Signals and Systems

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Lecture 1

1) General Informations about course
2) Signals
3) Systems
4) Examples

General Information (Cnt'd)

Course Material:

Lecture Notes

Oppenheim, A. V., and A. S. Willsky, with S. H. Nawab. ***Signals and Systems***. 2nd ed. New Jersey: Prentice-Hall, 1997.

ISBN: 0138147574.

General Information

Evaluation:

2 Homeworks %20

Mid-Term → %20

Final → %60

Calendar

Week 1 Introduction
Week 2 Continuous-Time and Discrete-Time Signals and Systems. System Properties. Singular functions.
Week 3 Convolution. Periodic Signals.
Week 4 Continuous- and Discrete-Time Fourier Series.
Week 5 Continuous-Time Fourier Transform.
Week 6 Continuous-Time Fourier Transform (cont.). Discrete-Time Fourier Transform.
Week 7 Discrete-Time Fourier Transform (cont.).
Week 8 First and Second Order Continuous- and Discrete-Time Systems. Ideal and Non-Ideal Filters.
Week 9 Midterm Exam
Week 10 Sampling. Impulse-Train Sampling. Sampling Theorem and Aliasing. Zero and First Order Hold. Analog-to-Digital and Digital-to-Analog Conversions.
Week 11 Laplace Transforms, Unilateral and Bilateral z-Transforms, Region of Convergence (ROC). The relationships between Laplace Transform, (Continuous and Discrete) Fourier Transforms and z-Transform.
Week 12 Transfer Functions using the Laplace- and z-Transforms, Pole-Zero Plot in s- and z-planes, Stability.
Week 13 Constant Coefficient Linear Differential and Difference Equations.
Week 14 Block Diagram Representation of Continuous- and Discrete-Time Systems. Direct Form, Series and Cascade Filter Realizations. Feedback Structure in s-Domain.

Course Outline (Tentative)

- Fundamental Concepts of Signals and Systems
  - Signals
  - Systems
- Linear Time-Invariant (LTI) Systems
  - Convolution integral and sum
  - Properties of LTI Systems …
- Fourier Series
  - Response to complex exponentials
  - Harmonically related complex exponentials …
- Fourier Integral
  - Fourier Transform & Properties …
  - Modulation (An application example)
- Discrete-Time Frequency Domain Methods
  - DT Fourier Series
  - DT Fourier Transform
  - Sampling Theorem
- Z Transform
  - Stability analysis in z domain

Chapter I

Signals and Systems

SIGNALS

Signals are functions of independent variables that carry information about the behavior or nature of some phenomenon

For example:

• Electrical signals --- voltages and currents in a circuit

• Acoustic signals --- audio or speech signals (analog or digital)

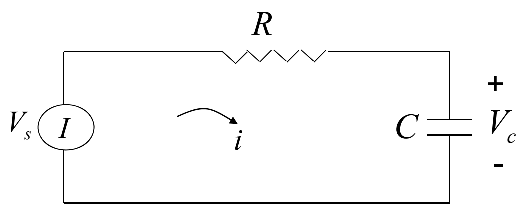
• Video signals --- intensity variations in an image (e.g. a CAT scan)

• Biological signals --- sequence of bases in a gene

What is Signal?

- **Signal** is the variation of a physical phenomenon / quantity with respect to one or more independent variable
- A signal is a function.

Example 1: Voltage on a capacitor as a function of time.



RC circuit

What is Signal?

Example 2 : Closing value of the stock exchange index as a function of days

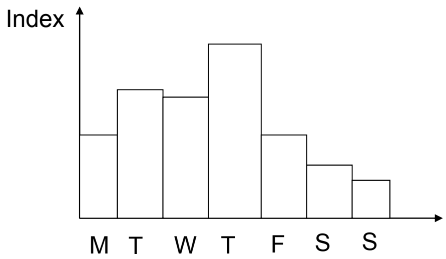
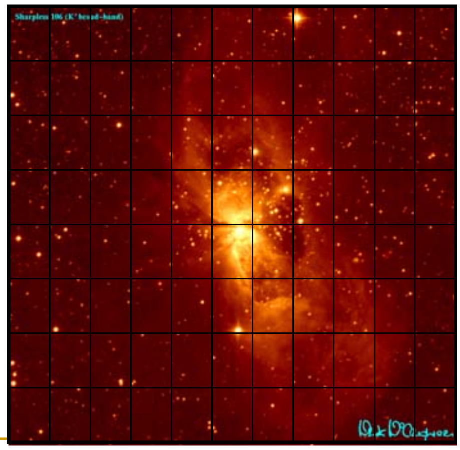


Fig. Stock exchange

Example 3:Image as a function of x-y coordinates (e.g. 256 X 256 pixel image)

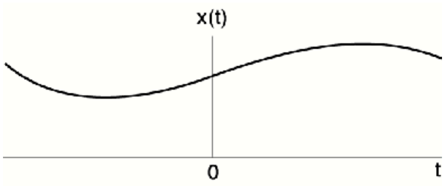


THE INDEPENDENT VARIABLES

- Can be continuous
  - — Trajectory of a space shuttle
  - — Mass density in a cross-section of a brain
- Can be discrete
  - — DNA base sequence
  - — Digital image pixels
- Can be 1-D, 2-D, ••• N-D
- For this course: Focus on a single (1-D) independent variable which we call “time”.

  Continuous-Time (CT) signals: $x(t)$, t — continuous values
  Discrete-Time (DT) signals: $x[n]$, n — integer values only

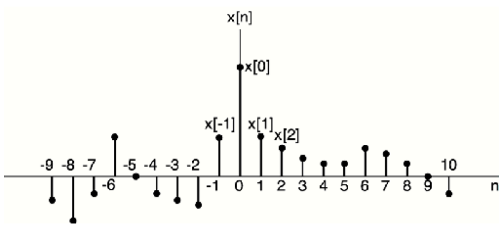
CT Signals



Most of the signals in the physical world are CT signals—E.g. voltage & current, pressure, temperature, velocity, etc.

DT Signals

- x[n], n — integer, time varies discretely



- Examples of DT signals in nature:
  - — DNA base sequence
  - — Population of the nth generation of certain species

Many human-made DT Signals

Ex.#1 Weekly Dow-Jones industrial average

Ex.#2 digital image

Courtesy of Jason Oppenheim.
Used with permission.

Why DT? — Can be processed by modern digital computers and digital signal processors (DSPs).

Continuous-Time vs. Discrete Time

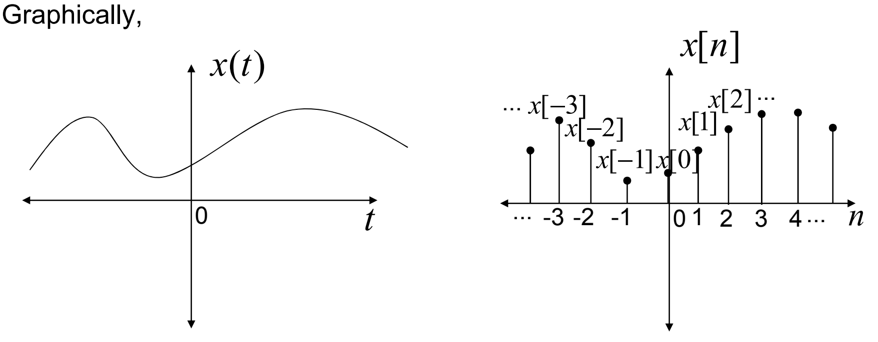
- Signals are classified as continuous-time (CT) signals and discrete-time (DT) signals based on the continuity of the independent variable!
- In CT signals, the independent variable is continuous (See Example 1 (Time))
- In DT signals, the independent variable is discrete (See Ex 2 (Days), Example 3 (x-y coordinates, also a 2-D signal))
  - DT signal is defined only for specified time instants!
  - also referred as DT sequence!

Continuous-Time vs. Discrete Time

- The postfix (-time) is accepted as a convention, although some independent variables are not time
- To distinguish CT and DT signals, t is used to denote CT independent variable in (.), and n is used to denote DT independent variable in [.]
  - Discrete $x[n]$, n is integer
  - Continuous $x(t)$, t is real
- Signals can be represented in mathematical form:
  - $x(t) = e^t$, $x[n] = n/2$
  - $y(t) = \begin{cases} 0 & ,t < 5 \\ -t^2 & ,t \geq 5 \end{cases}$
- Discrete signals can also be represented as sequences:
  - $\{y[n]\} = \{\ldots,1,0,1,0,\underline{1},0,1,0,1,0,\ldots\}$

Continuous-Time vs. Discrete Time

Graphically,



(Fig.1.7 Oppenheim)

- It is meaningless to say 3/2th sample of a DT signal because it is not defined.
- The signal values may well also be complex numbers (e.g. Phasor of the capacitor voltage in Example 1 when the input is sinusoidal and R is time varying)

Signal Energy and Power

- In many applications, signals are directly related to physical quantities capturing power and energy in a physical systems
- Total energy of a CT signal $x(t)$ over $t_1 \le t \le t_2$ is $\int_{t_1}^{t_2} |x(t)|^2 dt$
- The time average of total energy is $\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} |x(t)|^2 dt$ and referred to as average power of $x(t)$ over $t_1 \le t \le t_2$
- Similarly, total energy of a DT signal $x[n]$ over $n_1 \le n \le n_2$ is $\sum_{n_1}^{n_2} |x[n]|^2$
- Average power of $x[n]$ over $n_1 \le n \le n_2$ is

$$\frac{1}{(n_2 - n_1 + 1)} \sum_{n_1}^{n_2} |x[n]|^2$$

Signal Energy and Power

- For infinite time intervals:
  - Energy: accumulation of absolute of the signal

$$E_\infty \stackrel{\Delta}{=} \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt \qquad \text{Total energy in CT signal}$$

$$E_\infty \stackrel{\Delta}{=} \lim_{T\to\infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 \qquad \text{Total energy in DT signal}$$

  - Signals with $E_\infty < \infty$ are of finite energy
  - In order to define the power over infinite intervals we need to take limit of the average:

$$P_\infty \stackrel{\Delta}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{E_\infty}{2T}$$

$$P_\infty \stackrel{\Delta}{=} \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 = \lim_{N\to\infty} \frac{E_\infty}{2N+1}$$

Note: Signals with $E_\infty < \infty$ have $P_\infty = 0$

Signal Energy and Power

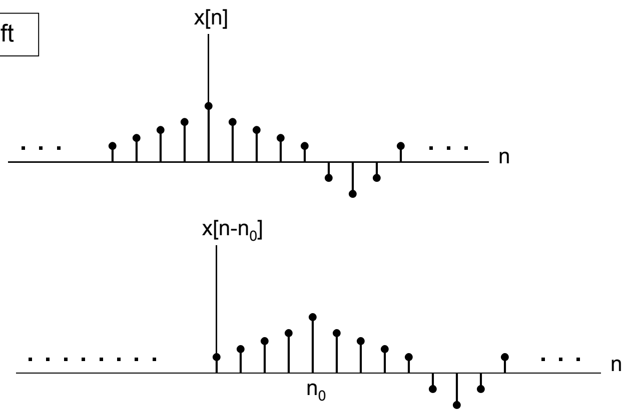
- *Energy signal* iff $0<E<\infty$, and so $P=0$
  - e.g: $$x(t) = \begin{cases} 0, & t<0 \\ e^{-t}, & t \ge 0 \end{cases}$$
- *Power signal* iff $0<P<\infty$, and so $E=\infty$
  - e.g: $\{x[n]\} = \{\ldots -1,1,\underline{-1},1,-1,1\ldots\}$
- Neither energy nor power, when both E and P are infinite
  - e.g: $x(t) = e^{t}$
- Exercise: Calculate power and energy for the above signals

Transformation of Independent Variable

- Sometimes we need to change the independent variable axis for teoretical analysis or for just practical purposes (both in CT and DT signals)
  - Time shift $x[n] \to x[n-n_0]$
  - Time reversal $x(t) \to x(-t)$ (reverse playing of magnetic tape)
  - Time scaling $x(t) \to x(t/2)$ (slow playing, fast playing)

Examples of Transformations

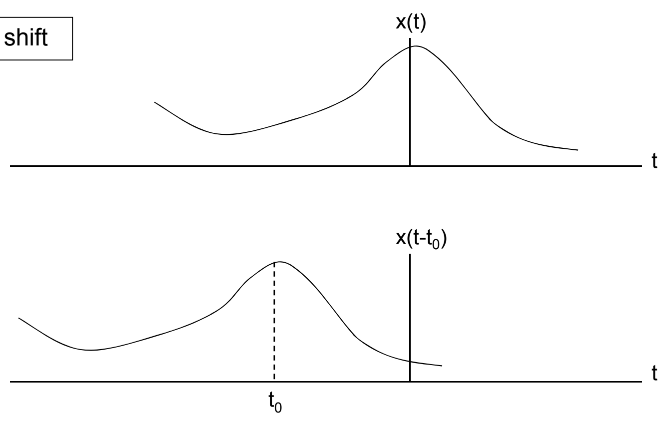
Time shift



If $n_0 > 0 \rightarrow x[n-n_0]$ is the delayed version of x[n]
(Each point in x[n] occurs later in $x[n-n_0]$)

Examples of Transformations

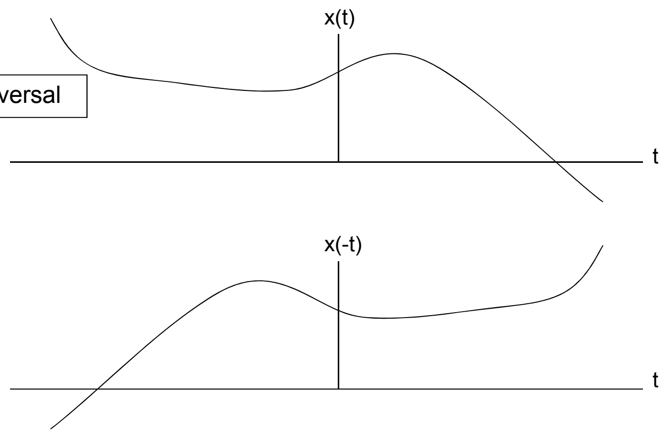
Time shift



$t_0 < 0 \rightarrow x(t-t_0)$ is an advanced version of x(t)

Examples of Transformations

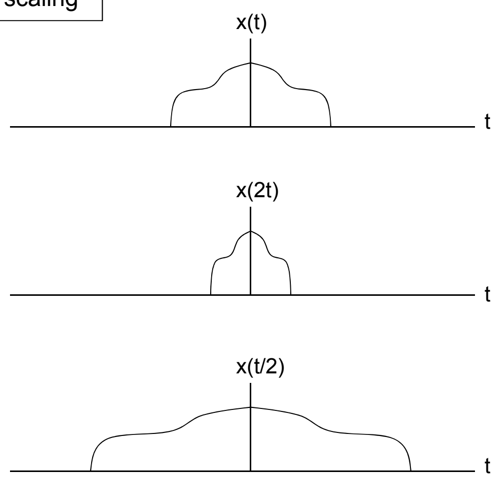
Time reversal



Reflection about t=0

Examples of Transformations

Time scaling



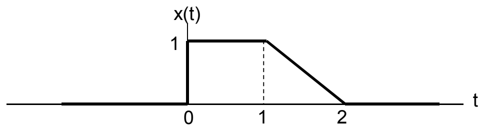
compressed!

stretched!

Examples of Transformations

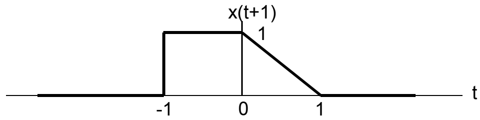
- It is possible to transform the independent variable with a general nonlinear function h(t) (we can find x(h(t)))
- However, we are interested in 1st order polynomial transforms of *t*, i.e., $x(\alpha t+\beta)$

Given the signal x(t):



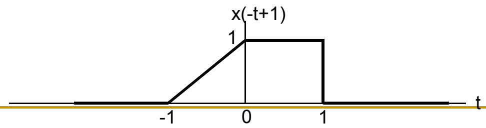
Let us find x(t+1):

(It is a time shift to the left)



Let us find x(-t+1):

(Time reversal of x(t+1))

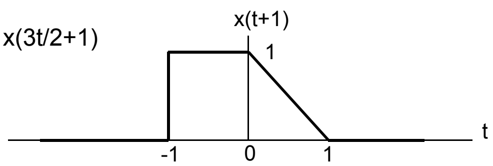


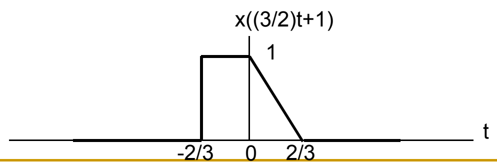
Examples of Transformations

For the general case, i.e., $x(\alpha t+\beta)$,

1. first apply the shift (β),
2. and then perform time scaling (or reversal) based on α.

Example: Find x(3t/2+1)

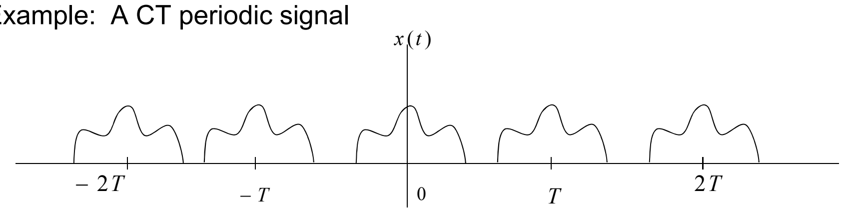




Periodic Signals

- A periodic signal satisfies: $x(t) = x(t+T)\ \forall t, \quad T > 0$

Example: A CT periodic signal



- If *x(t)* is periodic with *T* then $x(t) = x(t+mT)$ for $m \in Z^+$
- Thus, *x(t)* is also periodic with *2T, 3T, 4T, ...*
- The fundamental period T_0 of *x(t)* is the smallest value of T for which $x(t) = x(t+T)\ \forall t, \quad T > 0$ holds

Periodic Signals

- A non-periodic signal is called aperiodic.
- For DT we must have

$$x[n+N] = x[n] \quad \forall n, N > 0$$

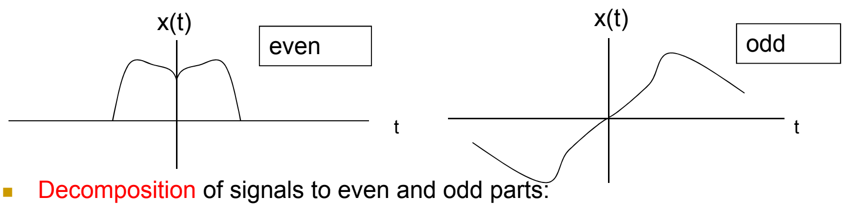
Period must be integer!

Here the smallest N can be 1, → a constant signal

- The smallest positive value N_0 of N is the fundamental period

Even and Odd Signals

- If $x(-t) = x(t)$ or $x[-n] = x[n]$ even signal (symmetric wrt y-axis)
- If $x(-t) = -x(t)$ or $x[-n] = -x[n]$ odd signal (symmetric wrt origin)



- Decomposition of signals to even and odd parts:

$$\left.\begin{aligned} EV\{x(t)\} &= \frac{1}{2}[x(t) + x(-t)] \\ OD\{x(t)\} &= \frac{1}{2}[x(t) - x(-t)] \end{aligned}\right\} \quad x(t) = EV\{x(t)\} + OD\{x(t)\}$$

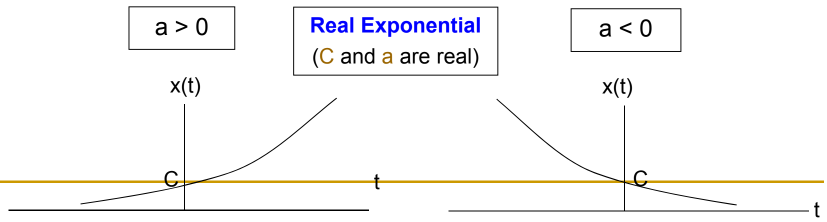
Exponential and Sinusoidal Signals

- Occur frequently and serve as building blocks to construct many other signals
- CT Complex Exponential:

$$x(t) = Ce^{at}$$

where a and C are in general complex.

- Depending on the values of these parameters, the complex exponential can exhibit several different characteristics



Exponential and Sinusoidal Signals

- Periodic Complex Exponential (C real, a purely imaginary)

$$x(t) = e^{jw_0 t}$$

- Is this function periodic?

$$x(t) = \underbrace{e^{jw_0 t} = e^{jw_0(t+T)}}_{\text{for periodicity}} = e^{jw_0 t}.\underbrace{e^{jw_0 T}}_{=1} \longrightarrow T = \frac{2\pi n}{|\omega_0|} \qquad n \in Z^+$$

- The fundamental period is $T_0 = \dfrac{2\pi}{|\omega_0|}$
- Thus, the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ have the same fundamental period

Exponential and Sinusoidal Signals

- By using the Euler's relations:

$$\boxed{\begin{aligned} e^{j\theta} &= \cos\theta + j\sin\theta \\ e^{-j\theta} &= \cos\theta - j\sin\theta \end{aligned}}$$

We can express: (put $\theta = \omega_0 t + \phi$

$$e^{j\omega_0 t} = \cos\omega_0 t + j\sin\omega_0 t$$

$$A\cos(\omega_0 t + \phi) = \frac{A}{2}\left(e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}\right) = \frac{A}{2}\left(e^{j\phi}e^{j\omega_0 t} + e^{-j\phi}e^{-j\omega_0 t}\right)$$

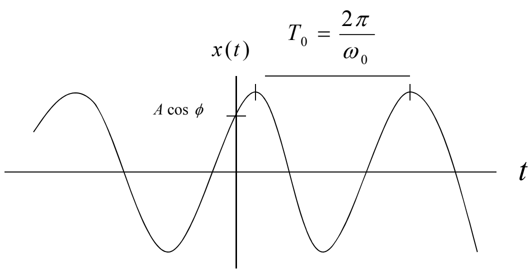
$$A\sin(\omega_0 t + \phi) = \frac{A}{2j}\left(e^{j(\omega_0 t+\phi)} - e^{-j(\omega_0 t+\phi)}\right) = \frac{A}{2j}\left(e^{j\phi}e^{j\omega_0 t} - e^{-j\phi}e^{-j\omega_0 t}\right)$$

Sinusoidals in terms of complex exponentials

Exponential and Sinusoidal Signals

Alternatively,

$$A\cos(\omega_0 t+\phi) = A\,\mathrm{Re}\left(e^{j(\omega_0 t+\phi)}\right)$$
$$A\sin(\omega_0 t+\phi) = A\,\mathrm{Im}\left(e^{j(\omega_0 t+\phi)}\right)$$



$x(t) = A\cos(\omega_0 t+\theta)$

CT sinusoidal signal

$A\cos(\omega_0 t+\phi)$

Exponential and Sinusoidal Signals

- Complex periodic exponential and sinusoidal signals are of infinite total energy but finite average power

$$E_{period} = \int_T^{T+T_0} \left|e^{j\omega_0 t}\right|^2 dt = \int_T^{T+T_0} 1\cdot dt = (T+T_0)-T = T_0$$

$$P_{period} = \frac{1}{(T+T_0)-T} E_{period} = 1$$

- As the upper limit of integrand is increased as $T+2T_0, T+3T_0, \ldots E_{period} \uparrow$
- However, always $P_{period} = 1$
- Thus,

Finite average power!

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T}\cdot \int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt = 1 \text{ IMPORTANT}$$

Harmonically Related Complex Exponentials

- Set of periodic exponentials with fundamental frequencies that are multiples of a single positive frequency ω_0

$$\boxed{x_k(t) = e^{jk\omega_0 t} \text{ for } k = 0, \mp 1, \mp 2, \ldots}$$

$k = 0 \Rightarrow x_k(t)$ is a constant

$k \neq 0 \Rightarrow x_k(t)$ is periodic with fundamental frequency $|k|\omega_0$

and fundamental period $\dfrac{2\pi}{|k|\omega_0} = \dfrac{T_0}{|k|}$, where $T_0 = \dfrac{2\pi}{\omega_0}$

Harmonically Related Complex Exponentials

- k^{th} harmonic $x_k(t)$ is still periodic with T_0 as well
- Harmonic (from music): tones resulting from variations in acoustic pressures that are integer multiples of a fundamental frequency
- *Used to build very rich class of periodic signals*

General Complex Exponential Signals

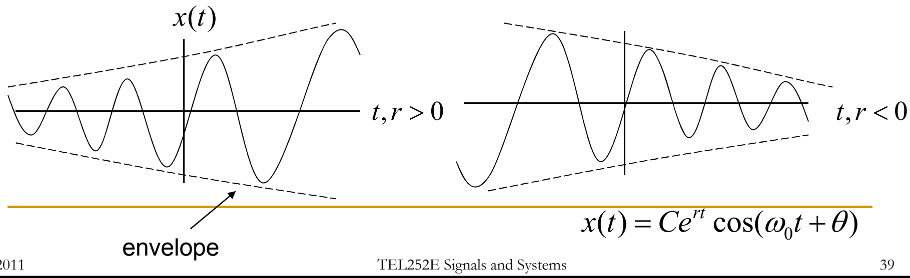
Here, C and a are general complex numbers

Say, $C = |C|e^{j\theta}$ and $a = r + j\omega_0$ $x(t) = Ce^{at}$

Then

$$x(t) = Ce^{at} = |C|e^{rt}e^{j(\omega_0 t+\theta)} = |C|e^{rt}\cos(\omega_0 t+\theta) + j|C|e^{rt}\sin(\omega_0 t+\theta)$$

(Real and imaginary parts) Growing and damping sinusoids for r>0 and r<0



$x(t) = Ce^{rt}\cos(\omega_0 t+\theta)$

DT Complex Exponential and Sinusoidal Signals

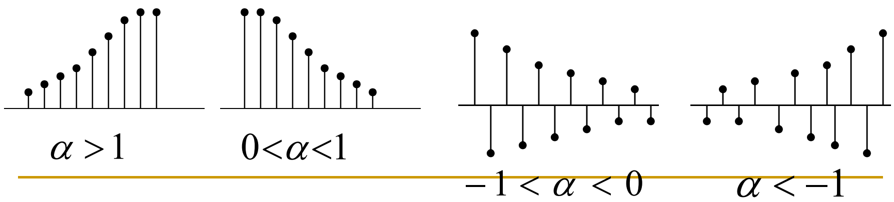
$x[n] = C\alpha^n$ where C and α are in general complex numbers

It is more convenient and customary to use α instead of $e^{\alpha n}$

Real exponential signals : C and α are real

for $\alpha > 1, 0 < \alpha < 1, -1 < \alpha < 0, \alpha < -1$

(Sign alternation for $\alpha < 0$)

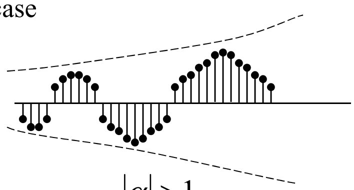


DT Sinusoidal Signals

Consider $e^{j\omega_0 n}$ we then have similar to the CT case

$$A\cos(\omega_0 n+\phi)=\frac{A}{2}(e^{j\phi}e^{j\omega_0 n}+e^{-j\phi}e^{-j\omega_0 n})$$

Infinite energy, finite average power with 1.



$|\alpha|>1$

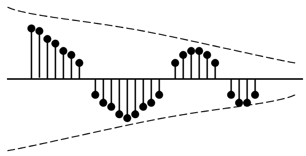
<u>General complex exp signals</u>

If C and α are in polar form as

$C=|C|e^{j\theta},\ \alpha=|\alpha|e^{j\omega_0}$

then

$$C\alpha^n=|C||\alpha|^n\cos(\omega_0 n+\theta)+j|C||\alpha|^n\sin(\omega_0 n+\theta)$$



$|\alpha|<1$

Real and imaginary parts of DT general complex exp are sinusoidals (growing $|\alpha|>1$, and decaying $|\alpha|<1$)

Periodicity Properties of DT Signals

Consider the DT complex exp : $e^{j\omega_0 n}$

Let's find $e^{j(\omega_0+2\pi)n}=e^{j\omega_0 n}\underbrace{e^{j2\pi n}}_{=1}$

SO THE FN WITH FREQ ω_0 IS THE SAME AS THE FN WITH $\omega_0+2\pi$

This is very different from CT complex exp.

CT exp has distinct freq values ω_0

DT exp has identical freq values $\omega_0+2k\pi,\ k\in Z$

<u>Result:</u>

It is sufficient to consider an interval from ω_0 to ω0 +2π to completely characterize the DT complex exponential!

Periodicity Properties of DT Signals

One usually takes $0 \le \omega_0 < 2\pi$ or $-\pi \le \omega_0 < \pi$

– For CT exp as $\omega_0 \uparrow$ the rate of oscillation $\uparrow$ indefinitely

– For DT exp as $\omega_0 \uparrow$ from 0 to π, the rate of oscillation $\uparrow$,

as $\omega_0 \uparrow$ more until 2π, the rate of oscillation $\downarrow$ to zero.

Hence, low freq (slow varying) DT complex exp is around $\omega_0 = 0$ and $\omega_0 = 2\pi$

High freq (rapidly varying) DT complex exp is around $\omega_0 = \pi$

What about the periodicity of DT complex exp?

Periodicity Properties of DT Signals

Periodicity condition : $e^{j\omega_0(n+N)} = e^{j\omega_0 n} \underbrace{e^{j\omega_0 N}}_{\text{must be unity}}$ (*)

This holds if $\omega_0 N$ is an integer multiple of 2π. (**)

In other words some integer m we must have $\omega_0 N = 2\pi m$

Or equivalently $\dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$ (***)

We have the conditions from (*) and (**) that m and N must be integers.

So DT exp is periodic when $\dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$ is a rational number, not periodic otherwise!!!

Periodicity Properties of DT Signals

Take the common factor out

The fundamental frequency is then $\frac{2\pi}{N} = \frac{\omega_0}{m}$

The fundamental period is then $N = m\left(\frac{2\pi}{\omega_0}\right)$ (****)

Therefore to find the fund freq of an complex exp we need to express

$\frac{\omega_0}{2\pi}$ as in (***)

(The same development is also valid for DT sinusoidal signals.)!!

Periodicity Properties of DT Signals

Examples

Ex : $x[n] = \cos(2\pi n/12)$ is periodic with fund period 12.

$x[n] = \cos(2\pi n/12) = \cos(\omega_0 n) \quad \omega_0 = \frac{2\pi}{12} \rightarrow \frac{\omega_0}{2\pi} = \frac{1}{12}$ no factors in common,

so by using (****), $N = 1\left(\frac{12}{1}\right) = 12$

Ex : $x[n] = \cos(4\pi n/12)$ is periodic with fundamental period 6.

$x[n] = \cos(4\pi n/12) = \cos(\omega_0 n) \quad \omega_0 = \frac{4\pi}{12} \rightarrow \frac{\omega_0}{2\pi} = \frac{2}{12} = \frac{(n=1)}{(N=6)}$,

then using (****), $N = 1\left(\frac{12}{2}\right) = 6$

Periodicity Properties of DT Signals
Examples

- OBSERVATION:
  - With no common factors between N and m, N in (***) is the fundamental period of the signal
  - Hence, if we take common factors out

$$\frac{\omega_0}{2\pi} = \frac{1}{6} \quad \rightarrow N = 6$$

- *Comparison of Periodicity of CT and DT Signals:*
  - Consider x(t) and x[n]

$$x(t) = \cos(2\pi t/12) \quad x[n] = \cos(2\pi n/12)$$

x(t) is periodic with T=12, x[n] is periodic with N=12.

Periodicity Properties of DT Signals
Examples

- But, if $x(t) = \cos(8\pi t/31)$ and $x[n] = \cos(8\pi n/31)$

  x(t) is periodic with 31/4.
- In DT there can be no fractional periods, for x[n] we have $\frac{\omega_0}{2\pi} = \frac{4}{31}$

  then N=31.

If $x(t) = \cos(t/6)$ and $x[n] = \cos(n/6)$

x(t) is periodic with 12π, but x[n] is not periodic, because $\frac{\omega_0}{2\pi} = \frac{1}{12\pi}$ there is no way to express it as in (***)

Study Fig.1.27 page 27, Table 1.1 in Opp. Example 1.6 as well

Harmonically Related Complex Exponentials *(Discrete Time)*

- Set of periodic exponentials with a common period *N*
- Signals at frequencies multiples of $\frac{2\pi}{N}$ (from $\omega_0 N = 2\pi m$)

$$\boxed{\phi_k[n] = e^{jk\left(\frac{2\pi}{N}\right)n} \quad \text{for } k = 0, \mp 1, \mp 2, \ldots}$$

- In CT, all of the HRCE, $e^{jk\omega_0 t}$ for $k = 0, \mp 1, \mp 2, \ldots$ are distinct
- *Different in DT case!*

Harmonically Related Complex Exponentials *(Discrete Time)*

- Let's look at $(k+N)^{th}$ harmonic:

$$\phi_{k+N}[n] = e^{j(k+N)\left(\frac{2\pi}{N}\right)n} = e^{jk\left(\frac{2\pi}{N}\right)n} . \underbrace{e^{j2\pi n}}_{=1} = \phi_k[n]$$

- Only N distinct periodic exponentials in $\phi_k[n]$!!
- That is,

$$\phi_0[n] = 1, \phi_1[n] = e^{j\frac{2\pi}{N}n}, \phi_2[n] = e^{j\frac{4\pi}{N}n}, \ldots, \phi_{N-1}[n] = e^{j(N-1)\frac{2\pi}{N}n}$$

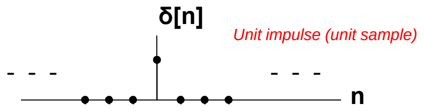
$$\phi_0[n] = \phi_N[n], \quad \phi_{-1}[n] = \phi_{N-1}[n]$$

Unit Impulse and Unit Step Functions

- Basic signals used to construct and represent other signals

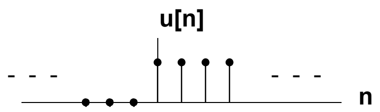
DT unit impulse:

$$\delta[n] = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases}$$



DT unit step:

$$u[n] = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}$$



Relation between DT unit impulse and unit step (?):

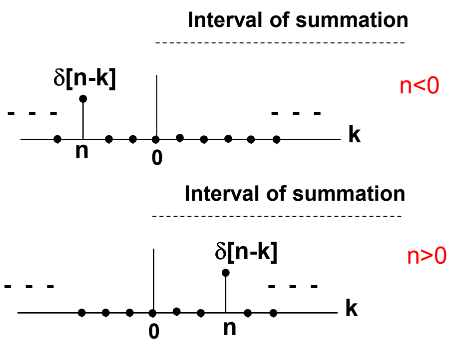
$$\boxed{\delta[n] = u[n] - u[n-1]}$$

(DT unit impulse is the first difference of the DT step)

Unit Impulse and Unit Step Functions

$$\boxed{u[n] = \sum_{0}^{\infty} \delta[\mathrm{n-k}]}$$

(DT step is the running sum of DT unit sample)



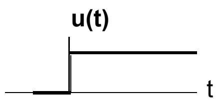
$$x[n]\delta[n] = x[0]\delta[n]$$

More generally for a unit impulse $\delta[n-n_0]$ at n_0 :

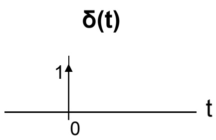
$$\boxed{x[n]\delta[n-n_0] = x[n_0]\delta[n-n_0]} \longrightarrow \text{Sampling property}$$

Unit Impulse and Unit Step Functions (Continuous-Time)

CT unit step: $u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$



CT impulse: $\delta(t) = \dfrac{du(t)}{dt}$



CT unit impulse is the 1st derivative of the unit sample

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

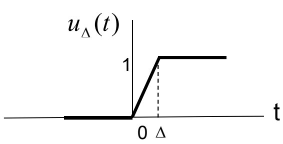
CT unit step is the running integral of the unit impulse

Continuous-Time Impulse

- CT impulse is the 1st derivative of unit step

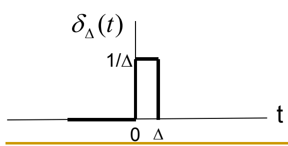
$$\delta(t) = \frac{du(t)}{dt}$$

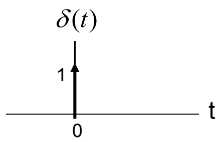
- There is discontinuity at t=0, therefore we define $u_\Delta(t)$ as



$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt} \Rightarrow \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$





Continuous-Time Impulse

REMARKS:

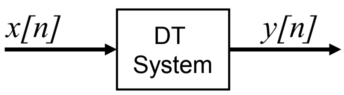
- Signal of a unit area
- Derivative of unit step function
- Sampling property $\boxed{x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)}$
- The integral of product of φ(t) and δ(t) equals φ(0) for any φ(t) continuous at the origin and if the interval of integration includes the origin, i.e.,

$$\boxed{\int_{t_1}^{t_2} \varphi(\tau)\delta(\tau)d\tau = \varphi(0) \quad for \;\; t_1 < 0 < t_2}$$

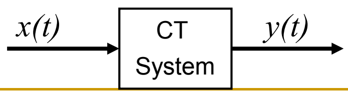
CT and DT Systems

What is a system?

- A system: any process that results in the transformation of signals
- A system has an input-output relationship
- Discrete-Time System: *x[n] → y[n] : y[n] = H[x[n]]*



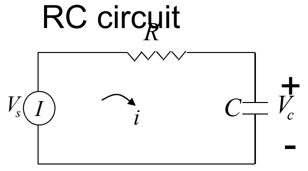
- Continuous-Time System: *x(t) → y(t) : y(t) = H(x(t))*



CT and DT Systems

Examples

- In CT, differential equations are examples of systems
- Zero state response of the capacitor voltage in a series RC circuit



RC circuit

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

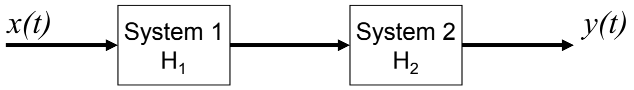
$v_c(t)$: output, $v_s(t)$: input

- In DT, we have difference equations
- Consider a bank account with %1 monthly interest rate added on: $y[n] = 1.01y[n-1] + x[n]$

y[n]: output: account balance at the end of each month
x[n]: input: net deposit (deposits-withdrawals)

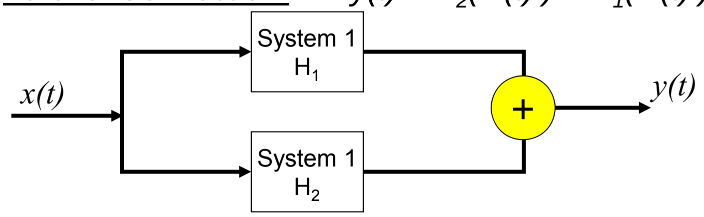
Interconnection of Systems

- *Series (or cascade) Connection:* $y(t) = H_2(H_1(x(t)))$



  - e.g. radio receiver followed by an amplifier

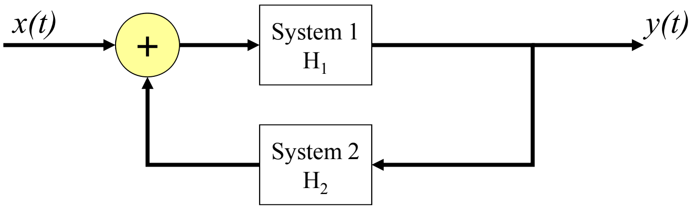
- *Parallel Connection:* $y(t) = H_2(x(t)) + H_1(x(t))$



  - e.g. phone line connecting parallel phone microphones

Interconnection of Systems

- Previous interconnections were "feedforward systems"
  - The systems has no idea what the output is
- <u>*Feedback Connection:*</u> *$y(t) = H_2(\, y(t)\,) + H_1(\, x(t)\,)$*



  - In feedback connection, the system has the knowledge of output
  - e.g. cruise control
- Possible to have combinations of connections..

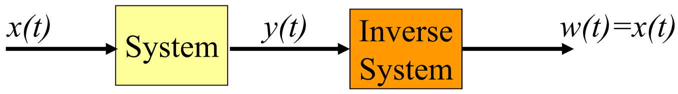
System Properties
Memory vs. Memoryless Systems

- Memoryless Systems: System output *y(t)* depends only on the input at time *t*, i.e. y(t) is a function of *x(t)*.
  - e.g. *y(t)=2x(t)*
- Memory Systems: System output *y(t)* depends on input at past or future of the current time *t*, i.e. *y(t)* is a function of $x(\tau)$ where $-\infty < \tau < \infty$.
  - Examples:
    - A resistor: *y(t) = R x(t)*
    - A capacitor: $y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$
    - A one unit delayer: *y[n] = x[n-1]*
    - An accumulator: $y[n] = \sum_{k=-\infty}^{n} x[k]$

System Properties

Invertibility

- A system is invertible if distinct inputs result in distinct outputs.
- If a system is invertible, then there exists an inverse system which converts output of the original system to the original input.
  - Examples:



$$y(t) = 4x(t) \qquad y[n] = \sum_{k=-\infty}^{n} x[k] \qquad y(t) = \int_{-\infty}^{t} x(t)dt$$

$$w(t) = \frac{1}{4} y(t) \qquad w[n] = y[n] - y[n-1] \qquad w(t) = \frac{dy(t)}{dt}$$

$y(t) = x^4(t)$ → Not invertible

System Properties

Causality

- A system is causal if the output at any time depends only on values of the input at the present time and in the past
- Examples:
  - Capacitor voltage in series RC circuit (casual)

$y(t) = 2x(t+4)$ → Non-causal

$y[n] = x[-n]$ → Non-causal (why?) (For n<0, system requires future inputs)

$y(t) = 2x(t-4)\cos(t+1)$ → Causal (why?)

- Systems of practical importance are usually casual
- However, with pre-recorded data available we do not constrain ourselves to causal systems (or if independent variable is not time, any example??)

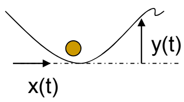
$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{M} x[n-k]$ Averaging system in a block of data

System Properties
Stability

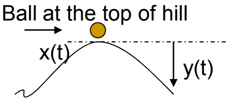
- A system is stable if small inputs lead to responses that do not diverge
- More formally, a system is stable if it results in a bounded output for any bounded input, i.e. bounded-input/bounded-output (BIBO).
  - If $|x(t)| < k_1$, then $|y(t)| < k_2$.
- Example:

$$y(t) = \int_0^t x(t)dt \qquad y[n] = 100x[n] \quad \rightarrow \text{stable}$$

Ball at the base of valley



Ball at the top of hill



  - Averaging system: $y[n] = \frac{1}{2M+1}\sum_{k=-M}^{M} x[n-k]$ →stable
  - Interest system: $y[n] = 1.01y[n-1] + x[n]$ →unstable (say x[n]=δ[n], y[n] grows without bound

System Properties
Time-Invariance

- A system is time-invariant if the behavior and characteristics of the system are fixed over time
- More formally: A system is time-invariant if a delay (or a time-shift) in the input signal causes the same amount of delay (or time-shift) in the output signal, i.e.:

$$x(t) = x_1(t-t_0) \rightarrow y(t) = y_1(t-t_0)$$
$$x[n] = x_1[n-n_0] \rightarrow y[n] = y_1[n-n_0]$$

- Examples:

$$y[n] = nx[n]$$
$$x_1[n] \rightarrow y_1[n] = nx_1[n]$$
$$x_2[n] = x_1[n-n_0] \rightarrow y_2[n] = nx_1[n-n_0]$$
$$y_1[n-n_0] = (n-n_0)x_1[n-n_0] \neq y_2[n]$$

(Ex. 1.15)

Not TIV

$$y(t) = x(2t)$$

Not TIV (explicit operation on time)

(Ex. 1.16)

$$y(t) = \sin|x(t)|$$

TIV

(Ex. 1.14)

When showing a system is not TIV, try to find counter examples…

System Properties
Linearity

- A system is linear if it possesses superposition property, i.e., weighted sum of inputs lead to weighted sum of responses of the system to those inputs
- In other words, a system is linear if it satisfies the properties:
  - It is *additivity*: $x(t) = x_1(t) + x_2(t) \Rightarrow y(t) = y_1(t) + y_2(t)$
  - And it is *homogeneity* (or *scaling*): $x(t) = a\,x_1(t) \Rightarrow y(t) = a\,y_1(t)$, for a any complex constant.
- The two properties can be combined into a single property:
  - Superposition:

    $x(t) = a\,x_1(t) + b\,x_2(t) \Rightarrow y(t) = a\,y_1(t) + b\,y_2(t)$

    $x[n] = a\,x_1[n] + b\,x_2[n] \Rightarrow y[n] = a\,y_1[n] + b\,y_2[n]$
- How do you check linearity of a given system?

System Properties
Linearity

- Examples:

$y(t) = x^2(t)$

nonlinear

$y[n] = \mathrm{Re}\{x[n]\}$

$x_1[n] = r[n] + js[n] \rightarrow y_1[n] = r[n]$

$x_2[n] = ax_1[n] = j(r[n] + js[n]) \quad for\ a = j$

$\rightarrow x_2[n] = -s[n] + jr[n]$

$x_2[n] \rightarrow y_2[n] = -s[n] \neq ay_1[n]$

nonlinear

$y[n] = 2x[n] + 3$

$x_1[n] = 2,\ x_2[n] = 3$

$x_1[n] \rightarrow y_1[n] = 2.2 + 3 = 7$

$x_2[n] \rightarrow y_2[n] = 2.3 + 3 = 9$

$x_1[n] + x_2[n] = 2 + 3 = 5$

$x_1[n] + x_2[n] \rightarrow y_{1+2}[n] = 2.5 + 3 = 13$

$y_1[n] + y_2[n] = 7 + 9 = 16 \neq y_{1+2}[n]$

nonlinear

$y[n] = 2x[n-1]$

linear

Superposition in LTI Systems

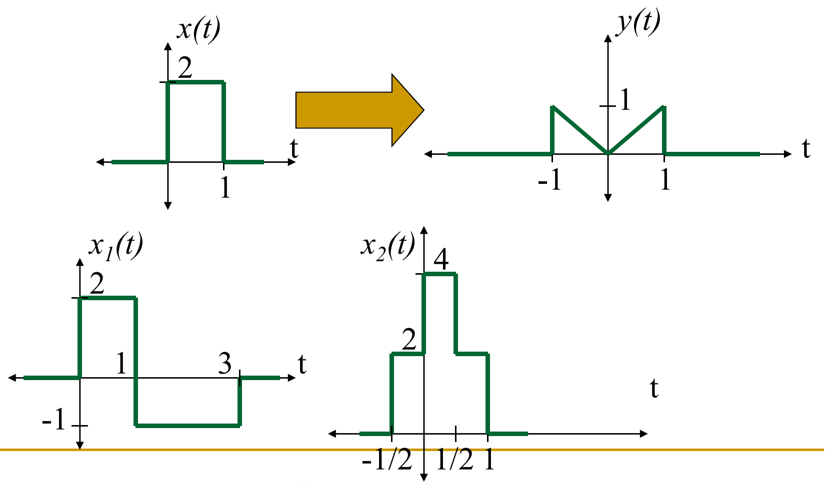
- For an LTI system:
  - given response *y(t)* of the system to an input signal *x(t)*
  - it is possible to figure out response of the system to any signal $x_1(t)$ that can be obtained by "scaling" or "time-shifting" the input signal *x(t)*, i.e.:

$$x_1(t) = a_0 x(t-t_0) + a_1 x(t-t_1) + a_2 x(t-t_2) + \ldots \Rightarrow$$
$$y_1(t) = a_0 y(t-t_0) + a_1 y(t-t_1) + a_2 y(t-t_2) + \ldots$$

- Very useful property since it becomes possible to solve a wider range of problems.
- This property will be basis for many other techniques that we will cover throughout the rest of the course.

Superposition in LTI Systems

- Exercise: Given response y(t) of an LTI system to the input signal x(t) below, find response of that system to the input signals $x_1(t)$ and $x_2(t)$ shown below.

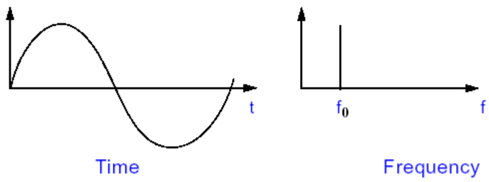


3 Audio

Physics Of Acoustics

Sound can be thought of in one of two ways:

- In the time domain
- In the frequency domain
- Transformation is accomplished by a Fourier transform



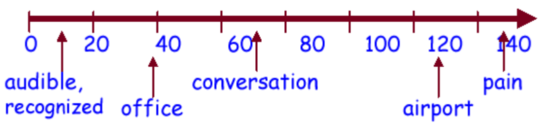
Fourier:

- Any waveform can be represented by a sum of sine waves

3 Audio

Physics Of Acoustics con’t

Loudness measured in dB (decibel): logarithm



+10 dB perceived as „double“ loudness β

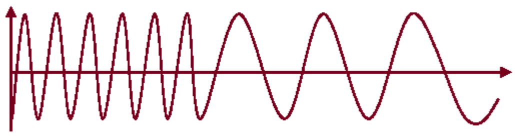
loudness $\beta = 10\log\left(I/I_0 \right)$

- I_0 threshold (just perceived) (10^{-12} Joule/s . m²), OdB
- +10 dB (double loudness) equals 10-fold energy

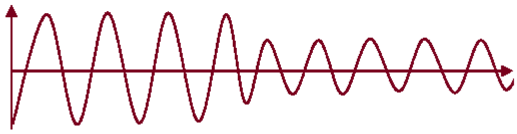
Physics Of Acoustics

con’t

Frequency — Pitch



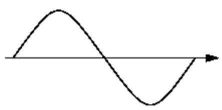
Amplitude — Loudness



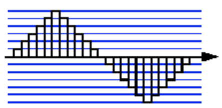
Analog to Digital: Theory of Sampling

From analog to digital:

0. Filter (low-pass): Remove frequencies larger than f_{max}
1. Sample: Measure signal value (1a: hold=store temporarily)
2. Quantize: Relate value to interval (A/D conv. quantizes, too)
3. Encode: Assign binary code (e.g., integer no. of interval)



analog signal



digital signal

Steps 1,2,3 standardized as "*Pulse Code Modulation*" PCM

Important factors:

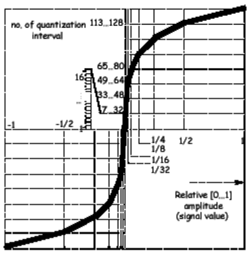
- Sampling rate: Number of sampled values per second
- Quantization depth: Number of bits per digitized value

3 Audio

Analog to Digital: Theory of Sampling con't

Sampling rate determined by properties of recorded sound:
- Nyquist: "For lossless digitization, the sampling rate (frequency f_s) should be at least twice the maximum frequency considered, fmax"
- Mathematically precise: "any epsilon larger than...", i.e. $f_s > 2f_{max}$
- Attention: assumes 0 quantization error, unrealistic → quantization noise
- Music typically extends from 20 Hz to 20 kHz
- Speech 100 Hz to 10 kHz, major energy in band from 200Hz to 4kHz

Quantization depth determined by desired sound quality (quant. noise): Typically 8 (256 levels) or 16 (65,536 levels)
Samples always "per channel": (e.g., 2x for stereo)
Logarithmic Quantization:
compensates for fact that quantization error much more audible around 0 amplitude



3 Audio

Audio Quality of Common Appliances

Audio Device	Frequency Response (Bandwidth)	Signal-to-Noise Ratio	Total Harmonic Distortion
CD	20 Hz - 20,000 Hz	98dB	0.005%
Cassette tape	20 Hz - 17,000 Hz	75dB	0.01%
FM Radio	20 Hz - 15,000 Hz	75dB	0.01%
AM Radio	50 Hz - 5,000 Hz	60dB	0.1%
Telephone	300 Hz - 3400 Hz	42dB	Poor

3 Audio

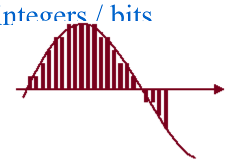
Sampling Rates

Sampling Rates	Used As...
8000	Telephony Standard, Popular in UNIX Workstations
11000	Quarter of CD rate, Popular on Macintosh
16000	G.722 Standard (Federal Standard)
18900	CD-ROM XA Rate
22000	Half CD rate, Macintosh rate
32000	Japanese HDTV, British TV audio, Long play DAT
37800	CD XA Standard
44056	Professional audio industry
44100	CD Rate
48000	DAT Rate

3 Audio

Common Coding Methods

Coding = representation of sampled values by integers / bits



PCM (Pulse Code Modulation)
- integer value = (quantized) sampled value
- simple but requires high number of bits

DPCM (Differential PCM)
- integer value = difference between current value and predicted value
- prediction based on previous values
- requires less bits than PCM for same quality

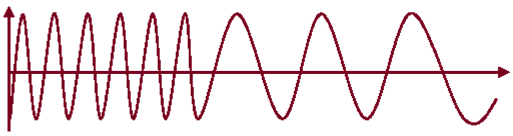
DM (Delta Modulation)
- as DPCM but only differences of 1 and -1 allowed
- requires minimal number of bits but quality can be poor

ADPCM (Adaptive Differential PCM):
- as DPCM but adapts predictor to signal characteristics
- simplest prediction: "difference remains"
- also adapts width of quantization steps to signal characteristics
- better quality than DPCM with same storage requirements

3 Audio

Physics Of Acoustics
con’t

Frequency — Pitch



Amplitude — Loudness

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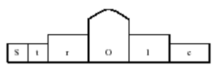
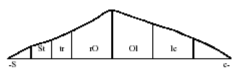
3 Audio

Example

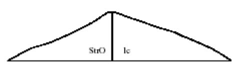
Phonem: Stairs



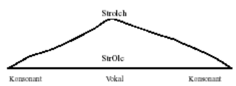
Diphon



Half syllable



Syllable



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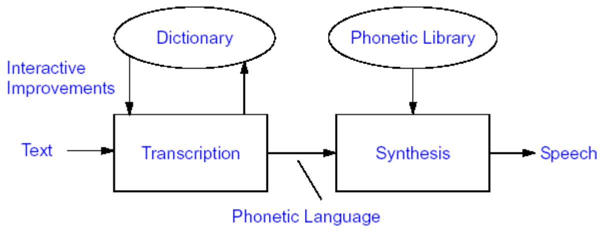
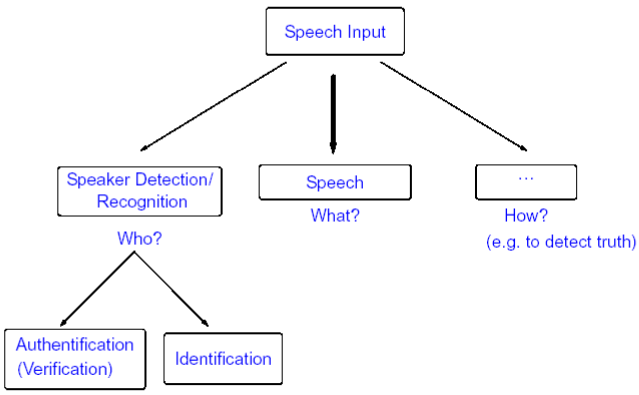
3 Audio

Speech Synthesis in Frequency Domain



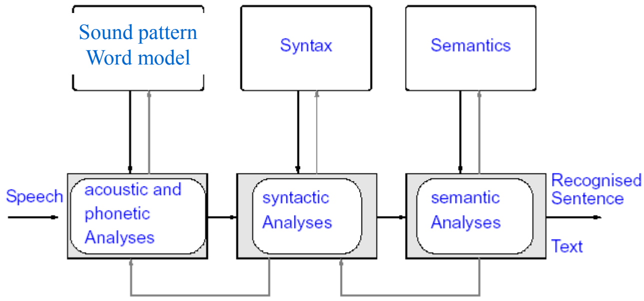
3 Audio

Speech Input



3 Audio

Speech Recognition



3 Audio

Sound Enhancement

- Declicking
- Noise Reduction
- Echo suppression

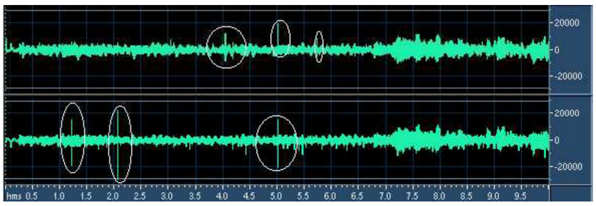
3 Audio

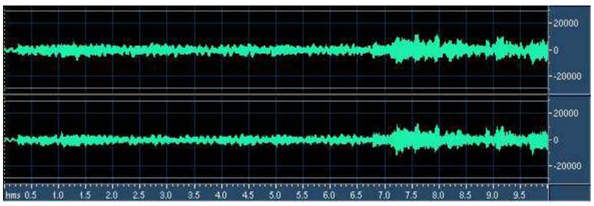
Declicking

Removing impulse distortions, also called declicking of recorded sound, is performed in two steps. In a first step impulse distortions - **clicks** - are detected within a signal, which are going to be removed from the signal in the second step.

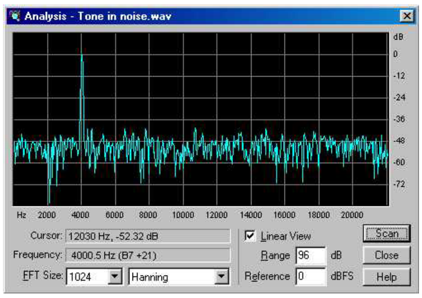
3 Audio

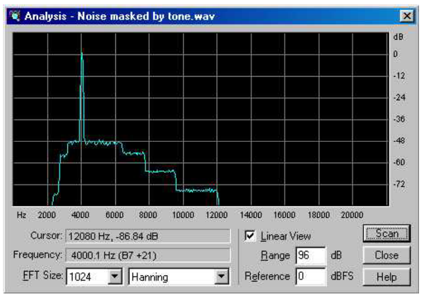
Declicking con't

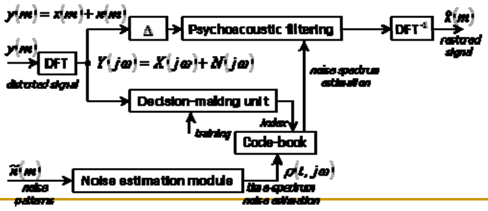




Noise Reduction

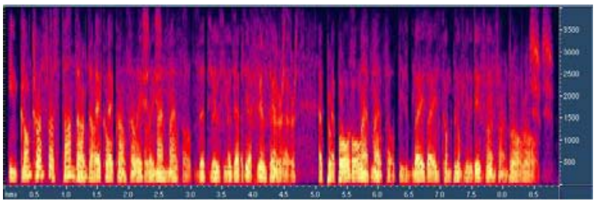


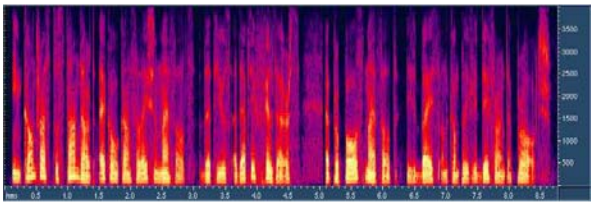




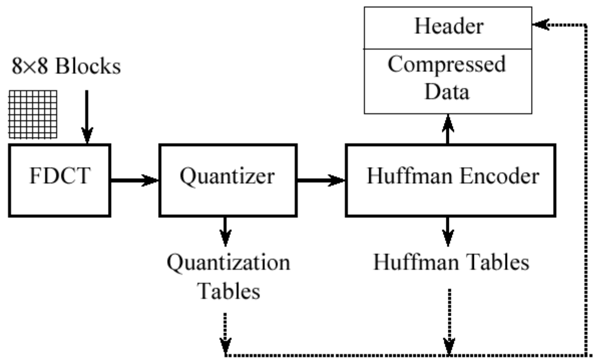
Echo Suppression

sonogram

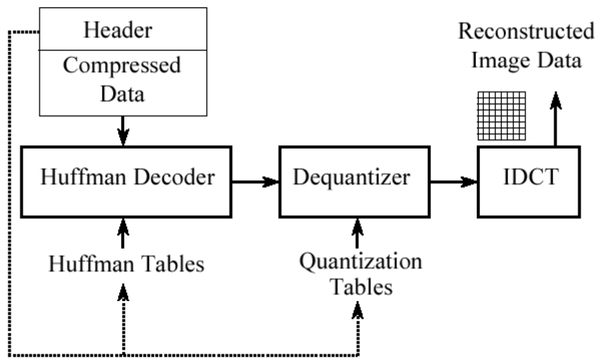




Baseline JPEG Encoder Block Diagram



Baseline JPEG Decoder Block Diagram



Baseline JPEG Pros and Cons

- Advantages
  - Memory Efficient
  - Low complexity
  - Compression efficiency
  - Visual model utilization
  - Robustness
- Disadvantages
  - Single resolution
  - Single quality
  - No target bit rate
  - No lossless capability
  - No tiling
  - No ROI
  - Blocking artifacts
  - Poor error resilience

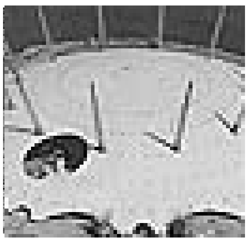
JPEG at 0.125 bpp

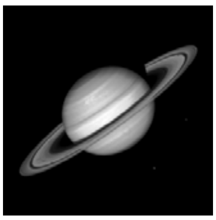
JPEG2000 at 0.125 bpp

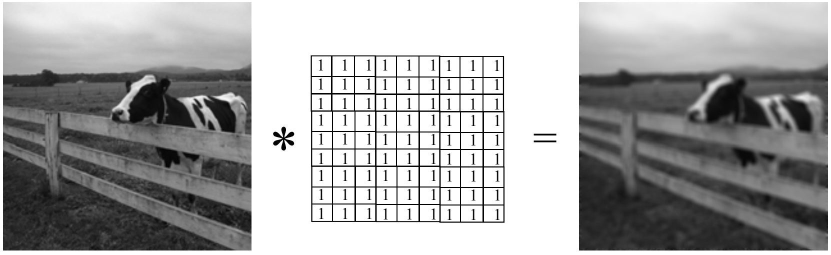
JPEG at 0.25 bpp JPEG2000 at 0.25 bpp

Noise reduction Edge Enhancement

Zooming







Median Operation

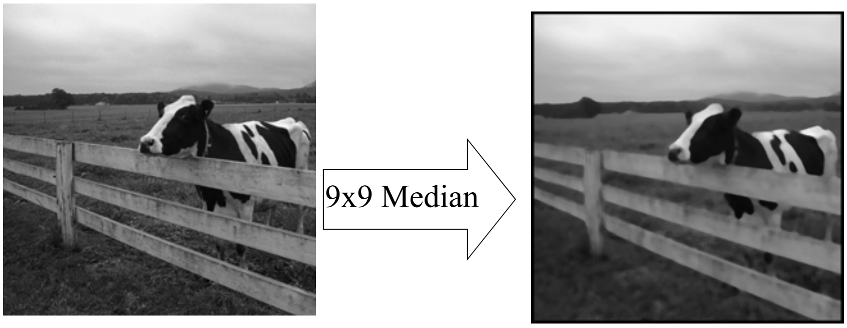
61	62	59	65
57	60	63	56
59	55	58	57
49	53	55	45

62	59	65
60	63	56
55	58	57

rank →

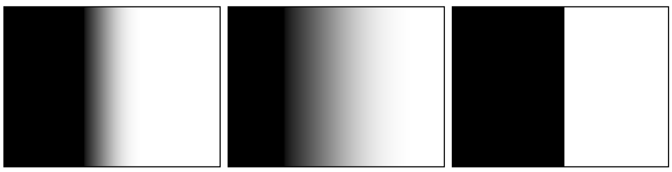
9	65
8	63
7	62
6	60
5	59
4	58
3	57
2	56
1	55

$C_{1,2} = \boxed{59}$



Edge Detection

- What is an edge
  - A large change in image brightness of a short spatial distance
  - Edge strength = (I(x,y)-I(x+dx,y))/dx



Roberts Operator

$$\sqrt{[I(x,y)-I(x+1,y+1)]^2+[I(x,y+1)-I(x+1,y)]^2}$$

or

$$|I(x,y)-I(x+1,y+1)|+|I(x,y+1)-I(x+1,y)|$$

$$\left|\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & -1 \\ \hline\end{array}\right| + \left|\begin{array}{|c|c|}\hline 0 & 1 \\ \hline -1 & 0 \\ \hline\end{array}\right|$$

- Does not return any information about the orientation of the edge

Prewitt Operator

$$P_1 = \begin{array}{|c|c|c|}\hline -1 & -1 & -1 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline\end{array} \qquad P_2 = \begin{array}{|c|c|c|}\hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline -1 & 0 & 1 \\ \hline\end{array}$$

$$\text{Edge Magnitude} = \sqrt{P_1^2 + P_2^2}$$

$$\text{Edge Direction} = \tan^{-1}\left[\frac{P_1}{P_2}\right]$$

Robinson Compass Masks

-1	0	1
-2	0	2
-1	0	1

0	1	2
-1	0	1
-2	-1	0

1	2	1
0	0	0
-1	-2	-1

2	1	0
1	0	-1
0	-1	-2

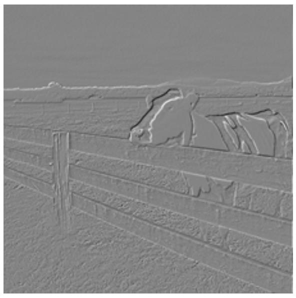
1	0	-1
2	0	-2
1	1	-1

0	-1	-2
-1	0	-1
2	1	0

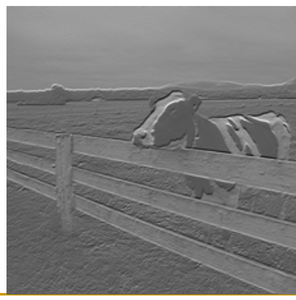
-1	-2	-1
0	0	0
1	2	1

-2	-1	0
-1	0	1
0	1	2

0	1	2
-1	0	1
-2	-1	0



1	2	1
0	0	0
-1	-2	-1



2D Laplacian Operator

$$\nabla^2 f(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2}$$

Convolution masks approximating a Laplacian

0	-1	0
-1	4	-1
0	-1	0

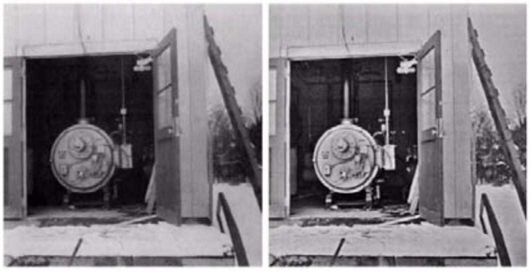
1	-2	1
-2	4	-2
1	-2	1

-1	-1	-1
-1	8	-1
-1	-1	-1

0	-1	0
-1	4	-1
0	-1	0

Input Mask Output

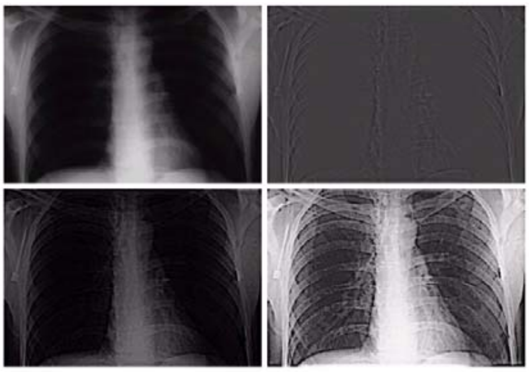
Chapter 4
Image Enhancement in the Frequency Domain



a b

FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)

Chapter 4
Image Enhancement in the Frequency Domain



a b
c d

FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

Chapter 3
Image Enhancement in the Spatial Domain

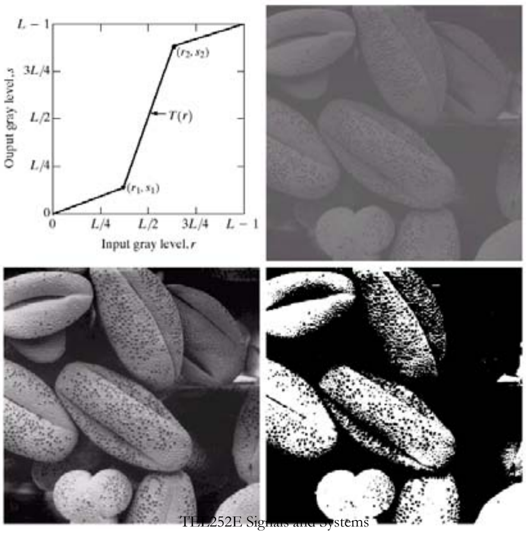


FIGURE 3.10 Contrast stretching. (a) Form of transformation function. (b) A low-contrast image. (c) Result of contrast stretching. (d) Result of thresholding. (Original image courtesy of Dr. Roger Heady, Research School of Biological Sciences, Australian National University, Canberra, Australia.)

Chapter 3
Image Enhancement in the Spatial Domain

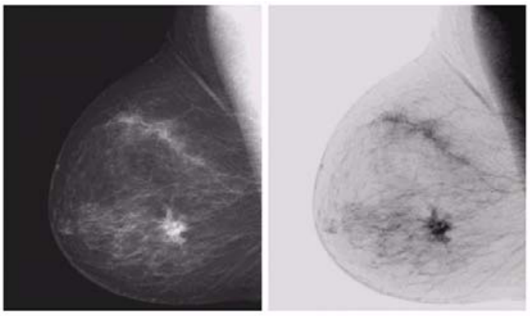


FIGURE 3.4 (a) Original digital mammogram. (b) Negative image obtained using the negative transformation in Eq. (3.2-1). (Courtesy of G.E. Medical Systems.)