Z-transform and Inverse Z-transform Analysis

<u>Objective:</u> To study the Z-transform and Inverse Z-transform practically using MATLAB.

Tool Used: MATLAB

<u>Description:</u>

In mathematics and signal processing, the **Z-transform** converts a discrete time-domain signal, which is a sequence of real or complex numbers, into a complex frequency-domain representation.

The Z-transform, like many other integral transforms, can be defined as either a *one-sided* or *two-sided* transform.

Bilateral Z-transform

The *bilateral* or *two-sided* Z-transform of a discrete-time signal *x[n]* is the function *X(z)* defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

.

Unilateral Z-transform

Alternatively, in cases where $x[n]$ is defined only for $n \geq 0$, the *single-sided* or *unilateral* Z-transform is defined as

$$X(z) = \mathcal{Z}\{x[n]\} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

In signal processing, this definition is used when the signal is causal.

As analog filters are designed using the Laplace transform, recursive digital filters are developed with a parallel technique called the z-transform. The overall strategy of these two transforms is the same: probe the impulse response with sinusoids and exponentials to find the system's poles and zeros. The Laplace transforms deals with differential equations, the s-domain, and the s-plane. Correspondingly, the z-transform deals with difference equations, the z-domain, and the z-plane. However, the two techniques are not a mirror image of each other; the s-plane is arranged in a rectangular coordinate system, while the z-plane uses a polar format. Recursive digital filters are often designed by starting with one of the classic analog filters, such as the Butterworth, Chebyshev, or elliptic. A series of mathematical conversions are then used to obtain the desired digital filter. The Z transform of a discrete time system X[n] is defined as Power Series.

Rational Z-transform to factored Z-transform:

Example:

Let the given transfer function be in the rational form,

$$G(z)= \frac{2z^4+16z^3+44z^2+56z+32}{3z^4+3z^3-15z^2+18z-12}$$

It is required to convert it into factored form, so that we can find the poles and zeros mathematically by applying quadratic equation.

Matlab command required for converting rational form to factored form be

'Zp2sos'

The factored form of G(z) as evaluated by 'zp2sos' be,

$$G(z)=\frac{(0.6667 + 0.4z^{-1} + 0.5333 z^{-2}) (1.000 + 2.000 z^{-1} +2.000 z^{-2})}{(1.000 + 2.000z^{-1} -4.000z^{-2})(1.000 - 1.000 z^{-1} + 1.000 z^{-2})}$$

Factored Z-transform / zeros,poles to rational Z-transform:

It is the inverse of the above case, when the transfer function is given in factored form and it is required to convert in rational form then a single 'matlab' command can serve the purpose.

Example:

Lets use the above result i-e;transfer function in factored for,

$$G(z)=\frac{(0.6667 + 0.4z^{-1} + 0.5333 z^{-2}) (1.000 + 2.000 z^{-1} +2.000 z^{-2})}{(1.000 + 2.000z^{-1} -4.000z^{-2})(1.000 - 1.000 z^{-1} + 1.000 z^{-2})}$$

For building up transfer function in rational form we find the poles and zers of above system simply by using matlab 'root' command or by hand. Or simply we have poles and zeros of the given system we can find the transfer function in factored form.

Matlab command that converts poles and zeros of the system in to transfer function is **'zp2tf'** *.*

Rational Z-transform to partial fraction form:

This technique is usually used , while taking the inverse Z-transform and when the order 'H(z)' is high so that it is quite difficult to solve it mathematically.

Example:

Consider the transfer function in the rational form i-e;

$$G(z)= \frac{18z^3}{18z^3+3z^2-4z-1}$$

We can evaluate the partial fraction form of the above system using matlab command. The partial fraction form be,

$$G(z)= \frac{0.36}{1 - 0.5z^{-1}} + \frac{0.24}{1+0.33 z^{-1}} + \frac{0.4}{(1+0.33 z^{-1})}$$

Matlab command that converts rational z-transform in to partial fraction form is

'residuez'.

Partial fraction form to Z-transform:

This technique is used when it is required to convert partial fraction expression in to rational Z-transform.

Example:

Take the partial fraction form of above ,

The partial fraction form be,

$$G(z) = \frac{0.36}{1 - 0.5z^{-1}} + \frac{0.24}{1+0.33\, z^{-1}} + \frac{0.4}{(1+0.33\, z^{-1})}$$

Matlab command that converts partial fraction form into rational z-transform is

'residuez'

Zplane:

Zero-pole plot

zplane(b,a)

This function displays the poles and zeros of discrete-time systems.

MATLAB:

```
syms z n
a=ztrans(1/16^n)
```

Inverse Z-Transform:

MATLAB:

```
syms Z n
iztrans(3*Z/(Z+1))
```

Pole Zero Diagrams For A Function In Z Domain:

Z plane command computes and display the pole-zero diagram of Z function.

The Command is

Zplane(b,a)

To display the pole value, use **root(a)**

To display the zero value, use **root(b)**

$$X(Z) = [Z^{-2} + Z^{-1}\ ] / [1-2Z^{-1}+3Z^{-2}]$$

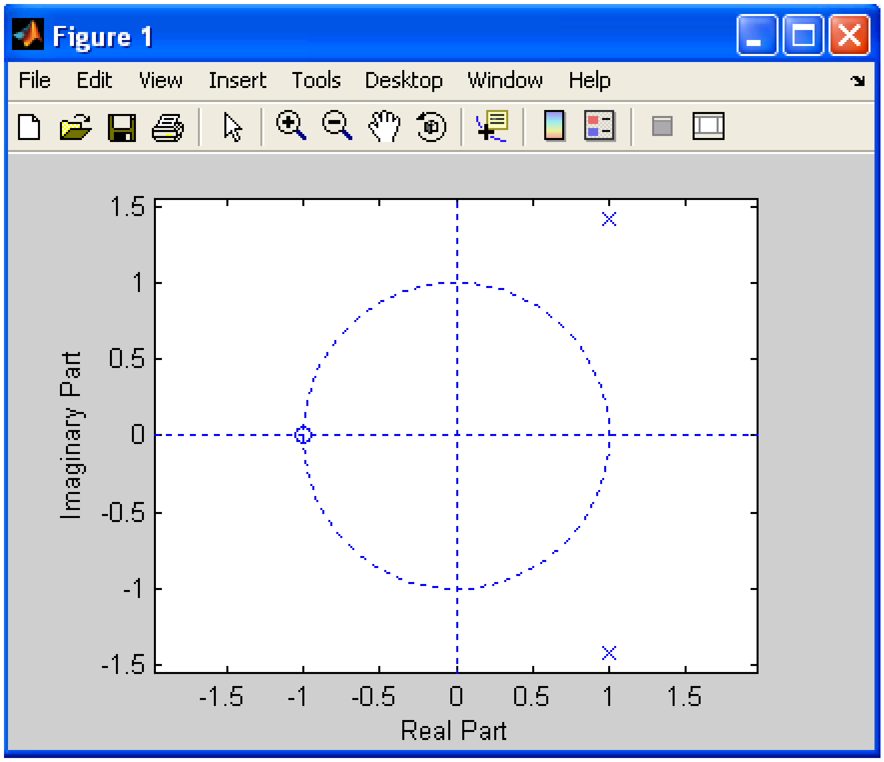
Matlab Code:

```
b=[0 1 1 ]
a= [1 -2 +3]
roots(a)
roots(b)
zplane(b,a);

ans =
1.0000 + 1.4142i
```

1.0000 - 1.4142i

ans=
-1



Frequency Response:

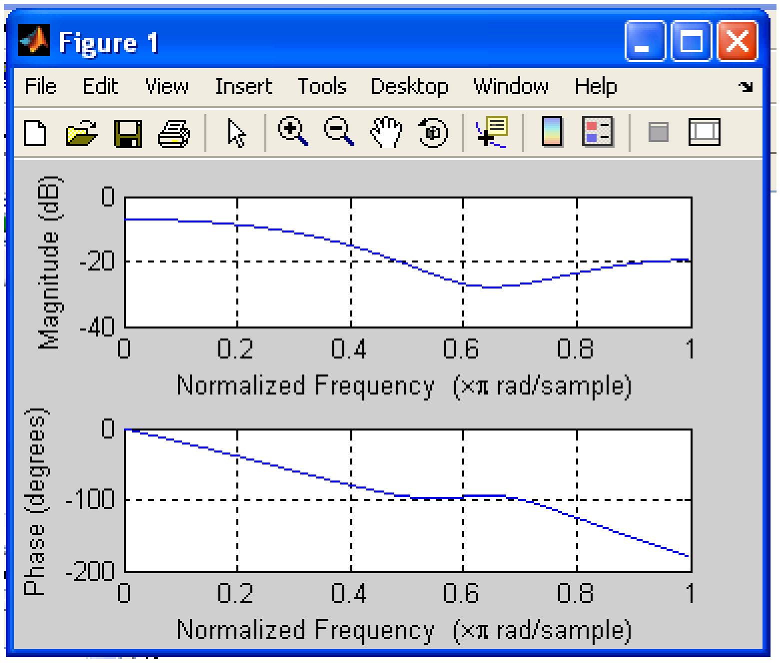
The Freqz function computes and display the frequency response of given Z- Transform of the function

freqz(b,a,Fs)

b= Coeff. Of Numerator
a= Coeff. Of Denominator
Fs= Sampling Frequency

Matlab Code:

```
b=[2 5 9 5 3]
a= [5 45 2 1 1]
freqz(b,a);
```



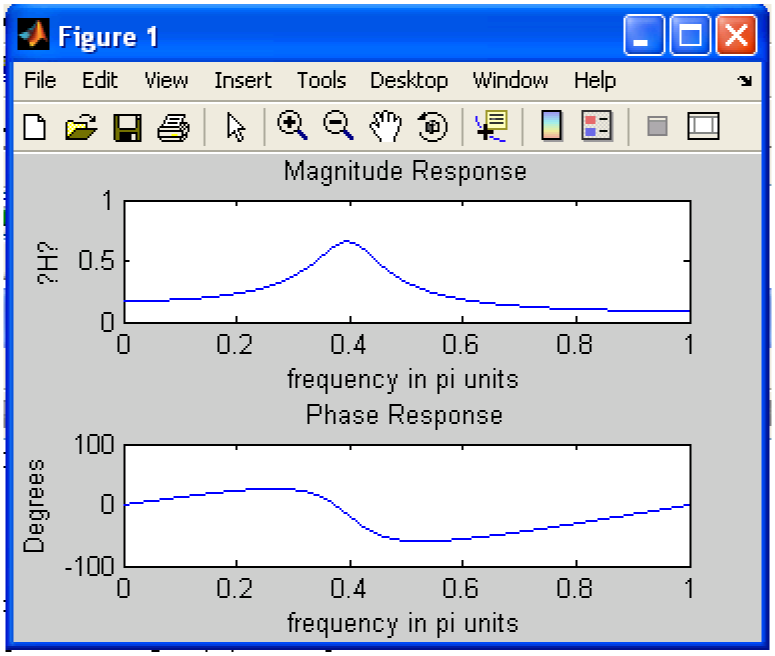
Example:

Plot the magnitude and phase of the frequency response of the given digital filter Using freqz function:

$$Y(n) = 0.2x(n) + 0.52y(n-1) - 0.68(y(n-2)$$

Matlab Code:

```
b = [0.2];
a= [1, -0.52, 0.68];
w = [0:1:500]*pi/500;
H=freqz(b,a,w);
magH = abs(H);
phaH = angle(H)*180/pi;
subplot(2,1,1);
plot(w/pi,magH);
title('Magnitude Response');
xlabel('frequency in pi units');
ylabel('|H|');
subplot(2,1,2);
plot(w/pi,phaH);
title('Phase Response');
xlabel('frequency in pi units');
ylabel('Degrees');
```



Examples

1-

Let $X_1(z) = 2 + 3z^{-1} + 4z^{-2}$ *and* $X_2(z) = 3 + 4z^{-1} + 5z^{-2} + 6z^{-3}$.
Determine $X_3(z) = X_1(z)\, X_2(z)$

Solution:
From the definition of the z-transform we observe that x1(n) = {2,3,4} and x2(n) = {3,4,5,6}
Then the convolution of the above two sequences will give the coefficients of the required polynomial product.

```
>> x1=[2,3,4]; x2=[3,4,5,6];
>> x3= conv(x1,x2)
```

```
x3 = 6     17     34     43     38     24
```

Hence
$X_3(z) = 6 + 17z^{-1} + 34z^{-2} + 43z^{-3} + 38z^{-4} + 24z^{-5}$

2-

Compute the inverse z-transform of

$$X(z) = \frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})} \quad , \qquad |z| > 0.9$$

We can evaluate the denominator polynomial as well as the residues using MATLAB.

```
>> b=[1 0 0 0]; a= poly([0.9,0.9,-0.9])
>> [R,p,C] = residuez(b,a)
a =
        1.0000    -0.9000    -0.8100     0.7290

R =
     0.2500
     0.2500
     0.5000
p =
    -0.9000
     0.9000
     0.9000
C =
      0
```

Note that the denominator polynomial is computed using MATLAB's polynomial function poly, which computes the polynomial coefficients, given its roots. We could have used the conv function, but the use of the poly function is more convenient for this purpose. From the residue calculations and using the order of residues given in (4.16), we have

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

$$X(z) = \frac{0.25}{1-0.9z^{-1}} + \frac{0.5}{0.9} z \frac{0.9z^{-1}}{(1-0.9z^{-1})^2} + \frac{0.25}{1+0.9z^{-1}}, \quad |z| > 0.9$$

Hence from Table 4.1 and using the z-transform property of time-shift,

$$x(n) = 0.25(0.9)^n u(n) + \frac{5}{9}(n+1)(0.9)^{n+1}u(n+1) + 0.25(-0.9)^n u(n)$$

which upon simplification becomes

$$x(n) = 0.75(0.9)^n u(n) + 0.5n(0.9)^n u(n) + 0.25(-0.9)^n u(n)$$

```
>> b=[1]; a =[1,-0.9,-0.81,0.729]
>>[delta,n]= impseq(0,0,7);
>> x=filter(b,a,delta) % check sequence
```

```
x =1   0.9    1.62    1.458    1.9683    1.7715    2.1258     1.9132
```

```
% answer sequence
>> x=(0.75)*(0.9).^n + (0.5)*n.*(0.9).^n + (0.25)*(-0.9).^n
```

```
x =1   0.9    1.62    1.458    1.9683    1.7715    2.1258     1.9132
```

Other verification:

```
syms z
f=1/(((1-0.9*z^-1)^2)*(1+0.9*z^-1))
iztrans(f)
```

```
ans =

(-9/10)^n/4 + (5*(9/10)^n)/4 + ((9/10)^n*(n - 1))/2
```

4-
Solve:

$$y(n) - \frac{3}{2}y(n-1) + \frac{1}{2}y(n-2) = x(n), \qquad n \geq 0$$

$$where:\ y(-1) = 4\ \ and\ \ y(-2) = 10. \qquad x(n) = \left(\frac{1}{4}\right)^n u(n)$$

Solution:

```
>> a= [1,-1.5, 0.5]; b=1;
>> n= [0:7]; x= (1/4). ^n;
>> Y= [4, 10];
>> xic=filtic(b,a,Y);
>> y1=filter (b,a,x,xic)
y1 = 2   1.25    0.9375     0.7969     0.7305    0.6982    0.6824     0.6745
>> y2= (1/3)*(1/4). ^n+(1/2).^n+(2/3)*ones(1,8)   % Matlab Check

y2 = 2   1.25    0.9375     0.7969     0.7305    0.6982    0.6824     0.6745
```

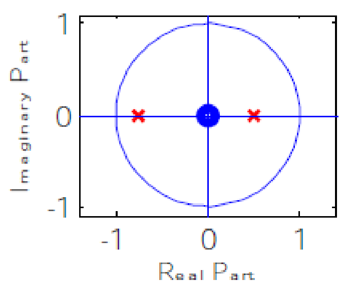
5-

Plot the zero-pole diagram of

$$X(z) = \frac{z}{(z-0.5)\,(z+0.75)}$$

```
>> help zplane
   ZPLANE Z-plane zero-pole plot.
   ZPLANE(Z,P) plots the zeros Z and poles P (in column vectors)
   with the unit circle for reference.
```

```
z=[0 ]
p=[0.5;-75]
zplane(z,p)
```



Assignments:

1-

Consider the system:

$$H(z) = \frac{1 - 2z^{-1} + 2z^{-2} - z^{-3}}{(1 - z^{-1})(1 - 0.5z^{-1})(1 - 0.2z^{-1})} \qquad ROC: 0.5 < |z| < 1$$

(a) Sketch the pole-zero pattern. Is system stable? (search and Use "tf" and "pzmap" commands).

(b) Determine impulse response of system.(you may use "filter" command with input [1 zeros(1,49)].

2-

A discrete time control system is characterized by difference equation:

$$y[n] - 2.8y[n-1] + 3.02y[n-2] - 1.468y[n-3] + 0.27y[n-4]$$
$$= 0.03x[n] - 0.02x[n-1] + 0.01x[n-2].$$

Use the Z-transform to find system transfer function $H(z)$.Find the pole-zero plot and discuss the stability. Then determine and plot the system output when $x[n] = 5u[n]$. If:

i. System is relaxed.

ii. $y[-1] = -0.2, y[-2] = 0.3, y[-3] = y[-4] = 0.$

3-

A system is characterized by the transfer function:

$$H(z) = \frac{0.74z^2 - 2.544z + 2.5216}{z^2 + 0.64}$$

(a) Draw the pole-zero plot for the transfer function. Also find and plot the impulse response .

(b) Find the output signal for the input $x[n] = (-2(0.3)^n + 2(0.8)^n)u(n)$and draw the output signal pole-zero plot. Identify the output signal poles contributed TF or the input