Leontief's Economic Input-Output Model For Analyzing Inter-Sectoral Dependencies in the Economy

Project Report by
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Abstract:

Leontief is an economic model which has wide applications as a basis for more advanced models. In this project we simulate this model under different setups using different numerical approaches and preform some basic case studies which could be scaled up to larger systems and economies. We will also visually represent the correlations between these industries based on the relative prices/units produced.

1. Introduction:

The Leontief input-output model is an economic model which utilizes matrices to quantitatively analyze a given system for any arbitrary ecosystem of industries and sectors. It was a revolutionary model which awarded it's developer Wassily Leontief the Nobel prize in Economics in 1973.

It considers two forms, the closed model and the open model.

The Closed Model:

The closed model considers a closed system where there is no external demand and the production of one industry is consumed internally in the same system by other industries. This works to get the relative price of the products, as the net of all products produced should be the same as the net products consumed.

$$X = AX$$

Where A is the input-output (consumption) matrix (where element a_{ij} represent the number of units produced by industry S_i needed by industry S_j to make one unit where S is the set of all industries in the system) and X is the production level vector.

The closed model considers a consumption matrix where all columns add up to 1, which is equivalent to a Markov matrix. If we are to start from any given X we will be able to see after some repeated cycles that the system approaches a steady state where the prices tend towards an equilibrium.

Alternatively, we may solve the equation

$$(A - I)X = 0$$

Where I is the identity matrix to get the relative prices of the system to be scaled by arbitrary factor of inflation or currency value.

The Open Model:

The open model considers an open system where there is external demand and the production of industries must meet the external demand. The problem is to get the production level necessary if this external demand is given.

$$X = AX + D$$

Where D is the non-zero demand vector.

By performing some transformations

$$(I - A)X = D$$

$$X = (I - A)^{-1}D$$

Hence, our problem is limited to either finding if $(I - A)^{-1}$ exists and is nonnegative since there can be no negative scaling in a real economic system, or to solve for the reduced echelon form for $(I - A)X$ and solve the augmented matrix for X for the given D.

An important point of consideration is assessing when the $(I - A)^{-1}$ exists and is nonnegative. If we are to take the geometric series of $(I - A)^{-1}$

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

According to the Cayley-Hamilton theorem, it's largest eigenvalue must also satisfy this equation and hence we can state that the largest eigenvalue λ in the equation $(1 - \lambda)^{-1}$ must converge and for this condition to be satisfied, λ must not equal one or the solution to the inverse will not exist. And by considering the case of $1/(1 - \lambda)$ the solution fails to be nonnegative if λ is greater than one.

We may use this as a test to see if an economic system is consistent. This can also act as a basis for studying systems where the resources being used are nonrenewable such as rare materials or cost limitations for more advanced models where optimizations are important to consider.

2. **Problem Description:**

We want to model both of these models to see how they perform including their properties as well as their different method of solution. We would also like to translate this math to a real micro-economic system which we will construct and study how we may value their initial prices and how their interdependencies can be changed by their respective costs.

3. **Methodology:**

We will use MATLAB to simulate these setups and plot their respective graphs and results.

We will be using a 5x5 matrix as a basis for our analysis for the sake of simplicity since no real world data is being input.

Assumptions will be arbitrarily set for the relative product consumption.

4. **Results:**

Closed Model:

We begin by studying the initial closed system where the net of each column is 1. We have two paths to attempt, getting the analytical solution of relative prices

$$(A - I)X = 0$$

Or, to start from a set of existing prices and see how the matrix causes the prices to fluctuate as a function of cycles of transactions

$$X = AX$$

Let A be a consumption matrix such that

```
A=[0.1   0.2    0.3    0.2    0.05;
0.2    0.15   0.3    0.05   0.15;
0.151  0.3    0.125  0.05   0.45;
0.159  0.25   0.075  0.2    0.15;
0.39   0.1    0.2    0.5    0.2]
```

And letting the initial X be [3;2;5;6;2] We find that

null$(A - I)$=[0.3667;0.3889;0.5119;0.3501;0.5742]

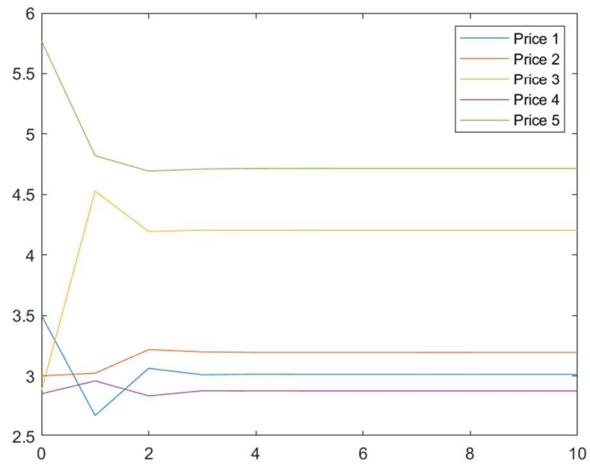

Figure 1: Prices of the products

By plotting the prices of the products X as we increase the number of cycles we see that the prices initially tend to fluctuate. But, as the number iterations increases it converges rather quickly to a stable value.

When we take the ratio of these prices to the ideal relative costs calculated from the null space, we see that they all converge to the same ratio which shows that it scales the relative prices such that the economy we are modeling reaches a kind of equilibrium where the inputs and outputs are perfectly balanced.

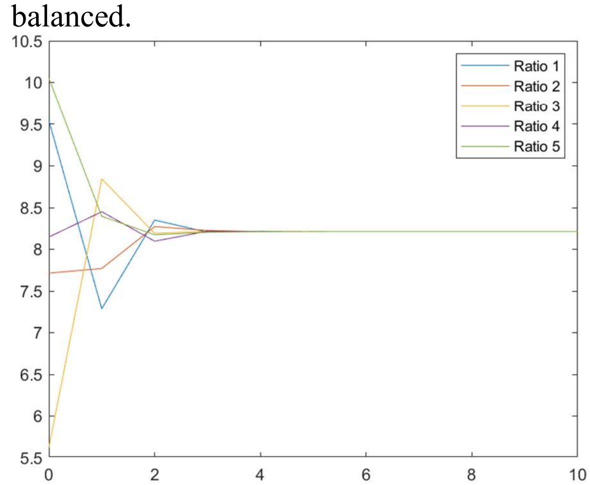

Figure 2: Ratios of the real prices to the ideal relative prices

Now, we introduce a parameter y such that it depicts the 'profit' or the amount of product being used internally by the industry per product made. We will inject this parameter into the 5^{th} column and adjust the remaining elements such that the net of the column remains 1.

```
A=[0.1   0.2   0.3   0.2    0.25*(1-y);
0.2     0.15  0.3   0.05   0.15*(1-y);
0.151   0.3   0.125 0.05   0.45*(1-y);
0.159   0.25  0.075 0.2    0.15*(1-y);
0.39    0.1   0.2   0.5    y]
```

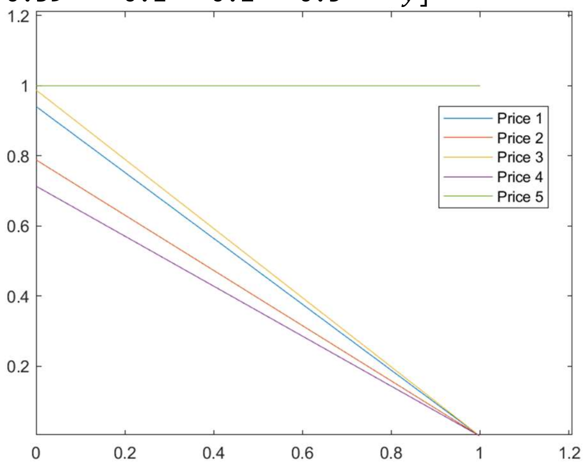

Figure 3: Relative Prices as a function of a changing parameter y the profit parameter

As we can see in Figure 3, the change in the relative prices is a constant linear function of the parameter y for the prices of all products and as the margin of profit increases, so does the relative price with respect to the remaining products. By setting the price of product 5 as the free variable in the equation we have derived the setup of a system with inelastic demand where the economy is depend mostly on one sole industry as its basis and the rest respond to its change in price.

We now choose to set this parameter as the effective efficiency of a product by instead placing the y parameter in one of the external industry product inputs which should give us an interesting result.

```
A=[0.1   0.2   0.3   0.2    0.25*(1-y);
0.2     0.15  0.3   0.05   0.15*(1-y);
0.151   0.3   0.125 0.05   0.45*(1-y);
0.159   0.25  0.075 0.2    y;
0.39    0.1   0.2   0.5    0.15*(1-y)]
```

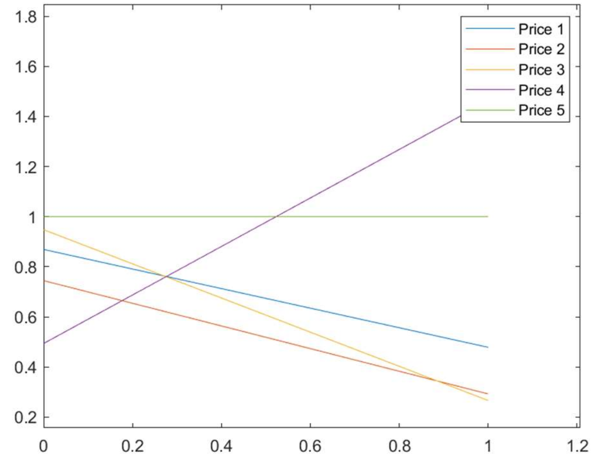

Figure 4: Relative Prices as a function of y the efficiency parameter

Unlike the previous example, it is expectedly not as debilitating as the change in the profit parameter since, at worst, the products cost is completely made up by the product who's parameter we chose to alter. This shows us that the economy we have modeled is not built on a necessity for the efficiency parameter yet it strongly affects the relative prices. As we can see the price of product 3 plummets which causes it to have a lower price than both product 2 and 4 which would be important to consider if product 3 was competing with 2 and 4.

Open Model:

Moving on to the open model, we must construct a consumption matrix by ensuring the output is greater than the inputs into the system. This can be done by ensuring that the products do not have an extreme codependence where, for example, you need multiple of product 1 to produce one of product 2 which is itself used in similar amounts to produce one of product 1. This is illustrated in the following matrix

$$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$$

The eigenvalue $\lambda = 3$ in this system which means there is no production level which could satisfy any demand in this system since the inverse is negative.

To model A, it will be easiest to model it based on a case study of a realistic system of industries.
Hence, let A be a consumption matrix such that

```
A=[0      1      0.1    1          0.05;
   0.05   0.05   0.1    0.0125     0.0125;
   0.05   0.1    0      0.0125     0.0125;
   0.05   0.05   0.3    0.1        0.15;
   0.05   0.15   0.2    0.3        0.1]
```

Letting S_i be the industry in column i:
S_1 is the government (GVMNT) collecting taxes with the output products being subsidies,
S_2 is the food industry (FOOD) which includes farming and processing,
S_3 is the health industry (HLTH) which includes pharmaceuticals and healthcare,
S_4 is the materials industry (MTRL) which includes mining and materials processing and manufacturing,
S_5 is the technology industry (TECH) which includes research, design, and manufacturing of technology.

The largest λ of this system is 0.5534, hence this system has a real nonnegative inverse and as a result is consistent. For an arbitrary demand vector D there will be a production vector X to satisfy this demand.

We initially start by arbitrarily setting D the demand vector as `D=[200;300;50;40;100]`, this gives us three paths to go, solving $(I - A)X = D$, solving for the inverse $(I - A)^{-1}$ then solving and hence solve $X = (I - A)^{-1}D$, or solving for $(I - A)^{-1}$ numerically by testing the convergence of the geometric series and applying the same equation in the previous method.

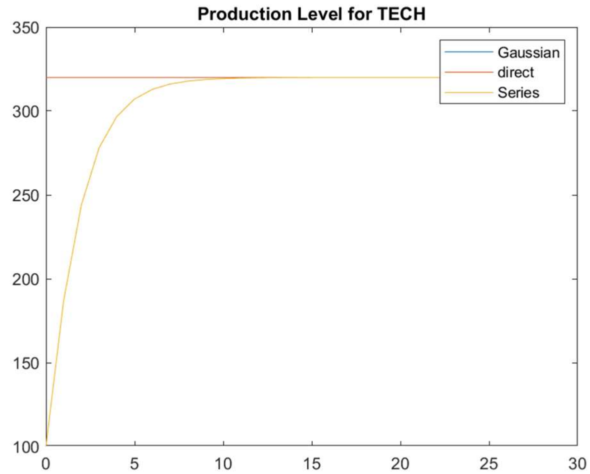

Figure 5: Sample production level for TECH industry

The Gaussian method had resulted in the same solution as the direct inverse method using MATLAB, however we can see that it takes around 10 terms for the series method to converge to the same result as the other two methods.

The same pattern was seen for the rest of the industries and there was nothing additional to analyze.

We may now try to set a parameter y and analyze how it can affect the system.

Let T be the governments tax rate, how will this affect the economy we have constructed?
Let it be such that getting λ and consequentially knowing the properties of $(I - A)^{-1}$ of the system represent the consistency of the economy. Let our A then be

```
A=[0   1      0.1    1       0.05;
   T   0.05   0.1    0.0125  0.0125;
   T   0.1    0      0.0125  0.0125;
   T   0.05   0.3    0.1     0.15;
   T   0.15   0.2    0.3     0.1]
```

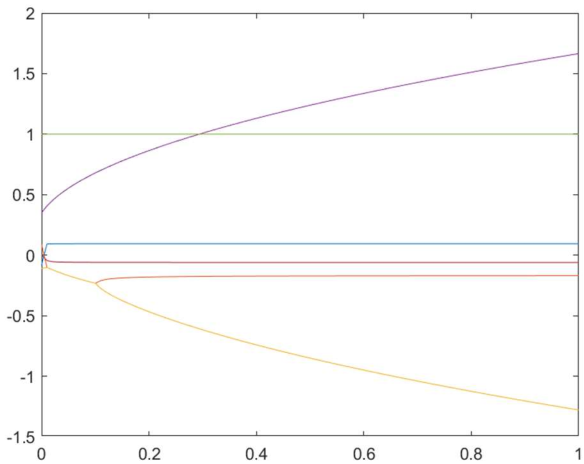

Figure 6: λ eigenvalue as a function of T

Given that the green line is a constant 1, all the other lines represent the various eigenvalues of the system, ignoring the complex values. Since we only consider the largest eigenvalues, we can see there is a distinct point where the system stops being consistent, at $T \cong 0.29$ particular. If we are to go back to the system's consumption matrix and replace this value, we would no longer be able to get an inverse and hence there would be no feasible solution to

$$X = AX + D$$

5. Conclusion:

In conclusion, we have studied this model and have taken it to its extremes to study its behavior and properties when placed under different circumstances. We constructed a case study as well to see how this model can be used in a real world system and how it could potentially be used in price setting as well as tax modeling.

Of course, there are many factors which we are not taking into account in this simple demonstration and economies are not always expressible in linear models. This however shows us a starting point to build more advanced models and could still be used for simple systems which do not hold as much importance to be accurate. Some example of such a system include item pricing in multiplayer games to balance their usage, small businesses to track the inflow and outflow of products and to model the expected products needed to satisfy the average monthly demand, and other various systems.

References:

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