On the magentic field of a spinning charge

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Introduction

Consider a spherical shell of radius R ,carrying a uniform surface charge density σ , spinning with angular velocity ω. We would like to find **B(r)** the magnetic field produced, and we use it to demonstrate the Helmholtz theorem.

Biot-Savart

The most obvious and direct approach is the use of Biot-Savart law, which gives us the following integral

$$B_{sphere} = \int_0^{\pi} B_{loop}(r)d\theta$$

where B_loop is defined by the following

$$\frac{\mu_0 \omega\sigma\rho^2 sin^2\theta}{4\pi}\int_0^{2\pi}\frac{(z-\rho cos\theta)(\rho sin\theta)(cos\phi\hat{i}-sin\phi\hat{j})+(\rho^2 sin^2\theta-\rho sin\theta(ysin\phi+xcos\phi))\hat{k}}{(x^2+y^2+(z-\rho cos\theta)^2+\rho^2 sin^2\theta-2\rho sin\theta(ysin\phi+xcos\phi))^{\frac{3}{2}}}d\phi$$

Helmholtz

As we may notice that Biot-Savart is a very cumbersone approach, we seek an analytic solution for B(r).

We make use of the Helmholtz theorem, which states that for any vector field $F(r)$, there exists a scalar field $U(r)$ and a solinoidal vector field $W(r)$ such that

$$F(r) = \nabla U + \nabla \times W$$

we can see right away that $\nabla \cdot F = \nabla \cdot (\nabla U)$, and $\nabla \times F = \nabla \times (\nabla \times W)$
Now, returning back $B(r)$, we know that

$$\nabla \cdot B = 0$$

$$\nabla \times B = \mu_0 \mathbf{J} \ Ampere's law$$

hence,

$$U = 0$$

therefore,

$$B(r) = \nabla \times W$$

If we know W we can easily find B.

$$\nabla \times B = \nabla \times (\nabla \times W) = \nabla(\nabla \cdot W) - \nabla^2 W = \mu_0 \mathbf{J}$$

as W is solinoidal, $\nabla \cdot W = 0$, hence

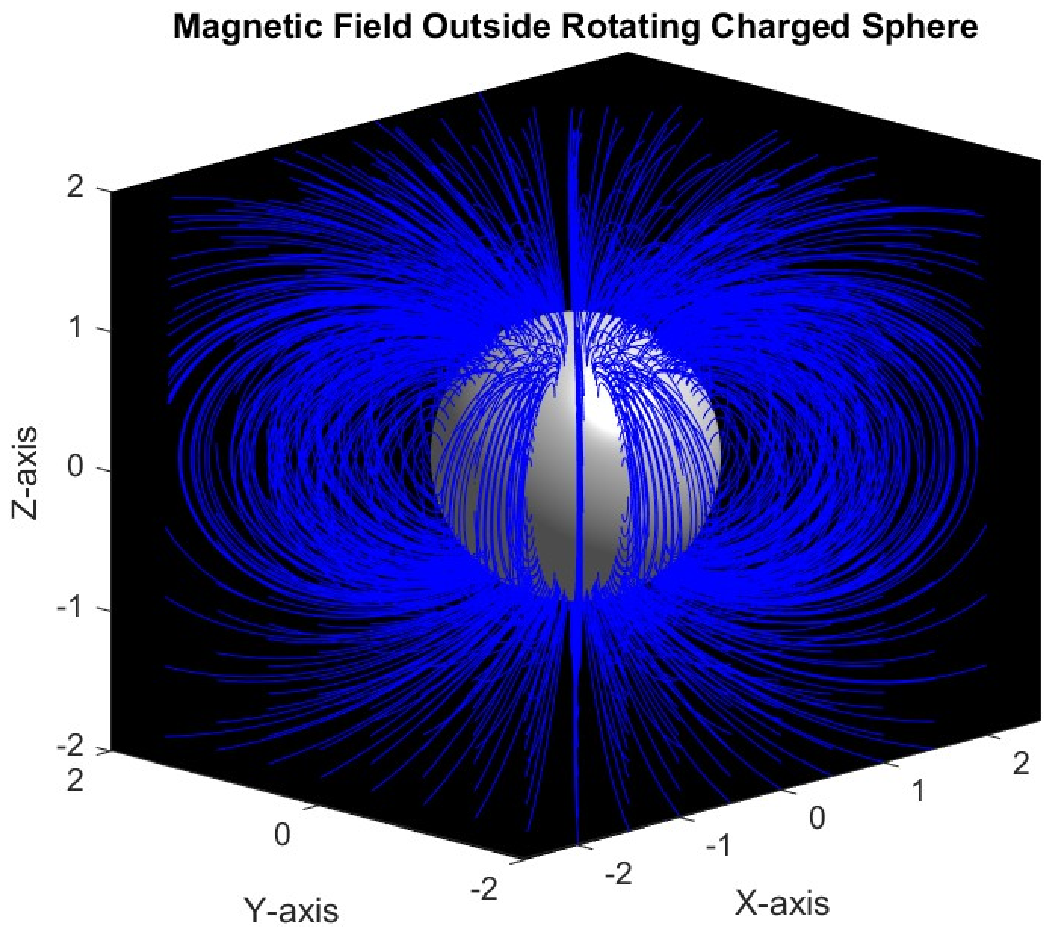
$$\nabla^2 W = -\mu_0 \mathbf{J}$$

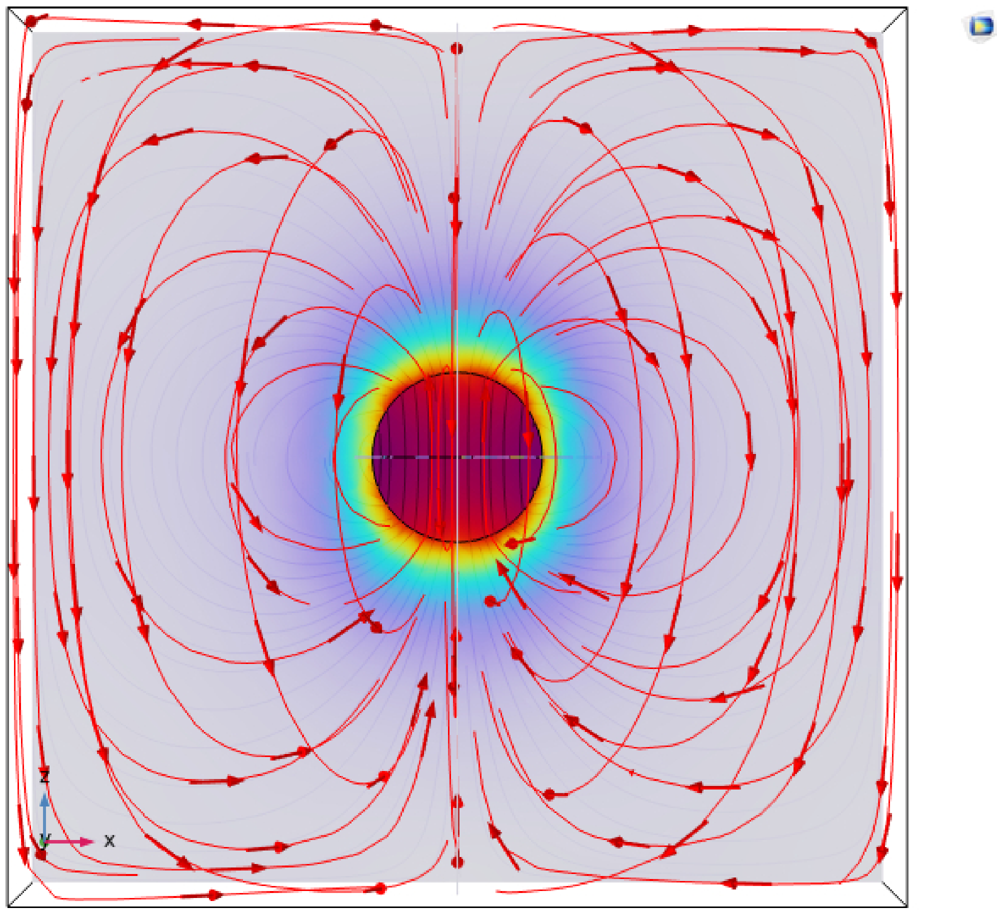
The solution will be

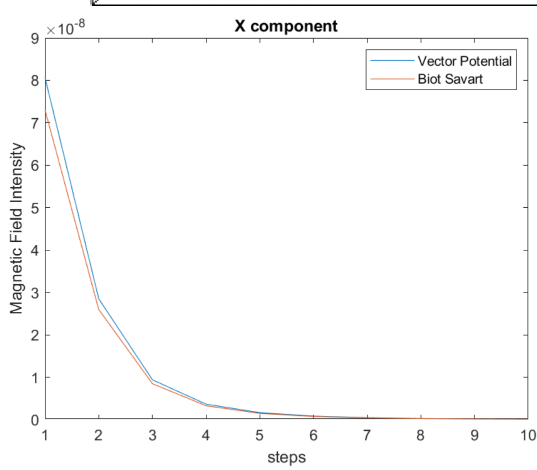
$$W(r, \theta, \phi) = \begin{cases} \frac{\mu_0 R \omega \sigma}{3} r sin\theta \, \hat{\phi} & r \leq R \\ \frac{\mu_0 R \omega \sigma}{3} \frac{sin\theta}{r^2} \, \hat{\phi} & r \geq R \end{cases}$$

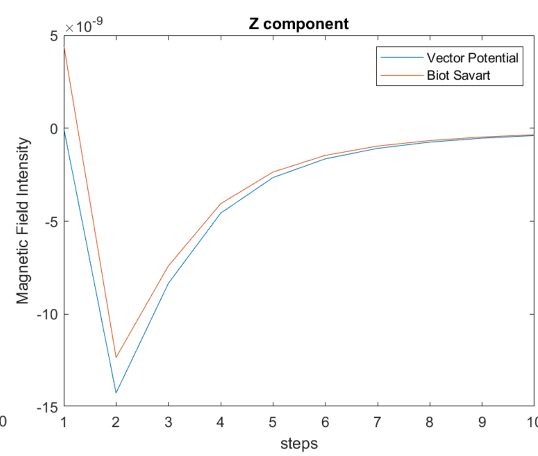
We call W the vector potential of the magnetic field. from here we can easily calculate $\mathbf{B}$

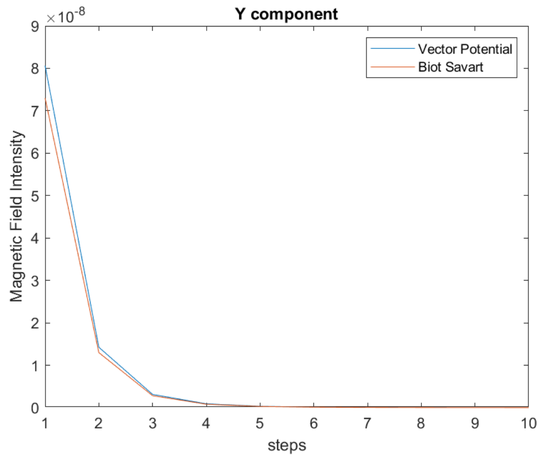
$$\mathbf{B} = \begin{cases} \frac{2\mu_0 R \omega \sigma}{3} (cos\theta \, \hat{r} - sin\theta \, \hat{\theta}) = \frac{2\mu_0 R \omega \sigma}{3} \hat{z} & r \leq R \\ \frac{\mu_0 R^4 \omega \sigma}{3} (2cos\theta \, \hat{r} + sin\theta \, \hat{\theta}) & r \geq R \end{cases}$$


Magnetic Field Outside Rotating Charged Sphere
Z-axis
2
1
0
-1
-2
Y-axis
2
0
-2
X-axis
-2
-1
0
1
2


Z
X


X component
×10^-8
Vector Potential
Biot Savart
Magnetic Field Intensity
steps


Z component
×10^-9
Vector Potential
Biot Savart
Magnetic Field Intensity
steps



conclusion

As you have seen in the discussion above, the Helmholtz theorem provides a very powerful tool for tackling problems dealing with vector fields, which otherwise would be almost impossible to solve analytically, we also suspect that Helmholtz may be more efficient to use numerically to approximate the magnetic field.