Faculty of Engineering
Cairo University

CAIRO UNIVERSITY

Image Denoising
Mathematical modelling

Team name: Pixels

	Name	Sec.	B.N.
1	Ahmed Mohamed Ibrahim Mahmoud	1	18
2	Hatem Essam Eldein Amin	1	52
3	Ramy Atef Mohamed AbdelRahman	2	09
4	Zeyad Ahmed Fawzy Hefny	2	13
5	Saif El-Din Ayman AbdelFatah Mohamed	2	23
6	Omar Amr Mahmoud Hafez	2	55
7	Omar Mohamed Tolba Mohamed	3	03
8	Omar Muhammad Mustafa AbdelLatif	3	05

Submitted to:

Dr. Samah Elshafiey

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Methodology

There are a variety of denoising methods available, but partial differential equations (PDEs) play a key role in the process because to their great efficiency and the fact that they require no prior information.

Second order partial differential equation:

The second order partial differential equations are used widely for image enhancement and denoising and it is based on the heat equation. These techniques have been shown that these approaches can provide a good trade-off between noise removal and edge retention. The problem is that they caused the image to be blocky and this result is unsightly and it may cause a computer vision system to mistakenly detect the edges. That is why it is considered the biggest drawbacks of utilizing a second order PDE.

The equation that governs this process for an image (I) is given by:

$$\frac{\partial I(x,y,t)}{\partial t} = \nabla \cdot \big(c(x,y,t)\nabla I(x,y,t)\big)$$

Where:

- $c(x,y,t)$ is the diffusion coefficient
- t is the time
- $\nabla \cdot$ is the divergence operator
- ∇ is the gradient operator

The diffusion coefficient could be either constant or variable. In case of being a constant value (i.e. is independent of x, y, and t) all pixels of the image will be smoothen by the same way without differentiating between the pixels in the middle and the edge pixels. Clearly, this isn't the best option and in order to correct this flaw, it is preferable to choose a diffusion coefficient that depends on x and y which will lead the above equation to be a linear one. Yet, if the diffusion coefficient $c(.)$ depends on each and every image the equation becomes nonlinear.

The diffusion coefficient is given by:

$$c(x,y,t) = \frac{1}{\left(1 + \frac{|\nabla I|^2}{k^2}\right)}$$

Where k controls the diffusion factor.

The diffusion factor c in the above equation varies at different spots in the image. This component has a small value in points where the image's gradient is large. As a result, the diffusion factor at the margins would be low, and the edges would be spared from smoothing. However, as it was mentioned before that the second order partial differential equations causes blocky effect to the image which is undesired, and to overcome this flaw, You and Kaveh established a fourth-order partial differential equations using the Laplacian operator instead of the gradient of picture intensity (PDEs). Since then, fourth order PDEs have been widely used for image denoising. Despite the fact that the model suggested by You and Kaveh has had a lot of success in image processing, it has certain flaws. On the one hand, isolated white and black speckles would be added to the denoised image. The approach, on the other hand, does not include an automated stopping device in the iteration process, thus users must determine a maximum number of iterations by trial and error. As a result, the denoising quality cannot be totally regulated. Because fourth-order PDEs are more effective at suppressing oscillation at high frequencies than second-order PDEs, the evolution of second-order PDEs becomes weak in high-frequency zones.

(a) (b) (c)

(a) Original image, (b) image processed with a constant c, (c) the blocky effect due to 2nd order PDE

Fourth order partial differential equation:

The noise that could be a Gaussian or Poisson noise could be removed by solving the partial differential equation that is represented by Euler equation

$$\frac{\partial u}{\partial t} = -\nabla^2[c(|\nabla^2 u|)\nabla^2 u] \qquad (x, y) \in \Omega, t > 0$$

Where:

- u is the grey-level function at scale t
- c is the diffusion coefficient
- ∇^2 denotes the Laplacian operator
- Ω stands for the image domain

The process aims to increasingly smoothen the image till it becomes a planar image. The step picture used by second-order PDEs to approximate the original image seems less natural than the piecewise planar image. This is why this model is able to avoid the 'block effect,' which is a common occurrence in all second-order PDE models. Yet, the solution is proposed at a time which achieves the optimal tradeoff between both the noise removal and the edge preservation. The partial differential equation is solved using the initial condition of the observed image and the PDE equation is associated with:

$$E(u) = \int_{\Omega} f(|\nabla^2 u|)d\,xdy$$

Which is the energy function of the image that represents the intensity of the pixels and from which the PDE was derived by You and Kaveh in 2000 and that model uses piecewise planar image in order to approximate to an original pure, smooth image. The differential equation could be solved numerically using an iterative approach (Finite difference method) Firstly time size (Δt) and a space grid size (h) are assumed and based on experimental approach, it was found that the best time size is 0.25 and (h) is chosen depending on the input image and the noise scale. The time and space coordinates are quantized as:

$$t = n\Delta t,\ n = 0,1,2,\cdots \qquad x = ih,\ i = 0,1,2,\cdots I \qquad y = jh,\ j = 0,1,2,\cdots J$$

Where the $Ih\ x\ Jh$ is the image domain. Consequently, we follow three-stage approach to calculate the right-hand side of Euler equation so to denoise the image. The first stage is calculating the Laplacian of the image intensity function (u)

$$\nabla^2 u_{i,j}^n = \frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n}{h^2}$$

- The number h is called the mesh size.
- The points $u_{i,j}$ = *(ih, jh),* i and j are integers called mesh points

With the boundary conditions

$$u_{-1,j}^n = u_{0,j}^n, \quad u_{I+1,j}^n = u_{I,j}^n, \quad j = 0,1,2,\cdots,J$$

$$u_{i,-1}^n = u_{i,0}^n, \quad u_{i,J+1}^n = u_{i,J}^n, \quad i = 0,1,2,\cdots,I$$

On the second stage we calculate the g function (it is calculated separately to ease the calculations)

$$g(\nabla^2 u) = c(|\nabla^2 u|)\nabla^2 u$$

$$g_{i,i}^n = g\left(\nabla^2 u_{i,i}^n\right).$$

On the third stage we calculate the Laplacian of the g function:

$$\nabla^2 g_{i,j}^n = \frac{g_{i+1,j}^n + g_{i-1,j}^n + g_{i,j+1}^n + g_{i,j-1}^n - 4g_{i,j}^n}{h^2}$$

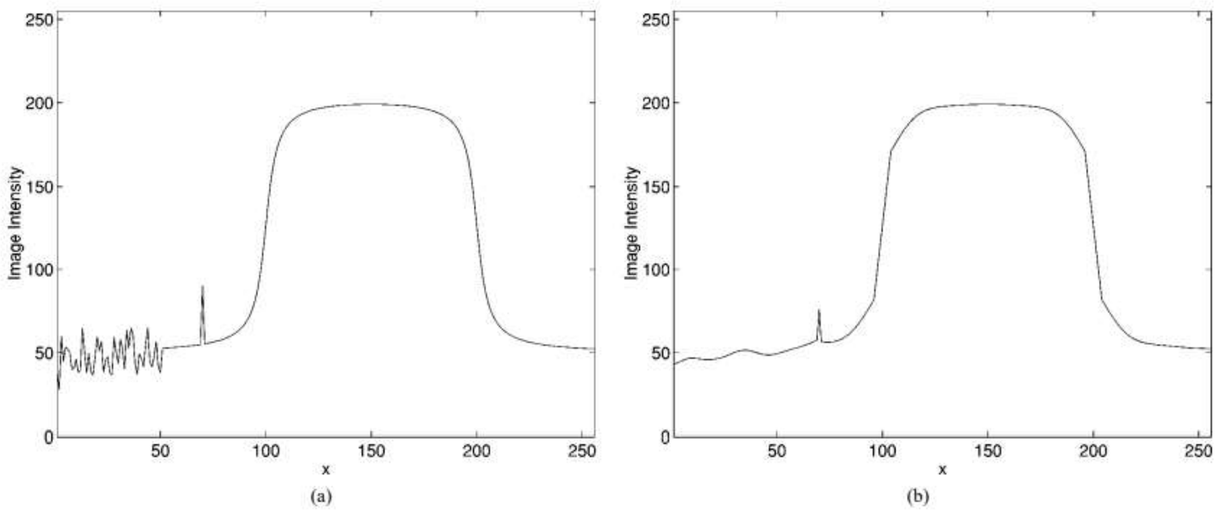
With the boundary conditions

$$g_{-1,j}^n = g_{0,j}^n, \quad g_{I+1,j}^n = g_{I,j}^n, \quad j = 0,1,2,\cdots,J$$

$$g_{i,-1}^n = g_{i,0}^n, \quad g_{i,J+1}^n = g_{i,J}^n, \quad i = 0,1,2,\cdots,I.$$

Finally, the numerical approximation to the differential equation

$$u_{i,j}^{n+1} = u_{i,j}^n - \Delta t \nabla^2 g_{i,j}^n$$



(a) Original image, (b) image processed by the proposed fourth-order PDE at t=500

Conclusion:

There are a lot of useful points or benefits in the image processing techniques whish are develop or relay on the Fourth Order PDEs, Frist of all fourth order PDEs prevent the oscillation at the high frequencies, And there is some kind of ease and flexibility to add a lot of different functional behaviors in the formulation, The advantage of using the Fourth Order PDE in image denoising is that it manages to improve the noise removing of the images and at the same time it maintain the edges not affected and that by approximating an observed image with a piecewise planar image because the piecewise planar images have less masking capability and looks more natural than step images (which are used in the second order partial differential equations) and anisotropic diffusion which result in more and multiple false edges, All the second order PDE models have many problems like “Block effect “ and that make the world search for another model can prevent that effect which had been found in the fourth order PDEs (Developed by You and Kaveh), Beside that the Fourth Order PDEs reduces fluctuation faster than the Second Orders, Also the Fourth Order PDEs can find or get a lot of image features in comparison with the Second Orders.

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