The Age of Information in Multihop Networks

Ahmed M. Bedewy†, Yin Sun⋆, and Ness B. Shroff†‡

†Dept. of ECE, ‡Dept. of CSE, The Ohio State University, Columbus, OH.

⋆Dept. of ECE, Auburn University, Auburn, Alabama.

emails: bedewy.2@osu.edu, yzs0078@auburn.edu, shroff.11@osu.edu

Abstract

The problem of minimizing the age-of-information has been recently resolved in single-hop networks. In this paper, we minimize the the age of a single information flow in multihop networks. If the packet transmission times over the network links are exponentially distributed, we prove that a preemptive Last-Generated, First-Serve (LGFS) policy results in smaller age processes at all nodes of the network (in a stochastic ordering sense) than any other causal policy. In addition, for arbitrary distributions of packet transmission times, the non-preemptive LGFS policy is shown to minimize the age processes at all nodes among all non-preemptive work-conserving policies (again in a stochastic ordering sense). Interestingly, these simple policies can achieve optimality of the joint distribution of the age processes at all nodes even under arbitrary network topologies, as well as arbitrary packet generation and arrival times. These optimality results not only hold for the age processes, but also for any non-decreasing functional of the age processes. Finally, we investigate the class of New-Better-than-Used (NBU) packet transmission time distributions and show that the non-preemptive LGFS policy can come close to age-optimality.

This paper was presented in part at IEEE ISIT 2017 [1].

This work has been supported in part by ONR grants N00014-17-1-2417 and N00014-15-1-2166, Army Research Office grants W911NF-14-1-0368 and MURI W911NF-12-1-0385, National Science Foundation grants CNS-1446582, CNS-1421576, and CNS-1518829, and a grant from the Defense Thrust Reduction Agency HDTRA1-14-1-0058.

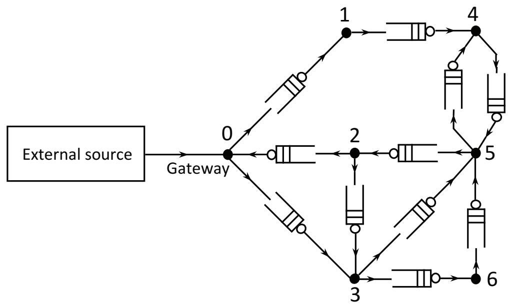


Figure 1: Information updates in a multihop network.

I. Introduction

There has been a growing interest in applications that require real-time information updates, such as news, weather reports, email notifications, stock quotes, social updates, mobile ads, etc. The freshness of the information is also crucial in other systems, e.g., monitoring systems that obtain information from environmental sensors, wireless systems that need rapid updates of channel state information, etc.

As a metric of data freshness, the *age-of-information*, or simply *age*, was defined in [2]–[5]. At time t, if the freshest update at the destination was generated at time $U(t)$, the age $\Delta(t)$ is defined as $\Delta(t) = t - U(t)$. Hence, age is the time elapsed since the freshest packet was generated.

In our recent work [6], [7], we have shown that the LGFS scheduling policy minimizes the age-of-information in single-hop networks in a quite strong sense. In this paper, we further investigate age-optimal scheduling in multihop networks. We consider an interference free network represented by a directed graph, as shown in Fig. 1, where the update packets are generated at an external source and are then dispersed throughout the network via a gateway node. It is well known that delay-optimality is notoriously difficult in multihop networks, except for some special network settings (e.g., tandem networks) [8], [9]. This difficulty stems from the fact that the packet scheduling decisions at each hop are influenced by the decisions on other hops and vise versa. Somewhat to our surprise, it turns out that age minimization has very different features from delay minimization. In particular, we find that some simple policies can achieve optimality of the joint distribution of the age processes at all nodes, even under arbitrary network topologies. The following summarizes our main contributions in this paper:

- If the packet transmission times over the network links are exponentially distributed, then for arbitrary arrival process, network topology, and buffer sizes, the preemptive LGFS policy minimizes the age processes at all nodes in the network among all causal policies in a stochastic ordering sense (Theorem 1). In other words, the preemptive LGFS policy minimizes any *non-decreasing functional of the age processes at all nodes* in a stochastic ordering sense. Note that this age penalty model is very general. Many age penalty metrics studied in the literature, such as the time-average age [5], [10]–[18], average peak age [11], [14], [18]–[20], time-average age penalty function [21], [22], and age penalty functional at single-hop network [6], [7], are special cases of the general age functional model that we consider in this paper.
- We then prove that, for arbitrary distributions of packet transmission times, the non-preemptive LGFS policy minimizes the age processes at all nodes among all non-preemptive work-conserving

policies in the sense of stochastic ordering (Theorem 2). It is interesting to note that age-optimality here can be achieved even if the transmission time distribution differs from one link to another, i.e., the transmission time distributions are heterogeneous.

- Finally, we investigate an important class of packet transmission time distributions called New-Better-than-Used (NBU) distributions, which are more general than exponential. We generalize the policy space (compared to that of the general transmission time case) such that it includes all causal policies. The network topology we consider here is somewhat more restrictive in the sense that each node can have one incoming link only. We show that the non-preemptive LGFS policy is within a constant age gap from the optimum average age, and that the gap is independent of the system parameters (Theorem 4).

To the best of our knowledge, these are the first optimal results on minimizing the age-of-information in multihop networks for given generation and arrival times of the update packets.

The remainder of this paper is organized as follows. After a brief overview of related work in Section II, we describe the model and problem formulation in Section III. The age-optimality results are presented in Section IV, and near age-optimality results are developed in Section V. Finally, the conclusion is drawn in Section VI.

II. Related Work

There are a number of studies that have focused on the analysis of the age and figuring out ways to reduce it in single-hop networks [5], [10], [11], [13]–[20], [23]. In [5], [10], the update frequency was optimized to minimize the age in First-Come, First-Serve (FCFS) queueing systems when the service time is exponentially distributed. It was found that this frequency differs from those that minimize the delay or maximize the throughput. Extending the analysis to multi-class M/G/1 queue was considered in [19]. In [11]–[13], it was shown that age can be improved by discarding old packets waiting in the queue when a new sample arrives. Analyzing the age-of-information in the existence of the energy replenishment constraints was considered in [14], [15]. In [16], the time-average age was characterized for Last-Come, First-Serve (LCFS) information-update systems with and without preemption. Expanding the analysis to multiple sources was considered in [17], where sharing service facility among Poisson sources was found to improve the total age. The work in [20] analyzed the age in the presence of errors when the service times are exponentially distributed. Gamma-distributed service times was considered in [18]. The studies in [20], [18] were carried out for LCFS queueing systems with and without preemption. Wireless scheduling for minimizing the age in single-hop networks was investigated in [23], where the minimum age scheduling problem is shown to be NP-hard in general.

Age-optimal generation of update packets was studied for single-hop networks in [14], [15], [21], [22]. A general class of non-negative, non-decreasing age penalty functions was minimized in [21], [22]. Later, a real-time sampling problem of the Wiener process is solved in [24]: If the sampling times are independent of the observed Wiener process, the optimal sampling problem in [24] reduces to an age-of-information optimization problem; otherwise, the optimal sampling policy can use knowledge of the Wiener process to achieve better performance than age-of-information optimization.

Quite recently, it was shown in [6], [7] that for arbitrary packet generation times, arrival times, and queue buffer size, a preemptive LGFS policy simultaneously optimizes the age, throughput, and delay in multi-server single-hop networks with exponential service times. The study was extended to the class of NBU packet service time distributions in [7], where it was shown that near age-optimality can be achieved. In this study [7], packet replication technique was considered and an interesting result was revealed: While it is known that replication worsens the delay and decreases the throughput for most NBU service time distributions, surprisingly, it was found that replication can in fact reduce the age for a variety of NBU service time distributions.

There exist a few studies on the age-of-information in multihop networks [25]–[27]. In [25], the age-of-information was characterized in multihop networks with special topologies. Minimizing the age-of-information in energy harvesting two hop network by controlling the update transmissions at each node was considered in [26]. The authors of [27] addressed the problem of scheduling in wireless multihop networks, where all network queues are served in FCFS manner. An age-based scheduling algorithm was developed to maintains the freshness of the information at the receiver. The performance of the proposed scheduler was characterized by the heavy-traffic analysis. This paper differs from these studies in two aspects: First, our study is carried out for very general age metric and system settings in a multihop network with a single source. Second, in our study, age-optimality results are established when it is possible (exponential transmission time and general transmission time for restrictive policy space); and in some more general scenarios where age-optimality is notoriously difficult to be achieved (NBU transmission time), alternative near age-optimality results are obtained.

III. Model and Formulation

A. Notations and Definitions

For any random variable Z and an event A, let $[Z|A]$ denote a random variable with the conditional distribution of Z for given A, and $\mathbb{E}[Z|A]$ denote the conditional expectation of Z for given A.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two vectors in $\mathbb{R}^n$, then we denote $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. A set $U \subseteq \mathbb{R}^n$ is called upper if $\mathbf{y} \in U$ whenever $\mathbf{y} \geq \mathbf{x}$ and $\mathbf{x} \in U$. We will need the following definitions:

Definition III.1. Univariate Stochastic Ordering: [28] Let X and Y be two random variables. Then, X is said to be stochastically smaller than Y (denoted as $X \leq_{\text{st}} Y$), if

$$\mathbb{P}\{X > x\} \leq \mathbb{P}\{Y > x\}, \quad \forall x \in \mathbb{R}.$$

Definition III.2. Multivariate Stochastic Ordering: [28] Let $\mathbf{X}$ and $\mathbf{Y}$ be two random vectors. Then, $\mathbf{X}$ is said to be stochastically smaller than $\mathbf{Y}$ (denoted as $\mathbf{X} \leq_{\text{st}} \mathbf{Y}$), if

$$\mathbb{P}\{\mathbf{X} \in U\} \leq \mathbb{P}\{\mathbf{Y} \in U\}, \quad \text{for all upper sets} \quad U \subseteq \mathbb{R}^n.$$

Definition III.3. Stochastic Ordering of Stochastic Processes: [28] Let $\{X(t), t \in [0, \infty)\}$ and $\{Y(t), t \in [0, \infty)\}$ be two stochastic processes. Then, $\{X(t), t \in [0, \infty)\}$ is said to be stochastically smaller than $\{Y(t), t \in [0,\infty)\}$ (denoted by $\{X(t), t \in [0,\infty)\} \leq_{\text{st}} \{Y(t), t \in [0,\infty)\}$), if, for all choices of an integer n and $t_1 < t_2 < \ldots < t_n$ in $[0,\infty)$, it holds that

$$(X(t_1), X(t_2), \ldots, X(t_n)) \leq_{\text{st}} (Y(t_1), Y(t_2), \ldots, Y(t_n)), \tag{1}$$

where the multivariate stochastic ordering in (1) was defined in Definition III.2.

B. Network Model

We consider a multihop network represented by a directed graph $\mathcal{G}(\mathcal{V}, \mathcal{L})$ where $\mathcal{V}$ is the set of nodes and $\mathcal{L}$ is the set of links, as shown in Fig. 1. The network topology is arbitrary and the number of nodes in the network is $|\mathcal{V}| = N$. The system starts to operate at time $t = 0$. A sequence of n update packets are generated at the external source, where n can be an arbitrary finite or infinite number. The generation time of the l-th packet is s_l, such that $0 \leq s_1 \leq s_2 \leq \ldots \leq s_n$. The external source is connected to the network through a gateway node 0. The update packets are firstly forwarded to node 0, from which they are dispersed throughout the network. Let a_{lj} denote the arrival time of packet l to node j. Then, $s_l \leq a_{l0} \leq a_{lj}$ for all $j = 1, \ldots, N-1$. Note that the update packets may arrive at node 0 *out of the order* of their generation times. For example, packet $l+1$ may arrive at node 0 earlier than packet l such that $s_l \leq s_{l+1}$ but $a_{l0} \geq a_{(l+1)0}$. Define $(i,j) \in \mathcal{L}$ as a link from node i to node j, where i is the origin node and j is the destination node. Once a packet arrives at node i, it is immediately available to all the outgoing links from node i. Each link (i,j) has a queue of buffer size B_{ij} to store the incoming packets. If B_{ij} is finite, the queue buffer may overflow and some packets are dropped. The packet transmission time on each link (i,j) is random and the network is interference free.

C. Scheduling Policy

We let π denote a scheduling policy that determines when to send the packets on each link and in which order. The packet generation times $(s_1, s_2, \ldots, s_n)$ and packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ at node 0 are arbitrary and do not change according to the scheduling policy, while the packet arrival times $(a_{1j}, a_{2j}, \ldots, a_{nj})$ for $j \geq 1$ are functions of the scheduling policy π. We assume that the packet transmission times are invariant of the scheduling policy.

Let Π denote the set of all *causal* policies, in which scheduling decisions are made based on the history and current state of the system. We define several types of policies in Π:

A policy is said to be **preemptive**, if a link can switch to send another packet at any time; the preempted packets will be stored back into the queue if there is enough buffer space and then sent out at a later time when the link is available again. In contrast, in a **non-preemptive** policy, a link must complete sending the current packet before starting to send another packet. A policy is said to be **work-conserving**, if each link is busy whenever there are packets waiting in the queue feeding this link.

D. Age Performance Metric

Let $U_j(t) = \max\{s_l : a_{lj} \leq t\}$ be the generation time of the freshest packet arrived at node j before time t. The *age-of-information*, or simply the *age*, at node j is defined as

$$\Delta_j(t) = t - U_j(t). \tag{2}$$

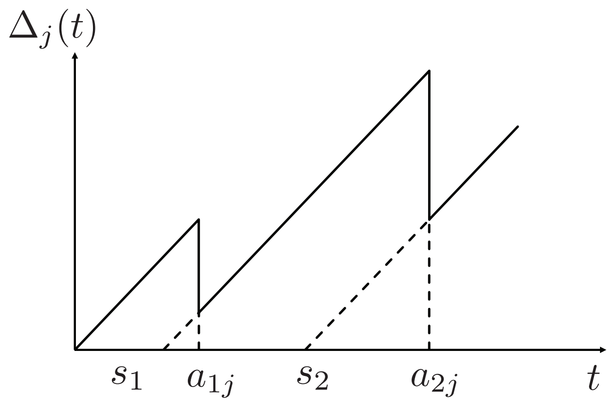


Figure 2: Sample path of the age process $\Delta_j(t)$ at node j.

The process of $\Delta_j(t)$ is given by $\Delta_j = \{\Delta_j(t), t \in [0, \infty)\}$. The initial state $U_j(0^-)$ at time $t = 0^-$ is invariant of the scheduling policy $\pi \in \Pi$, where we assume that $U_j(0^-) = 0 = s_0$ for all $j \in \mathcal{V}$. As shown in Fig. 2, the age increases linearly with t but is reset to a smaller value with the arrival of a fresher packet. The age vector of all the network nodes at time t is

$$\boldsymbol{\Delta}(t) = (\Delta_0(t), \Delta_1(t), \ldots, \Delta_{N-1}(t)). \tag{3}$$

The age process of all the network nodes is given by

$$\boldsymbol{\Delta} = \{\boldsymbol{\Delta}(t), t \in [0, \infty)\}. \tag{4}$$

In this paper, we introduce a general *age penalty functional* $g(\boldsymbol{\Delta})$ to represent the level of dissatisfaction for data staleness at all the network nodes.

Definition III.4. Age Penalty Functional: Let $\mathbf{V}$ be the set of N-dimensional Lebesgue measurable functions, i.e.,

$$\mathbf{V} = \{f : [0, \infty)^N \mapsto \mathbb{R} \text{ such that } f \text{ is Lebesgue measurable}\}.$$

A functional $g : \mathbf{V} \mapsto \mathbb{R}$ is said to be an *age penalty functional* if g is *non-decreasing* in the following sense:

$$g(\boldsymbol{\Delta}_1) \leq g(\boldsymbol{\Delta}_2), \text{ whenever } \boldsymbol{\Delta}_1(t) \leq \boldsymbol{\Delta}_2(t), \forall t \in [0, \infty). \tag{5}$$

The age penalty functionals used in prior studies include:

- *Time-average age [5], [10]–[18]:* The time-average age of node j is defined as

$$g_1(\boldsymbol{\Delta}) = \frac{1}{T} \int_0^T \Delta_j(t) dt, \tag{6}$$

- *Average peak age [11], [14], [18]–[20]:* The average peak age of node j is defined as

$$g_2(\boldsymbol{\Delta}) = \frac{1}{K} \sum_{k=1}^{K} A_{kj}, \tag{7}$$

where A_{kj} denotes the k-th peak value of $\Delta_j(t)$ since time $t = 0$.

- *Time-average age penalty function [21], [22]:* The average age penalty function of node j is

$$g_3(\mathbf{\Delta}) = \frac{1}{T}\int_0^T h(\Delta_j(t))dt, \tag{8}$$

where $h : [0, \infty) \to [0, \infty)$ can be any non-negative and non-decreasing function. As pointed out in [22], a stair-shape function $h(\Delta) = \lfloor \Delta \rfloor$ can be used to characterize the dissatisfaction of data staleness when the information of interests is checked periodically, and an exponential function $h(\Delta) = e^{\Delta}$ is appropriate for online learning and control applications where the desire for data refreshing grows quickly with respect to the age.

- *Age penalty functional at single-hop network [6], [7].*

IV. Age-optimality Results

In this section, we present our age-optimality results for multihop networks. We prove our results in a stochastic ordering sense.

A. Exponential Transmission Time Distributions

Algorithm 1: The preemptive Last-Generated, First-Serve (LGFS) policy.

```
α_ij := 0;
while the system is ON do
    if a new packet with generation time s arrives to node i then
        if the link is busy then
            if s ≤ α_ij then
                Store the packet in the queue;
            else        // The packet carries fresh information.
                Send the packet over the link by preempting the packet being transmitted;
                The preempted packet is stored back to the queue;
                α_ij = s;
            end
        else        // The link is idle.
            The new packet is sent over the link;
        end
    end
    if a packet is delivered then
        if the queue is not empty then
            The freshest packet in the queue is sent over the link;
        end
    end
end
```

We study the age-optimal packet scheduling when the packet transmission times are exponentially distributed, *independent* across the links and *i.i.d.* across time. We consider a LGFS scheduling principle in which the packet being transmitted at each link is generated the latest (i.e., the freshest) one among all packets in the queue; after transmission, the link starts to send the next freshest packet in its queue. We consider a preemptive LGFS (prmp-LGFS) policy at each link $(i,j) \in \mathcal{L}$. The implementation details of this policy are depicted in Algorithm 1. Throughout Algorithm 1, we use α_{ij} to denote the generation time of the packet being transmitted on the link (i,j).

Define a set of parameters $\mathcal{I} = \{n, (s_l, a_{l0})_{l=1}^{n}, \mathcal{G}(\mathcal{V},\mathcal{L}), (B_{ij}, (i,j) \in \mathcal{L})\}$, where n is the total number of packets, s_l is the generation time of packet l, a_{l0} is the arrival time of packet l to node 0, $\mathcal{G}(\mathcal{V},\mathcal{L})$ is the network graph, and B_{ij} is the queue buffer size of link (i,j). Let $\mathbf{\Delta}_{\pi} = \{\mathbf{\Delta}_{\pi}(t), t \in [0,\infty)\}$ be the age processes of all nodes in the network under policy π. The age optimality of prmp-LGFS policy is provided in the following theorem.

Theorem 1. If the packet transmission times are exponentially distributed, *independent* across links and *i.i.d.* across time, then for all $\mathcal{I}$ and $\pi \in \Pi$

$$[\mathbf{\Delta}_{\text{prmp-LGFS}}|\mathcal{I}] \leq_{\text{st}} [\mathbf{\Delta}_{\pi}|\mathcal{I}], \tag{9}$$

or equivalently, for all $\mathcal{I}$ and non-decreasing functional g

$$\mathbb{E}[g(\mathbf{\Delta}_{\text{prmp-LGFS}})|\mathcal{I}] = \min_{\pi \in \Pi} \mathbb{E}[g(\mathbf{\Delta}_{\pi})|\mathcal{I}], \tag{10}$$

provided the expectations in (10) exist.

Proof. See Appendix A. □

Theorem 1 tells us that for arbitrary number n, packet generation times $(s_1, s_2, \ldots, s_n)$ and arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ at node 0, network topology $\mathcal{G}(\mathcal{V},\mathcal{L})$, and buffer sizes $(B_{ij}, (i,j) \in \mathcal{L})$, the prmp-LGFS policy achieves optimality of the joint distribution of the age processes at the network nodes within the policy space Π. In addition, (10) tells us that the prmp-LGFS policy minimizes any non-decreasing age penalty functional g, including the time-average age (6), average peak age (7), and average age penalty (8).

B. General Transmission Time Distributions

Now, we study the age-optimal packet scheduling for *arbitrary* general packet transmission time distributions which are *independent* across the links and *i.i.d.* across time. We consider the set of non-preemptive work-conserving policies, denoted by $\Pi_{npwc} \subset \Pi$. We propose a non-preemptive LGFS (non-prmp-LGFS) policy. It is important to note that under non-prmp-LGFS policy, the fresh packet replaces the oldest packet in a link's queue when the queue has a finite buffer size and full. The description of non-preemptive LGFS policy can be obtained from Algorithm 1 by replacing Steps 5-11 by Step 6. We next show that the non-preemptive LGFS policy is age-optimal among the policies in Π_{npwc}.

Theorem 2. If the packet transmission times are *independent* across the links and *i.i.d.* across time, then for all $\mathcal{I}$ and $\pi \in \Pi_{npwc}$

$$[\mathbf{\Delta}_{\text{non-prmp-LGFS}}|\mathcal{I}] \leq_{\text{st}} [\mathbf{\Delta}_{\pi}|\mathcal{I}], \tag{11}$$

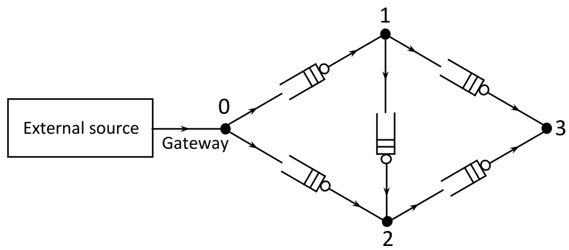


Figure 3: A multihop network.

or equivalently, for all $\mathcal{I}$ and non-decreasing functional g

$$\mathbb{E}[g(\mathbf{\Delta}_{\text{non-prmp-LGFS}})|\mathcal{I}] = \min_{\pi \in \Pi_{npwc}} \mathbb{E}[g(\mathbf{\Delta}_{\pi})|\mathcal{I}], \tag{12}$$

provided the expectations in (12) exist.

Proof. The proof of Theorem 2 is similar to that of Theorem 1. The difference is that preemption is not allowed here. See Appendix B for more details. □

It is interesting to note from Theorem 2 that, age-optimality can be achieved for arbitrary transmission time distributions, even if the transmission time distribution differs from a link to another.

C. Simulation Results

We present some numerical results to illustrate the age performance of different policies and validate the theoretical results. We consider the network in Fig. 3. The inter-generation times are *i.i.d.* Erlang-2 distribution with mean $1/\lambda$. The time difference between packet generation and arrival to node 0, i.e., $a_{i0} - s_i$, is modeled to be either 1 or 100, with equal probability. This means that the update packets may arrive to node 0 out of order of their generation time.

Figure 4 illustrates the average peak age at node 2 versus the packet generation rate λ for the multihop network in Fig. 3. The packet transmission times are exponentially distributed with mean 1 at links $(0, 1)$ and $(1, 2)$, and mean 0.5 at link $(0, 2)$. One can observe that the preemptive LGFS policy achieves a better (smaller) peak age at node 2 than the non-preemptive LGFS policy, non-preemptive LCFS policy, and FCFS policy, where the buffer sizes are either 1 or infinity. It is important to emphasize that the peak age is minimized by preemptive LGFS policy for out of order packet receptions at node 0, and general network topology. This numerical result shows agreement with Theorem 1.

Figure 5 plots the time-average age at node 3 versus the packets generation rate λ for the multihop network in Fig. 3. The plotted policies are FCFS policy, non-preemptive LCFS, and non-preemptive LGFS policy, where the buffer sizes are either 1 or infinity. The packet transmission times at links $(0, 1)$ and $(1, 3)$ follow a gamma distribution with mean 1. The packet transmission times at links $(0, 2)$, $(1, 2)$, and $(2, 3)$ are distributed as the sum of a constant with value 0.5 and a value drawn from an exponential distribution with mean 0.5. We find that the non-preemptive LGFS policy achieves the best age performance among all plotted policies. By comparing the age performance of the non-preemptive LGFS and non-preemptive LCFS policies, we observe that the LGFS scheduling principle improves the

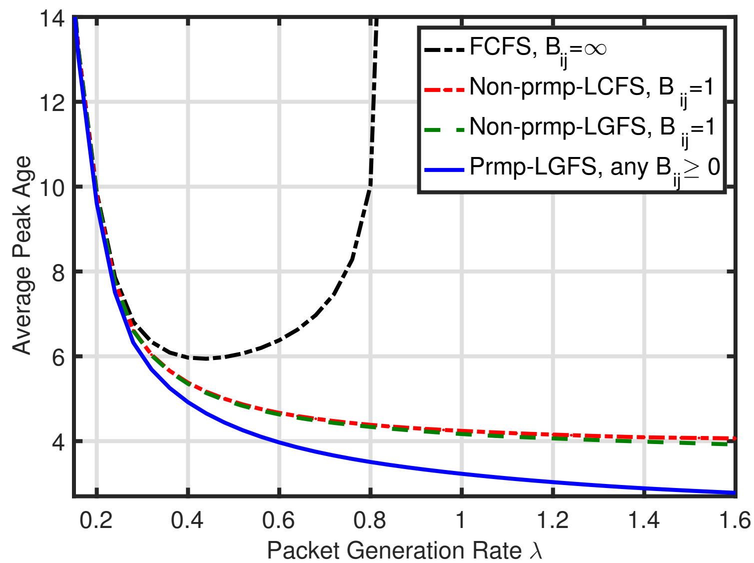


Figure 4: Average peak age at node 2 versus packets generation rate λ for exponential packet transmission times.

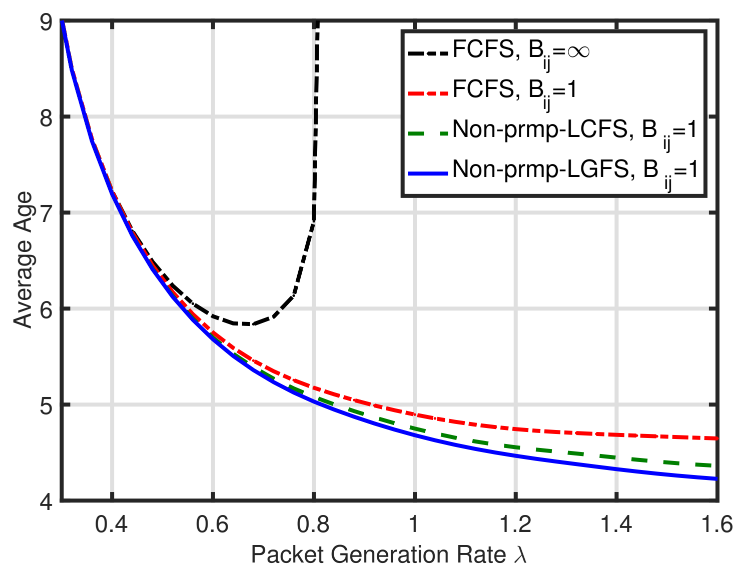


Figure 5: Average age at node 3 versus packets generation rate λ for general packet transmission time distributions.

age performance when the update packets arrive to node 0 out of the order of their generation times. It is important to note that the non-preemptive LGFS policy minimizes the age among the non-preemptive work-conserving policies even if the packet transmission time distributions are heterogeneous across the links. We also observe that the average age of FCFS policy with $B_{ij} = \infty$ blows up when the traffic intensity is high. This is due to the increased congestion in the network which leads to a delivery of stale packets. Moreover, in case of the FCFS policy with $B_{ij} = 1$, the average age is finite at high

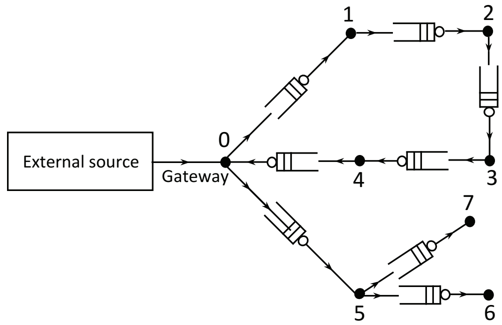


Figure 6: Information updates over a multihop network, where each node in the network (except the gateway) is restricted to receive data from only one node.

traffic intensity, since the fresh packet has a better opportunity to be delivered in a relatively short period compared with FCFS policy with $B_{ij} = \infty$. This numerical result agrees with Theorem 2.

V. Near Age-optimality Results

While the policy space being considered in Theorem 1 (for exponential transmission distributions) is very general, the policy space considered in Theorem 2 (for general transmission distributions) is restricted to only non-preemptive work-conserving policies. The next question we answer is whether for an important class of distributions that are more general than exponential, optimality or near-optimality can be achieved without the non-preemptive work-conserving policy space restriction. We propose non-preemptive LGFS policy and show that it is within small age gap of the optimum average age among all policies in Π. However, we are able to prove this result for a somewhat more restrictive network than the general topology $\mathcal{G}(\mathcal{V}, \mathcal{L})$. The network here is represented by a directed graph $\mathcal{G}'(\mathcal{V}, \mathcal{L})$, in which each node $j \in \mathcal{V}\backslash\{0\}$ is allowed to receive data from only one node (the total number of incoming links to each node $j \in \mathcal{V}\backslash\{0\}$ is at most one), as shown in Fig. 6. The packet transmission times are assumed to be *i.i.d.* across time and links. We consider the classes of New-Better-than-Used (NBU) packet transmission time distributions, which are defined as follows.

Definition V.1. New-Better-than-Used distributions: Consider a non-negative random variable X with complementary cumulative distribution function (CCDF) $\bar{F}(x) = \mathbb{P}[X > x]$. Then, X is **New-Better-than-Used (NBU)** if for all $t, \tau \geq 0$

$$\bar{F}(\tau + t) \leq \bar{F}(\tau)\bar{F}(t). \tag{13}$$

Examples of NBU distributions include constant transmission time, (shifted) exponential distribution, geometrical distribution, Erlang distribution, negative binomial distribution, etc.

Define a set of the parameters $\mathcal{I}' = \{n, (s_l, a_{l0})_{l=1}^{n}, \mathcal{G}'(\mathcal{V}, \mathcal{L}), (B_{ij}, (i,j) \in \mathcal{L})\}$, where n is the total number of packets, s_l is the generation time of packet l, a_{l0} is the arrival time of packet l to node 0, $\mathcal{G}'(\mathcal{V}, \mathcal{L})$ is the network graph with the new restriction, and B_{ij} is the queue buffer size of link (i, j). Define $\mathcal{H}_k$ as the set of nodes in the k-th hop, i.e., $\mathcal{H}_k$ is the set of nodes that are separated by k links

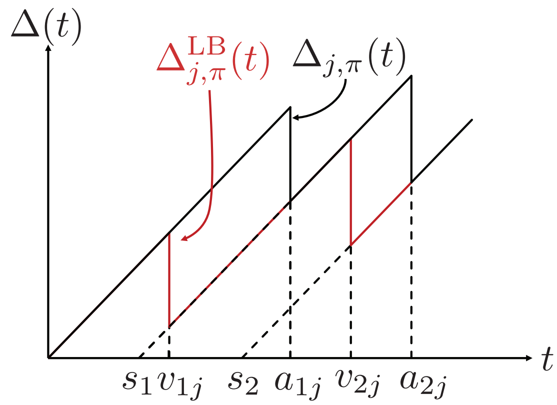


Figure 7: The evolution of $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ and $\Delta_{j,\pi}(t)$ at node $j \in \mathcal{H}_1$.

from node 0 [1]. To show that policy non-prmp-LGFS can come close to age-optimal, we need to construct an age lower bound. We first construct a lower bound for each policy π. Then, we show that the lower bound of policy non-prmp-LGFS is an age lower bound for all policies in Π. The age lower bound of policy π is defined as follows:

Since the packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ at node 0 are invariant of the scheduling policy π, the age at node 0 is invariant of the scheduling policy too. Hence, we define a function $\Delta^{\mathrm{LB}}_{0,\pi}(t) = \Delta_{0,\pi}(t)$. Let v_{lj} denote the transmission starting time of packet l over the incoming link to node j, which is a function of the scheduling policy π. For each node $j \in \mathcal{H}_1$, define a function $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ as

$$\Delta^{\mathrm{LB}}_{j,\pi}(t) = t - \max\{s_l : v_{lj}(\pi) \leq t\}. \tag{14}$$

The definition of the $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ is similar to that of the age of policy π at node j except that the packets arrival times to node j are replaced by their transmission starting times at the link $(0, j)$. In this case, $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ increases linearly with t but is reset to a smaller value with the transmission start of a fresher packet over the link $(0, j)$, as shown in Fig. 7. For the nodes in the subsequent hops (the nodes in $\mathcal{H}_k$ such that $k \geq 2$), the lower bound of policy π (LB_π) is constructed as follows. The packets at each link queue are served in LGFS manner. Each packet l is considered to be delivered from node i to node j once the transmission of packet l starts over the link (i, j); however, after the transmission of packet l starts over the link (i, j), the link (i, j) becomes busy in sending a virtual copy of packet l for a duration equal to the transmission time of packet l at the link (i, j). Thus, we have $a_{lj}(\mathrm{LB}_\pi) = v_{lj}(\mathrm{LB}_\pi)$ for all $l = 1, \ldots, n$ and all $j \in \mathcal{H}_k$ such that $k \geq 2$. The function $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ is defined at each node $j \in \mathcal{H}_k$ for $k \geq 2$ as

$$\Delta^{\mathrm{LB}}_{j,\pi}(t) = t - \max\{s_l : v_{lj}(\mathrm{LB}_\pi) \leq t\}. \tag{15}$$

The process of $\Delta^{\mathrm{LB}}_{j,\pi}(t)$ is given by $\Delta^{\mathrm{LB}}_{j,\pi} = \{\Delta^{\mathrm{LB}}_{j,\pi}(t), t \in [0, \infty)\}$ for each $j \in \mathcal{V}$. Policy π lower bound

[1]Node 0 is always considered to be in $\mathcal{H}_0$.

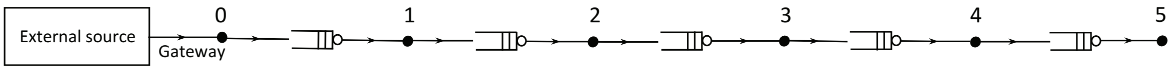


Figure 8: A multihop network.

vector of all the network nodes is

$$\mathbf{\Delta}_{\pi}^{\text{LB}}(t) =(\Delta_{0,\pi}^{\text{LB}}(t), \Delta_{1,\pi}^{\text{LB}}(t), \ldots, \Delta_{N-1,\pi}^{\text{LB}}(t)). \tag{16}$$

Policy π lower bound process of all the network nodes is given by

$$\mathbf{\Delta}_{\pi}^{\text{LB}} = \{\mathbf{\Delta}_{\pi}^{\text{LB}}(t), t \in [0, \infty)\}. \tag{17}$$

The process $\mathbf{\Delta}_{\text{non-prmp-LGFS}}^{\text{LB}}$ is age lower bound of all policies in Π in the following sense.

Lemma 3. Suppose that the packet transmission times are NBU and *i.i.d.* across time and links, then for all $\mathcal{I}'$ satisfying $B_{ij} \geq 1$ for each $(i, j) \in \mathcal{L}$, and $\pi \in \Pi$

$$[\mathbf{\Delta}_{\text{non-prmp-LGFS}}^{\text{LB}}|\mathcal{I}'] \leq_{\text{st}} [\mathbf{\Delta}_{\pi}|\mathcal{I}']. \tag{18}$$

Proof. See Appendix C. □

We can now proceed to characterize the age performance of policy non-prmp-LGFS among policies in Π. Let X_{ij} denote the packet transmission time over the link (i, j), with mean $\mathbb{E}[X_{ij}] = \mathbb{E}[X] < \infty$ for all $(i, j) \in \mathcal{L}$. We use Lemma 3 to prove the following theorem.

Theorem 4. Suppose that the packet transmission times are NBU and *i.i.d.* across time and links, then for all $\mathcal{I}'$ satisfying $B_{ij} \geq 1$ for each $(i, j) \in \mathcal{L}$

$$\min_{\pi\in\Pi}[\bar{\Delta}_{j,\pi}|\mathcal{I}'] \leq [\bar{\Delta}_{j,\text{non-prmp-LGFS}}|\mathcal{I}'] \leq \min_{\pi\in\Pi}[\bar{\Delta}_{j,\pi}|\mathcal{I}'] + (2k-1)^{+}\mathbb{E}[X],\ \forall j \in \mathcal{H}_k, \forall k \geq 0, \tag{19}$$

where $(x)^+ = \max\{0, x\}$ and $\bar{\Delta}_{j,\pi} = \limsup_{T\to\infty} \frac{\mathbb{E}[\int_0^T \Delta_{j,\pi}(t)dt]}{T}$ is the average age at node j under policy π.

Proof. See Appendix E. □

Theorem 4 tells us that for arbitrary number n, packet generation times $(s_1, s_2, \ldots, s_n)$ and arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ at node 0, and buffer sizes $(B_{ij} \geq 1, (i, j) \in \mathcal{L})$, the non-prmp-LGFS policy is within a constant age gap from the optimum average age among policies in Π.

A. Simulation Results

We now provide a simulation result to illustrate the age performance of different policies when the transmission times are NBU. We consider a tandem network consisting of 6 nodes as shown in Fig. 8. Figure 9 illustrates the average age at node 5 under gamma transmission time distributions at each link with different shape parameter β, where the buffer sizes are 1. The mean of the gamma transmission time

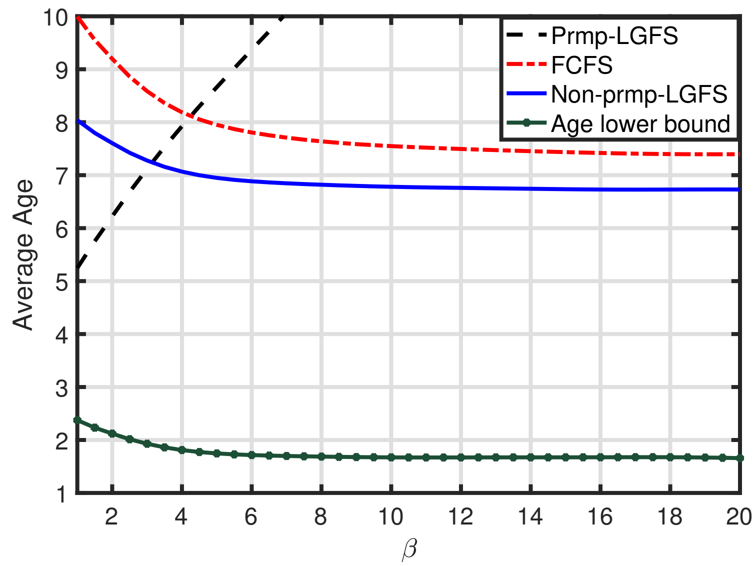


Figure 9: Average age at node 5 under gamma transmission time distributions at each link with different shape parameter β

distributions at each link are normalized to 1. The inter-generation times are *i.i.d.* Erlang-2 distribution with mean $1/\lambda = 1/3$. The time difference $(a_{i0} - s_i)$ between packet generation and arrival to node 0 is Zero. Note that the average age of the FCFS policy with infinite buffer sizes is extremely high in this case and hence is not plotted in this figure. The "Age lower bound" curve is generated by using $\frac{\int_0^T \Delta^{LB}_{5,\text{non-prmp-LGFS}(t)}}{T}$ when the buffer sizes are 1 which, according to Lemma 3, is a lower bound of the optimum average age at node 5. We can observe that the gap between the "Age lower bound" curve and the average age of the non-prmp-LGFS policy at node 5 is no larger than $9E[X] = 9$, which agrees with Theorem 4. In addition, we can observe that prmp-LGFS policy achieves the best age performance among all plotted policies when $\beta = 1$. This is because a gamma distribution with shape parameter $\beta = 1$ is an exponential distribution. Thus, age-optimality can be achieved in this case by policy prmp-LGFS as stated in Theorem 1. However, as can be seen in the figure, the average age at node 5 of the prmp-LGFS policy blows up as the shape parameter β increases and the non-prmp-LGFS policy achieves the best age performance among all plotted policies when $\beta > 3$. The reason of this phenomenon is as follows: As β increases, the variance (variability) of normalized gamma distribution decreases. Hence, when a packet is preempted, the service time of a new packet is probably longer than the remaining service time of the preempted packet. Because the generation rate is high, packet preemption happens frequently, which leads to infrequent packet delivery and increase the age. This phenomenon occurs heavily at the first link (link (0,1)) which, in turn, affects the age at the subsequent nodes.

VI. Conclusion

In this paper, we made the first attempt to minimize the age-of-information in multihop interference free networks. We considered general system settings including arbitrary network topology, packet generation and arrival times to node 0, and queue buffer sizes. A number of scheduling policies are developed and are proven to be age-optimal in stochastic ordering sense for (i) exponential transmission times and general policy space, (ii) general transmission times and restrictive policy space (non-preemptive work-conserving policies). These optimality results not only hold for the age processes, but also for any non-decreasing functional of the age processes. Finally, we investigated the class of NBU packet transmission time distributions in a somewhat more restrictive network topology and showed that the non-preemptive LGFS policy is near age-optimal.

REFERENCES

[1] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Age-optimal information updates in multihop networks," in *Proc. IEEE ISIT*, June 2017, pp. 576–580.

[2] B. Adelberg, H. Garcia-Molina, and B. Kao, "Applying update streams in a soft real-time database system," in *ACM SIGMOD Record*, 1995, vol. 24, pp. 245–256.

[3] J. Cho and H. Garcia-Molina, "Synchronizing a database to improve freshness," in *ACM SIGMOD Record*, 2000, vol. 29, pp. 117–128.

[4] L. Golab, T. Johnson, and V. Shkapenyuk, "Scheduling updates in a real-time stream warehouse," in *Proc. IEEE 25th Int'l Conf. Data Eng. (ICDE)*, March 2009, pp. 1207–1210.

[5] S. Kaul, R. D. Yates, and M. Gruteser, "Real-time status: How often should one update?," in *Proc. IEEE INFOCOM*, 2012, pp. 2731–2735.

[6] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Optimizing data freshness, throughput, and delay in multi-server information-update systems," in *Proc. IEEE ISIT*, July 2016, pp. 2569–2573.

[7] A. M. Bedewy, Y. Sun, and N. B. Shroff, "Minimizing the age of the information through queues," submitted to *IEEE Trans. Inf. Theory*, 2017, https://arxiv.org/abs/1709.04956.

[8] G. R. Gupta and N. B. Shroff, "Delay analysis and optimality of scheduling policies for multihop wireless networks," *IEEE/ACM Trans. Netw.*, vol. 19, no. 1, pp. 129–141, 2011.

[9] L. Tassiulas and A. Ephremides, "Dynamic scheduling for minimum delay in tandem and parallel constrained queueing models," *Ann. Oper. Res.*, vol. 48, no. 4, pp. 333–355, 1994.

[10] R. D. Yates and S. Kaul, "Real-time status updating: Multiple sources," in *Proc. IEEE ISIT*, July 2012, pp. 2666–2670.

[11] M. Costa, M. Codreanu, and A. Ephremides, "Age of information with packet management," in *Proc. IEEE ISIT*, June 2014, pp. 1583–1587.

[12] M. Costa, M. Codreanu, and A. Ephremides, "On the age of information in status update systems with packet management," *IEEE Trans. Inf. Theory*, vol. 62, no. 4, pp. 1897–1910, April 2016.

[13] N. Pappas, J. Gunnarsson, L. Kratz, M. Kountouris, and V. Angelakis, "Age of information of multiple sources with queue management," in *Proc. IEEE ICC*, June 2015, pp. 5935–5940.

[14] B. T. Bacinoglu, E. T. Ceran, and E. Uysal-Biyikoglu, "Age of information under energy replenishment constraints," in *Proc. ITA*, Feb. 2015.

[15] R. D. Yates, "Lazy is timely: Status updates by an energy harvesting source," in *Proc. IEEE ISIT*, 2015.

[16] S. K. Kaul, R. D. Yates, and M. Gruteser, "Status updates through queues," in *2012 46th Annu. Conf. Inf. Sci. Syst.*, March 2012, pp. 1–6.

[17] R. D. Yates and S. K. Kaul, "The age of information: Real-time status updating by multiple sources," submitted to *IEEE Trans. Inf. Theory*, 2016, https://arxiv.org/abs/1608.08622.

[18] E. Najm and R. Nasser, "Age of information: The gamma awakening," in *Proc. IEEE ISIT*, July 2016, pp. 2574–2578.

[19] L. Huang and E. Modiano, "Optimizing age-of-information in a multi-class queueing system," in *Proc. IEEE ISIT*, June 2015, pp. 1681–1685.

[20] K. Chen and L. Huang, "Age-of-information in the presence of error," in *Proc. IEEE ISIT*, July 2016, pp. 2579–2583.

[21] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," in *Proc. IEEE INFOCOM*, April 2016.

[22] Y. Sun, E. Uysal-Biyikoglu, R. D. Yates, C. E. Koksal, and N. B. Shroff, "Update or wait: How to keep your data fresh," *IEEE Trans. Inf. Theory*, vol. PP, no. 99, pp. 1–1, 2017.

[23] Q. He, D. Yuan, and A. Ephremides, "Optimal link scheduling for age minimization in wireless systems," *IEEE Trans. Inf. Theory*, vol. PP, no. 99, pp. 1–1, 2017.

[24] Y. Sun, Y. Polyanskiy, and E. Uysal-Biyikoglu, "Remote estimation of the Wiener process over a channel with random delay," in *Proc. IEEE ISIT*, 2017.

[25] J. Selen, Y. Nazarathy, L. LH Andrew, and H. L. Vu, "The age of information in gossip networks," in *International Conference on Analytical and Stochastic Modeling Techniques and Applications*. Springer, 2013, pp. 364–379.

[26] A. Araf and S. Ulukus, "Age-minimal transmission in energy harvesting two-hop networks," CoRR, abs/1704.08679, Apr 2017.

[27] C. Joo and A. Eryilmaz, "Wireless scheduling for information freshness and synchrony: Drift-based design and heavy-traffic analysis," in *2017 15th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks (WiOpt)*, May 2017, pp. 1–8.

[28] M. Shaked and J. G. Shanthikumar, *Stochastic orders*, Springer Science & Business Media, 2007.

[29] Y. Sun, C. E. Koksal, and N. B. Shroff, "On delay-optimal scheduling in queueing systems with replications," CoRR, abs/1603.07322, Jan. 2016.

APPENDIX A
PROOF OF THEOREM 1

Let us define the system state of a policy π:

Definition A.1. At any time t, the *system state* of policy π is specified by $\mathbf{U}_\pi(t) = (U_{0,\pi}(t), U_{2,\pi}(t), \ldots, U_{N-1,\pi}(t))$, where $U_{j,\pi}(t)$ is the generation time of the freshest packet that arrived at node j by time t. Let $\{\mathbf{U}_\pi(t), t \in [0,\infty)\}$ be the state process of policy π, which is assumed to be right-continuous. For notational simplicity, let policy P represent the preemptive LGFS policy.

The key step in the proof of Theorem 1 is the following lemma, where we compare policy P with any work-conserving policy π.

Lemma 5. Suppose that $\mathbf{U}_P(0^-) = \mathbf{U}_\pi(0^-)$ for all work conserving policies π, then for all $\mathcal{I}$,

$$[\{\mathbf{U}_P(t), t \in [0,\infty)\}|\mathcal{I}] \geq_{\text{st}} [\{\mathbf{U}_\pi(t), t \in [0,\infty)\}|\mathcal{I}]. \tag{20}$$

We use coupling and forward induction to prove Lemma 5. For any work-conserving policy π, suppose that stochastic processes $\widetilde{\mathbf{U}}_P(t)$ and $\widetilde{\mathbf{U}}_\pi(t)$ have the same distributions with $\mathbf{U}_P(t)$ and $\mathbf{U}_\pi(t)$, respectively. The state processes $\widetilde{\mathbf{U}}_P(t)$ and $\widetilde{\mathbf{U}}_\pi(t)$ are coupled in the following manner: If a packet is delivered from node i to node j at time t as $\widetilde{\mathbf{U}}_P(t)$ evolves in policy P, then there exists a packet delivery from node i to node j at time t as $\widetilde{\mathbf{U}}_\pi(t)$ evolves in policy π. Such a coupling is valid since the transmission time is exponentially distributed and thus memoryless. Moreover, policy P and policy π have identical packet generation times $(s_1, s_2, \ldots, s_n)$ at the external source and packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ to node 0. According to Theorem 6.B.30 in [28], if we can show

$$\mathbb{P}[\widetilde{\mathbf{U}}_P(t) \geq \widetilde{\mathbf{U}}_\pi(t), t \in [0,\infty)|\mathcal{I}] = 1, \tag{21}$$

then (20) is proven.

To ease the notational burden, we will omit the tildes in this proof on the coupled versions and just use $\mathbf{U}_P(t)$ and $\mathbf{U}_\pi(t)$. Next, we use the following lemmas to prove (21):

Lemma 6. Suppose that under policy P, $\mathbf{U}'_P$ is obtained by a packet delivery over the link (i, j) in the system whose state is $\mathbf{U}_P$. Further, suppose that under policy π, $\mathbf{U}'_\pi$ is obtained by a packet delivery over the link (i, j) in the system whose state is $\mathbf{U}_\pi$. If

$$\mathbf{U}_P \geq \mathbf{U}_\pi, \tag{22}$$

then,

$$\mathbf{U}'_P \geq \mathbf{U}'_\pi. \tag{23}$$

Proof. Let s_P and s_π denote the generation times of the packets that are delivered over the link (i, j) under policy P and policy π, respectively. From the definition of the system state, we can deduce that

$$\begin{aligned} U'_{j,P} &= \max\{U_{j,P}, s_P\}, \\ U'_{j,\pi} &= \max\{U_{j,\pi}, s_\pi\}. \end{aligned} \tag{24}$$

Hence, we have two cases:

Case 1: If $s_P \geq s_\pi$. From (22), we have

$$U_{j,P} \geq U_{j,\pi}. \tag{25}$$

Also, $s_P \geq s_\pi$, together with (24) and (25) imply

$$U'_{j,P} \geq U'_{j,\pi}. \tag{26}$$

Since there is no packet delivery under other links, we get

$$U'_{k,P} = U_{k,P} \geq U_{k,\pi} = U'_{k,\pi}, \quad \forall k \neq j. \tag{27}$$

Hence, we have

$$\mathbf{U}'_P \geq \mathbf{U}'_\pi. \tag{28}$$

Case 2: If $s_P < s_\pi$. By the definition of the system state, $s_P \leq U_{i,P}$ and $s_\pi \leq U_{i,\pi}$. Then, using $U_{i,P} \geq U_{i,\pi}$, we obtain

$$s_P < s_\pi \leq U_{i,\pi} \leq U_{i,P}. \tag{29}$$

Because $s_P < U_{i,P}$, policy P is sending a stale packet on link (i, j). By the definition of policy P, this happens only when all packets generated after s_P in the queue of the link (i, j) have been delivered to node j. Since $s_\pi \leq U_{i,P}$, node i has already received a packet (say packet w) generated no earlier than s_π in policy P. Because $s_P < s_\pi$, packet w is generated after s_P. Hence, packet w must have been delivered to node j in policy P such that

$$s_\pi \leq U_{j,P}. \tag{30}$$

Also, from (22), we have

$$U_{j,\pi} \leq U_{j,P}. \tag{31}$$

Combining (30) and (31) with (24), we obtain

$$U'_{j,P} \geq U'_{j,\pi}. \tag{32}$$

Since there is no packet delivery under other links, we get

$$U'_{k,P} = U_{k,P} \geq U_{k,\pi} = U'_{k,\pi}, \quad \forall k \neq j. \tag{33}$$

Hence, we have

$$\mathbf{U}'_P \geq \mathbf{U}'_\pi, \tag{34}$$

which complete the proof. □

Lemma 7. Suppose that under policy P, $\mathbf{U}'_P$ is obtained by the arrival of a new packet to node 0 in the system whose state is $\mathbf{U}_P$. Further, suppose that under policy π, $\mathbf{U}'_\pi$ is obtained by the arrival of a new packet to node 0 in the system whose state is $\mathbf{U}_\pi$. If

$$\mathbf{U}_P \geq \mathbf{U}_\pi, \tag{35}$$

then,

$$\mathbf{U}'_P \geq \mathbf{U}'_\pi. \tag{36}$$

Proof. Let s denote the generation time of the new arrived packet. From the definition of the system state, we can deduce that

$$\begin{aligned} U'_{0,P} &= \max\{U_{0,P}, s\}, \\ U'_{0,\pi} &= \max\{U_{0,\pi}, s\}. \end{aligned} \tag{37}$$

Combining this with (35), we obtain

$$U'_{0,P} \geq U'_{0,\pi}. \tag{38}$$

Since there is no packet delivery under other links, we get

$$U'_{k,P} = U_{k,P} \geq U_{k,\pi} = U'_{k,\pi}, \quad \forall k \neq 0. \tag{39}$$

Hence, we have

$$\mathbf{U}'_P \geq \mathbf{U}'_\pi, \tag{40}$$

which complete the proof. □

Proof of Lemma 5. For any sample path, we have that $\mathbf{U}_P(0^-) = \mathbf{U}_\pi(0^-)$. This, together with Lemma 6 and Lemma 7, implies that

$$[\mathbf{U}_P(t)|\mathcal{I}] \geq [\mathbf{U}_\pi(t)|\mathcal{I}],$$

holds for all $t \in [0, \infty)$. Hence, (21) holds which implies (20) by Theorem 6.B.30 in [28]. This completes the proof. □

Proof of Theorem 1. According to Lemma 5, we have

$$[\{\mathbf{U}_P(t), t \in [0,\infty)\}|\mathcal{I}] \geq_{\text{st}} [\{\mathbf{U}_\pi(t), t \in [0,\infty)\}|\mathcal{I}],$$

holds for all work-conserving policies π, which implies

$$[\{\mathbf{\Delta}_P(t), t \in [0,\infty)\}|\mathcal{I}] \leq_{\text{st}} [\{\mathbf{\Delta}_\pi(t), t \in [0,\infty)\}|\mathcal{I}],$$

holds for all work-conserving policies π.

Finally, transmission idling only postpones the delivery of fresh packets. Therefore, the age under non-work-conserving policies will be greater. As a result,

$$[\{\mathbf{\Delta}_P(t), t \in [0,\infty)\}|\mathcal{I}] \leq_{\text{st}} [\{\mathbf{\Delta}_\pi(t), t \in [0,\infty)\}|\mathcal{I}],$$

holds for all $\pi \in \Pi$. This completes the proof. □

APPENDIX B
PROOF OF THEOREM 2

This proof is similar to that of Theorem 1. The difference between this proof and the proof of Theorem 1 is that policy π cannot be a preemptive policy here. We will use the same definition of the system state of policy π used in Theorem 1. For notational simplicity, let policy P represent the non-preemptive LGFS policy.

The key step in the proof of Theorem 2 is the following lemma, where we compare policy P with an arbitrary policy $\pi \in \Pi_{npwc}$.

Lemma 8. Suppose that $\mathbf{U}_P(0^-) = \mathbf{U}_\pi(0^-)$ for all $\pi \in \Pi_{npwc}$, then for all $\mathcal{I}$,

$$[\{\mathbf{U}_P(t), t \in [0,\infty)\}|\mathcal{I}] \geq_{\text{st}} [\{\mathbf{U}_\pi(t), t \in [0,\infty)\}|\mathcal{I}]. \tag{41}$$

We use coupling and forward induction to prove Lemma 8. For any work-conserving policy π, suppose that stochastic processes $\widetilde{\mathbf{U}}_P(t)$ and $\widetilde{\mathbf{U}}_\pi(t)$ have the same distributions with $\mathbf{U}_P(t)$ and $\mathbf{U}_\pi(t)$, respectively. The state processes $\widetilde{\mathbf{U}}_P(t)$ and $\widetilde{\mathbf{U}}_\pi(t)$ are coupled in the following manner: If a packet is delivered from node i to node j at time t as $\widetilde{\mathbf{U}}_P(t)$ evolves in policy prmp-LGFS, then there exists a packet delivery from node i to node j at time t as $\widetilde{\mathbf{U}}_\pi(t)$ evolves in policy π. Such a coupling is valid since the transmission time distribution at each link is identical under all policies. Moreover, policy π can not be either preemptive or non-work-conserving policy, and both policies have the same packets generation times $(s_1, s_2, \ldots, s_n)$ at the exterenal source and packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ to node 0. According to Theorem 6.B.30 in [28], if we can show

$$\mathbb{P}[\widetilde{\mathbf{U}}_P(t) \geq \widetilde{\mathbf{U}}_\pi(t), t \in [0,\infty)|\mathcal{I}] = 1, \tag{42}$$

then (41) is proven.

To ease the notational burden, we will omit the tildes henceforth on the coupled versions and just use $\mathbf{U}_P(t)$ and $\mathbf{U}_\pi(t)$.

Next, we use the following lemmas to prove (42):

Lemma 9. Suppose that under policy P, $\mathbf{U}_P(\nu)$ is obtained by a packet delivery over the link (i, j) at time ν in the system whose state is $\mathbf{U}_P(\nu^-)$. Further, suppose that under policy π, $\mathbf{U}_\pi(\nu)$ is obtained by a packet delivery over the link (i, j) at time ν in the system whose state is $\mathbf{U}_\pi(\nu^-)$. If

$$\mathbf{U}_P(t) \geq \mathbf{U}_\pi(t), \tag{43}$$

holds for all $t \in [0, \nu^-]$, then

$$\mathbf{U}_P(\nu) \geq \mathbf{U}_\pi(\nu). \tag{44}$$

Proof. Let s_P and s_π denote the packet indexes and the generation times of the delivered packets over the link (i, j) at time ν under policy P and policy π, respectively. From the definition of the system state, we can deduce that

$$\begin{aligned} U_{j,P}(\nu) &= \max\{U_{j,P}(\nu^-), s_P\}, \\ U_{j,\pi}(\nu) &= \max\{U_{j,\pi}(\nu^-), s_\pi\}. \end{aligned} \tag{45}$$

Hence, we have two cases:

Case 1: If $s_P \geq s_\pi$. From (43), we have

$$U_{j,P}(\nu^-) \geq U_{j,\pi}(\nu^-). \tag{46}$$

By $s_P \geq s_\pi$, (45), and (46), we have

$$U_{j,P}(\nu) \geq U_{j,\pi}(\nu). \tag{47}$$

Since there is no packet delivery under other links, we get

$$U_{k,P}(\nu) = U_{k,P}(\nu^-) \geq U_{k,\pi}(\nu^-) = U_{k,\pi}(\nu), \quad \forall k \neq j. \tag{48}$$

Hence, we have

$$\mathbf{U}_P(\nu) \geq \mathbf{U}_\pi(\nu). \tag{49}$$

Case 2: If $s_P < s_\pi$. Let a_π represent the arrival time of packet s_π to node i under policy π. The transmission starting time of the delivered packets over the link (i, j) is denoted by τ under both policies. Apparently, $a_\pi \leq \tau \leq \nu^-$. Since packet s_π arrived to node i at time a_π in policy π, we get

$$s_\pi \leq U_{i,\pi}(a_\pi). \tag{50}$$

From (43), we obtain

$$U_{i,\pi}(a_\pi) \leq U_{i,P}(a_\pi). \tag{51}$$

Combining (50) and (51), yields

$$s_\pi \leq U_{i,P}(a_\pi). \tag{52}$$

Hence, in policy P, node i has a packet with generation time no smaller than s_π by the time a_π. Because the $U_{i,P}(t)$ is a non-decreasing function of t and $a_\pi \le \tau$, we have

$$U_{i,P}(a_\pi) \le U_{i,P}(\tau). \tag{53}$$

Then, (52) and (53) imply

$$s_\pi \le U_{i,P}(\tau). \tag{54}$$

Since $s_P < s_\pi$, (54) tells us

$$s_P < U_{i,P}(\tau), \tag{55}$$

and hence policy P is sending a stale packet on link (i,j). By the definition of policy P, this happens only when all packets generated after s_P in the queue of the link (i,j) have been delivered to node j by time τ. In addition, (54) tells us that by time τ, node i has already received a packet (say packet h) generated no earlier than s_π in policy P. By $s_P < s_\pi$, packet h is generated after s_P. Hence, packet h must have been delivered to node j by time τ in policy P such that

$$s_\pi \le U_{j,P}(\tau). \tag{56}$$

Because the $U_{j,P}(t)$ is a non-decreasing function of t, and $\tau \le \nu^-$, (56) implies

$$s_\pi \le U_{j,P}(\nu^-). \tag{57}$$

Also, from (43), we have

$$U_{j,\pi}(\nu^-) \le U_{j,P}(\nu^-). \tag{58}$$

Combining (57) and (58) with (45), we obtain

$$U_{j,P}(\nu) \ge U_{j,\pi}(\nu). \tag{59}$$

Since there is no packet delivery under other links, we get

$$U_{k,P}(\nu) = U_{k,P}(\nu^-) \ge U_{k,\pi}(\nu^-) = U_{k,\pi}(\nu), \quad \forall k \ne j. \tag{60}$$

Hence, we have

$$\mathbf{U}_P(\nu) \ge \mathbf{U}_\pi(\nu), \tag{61}$$

which complete the proof. □

Lemma 10. Suppose that under policy P, $\mathbf{U}'_P$ is obtained by the arrival of a new packet to node 0 in the system whose state is $\mathbf{U}_P$. Further, suppose that under policy π, $\mathbf{U}'_\pi$ is obtained by the arrival of a new packet to node 0 in the system whose state is $\mathbf{U}_\pi$. If

$$\mathbf{U}_P \ge \mathbf{U}_\pi, \tag{62}$$

then,

$$\mathbf{U}'_P \ge \mathbf{U}'_\pi. \tag{63}$$

Proof. The proof of Lemma 10 is similar to that of Lemma 7, and hence is not provided. □

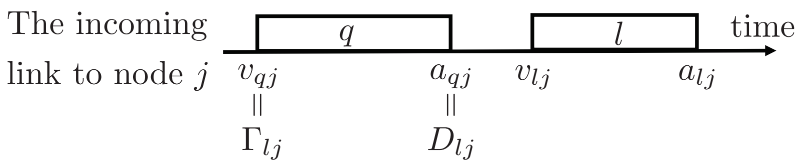


Figure 10: An illustration of v_{lj}, a_{lj}, Γ_{lj} and D_{lj}. We consider the incoming link to node j, and $s_q > s_l$. The transmission starting time over this link and the arrival time to node j of packet q are earlier than those of packet l. Thus, we have $\Gamma_{lj} = v_{qj}$ and $D_{lj} = a_{qj}$.

Proof of Lemma 8. For any sample path, we have that $\mathbf{U}_P(0^-) = \mathbf{U}_\pi(0^-)$. This, together with Lemma 9 and Lemma 10, implies that

$$[\mathbf{U}_P(t)|\mathcal{I}] \geq [\mathbf{U}_\pi(t)|\mathcal{I}],$$

holds for all $t \in [0, \infty)$. Hence, (42) holds which implies (41) by Theorem 6.B.30 in [28]. This completes the proof. $\square$

Proof of Theorem 2. According to Lemma 8, we have

$$[\{\mathbf{U}_P(t), t \in [0,\infty)\}|\mathcal{I}] \geq_{\text{st}} [\{\mathbf{U}_\pi(t), t \in [0,\infty)\}|\mathcal{I}],$$

holds for all $\pi \in \Pi_{npwc}$, which implies

$$[\{\boldsymbol{\Delta}_P(t), t \in [0,\infty)\}|\mathcal{I}] \leq_{\text{st}} [\{\boldsymbol{\Delta}_\pi(t), t \in [0,\infty)\}|\mathcal{I}],$$

holds for all $\pi \in \Pi_{npwc}$. This completes the proof. $\square$

Appendix C
Proof of Lemma 3

For notation simplicity, let policy P represent the non-prmp-LGFS policy and LB represent its lower bound. We need to define the following parameters: Recall that v_{lj} denotes the transmission starting time of packet l over the incoming link to node j and a_{lj} denotes the arrival time of packet l to node j. We define Γ_{lj} and D_{lj} as

$$\Gamma_{lj} = \min_{q \geq l} \{v_{qj}\}, \tag{64}$$

$$D_{lj} = \min_{q \geq l} \{a_{qj}\}, \tag{65}$$

where Γ_{lj} and D_{lj} are the smallest transmission starting time over the incoming link to node j and arrival time to node j, respectively, of all packets that are fresher than the packet l. An illustration of these parameters is provided in Fig. 10. Define the vectors $\mathbf{\Gamma}_j = (\Gamma_{1j}, \ldots, \Gamma_{nj})$, and $\mathbf{D}_j = (D_{1j}, \ldots, D_{nj})$. All these quantities are functions of the scheduling policy π (except the packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ to node 0 which are invariant of the the scheduling policy).

To prove (18), we need to show that

$$[\mathbf{\Gamma}_1(\text{LB}), \ldots, \mathbf{\Gamma}_{N-1}(\text{LB})|\mathcal{I}'] \leq_{\text{st}} [\mathbf{D}_1(\pi), \ldots, \mathbf{D}_{N-1}(\pi)|\mathcal{I}'], \tag{66}$$

holds for all $\pi \in \Pi$. To prove (66), we need the following lemma.

Lemma 11. For any link $(i, j) \in \mathcal{L}$ and policy $\pi \in \Pi$, if (i) the packet transmission times are NBU, and (ii) $\Gamma_i(\text{LB}) \leq D_i(\pi)$, then

$$[\mathbf{\Gamma}_j(\text{LB})|\mathcal{I}'] \leq_{\text{st}} [\mathbf{D}_j(\pi)|\mathcal{I}'], \tag{67}$$

holds for all $\mathcal{I}'$ satisfying $B_{ij} \geq 1$.

Proof. See Appendix D □

Proof of (66). Pick any policy $\pi \in \Pi$. We use m_j^k to represent the index of the j-th node in $\mathcal{H}_k$[2]. We prove (18) using Theorem 6.B.3 of [28] into two steps

Step 1: Consider node m_1^1. Since the packet arrival times $(a_{10}, \ldots, a_{n0})$ to node 0 are invariant of the scheduling policy, both conditions of Lemma 11 are satisfied and we can apply it on the link $(0, m_1^1)$ to obtain

$$[\mathbf{\Gamma}_{m_1^1}(\text{LB})|\mathcal{I}'] \leq_{\text{st}} [\mathbf{D}_{m_1^1}(\pi)|\mathcal{I}']. \tag{68}$$

Step 2: Consider a node m_J^K, where $J \geq 2$ if $K = 1$ and $J \geq 1$ if $K \geq 2$. We need to prove that

$$\begin{aligned}
&[\mathbf{\Gamma}_{m_J^K}(\text{LB})|\mathcal{I}', \mathbf{\Gamma}_{m_1^1}(\text{LB}) = \gamma_{m_1^1}, \mathbf{\Gamma}_{m_2^1}(\text{LB}) = \gamma_{m_2^1}, \ldots, \mathbf{\Gamma}_{m_{J-1}^K}(\text{LB}) = \gamma_{m_{J-1}^K}] \\
&\leq_{\text{st}} [\mathbf{D}_{m_J^K}(\pi)|\mathcal{I}', \mathbf{D}_{m_1^1}(\pi) = \mathbf{d}_{m_1^1}, \mathbf{D}_{m_2^1}(\pi) = \mathbf{d}_{m_2^1}, \ldots, \mathbf{D}_{m_{J-1}^K}(\pi) = \mathbf{d}_{m_{J-1}^K}], \\
&\qquad\qquad \text{whenever} \quad \gamma_{m_j^k} \leq \mathbf{d}_{m_j^k}, m_j^k = m_1^1, m_2^1, \ldots, m_{J-1}^K.
\end{aligned} \tag{69}$$

Note that node m_J^K receives data from some node in $\mathcal{H}_{K-1}$. Since the both conditions of Lemma 11 are satisfied in this case too, we can use it to prove (69).

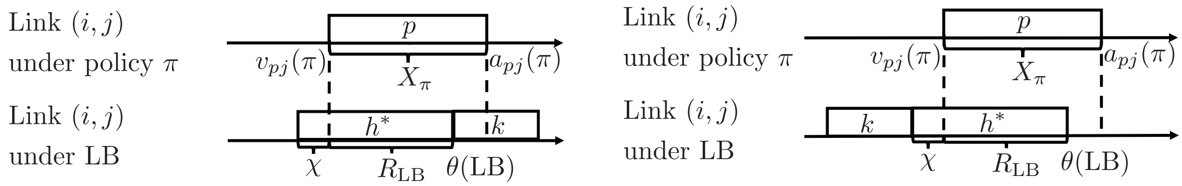
Finally, using (68) and (69) with Theorem 6.B.3 and Theorem 6.B.16.(c) of [28], we prove (66). This completes the proof. □

Proof of Lemma 3. We can deduce from (2) that the age process $\mathbf{\Delta}_\pi$ under any policy π is an increasing function of $[\mathbf{D}_1(\pi), \ldots, \mathbf{D}_{N-1}(\pi)]$. Moreover, we can deduce from (14) and (15) that the process $\mathbf{\Delta}_P^{\text{LB}}$ is an increasing function of $[\mathbf{\Gamma}_1(\text{LB}), \ldots, \mathbf{\Gamma}_{N-1}(\text{LB})]$. By using Theorem 6.B.16.(a) of [28], (18) follows directly from (66). This completes the proof. □

APPENDIX D
PROOF OF LEMMA 11

The proof of Lemma 11 is motivated by [29]. Note that if $j = 0$, then (67) holds trivially. In what follows we assume that $j \neq 0$. Also, note that if $j \in \mathcal{H}_1$ (i.e., $i = 0$), then we have $v_{lj}(\text{LB}) = v_{lj}(P)$

[2]The nodes in $\mathcal{H}_k$ are ordered rondamly.



(a) Case 1: Link (i,j) sends packet k after the time $\theta(\text{LB})$. (b) Case 2: Link (i,j) sends packet k before the time $\theta(\text{LB})$.

Figure 11: Illustration of packet transmissions under policy π and LB. In policy π, link (i,j) starts to send packet p at time $v_{pj}(\pi)$ and will complete its transmission at time $a_{pj}(\pi)$. Hence, the transmission duration of packet p is $[v_{pj}(\pi), a_{pj}(\pi)]$ in policy π. Under LB, link (i,j) starts sending packet h^* before time $v_{pj}(\pi)$ and becomes available to send a new packet at time $\theta(\text{LB}) < a_{pj}(\pi)$.

for all $l = 1, \ldots, n$ (from (14)), and the following argument still holds in this case too. We prove (67) using Theorem 6.B.3 of [28] into two steps.

Step 1: Consider packet 1. Define $l^* = \max\{l : a_{li}(\text{LB}) = \Gamma_{1i}(\text{LB})\}$, where $s_{l^*} \geq s_1$. From the construction of the LB and (64), $\Gamma_{1i}(\text{LB})$ is the arrival time of the first packet that arrives to node i under LB. Since the link (i,j) is idle by time $\Gamma_{1i}(\text{LB})$ and LB is constructed by a work-conserving policy, packet l^* will start its transmission under LB over the link (i,j) once it arrives to node i. Thus, from (64), we obtain

$$[\Gamma_1 j(\text{LB})|\mathcal{I}'] = [v_{l^*j}(\text{LB})|\mathcal{I}'] = [\Gamma_{1i}(\text{LB})|\mathcal{I}']. \tag{70}$$

Since we have $[\Gamma_{1i}(\text{LB})|\mathcal{I}'] \leq [D_{1i}(\pi)|\mathcal{I}']$ and $[D_{1i}(\pi)|\mathcal{I}'] \leq [D_{1j}(\pi)|\mathcal{I}']$, it follows that

$$[\Gamma_{1j}(\text{LB})|\mathcal{I}'] = [\Gamma_{1i}(\text{LB})|\mathcal{I}'] \leq [D_{1i}(\pi)|\mathcal{I}'] \leq [D_{1j}(\pi)|\mathcal{I}']. \tag{71}$$

Step 2: Consider a packet p, where $2 \leq p \leq n$. We suppose that, both in math description and in words, there is no packet with generation time greater than s_p has arrived to node j before packet p under policy π; otherwise, if there is a fresher packet y with $s_y > s_p$ and $a_{yj}(\pi) < a_{pj}(\pi)$, then we replace packet p by packet y in the following argument. We need to prove that

$$\begin{aligned} &[\Gamma_{pj}(\text{LB})|\mathcal{I}', \Gamma_{1j}(\text{LB}) = \gamma_1, \ldots, \Gamma_{(p-1)j}(\text{LB}) = \gamma_{p-1}] \\ &\leq_{\text{st}} [D_{pj}(\pi)|\mathcal{I}', D_{1j}(\pi) = d_1, \ldots, D_{(p-1)j}(\pi) = d_{p-1}], \\ &\qquad\qquad \text{whenever} \quad \gamma_l \leq d_l, l = 1, 2, \ldots, p-1. \end{aligned} \tag{72}$$

For notation simplicity, Define $\Gamma^{p-1} \triangleq \{\Gamma_{1j}(\text{LB}) = \gamma_1, \ldots, \Gamma_{(p-1)j}(\text{LB}) = \gamma_{p-1}\}$ and $D^{p-1} \triangleq \{D_{1j}(\pi) = d_1, \ldots, D_{(p-1)j}(\pi) = d_{p-1}\}$.

As illustrated in Fig. 11, suppose that under policy π, link (i,j) starts to send packet p at time $v_{pj}(\pi)$ and will complete its transmission at time $a_{pj}(\pi)$. Under LB, define $h^* = \text{argmax}_h\{v_{hj}(\text{LB}) : v_{hj}(\text{LB}) \leq v_{pj}(\pi)\}$ as the index of the last packet that its transmission starts over the link (i,j) before time $v_{pj}(\pi)$. Note that the link (i,j) under LB becomes busy after time $v_{h^*j}(\text{LB})$ in sending a virtual copy of packet h^* for a duration equal to the transmission time of packet h^* at the link (i,j). Suppose that under LB, link (i,j) has spent χ ($\chi \geq 0$) seconds on sending the virtual copy of packet h^* before time $v_{pj}(\pi)$.

Let R_{LB} denote the remaining transmission time of the virtual copy of packet h^* after time $v_{pj}(\pi)$ at the link (i,j) under LB. Hence, link (i,j) becomes available to send a new packet at time $v_{pj}(\pi)+R_{\text{LB}}$. Let $X_\pi = a_{pj}(\pi) - v_{pj}(\pi)$ denote the transmission time of packet p under policy π and $X_{\text{LB}} = \chi + R_{\text{LB}}$ denote the transmission time of the virtual copy of packet h^*. Then, the CCDF of R_{LB} is given by

$$\mathbb{P}[R_{\text{LB}} > s] = \mathbb{P}[X_{\text{LB}} - \chi > s | X_{\text{LB}} > \chi]. \tag{73}$$

Because the packet transmission times are NBU, we can obtain that for all $s, \chi \geq 0$

$$\mathbb{P}[X_{\text{LB}} - \chi > s | X_{\text{LB}} > \chi] = \mathbb{P}[X_\pi - \chi > s | X_\pi > \chi] \leq \mathbb{P}[X_\pi > s]. \tag{74}$$

By combining (73) and (74), we obtain

$$R_{\text{LB}} \leq_{\text{st}} X_\pi, \tag{75}$$

which implies

$$v_{pj}(\pi) + R_{\text{LB}} \leq_{\text{st}} v_{pj}(\pi) + X_\pi = a_{pj}(\pi). \tag{76}$$

From (76), we can deduce that link (i,j) becomes available to send a new packet under LB at a time that is stochastically smaller than the time $a_{pj}(\pi)$. Let $\theta(\text{LB})$ denote the time that link (i,j) becomes available to send a new packet under LB. According to (76), we have

$$\begin{aligned} &[\theta(\text{LB}) | \mathcal{I}', \Gamma^{p-1}] \leq_{\text{st}} [a_{pj}(\pi) | \mathcal{I}', D^{p-1}], \\ &\text{whenever} \quad \gamma_l \leq d_l, l = 1, 2, \ldots, p-1. \end{aligned} \tag{77}$$

It is important to note that, since we have $[\Gamma_{pi}(\text{LB})|\mathcal{I}'] \leq [D_{pi}(\pi)|\mathcal{I}']$, there is a packet with generation time greater than s_p is available to the link (i,j) before time $v_{pj}(\pi)$ under LB. At time $\theta(\text{LB})$, we have to possible cases under LB:

Case 1: link (i,j) starts to send a fresh packet k with $k \geq p$ at time $\theta(\text{LB})$ under LB, as shown in Fig. 11(a). Hence we obtain

$$\begin{aligned} [v_{kj}(\text{LB}) | \mathcal{I}', \Gamma^{p-1}] = [\theta(\text{LB}) | \mathcal{I}', \Gamma^{p-1}] &\leq_{\text{st}} [a_{pj}(\pi) | \mathcal{I}', D^{p-1}] \\ \text{whenever} \quad \gamma_l &\leq d_l, l = 1, 2, \ldots, p-1. \end{aligned} \tag{78}$$

Since $s_k \geq s_p$, (64) implies

$$[\Gamma_{pj}(\text{LB}) | \mathcal{I}', \Gamma^{p-1}] \leq [v_{kj}(\text{LB}) | \mathcal{I}', \Gamma^{p-1}]. \tag{79}$$

Since there is no packet with generation time greater than s_p has been arrived to node j before packet p under policy π, (65) implies

$$[D_{pj}(\pi) | \mathcal{I}', D^{p-1}] = [a_{pj}(\pi) | \mathcal{I}', D^{p-1}]. \tag{80}$$

By combining (78), (79), and (80), (72) follows.

Case 2: Link (i,j) starts to send a stale packet (with generation time smaller than s_p) or there is no packet transmission over the link (i,j) at time $\theta(\text{LB})$ under LB. Since LB is constructed by a work-conserving policy, the packets in the link (i,j) queue under LB are served in LGFS manner, and a packet with generation time greater than s_p is available to the link (i,j) before time $v_{pj}(\pi)$ under LB, the link (i,j) must have sent a fresh packet k with $k \geq p$ before time $\theta(\text{LB})$, as shown in Fig. 11(b). Hence, we have

$$\begin{aligned}[v_{kj}(\text{LB})|\mathcal{I}', \Gamma^{p-1}] \leq [\theta(\text{LB})|\mathcal{I}', \Gamma^{p-1}] \leq_{\text{st}} [a_{pj}(\pi)|\mathcal{I}', D^{p-1}]\\ \text{whenever} \quad \gamma_l \leq d_l, l = 1, 2, \ldots, p-1.\end{aligned} \tag{81}$$

Similar to Case 1, we can use (64), (65), and (81) to show that (72) holds in this case.

As we mentioned before, if there is a fresher packet y with $s_y > s_p$ and $a_{yj}(\pi) < a_{pj}(\pi)$, then we replace packet p by packet y in the previous argument to obtain

$$\begin{aligned}[\Gamma_{yj}(\text{LB})|\mathcal{I}', \Gamma^{p-1}] \leq [D_{yj}(\pi)|\mathcal{I}', D^{p-1}]\\ \text{whenever} \quad \gamma_l \leq d_l, l = 1, 2, \ldots, p-1.\end{aligned} \tag{82}$$

Observing that $s_y > s_p$, (64) implies

$$[\Gamma_{pj}(\text{LB})|\mathcal{I}', \Gamma^{p-1}] \leq [\Gamma_{yj}(\text{LB})|\mathcal{I}', \Gamma^{p-1}]. \tag{83}$$

Since $a_{yj}(\pi) < a_{pj}(\pi)$ and $s_y > s_p$, (65) implies

$$[D_{pj}(\pi)|\mathcal{I}', D^{p-1}] = [D_{yj}(\pi)|\mathcal{I}', D^{p-1}]. \tag{84}$$

By combining (82), (83), and (84), we can prove (72) in this case too. Finally, substitute (71) and (72) into Theorem 6.B.3 of [28], (67) is proven.

APPENDIX E
PROOF OF THEOREM 4

For notation simplicity, let policy P represent the non-prmp-LGFS policy and LB represent its lower bound. Note that (19) holds trivially at node 0. We prove Theorem 4 into two steps:

Step 1: Consider a node $j \in \mathcal{H}_1$. We identify the average gap between $\Delta^{\text{LB}}_{j,P}$ and $\Delta_{j,P}$. According to (15), we have $v_{lj}(\text{LB}) = v_{lj}(P)$ and $\Gamma_{lj}(\text{LB}) = \Gamma_{lj}(P)$ (recall the definition of Γ_{lj} from (64)) for all $l = 1, \ldots, n$. A packet l is said to be informative packet at node j under policy P if every packet starting its transmission over the link $(0,j)$ before packet l are staler than packet l, i.e., $s_q \leq s_l$ for all packet q satisfying $v_{qj}(P) \leq v_{lj}(P)$. The informative packets at node j under policy P are those packets affecting the process $\Delta^{\text{LB}}_{j,P}$ defined in (14). We use l_m to represent the index of the m-th informative packet at node j under policy P, where we have $s_{l_0} = s_0 = 0$. For example, in Fig. 12, the first informative packet at node j under policy P is packet 2, so $l_1 = 2$. Define $z_l = D_{lj}(P) - \Gamma_{lj}(\text{LB})$ (recall the definition of D_{lj} from (65)). Since we have $\Gamma_{lj}(\text{LB}) = \Gamma_{lj}(P)$ for all $l = 1, \ldots, n$, we have $z_l = D_{lj}(P) - \Gamma_{lj}(P) = X_{0j}$ (the packet that starts its transmission at time $\Gamma_{lj}(P)$ must arrive to node j at time $D_{lj}(P) = \Gamma_{lj}(P) + X_{0j}$, because each link can send only one packet at a time). Hence, we have $\mathbb{E}[z_l] = \mathbb{E}[X]$ for all $l = 1, \ldots, n$.

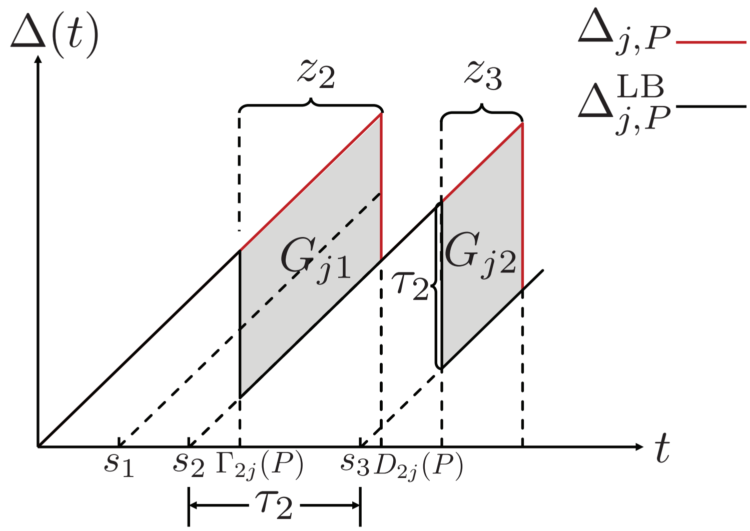


Figure 12: The evolution of $\Delta_{j,P}^{\mathrm{LB}}$ and $\Delta_{j,P}$.

We use $\{G_j(t), t \in [0, \infty)\}$ to denote the gap process between $\Delta_{j,P}^{\mathrm{LB}}$ and $\Delta_{j,P}$. The average age is given by

$$[\bar{G}_j|\mathcal{I}'] = \limsup_{T\to\infty} \frac{\int_0^T G_j(t)dt}{T}. \tag{85}$$

Let τ_m denote the inter-generation time between packet l_m and packet l_{m-1} (i.e., $\tau_m = s_{l_m} - s_{l_{m-1}}$), where $\tau = \{\tau_m, m \geq 1\}$. Define $N(T) = \max\{m : s_{l_m} \leq T\}$ as the number of informative packets at node j under policy P by time T. Note that $[0, s_{l_{N(T)}}] \subseteq [0, T]$, where the length of the interval $[0, s_{l_{N(T)}}]$ is $\sum_{m=1}^{N(T)} \tau_m$. Thus, we have

$$\sum_{m=1}^{N(T)} \tau_m \leq T. \tag{86}$$

The area defined by the integral in (85) can be decomposed into a sum of disjoint geometric parts. Observing Fig. 12, the area can be approximated to the concatenation of the parallelograms $G_{j1}, G_{j2}, \ldots$ (G_{jm}'s are highlighted in Fig. 12). Note that the parallelogram G_{jm} results from the transmitting of the informative packet l_m. As a result, each parallelogram G_{jm} comes after the time s_{l_m}. Since the observing time T is chosen arbitrary, when $T \geq s_{l_m}$, the total area of the parallelogram G_{jm} is accounted in the summation $\sum_{m=1}^{N(T)} G_{jm}$, while it may not be accounted in the integral $\int_0^T G_j(t)dt$. This implies that

$$\sum_{m=1}^{N(T)} G_{jm} \geq \int_0^T G_j(t)dt. \tag{87}$$

Combining (86) and (87), we get

$$\frac{\int_0^T G_j(t)dt}{T} \leq \frac{\sum_{m=1}^{N(T)} G_{jm}}{\sum_{m=1}^{N(T)} \tau_m}. \tag{88}$$

Then, take conditional expectation given τ and $N(T)$ on both sides of (88), we obtain

$$\frac{\mathbb{E}[\int_0^T G_j(t)dt|\tau, N(T)]}{T} \leq \frac{\mathbb{E}[\sum_{m=1}^{N(T)} G_{jm}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m} = \frac{\sum_{m=1}^{N(T)} \mathbb{E}[G_{jm}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m}, \tag{89}$$

where the second equality follows from the linearity of the expectation. From Fig. 12, G_{jm} can be calculated as

$$G_{jm} = \tau_m z_{l_m}. \tag{90}$$

substituting by (90) into (89), yields

$$\frac{\mathbb{E}[\int_0^T G_j(t)dt|\tau, N(T)]}{T} \leq \frac{\sum_{m=1}^{N(T)} \mathbb{E}[\tau_m z_{l_m}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m} = \frac{\sum_{m=1}^{N(T)} \tau_m \mathbb{E}[z_{l_m}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m}. \tag{91}$$

Note that the packet transmission times over the link $(0, j)$ are independent of the packet generation process. Thus, we have $\mathbb{E}[z_{l_m}|\tau, N(T)] = \mathbb{E}[X_{0j}] = \mathbb{E}[X]$ for all l_m. Substituting this into (91), yields

$$\frac{\mathbb{E}[\int_0^T G_j(t)dt|\tau, N(T)]}{T} \leq \frac{\sum_{m=1}^{N(T)} \tau_m \mathbb{E}[X]}{\sum_{m=1}^{N(T)} \tau_m} = \mathbb{E}[X], \tag{92}$$

by the law of iterated expectations, we have

$$\frac{\mathbb{E}[\int_0^T G_j(t)dt]}{T} \leq \mathbb{E}[X]. \tag{93}$$

Taking $\limsup$ of both sides of (93) when $T \to \infty$, yields

$$\limsup_{T\to\infty} \frac{\mathbb{E}[\int_0^T G_j(t)dt]}{T} \leq \mathbb{E}[X]. \tag{94}$$

Equation (94) tells us that the average gap between $\Delta_{j,P}^{\text{LB}}$ and $\Delta_{j,P}$ is no larger than $\mathbb{E}[X]$. Since, $\Delta_{j,P}^{\text{LB}}$ is a lower bound of $\Delta_{j,P}$, we obtain

$$[\bar{\Delta}_{j,P}^{\text{LB}}|\mathcal{I}'] \leq [\bar{\Delta}_{j,P}|\mathcal{I}'] \leq [\bar{\Delta}_{j,P}^{\text{LB}}|\mathcal{I}'] + \mathbb{E}[X], \tag{95}$$

where $\bar{\Delta}_{j,P}^{\text{LB}} = \limsup_{T\to\infty} \frac{\mathbb{E}[\int_0^T \Delta_{j,P}^{\text{LB}}(t)dt]}{T}$. From Lemma 3, we have for all $\mathcal{I}'$ satisfying $B_{ij} \geq 1$, and $\pi \in \Pi$

$$[\Delta_{j,P}^{\text{LB}}|\mathcal{I}'] \leq_{\text{st}} [\Delta_{j,\pi}|\mathcal{I}'], \tag{96}$$

which implies that

$$[\bar{\Delta}_{j,P}^{\text{LB}}|\mathcal{I}'] \leq [\bar{\Delta}_{j,\pi}|\mathcal{I}'], \tag{97}$$

holds for all $\pi \in \Pi$. As a result, we get

$$[\bar{\Delta}_{j,P}^{\text{LB}}|\mathcal{I}'] \leq \min_{\pi\in\Pi} [\bar{\Delta}_{j,\pi}|\mathcal{I}']. \tag{98}$$

Since policy non-prmp-LGFS is a feasible policy, we get

$$\min_{\pi\in\Pi} [\bar{\Delta}_{j,\pi}|\mathcal{I}'] \leq [\bar{\Delta}_{j,P}|\mathcal{I}']. \tag{99}$$

Combining (95), (98), and (99), we get

$$\min_{\pi\in\Pi}[\bar{\Delta}_{j,\pi}|\mathcal{I}'] \leq [\bar{\Delta}_{j,P}|\mathcal{I}'] \leq \min_{\pi\in\Pi}[\bar{\Delta}_{j,\pi}|\mathcal{I}'] + \mathbb{E}[X]. \tag{100}$$

Following the previous argument, we can show that (100) holds for all $j \in \mathcal{H}_1$.

Step 2: Consider a node $f \in \mathcal{H}_k$ with $k \geq 2$. We first identify the average gap between $\Delta^{\text{LB}}_{f,P}$ and $\Delta_{f,P}$. We use l_m to represent the index of the m-th informative packet at node f under LB. It is important to note that, in this case, the informative packets under LB are different from those under policy P because we may not have $\mathbf{\Gamma}_f(\text{LB}) = \mathbf{\Gamma}_f(P)$ (this is because the arrival times of the packets to the nodes in $\mathcal{H}_k$ with $k \geq 2$ are earlier under LB). For each link (i,j), define $R_{lj} = \Gamma_{lj} - D_{li}$ as the time spent in the queue of the link (i,j) by the first packet which has a generation time greater than s_l and arrives to node i at time D_{li} until this packet (or a fresher one) starts its transmission at the link (i,j). Let χ_{lj} $(\chi_{lj} \geq 0)$ denote the amount of time that the link (i,j) has spent on sending a packet being transmitted by the time D_{li}. Note that R_{lj} and χ_{lj} are functions of the scheduling policy π. Since policy P is a LGFS work-conserving policy and LB is constructed by a LGFS work-conserving policy, we can express R_{lj} as $R_{lj} = [X_{ij} - \chi_{lj}|X_{ij} > \chi_{lj}]$. Because the packet transmission times are NBU and *i.i.d.* across time and links, for all realization of χ_{lj}

$$[R_{lj}|\chi_{lj}] \leq_{\text{st}} X_{ij},\ \forall j \in \mathcal{V},\ \forall l = 1, \ldots, n, \tag{101}$$

which implies that

$$\mathbb{E}[R_{lj}|\chi_{lj}] \leq \mathbb{E}[X],\ \forall j \in \mathcal{V},\ \forall l = 1, \ldots, n. \tag{102}$$

Let $\mathcal{L}_f$ denote the set of links that connect node 0 to node f (i.e., the set of links in the path from node 0 to node f), where $|\mathcal{L}_f| = k$. Define $z_l = D_{lf}(P) - \Gamma_{lf}(\text{LB})$. Invoking the definition of LB, we have $\mathbf{D}_j(\text{LB}) = \mathbf{\Gamma}_j(\text{LB})$ for all $j \in \mathcal{V}$. Using this with the definition of R_{lj}, we can express $\Gamma_{lf}(\text{LB})$ as

$$\Gamma_{lf}(\text{LB}) = a_{l0} + \sum_{j:(i,j)\in\mathcal{L}_f} [R_{lj}(\text{LB})|\chi_{lj}(\text{LB})], \tag{103}$$

where $\Gamma_{lf}(\text{LB})$ is considered as the arrival time of the first packet with generation time greater than s_l to node f under LB. Also, we can express $D_{lf}(P)$ as

$$D_{lf}(P) = a_{l0} + \sum_{j:(i,j)\in\mathcal{L}_f} [R_{lj}(P)|\chi_{lj}(P)] + \sum_{(i,j)\in\mathcal{L}_f} X_{ij}. \tag{104}$$

Observing that packet arrival times $(a_{10}, a_{20}, \ldots, a_{n0})$ at node 0 are invariant of the scheduling policy π, then, from the definition of LB in (14), we have $[R_{lj}(\text{LB})|\chi_{lj}(\text{LB})] = [R_{lj}(P)|\chi_{lj}(P)]$ for all $l = 1, \ldots, n$ and all $j \in \mathcal{H}_1$. Using this with (103) and (104), we can obtain

$$\begin{aligned} z_l = D_{lf}(P) - \Gamma_{lf}(\text{LB}) =& \sum_{j:(i,j)\in\mathcal{L}_f, j\notin\mathcal{H}_1} [R_{lj}(P)|\chi_{lj}(P)] + \sum_{(i,j)\in\mathcal{L}_f} X_{ij} - \sum_{j:(i,j)\in\mathcal{L}_f, j\notin\mathcal{H}_1} [R_{lj}(\text{LB})|\chi_{lj}(\text{LB})] \\ \leq& \sum_{j:(i,j)\in\mathcal{L}_f, j\notin\mathcal{H}_1} [R_{lj}(P)|\chi_{lj}(P)] + \sum_{(i,j)\in\mathcal{L}_f} X_{ij} = z_l^{'}. \end{aligned} \tag{105}$$

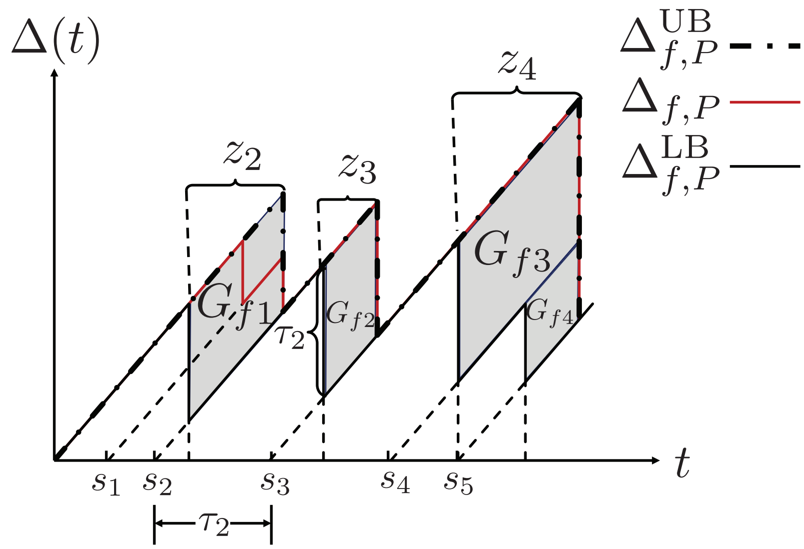


Figure 13: The evolution $\Delta^{\mathrm{LB}}_{f,P}$, $\Delta_{f,P}$, and $\Delta^{\mathrm{UB}}_{f,P}$.

Since the packet transmission times are independent of the packet generation process, we also have $z_l^{'}$'s are independent of the packet generation process. In addition, from (102) and the fact that $|\mathcal{L}_f| = k$, we have

$$\mathbb{E}[z_l^{'}] \leq (2k-1)\mathbb{E}[X]. \tag{106}$$

To identify the average gap between $\Delta^{\mathrm{LB}}_{f,P}$ and $\Delta_{f,P}$, we need to construct an upper bound of the age process at node f under policy P. The upper bound, denoted by $\Delta^{\mathrm{UB}}_{f,P}(t)$, is constructed from the lower bound $\Delta^{\mathrm{LB}}_{f,P}(t)$ as follows: We know that the m-th informative packet at node f under LB (i.e., packet l_m) arrives to node f under LB at time $\Gamma_{l_m f}(\mathrm{LB})$. This packet (or a fresher one) arrives to node f under policy P at time $D_{l_m f}(P) = z_{l_m} + \Gamma_{l_m f}(\mathrm{LB})$. Thus, the upper bound $\Delta^{\mathrm{UB}}_{f,P}$ is constructed by postpone the drops in the graph of $\Delta^{\mathrm{LB}}_{f,P}(t)$ that occur at times $\Gamma_{l_1 f}(\mathrm{LB}), \Gamma_{l_2 f}(\mathrm{LB}), \ldots$ by $z_{l_1}, z_{l_2}, \ldots$, respectively, as shown in Fig. 13. In other words, we have

$$\Delta^{\mathrm{UB}}_{f,P}(t) = t - \max\{s_{l_m} : D_{l_m f}(P) \leq t\}, \tag{107}$$

where the process of $\Delta^{\mathrm{UB}}_{f,P}(t)$ is given by $\Delta^{\mathrm{UB}}_{f,P} = \{\Delta^{\mathrm{UB}}_{f,P}(t), t \in [0,\infty)\}$. Now, we identify the average gap between $\Delta^{\mathrm{LB}}_{f,P}$ and $\Delta^{\mathrm{UB}}_{f,P}$. We use $\{G_f(t), t \in [0,\infty)\}$ to denote the gap process between $\Delta^{\mathrm{LB}}_{f,P}$ and $\Delta^{\mathrm{UB}}_{f,P}$. The average gap is given by

$$[\bar{G}_f|\mathcal{I}'] = \limsup_{T\to\infty} \frac{\int_0^T G_f(t)dt}{T}. \tag{108}$$

Similarly to Step 1, define τ_m as the inter-generation time between packet l_m and packet l_{m-1} (i.e., $\tau_m = s_{l_m} - s_{l_{m-1}}$), where $\tau = \{\tau_m, m \geq 1\}$. Define $N(T) = \max\{m : s_{l_m} \leq T\}$ as the number of informative packets at node f under LB by time T. Following the same steps as in Step 1, we can show that

$$\frac{\mathbb{E}[\int_0^T G_f(t)dt|\tau, N(T)]}{T} \leq \frac{\sum_{m=1}^{N(T)} \tau_m \mathbb{E}[z_{l_m}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m} \leq \frac{\sum_{m=1}^{N(T)} \tau_m \mathbb{E}[z'_{l_m}|\tau, N(T)]}{\sum_{m=1}^{N(T)} \tau_m}. \tag{109}$$

Note that z'_{l_m}'s are independent of the packet generation process. Thus, we have $\mathbb{E}[z'_{l_m}|\tau, N(T)] = \mathbb{E}[z'_{l_m}] \leq (2k-1)\mathbb{E}[X]$ for all l_m. Using this with the law of iterated expectations, we can show, in a similar way as in Step 1, that

$$\limsup_{T\rightarrow\infty} \frac{\mathbb{E}[\int_0^T G_f(t)dt]}{T} \leq (2k-1)\mathbb{E}[X]. \tag{110}$$

Equation (110) tells us that the average gap between $\Delta^{\text{LB}}_{f,P}$ and $\Delta^{\text{UB}}_{f,P}$ is no larger than $(2k-1)\mathbb{E}[X]$. Since, $\Delta^{\text{LB}}_{f,P}$ and $\Delta^{\text{UB}}_{f,P}$ are lower and upper bounds, respectively, of the age process of policy P at node f, we obtain

$$[\bar{\Delta}^{\text{LB}}_{f,P}|\mathcal{I}'] \leq [\bar{\Delta}_{f,P}|\mathcal{I}'] \leq [\bar{\Delta}^{\text{LB}}_{f,P}|\mathcal{I}'] + (2k-1)\mathbb{E}[X]. \tag{111}$$

Similar to Step 1, we can use (111) and Lemma 3 to show that

$$\min_{\pi\in\Pi}[\bar{\Delta}_{f,\pi}|\mathcal{I}'] \leq [\bar{\Delta}_{f,P}|\mathcal{I}'] \leq \min_{\pi\in\Pi}[\bar{\Delta}_{f,\pi}|\mathcal{I}'] + (2k-1)\mathbb{E}[X]. \tag{112}$$

Following the previous argument, we can show that (112) holds for all $f \in \mathcal{H}_k$, with $k \geq 2$. This with (100) prove (19), which complete the proof.