proof of Theorem 3.4 Theorem 3.4 coincides with Condition C.2(b) in Chernozhukov et al. (2013) with $\mathcal{V} = \mathcal{X}$ and

$$
Z_N^\star(x) = \frac{p(x)'\widehat{\Omega}^{1/2}}{\|p(x)'\widehat{\Omega}^{1/2}\|}\mathcal{N}_d^b, \quad Z_N^*(x) = \frac{p(x)'\Omega^{1/2}}{\|p(x)'\Omega^{1/2}\|}\mathcal{N}_d^b, \quad \delta_N = o(1/\log N)
$$

for some ℓ_N chosen below. Condition C.2(b) holds by Lemma 5 of Chernozhukov et al. (2013), whose assumptions we verify below. First, condition NS 1 (a) of Lemma 5 is verified by Theorem 3.2 with $\bar{a}_N = \log N$. Second, the eigenvalues of matrix Ω are bounded below by the minimal eigenvalue of $\underline{\sigma}^2 Q^{-1}$, which is bounded away from zero by Assumption 3.1. Finally, the last two conditions of C.2 are directly assumed in the theorem. Therefore, all conditions of Lemma 5 are verified, and the statement of Theorem holds. $\qquad \square$

We approximate the t-statistic process in (10) by the following Gaussian coupling process conditional on the data:

$$\left\{ t_N^*(x) = \frac{p(x)'\Omega^{1/2}\mathcal{N}_d/\sqrt{N}}{\sigma_N(x)}, x \in \mathcal{X} \right\} \tag{A.2}$$

LEMMA A.4 (STRONG APPROXIMATION OF THE T-STATISTIC PROCESS). *Suppose Assumptions 3.1 through 3.8 hold. In addition, suppose the assumptions of Theorem 3.2 hold with $\bar{a}_N^{-1} = o(1)$. Then,*

$$t_N(x) = t_N^*(x) + o_P(\bar{a}_N) \text{ in } \ell^\infty(\mathcal{X}),$$

where $t_N(x)$ is as in (10) and $t_N^*(x)$ is as in (2).

The statement of Lemma A.4 follows from the proof of Theorem 5.4 from Belloni et al. (2015). Specifically, the proof of Theorem 5.4 is invoked with $a_N = \bar{a}_N$, $\ell(w) = p(x)$, $\mathcal{I} = \mathcal{X}$, Theorem 3.3(b) (in place of Lemma 5.1), Lemma 3.2(a) (in place of Lemma 5.2), Lemma 3.2(b) (in place of Theorem 5.3), and Theorem 3.2 (in place of Theorem 4.4), where in parentheses we refer to statements from Belloni et al. (2015).

LEMMA A.5 (STRONG APPROXIMATION OF GAUSSIAN PROCESS SUPREMA). *Suppose Assumptions 3.1 through 3.8 hold with $m \geq 4$. Assume that (i) $\bar{R}_{1N} + \bar{R}_{2N} \lesssim \log^{-1/2}(N)$, (ii)*

$$\xi_d \log^2 N / N^{1/2-1/m} = o(1),$$

(iii) $1 \lesssim \underline{\sigma}^2$, and (iv) $\sup_{x \in \mathcal{X}} \sqrt{N}|r(x)|/\|p(x)\| = o(1/\sqrt{\log N})$. Then,

$$\sup_{x \in \mathcal{X}} |t_N(x)| =_d \sup_{x \in \mathcal{X}} |t_N^*(x)| + o_P(1/\sqrt{\log N}).$$

The statement of Lemma A.5 follows from the proof of Theorem 5.5 from Belloni et al. (2015) with $a_N = \sqrt{\log N}$, $\ell(w) = p(x)$, $\mathcal{I} = \mathcal{X}$. Specifically, the proof of Theorem 5.5 is invoked with $a_N = \bar{a}_N$, $\ell(w) = p(x)$, $\mathcal{I} = \mathcal{X}$, Lemma A.4 (in place of Theorem 5.4), Lemma 3.2(a) (in place of Lemma 5.2), where in parentheses we refer to statements from Belloni et al. (2015).

Proof of Theorem 3.5 The proof follows by invoking Lemma A.5 and the same steps as the proof of Theorem 5.5 in Belloni et al. (2015), replacing the bound on $\|\widehat{\Omega} - \Omega\|$ by the bound from Theorem 3.3. The validity of the critical value obtained by Gaussian bootstrap follows exactly the same steps as in the proof of Theorem 5.6 in Belloni et al. (2015). $\qquad\square$

A.4. Proofs from Section 4

LEMMA A.6 (MAXIMAL INEQUALITY FOR CANONICAL U-STATISTICS). *Let $(V_i)_{i=1}^N$ be a sample of independent and indentically distributed (i.i.d) random variables. Let $\tilde{\tau}(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathcal{R}^d$ be a symmetric d-dimensional kernel function whose m^{th} coordinate is denoted by $\tilde{\tau}_m(\cdot, \cdot)$. The kernel function is canonical:*

$$\mathbb{E}[\tilde{\tau}_m(V_1, V_2)] = 0 \quad \text{for any } m \in \{1, 2, \dots, d\}.$$

Assume that $\mathbb{E}|\tilde{\tau}_m(V_1, V_2)| < \infty$ for all $m \in \{1, 2, \dots, d\}$. Let

$$\bar{V}_N = \frac{1}{N(N-1)} \sum_{1 \leq i \neq j \leq N} \tilde{\tau}(V_i, V_j),$$

$$M = \max_{1 \leq i \neq j \leq N} \max_{1 \leq m \leq d} |\tilde{\tau}_m(V_i, V_j)|,$$

and $D_q = \max_{1 \le m \le d} (\mathbb{E}|\tilde{\tau}_m(V_1, V_2)|^q)^{1/q}$, $q > 0$. If $2 \le d \le \exp(bN)$ for some constant $b > 0$, then there exists an absolute constant $K > 0$ such that

$$\mathbb{E}[\|\bar{V}_N\|_\infty] \le K(1 + b^{1/2}) \left\{ \frac{\log d}{N} D_2 + \left(\frac{\log d}{N} \right)^{5/4} D_4 + \left(\frac{\log d}{N} \right)^{3/2} \|M\|_4 \right\}$$

Lemma A.6 is a restatement of Theorem 5.1 of Chen (2018). In the present paper, d is less than N. Therefore, the right-hand side is bounded by $K(1 + b^{1/2})(D_2 \vee D_4 \vee \|M\|_4) \frac{\log d}{N}$.

For a symmetric function $f : \mathcal{V}^r \to \mathcal{R}$, denote

$$P^{r-k} f(x_1, x_2, \ldots, x_k) = \mathbb{E} f(x_1, x_2, \ldots, x_k, X_{k+1}, \ldots, X_r)$$

and let $P^0 f = f$. Let $\mathcal{F}$ be a class of symmetric functions, and denote $\| \cdot \|_\mathcal{F} = \sup_{f \in \mathcal{F}} | \cdot |$. Assume that there exists an envelope F for $\mathcal{F}$ such that $P^r F^2 < \infty$. Consider the associated U-process

$$U_N^{(r)}(f) = \frac{1}{|I_{N,r}|} \sum_{(i_1, i_2, \ldots, i_r) \in I_{N,r}} f(V_{i,1}, V_{i,2}, \ldots, V_{i,r}), \quad f \in \mathcal{F},$$

where $I_{N,r}$ is a collection of permutations. For each $k \in \{1, 2, \ldots, r\}$, the Hoeffding projection (with respect to P) is defined by

$$\pi_k(f)(x_1, x_2, \ldots x_k) := (\delta_{x_1} - P) \ldots (\delta_{x_k} - P) P^{r-k} f,$$

where δ_x is the delta-function at x. Let σ_k be any positive constant such that $\sup_{f \in \mathcal{F}} \|P^{r-k} f\|_{P^k, 2} \le \sigma_k \le \|P^{r-k} F\|_{P^k, 2}$ whenever $\|P^{r-k} F\|_{P^k, 2} > 0$ and let $\sigma_k = 0$ otherwise. Define M_k as

$$M_k = \max_{1 \le i \le [N/k]} (P^{r-k} F)(V_{i(k-1)+1}^{ik}),$$

where $V_{i(k-1)+1}^{ik} = (V_{i(k-1)+1}, V_{i(k-1)+2}, \ldots V_{ik})$. Finally, let $J_k(\delta)$ be a uniform entropy integral

$$J_k(\delta) := \int_0^\delta \sup_Q \left[(1 + \log N(P^{r-k} \mathcal{F}, \| \cdot \|_{Q,2}, \|P^{r-k} F\|_{Q,2})) \right]^{k/2} d\tau,$$

where $P^{r-k} \mathcal{F} = \{P^{r-k} f : f \in \mathcal{F}\}$ and $\sup_Q$ is taken over all finitely discrete distributions on $\mathcal{V}^r$. A function class $\mathcal{F}$ with envelope F is said to be Vapnik–Chervonenkis-type with characteristics (A, v) if $\sup_Q N(\mathcal{F}, \|\|_{Q,2}, \epsilon \|F\|_{Q,2}) \le (A/\epsilon)^v$ for all $0 < \epsilon \le 1$, where $\sup_Q$ is taken over all finitely discrete distributions on $\mathcal{V}^r$.

LEMMA A.7 (LOCAL MAXIMAL INEQUALITIES FOR U-PROCESSES). *Suppose $J_k(1) < \infty$ for $k = 1, 2, \ldots, r$. Let $\delta_k = \sigma_k / \|P^{r-k} F\|_{P^k, 2}$ for $k = 1, 2, \ldots, r$. Then,*

$$N^{k/2} \mathbb{E}[\|U_N^{(k)}(\pi_k f)\|_\mathcal{F}] \lesssim J_k(\delta_k) \|P^{r-k} F\|_{P^k, 2} + \frac{J_k^2(\delta_k) \|M_k\|_{P,2}}{\delta_k^2 \sqrt{N}}.$$

Lemma A.7 is a restatement of Theorem 5.1 of Chen and Kato (2019).

LEMMA A.8 (LOCAL MAXIMAL INEQUALITIES FOR U-PROCESSES INDEXED BY VC-TYPE CLASSES). *Let $\mathcal{F}$ be VC-type with characteristics $A \ge e^{2(r-1)}/16 \vee e$ and $v \ge 1$. Then, for $k = 1, 2, \ldots, r$,*

$$N^{k/2} \mathbb{E}[\|U_N^{(k)}(\pi_k f)\|_\mathcal{F}] \lesssim \sigma_k \left\{ v \log(A \|P^{r-k} F\|_{P^k, 2} / \sigma_k) \right\}^{k/2}$$

$$+ \frac{\|M_k\|_{P,2}}{\sqrt{N}} \left\{ v \log(A \|P^{r-k} F\|_{P^k, 2} / \sigma_k) \right\}^k.$$

Lemma A.8 is a restatement of Corollary 5.3 of Chen and Kato (2019).

LEMMA A.9 (USEFUL PROPERTIES FOR CTEs). *Suppose Assumptions 3.1 and 4.9 hold. Let α be a vector on a unit sphere $\mathcal{S}^{d-1} = \{\alpha \in \mathcal{R}^d : \|\alpha\| = 1\}$ and let $v = Q^{-1} \alpha$. Then, the following properties hold for $\tau_1(z, \mu)$ in (15):*

(a) *The function $\mu^0(x, z)$ is bounded uniformly over its domain:*

$$\sup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} |\mu^0(x, z)| \leq 2\bar{\mathcal{C}}.$$

(b) *The function $\tau_1(z; \mu)$ is linear in μ for each $z \in \mathcal{Z}$*

$$\tau_1(z; \mu - \mu_0) = \tau_1(z; \mu) - \tau_1(z; \mu_0).$$

(c) *For each $z \in \mathcal{Z}$ the following inequality holds:*

$$|v'\tau_1(z; \mu)| \leq 2C_{\min}^{-1}\bar{\mathcal{C}}.$$

(d) *For any non-negative function $\phi(x, z)$,*

$$\int_{z\in\mathcal{Z}} \phi(x, z)w_Z(z)dz \lesssim \mathbb{E}[\phi(X, Z)|X = x] \quad \text{for any } x,$$

and

$$\int_{x\in\mathcal{X}} \left(\int_{z\in\mathcal{Z}} \phi(x, z)w_Z(z)\right)^2 w_X(x)dz \leq \int_{x\in\mathcal{X}}\int_{z\in\mathcal{Z}} \phi^2(x, z)w_Z(z)w_X(x)dzdx$$

$$\lesssim \mathbb{E}\phi^2(X, Z).$$

Proof of Lemma A.9 Lemma A.9(a). Because $\sup_{(x,z)\in\mathcal{X}\times\mathcal{Z}} |\mu(x, z)| \leq \bar{\mathcal{C}}$,

$$|\mu^0(x, z)| = |\mu(x, z) - \mathbb{E}\mu(x, Z)| \leq |\mu(x, z)| + |\mathbb{E}\mu(x, Z)| \leq 2\bar{\mathcal{C}}.$$

Lemma A.9(b) follows from the definition of $\tau_1(z; \mu)$. Lemma A.9(c) follows from the following inequality:

$$|v'\tau_1(z; \mu)| = |\mathbb{E}_X[(v'p(X))\cdot(\mu^0(X; z)))]|$$

$$\leq v'Qv(\mathbb{E}_X[\mu^0(X; z)]^2)^{1/2} \leq 2C_{\min}^{-1}\bar{\mathcal{C}},$$

where the last inequality follows from Lemma A.9(a) and $v'Qv = \alpha'Q^{-1}\alpha$. Lemma A.9(d) holds because

$$\int_{z\in\mathcal{Z}} \phi(x, z)w_Z(z)dz = \mathbb{E}\left[\phi(X, Z)\frac{w_X(X)}{s(X|Z)}\middle| X = x\right] \leq \bar{\mathcal{C}}^2\mathbb{E}[\phi(X, Z)|X = x]$$

$$\int_{x\in\mathcal{X}} \left(\int_{z\in\mathcal{Z}} \phi(x, z)w_Z(z)\right)^2 w_X(x)dz \leq \int_{x\in\mathcal{X}}\int_{z\in\mathcal{Z}} \phi^2(x, z)w_Z(z)w_X(x)dzdx$$

$$\leq \int_{x\in\mathcal{X}}\int_{z\in\mathcal{Z}} \phi^2(x, z)w_{Z,X}(x, z)\frac{w_X(x)}{s(x|z)}dxdz$$

$$\leq \bar{\mathcal{C}}^2\mathbb{E}\phi^2(X, Z). \qquad \square$$

Proof of Lemma 4.3 **Step 1**. Let us show that $Y(\eta)$ given by Equation (5) satisfies Assumption 3.5. Bayes' rule implies that $w(z|x) = \dfrac{s_0(x|z)w_Z(z)}{w_X(x)}$, where $w_X(x) = w_0(x)$ denotes the true values the marginal density of X, and $w(x)$ is a candidate value of the nuisance parameter. Likewise, let $s(x|z)$ and $s_0(x|z)$ be the true and the candidate value of the conditional density. Observe that

$$\mathbb{E}[Y(\eta_0)|X = x] = \int_{z\in\mathcal{Z}} \mu_0(x, z)dP_Z(z)$$

$$\mathbb{E}[Y(\eta)|X = x] = \mathbb{E}[\mathbb{E}[Y(\eta)|X, Z]|X = x] = \int_{z\in\mathcal{Z}} \mu(x, z)dP_Z(z)$$

$$+ \mathbb{E}\left[\frac{w(x)(\mu_0(x, z) - \mu(x, z))}{s(x|z)}\middle| X = x\right].$$

Plugging both statements in $\mathbb{E}(Y(\eta) - Y(\eta_0)|X = x)$ gives

$$
\begin{aligned}
\mathbb{E}\left(Y(\eta) - Y(\eta_0)|X = x\right) &= \int_{z \in \mathcal{Z}} \frac{\mu_0(x, z) - \mu(x, z)}{s(x|z)/w(x)} \frac{s_0(x|z)}{w_X(x)} w_Z(z) dz \\
&\quad + \int_{z \in \mathcal{Z}} [\mu(x, z) - \mu_0(x, z)] w_Z(z) dz \\
&= \frac{w(x)}{w_X(x)} \int_z \frac{(\mu_0(x, z) - \mu(x, z))(s_0(x|z) - s(x|z))}{s(x|z)} w_Z(z) dz \\
&\quad + \frac{w(x) - w_0(x)}{w_0(x)} \int_z [\mu_0(x, z) - \mu(x, z)] w_Z(z) dz \\
&\quad + \int_{z \in \mathcal{Z}} [\mu_0(x, z) - \mu(x, z)] w_Z(z) dz \\
&\quad + \int_{z \in \mathcal{Z}} [\mu(x, z) - \mu_0(x, z)] w_Z(z) dz \\
&=: S_1(x) + S_2(x) + 0.
\end{aligned}
$$

Recognise that

$$
\|\mathbb{E} p(X)(S_1(X) + S_2(X))\| \leq \|\mathbb{E} p(X) S_1(X)\| + \|\mathbb{E} p(X) S_2(X)\|.
$$

For each $j \in \{1, 2, \ldots, d\}$, Cauchy-Scwartz inequality implies

$$
(\mathbb{E}|p_j(X) S_k(X)|)^2 \leq \sup_{x \in \mathcal{X}} p_j^2(x) (\mathbb{E}|S_k(X)|)^2, \quad k \in \{1, 2\}.
$$

Therefore,

$$
\begin{aligned}
\|\mathbb{E} p(X)(S_1(X) + S_2(X))\|^2 &\leq 2(\|\mathbb{E} p(X) S_1(X)\|^2 + \|\mathbb{E} p(X) S_2(X)\|^2) \\
&\leq 2 \sum_{j=1}^d \sup_{x \in \mathcal{X}} p_j(x)^2 ((\mathbb{E}|S_1(X)|)^2 + (\mathbb{E}|S_2(X)|)^2) \\
&\lesssim d((\mathbb{E}|S_1(X)|)^2 + (\mathbb{E}|S_2(X)|)^2)
\end{aligned}
$$

if the dictionary of basis function is bounded. Alternatively,

$$
\begin{aligned}
\|\mathbb{E} p(X)(S_1(X) + S_2(X))\|^2 &\leq 2(\|\mathbb{E} p(X) S_1(X)\|^2 + \|\mathbb{E} p(X) S_2(X)\|^2) \\
&\lesssim d(\mathbb{E} S_1^2(X) + \mathbb{E} S_2^2(X)),
\end{aligned}
$$

where the bounds on $(\mathbb{E}|S_k(X)|)^2$ and $\mathbb{E} S_k^2(X)$ are established in Steps 2 and 3. Finally, Assumption 4.9 implies the following inequalities:

$$
\sup_{w \in \mathcal{W}_N} \sup_{x \in \mathcal{X}} \frac{w(x)}{w_0(x)} \leq \bar{C}^2, \tag{A.3}
$$

$$
\sup_{s \in \mathcal{S}_N} \sup_{(x, z) \in (\mathcal{X} \times \mathcal{Z})} s^{-1}(x|z) \leq \bar{C}, \tag{A.4}
$$

$$
\sup_{s \in \mathcal{S}_N} \sup_{w \in \mathcal{W}_N} \sup_{(x, z) \in \mathcal{X} \times \mathcal{Z}} w(x) s^{-1}(x, z) \leq \bar{C}^2. \tag{A.5}
$$

Joint, conditional, and marginal densities are related as

$$
s_0(x|z) w_Z(z) = w_{X,Z}(x, z) = w(z|x) w_X(x). \tag{A.6}
$$

Step 2. The following bound holds:

$$|S_1(x)| \leq \sup_{x \in \mathcal{X}} \frac{w(x)}{w_0(x)} \sup_{(x,z) \in \mathcal{X} \times \mathcal{Z}} s^{-1}(x|z) \int_{z \in \mathcal{Z}} |\mu_0(x,z) - \mu(x,z)||s_0(x|z) - s(x|z)|w_Z(z)dz$$

$$\leq^i \bar{\mathcal{C}}^3 \int_{z \in \mathcal{Z}} |\mu_0(x,z) - \mu(x,z)||s_0(x|z) - s(x|z)|w_Z(z)dz$$

$$\leq^{ii} \bar{\mathcal{C}}^5 \mathbb{E}[|\mu_0(X,Z) - \mu(X,Z)||s_0(X|Z) - s(X|Z)||X = x],$$

where i holds by (3) and ii holds by Lemma A.9(d). Therefore, $\mathbb{E}|S_1(X)| \lesssim \mathbf{s}_N \mathbf{m}_N$ and $\mathbb{E}S_1^2(X) \lesssim \mathbf{s}_{N,2\kappa} \mathbf{m}_{N,2\gamma}$ for any $\kappa, \gamma \geq 1$ so that $1/\kappa + 1/\gamma = 1$.

Step 3. By (3) and (6),

$$|S_2(x)| \leq^i \bar{\mathcal{C}}|w(x) - w_0(x)| \int_{z \in \mathcal{Z}} |\mu(x,z) - \mu_0(x,z)|w_Z(z)dz$$

$$\lesssim^{ii} \bar{\mathcal{C}}^3 |w(x) - w_0(x)|\mathbb{E}[|\mu(X,Z) - \mu_0(X,Z)||X = x].$$

where i holds by (3) and ii holds by Lemma A.9(d). Therefore, $\mathbb{E}|S_2(X)| \lesssim \mathbf{w}_N \mathbf{m}_N$ and $\mathbb{E}S_2^2(X) \lesssim \mathbf{w}_{N,2\kappa} \mathbf{m}_{N,2\gamma}$ for any $\kappa, \gamma \geq 1$ so that $1/\kappa + 1/\gamma = 1$.

Step 4. To choose Λ_N, we decompose $Y(\eta) - Y(\eta_0)$ as follows

$$Y(\eta) - Y(\eta_0) = (\mu_0(X,Z) - \mu(X,Z))\left(\frac{w(X)}{s(X|Z)}\right)$$

$$+ (Y - \mu_0(X,Z))\left(\frac{w(X)}{s(X|Z)} - \frac{w_0(X)}{s_0(X|Z)}\right)$$

$$+ \int_{z \in Z} (\mu(X,z) - \mu_0(X,z))dP_Z(z)$$

$$= M_1 + M_2 + M_3.$$

Recall that $\Lambda_N \leq \xi_d \sup_{\eta \in \mathcal{T}_N} (\mathbb{E}(Y(\eta) - Y(\eta_0))^2)^{1/2}$. Therefore, it suffices to bound $\mathbb{E}M_k^2$ for $k = 1, 2, 3$. By (5),

$$\mathbb{E}M_1^2 = \mathbb{E}(\mu_0(X,Z) - \mu(X,Z))^2 \sup_{(x,z) \in \mathcal{X} \times \mathcal{Z}} \left(\frac{w(x)}{s(x|z)}\right)^2 \lesssim \mathbf{m}_N^2 \bar{\mathcal{C}}^4.$$

By (3) and (4),

$$\mathbb{E}M_2^2 \leq \bar{\sigma}^2 \mathbb{E}\left[\frac{w(X) - w_0(X)}{s(X|Z)} + \frac{w_0(X)}{s_0(X|Z)s(X|Z)}(s_0(X|Z) - s(X|Z))\right]^2$$

$$\lesssim 2\bar{\sigma}^2 \bar{\mathcal{C}}^2 \mathbf{w}_N^2 + 2\bar{\sigma}^2 \bar{\mathcal{C}}^6 \mathbf{s}_N^2.$$

By Lemma A.9(d) and Jensen inequality,

$$\mathbb{E}M_3^2 = \mathbb{E}_X\left[\int_{z \in Z} (\mu(X,z) - \mu_0(X,z))dP_Z(z)\right]^2$$

$$\leq \mathbb{E}_X\left[\mathbb{E}[(\mu(X,Z) - \mu_0(X,Z))\frac{w_0(X)}{s_0(X|Z)}|X]\right]^2 \lesssim \mathbf{m}_N^2.$$

Therefore, Assumption 3.5 holds with $B_N := \sqrt{N}\sqrt{d}(\mathbf{w}_N \mathbf{m}_N + \mathbf{s}_N \mathbf{m}_N)$ if the dictionary is bounded or $B_N := \sqrt{N}\sqrt{d}(\mathbf{w}_{N,2\kappa} \mathbf{m}_{N,2\gamma} + \mathbf{s}_{N,2\kappa} \mathbf{m}_{N,2\gamma})$ for any κ, γ so that $1/\kappa + 1/\gamma = 1$. Furthermore, $\Lambda_N := \xi_d(\mathbf{m}_N \vee \mathbf{w}_N \vee \mathbf{s}_N)$ obeys Assumption 3.5. By Assumption 4.9, $\Lambda_N = o(1)$ and $B_N = o(1)$. $\qquad\square$

LEMMA A.10 (POINTWISE LINEARIZATION OF ORTHOGONAL ESTIMATOR). *Suppose Assumptions 3.1 through 3.4 and 4.9 hold, and $\xi_d N^{-1} = o(1)$. Then, for any $\alpha \in \mathcal{S}^{d-1} := \{\alpha \in \mathbb{R}^d : \|\alpha\| = 1\}$ the estimator*

$\widehat{\beta}^\dagger$ *is approximately linear:*

$$\sqrt{N}\alpha'(\widehat{\beta}^\dagger - \beta) = \alpha' Q^{-1}\mathbb{G}_N[p(X_i)(U_i + r_g(X_i)) + \tau_1(Z_i; \mu_0)] + R_{1,N}(\alpha) + R_{1,N}^\dagger(\alpha),$$

where $R_{1,N}(\alpha)$ is defined in Lemma 3.1 and

$$R_{1,N}^\dagger(\alpha) := \frac{1}{n(n-1)}\frac{1}{K}\sum_{k=1}^{K}\sum_{i\in J_k}\sum_{i\neq j}\tau(V_i, V_j; \widehat{\mu}) - \frac{1}{N}\sum_{i=1}^{N}\tau_1(V_i; \mu_0),$$

summarising the remainder of the U-statistic projection, obeys

$$R_{1,N}^\dagger(\alpha) \lesssim_P \sqrt{d/N} + \sqrt{d\xi_d^4 \log^3 N/N^2} + \sqrt{d}\mathbf{m}_N(1 + \sqrt{\xi_d^2 \log N/N}).$$

Proof of Lemma A.10   Step 1. We decompose the remainder term $R_{1,N}^\dagger(\alpha)$ into $R_{1,k}^\dagger$ and $R_{2,k}^\dagger$:

$$R_{1,N}^\dagger(\alpha) := \sqrt{N}\alpha'(\widehat{\beta}^\dagger - \widehat{\beta} - \frac{1}{N}\sum_{i=1}^{N}\tau_1(Z_i; \mu_0))$$

$$= \sqrt{N}\alpha'\widehat{Q}^{-1}\frac{1}{K}\sum_{k=1}^{K}\left[\frac{1}{n(n-1)}\sum_{i,j\in J_k, i\neq j}\tau(V_i, V_j; \widehat{\mu}) - \frac{1}{n}\sum_{i\in J_k}\tau_1(Z_i; \widehat{\mu})\right]$$

$$+ \sqrt{N}\alpha'\widehat{Q}^{-1}\frac{1}{K}\sum_{k=1}^{K}\left[\frac{1}{n}\sum_{i\in J_k}\tau_1(Z_i; \widehat{\mu}) - \tau_1(Z_i; \mu_0)\right]$$

$$=: \sqrt{N}\alpha'\widehat{Q}^{-1}\frac{1}{K}\sum_{k=1}^{K}[R_{1,k}^\dagger + R_{2,k}^\dagger]$$

$$= \frac{1}{K}\sum_{k=1}^{K}[\mathcal{R}_{1,k} + \mathcal{R}_{2,k} + \mathcal{R}_{1,k}' + \mathcal{R}_{2,k}'],$$

where the latter terms are defined as:

$$\mathcal{R}_{1,k} := \sqrt{N}\alpha' Q^{-1} R_{1,k}^\dagger, \quad \mathcal{R}_{2,k} := \sqrt{N}\alpha' Q^{-1} R_{2,k}^\dagger,$$

$$\mathcal{R}_{1,k}' := \sqrt{N}\alpha'(\widehat{Q}^{-1} - Q^{-1})R_{1,k}^\dagger, \quad \mathcal{R}_{2,k}' := \sqrt{N}\alpha'(\widehat{Q}^{-1} - Q^{-1})R_{2,k}^\dagger$$

Step 2. Term $\mathcal{R}_{1,k}$. Conditional on the data $(V_j)_{j\in J_k^c}$, $\widehat{\mu}$ is treated as fixed, and $R_{1,k}^\dagger$ is a degenerate U-statistic of order 2. To bound $R_{1,k}^\dagger$, we invoke Lemma A.7 with the function class

$$\mathcal{F}_\alpha = \{\alpha' Q^{-1}\tau(\cdot, \cdot; \widehat{\mu})\}$$

consisting of a single element. The entropy integral of $\mathcal{F}_\alpha$ $\mathcal{J}_2(\delta) = \delta$. Its envelope $F_\alpha(\cdot, \cdot) : (V, V') \to C_{\min}^{-1}\|\tau(V, V'; \widehat{\mu})\|$ is bounded in $L_{P,2}$ norm:

$$\mathbb{E}\left[\|\tau(V, V'; \widehat{\mu})\|^2 | \mathcal{E}_N, (V_j)_{j\in J_k^c}\right] \leq \sup_{\mu\in M_N}\mathbb{E}\|\tau(V, V'; \mu)\|^2$$

$$\leq \bar{\mathcal{C}}^2\frac{1}{2}(\|p(X)\|_{P,2}^2 + \|p(X')\|_{P,2}^2) \leq d\bar{\mathcal{C}}^2.$$

Furthermore, for each $k \in \{1, 2, \ldots, K\}$

$$\max_{i\in J_k}\|\tau(V_{2(i-1)+1}, V_{2i}; \widehat{\mu})\| \leq \sup_{x\in\mathcal{X}}\|p(x)\|\bar{\mathcal{C}} = \xi_d\bar{\mathcal{C}}.$$

Applying Lemma A.7 to $\alpha' Q^{-1} R_{1,k}^\dagger$ with $r = k = 2$ and $\sigma_2 = \|F\|_{P,2} \lesssim d\bar{\mathcal{C}}$ gives

$$n \mathbb{E}\left[|\alpha' Q^{-1} R_{1,k}^\dagger| \middle| \mathcal{E}_N, (V_j)_{j \in J_k^c} \right] \lesssim \bar{\mathcal{C}}\sqrt{d} + \bar{\mathcal{C}}n^{-1/2}\xi_d.$$

By Markov inequality, $\mathcal{R}_1 := \sqrt{N}\alpha' Q^{-1} R_{1,k}^\dagger \lesssim_P N^{-1/2}\bar{\mathcal{C}}\sqrt{d} + \xi_d N^{-1}\bar{\mathcal{C}} = o_P(1)$ conditional on $(V_j)_{j \in J_k^c}$. By Lemma A.2, $\mathcal{R}_{1,k} = \sqrt{N}\alpha' Q^{-1} R_{1,k}^\dagger = o_P(1)$ unconditionally.

Step 3. Term $\mathcal{R}'_{1,k}$. By Assumptions 3.1 and 3.2, $\|\widehat{Q}^{-1} - Q^{-1}\| \lesssim_P \sqrt{\dfrac{\xi_d^2 \log N}{N}}$. Therefore, $\mathcal{R}'_{1,k} \lesssim_P$
$\sqrt{\dfrac{\xi_d^2 \log N}{N}}\|R_{1,k}^\dagger\|$, where

$$\|R_{1,k}^\dagger\| \le \sqrt{d}\|R_{1,k}^\dagger\|_\infty.$$

We invoke Lemma A.6 for the canonical two-sample U-statistic $R_{1,k}^\dagger$ conditional on the event $\mathcal{E}_N$ and the sample $(V_j)_{j \in J_k^c}$. On this event, $\tau(V_i, V_j; \widehat{\mu})$ is bounded in absolute norm:

$$\|R_{1,k}^\dagger\|_\infty \le 2 \max_{1 \le j \le d} \sup_{x \in \mathcal{X}} |p_j(x)|\mathcal{C} \le 2\xi_d \mathcal{C}.$$

Therefore, the constants D_2, D_4, M_4 are bounded by $\xi_d \mathcal{C}$. Thus,

$$\mathbb{E}[\|R_{1,k}^\dagger\|_\infty] \lesssim \xi_d \mathcal{C} \frac{\log d}{N}$$

and $\|R_{1,k}^\dagger\|_\infty = O_P\left(\dfrac{\xi_d \log d}{N}\right) = o(1)$. Invoking $\log d \le \log N$, we obtain

$$\mathcal{R}'_{1,k} \lesssim_P \sqrt{N}\sqrt{\frac{d\xi_d^2 \log N}{N}} \frac{\xi_d \log d}{N} = O\left(\sqrt{\frac{d\xi_d^4 \log^2 N}{N^2}}\right).$$

Step 4. Term $\mathcal{R}_{2,k}$. Conditional on the data $(V_j)_{j \in J_k^c}$, the remainder term $R_{2,k}^\dagger$ is a sample average of mean zero i.i.d random variables:

$$R_{2,k}^\dagger := \mathbb{E}_{n,k}[\tau_1(Z_i; \widehat{\mu}) - \tau_1(Z_i; \mu_0)] =^i \mathbb{E}_{n,k}\tau_1(Z_i; \widehat{\mu} - \mu_0),$$

where i follows from Lemma A.9(b). The bound on the norm follows as

$$\mathbb{E}[\|\sqrt{n}R_{2,k}^\dagger\|^2 | \mathcal{E}_N, (V_j)_{j \in J_k^c}] \le \mathbb{E}[\|\tau_1(Z_i; \widehat{\mu}) - \tau_1(Z_i; \mu_0)\|^2 | \mathcal{E}_N, (V_j)_{j \in J_k^c}]$$

$$\le \sup_{\mu \in M_N} \mathbb{E}\|\tau_1(Z_i; \mu - \mu_0)\|^2$$

$$= \sup_{\mu \in M_N} \sum_{j=1}^d \mathbb{E}_Z(\mathbb{E}_X[p_j(X)(\mu^0(X; z) - \mu_0^0(X; z))]|_{z=Z})^2$$

$$\le \sup_{\mu \in M_N} \sum_{j=1}^d \mathbb{E}_Z \mathbb{E}_X p_j^2(X)\mathbb{E}_X[(\mu^0(X; z) - \mu_0^0(X; z))^2]|_{z=Z} \lesssim d\mathbf{m}_N^2.$$

Therefore, $\|\sqrt{n}R_{2,k}^\dagger\| = O_P(\sqrt{d}\mathbf{m}_N)$ and $\mathcal{R}'_{2,k} \lesssim_P \sqrt{d}\mathbf{m}_N \sqrt{\dfrac{\xi_d^2 \log N}{N}}$.

Step 5. Collecting the terms gives the bound in Lemma A.10. The bound on $R_{1,N}(\alpha)$ is established in Lemma 3.1. □

LEMMA A.11 (UNIFORM LINEARIZATION OF ORTHOGONAL ESTIMATOR). *Suppose Assumptions 3.1 through 3.7 hold with $\xi_d^L/C_{min} \ge e^2/16 \vee e$. Suppose $N^{-1}\xi_d \log N = o(1)$. Then, the estimator $\widehat{\beta}^\dagger$ is approximately linear uniformly over the domain $\mathcal{X}$:*

$$|\sqrt{N}\alpha(x)'(\widehat{\beta}^\dagger - \beta_0) - \alpha'(x)Q^{-1}\mathbb{G}_N[p(X_i)(U_i + r_g(X_i)) + \tau_1(Z_i; \mu_0)]|$$

$$\le R_{1,N}(\alpha(x)) + R_{1,N}^\dagger(\alpha(x)),$$

where $R_{1,N}(\alpha(x))$ is defined in Lemma 3.2 and $R_{1,N}^{\dagger}(\alpha(x))$, summarising the remainder of the U-statistic projection

$$R_{1,N}^{\dagger}(\alpha(x)) \lesssim_P N^{-1/2}\bar{\mathcal{C}}(\sqrt{d}\log N + \xi_d N^{-1}\log^2 N)$$

$$+ \sqrt{\frac{d\xi_d^4 \log^3 N}{N^2}} + \sqrt{d}\mathbf{m}_N\left(1 + \sqrt{\frac{\xi_d^2 \log N}{N}}\right) =: \bar{R}_{1,N}^{\dagger}$$

Proof of Lemma A.11 **Step 1**. Conditional on the data $(V_j)_{j \in J_k^c}$, $\widehat{\mu}$ is treated as fixed, and $R_{1,k}^{\dagger}(\alpha(x))$ is a degenerate U-statistic of order 2. Consider a function class

$$\mathcal{F} = \{\alpha(x)'Q^{-1}\tau(\cdot,\cdot;\widehat{\mu}), \quad x \in \mathcal{X}\}$$

whose envelope $F := C_{\min}^{-1}\|\tau(\cdot,\cdot;\widehat{\mu})\|$ is square integrable (Step 2, Proof of Lemma A.10). We determine the bracket size. Recognise that

$$|[\alpha(x) - \alpha(x')]^{\top}Q^{-1}\tau(V,V';\widehat{\mu})| \leq \xi_d^L\|x - x'\|\|Q^{-1}\|\|\tau(V,V';\widehat{\mu})\|$$

and therefore

$$\sup_Q N(\mathcal{F}, L^2(Q), \epsilon\|F\|_{L_Q^2}) \leq \left(\frac{\xi_d^L/C_{\min}}{\epsilon}\right)^r$$

Plugging in $\sigma = \|F\|_{L_P^2} \leq \sqrt{d}\bar{\mathcal{C}}$, $A = \xi_d^L/C_{\min}$, $V = r$ into Lemma A.8, we obtain:

$$\sup_{x \in \mathcal{X}}|\sqrt{N}\alpha(x)^{\top}Q^{-1}R_{1,k}^{\dagger}| \lesssim_P N^{-1/2}\sqrt{d}\bar{\mathcal{C}}\log \xi_d^L + N^{-1}\xi_d \log^2 \xi_d^L \bar{\mathcal{C}}$$

$$\lesssim_P N^{-1/2}\sqrt{d}\bar{\mathcal{C}}\log N + N^{-1}\xi_d \log^2 N \bar{\mathcal{C}}$$

Step 2. Let $R_{1,k}^{\dagger}$ and $R_{2,k}^{\dagger}$ be as defined in the proof of Lemma A.10. We establish the following bound

$$\sup_{x \in \mathcal{X}}|\sqrt{N}\alpha(x)^{\top}(Q^{-1} - \widehat{Q}^{-1})R_{1,k}^{\dagger}| \lesssim_P \|\widehat{Q} - Q\|\|R_{1,k}^{\dagger}\| = O_P\left(\sqrt{\frac{\xi_d^2 \log N}{N}}\sqrt{d}\|R_{1,k}^{\dagger}\|_{\infty}\right)$$

$$=^i O_P\left(\sqrt{\frac{\xi_d^4 \log^3 N}{N^2}}\sqrt{d}\right),$$

where i follows from Step 3 of the Proof of Lemma A.10.

Step 3. We establish the following bound

$$\sup_{x \in \mathcal{X}}|\sqrt{N}\alpha(x)^{\top}Q^{-1}R_{2,k}^{\dagger}| \lesssim_P \|Q^{-1}\|\|R_{2,k}^{\dagger}\| = O_P(\sqrt{d}\mathbf{m}_N)$$

$$\sup_{x \in \mathcal{X}}|\sqrt{N}\alpha(x)^{\top}(Q^{-1} - \widehat{Q}^{-1})R_{2,k}^{\dagger}| \lesssim_P \|\widehat{Q} - Q\|\|R_{2,k}^{\dagger}\| = O_P\left(\sqrt{\frac{\xi_d^2 \log N}{N}}\sqrt{d}\mathbf{m}_N\right),$$

where i and ii hold by Step 4 of the proof of Lemma A.10. □

Proof of Theorem 4.6 In Lemma A.10, we have shown the pointwise linearisation of $\widehat{\beta}^{\dagger}$. Consider a triangular array $\{(\chi_{Ni})_{i=1}^N, \quad i = 1, 2, \ldots, N\}$, where χ_{Ni} is defined as

$$\chi_{Ni} = \frac{\alpha'Q^{-1}[p_i(U_i + r_i) + \tau_1(Z_i; \mu_0)]}{\sqrt{N}\|\alpha'(\Omega^{\dagger})^{1/2}\|}.$$

We verify the conditions of Lindeberg's CLT for χ_{Ni}. Assumption 3.3 and Lemma A.9(a) imply

$$\text{var}\left(\sum_{i=1}^N \chi_{Ni}\right) = 1 \text{ and } |\chi_{Ni}| \leq (\xi_d|U_i| + \xi_d l_d r_d + \bar{\mathcal{C}}\sqrt{d})/\sqrt{N}.$$

Fix $\delta > 0$ and $C > 0$. Observe that

$$
N\mathbb{E}\chi_{Ni}^2 1_{\{|\chi_{Ni}|>\delta\}}
$$

$$
\leq \frac{2}{\alpha'\Omega^\dagger\alpha}\left(\mathbb{E}(\alpha'Q^{-1}p_i(U_i+r_i))^2 1_{\{|\chi_{Ni}|>\delta\}} + \mathbb{E}(\alpha'Q^{-1}\tau_1(Z_i;\mu_0))^2 1_{\{|\chi_{Ni}|>\delta\}}\right) = L_1 + L_2.
$$

Observe that the event $|\chi_{Ni}| > \delta$ implies a larger event

$$
\left\{|\chi_{Ni}| > \delta\right\} \Rightarrow \left\{|U_i| > \delta\sqrt{N}/\xi_d - \bar{\mathcal{C}}\sqrt{d}/\xi_d - l_d r_d\right\} =: \left\{|U_i| > M\right\},
$$

where $M = \delta\sqrt{N}/\xi_d - \bar{\mathcal{C}}\sqrt{d}/\xi_d - l_d r_d \to \infty$ by assumption of the Theorem. The numerator of the term L_1 is bounded as

$$
\mathbb{E}(\alpha'Q^{-1}p_i(U_i+r_i))^2 \leq (\alpha'Q^{-1}QQ^{-1}\alpha)\sup_{x\in\mathcal{X}}\mathbb{E}[(U_i+r_i)^2|X_i=x]1_{\{|U_i|>M\}}
$$

$$
\leq 2\alpha'Q^{-1}\alpha\left(\sup_{x\in\mathcal{X}}\mathbb{E}[U_i^2|X_i=x]1_{\{|U_i|>M\}}\right) + l_d^2 r_d^2 \mathrm{P}(|U_i| > M) \to 0,
$$

where the last inequality holds by the Lindeberg condition assumed in the Theorem. To bound the numerator of term L_2, observe that

$$
\mathbb{E}(\alpha'Q^{-1}\tau_1(Z_i;\mu_0))^2 1_{\{|U_i|>M\}} \lesssim d\bar{\mathcal{C}}^2 \mathrm{P}(|U_i| > M) \to 0.
$$

Finally, the common denominator of L_1 and L_2, equal to $(\alpha'\Omega^\dagger\alpha)^{-1}$, is bounded by $(C_{\min}^\dagger)^{-1} < \infty$ by assumption of the Theorem. □

Proof of Theorem 4.7   Proof of Theorem 4.7 (a). The bound on

$$
\sup_{x\in\mathcal{X}}|\alpha(x)'\mathbb{E}_N\tau_1(Z_i;\mu_0)| \leq \sup_{x\in\mathcal{X}}\|\alpha(x)\|\|\mathbb{E}_N\tau_1(Z_i;\mu_0)\|
$$

$$
\leq \sqrt{d}\|\mathbb{E}_N\tau_1(Z_i;\mu_0)\|_\infty \lesssim_P \sqrt{d\log d/N}, \tag{A.7}
$$

where the last inequality follows from McDiarmid inequality for bounded sample averages of bounded r.v. Therefore, the total bound on

$$
\sup_{x\in\mathcal{X}}|\alpha(x)'(\widehat{\beta}^\dagger - \beta_0)| \leq^i \sup_{x\in\mathcal{X}}|\alpha(x)'Q^{-1}[\mathbb{E}_N p_i(U_i+r_i) + \tau_1(Z_i;\mu_0)]| + \bar{R}_{1N} + \bar{R}_{2N} + \bar{R}_{1N}^\dagger
$$

$$
\lesssim_P N^{-1/2}(\sqrt{\log N} + \bar{R}_{1N} + \bar{R}_{2N} + \sqrt{d\log d} + \bar{R}_{1,N}^\dagger),
$$

where i follows from Lemmas 3.2 and A.11 and ii follows from (7). By assumption of the Theorem, $\sqrt{d\xi_d^4 \log^3 N/N^2} = o(\sqrt{\xi_d^2 \log^2 N/N}) = o(\sqrt{\log N})$. Furthermore, $(d\log N/N)^{1/2} = o(1)$ and $\xi_d \log^2 N/N = o(1)$. Therefore, the only non-negligible term in $\bar{R}_{1,N}^\dagger$ is $\sqrt{d}\mathbf{m}_N$.

Proof of Theorem 4.7 (b). We shall apply Yurinskii's coupling (see Theorem 4.4 in Belloni et al., 2015). To apply the coupling, we need to verify that $\mathbb{E}\|\tau_1(Z_i;\mu_0)\|^3 \lesssim d^{3/2}$, which holds by assumption of the Theorem. The rest of the Theorem follows from the proof of Theorem 4.4 of Belloni et al. (2015). □

Proof of Corollary 4.1 Let us show that $Y(\eta)$ given by Equation (6) satisfies Assumption 3.5. **Step 1. Choice of B_N.**

$$Y(\eta) - Y(\eta_0) = [\mu(1, Z) - \mu_0(1, Z)][1 - \frac{D}{s_0(Z)}] - [\mu(0, Z) - \mu_0(0, Z)][1 - \frac{1 - D}{1 - s_0(Z)}]$$

$$+ [s_0(Z) - s(Z)]\Big[[Y - \mu_0(1, Z)]\frac{D}{s(Z)s_0(Z)} - [Y - \mu_0(0, Z)]\frac{1 - D}{(1 - s(Z))(1 - s_0(Z))}\Big]$$

$$+ [s_0(Z) - s(Z)]\Big[[\mu(1, Z) - \mu_0(1, Z)]\frac{D}{s(Z)s_0(Z)}$$

$$- [\mu(0, Z) - \mu_0(0, Z)]\frac{1 - D}{(1 - s(Z))(1 - s_0(Z))}\Big]$$

$$= S_1 + S_1' + S_2 + S_2' + S_3 + S_3'.$$

We will bound the terms S_1, S_2, S_3. The bounds on the terms S_1', S_2', S_3' follow by a similar argument. To choose B_N in Assumption 3.5, observe that

$$\mathbb{E}p(X)S_1 := \mathbb{E}p(X)[\mu(1, Z) - \mu_0(1, Z)][1 - \frac{D}{s_0(Z)}]$$

$$=^i \mathbb{E}\Big[p(X)[\mu(1, Z) - \mu_0(1, Z)]\mathbb{E}\Big[1 - \frac{D}{s_0(Z)}\Big|Z\Big]\Big] =^{ii} 0,$$

where i holds because X is a subvector of Z and ii holds by definition of the propensity score. Likewise,

$$\mathbb{E}p(X)S_2 := \mathbb{E}p(X)[s_0(Z) - s(Z)]\big[[Y - \mu_0(1, Z)]$$

$$= \mathbb{E}\Big[p(X)[s_0(Z) - s(Z)]\mathbb{E}[Y - \mu_0(1, Z)|Z]\Big] =^{ii} 0,$$

where ii holds by the definition of regression function. Observe that $P(D^2 = 1|Z) = P(D = 1|Z) = s_0(Z)$ and

$$\mathbb{E}[D^2 s^{-2}(Z)s_0^{-2}(Z)|Z = z] \leq \bar{\mathcal{C}}^3 \text{ for any } z \in \mathcal{Z}, \tag{A.8}$$

$$\mathbb{E}\Big[\Big[1 - \frac{D}{s_0(Z)}\Big]^2\Big|Z = z\Big] = 1 - 2 + s_0^{-1}(z) \leq \bar{\mathcal{C}} \quad \text{for any } z \in \mathcal{Z} \tag{A.9}$$

Therefore,

$$\|\mathbb{E}p(X)S_3\|^2 \leq \sum_{j=1}^d (\mathbb{E}p_j(X)S_3(X))^2 \lesssim \bar{\mathcal{C}}^3 d\mathbf{m}_N^2 \mathbf{s}_N^2$$

if the basis is bounded, in which case $B_N := \sqrt{N}d\mathbf{m}_N\mathbf{s}_N$ satisfies Assumption 3.5. Otherwise,

$$\|\mathbb{E}p(X)S_3\|^2 \lesssim d\mathbf{m}_{N,2\gamma}^2 \mathbf{s}_{N,2\kappa}^2$$

for any κ, γ so that $1/\kappa + 1/\gamma = 1$, in which case $B_N := \sqrt{N}d\mathbf{m}_{N,2\gamma}\mathbf{s}_{N,2\kappa}$ satisfies Assumption 3.5. By Assumption 4.11, $B_N = o(1)$ for either choice of B_N.

Step 2. *Choice of Λ_N.* As stated in Assumption 3.5, $\Lambda_N \leq \xi_d \sup_{\eta \in \mathcal{T}_N}(\mathbb{E}(Y(\eta) - Y(\eta_0))^2)^{1/2}$. Therefore, it suffices to bound $\mathbb{E}S_k^2$ for $k = 1, 2, 3$. (9) implies the following inequalities

$$\mathbb{E}S_1^2 = \mathbb{E}[[\mu(1, Z) - \mu_0(1, Z)]^2 \mathbb{E}[[1 - D/s_0(Z)]^2|Z]] \lesssim \mathbf{m}_N^2 \bar{\mathcal{C}},$$

and

$$\mathbb{E}S_2^2 =^i \mathbb{E}[s(Z) - s_0(Z)]^2 \mathbb{E}[[Y - \mu_0(1, Z)]^2|Z, D = 1]\frac{1}{s_0(Z)s^2(Z)} \lesssim \mathbf{s}_N^2 \bar{\mathcal{C}}^3.$$

Because one of the terms $(s_0(Z) - s(Z))^2$ or $(\mu(1, Z) - \mu_0(1, Z))^2$ is bounded by $\bar{\mathcal{C}}^2$ almost surely in Z,

$$\mathbb{E}S_3^2 = \mathbb{E}[s_0(Z) - s(Z)]^2[\mu(1, Z) - \mu_0(1, Z)]^2 s^{-2}(Z)s_0^{-1}(Z) \leq \min(\mathbf{m}_N^2, \mathbf{s}_N^2)\bar{\mathcal{C}}^5.$$

Therefore, $\Lambda_N = \xi_d(\mathbf{m}_N \vee \mathbf{s}_N)$ satisfies Assumption 3.5. By Assumption 4.11, $\Lambda_N = o(1)$. □

Proof of Corollary 4.2   Let us show that $Y(\eta)$ given by Equation (7) satisfies Assumption 3.5.

$$Y(\eta) - Y(\eta_0) = [\mu(Z) - \mu_0(Z)][1 - \frac{D}{s_0(Z)}] + [s_0(Z) - s(Z)][Y - \mu_0(Z)]\frac{D}{s(Z)s_0(Z)}$$

$$+ [s_0(Z) - s(Z)][\mu(Z) - \mu_0(Z)]\frac{D}{s(Z)s_0(Z)}$$

$$= S_1 + S_2 + S_3.$$

To choose B_N in Assumption 3.5, observe that

$$\mathbb{E}p(X)S_1 = 0 \quad \text{and} \quad \mathbb{E}p(X)S_2 = 0$$

by Step 1 of the proof of Corollary 4.1. Invoking (8),

$$\|\mathbb{E}p(X)S_3\|^2 \leq \sum_{j=1}^d (\mathbb{E}p_j(X)S_3)^2 \lesssim d\mathbf{m}_N^2\mathbf{s}_N^2$$

if the basis is bounded, in which case $B_N = \sqrt{d}\mathbf{m}_N\mathbf{s}_N$ satisfies Assumption 3.5. Otherwise,

$$\|\mathbb{E}p(X)S_3\|^2 \leq \sum_{j=1}^d (\mathbb{E}p_j(X)S_3)^2 \lesssim d\mathbf{m}_{N,2\gamma}^2\mathbf{s}_{N,2\kappa}^2,$$

where $1/\kappa + 1/\gamma = 1$, in which case $B_N = \sqrt{d}\mathbf{m}_{N,2\gamma}\mathbf{s}_{N,2\kappa}$ satisfies Assumption 3.5. By Assumption 4.11, $B_N = o(1)$.

As stated in Assumption 3.5, $\Lambda_N \leq \xi_d \sup_{\eta \in \mathcal{T}_N}(\mathbb{E}(Y(\eta) - Y(\eta_0))^2)^{1/2}$. Similar to Step 2 in the proof of Corollary 4.1,

$$\mathbb{E}S_1^2 = \mathbb{E}[\mu(Z) - \mu_0(Z)]^2\mathbb{E}[[1 - \frac{D}{s_0(Z)}]^2|Z = z] \lesssim \mathbf{m}_N^2\bar{\mathcal{C}}$$

$$\mathbb{E}S_2^2 = \mathbb{E}[s(Z) - s_0(Z)]^2\mathbb{E}[[Y - \mu_0(Z)]^2|Z = z, D = 1]\frac{1}{s_0(Z)s^2(Z)} \lesssim \mathbf{s}_N^2\bar{\mathcal{C}}^3$$

$$\mathbb{E}S_3^2 = \mathbb{E}[s_0(Z) - s(Z)]^2[\mu(Z) - \mu_0(Z)]^2\frac{1}{s^2(Z)s_0(Z)} \lesssim \min(\mathbf{m}_N^2, \mathbf{s}_N^2)\bar{\mathcal{C}}^3.$$

Therefore, $\Lambda_N = \xi_d(\mathbf{m}_N \vee \mathbf{s}_N)$ satisfies Assumption 3.5. By Assumption 4.11, $\Lambda_N = o(1)$. □

Proof of Corollary 4.3   Let us show that $Y(\eta)$ given by Equation (8) satisfies Assumption 3.5.

$$Y(\eta) - Y(\eta_0) = -\partial_d \log s_0(D|Z)[\mu(D, Z) - \mu_0(D, Z)] + \partial_d[\mu(D, Z) - \mu_0(D, Z)]$$

$$+ [\partial_d \log s_0(D|Z) - \partial_d \log s(D|Z)][Y - \mu_0(D, Z)]$$

$$+ [\partial_d \log s_0(D|Z) - \partial_d \log s(D|Z)][\mu(D, Z) - \mu_0(D, Z)]$$

$$= S_1 + S_2 + S_3.$$

Step 1. *Choice of B_N.* Suppose the conditional density $s_0(d|z)$ of D, Z has bounded support. Integration by parts implies

$$\mathbb{E}\partial_d[\mu(D, Z) - \mu_0(D, Z)]|Z = z] = \int_{t \in \mathbb{R}} \partial_t(\mu(t, z) - \mu_0(t, z))s(t|z)dt$$

$$= -\int_{t \in \mathbb{R}} (\mu(t, z) - \mu_0(t, z))\partial_t s(t|z)dt$$

$$= \int_{t \in \mathbb{R}} (\mu(t, z) - \mu_0(t, z))\partial_t \log s(t|z)s(t|z)dt$$

$$= \mathbb{E}[\partial_d \log s_0(D|Z)(\mu(D, Z) - \mu_0(D, Z))|Z = z].$$

Therefore, $\mathbb{E}[S_1|Z = z] = 0$ for any $z \in \mathcal{Z}$. Because X is a subvector of Z, for any function of X,

$$\mathbb{E}p(X)S_1 = 0.$$

By definition of regression function $\mu_0(D, Z) = \mathbb{E}[Y|D, Z]$,

$$\mathbb{E}p(X)S_2 = \mathbb{E}p(X)[\partial_d \log s_0(D|Z) - \partial_d \log s(D|Z)]\mathbb{E}[Y - \mu_0(D, Z)] = 0.$$

Finally,

$$\|\mathbb{E}p(X)S_3\|^2 \leq \sum_{j=1}^{d}(\mathbb{E}p_j(X)S_3)^2 \leq d\mathbf{s}_N^2\mathbf{m}_N^2$$

if the basis is bounded, in which case $B_N = \sqrt{d}\mathbf{s}_N\mathbf{m}_N$ obeys Assumption 3.5. Otherwise, $B_N = \sqrt{d}\mathbf{s}_{N,2\kappa}\mathbf{m}_{N,2\gamma}$ obeys Assumption 3.5.

$$\mathbb{E}S_1^2 \leq \mathbb{E}(-\partial_d \log s_0(D|Z)[\mu(D, Z) - \mu_0(D, Z)])^2$$

$$+ \mathbb{E}(\partial_d[\mu(D, Z) - \mu_0(D, Z)])^2 \lesssim \mathbf{m}_N^2$$

$$\mathbb{E}S_2^2 \leq (\mathbb{E}[[Y - \mu_0(D, Z)]^2|D, Z])\mathbf{s}_N^2 \lesssim \mathbf{s}_N^2$$

$$\mathbb{E}S_3^2 \leq (\mathbb{E}[\mu(D, Z) - \mu_0(D, Z)]^2)\mathbf{s}_N^2 \lesssim \min[\mathbf{f}_N^2 \vee \mathbf{m}_N^2]$$

Therefore, $\Lambda_N = \xi_d(\mathbb{E}(Y(\eta) - Y(\eta_0))^2)^{1/2} \lesssim \xi_d[\mathbf{m}_N \vee \mathbf{s}_N]$ obeys Assumption 3.5. $\qquad\square$

APPENDIX B: VERIFICATION OF HIGH-LEVEL CONDITIONS AND MONTE-CARLO EVIDENCE

Here, we give an example of a model and and a first-stage estimator that satisfy the small bias condition from the main text.

EXAMPLE B.1 (PARTIALLY MISSING OUTCOME WITH HIGH-DIMENSIONAL SPARSE DESIGN). Consider the setup of Example 2.3. Let the observable vector (D, X, DY^*) consist of the covariate vector of interest X and a partially observed variable Y^*, whose presence is indicated by $D \in \{1, 0\}$. In addition, suppose there exists an observable vector Z such that missingnesss at random is satisfied conditional on Z. Let $p_\mu(z)$, $p_s(z)$ be high-dimensional basis functions of the vector Z that approximate the conditional expectation functions $\mu_0(z)$, $s_0(z)$ using the linear and logistic links, respectively:

$$\mu_0(z) = p_\mu(z)'\theta_0 + r_\mu(z) \tag{B.1}$$

$$s_0(z) = \mathcal{L}(p_s(z)'\delta_0) + r_s(z) := \frac{\exp(p_s(z)'\delta)}{\exp(p_s(z)'\delta) + 1} + r_s(z)$$

where θ, δ are the vectors in $\mathcal{R}^{p_\theta}$, $\mathcal{R}^{p_\delta}$ whose dimensions are allowed to be larger than the sample size N, and $r_\mu(z)$, $r_s(z)$ are the misspecification errors of the respective link functions that vanish as described in

Assumptions B.13 and B.14. For each $\gamma \in \{\theta, \delta\}$, denote a support set

$$T_\gamma := \{j : \gamma_j \neq 0, j \in \{1, 2, .., p_\gamma\}\}$$

and its cardinality, which we refer to as sparsity index of γ,

$$s_\gamma := |T| = \|\gamma\|_0 \quad \forall \quad \gamma \in \{\theta, \delta\}.$$

We allow the cardinality of s_δ, s_θ to grow with N.

Let $r_N, \Delta_N \to 0$ be a fixed sequence of constants approaching zero from above at a speed at most polynomial in N: for example, $r_N \geq \frac{1}{N^c}$ for some $c > 0$, $\ell_N = \log N$. For $\gamma \in \{\theta, \delta\}$, let $c_\gamma, C_\gamma, \kappa'_\gamma, \kappa''_\gamma$ and $\nu \in [0, 1]$ be positive constants that do not depend on N. Finally, let $\|V\|_{P_N, 2} := (N^{-1} \sum_{i=1}^{N} V_i^2)^{1/2}$.

ASSUMPTION B.13 (REGULARITY CONDITIONS FOR LINEAR LINK). *We assume that the following standard conditions hold. (a) With probability $1 - \Delta_N$, the minimal and maximal empirical Restricted Sparse Eigenvalue are bounded from below by κ'_μ and from above by κ''_μ:*

$$\kappa'_\mu \leq \inf_{\|\delta\|_0 \leq s_\theta \ell_N, \|\delta\|=1} \|Dp_\mu(Z)\|_{P_N, 2} \leq \sup_{\|\delta\|_0 \leq s_\theta \ell_N, \|\delta\|=1} \|Dp_\mu(Z)\|_{P_N, 2} \leq \kappa''_\mu.$$

(b) There exists absolute constants $B_\theta, C_\theta, c_\theta > 0$: regressors $\max_{1 \leq j \leq p_\theta} |p_{\mu,j}(Z)| \leq B_\theta$ a.s. and $c_\theta \leq \max_{1 \leq j \leq p_\theta} \mathbb{E}p_{\mu,j}(Z)^2 \leq C_\theta$ (c) With probability $1 - \Delta_N$, $\mathbb{E}_N r_\mu^2(Z_i) \leq C_\theta s_\theta \log(p_\theta \vee N)/N$. (d) Growth restriction: for some $r_N = o(1)$, $\log(p_\theta \wedge N) \lesssim r_N N^{1/3}$. (e) The moments of the model are boundedly heteroscedastic: $c_\theta \leq \mathbb{E}[(Y - \mu_0(Z))^2 | Z] \leq C_\theta$, $\max_{1 \leq j \leq p_\theta} \mathbb{E}[|p_{\mu,j}(Z)(Y - \mu_0(Z))|^3 + |p_{\mu,j}(Z)Y|^3] \leq C_\theta$. (f) With probability $1 - \Delta_N$,

$$\max_{1 \leq j \leq p_\theta} [\mathbb{E}_N - \mathbb{E}][p_{\mu,j}^2(Z)(Y - \mu_0(Z))^2 + p_{\mu,j}^2(Z)Y^2] \leq r_N N^{-1/2}.$$

ASSUMPTION B.14 (REGULARITY CONDITIONS FOR LOGISTIC LINK). *We assume that the following standard conditions hold. With probability $1 - \Delta_N$, the minimal and maximal empirical RSE are bounded from below by κ'_s and from above by κ''_s:*

$$\kappa'_s \leq \inf_{\|\delta\|_0 \leq s_\delta \ell_N, \|\delta\|=1} \|p_s(Z)'\delta\|_{P_N, 2} \leq \sup_{\|\delta\|_0 \leq s_\delta \ell_N, \|\delta\|=1} \|p_s(Z)'\delta\|_{P_N, 2} \leq \kappa''_s.$$

(b) There exist absolute constants $B_\delta, C_\delta, c_\delta > 0$: regressors $\max_{1 \leq j \leq p_\delta} |p_{s,j}(Z)| \leq B_\delta$ a.s. and $c_\delta \leq \max_{1 \leq j \leq p_\delta} \mathbb{E}p_{s,j}(Z)^2 \leq C_\delta$ (c) With probability $1 - \Delta_N$, $\mathbb{E}_N r_s^2(Z_i) \leq C_\delta s_\delta \log(p_\delta \vee N)/N$. (d) Growth restriction: for some $r_N = o(1)$, $\log(p_\delta \wedge N) \lesssim r_N N^{1/3}$. (e) With probability $1 - \Delta_N$,

$$\max_{1 \leq j \leq p_\delta} [\mathbb{E}_N - \mathbb{E}][p_{\delta,j}^2(Z)(D - s_0(Z))^2] \leq r_N.$$

Assumptions B.13 and B.14 are a simplification of the Assumption 6.1-6.2 in Belloni et al. (2017). The following estimators of $\mu_0(Z)$ and $s_0(Z)$ are available.

DEFINITION B.1 (LASSO ESTIMATOR OF THE REGRESSION FUNCTION). *Let $\lambda = 1.1\sqrt{N}\Phi^{-1}(1 - 0.05/(N \vee p_\theta \log N))$ and $\widehat{\Psi}_\theta = \text{diag}(\widehat{l}_1, \widehat{l}_2, \ldots, \widehat{l}_{p_\theta})$ be a diagonal matrix of data-dependent penalty loadings chosen as in Algorithm 6.1 in Belloni et al. (2017). Define $\widehat{\theta}$ as a solution to the following optimisation problem:*

$$\widehat{\theta} := \arg \min_{\theta \in \mathcal{R}^{p_\theta}} \mathbb{E}_N D_i(Y_i - p_\mu(Z_i)'\theta)^2 + \lambda\|\widehat{\Psi}_\theta \theta\|_1$$

and a first-stage estimate of μ as

$$\widehat{\mu}(z) := p_\mu(z)\widehat{\theta}.$$

DEFINITION B.2 (LASSO ESTIMATOR OF THE PROPENSITY SCORE). *Let $\lambda = 1.1\sqrt{N}\Phi^{-1}(1 - 0.05/(N \vee p_\delta \log N))$ and $\widehat{\Psi}_\delta = \text{diag}(\widetilde{l}_1, \widetilde{l}_2, \ldots, \widetilde{l}_{p_\delta})$ be a diagonal matrix of data-dependent penalty loadings chosen as in Algorithm 6.1 in Belloni et al. (2017). Let $\underline{s} > 0$ be a positive constant. Define $\widehat{\delta}$ as a*

solution to the following optimisation problem:

$$\widehat{\delta} := \arg\min_{\delta \in \mathcal{R}^{p_\delta}} \mathbb{E}_N[\log(1 + \exp(p_s(Z_i)'\delta)) - D_i p_s(Z_i)'\delta] + \lambda \|\widehat{\Psi}_\delta \delta\|_1$$

and a first-stage estimate of s_0 as

$$\widehat{s}(z) := \max(\underline{s}/2, \mathcal{L}(p_s(z)'\widehat{\delta})).$$

LEMMA B.1 (SUFFICIENT CONDITIONS FOR ASSUMPTION 4.11) *Suppose Assumptions B.13 and B.14 hold. Then, the following statements hold. (1) There exists $C_\theta < \infty$ be such that w.p. $1 - o(1)$:*
$\|p_\mu(Z)'(\widehat{\theta} - \theta_0)\|_{P_{N,2}} \le C_\theta \sqrt{\dfrac{s_\theta \log p_\theta}{N}}$ *and* $\|\widehat{\theta} - \theta_0\|_1 \le C_\theta \sqrt{\dfrac{s_\theta^2 \log p_\theta}{N}}$. *There exists $C_\delta < \infty$ be such that*

w.p. $1 - o(1)$: $\|p_s(Z)'(\widehat{\delta} - \delta_0)\|_{P_{N,2}} \le C_\delta \sqrt{\dfrac{s_\delta \log p_\delta}{N}}$ *and* $\|\widehat{\delta} - \delta_0\|_1 \le C_\delta \sqrt{\dfrac{s_\delta^2 \log p_\delta}{N}}$.

(2) Define the nuisance realisation sets M_N and S_N as:

$$M_N := \left\{\mu(z) = p_\mu(z)'\theta : \theta \in \mathcal{R}^{p_\theta} : \quad \|p_\mu(Z)'(\theta - \theta_0)\|_{P_{N,2}} \le C_\theta \sqrt{\frac{s_\theta \log p_\theta}{N}}, \right.$$

$$\left. \|\widehat{\theta} - \theta_0\|_1 \le C_\theta \sqrt{\frac{s_\theta^2 \log p_\theta}{N}} \right\}$$

$$S_N := \left\{s(z) = \mathcal{L}(p_s(z)'\delta) : \delta \in \mathcal{R}^{p_\delta} : \quad \|p_\delta(Z)'(\delta - \delta_0)\|_{P_{N,2}} \le C_\delta \sqrt{\frac{s_\delta \log p_\delta}{N}}, \right.$$

$$\left. \|\widehat{\delta} - \delta_0\|_1 \le C_\delta' \sqrt{\frac{s_\delta^2 \log p_\delta}{N}} \right\}$$

Then, w.p. $1 - o(1)$, $\widehat{\mu}(\cdot) \in M_N$ and the set M_N shrinks at rate $\mathbf{m}_N := \sqrt{\dfrac{s_\theta \log p_\theta}{N}}$. Then, w.p. $1 - o(1)$,

$\widehat{s}(\cdot) \in S_N$ and the set S_N shrinks at rate $\mathbf{s}_N := \sqrt{\dfrac{s_\delta \log p_\delta}{N}}$.

(3) Suppose $\xi_d(\mathbf{m}_N \vee \mathbf{s}_N) = o(1)$ and the product of sparsity indices $s_\theta s_\delta$ grows sufficiently slow:

$$\sqrt{N}\sqrt{d}\,\mathbf{m}_N\mathbf{s}_N = \sqrt{d}\sqrt{\frac{s_\theta s_\delta \log p_\theta \log p_\delta}{N}} = o(1).$$

Then, Assumption 4.11 holds.

Proof of Lemma B.1.

Step 1.Define $\widetilde{Y} = DY$, $\widetilde{p}_\mu(Z) = Dp_\mu(Z)$, $\widetilde{r}_\mu(Z) = Dr_\mu(Z)$, and $\widetilde{\epsilon} = D[Y - \mu_0(Z)]$. Here we verify the conditions of Theorem 6.1 from Belloni et al. (2017) for the model $\widetilde{Y}, \widetilde{p}_\mu(Z), \widetilde{r}_\mu(Z), \widetilde{\epsilon}$. Let us show that the original coefficient θ_0, defined in Equation (1), satisfies

$$\widetilde{Y} = \widetilde{p}_\mu(Z)'\theta_0 + \widetilde{r}_\mu(Z) + \widetilde{\epsilon}, \quad \mathbb{E}[\widetilde{\epsilon}|\widetilde{D}] = 0$$

Indeed,

$$\mathbb{E}[\widetilde{Y}|D, Z] = \mathbb{E}[DY|D, Z] = D\mathbb{E}[Y|Z, D] = D[\mu_0(Z)] = Dp_\mu(Z)'\theta_0 + Dr(Z)$$

$$= \widetilde{p}_\mu(Z)'\theta_0 + \widetilde{r}_\mu(Z).$$

$$\mathbb{E}[\widetilde{\epsilon}|D] = \mathbb{E}[\mathbb{E}[[\widetilde{Y} - \widetilde{p}_\mu(Z)'\theta_0 - \widetilde{r}_\mu(Z)]|D, Z]|D] = 0$$

Recognise that Assumption B.13 implies that an analog of Assumption B.13 holds for $\widetilde{p}_\mu(Z), \widetilde{Y}$. Assumption B.13 (a) directly assumes bounded restricted sparse eigenvalues for observed regressors $\widetilde{p}_\mu(Z) = p_\mu(Z)D$. Assumption B.13 (b) is satisfied with $\mathbb{E}\widetilde{p}_{\mu,j}(Z)^2 = \mathbb{E}Dp_{\mu,j}^2(Z) \ge \underline{s}\mathbb{E}p_{\mu,j}^2(Z) =: c'$ where $c' := c\underline{s}$ is the new lower bound on the moments of $\mathbb{E}\widetilde{p}_{\mu,j}(Z)$ for observed regressors. Assumption B.13 (c) is satified with the $\mathbb{E}\widetilde{r}_\mu(Z)^2 \le \mathbb{E}r_\mu(Z)^2 \le Cs_\theta \log(p_\theta \vee N)/N$. By Theorem 6.1 of Belloni et al. (2017), w.p. $1 - o(1)$,

$p_\mu(\cdot)'\widehat{\theta} \in M_N$. Under Assumption B.14, Theorem 6.2 of Belloni et al. (2017) implies that w.p. $1 - o(1)$, $\mathcal{L}(p_s(\cdot)'\widehat{\delta}) \in S_N$.

Step 2. Let $C_\theta' = (C_\theta^2 + \sqrt{2}B_\theta)^{1/2}$. For any $\theta \in \mathcal{R}^{p_\theta}$ such that $p_\mu(\cdot)'\theta \in M_N$ the following inequality holds:

$$
\mathbb{E}_Z(p_\mu(Z)'(\theta - \theta_0))^2 \leq \|\theta - \theta_0\|^2_{P_{N,2}}
$$

$$
+ \left|(\theta - \theta_0)^\top [\mathbb{E}_N p_\mu(Z_i)p_\mu^\top(Z_i) - \mathbb{E}p_\mu(Z_i)p_\mu^\top(Z_i)](\theta - \theta_0)\right|
$$

$$
\leq \|\theta - \theta_0\|^2_{P_{N,2}} + \|\theta - \theta_0\|^2_1 \max_{1 \leq i,j \leq p_\theta} \left|\mathbb{E}_N p_{i,\mu}(Z_i)p_{j,\mu}^\top(Z_i) - \mathbb{E}p_{i,\mu}(Z_i)p_{j,\mu}^\top(Z_i)\right|
$$

$$
\lesssim_P^i C_\theta^2 \mathbf{m}_N^2 + B_\theta \sqrt{\frac{s_\theta^2 \log p_\theta}{N} \frac{2\log p_\theta}{N}} \lesssim_P (C_\theta')^2 \mathbf{m}_N^2,
$$

where i follows by McDiarmid maximal inequality for bounded random variables. Therefore, the nuisance realisation set M_N shrinks at rate $\mathbf{m}_N$.

Step 3. Observe that $(\mathbb{E}_Z(\mathcal{L}(p_s(Z)'\delta) - \mathcal{L}(p_s(Z)'\delta_0))^2)^{1/2} \leq \sup_{t \in \mathcal{R}} \mathcal{L}'(t)(\mathbb{E}_Z(p_s(Z)'(\delta - \delta_0))^2)^{1/2} \leq \frac{1}{4}(\mathbb{E}_Z(p_s(Z)'(\delta - \delta_0))^2)^{1/2}$. By the argument similar to Step 2, the nuisance realisation set S_N shrinks at rate $\mathbf{s}_N$.

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