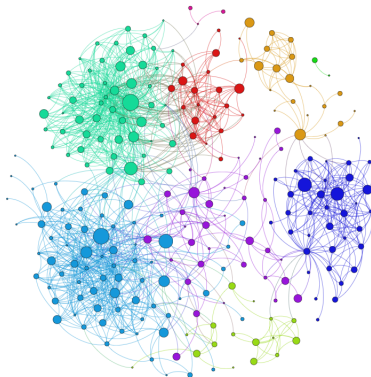


# Classification

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# Classification

Classification is a predictive task in which the response variable  $y$  is discrete or categorical<sup>1</sup>.

Examples:

- Is a credit card user going to default?
- Is a project going to be successful?
- Which product will a consumer buy?
- Which market will a firm enter?
- Which political candidate will an individual vote for?

---

<sup>1</sup> $y$  is **discrete** if it takes on a set of discrete numerical values.  $y$  is **categorical** if it belongs to a set of **categories** (also called **classes**).

# Binary Classification

For binary classification problems, let  $y \in \{0, 1\}$ . One approach to predict  $y$  is to first estimate  $p(y|x)$  and then let

$$\begin{aligned}\hat{y}(x) &= \arg \max_{c \in \{0,1\}} \{\hat{p}(y = c|x)\} \\ &= \begin{cases} 1 & \text{if } \hat{p}(y = 1|x) > 0.5 \\ 0 & \text{o.w.} \end{cases}\end{aligned}\tag{1}$$

(1) is the Bayes classifier.

# Linear Regression

Notice that when  $y \in \{0, 1\}$ ,

$$E(y|x) = 1 \cdot \Pr(y = 1|x) + 0 \cdot \Pr(y = 0|x) = \Pr(y = 1|x)$$

Hence if we assume

$$\Pr(y = 1|x) = x'\beta \quad (2)$$

, then we obtain the linear regression model:

$$y = x'\beta + e \quad (3)$$

Is the model reasonable?  $p(y = 1|x)$  is bounded by  $[0, 1]$ , but  $x'\beta$  is not. So (2) may not be a good assumption. But let's see how well the model performs...

# Linear Regression

Given data  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$ , we have:

$$y_i = x_i' \beta + e_i \quad (4)$$

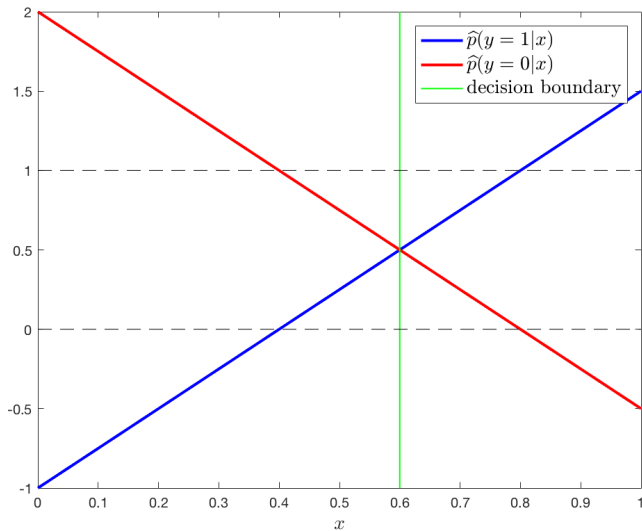
Fitting (4)  $\Rightarrow \hat{\beta}$ .  $x_i' \hat{\beta}$  gives us an estimate of  $\hat{E}(y_i | x_i) = \hat{p}(y_i = 1 | x_i)$ .

Therefore, given a new data point  $x_0$ , we predict  $y_0$  to be

$$\hat{y}_0 = \begin{cases} 1 & \text{if } x_0' \hat{\beta} > 0.5 \\ 0 & \text{o.w.} \end{cases}$$

, which yields the decision boundary:  $x' \hat{\beta} - 0.5 = 0$ .

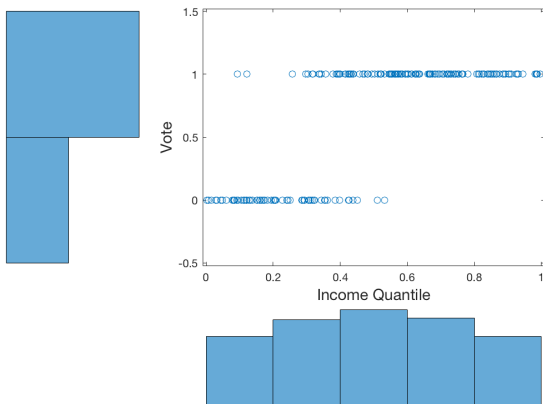
# Linear Regression



# Income and Voting

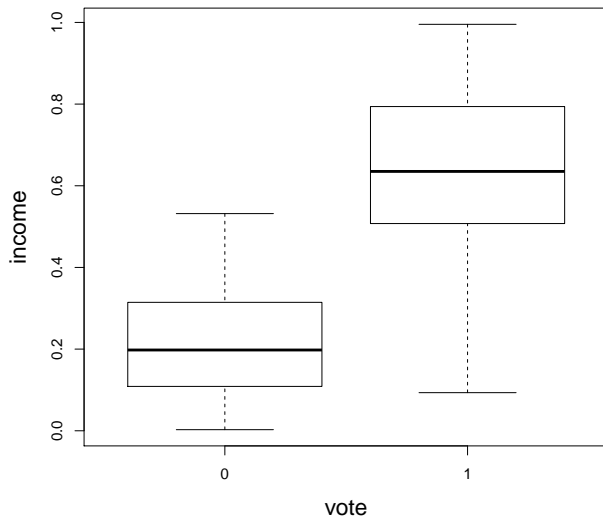
Data: income and voting records of 200 voters

- income: income quantile
- vote: whether voted in the last election





# Income and Voting



# Income and Voting

```
require(AER)
attach(read.csv("voting.txt"))
coeftest(lm(vote ~ income))

##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 0.020419   0.047237   0.4323   0.666
## income      1.310588   0.083000  15.7902  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Income and Voting

To predict vote at  $\text{income} = 0.5$ :

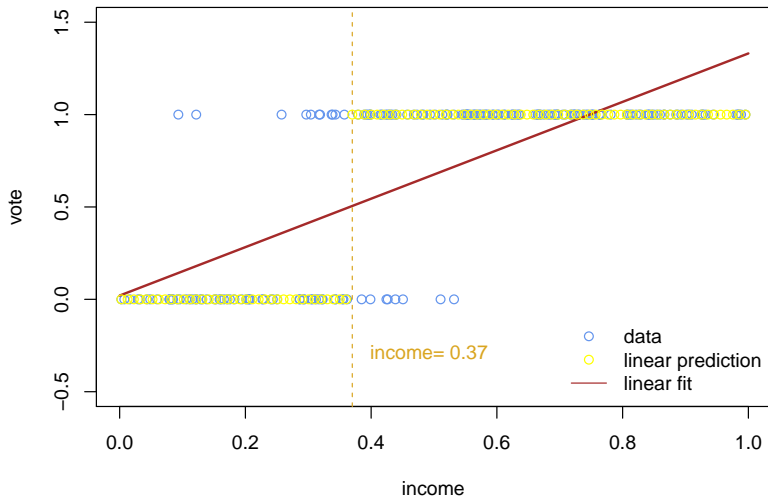
```
x0 = data.frame(income=.5)
p_hat <- predict(lm(vote ~ income),x0)
p_hat

##           1
## 0.6757133

vote_hat <- as.numeric(p_hat>.5)
vote_hat

## [1] 1
```

# Income and Voting



# Logistic Regression

$$\Pr(y = 1|x) = \sigma(x'\beta) = \frac{\exp(x'\beta)}{1 + \exp(x'\beta)} \quad (5)$$

, where  $\sigma(z) \equiv (1 + e^{-z})^{-1}$  is called the **logistic function** or **sigmoid function**<sup>2,3</sup>.

---

<sup>2</sup>More precisely, the logistic regression model is a discriminative probabilistic model with  $p(y|x)$  as the target function and  $\mathcal{H} = \{q(y|x) : q(y = 1|x) = \sigma(x'\beta)\}$ , i.e.,

$$\begin{array}{ll} \Pr(y|x) = p(y|x) & \text{true distribution} \\ \Pr(y|x) = q(y|x) = \begin{cases} \sigma(x'\beta) & y = 1 \\ 1 - \sigma(x'\beta) & y = 0 \end{cases} & \text{hypothesis} \end{array}$$

<sup>3</sup>The logistic function defines the CDF of the *standard logistic distribution*:

$$\mathcal{F}(x) = \frac{\exp(x)}{1 + \exp(x)}$$

# Logistic Regression

(5)  $\Rightarrow$

$$\log \frac{\Pr(y = 1|x)}{\Pr(y = 0|x)} = x'\beta$$

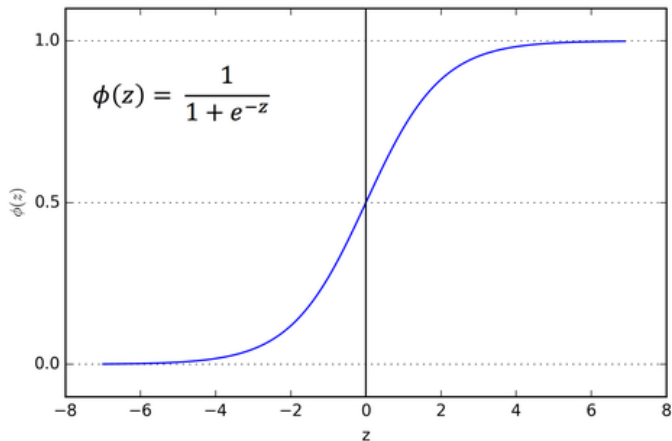
, i.e., logistic regression assumes that the log odds is a linear function<sup>4</sup>.

- The function  $g(p) = \log \frac{p}{1-p}$  – inverse of the sigmoid – is called the **logit function**.

---

<sup>4</sup>If  $p$  denotes the probability of “success”, then  $\frac{p}{1-p}$  is the *odds* of success.

# Logistic Regression



Sigmoid Function

# Logistic Regression

The logistic regression model can be estimated by maximum likelihood.

Given data  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$ ,

$$\hat{\beta} = \arg \max_{\beta} \log \mathcal{L}(\beta)$$

, where

$$\begin{aligned} \log \mathcal{L}(\beta) &= \sum_{i=1}^N \log \Pr(y_i | x_i; \beta) \\ &= \sum_{i=1}^N [y_i \log \sigma(x_i' \beta) + (1 - y_i) \log (1 - \sigma(x_i' \beta))] \\ &= \sum_{i=1}^N [y_i x_i' \beta - \log (1 + \exp(x_i' \beta))] \end{aligned}$$



# Logistic Regression

Equivalently, logistic regression minimizes the cross-entropy training error (also called **deviance**)<sup>5,6</sup>:

$$E_{in}(\beta) = -\frac{1}{N} \sum_{i=1}^N [y_i \log \sigma(x_i' \beta) + (1 - y_i) \log (1 - \sigma(x_i' \beta))] \quad (6)$$

---

<sup>5</sup>Recall that given true distribution  $p(y|x)$  and hypothesis  $q(y|x)$ , cross-entropy

$$\mathbb{H}(p, q) = - \sum_x p(y|x) \log q(y|x)$$

, with the in-sample expression being  $-\frac{1}{N} \sum_{i=1}^N \log q(y_i | x_i)$ .

<sup>6</sup>If we let  $y \in \{-1, 1\}$ , then (6) can be written as

$$E_{in}(\beta) = -\frac{1}{N} \sum_{i=1}^N \log \sigma(y_i x_i' \beta) = \frac{1}{N} \sum_{i=1}^N \log (1 + \exp(-y_i x_i' \beta))$$

, where we use the fact that  $\sigma(-z) = 1 - \sigma(z)$ .

# Logistic Regression

Then, given a new data point  $x_0$ , we predict  $y_0$  to be

$$\hat{y}_0 = \begin{cases} 1 & \text{if } \hat{p}(y_0 = 1 | x_0) = \frac{\exp(x_0' \hat{\beta})}{1 + \exp(x_0' \hat{\beta})} > 0.5 \\ 0 & \text{o.w.} \end{cases}$$

Note that this is equivalent to the decision rule:

$$\hat{y}_0 = \begin{cases} 1 & \text{if } \log \frac{\hat{p}(y_0=1|x_0)}{\hat{p}(y_0=0|x_0)} = x_0' \hat{\beta} > 0 \\ 0 & \text{o.w.} \end{cases}$$

, i.e., logistic regression yields the decision boundary:  $x' \hat{\beta} = 0$ .

# Income and Voting

```
logitfit <- glm(vote ~ income, family=binomial)
coeftest(logitfit)
```

```
##
```

```
## z test of coefficients:
```

```
##
```

```
##           Estimate Std. Error z value Pr(>|z|)
## (Intercept) -5.08565    0.86061 -5.9093 3.435e-09 ***
## income      14.53879    2.24278  6.4825 9.023e-11 ***
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Income and Voting

To predict vote at  $\text{income} = 0.5$ :

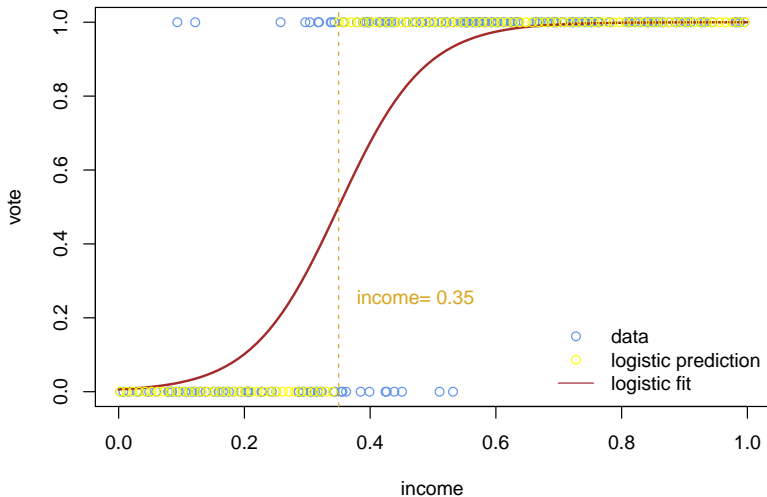
```
x0 = data.frame(income=.5)
p_hat <- predict(logitfit,x0,type="response")
p_hat

##           1
## 0.8987804

vote_hat <- as.numeric(p_hat>.5)
vote_hat

## [1] 1
```

# Income and Voting



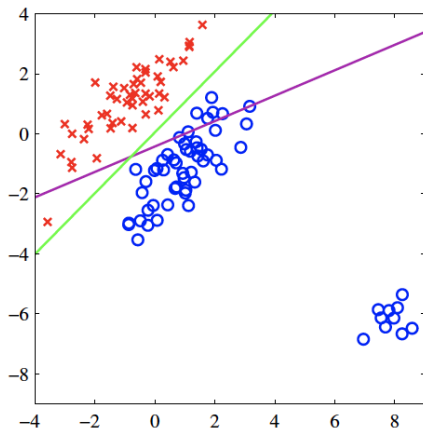
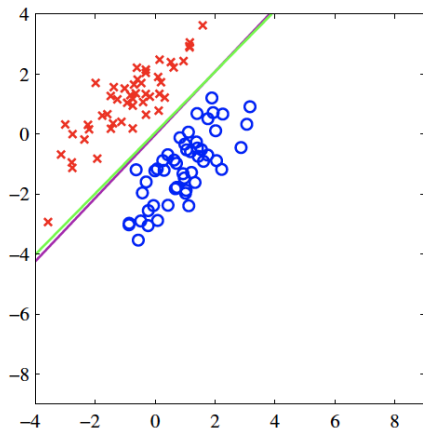
# Linear vs. Logistic Regression

- For many binary classification problems, linear regression and logistic regression yield similar results.
- However, linear regression can be less robust due to the error measure it uses, which is based on the squared-error loss<sup>7</sup>.
- When estimating (3) using least squares, the method seeks to find  $\hat{\beta}$  such that each  $x_i' \hat{\beta}$  is as close to  $y_i$  as possible, even though all we need is for  $\mathcal{I}(x_i' \hat{\beta} - 0.5)$  to be the same as  $y_i$ .
- In particular, the L2 loss penalizes cases in which  $y_i = 1$  and  $x_i' \hat{\beta} \gg 1$ , or  $y_i = 0$  and  $x_i' \hat{\beta} \ll 0$ , even though they lead to correct classification.
- In a sense, the L2 loss penalizes predictions that are “too correct” in that they lie a long way on the correct side of the decision boundary.

---

<sup>7</sup>We will talk about additional problems linear regression has for multiclass classification.

# Linear vs. Logistic Regression



Data from two classes are denoted by red crosses and blue circles, with decision boundaries found by least squares (magenta) and logistic regression (green). Least squares can be highly sensitive to outliers, unlike logistic regression.

## Dose Response

Five groups of animals were exposed to a dangerous substance in varying concentrations. Let  $n_i$  be the number of animals and  $y_i$  the number that died in group  $i$ .

Concentration	$\log_{10} \text{conc}$	$n_i$	$y_i$	$p_i$
$1 \times 10^{-5}$	-5	6	0	0.000
$1 \times 10^{-4}$	-4	6	1	0.167
$1 \times 10^{-3}$	-3	6	4	0.667
$1 \times 10^{-2}$	-2	6	6	1.000
$1 \times 10^{-1}$	-1	6	6	1.000

How to model  $y_i$  as a function of  $\log \text{conc}$ ?



# Logistic Regression

In addition to binary classification, logistic regression is suitable for regression problems where the response variable is the sum of individual binary outcomes.

The model is<sup>8</sup>:

$$\begin{aligned} y_i &\sim \text{Binomial}(n_i, \pi_i) \\ \pi_i &= \sigma(x_i' \beta) \end{aligned} \tag{7}$$

---

<sup>8</sup>The logistic model for binary classification can be similarly written as:

$$y_i \sim \text{Binomial}(1, \sigma(x_i' \beta)) = \text{Bernoulli}(\sigma(x_i' \beta))$$

# Logistic Regression

The log likelihood function is:

$$\begin{aligned}\log \mathcal{L}(\beta) &= \sum_{i=1}^N \log \left( \binom{n_i}{y_i} [\pi_i(\beta)]^{y_i} [1 - \pi_i(\beta)]^{n_i - y_i} \right) \\ &\propto \sum_{i=1}^N [y_i \log \pi_i(\beta) + (n_i - y_i) \log (1 - \pi_i(\beta))] \\ &= \sum_{i=1}^N [y_i (x_i' \beta) - n_i \log (1 + \exp(x_i' \beta))]\end{aligned}$$

# Generalized Linear Models

The logistic regression model belongs to a class of **generalized linear models (GLM)**. A GLM assumes that the response variable  $y$  comes from a known exponential family with mean  $\mu$ , and

$$g(\mu) = x'\beta$$

, where  $g(\cdot)$  is a *monotonic* function called the **link function**.

# Generalized Linear Models

- Normal linear model: Normal distribution with the identity link

$$y_i \sim \mathcal{N}(\mu_i, \sigma^2)$$

$$\mu_i = x_i' \beta$$

- Logistic model: Binomial distribution with the logit link

$$y_i \sim \text{Binomial}(n_i, \pi_i)$$

$$\log\left(\frac{\pi_i}{1 - \pi_i}\right) = x_i' \beta$$

- Poisson model: Poisson distribution with the log link

$$y_i \sim \text{Poisson}(\mu_i)$$

$$\log \mu_i = x_i' \beta$$

# Dose Response

```
## Logistic Regression
require(AER)
y <- c(0,1,4,6,6)
n <- c(6,6,6,6,6)
logconc <- c(-5,-4,-3,-2,-1)
logitfit <- glm(cbind(y,n-y) ~ logconc, family=binomial)
coeftest(logitfit)
```

```
##
## z test of coefficients:
##
##           Estimate Std. Error z value Pr(>|z|)
## (Intercept)   9.5868     3.7067   2.5864 0.009699 **
## logconc       2.8792     1.1023   2.6121 0.008999 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Dose Response

Let  $p_i = y_i / n_i$  be the *observed* proportion that died in group  $i$ . Can we run linear regression of  $p_i$  on  $\log \text{conc}$ ? i.e.,

$$p_i = x_i' \beta + e_i$$

Yes, but the linear model may generate predictions outside the range of  $[0, 1]$  ...

# Dose Response

Better: let

$$z_i \equiv \log \frac{p_i}{1 - p_i}$$

and regress

$$z_i = x_i' \beta + e_i \tag{8}$$

When  $n_i$  is large, model (8)  $\rightarrow$  the logistic model (7).

# Dose Response

```
## Linear Regression:  $p = x' \cdot \text{beta} + e$ 
```

```
p <- y/n
```

```
lsfit1 <- lm(p ~ logconc)
```

```
coeftest(lsfit1)
```

```
##
```

```
## t test of coefficients:
```

```
##
```

```
##          Estimate Std. Error t value Pr(>|t|)
```

```
## (Intercept) 1.416667   0.153055   9.2559 0.002668 **
```

```
## logconc      0.283333   0.046148   6.1397 0.008690 **
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

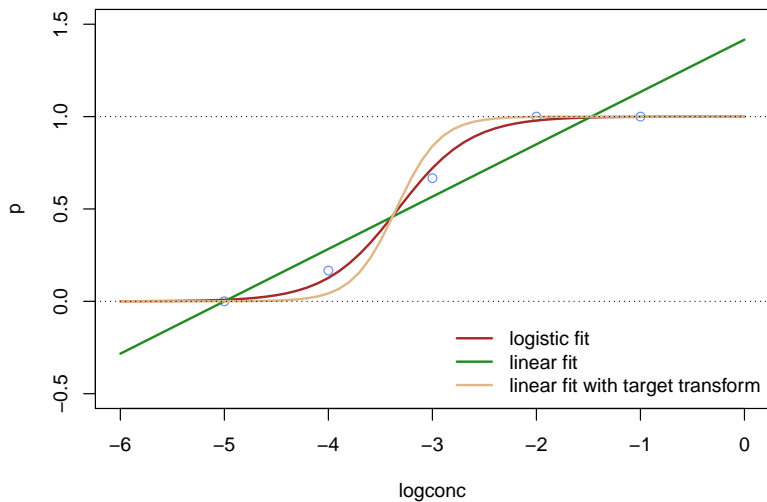


# Dose Response

```
## Linear regression with target transform:
#    $z = x' \beta + e$ , where  $z = \log(p/(1-p))$ 
# Since some  $p=0$  and some  $p=1$ , we add a small number  $\epsilon$  to  $p=0$ ,
# and subtract  $\epsilon$  from  $p=1$ , to avoid  $\log(p/(1-p))$  being undefined.
# Note: when  $n$  is small, regression results are highly sensitive to  $\epsilon$ 
eps <- 1e-4
p[p==0] <- p[p==0] + eps
p[p==1] <- p[p==1] - eps
z = log(p/(1-p))
lsfit2 <- lm(z ~ logconc)
coeftest(lsfit2)

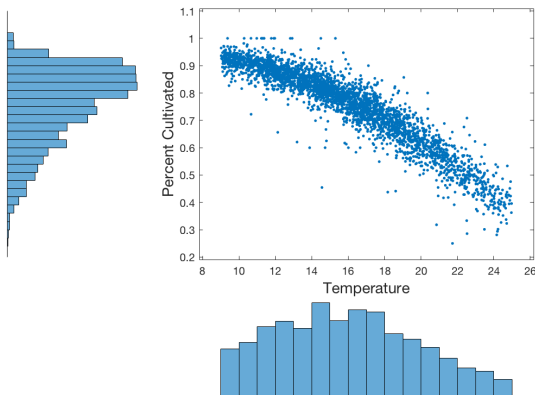
##
## t test of coefficients:
##
##           Estimate Std. Error t value Pr(>|t|)
## (Intercept) 15.95698    2.47044   6.4592 0.007528 **
## logconc      4.76606    0.74487   6.3986 0.007732 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Dose Response



# Cropland

Data on 3144 counties, including agricultural land (fields) available in each county, the number of fields that are being cultivated, and the annual average temperature of each county.



# Cropland

```
cropland <- read.csv("cropland.txt")  
attach(cropland)  
head(cropland)
```

##	temperature	fields	cultivated	percentCultivated
## 1	13.18475	63	49	0.7777778
## 2	12.35680	165	147	0.8909091
## 3	17.57882	38	30	0.7894737
## 4	20.86867	152	95	0.6250000
## 5	13.88084	88	69	0.7840909
## 6	17.18088	191	141	0.7382199

# Cropland

```
## Logistic Regression
require(AER)
logitfit <- glm(cbind(cultivated, fields-cultivated) ~ temperature,
               family=binomial)
coeftest(logitfit)

##
## z test of coefficients:
##
##           Estimate Std. Error z value Pr(>|z|)
## (Intercept)  4.266957   0.017392  245.34 < 2.2e-16 ***
## temperature -0.189233   0.000990 -191.14 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Cropland

```
## Linear Regression
lsfit <- lm(percentCultivated ~ temperature)
coeftest(lsfit)

##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  1.28838143  0.00383395  336.05 < 2.2e-16 ***
## temperature -0.03349385  0.00023385 -143.23 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Cropland

```
## Linear Regression with target transform
```

```
p <- percentCultivated
```

```
eps <- 1e-4
```

```
p[p==1] = p[p==1] - eps
```

```
lsfit2 <- lm(log(p/(1-p)) ~ temperature)
```

```
coeftest(lsfit2)
```

```
##
```

```
## t test of coefficients:
```

```
##
```

```
##           Estimate Std. Error t value Pr(>|t|)
```

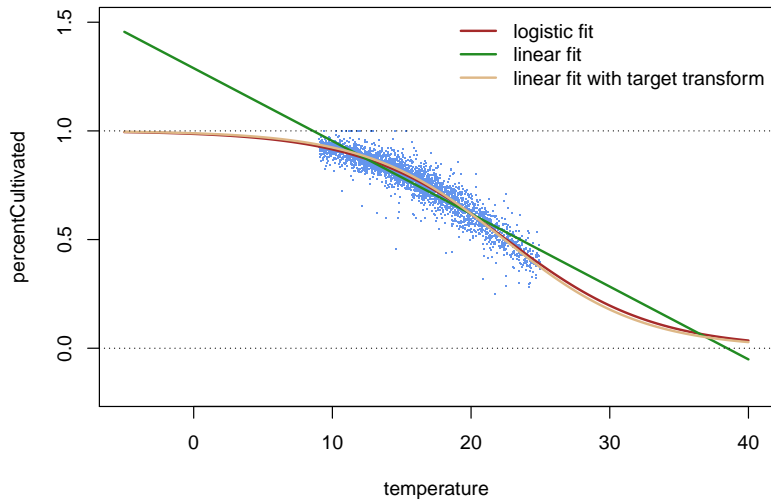
```
## (Intercept)  4.5086192  0.0430642 104.695 < 2.2e-16 ***
```

```
## temperature -0.2012857  0.0026266 -76.632 < 2.2e-16 ***
```

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Cropland





# Classification Error

A binary classifier can make two types of errors:

- **False positive rate (FPR):**  $\Pr(\hat{y} = 1 | y = 0)$
- **False negative rate (FNR):**  $\Pr(\hat{y} = 0 | y = 1)$

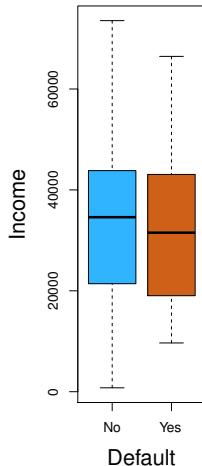
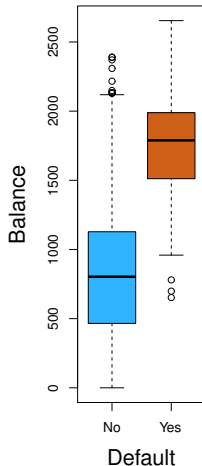
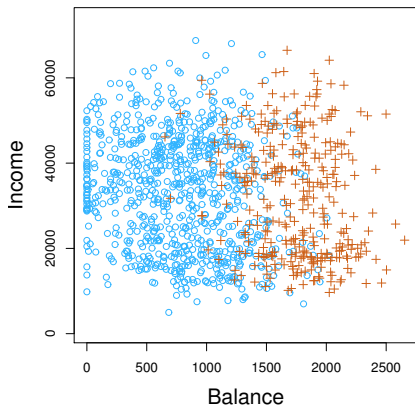
The **sensitivity** of the classifier is  $\Pr(\hat{y} = 1 | y = 1)$  and the **specificity** of the classifier is  $\Pr(\hat{y} = 0 | y = 0)$ .

# Classification Error

		<i>Predicted class</i>		
		– or Null	+ or Non-null	Total
<i>True class</i>	– or Null	True Neg. (TN)	False Pos. (FP)	N
	+ or Non-null	False Neg. (FN)	True Pos. (TP)	P
	Total	N*	P*	

Name	Definition	Synonyms
False Pos. rate	FP/N	Type I error, 1–Specificity
True Pos. rate	TP/P	1–Type II error, power, sensitivity, recall
Pos. Pred. value	TP/P*	Precision, 1–false discovery proportion
Neg. Pred. value	TN/N*	

# Credit Card Default



# Credit Card Default

```
require(ISLR) # contains the data set 'Default'  
attach(Default)  
Default <- Default[,-2]  
head(Default)
```

##	default	balance	income
## 1	No	729.5265	44361.625
## 2	No	817.1804	12106.135
## 3	No	1073.5492	31767.139
## 4	No	529.2506	35704.494
## 5	No	785.6559	38463.496
## 6	No	919.5885	7491.559

# Credit Card Default

```
require(AER)
logitfit <- glm(default ~., data=Default, family=binomial)
coeftest(logitfit)

##
## z test of coefficients:
##
##              Estimate Std. Error  z value  Pr(>|z|)
## (Intercept) -1.1540e+01  4.3476e-01 -26.5447 < 2.2e-16 ***
## balance      5.6471e-03  2.2737e-04  24.8363 < 2.2e-16 ***
## income       2.0809e-05  4.9852e-06   4.1742 2.991e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Credit Card Default

```
cutoff <- .5
logit.p <- logitfit$fit
logit.y <- as.factor(logit.p > cutoff)
levels(logit.y) <- c("No", "Yes")
t <- table(logit.y, default, dnn=c("predicted default", "true default"))
t
```

```
##               true default
## predicted default    No    Yes
##               No  9629  225
##               Yes   38  108
```

```
prop.table(t, 2)
```

```
##               true default
## predicted default          No          Yes
##               No  0.996069101 0.675675676
##               Yes 0.003930899 0.324324324
```

# Credit Card Default

- Overall training error rate:  $(225 + 38) / 10,000 = 2.63\%$
  - FPR: 0.39%. Specificity: 99.61%
  - FNR: 67.57%. Sensitivity: 32.43%
- 
- Note that only  $333 / 10,000 = 3.33\%$  individuals defaulted in the data. Hence a simple but useless *null* classifier that always predicts “No” will result in an error rate of 3.33%.
  - From the perspective of a credit card company that is trying to identify high-risk individuals, the FNR – not the overall error rate – is what's important.
    - ▶ Incorrectly classifying an individual who will not default, though still to be avoided, is less problematic.

## Credit Card Default

- In binary classification, the Bayes classifier assigns  $\hat{y} = 1$  if  $p(y = 1|x) > 0.5$  – here 0.5 is used as a **threshold** in order to classify  $\hat{y} = 1$  based on  $p(y = 1|x)$ .
- Recall that we can use different loss functions<sup>9</sup> to control which type of error we want to minimize: the overall error rate, FPR, or FNR. This is equivalent to changing the threshold for classifying  $\hat{y} = 1$ .
- If we are more concerned about FNR, then we can lower this threshold. For example, if we use 0.1 as the threshold, then we assign  $\hat{y} = 1$  if  $p(y = 1|x) > 0.1$ <sup>10</sup>.

---

<sup>9</sup>other than the 0 – 1 loss which gives us the Bayes classifier.

<sup>10</sup>This is equivalent to using the loss function:  $\ell(y, \hat{y}) = 9$  if  $(y, \hat{y}) = (1, 0)$ ,  $\ell(y, \hat{y}) = 1$  if  $(y, \hat{y}) = (0, 1)$ , and  $\ell(y, \hat{y}) = 0$  otherwise.



# Credit Card Default

```
cutoff <- .1
logit.y <- as.factor(logit.p > cutoff)
levels(logit.y) <- c("No", "Yes")
t <- table(logit.y, default, dnn=c("predicted default", "true default"))
t
```

```
##               true default
## predicted default    No    Yes
##               No  9105    90
##               Yes   562   243
```

```
prop.table(t, 2)
```

```
##               true default
## predicted default          No          Yes
##               No  0.94186407 0.27027027
##               Yes 0.05813593 0.72972973
```

# Credit Card Default

- Overall training error rate:  $(90 + 562) / 10,000 = 6.52\%$
- FPR: 5.81%. Specificity: 94.19%
- FNR: 27.03%. Sensitivity: 72.97%

# Credit Card Default

```
cutoff <- .01
logit.y <- as.factor(logit.p > cutoff)
levels(logit.y) <- c("No", "Yes")
t <- table(logit.y, default, dnn=c("predicted default", "true default"))
t
```

```
##               true default
## predicted default    No    Yes
##               No  7134    10
##               Yes 2533   323
```

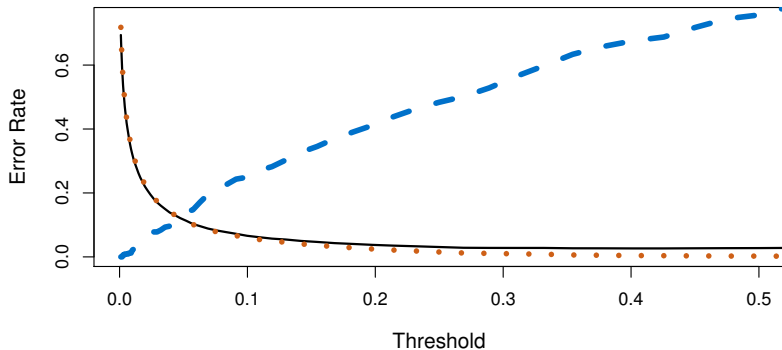
```
prop.table(t, 2)
```

```
##               true default
## predicted default          No          Yes
##               No  0.73797455 0.03003003
##               Yes 0.26202545 0.96996997
```

# Credit Card Default

- Overall training error rate:  $(10 + 2533) / 10,000 = 25.43\%$
- FPR: 26.20%. Specificity: 74.80%
- FNR: 3.00%. Sensitivity: 97.00%

# Credit Card Default

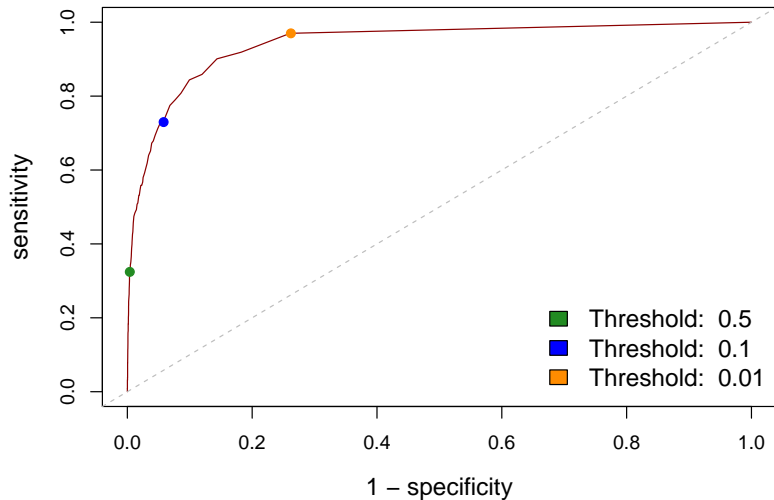


Black solid line: overall error rate; Orange dotted line: FPR;  
Blue dashed line: FNR.

# The ROC Curve

- The **ROC curve** displays **sensitivity** ( $1 - \text{FNR}$ ) vs  $1 - \text{specificity}$  (**FPR**) for *all* possible thresholds.
- The overall performance of a classifier, summarized over all possible thresholds, is given by the **area under the curve (AUC)**.
- An ideal ROC curve hugs the top left corner (high sensitivity, high specificity): *the larger the AUC the better the classifier*.
- ROC curves are useful for comparing different classifiers.

# Credit Card Default



# Credit Card Default

- The error rates we have calculated so far are *training errors*.
  - Now let's split our sample into a *training data set* and a *test data set*.
  - We are going to fit our models on the *training data* and test their performance on the *test data*.
- 

```
test <- sample(1:nrow(Default),2000) # sample 2000 random indices
TR.X <- Default[-test,-1] #training X
TE.X <- Default[test,-1] #test X
TR.y <- default[-test] #training y
TE.y <- default[test] #test y
```



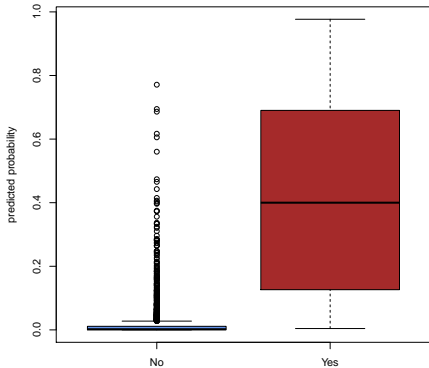
# Credit Card Default

```
cutoff <- .5 # threshold
logitfit <- glm(TR.y ~., data=TR.X, family=binomial)
logit.p <- predict(logitfit, TE.X, type="response")
logit.pred <- as.factor(logit.p > cutoff)
levels(logit.pred) <- c("No", "Yes")
table(logit.pred, TE.y, dnn=c("predicted default", "true default"))
```

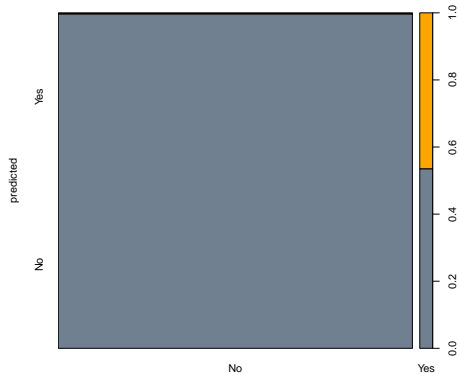
```
##                true default
## predicted default    No    Yes
##                No  1923    38
##                Yes    6    33
```

# Credit Card Default

Logistic Regression



Logistic Regression



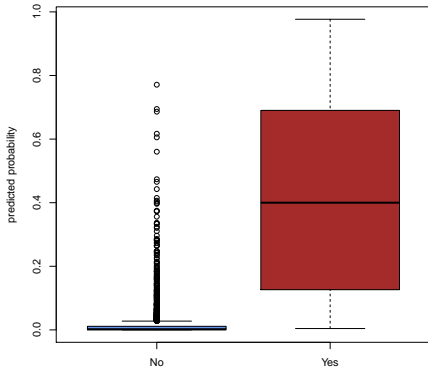
# Credit Card Default

```
cutoff <- .1 # threshold
logitfit <- glm(TR.y ~., data=TR.X, family=binomial)
logit.p <- predict(logitfit, TE.X, type="response")
logit.pred <- as.factor(logit.p > cutoff)
levels(logit.pred) <- c("No", "Yes")
table(logit.pred, TE.y, dnn=c("predicted default", "true default"))
```

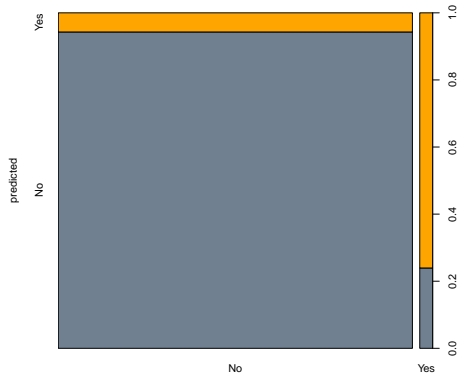
```
##                true default
## predicted default    No    Yes
##                No  1819    17
##                Yes   110    54
```

# Credit Card Default

Logistic Regression



Logistic Regression



# Similarity-Based Methods

- One way to classify data is to assign a new input the class of the most similar input(s) in the data. This is called the **nearest neighbor** method.
- The nearest neighbor method is a **similarity-based method**. These methods are *model free* and hence *nonparametric*.
  - ▶ If everyone around you is a republican, you are probably a republican.

- Given an input  $x$ , the **K-nearest neighbors (KNN)** classifier finds the  $K$  points that are closest in distance to  $x$ <sup>11</sup>, denoted by  $\mathcal{N}_K(x) = \{x_{(1)}, \dots, x_{(K)}\}$ , and then classify using **majority vote**: let  $y$  be the most common class among  $\{y_{(1)}, \dots, y_{(K)}\}$ <sup>12</sup>.
- Equivalently, the KNN classifier can be thought of as first estimating

$$\hat{p}(y = j|x) = \frac{1}{K} \sum_{i \in \mathcal{N}_K(x)} \mathcal{I}(y_i = j)$$

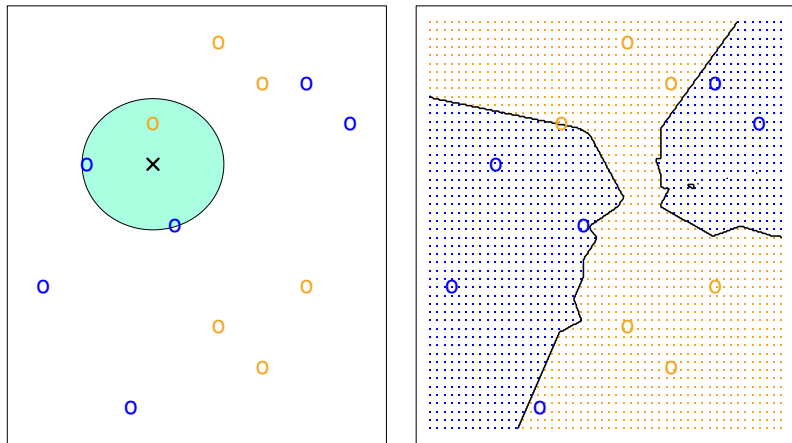
, where  $y \in \{1, \dots, J\}$ , and then applying the Bayes classifier.

---

<sup>11</sup>To do this, we need a **distance measure**, or **similarity measure**. For real-valued inputs, the common choice is to use the Euclidean distance:  $d(x, x') = \|x - x'\|$ .

<sup>12</sup>Ties are broken at random.

# KNN



KNN in two dimensions ( $K = 3$ )

# Credit Card Default

```
## KNN
# To perform KNN classification, we first standardize the x variables
# so that all variables have mean zero and standard deviation one.
s.balance <- scale(balance)
s.income <- scale(income)
SX <- data.frame(s.balance,s.income) #standardized x variables
TR.SX <- SX[-test,]
TE.SX <- SX[test,]
```



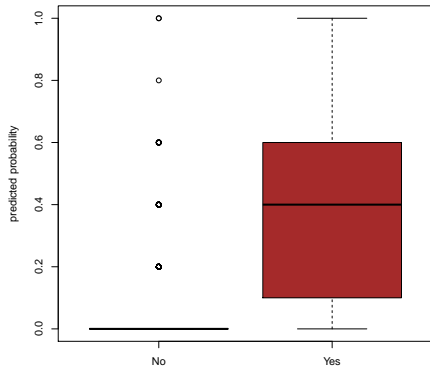
# Credit Card Default

```
require(class)
K <- 5 #K value
knn.pred <- knn(TR.SX,TE.SX,TR.y,k=K,prob=TRUE)
table(knn.pred,TE.y,dnn=c("predicted default","true default"))
```

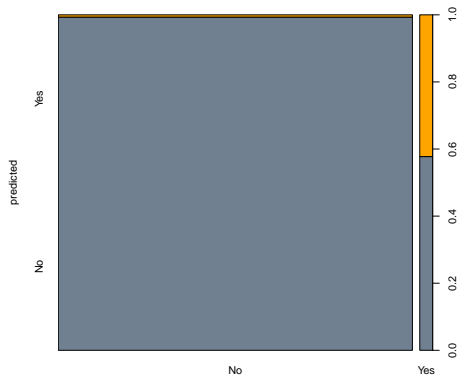
```
##                true default
## predicted default    No    Yes
##                No  1916    41
##                Yes   13    30
```

# Credit Card Default

KNN = 5



KNN = 5

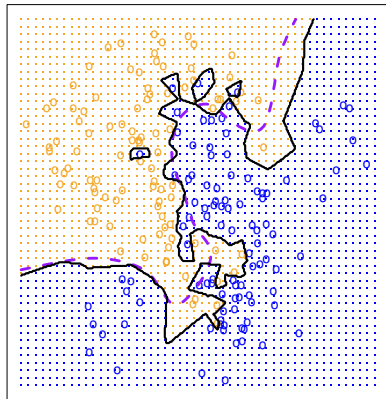


In choosing  $K$ , we face the bias-variance tradeoff:

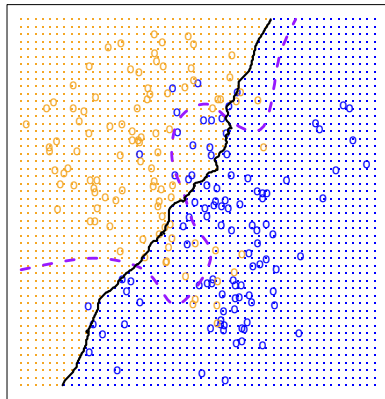
- With  $K = 1$ , the KNN training error rate is 0. Bias is low and variance is high.
- As  $K$  grows, the method becomes less flexible and produces a decision boundary that is closer to linear, with lower variance and higher bias.

# KNN

KNN: K=1

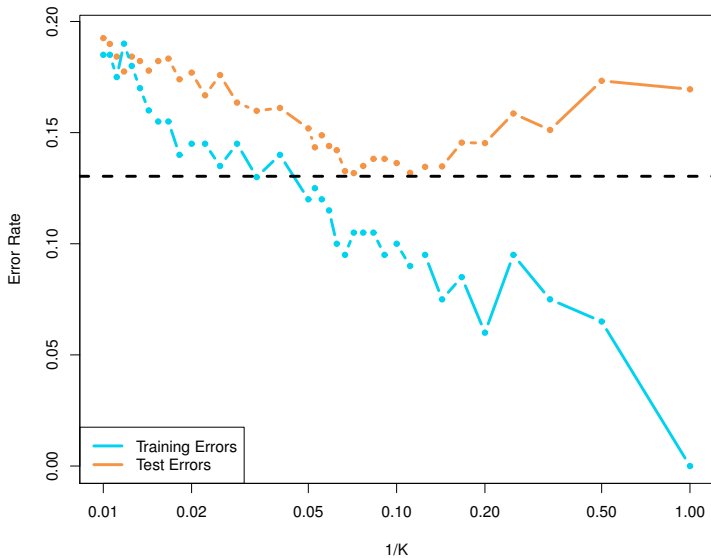


KNN: K=100



Black curve: KNN decision boundary. Purple curve: Bayes decision boundary (decision boundary based on the Bayes classifier and the true  $p(y|x)$ )

# KNN



# Parametric vs. Nonparametric Methods

- KNN is a nonparametric (model-free) method. In general, these methods can work well for prediction in a wide variety of situations, since they don't make any real assumptions.
- The downside is that they are essentially a black box and lack **interpretability**. They are also more *computationally expensive* since they typically need to store the *entire* data and use them whenever predicting on a new point.
  - ▶ In contrast, parametric methods summarize the data with a fixed set of parameters, which are sufficient for prediction.
- In addition, KNN suffers from the **curse of dimensionality**: given  $N$ , when  $p$  is large<sup>13</sup>, data become relatively *sparse*. In high dimensions, the neighborhood represented by the  $K$  nearest points may not be local.

---

<sup>13</sup> $p$  being the dimension of the input space.

# Multiclass Classification

For multiclass problems, let  $y \in \{1, \dots, J\}$ .

The Bayes classifier is:

$$\hat{y}(x) = \arg \max_{c \in \{1, \dots, J\}} \{\hat{p}(y = c|x)\}$$

# Linear Regression

$$\Pr(y = j|x) = x'\beta_j \quad (9)$$

Given data  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$ , let's define  $y_i^j = \mathcal{I}(y_i = j)$ . Then (9) implies the following  $J$  regression equations:

$$y_i^j = x_i'\beta_j + e_j, \quad j = 1, \dots, J \quad (10)$$

, where each  $y_i^j$  is a binary variable.

Estimating (10) gives us  $\{\hat{\beta}_j\}_{j=1}^J$ . Then, given a new data point  $x_0$ , we predict  $y_0$  to be:

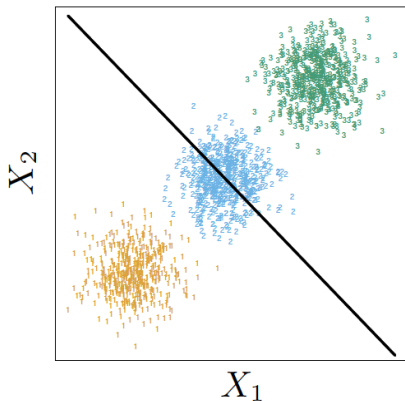
$$y_0 = \arg \max_j \{x_0'\hat{\beta}_j\} \quad (11)$$



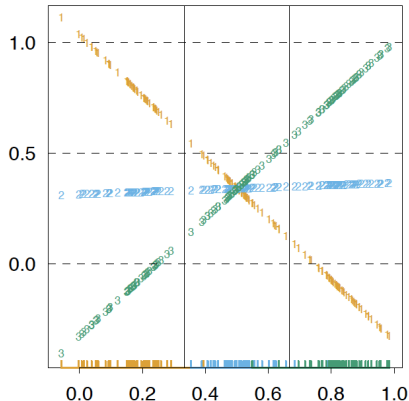
# Linear Regression

- In addition to a lack of robustness, the linear regression approach can have serious problems dealing with multiclass problems ( $J \geq 3$ ).
- Because of the rigid nature of the linear regression model, classes can be **masked** by others – particularly when  $J$  is large and  $p$  is small.

# Linear Regression



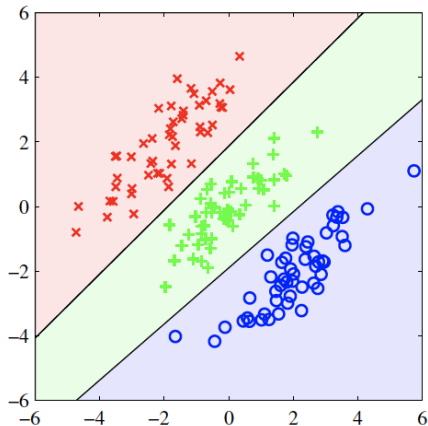
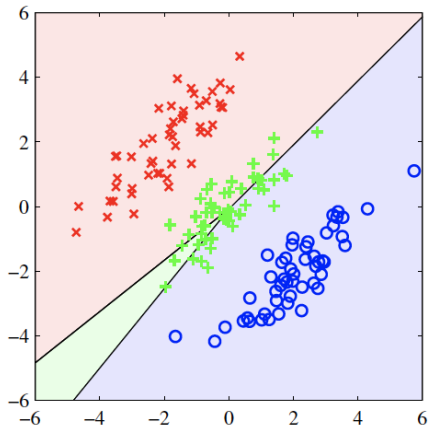
linear regression decision boundary



the three fitted regression lines

For this particular 3-class problem, linear regression misses the middle class completely. This problem is called **masking**. Projecting onto the line joining the three class centroids shows why this happened.

# Linear Regression



Left: linear regression; Right: logistic regression

# Multinomial Logistic Regression

$$\Pr(y = j|x) = \frac{\exp(x'\beta_j)}{\sum_{\ell=1}^J \exp(x'\beta_{\ell})} \quad (12)$$

(12)  $\Rightarrow$

$$\ln \frac{\Pr(y = j|x)}{\Pr(y = k|x)} = x'(\beta_j - \beta_k)$$

- The function  $\sigma_j(z) \equiv \frac{\exp(z_j)}{\sum_{\ell=1}^J \exp(z_{\ell})}$  is called the **softmax function**<sup>14</sup> – a generalization of the sigmoid.

---

<sup>14</sup>  $z = (z_1, \dots, z_J)$ .

# Multinomial Logistic Regression

Note that since  $\sum_{j=1}^J \Pr(y = j|x) = 1$ , we only need to estimate  $\Pr(y = j|x)$  for  $J - 1$  classes of  $y$ . Therefore, we can choose one class of  $y$ , say  $y = 1$ , to be the **reference level** and *normalize*  $\beta_1$  to 0.

This implies

$$\Pr(y = 1|x) = \frac{1}{1 + \sum_{\ell=2}^J \exp(x'\beta_{\ell})}$$
$$\Pr(y = j|x) = \frac{\exp(x'\beta_j)}{1 + \sum_{\ell=2}^J \exp(x'\beta_{\ell})}, \quad j = 2, \dots, J$$

, and

$$\ln \frac{\Pr(y = j|x)}{\Pr(y = 1|x)} = x'\beta_j$$

, i.e.,  $\exp(x'\beta_j)$  becomes the probability of  $y = j$  relative to  $y = 1$ .

# Mode of Transportation

Modes of transportation: {bus, car, subway}

Individual variables: log (annual) income, distance to work (from 0 to 1)

```
transport <- read.csv("Transport.txt")  
head(transport,3)
```

```
##      LogIncome DistanceToWork ModeOfTransportation  
## 1 11.777090      0.6454524      car  
## 2 11.130492      0.5135208      subway  
## 3  9.090856      0.8144265      subway
```

```
loginc <- transport$LogIncome  
distance <- transport$DistanceToWork  
y <- transport$ModeOfTransportation
```

# Mode of Transportation

```
prop.table(table(y))
```

```
## y
##      bus      car subway
## 0.22  0.31  0.47
```

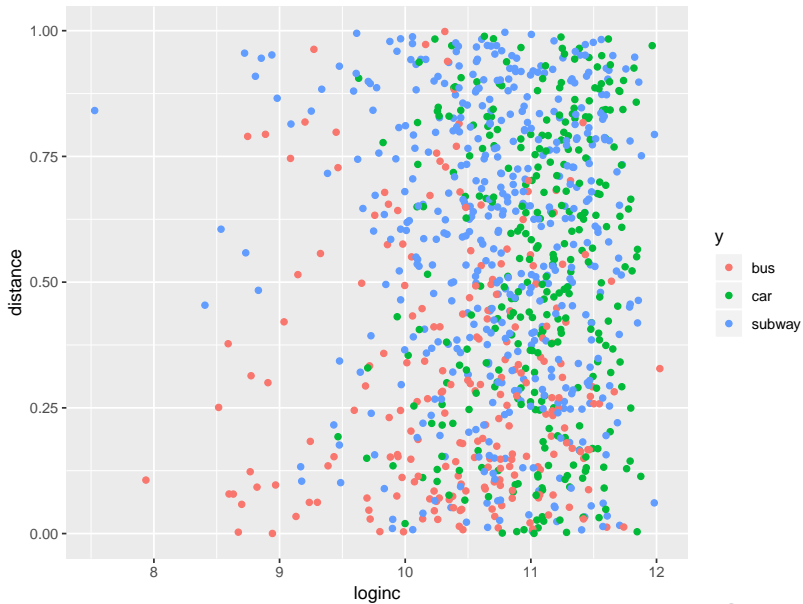
```
income <- exp(loginc)
cbind(mean(income[y=="bus"]),mean(income[y=="car"]),
mean(income[y=="subway"]))
```

```
##      [,1]      [,2]      [,3]
## [1,] 42792 70006.83 56048.95
```

```
cbind(mean(distance[y=="bus"]),mean(distance[y=="car"]),
mean(distance[y=="subway"]))
```

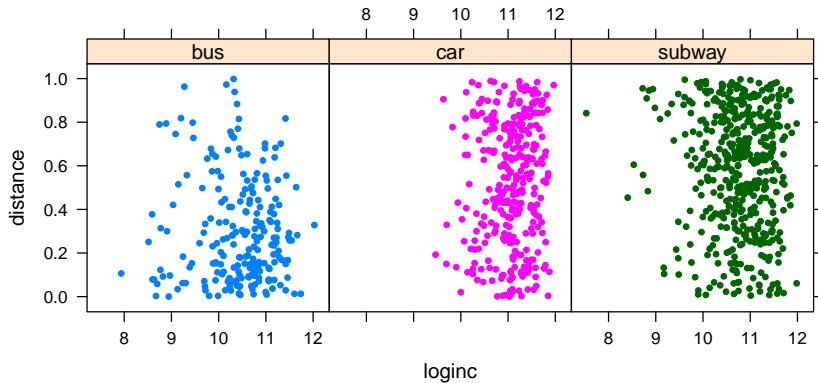
```
##      [,1]      [,2]      [,3]
## [1,] 0.3032989 0.5149095 0.580446
```

# Mode of Transportation

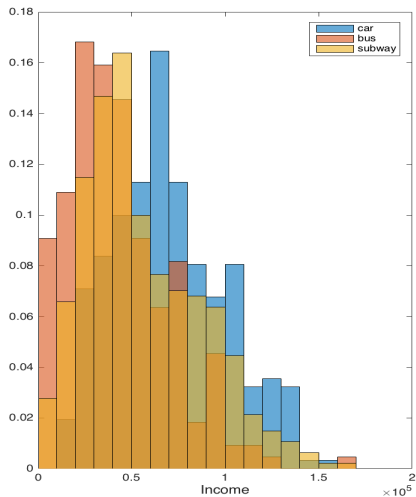
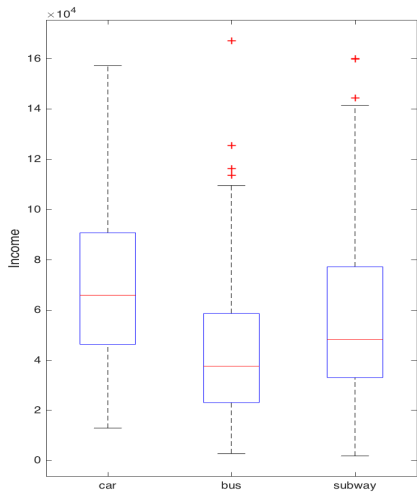




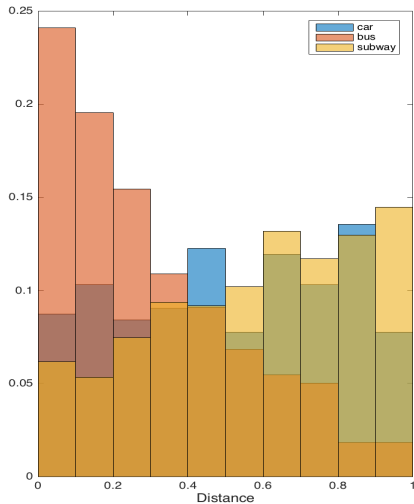
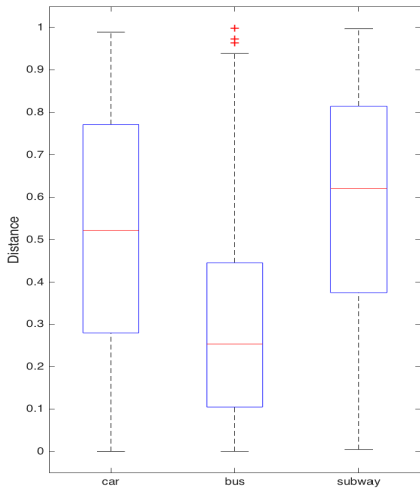
# Mode of Transportation



# Mode of Transportation



# Mode of Transportation



# Mode of Transportation

```
require(nnet)
logitfit <- multinom(y ~ loginc + distance)
```

```
require(AER)
coeftest(logitfit)
```

```
##
## z test of coefficients:
##
##              Estimate Std. Error  z value  Pr(>|z|)
## car:(Intercept)  -18.60894    1.85544 -10.0294 < 2.2e-16 ***
## car:loginc        1.64705     0.16969  9.7061 < 2.2e-16 ***
## car:distance      2.93996     0.37602  7.8187 5.339e-15 ***
## subway:(Intercept) -8.55927    1.45952 -5.8645 4.506e-09 ***
## subway:loginc      0.72359     0.13545  5.3421 9.189e-08 ***
## subway:distance    3.75524     0.35014 10.7248 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Mode of Transportation

Estimation results: (reference level: bus)

$$\log \frac{\hat{p}(\text{car}|x)}{\hat{p}(\text{bus}|x)} = -18.61 + 1.65 \times \text{loginc} + 2.94 \times \text{distance} \quad (13)$$
$$= x' \hat{\beta}_{\text{car}}$$

$$\log \frac{\hat{p}(\text{subway}|x)}{\hat{p}(\text{bus}|x)} = -8.56 + 0.72 \times \text{loginc} + 3.76 \times \text{distance}$$
$$= x' \hat{\beta}_{\text{subway}}$$

, where  $x = [1, \text{loginc}, \text{distance}]'$ ,  $\hat{\beta}_{\text{car}} = [-18.61, 1.65, 2.94]'$ , and  $\hat{\beta}_{\text{subway}} = [-8.56, 0.72, 3.76]'$ .

# Mode of Transportation

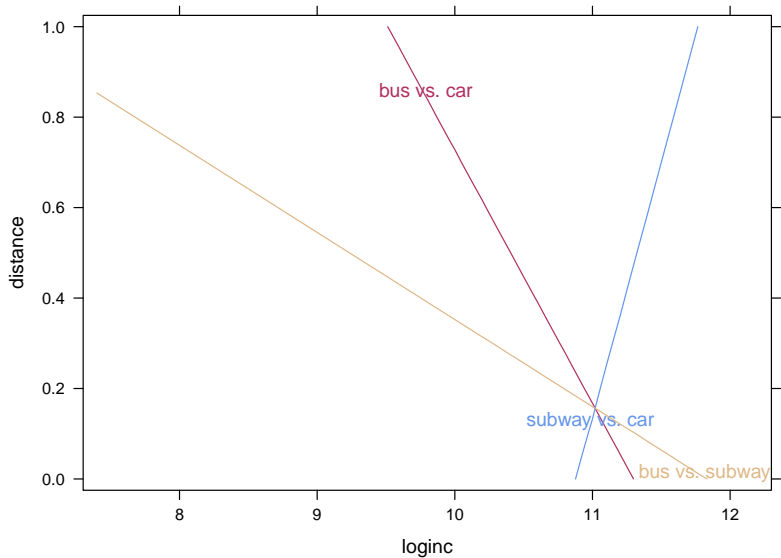
(13)  $\Rightarrow$

$$\begin{aligned}\hat{p}(\text{bus}|x) &= \frac{1}{1 + \exp(x'\hat{\beta}_{\text{car}}) + \exp(x'\hat{\beta}_{\text{subway}})} \\ \hat{p}(\text{car}|x) &= \frac{\exp(x'\hat{\beta}_{\text{car}})}{1 + \exp(x'\hat{\beta}_{\text{car}}) + \exp(x'\hat{\beta}_{\text{subway}})} \\ \hat{p}(\text{subway}|x) &= \frac{\exp(x'\hat{\beta}_{\text{subway}})}{1 + \exp(x'\hat{\beta}_{\text{car}}) + \exp(x'\hat{\beta}_{\text{subway}})}\end{aligned}\tag{14}$$

# Mode of Transportation

- Decision boundary between bus and car:  $x' \hat{\beta}_{\text{car}} = 0$
- Decision boundary between bus and subway:  $x' \hat{\beta}_{\text{subway}} = 0$
- Decision boundary between car and subway:  $x' (\hat{\beta}_{\text{subway}} - \hat{\beta}_{\text{car}}) = 0$

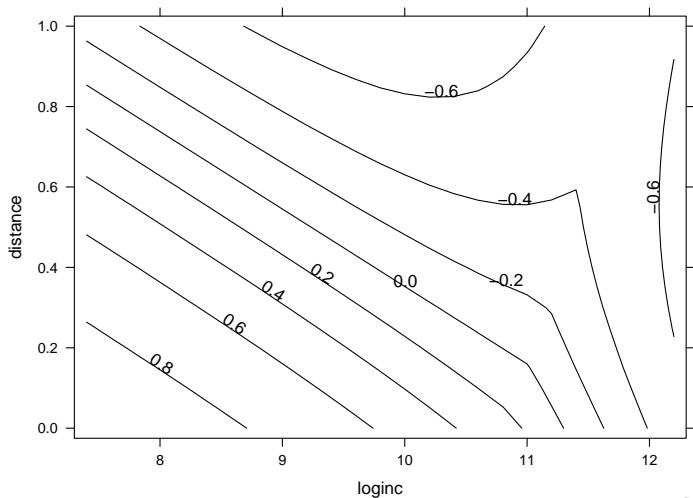
# Mode of Transportation





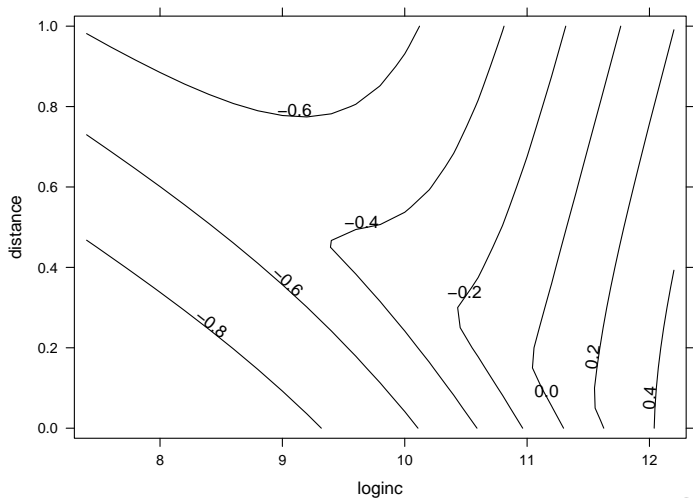
# Mode of Transportation

Contour plot of  $-\max(x'\hat{\beta}_{\text{car}}, x'\hat{\beta}_{\text{subway}})$ :



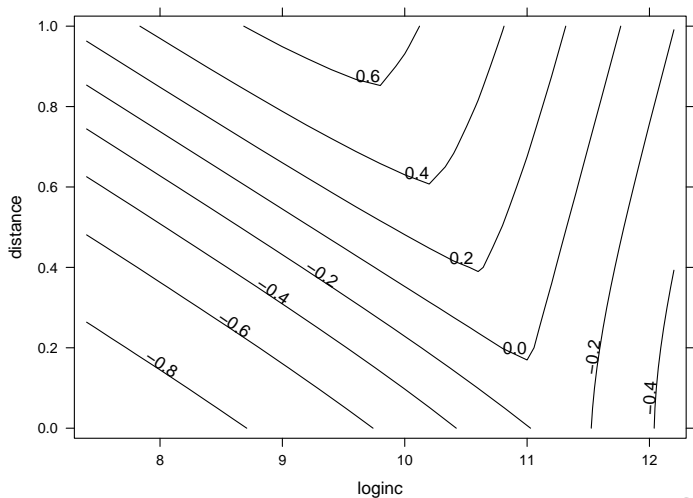
# Mode of Transportation

Contour plot of  $x'\hat{\beta}_{\text{car}} - \max(0, x'\hat{\beta}_{\text{subway}})$ :



# Mode of Transportation

Contour plot of  $x'\hat{\beta}_{\text{subway}} - \max(0, x'\hat{\beta}_{\text{car}})$ :



# Mode of Transportation

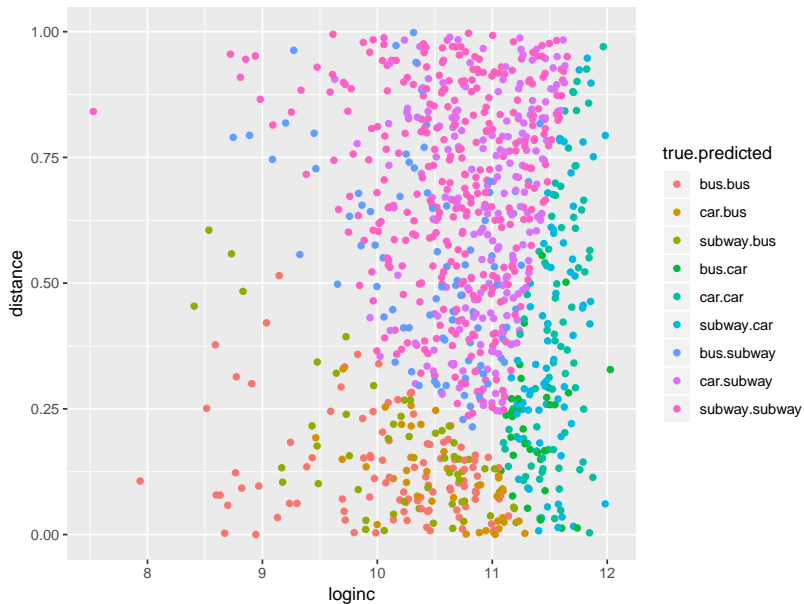
```
logit.yhat <- predict(logitfit)
t <- table(logit.yhat,y,dnn=c("predicted","true"))
t

##           true
## predicted bus car subway
##    bus    101  41    55
##    car     33  78    72
##    subway  86 191   343

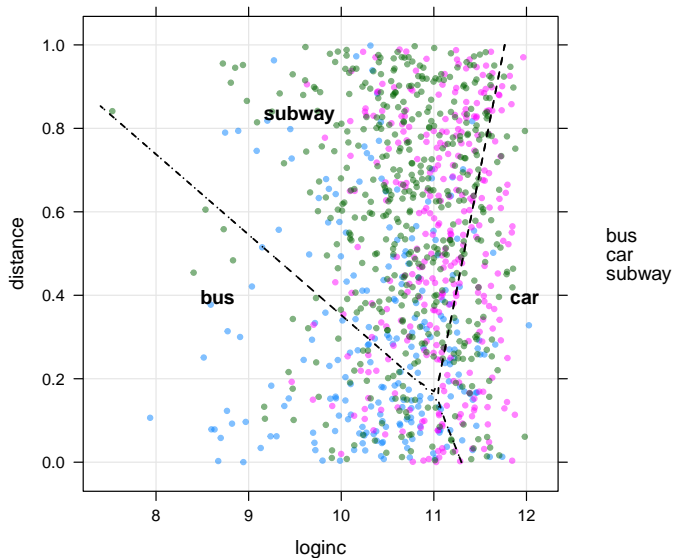
1 - sum(diag(t))/sum(t) # training error rate

## [1] 0.478
```

# Mode of Transportation



# Mode of Transportation



# Mode of Transportation

Now suppose there is no subway, what will be the share of bus and car as mode of transportation among the commuters?

From (13), we know that:

$$\log \frac{\hat{p}(\text{car}|x)}{\hat{p}(\text{bus}|x)} = -18.61 + 1.65 \times \log \text{inc} + 2.94 \times \text{distance}$$

The decision boundary between bus and car does *not* change whether there is subway or not.

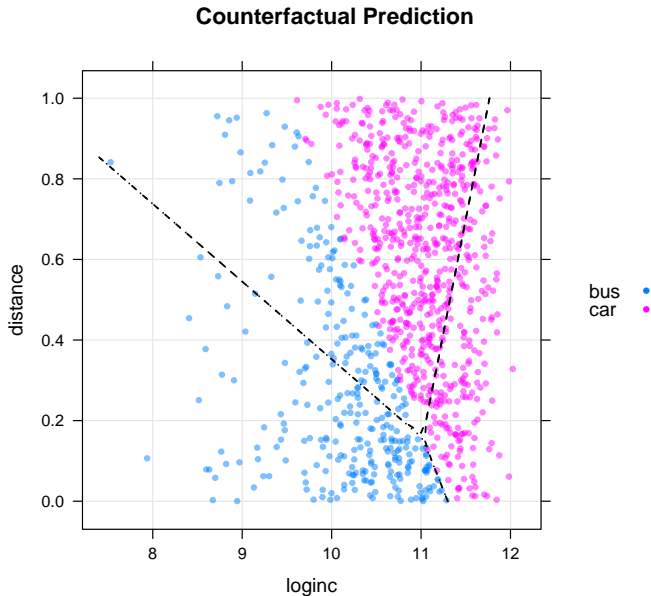
# Mode of Transportation

```
require(ramify)
logit.phat <- predict(logitfit,type="probs")
counterfactual.p <- logit.phat[,c(1,2)] # no subway
counterfactual.p <- counterfactual.p/rowSums(counterfactual.p)
counterfactual.y <- as.factor(argmax(counterfactual.p))
levels(counterfactual.y) <- c("bus","car")
table(counterfactual.y,logit.yhat)
```

```
##               logit.yhat
## counterfactual.y bus car subway
##               bus 197   0   116
##               car   0 183   504
```



# Mode of Transportation



# Calculating Market Share

Assume the observed data  $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$  is a random sample drawn from the underlying population. Then the “market share” of alternative  $j$  – the share of individuals in the population that choose  $j$  – is

$$\begin{aligned}\Pr(y_i = j) &= \int \Pr(y_i = j | x_i) f(x_i) dx_i \\ &\approx \frac{1}{N} \sum_{i=1}^N \Pr(y_i = j | x_i)\end{aligned}$$

, i.e., we can average individual conditional choice probabilities to get an estimate of the market share of each alternative in the population.

# Mode of Transportation

```
# note: average choice probabilities estimated by logistic regression  
# on the training data always match the observed shares of choices  
# (if intercepts are included in the model)
```

```
marketShare.subway = colMeans(logit.phat)  
marketShare.subway
```

```
##          bus          car      subway  
## 0.2199985 0.3100005 0.4700010
```

```
marketShare.nosubway = colMeans(counterfactual.p)  
marketShare.nosubway
```

```
##          bus          car  
## 0.3822821 0.6177179
```

# Mode of Transportation

predicted share	with subway	without subway
bus	22%	38%
car	31%	62%

Is this reasonable? Many people use subway not because of income or distance considerations, but because they cannot drive or they strongly prefer public transportation. For these people, if there is no subway, they would mostly switch to bus rather than car...

# Independence of Irrelevant Alternatives (IIA)

For the multinomial logistic regression model,

$$\log \frac{\Pr(y = j|x)}{\Pr(y = k|x)} = x'(\beta_j - \beta_k)$$

for any two classes  $j$  and  $k$ .

The probability of  $y = j$  relative to  $y = k$  depends *only* on  $x'\beta_j$  and  $x'\beta_k$  – in particular, it is *not* affected by the existence and the properties of other classes.

This is called the **independence of irrelevant alternatives (IIA)** property.

# Independence of Irrelevant Alternatives (IIA)

As an illustration of the IIA property (and why it can be undesirable in some cases), consider a more extreme example of the transportation problem:

## Blue bus, Red bus

A route is currently served by a blue bus. People traveling along this route can either take the blue bus or drive themselves.

Suppose we observe each traveler's transportation choice, but do not observe any other characteristics. Our logistic regression model is then simply:

$$\log \frac{\Pr(\text{blue bus}|x)}{\Pr(\text{car}|x)} = \beta_0 \quad (15)$$

, where  $x = 1$ . If currently 40% of the travelers take the blue bus, while 60% drive, then  $\hat{\beta}_0 = \log\left(\frac{2}{3}\right)$ .

# Independence of Irrelevant Alternatives (IIA)

## Blue bus, Red bus (cont.)

Note that (15) predicts the relative share of blue bus riders to car drivers to be 2 : 3 regardless of what other transportation options are available.

What if the government now decides to introduce a red bus to this route, which is identical to the blue bus except the color of the paint?

Suppose people do not care about color, so that  $\frac{\Pr(\text{red bus})}{\Pr(\text{blue bus})} = 1$ , then the model would predict the rider shares to be

$$\Pr(\text{blue bus}) : \Pr(\text{red bus}) : \Pr(\text{car}) = 2 : 2 : 3$$

$$\Rightarrow \Pr(\text{blue bus}) = \Pr(\text{red bus}) = 28.57\%, \Pr(\text{car}) = 42.86\% .$$

This is clearly unreasonable, since we should expect

$\Pr(\text{blue bus}) = \Pr(\text{red bus}) = 20\%$ ,  $\Pr(\text{car}) = 60\%$ , i.e., the bus riders would be split between the blue bus and the red bus, while the car drivers continue to drive.

# Independence of Irrelevant Alternatives (IIA)

The problem is due to *unobserved* variables. Suppose the true model is:

$$\Pr(y = j|x, z) = \frac{\exp(x'\beta_j + z'\gamma_j)}{\sum_{\ell} \exp(x'\beta_{\ell} + z'\gamma_{\ell})}$$

, where  $z$  is unobserved<sup>15</sup>. Then

$$\Pr(y = j|x) = \int \frac{\exp(x'\beta_j + z'\gamma_j)}{\sum_{\ell} \exp(x'\beta_{\ell} + z'\gamma_{\ell})} f(z) dz$$

In this case,  $\log \frac{\Pr(y=j|x)}{\Pr(y=k|x)}$  is in general no longer a function of  $x'\beta_j$  and  $x'\beta_k$  only, hence the IIA no longer holds.

---

<sup>15</sup>e.g., preference for public transportation.



# Multinomial Logistic Regression

As in the binary case, multinomial logistic regression can be used for problems where the response variable is the sum of individual discrete outcomes.

The model is:

$$y_i \sim \text{Multinomial}(n_i, \pi_i) \quad (16)$$

, where  $\pi_i = (\pi_{i1}, \dots, \pi_{iJ})$ ,  $\sum_{j=1}^J \pi_{ij} = 1$ , and

$$\pi_{ij} = \frac{\exp(x_i' \beta_j)}{\sum_{\ell=1}^J \exp(x_i' \beta_{\ell})}$$

- When  $n_i = 1$ , (16) becomes the multinomial logistic model for multiclass classification.

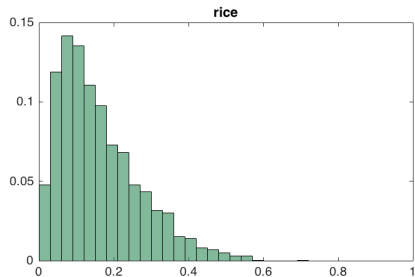
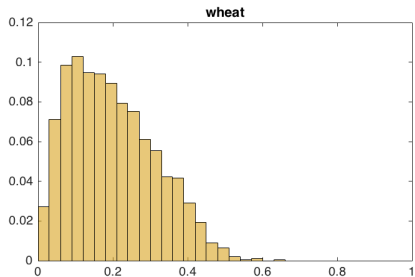
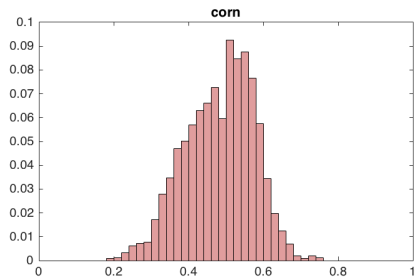
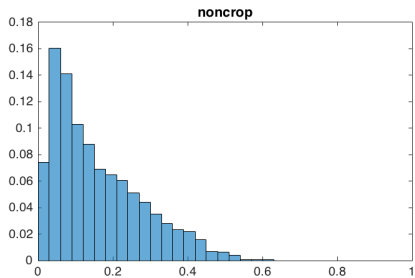
# Crop Choice

Crops: {corn, wheat, rice}

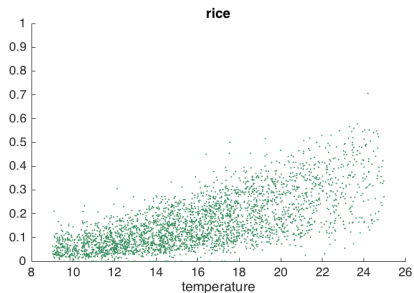
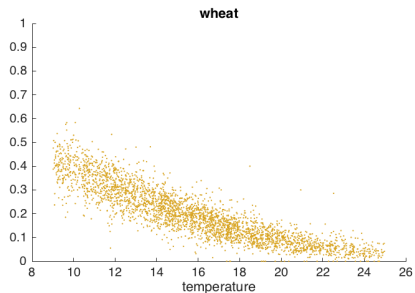
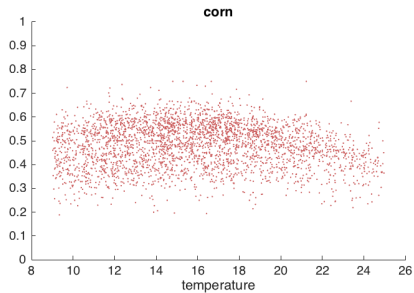
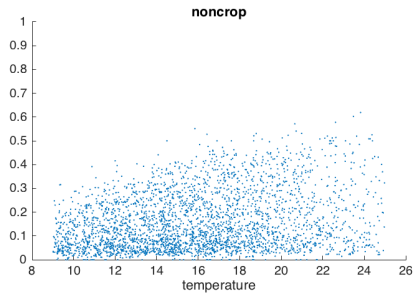
3144 counties, data on each county include number of agricultural land (fields) available, number of fields that are being cultivated for each crop, average temperature, and average monthly rainfall.

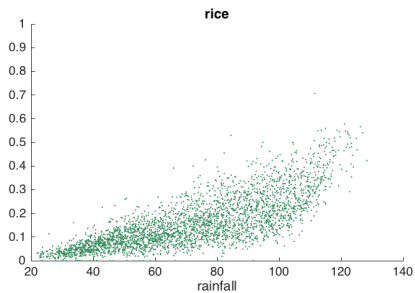
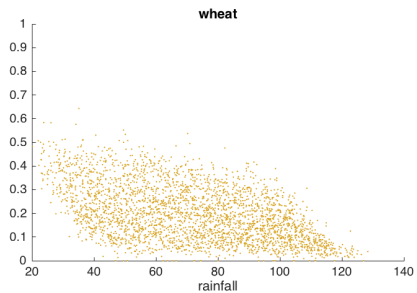
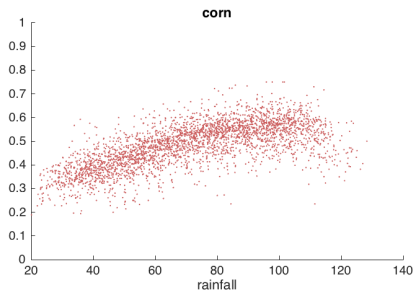
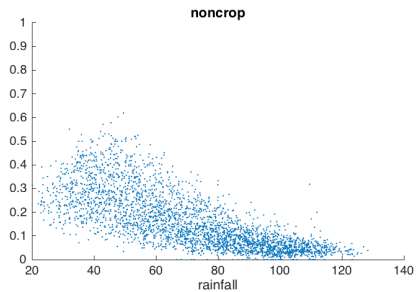
```
cropchoice <- read.csv("cropchoice.txt")
attach(cropchoice)
head(cropchoice,5)
```

##	temperature	rainfall	fields	noncrop	corn	wheat	rice
## 1	13.18475	75.26666	63	8	31	17	7
## 2	12.35680	102.37572	165	7	100	30	28
## 3	17.57882	101.61363	38	1	26	3	8
## 4	20.86867	64.35788	152	45	78	12	17
## 5	13.88084	107.54101	88	4	54	15	15



Distribution of percentage cultivated





# Crop Choice

```
require(nnet)
crops <- cbind(noncrop,corn,wheat,rice)
logitfit <- multinom(crops ~ temperature + rainfall)
```

```
require(AER)
coeftest(logitfit)
```

```
##
## z test of coefficients:
##
##              Estimate Std. Error  z value  Pr(>|z|)
## corn:(Intercept)   0.63814409  0.02120175   30.099 < 2.2e-16 ***
## corn:temperature  -0.12877826  0.00128084 -100.542 < 2.2e-16 ***
## corn:rainfall      0.03864995  0.00022141  174.564 < 2.2e-16 ***
## wheat:(Intercept)  2.57310771  0.02427508  105.998 < 2.2e-16 ***
## wheat:temperature -0.25688133  0.00156614 -164.022 < 2.2e-16 ***
## wheat:rainfall     0.02567228  0.00025031  102.563 < 2.2e-16 ***
## rice:(Intercept)  -3.26197982  0.02843702 -114.709 < 2.2e-16 ***
## rice:temperature  -0.02241833  0.00155758  -14.393 < 2.2e-16 ***
## rice:rainfall      0.05132472  0.00026986  190.187 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Crop Choice

Can we run linear regression instead?

Yes. Let  $y_{ij}$  be the number of fields used for crop  $j$  in county  $i$ , with  $j = 1$  denoting no cultivated crops. Let  $n_i$  be the number of fields in county  $i$ . Let  $p_{ij} = y_{ij} / n_i$  and  $z_{ij} = \log p_{ij} - \log p_{i1}$ . Then we can estimate the following  $J - 1$  linear regression equations:

$$z_i = x_i' \beta_j + e_j, \quad j = 2, \dots, J \quad (17)$$

, where  $x_i = [1, \text{temperature}_i, \text{rainfall}_i]$ .

When  $n_i$  is large, (17)  $\rightarrow$  the multinomial logistic model (16).

# Crop Choice

```
p <- crops/fields
eps <- 1e-4
p[p==0] = p[p==0] + eps
z.corn = log(p[,2]) - log(p[,1])
lsfit.corn <- lm(z.corn ~ temperature + rainfall)
coeftest(lsfit.corn)

##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.64378418  0.06253041  10.296 < 2.2e-16 ***
## temperature -0.14078836  0.00371590 -37.888 < 2.2e-16 ***
## rainfall      0.04268634  0.00059017  72.329 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```



# Crop Choice

```
z.wheat = log(p[,3]) - log(p[,1])
lsfit.wheat <- lm(z.wheat ~ temperature + rainfall)
coeftest(lsfit.wheat)

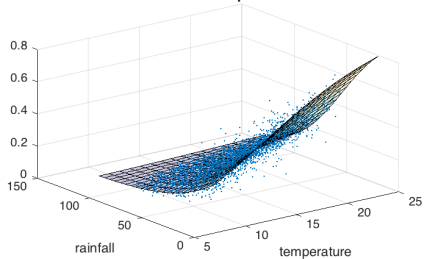
##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  2.85626917  0.07518212  37.991 < 2.2e-16 ***
## temperature -0.28943989  0.00446774 -64.784 < 2.2e-16 ***
## rainfall      0.02933530  0.00070958  41.342 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Crop Choice

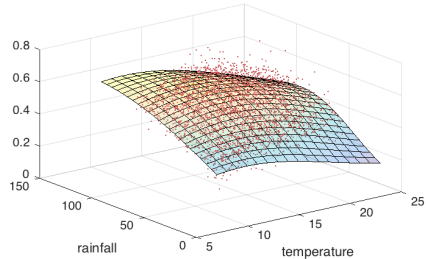
```
z.rice = log(p[,4]) - log(p[,1])
lsfit.rice <- lm(z.rice ~ temperature + rainfall)
coeftest(lsfit.rice)

##
## t test of coefficients:
##
##              Estimate Std. Error  t value Pr(>|t|)
## (Intercept) -3.68724544  0.07945515 -46.4066 < 2.2e-16 ***
## temperature -0.02622848  0.00472166  -5.5549 3.009e-08 ***
## rainfall      0.05834856  0.00074991  77.8074 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

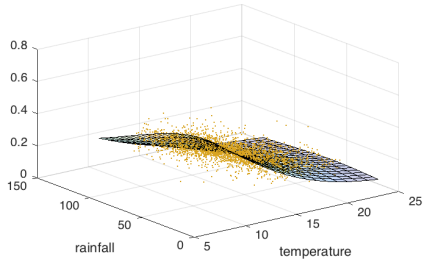
**noncrop**



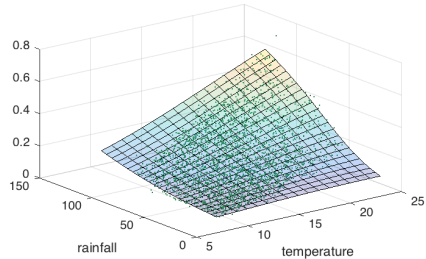
**corn**



**wheat**



**rice**



Multinomial Logistic Fit

# Discrete Choice Models

- In the econometrics literature, the response variables in classification problems are often individual choices.
  - ▶ Here “individuals” can refer to people, firms, governments – any unit of decision making.
- Discrete choice models are a class of econometric models of how individuals make choices.
  - ▶ These models can be considered **structural models** of decision making based on **utility maximization**.

# The Random Utility Framework

- Individual  $i$  faces a choice among  $J$  alternatives.
- The utility associated with alternative  $j$  is  $U_{ij}$ .
- The individual chooses the alternative that generates the highest utility, i.e., let  $y_i \in \{1, \dots, J\}$  denote the choice the individual makes, then

$$y_i = \arg \max_{j \in \{1, \dots, J\}} \{U_{ij}\} \quad (18)$$

# The Random Utility Framework

We do not observe  $U_{ij}$ <sup>16</sup>. Instead, we observe  $(x_{ij}, y_i)$ , where  $x_{ij}$  are characteristics associated with individual  $i$  and alternative  $j$ .

In general,  $x_{ij}$  may contain two types of variables:  $s_i$  and  $z_{ij}$

- $s_i$  : individual-specific variables (e.g., income)
- $z_{ij}$  : alternative-specific variables (e.g., price)<sup>17</sup>

---

<sup>16</sup>  $U_{ij}$  is called a **latent variable**.

<sup>17</sup> If  $z_{ij}$  is the same for all  $i$ , then we can denote it by  $z_i$ .

# The Random Utility Framework

Since we observe  $x_{ij}$  but not  $U_{ij}$ , we can write:

$$U_{ij} = f_j(x_{ij}) + e_{ij} \quad (19)$$

, where  $e_{ij}$  captures *unobserved* factors<sup>18</sup> that influence  $U_{ij}$ <sup>19</sup>.

Let  $e_i \equiv [e_{i1}, \dots, e_{iJ}]'$ . We assume

$$e_i \sim^{i.i.d.} \mathcal{F}_e(\cdot)$$

Different specifications of  $f_j(x_{ij})$  and  $\mathcal{F}_e(\cdot)$  lead to different discrete choice models.

---

<sup>18</sup>Unobserved to *us* not to individual  $i$

<sup>19</sup>One can think of  $f_j(x_{ij})$  as the **systematic** component of a decision maker's utility and  $e_{ij}$  as the **idiosyncratic** or **stochastic** component.

# The Random Utility Framework

Let  $x_i = \{x_{ij}\}_{j=1}^J$ . (18) and (19)  $\Rightarrow$

$$\begin{aligned}\Pr(y_i = j | x_i) &= \Pr(U_{ij} > U_{i\ell} \quad \forall \ell \neq j | x_i) \\ &= \Pr(f_j(x_i) + e_{ij} > f_\ell(x_i) + e_{i\ell} \quad \forall \ell \neq j | x_i) \\ &= \int \mathcal{I}(e_{i\ell} - e_{ij} < f_j(x_i) - f_\ell(x_i) \quad \forall \ell \neq j) d\mathcal{F}_e(e_i)\end{aligned}$$

, i.e., once we place assumptions on  $f_j(x_{ij})$  and  $\mathcal{F}_e(\cdot)$ , we can calculate  $\Pr(y_i = j | x_i)$ , which is called the **conditional choice probability (CCP)** in discrete choice models<sup>20</sup>.

---

<sup>20</sup>The random utility framework assumes that the individual knows her  $U_{ij}$ , so that her decision is *deterministic*. However, since we do not observe  $U_{ij}$ , we can only calculate the probability of her choosing each alternative conditional on the variables we observe.



# The Random Utility Framework

Discrete choice models derived from the random utility framework has the following features:

- ① The absolute level of utility is irrelevant. Only differences in utility matter.
- ② The overall scale of utility is irrelevant.

# Only Differences in Utility Matter

The absolute level of utility is irrelevant. If a constant is added to the utility of all alternatives, then the alternative with the highest utility does not change.

The following models are equivalent:

$$\text{Model 1: } U_{ij} = f_j(x_{ij}) + e_{ij}$$

$$\text{Model 2: } U_{ij} = \alpha + f_j(x_{ij}) + e_{ij}$$

, where  $\alpha$  is any constant.

# Only Differences in Utility Matter

## Example

Consider a binary choice problem:  $y \in \{A, B\}$ . The following models are equivalent:

- Model 1

$$U_{iA} = \mu_A + e_{iA}, \quad e_{iA} \sim \mathcal{N}(0, \sigma_A^2)$$

$$U_{iB} = \mu_B + e_{iB}, \quad e_{iB} \sim \mathcal{N}(0, \sigma_B^2)$$

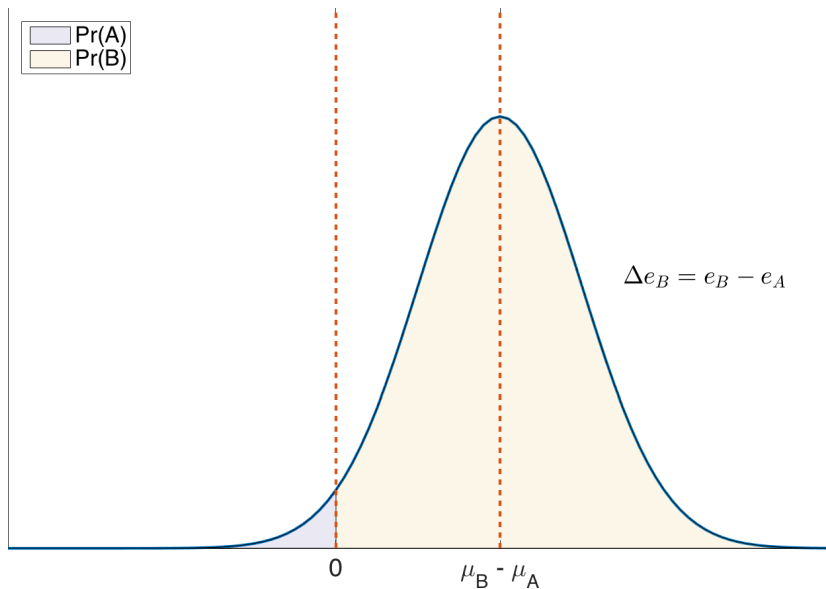
- Model 2

$$U_{iA} = 0$$

$$U_{iB} = \Delta\mu_B + \Delta e_{iB}, \quad \Delta e_{iB} \sim \mathcal{N}(0, \sigma_A^2 + \sigma_B^2)$$

, where  $\Delta\mu_B = \mu_B - \mu_A$  and  $\Delta e_{iB} = e_{iB} - e_{iA}$ .

# Only Differences in Utility Matter



# The Overall Scale of Utility is Irrelevant

The overall scale of utility is irrelevant. Multiplying the utility of all alternatives does not change individual choice: the alternative with the highest utility is the same irrespective of how utility is scaled.

The following models are equivalent:

$$\text{Model 1: } U_{ij} = f_j(x_{ij}) + e_{ij}$$

$$\text{Model 2: } U_{ij} = \lambda f_j(x_{ij}) + \lambda e_{ij}$$

, where  $\lambda$  is any positive constant.

# The Overall Scale of Utility is Irrelevant

## Example (cont.)

The following models are equivalent to Model 1 and Model 2:

- Model 3

$$U_{iA} = \tilde{\mu}_A + \tilde{e}_{iA}, \quad \tilde{e}_{iA} \sim \mathcal{N}\left(0, \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2}\right)$$

$$U_{iB} = \tilde{\mu}_B + \tilde{e}_{iB}, \quad \tilde{e}_{iB} \sim \mathcal{N}\left(0, \frac{\sigma_A^2}{\sigma_A^2 + \sigma_B^2}\right)$$

, where  $\tilde{\mu}_j = \lambda\mu_j$ ,  $\tilde{e}_{ij} = \lambda e_{ij}$ , and  $\lambda = 1 / \sqrt{\sigma_A^2 + \sigma_B^2}$ .

# The Overall Scale of Utility is Irrelevant

## Example (cont.)

- Model 4

$$U_{iA} = 0$$

$$U_{iB} = \Delta\tilde{\mu}_B + \Delta\tilde{e}_{iB}, \quad \Delta\tilde{e}_{iB} \sim \mathcal{N}(0, 1)$$

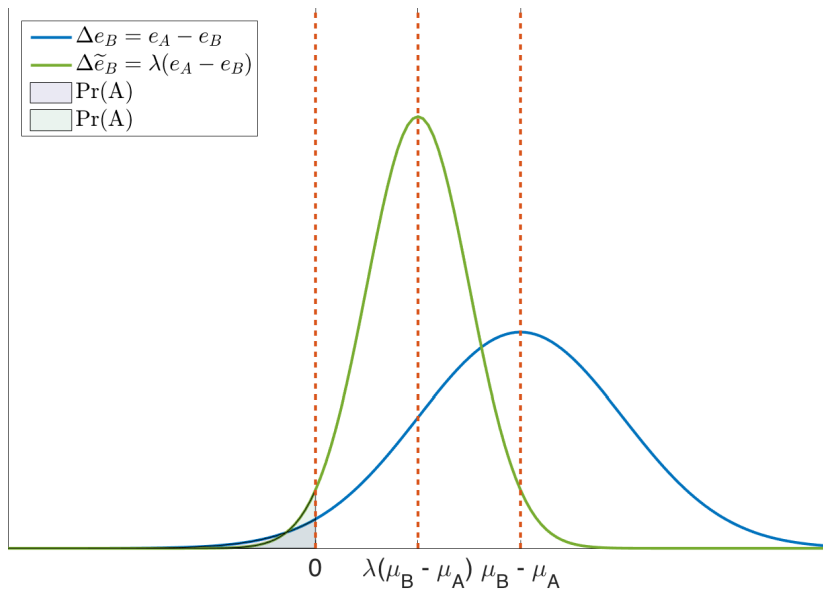
, where  $\Delta\tilde{\mu}_B = \tilde{\mu}_B - \tilde{\mu}_A$  and  $\Delta\tilde{e}_{iB} = \tilde{e}_{iB} - \tilde{e}_{iA}$ .

---

Therefore, in Model 1, the parameters  $\mu_A, \mu_B, \sigma_A, \sigma_B$  are not separately *identifiable*, because an infinite number of models (corresponding to different values of  $\alpha$  and  $\gamma$ ) are *consistent* with the same choice behavior.

To estimate the model, we need to *normalize* the **level** and **scale** of utility. What we can estimate as a result is  $\Delta\tilde{\mu}_B = \lambda(\mu_B - \mu_A)$  – the *scaled difference* between  $\mu_A$  and  $\mu_B$ .

# The Overall Scale of Utility is Irrelevant





# Probit

For  $j = 1, \dots, J$ ,

$$U_{ij} = x'_{ij}\beta_j + e_{ij}$$

, and

$$\mathbf{e}_i = \begin{bmatrix} e_{i1} \\ \vdots \\ e_{iJ} \end{bmatrix} \sim \mathcal{N}(0, \Sigma)$$

# Probit

For binary discrete choice problems, let  $y \in \{A, B\}$ . We have:

$$U_{iA} = x'_{iA}\beta_A + e_{iA} \quad (20)$$

$$U_{iB} = x'_{iB}\beta_B + e_{iB}$$

, and

$$e_i = \begin{bmatrix} e_{iA} \\ e_{iB} \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \cdot & \sigma_B^2 \end{bmatrix}\right) \quad (21)$$

# Probit

Note that (21)  $\Rightarrow$

$$e_{iA} - e_{iB} \sim \mathcal{N}\left(0, \sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}\right)$$

Normalizing (20)  $\Rightarrow$

$$U_{iA} = 0$$

$$U_{iB} = x'_{iB}\tilde{\beta}_B - x'_{iA}\tilde{\beta}_A + \Delta\tilde{e}_{iB}$$

, where, let  $\lambda = 1 / \sqrt{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$ , then  $\tilde{\beta}_A = \lambda\beta_A$ ,  $\tilde{\beta}_B = \lambda\beta_B$ , and  $\Delta\tilde{e}_{iB} = \lambda(e_{iB} - e_{iA}) \sim \mathcal{N}(0, 1)$ .

## Example 1

$$U_{iA} = \alpha_A + z_A' \delta_A + e_{iA}$$

$$U_{iB} = \alpha_B + z_B' \delta_B + e_{iB}$$

Here  $z_j' \delta_j$  and  $\alpha_j$  are both constants and hence cannot be separately identified.

- As long as there is an intercept term, alternative-specific variables  $z_{ij}$  *must* vary with  $i$  in order to be identified.

## Example 2

$$U_{iA} = \alpha_A + s_i' \gamma + e_{iA} \quad (22)$$

$$U_{iB} = \alpha_B + s_i' \gamma + e_{iB}$$

(22)  $\Rightarrow$

$$U_{iB} - U_{iA} = (\alpha_B - \alpha_A) + (e_{iB} - e_{iA})$$

Since only difference in utility matters,  $\gamma$  cannot be identified.

- The coefficients of individual-specific variables must be alternative-specific in order to be identified.

## Example 3

$$U_{iA} = \alpha_A + s_i' \gamma_A + e_{iA} \quad (23)$$

$$U_{iB} = \alpha_B + s_i' \gamma_B + e_{iB}$$

(23)  $\Rightarrow$

$$U_{iB} - U_{iA} = (\alpha_B - \alpha_A) + s_i' (\gamma_B - \gamma_A) + (e_{iB} - e_{iA})$$

- $\alpha_A$  and  $\alpha_B$  cannot be separately identified.
- $\gamma_A$  and  $\gamma_B$  cannot be separately identified.

## Example 3 (cont.)

### Normalization of the model:

- 1 normalize level

$$U_{iA} = 0$$

$$U_{iB} = \Delta\alpha_B + s'_i\Delta\gamma_B + \Delta e_{iB}$$

, where  $\Delta\alpha_B = \alpha_B - \alpha_A$ ,  $\Delta\gamma_B = \gamma_B - \gamma_A$ , and  $\Delta e_{iB} = e_{iB} - e_{iA}$ .

- 2 normalize scale

$$U_{iA} = 0$$

$$U_{iB} = \Delta\tilde{\alpha}_B + s'_i\Delta\tilde{\gamma}_B + \Delta\tilde{e}_{iB}$$

, where we divide  $\Delta\alpha_B$ ,  $\Delta\lambda_B$ , and  $\Delta e_{iB}$  by  $\sqrt{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$ .

## Example 4

$$U_{iA} = \alpha_A + s_i' \gamma_A + z_{iA}' \delta + e_{iA} \quad (24)$$

$$U_{iB} = \alpha_B + s_i' \gamma_B + z_{iB}' \delta + e_{iB}$$

$$U_{iA} = \alpha_A + s_i' \gamma_A + z_{iA}' \delta_A + e_{iA} \quad (25)$$

$$U_{iB} = \alpha_B + s_i' \gamma_B + z_{iB}' \delta_B + e_{iB}$$

Here we can specify either  $z_{ij}' \delta$  or  $z_{ij}' \delta_j$ .

- Alternative-specific variables can have either **alternative-specific coefficients** or **generic coefficients** that do not change with alternatives.



## Example 4 (cont.)

Normalizing (24)  $\Rightarrow^a$

$$U_{iA} = 0$$

$$U_{iB} = \Delta\tilde{\alpha}_B + s_i'\Delta\tilde{\gamma}_B + (z_{iB} - z_{iA})'\tilde{\delta} + \Delta\tilde{e}_{iB}$$

Normalizing (25)  $\Rightarrow$

$$U_{iA} = 0$$

$$U_{iB} = \Delta\tilde{\alpha}_B + s_i'\Delta\tilde{\gamma}_B + (z_{iB}'\tilde{\delta}_B - z_{iA}'\tilde{\delta}_A) + \Delta\tilde{e}_{iB}$$

---

<sup>a</sup>For both,  $\Delta\tilde{\alpha}_B, \Delta\tilde{\gamma}_B, \Delta\tilde{e}_{iB}$  are defined as before.

$\tilde{\delta} = \lambda\delta, \tilde{\delta}_A = \lambda\delta_A, \tilde{\delta}_B = \lambda\delta_B$ , and  $\lambda = 1 / \sqrt{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$ .

# Probit

## Simulation 1:

$$U_{iA} = 5 - 10s_i + e_{iA} \quad (26)$$

$$U_{iB} = -5 + 10s_i + e_{iB}$$

$$e_i = \begin{bmatrix} e_{iA} \\ e_{iB} \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \right)$$

Normalizing (26)  $\Rightarrow$

$$U_{iA} = 0$$

$$\begin{aligned} U_{iB} &= -\frac{12}{\sqrt{5}} + \frac{20}{\sqrt{5}}s_i + \epsilon_{iB} \\ &= -5.37 + 8.94s_i + \epsilon_{iB} \end{aligned}$$

, where  $\epsilon_{iB} = (e_{iB} - e_{iA})/\sqrt{5} \sim \mathcal{N}(0, 1)$ .

# Probit

```
require(ramify)
n = 1e3
s = runif(n)
e1 <- rnorm(n,mean=1,sd=1)
e2 <- rnorm(n,mean=-1,sd=2)
u1 <- 5 - 10*s + e1
u2 <- -5 + 10*s + e2
U <- cbind(u1,u2)
y <- as.factor(argmax(U))
mydata <- data.frame(s,y)
```

# Probit

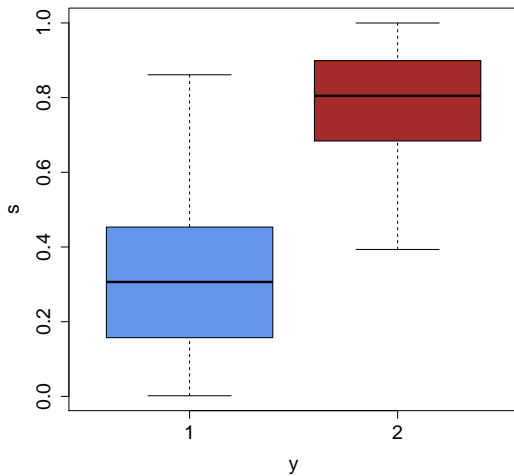
```
head(mydata,5)
```

```
##           s y
## 1 0.1680415 1
## 2 0.8075164 2
## 3 0.3849424 1
## 4 0.3277343 1
## 5 0.6021007 2
```

```
prop.table(table(y))
```

```
## y
##  1  2
## 0.586 0.414
```

# Probit



# Probit

```
require(AER)
probitfit <- glm(y ~ s, family=binomial(link="probit"))
coeftest(probitfit)

##
## z test of coefficients:
##
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -5.40721    0.36303 -14.895 < 2.2e-16 ***
## s            9.07978    0.59449  15.273 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Simulation 2:

$$U_{iA} = 5 - 10s_i + e_{iA} \quad (27)$$

$$U_{iB} = -5 + 10s_i + e_{iB}$$

$$e_i = \begin{bmatrix} e_{iA} \\ e_{iB} \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \right)$$

, where we let  $\rho(e_{iA}, e_{iB}) = 0.5$ , so that  $\sigma_{AB} = \rho\sigma_A\sigma_B = 1$ .

Normalizing (27)  $\Rightarrow$

$$U_{iA} = 0$$

$$\begin{aligned} U_{iB} &= -\frac{12}{\sqrt{3}} + \frac{20}{\sqrt{3}}s_i + \epsilon_{iB} \\ &= -6.93 + 11.55s_i + \epsilon_{iB} \end{aligned}$$

, where  $\epsilon_{iB} = (e_{iB} - e_{iA})/\sqrt{3} \sim \mathcal{N}(0, 1)$ .



# Probit

```
n = 1e3
s = runif(n)

## generating e
require(MASS)
mu <- c(1,-1) # mean
sig <- c(1,2) # s.t.d. of each dimension
rho <- .5 # correlation
Sigma <- matrix(c(sig[1]^2,rho*sig[1]*sig[2], # covariance matrix
                  rho*sig[1]*sig[2],sig[2]^2),2,2)
e <- mvrnorm(n,mu,Sigma)

## generating y
e1 <- e[,1]
e2 <- e[,2]
u1 <- 5 - 10*s + e1
u2 <- -5 + 10*s + e2
y <- as.factor(argmax(cbind(u1,u2)))
```

# Probit

```
head(e,4)
```

```
##           [,1]      [,2]
## [1,] -0.5750613 -5.1608065
## [2,]  0.1128529 -3.4697423
## [3,]  1.9516721 -1.1075891
## [4,]  0.6012319 -0.4711042
```

```
colMeans(e)
```

```
## [1]  0.9808862 -1.0736455
```

```
var(e)
```

```
##           [,1]      [,2]
## [1,] 1.0208563 0.9980833
## [2,] 0.9980833 3.7899603
```

# Probit

```
probitfit <- glm(y ~ s, family=binomial(link="probit"))
coeftest(probitfit)

##
## z test of coefficients:
##
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -7.22787    0.56184 -12.865 < 2.2e-16 ***
## s           11.96904    0.91906  13.023 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Probit

## Simulation 3:

$$U_{iA} = 5 - 10s_i - 0.1z_{iA} + e_{iA} \quad (28)$$

$$U_{iB} = -5 + 10s_i - 0.1z_{iB} + e_{iB}$$

$$e_i = \begin{bmatrix} e_{iA} \\ e_{iB} \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \right)$$

Normalizing (28)  $\Rightarrow$

$$U_{iA} = 0$$

$$\begin{aligned} U_{iB} &= -\frac{12}{\sqrt{5}} + \frac{20}{\sqrt{5}}s_i - \frac{0.1}{\sqrt{5}}(z_{iB} - z_{iA}) + \epsilon_{iB} \\ &= -5.37 + 8.94s_i - 0.045(z_{iB} - z_{iA}) + \epsilon_{iB} \end{aligned}$$

, where  $\epsilon_{iB} \sim \mathcal{N}(0, 1)$ .

# Probit

```
n = 1e3
s = runif(n)
z1 <- 100*runif(n)
z2 <- 50*runif(n)
e1 <- rnorm(n,mean=1,sd=1)
e2 <- rnorm(n,mean=-1,sd=2)
u1 <- 5 - 10*s -0.1*z1 + e1
u2 <- -5 + 10*s -0.1*z2 + e2
y <- as.factor(argmax(cbind(u1,u2)))
mydata <- data.frame(s,z1,z2,y)
```

# Probit

```
probitfit <- glm(y ~ s + z1 + z2, family=binomial(link="probit"))  
coeftest(probitfit)
```

```
##  
## z test of coefficients:  
##  
##           Estimate Std. Error  z value  Pr(>|z|)  
## (Intercept) -5.571587    0.431299 -12.9182 < 2.2e-16 ***  
## s           9.401062    0.621498  15.1265 < 2.2e-16 ***  
## z1          0.044736    0.003975  11.2544 < 2.2e-16 ***  
## z2         -0.046307    0.005858  -7.9049 2.682e-15 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Probit

Can we estimate the model with a *generic* coefficient for  $z_{ij}$  that does not change with  $j$ ? Yes!

```
dz = z2 - z1
probitfit <- glm(y ~ s + dz, family=binomial(link="probit"))
coeftest(probitfit)

##
## z test of coefficients:
##
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -5.623073   0.388506 -14.474 < 2.2e-16 ***
## s            9.407041   0.621340  15.140 < 2.2e-16 ***
## dz          -0.045071   0.003779 -11.927 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Multinomial Probit

Now consider  $J = 3$ .

$$U_{ij} = \mathbf{x}_{ij}'\beta_j + e_{ij}$$

, and

$$\mathbf{e}_i = \begin{bmatrix} e_{i1} \\ e_{i2} \\ e_{i3} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_2^2 & \sigma_{23} \\ \cdot & \cdot & \sigma_3^2 \end{bmatrix} \right)$$



# Multinomial Probit

Normalizing level  $\Rightarrow$

$$U_{i1} = 0$$

$$U_{i2} = (x'_{i2}\beta_2 - x'_{i1}\beta_1) + \Delta e_{i2}$$

$$U_{i3} = (x'_{i3}\beta_3 - x'_{i1}\beta_1) + \Delta e_{i3}$$

, where  $\Delta e_{ij} = e_{ij} - e_{i1}$ , and<sup>21</sup>

$$\begin{bmatrix} \Delta e_{i2} \\ \Delta e_{i3} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} \sigma_1^2 + \sigma_2^2 - 2\sigma_{12} & \sigma_1^2 + \sigma_{23} - \sigma_{12} - \sigma_{13} \\ \cdot & \sigma_1^2 + \sigma_3^2 - 2\sigma_{13} \end{bmatrix} \right)$$

---

21

$$\begin{aligned} \text{Cov}(\Delta e_{i2}, \Delta e_{i3}) &= \text{Cov}(e_{i2} - e_{i1}, e_{i3} - e_{i1}) \\ &= \sigma_{23} - \sigma_{21} - \sigma_{13} + \sigma_1^2 \end{aligned}$$

# Multinomial Probit

Normalizing scale  $\Rightarrow$

$$U_{i1} = 0$$

$$U_{i2} = (x'_{i2}\tilde{\beta}_2 - x'_{i1}\tilde{\beta}_1) + \Delta\tilde{e}_{i2}$$

$$U_{i3} = (x'_{i3}\tilde{\beta}_3 - x'_{i1}\tilde{\beta}_1) + \Delta\tilde{e}_{i3}$$

, where  $\tilde{\beta}_j = \lambda\beta_j$ ,  $\Delta\tilde{e}_{ij} = \lambda\Delta e_{ij}$ ,  $\lambda = 1 / \sqrt{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$ , and

$$\begin{bmatrix} \Delta\tilde{e}_{i2} \\ \Delta\tilde{e}_{i3} \end{bmatrix} \sim \mathcal{N}\left(0, \begin{bmatrix} 1 & \frac{\sigma_1^2 + \sigma_{23} - \sigma_{12} - \sigma_{13}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \\ \cdot & \frac{\sigma_1^2 + \sigma_3^2 - 2\sigma_{13}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}} \end{bmatrix}\right)$$

# Multinomial Probit

Thus, before normalization, the covariance matrix of the error term has 6 parameters:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ . & \sigma_2^2 & \sigma_{23} \\ . & . & \sigma_3^2 \end{bmatrix}$$

After normalization,

$$\tilde{\Sigma} = \begin{bmatrix} 1 & \omega_{12} \\ . & \omega_{22} \end{bmatrix}$$

, where  $\omega_{12} = \frac{\sigma_1^2 + \sigma_{23} - \sigma_{12} - \sigma_{13}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$ ,  $\omega_{22} = \frac{\sigma_1^2 + \sigma_3^2 - 2\sigma_{13}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$ .

The number of covariance parameters to estimate decreases from 6 to 2 after normalization.

# Multinomial Probit

In general, a model with  $J$  alternatives has *at most*  $\frac{1}{2}J(J-1) - 1$  covariance parameters after normalization.

# Ketchup

Brands: {Heinz, Hunt's, Del Monte, Store Brand}

Variables: the price of each brand, the income of the buyer (in \$1000), the brand purchased

```
ketchup = read.csv("Ketchup.csv")
head(ketchup, 2)

##      choice price.heinz price.hunts price.delmonte price.stb   income
## 1      stb         1.46         1.43           1.45         0.99 44.49198
## 2    heinz         0.99         1.39           1.49         0.89 59.26444

prop.table(table(ketchup$choice))

##
## delmonte    heinz    hunts      stb
## 0.05375    0.51125    0.21375    0.22125
```

Model 1:

$$\begin{aligned}U_{ij} &= \alpha_j + \delta \text{price}_{ij} + \gamma_j \text{income}_i + e_{ij} \\e_i &\sim \mathcal{N}(0, \Sigma)\end{aligned}\tag{29}$$

# Ketchup

```
require(mlogit)
ketchup.long <- mlogit.data(ketchup, shape="wide",
                           varying=2:5, choice="choice")
head(ketchup.long,8)
```

##		choice	income	alt	price	chid
##	1.delmonte	FALSE	44.49198	delmonte	1.45	1
##	1.heinz	FALSE	44.49198	heinz	1.46	1
##	1.hunts	FALSE	44.49198	hunts	1.43	1
##	1.stb	TRUE	44.49198	stb	0.99	1
##	2.delmonte	FALSE	59.26444	delmonte	1.49	2
##	2.heinz	TRUE	59.26444	heinz	0.99	2
##	2.hunts	FALSE	59.26444	hunts	1.39	2
##	2.stb	FALSE	59.26444	stb	0.89	2

# Ketchup

```
# mlogit(y ~ z/s/w,...)
# - s: individual-specific vars
# - z: alternative-specific vars with generic coeffs
# - w: alternative-specific vars with alternative-specific coeffs
probitfit1 <- mlogit(choice ~ price|income, ketchup.long,
                    refllevel="stb", probit=TRUE)
```

```
require(AER)
coeftest(probitfit1)[1:7,]
```

##	Estimate	Std. Error	t value	Pr(> t )
## delmonte:(intercept)	-1.13931111	1.16876911	-0.9747957	3.299608e-01
## heinz:(intercept)	-7.05610040	1.81280583	-3.8923641	1.076714e-04
## hunts:(intercept)	-4.32246680	1.33056061	-3.2486057	1.208861e-03
## price	-3.07882503	0.61797639	-4.9821078	7.733865e-07
## delmonte:income	0.03465121	0.02801584	1.2368435	2.165137e-01
## heinz:income	0.18002372	0.04398663	4.0926917	4.703326e-05
## hunts:income	0.11979359	0.03371002	3.5536490	4.025408e-04



# Ketchup

```
coeftest(probitfit1)[8:12,]
```

##		Estimate	Std. Error	t value	Pr(> t )
##	delmonte.heinz	-0.1258684	0.4446334	-0.2830836	0.7771870996
##	delmonte.hunts	-0.7047540	0.4977681	-1.4158280	0.1572209988
##	heinz.heinz	1.2623283	0.3342783	3.7762796	0.0001711608
##	heinz.hunts	0.6634783	0.3711713	1.7875259	0.0742367344
##	hunts.hunts	0.9704545	0.3640111	2.6660021	0.0078331911

So the estimated covariance matrix is ...

```
probitfit1$omega$stb #covariance matrix using "stb" as reference
```

##		delmonte	heinz	hunts
##	delmonte	1.0000000	-0.1258684	-0.7047540
##	heinz	-0.1258684	1.6093156	0.9262337
##	hunts	-0.7047540	0.9262337	1.8786636

$$(\hat{U}_{i,\text{stb}} = 0)$$

$$\hat{U}_{i,\text{delmonte}} = -1.14 - 3.08 \times \text{price}_{i,\text{delmonte}} + 0.035 \times \text{income}_i + \epsilon_{i,\text{delmonte}}$$

$$\hat{U}_{i,\text{heinz}} = -7.06 - 3.08 \times \text{price}_{i,\text{heinz}} + 0.18 \times \text{income}_i + \epsilon_{i,\text{heinz}}$$

$$\hat{U}_{i,\text{hunts}} = -4.32 - 3.08 \times \text{price}_{i,\text{hunts}} + 0.12 \times \text{income}_i + \epsilon_{i,\text{hunts}}$$

, where

$$\begin{bmatrix} \epsilon_{i,\text{delmonte}} \\ \epsilon_{i,\text{heinz}} \\ \epsilon_{i,\text{hunts}} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} 1 & -0.13 & -0.70 \\ . & 1.61 & 0.93 \\ . & . & 1.88 \end{bmatrix} \right)$$

Model 2:

$$\begin{aligned}U_{ij} &= \alpha_j + \delta_j \text{price}_{ij} + \gamma_j \text{income}_i + e_{ij} \\ e_i &\sim \mathcal{N}(0, \Sigma)\end{aligned}\tag{30}$$

# Ketchup

```
probitfit2 <- mlogit(choice ~ 0|income|price, ketchup.long,  
                    refllevel="stb", probit=TRUE)
```

```
coeftest(probitfit2)[1:10,]
```

##	Estimate	Std. Error	t value	Pr(> t )
## delmonte:(intercept)	-2.93786780	2.48851715	-1.180570	2.381313e-01
## heinz:(intercept)	-9.79073108	4.40531214	-2.222483	2.653490e-02
## hunts:(intercept)	-5.50096028	2.64999192	-2.075840	3.823372e-02
## delmonte:income	0.04822031	0.03350156	1.439345	1.504514e-01
## heinz:income	0.25532406	0.13070045	1.953506	5.111457e-02
## hunts:income	0.16927598	0.08526977	1.985182	4.747189e-02
## stb:price	-4.10482188	1.79833571	-2.282567	2.272253e-02
## delmonte:price	-2.85282115	0.64411156	-4.429079	1.080621e-05
## heinz:price	-4.37328318	2.49407340	-1.753470	7.991161e-02
## hunts:price	-4.71107228	2.57769096	-1.827633	6.798415e-02

# Ketchup

```
coeftest(probitfit2)[11:15,]
```

##		Estimate	Std. Error	t value	Pr(> t )
##	delmonte.heinz	-0.1424442	0.6506784	-0.2189165	0.82677203
##	delmonte.hunts	-1.0566066	0.8770324	-1.2047520	0.22866213
##	heinz.heinz	1.7969567	0.9806295	1.8324522	0.06726274
##	heinz.hunts	0.9872264	0.7128141	1.3849704	0.16645499
##	hunts.hunts	1.4535726	0.9021936	1.6111537	0.10754821

```
probitfit2$omega$stb
```

##		delmonte	heinz	hunts
##	delmonte	1.0000000	-0.1424442	-1.056607
##	heinz	-0.1424442	3.2493438	1.924511
##	hunts	-1.0566066	1.9245106	4.203907

$$\hat{U}_{i,\text{stb}} = -4.1 \times \text{price}_{i,\text{stb}}$$

$$\hat{U}_{i,\text{delmonte}} = -2.94 - 2.85 \times \text{price}_{i,\text{delmonte}} + 0.048 \times \text{income}_i + \epsilon_{i,\text{delmonte}}$$

$$\hat{U}_{i,\text{heinz}} = -9.79 - 4.37 \times \text{price}_{i,\text{heinz}} + 0.255 \times \text{income}_i + \epsilon_{i,\text{heinz}}$$

$$\hat{U}_{i,\text{hunts}} = -5.50 - 4.71 \times \text{price}_{i,\text{hunts}} + 0.169 \times \text{income}_i + \epsilon_{i,\text{hunts}}$$

, where

$$\begin{bmatrix} \epsilon_{i,\text{delmonte}} \\ \epsilon_{i,\text{heinz}} \\ \epsilon_{i,\text{hunts}} \end{bmatrix} \sim \mathcal{N} \left( 0, \begin{bmatrix} 1 & -0.14 & -1.06 \\ . & 3.25 & 1.92 \\ . & . & 4.20 \end{bmatrix} \right)$$

# Logistic Regression Revisited

Now let's assume the following model:

$$U_{ij} = x'_{ij}\beta_j + e_{ij} \quad (31)$$

, and

$$e_{ij} \sim^{i.i.d.} \text{Gumbel}(0, \sigma)$$

# Extreme Value Distribution

The Gumbel distribution, also called the Type I extreme value distribution, has the following CDF:

$$\mathcal{F}(e; \mu, \sigma) = \exp \left\{ -\exp \left( -\frac{e - \mu}{\sigma} \right) \right\}$$

- $\mu$  is the **location** parameter.
- $\sigma$  is the **scale** parameter

For  $e \sim \text{Gumbel}(\mu, \sigma)$ ,

$$E(e) = \mu + \sigma \gamma_e$$
$$\text{Var}(e) = \frac{\pi^2}{6} \sigma^2$$

, where  $\gamma_e \approx 0.577$  is the Euler constant.



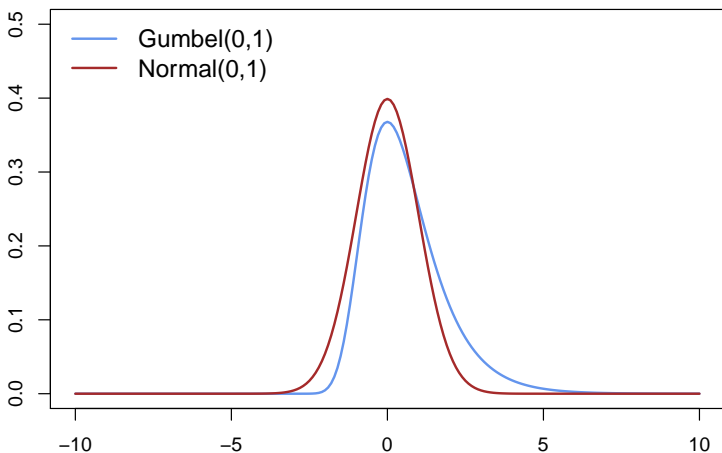
# Extreme Value Distribution

- The difference between two extreme value random variables is distributed as a logistic distribution. Let  $e_1, e_2 \sim \text{Gumbel}(0, 1)$  and let  $\Delta e = e_2 - e_1$ . Then the CDF of  $\Delta e$  is:

$$\mathcal{F}(\Delta e) = \frac{\exp(\Delta e)}{1 + \exp(\Delta e)}$$

- In practice, assuming  $e_{ij} \sim^{i.i.d.}$  Gumbel is nearly the same as assuming  $e_{ij} \sim^{i.i.d.}$  Normal.
  - ▶ The extreme value distribution has fatter tails than the normal, but the difference is small empirically.

# Extreme Value Distribution



# Logistic Regression Revisited

We can always normalize the scale of (31) so that  $\sigma = 1$ :

$$U_{ij} = x'_{ij}\beta_j + e_{ij}$$

, where

$$e_{ij} \sim^{i.i.d.} \text{Gumbel}(0, 1)$$

# Logistic Regression Revisited

Let  $x_i = \{x_{ij}\}_{j=1}^J$  and  $V_{ij} = x'_{ij}\beta_j$ . We have:

$$\begin{aligned}\Pr(y_i = j | x_i) &= \Pr(V_{ij} + e_{ij} > V_{i\ell} + e_{i\ell} \quad \forall \ell \neq j | x_i) \\ &= \Pr(e_{i\ell} < V_{ij} - V_{i\ell} + e_{ij} \quad \forall \ell \neq j | x_i) \\ &= \int \left[ \prod_{\ell \neq j} e^{-e^{-(V_{ij} - V_{i\ell} + e_{ij})}} \right] e^{-e_{ij}} e^{-e^{-e_{ij}}} de_{ij} \\ &= \frac{\exp(V_{ij})}{\sum_{\ell=1}^J \exp(V_{i\ell})}\end{aligned}$$

- Under the assumption of  $e_{ij} \sim^{i.i.d.}$  Gumbel(0, 1), the random utility framework gives rise to the logistic model.

# Logistic Regression Revisited

Let  $\overline{U}_i = E(U_i | x_i)$  be the expected utility of individual  $i$ <sup>22</sup>.

$$\begin{aligned}\overline{U}_i &= E[U_i | x_i] \\ &= E\left[\max_j \{U_{ij}\} \middle| x_i\right] \\ &= \log \left[ \sum_{j=1}^J \exp(V_{ij}) \right]\end{aligned}$$

---

<sup>22</sup>Technically,  $\overline{U}_i = \log \left[ \sum_{j=1}^J \exp(V_{ij}) \right] + C$ , where  $C$  is any constant. This is because we can add any  $C$  to  $(U_{i1}, \dots, U_{iJ})$  and the model would be the same. Therefore, we will not be able to learn the *level* of utilities associated with different alternatives, only the *difference* between them.

# Logistic Regression Revisited

- For binary problems, the probit model, after normalization, is

$$U_{iA} = x'_{iA}\beta_A$$

$$U_{iB} = x'_{iB}\beta_B + e_{iB}$$

, where  $e_{iB} \sim \mathcal{N}(0, 1)$ .

Therefore, the probit and the logistic model are basically the same for binary problems.

- For multinomial problems, the two types of models are different as probit allows  $e_i$  to have an arbitrary covariance structure<sup>23</sup>.

---

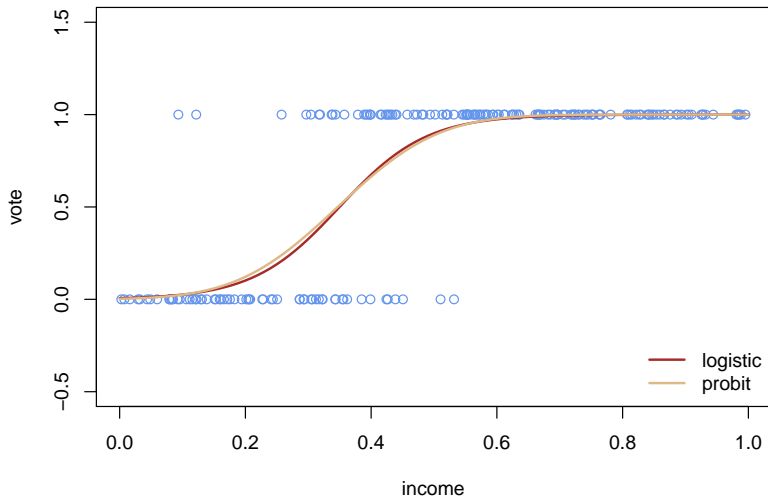
<sup>23</sup>In the econometrics literature, logistic and probit models with alternative-specific regressors are called **conditional logit** and **conditional probit models**, so as to be distinguished from logistic and probit models with only individual-specific regressors.

# Income and Voting

```
probitfit <- glm(vote ~ income, family=binomial(link="probit"))
coeftest(probitfit)

##
## z test of coefficients:
##
##              Estimate Std. Error z value  Pr(>|z|)
## (Intercept) -2.75277    0.41935  -6.5644 5.225e-11 ***
## income       7.93916    1.07686   7.3725 1.675e-13 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Income and Voting





## Marginal Effects

$$U_{ij} = \alpha_j + \gamma_j s_i + \delta z_{ij} + e_{ij}, \quad e_{ij} \sim^{i.i.d.} \text{Gumbel}(0, 1) \quad (32)$$

$\Rightarrow^{24}$

$$\begin{aligned} \frac{\partial \Pr(y_i = j | x_i)}{\partial s_i} &= \frac{\partial [e^{V_{ij}} / \sum_{\ell} e^{V_{i\ell}}]}{\partial s_i} \\ &= \Pr(y_i = j | x_i) \left( \gamma_j - \sum_{\ell} \gamma_{\ell} \Pr(y_i = \ell | x_i) \right) \\ \frac{\partial \Pr(y_i = j | x_i)}{\partial z_{ij}} &= \delta \Pr(y_i = j | x_i) (1 - \Pr(y_i = j | x_i)) \end{aligned}$$

---

<sup>24</sup> If  $\delta$  is alternative-specific, i.e.  $\delta_j$ , then  
 $\partial \Pr(y_i = j | x_i) / \partial z_{ij} = \delta_j \Pr(y_i = j | x_i) (1 - \Pr(y_i = j | x_i)).$

# Marginal Effects

- For alternative-specific variables, the sign of the coefficient is the sign of the marginal effect:  $\gamma > 0 \iff \partial \Pr(y_i = j | x_i) / \partial z_{ij} > 0$ .
- For individual-specific variables, the sign of the coefficient is not necessarily the sign of the marginal effect:  $\gamma_j > 0$  does not imply  $\partial \Pr(y_i = j | x_i) / \partial s_i > 0$ .

## Choice Probability Elasticity

Let  $\mathcal{E}_i^{jj}$  be the **own-elasticity** of the change in  $\Pr(y_i = j | x_i)$  given a change in  $z_{ij}$ . (32)  $\Rightarrow$

$$\begin{aligned}\mathcal{E}_i^{jj} &= \frac{\partial \Pr(y_i = j | x_i)}{\partial z_{ij}} \frac{z_{ij}}{\Pr(y_i = j | x_i)} \\ &= \delta z_{ij} [1 - \Pr(y_i = j | x_i)]\end{aligned}\tag{33}$$

Similarly, we can calculate the **cross-elasticity** of  $\Pr(y_i = j | x_i)$  given a change in  $z_{ik}$ ,  $k \neq j$ :

$$\begin{aligned}\mathcal{E}_i^{jk} &= \frac{\partial \Pr(y_i = j | x_i)}{\partial z_{ik}} \frac{z_{ik}}{\Pr(y_i = j | x_i)} \\ &= -\delta z_{ik} \Pr(y_i = k | x_i)\end{aligned}\tag{34}$$

# Choice Probability Elasticity

- Note that (34) does *not* depend on  $j$  – a percentage change in  $z_{ik}$  results in the *same* percentage change in all  $\Pr(y_i = j | x_i)$ ,  $j \neq k$ .
- For example, consider the car market. Suppose the choice set is  $\{\text{Honda, Toyota, Tesla}\}$ . Let  $z_{ij} = p_{ij}$  be the price of each car to each consumer. Then (34) says that, for each consumer, a 1% decrease in the price of Honda will result in the same percentage decrease in the probability of buying Toyota and the probability of buying Tesla.
- This property, which is called *proportional substitution*, is a manifestation of the IIA property of the logistic model.

# Independence of Irrelevant Alternatives (IIA)

- The IIA property is the result of assuming that errors are independent of each other.
  - ▶ Hence IIA holds not only for logistic models with *i.i.d.* extreme value distributed errors, but holds in general for discrete choice models with independently distributed errors.
- Multinomial probit models, by allowing for correlated errors, do not have the IIA property.

# Independence of Irrelevant Alternatives (IIA)

- Note that the IIA property should be a desirable property for well-specified models.
- Under independence, the error for one alternative provides no information about the error for another alternative. This should be the property of a well-specified model such that the unobserved portion of utility is essentially “white noise.”
- When a model omits important unobserved variables that explain individual choice patterns, however, the errors can become correlated over alternatives.
- In this sense, the ultimate goal of the researcher is to represent utility so well that the assumption of error independence is appropriate.
- In the absence of that, a discrete choice model that allows for correlated errors, such as the multinomial probit, can be used.

# Employment

- Sector of employment: Manufacturing, Retail, Education, Health, Personal Service, Professional Service
- Individual variables: sex, education (years of schooling), wage

```
emp <- read.csv("employment.csv")
emp$sex <- factor(emp$sex, labels=c("male", "female"))
head(emp, 4)
```

##	sex	education	wage	sector
## 1	female	15	32241.35	personal
## 2	female	16	70051.50	education
## 3	male	13	35248.51	manufacturing
## 4	female	12	15535.13	health

# Employment

```
require(descr)
freq(emp$sector, plot=FALSE)
```

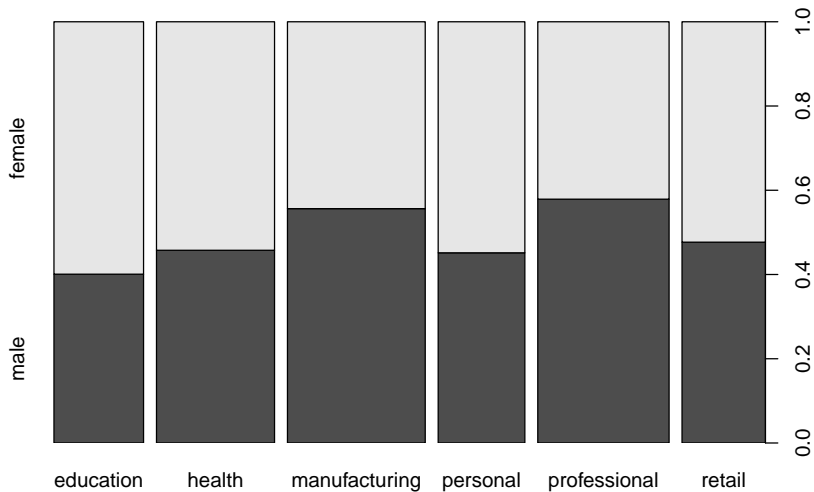
```
## emp$sector
##           Frequency Percent
## education         277    13.85
## health            365    18.25
## manufacturing     426    21.30
## personal          268    13.40
## professional      406    20.30
## retail            258    12.90
## Total             2000   100.00
```

```
aggregate(wage~sector, emp, mean)
```

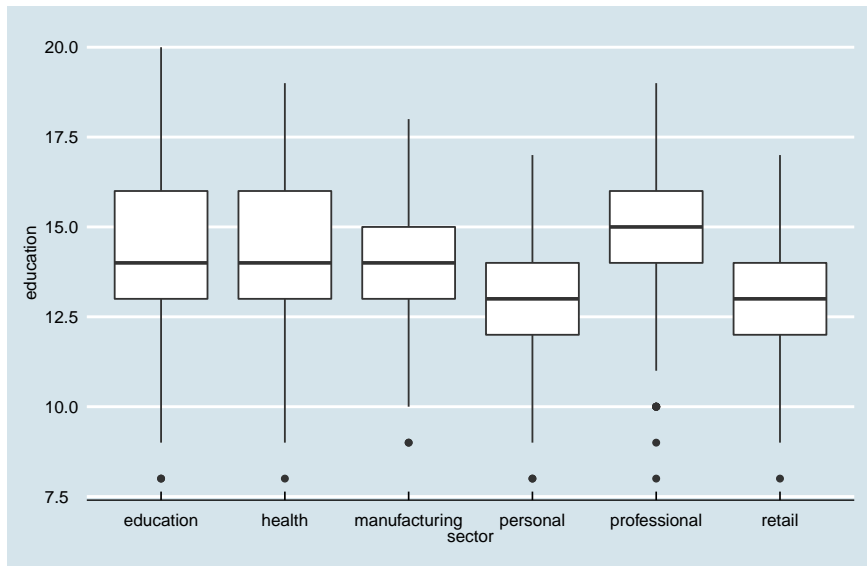
```
##           sector      wage
## 1 education 57134.48
## 2 health 50039.96
## 3 manufacturing 43630.54
## 4 personal 36799.96
## 5 professional 85319.71
## 6 retail 25460.33
```



# Employment



# Employment



Model:

$$\begin{aligned}U_{ij} &= \alpha_j + \beta w_{ij} + e_{ij} \\ e_{ij} &\sim \text{Gumbel}(0, 1)\end{aligned}\tag{35}$$

Let  $y_i$  be the observed sector of employment of individual  $i$ . To estimate the model, we need to construct *counterfactual wages*  $w_{ij}$  for each individual  $i$  and sector  $j \neq y_i$ .

# Employment

We can predict counterfactual wages by running the following regressions for each sector  $j$ :

$$\begin{aligned}\log w_{ij} = & \omega_{0j} + \omega_{1j}\text{Education}_i + \omega_{2j}\text{Female}_i \\ & + \omega_{3j}\text{Education}_i \times \text{Female}_i + \xi_{ij}\end{aligned}\tag{36}$$

, where  $\text{Female}_i$  is an indicator variable.

(36)  $\Rightarrow \hat{w}_{ij}$ . We then estimate:

$$\begin{aligned}U_{ij} &= \alpha_j + \beta \hat{w}_{ij} + e_{ij} \\ e_{ij} &\sim \text{Gumbel}(0, 1)\end{aligned}$$

# Employment

Constructed data set with counterfactual wages:

```
head(emp, 4)
```

```
##           sector wage.education wage.health wage.manufacturing
## 1      personal    36373.753    45757.89          37138.46
## 2      education    60971.110    69129.87          50215.49
## 3 manufacturing    15656.873    21219.96          33982.85
## 4         health     7722.895    13269.87          15023.89
## wage.personal wage.professional wage.retail
## 1      45022.19          54747.97    32333.67
## 2      50944.08          83152.41    40173.16
## 3      37336.32          33341.71    24485.34
## 4      31076.01          15625.99    16858.23
```

# Employment

```
## Estimating the discrete choice model
require(AER)
emp.long <- mlogit.data(emp, shape="wide", varying=2:7, choice="sector")
modelfit <- mlogit(sector ~ wage, emp.long)
coeftest(modelfit)

##
## t test of coefficients:
##
##              Estimate Std. Error t value Pr(>|t|)
## health:(intercept)   8.7959e-02  8.1429e-02  1.0802   0.28019
## manufacturing:(intercept) 1.7359e-01  8.2219e-02  2.1113   0.03487 *
## personal:(intercept)  -3.8266e-01  9.5724e-02 -3.9975  6.634e-05 ***
## professional:(intercept) -3.9360e-01  9.7211e-02 -4.0489  5.342e-05 ***
## retail:(intercept)     5.5781e-02  8.8256e-02  0.6320   0.52743
## wage                  3.7627e-05  2.6104e-06 14.4142 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Welfare Analysis

The expected utility of individual  $i$  is:

$$\bar{U}_i = \log \left[ \sum_j \exp(\alpha_j + \beta w_{ij}) \right] \quad (37)$$

Let  $\bar{U}_i^{\$}$  denote the utility of the individual *in monetary terms*. Since in model (35), each dollar in wage adds  $\beta$  to utility, each unit of utility is equivalent to  $1/\beta$  dollars. The expected utility of individual  $i$  in monetary terms is thus<sup>25</sup>:

$$\bar{U}_i^{\$} = \frac{1}{\beta} \log \left[ \sum_j \exp(\alpha_j + \beta w_{ij}) \right] \quad (38)$$

---

<sup>25</sup> More precisely, we can add any constant  $C$  to (37) and (38).

# Welfare Analysis

```
## Calculating expected utilities
J = 6 # number of sectors
N <- nrow(emp) # number of individuals
b <- coef(modelfit)["wage"]
X <- model.matrix(modelfit)
V <- X %*% coef(modelfit)
V <- matrix(V,N,J,byrow=TRUE)
U = log(rowSums(exp(V)))/b
summary(U)
```

##	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
##	52366	68246	84799	94882	101307	564822



## Counterfactual Experiment: 20% decrease in Manufacturing wages

Suppose trade liberalization causes a 20% decrease in the wages of manufacturing workers.

- How does the employment pattern change after trade liberalization?
- What are its welfare consequences?

# Counterfactual Experiment: 20% decrease in Manufacturing wages

```
emp2 <- emp
emp2$wage.manufacturing <- emp$wage.manufacturing*0.8
emp2.long <- mlogit.data(emp2,shape="wide",varying=2:7,choice="sector")
colMeans(predict(modelfit,emp2.long))
```

##	education	health	manufacturing	personal	professional
##	0.1464848	0.1937193	0.1657904	0.1406273	0.2176602
##	retail				
##	0.1357180				

## Counterfactual Experiment: 20% decrease in Manufacturing wages

Employment Share Before and After Trade Liberalization

Employment Share	Before	After
Manufacturing	21.35	16.63
Retail	12.75	13.41
Education	14.10	14.91
Health	18.40	19.52
Personal Service	12.85	13.49
Professional Service	20.55	22.05

# Counterfactual Experiment: 20% decrease in Manufacturing wages

```
## Calculating expected utilities
X2 <- X
X2[index(emp.long)$alt=="manufacturing","wage"] =
  X2[index(emp.long)$alt=="manufacturing","wage"]*.8
V2 <- X2 %*% coef(modelfit)
V2 <- matrix(V2,N,J,byrow=TRUE)
U2 = log(rowSums(exp(V2)))/b
summary(U2)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  52192   67498   82421   93295   97975   564818
```

# Counterfactual Experiment: 20% decrease in Manufacturing wages

```
## Change in expected utilities
```

```
dU = U2 - U
```

```
summary(dU)
```

```
##      Min.   1st Qu.   Median     Mean   3rd Qu.     Max.
## -4026.199 -2377.896 -1161.950 -1587.433 -755.802   -0.146
```

```
emp = data.frame(emp0,U,U2,dU)
```

```
# by gender
```

```
aggregate(dU ~ sex,emp,mean)
```

```
##      sex      dU
## 1  male -2342.3223
## 2 female -841.5479
```

# Counterfactual Experiment: 20% decrease in Manufacturing wages

```
# by education  
aggregate(dU ~ education, emp, mean)
```

##	education	dU
## 1	8	-199.81766
## 2	9	-268.37735
## 3	10	-424.76973
## 4	11	-587.90009
## 5	12	-818.89454
## 6	13	-1228.86410
## 7	14	-1643.87818
## 8	15	-2208.52149
## 9	16	-2637.40772
## 10	17	-2069.31717
## 11	18	-939.75103
## 12	19	-143.24862
## 13	20	-2.87952

# Ketchup

Let's take model (29) and compare logistic vs. probit counterfactual predictions:

```
logitfit <- mlogit(choice ~ price|income, ketchup.long, relevel="stb")
coeftest(logitfit)

##
## t test of coefficients:
##
##              Estimate Std. Error  t value  Pr(>|t|)
## delmonte:(intercept) -3.831626   1.169149  -3.2773  0.001094 **
## heinz:(intercept)    -10.888985   0.946463 -11.5049 < 2.2e-16 ***
## hunts:(intercept)    -6.305256   0.871547  -7.2346  1.103e-12 ***
## price                -4.418198   0.329590 -13.4051 < 2.2e-16 ***
## delmonte:income       0.107143   0.025841   4.1462  3.745e-05 ***
## heinz:income          0.276613   0.020943  13.2078 < 2.2e-16 ***
## hunts:income         0.180305   0.019794   9.1091 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Counterfactual Experiment: 20% price increase for Heinz

```
newdata <- ketchup.long
idx <- index(newdata)$alt == "heinz"
newdata[idx,"price"] <- newdata[idx,"price"]*1.2 # 20% price increase

# logistic prediction
logit.phat.new <- predict(logitfit,newdata)
logit.share.new <- colMeans(logit.phat.new)
logit.share.new

##           stb    delmonte        heinz        hunts
## 0.25132916 0.06914047 0.37982532 0.29970505

# probit prediction
probit.phat.new <- predict(probitfit1,newdata)
probit.share.new <- colMeans(probit.phat.new)
probit.share.new

##           stb    delmonte        heinz        hunts
## 0.22741067 0.07871089 0.37283446 0.32164539
```



## Counterfactual Experiment: 20% price increase for Heinz

market share	Heinz	Hunts	Del Monte	Store Brand
	51.13%	21.38%	5.38%	22.13%
After Heinz price increase:				
logistic	37.98%	29.97%	6.91%	25.13%
probit	37.28%	32.16%	7.87%	22.74%

# Mode of Transportation

```
## Probit Regression
transport.long <- mlogit.data(transport, shape="wide", choice="y")
probitfit <- mlogit(y ~ 0|loginc+distance, transport.long, probit=TRUE)
```

```
coeftest(probitfit)
```

```
##
```

```
## t test of coefficients:
```

```
##
```

##		Estimate	Std. Error	t value	Pr(> t )	
##	car:(intercept)	-8.925928	1.389554	-6.4236	2.062e-10	***
##	subway:(intercept)	-1.454769	1.609180	-0.9040	0.3662	
##	car:loginc	0.773128	0.128574	6.0131	2.555e-09	***
##	subway:loginc	0.118611	0.133202	0.8905	0.3734	
##	car:distance	0.557613	0.532888	1.0464	0.2956	
##	subway:distance	0.698667	0.772920	0.9039	0.3663	
##	car.subway	-0.013351	0.153096	-0.0872	0.9305	
##	subway.subway	0.315844	0.364598	0.8663	0.3865	

```
## ---
```

```
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Mode of Transportation

```
probitfit$omega
```

```
## $bus
```

```
##           car      subway
```

```
## car      1.00000000 -0.01335131
```

```
## subway -0.01335131  0.09993555
```

```
##
```

```
## $car
```

```
##           bus      subway
```

```
## bus      1.000000  1.013351
```

```
## subway 1.013351  1.126638
```

```
##
```

```
## $subway
```

```
##           bus      car
```

```
## bus 0.09993555 0.1132869
```

```
## car 0.11328686 1.1266382
```

# Counterfactual Experiment: No Subway

```
# To predict choice probabilities without one alternative,  
# one trick is to make the  $x_{ij}$  associated with that alternative  
# extremely large or small so that its predicted prob is always 0  
newdata <- transport.long  
idx <- index(newdata)$alt == "subway"  
newdata[idx,"loginc"] <- -1e10  
newdata[idx,"distance"] <- -1e10  
probit.phat.new <- predict(probitfit,newdata)  
probit.share.new <- colMeans(probit.phat.new)
```

```
probit.share.new
```

```
##          bus          car          subway  
## 0.6047072 0.3952928 0.0000000
```

## Counterfactual Experiment: No Subway

Observed Market Share		
bus	car	subway
22%	31%	47%

### Predicted Market Share without Subway

	bus	car
logistic	38%	62%
probit	60%	40%

# Acknowledgement

Part of this lecture is adapted from the following sources:

- Bishop, C. M. 2011. *Pattern Recognition and Machine Learning*. Springer.
- Hastie, T., R. Tibshirani, and J. Friedmand. 2008. *The Elements of Statistical Learning* (2<sup>nd</sup> ed.). Springer.
- James, G., D. Witten, T. Hastie, and R. Tibshirani. 2013. *An Introduction to Statistical Learning: with Applications in R*. Springer.
- Schafer, J. S. *Regression Analysis and Modeling*. Lecture at Penn State University, personal copy.
- Train, K. E. 2009. *Discrete Choice Methods with Simulation* (2<sup>nd</sup> ed.). Cambridge University Press.