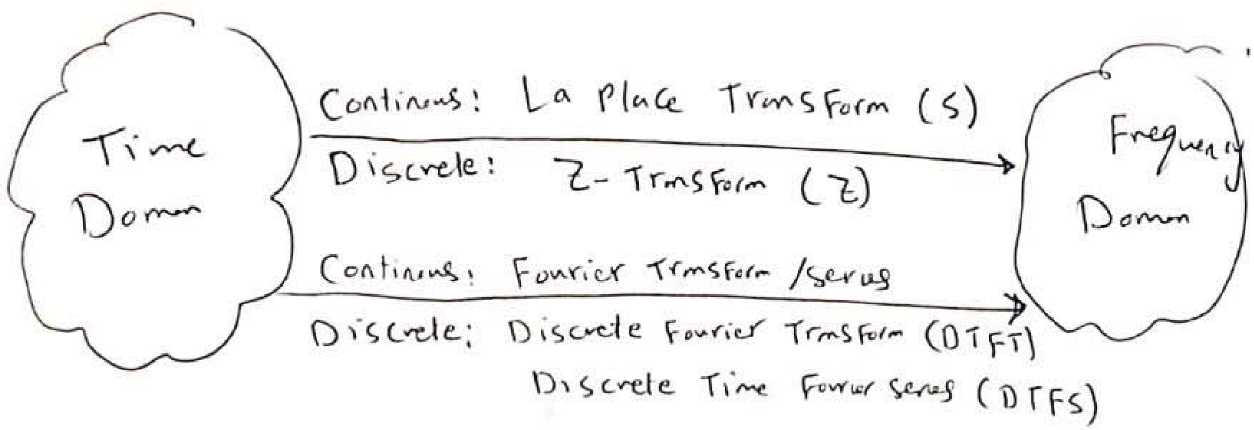
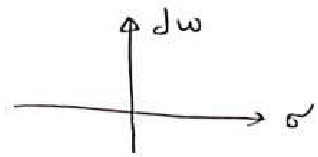
Frequency Domain Representation of LTI discrete-time systems

In some operations, Frequency domain representation is easier than time Domain. Also, we change to frequency domain to check some properties that is not seen in time domain



Recall: $s = \sigma + j\omega$



IF $\sigma = 0 \Rightarrow$ use Fourier representation

IF $\sigma \neq 0 \Rightarrow$ use Laplace / Z-transform representation

Z - Trans Form

Mapping between S-plane & Z-plane:

$$f(t) \xrightarrow[t \to nT]{\text{Sampling}} f(n)$$

$$e^{-st} \longrightarrow e^{-snT}$$

La place Trans Form: $F(S) = L\{f(t)\}$

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

After sampling $t \longrightarrow nT$ & $f(t) \longrightarrow f(n)$

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)\, e^{-snT}$$

$$F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}, \text{ where } z = e^{ST}$$

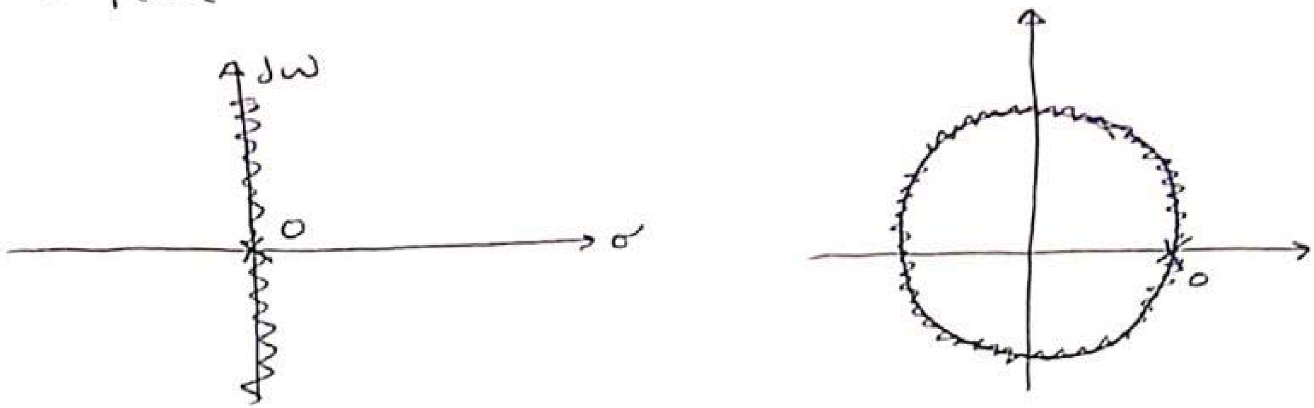
$$\therefore F(z) = \sum_{n=-\infty}^{\infty} f(n)\, z^{-n}$$

General Form of Z-trans form

Note

$$z = e^{ST} \quad , \quad s = \sigma + j\omega$$

S-plane



In the s-plane, ($j\omega$-axis) is mapped as a unit circle in the z-plane.

$$z = e^{ST} = e^{(\sigma + j\omega)T} \quad \text{but } \sigma = 0 \text{ For } j\omega\text{-axis}$$

$$z = e^{j\omega T} = \cos \omega T + j \sin \omega T = 1 \angle \omega T$$

phase

magnitude $= \sqrt{\sin^2 \omega T + \cos^2 \omega T}$

$\therefore$ $j\omega$-axis (s-plane) $\longrightarrow$ $1 \angle \omega T$ (z-plane) unit circle with different phases

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Q: Discuss the mapping between s-plane & z-plane, then explain the stability for both

Solution

$Z = e^{ST}$ where S: variable of Laplace

$$S = \sigma + J\omega$$

Z: variable of Z-Transform

T: Sampling time.

$$\therefore Z = e^{(\sigma + J\omega)T} = e^{\sigma T} e^{J\omega T} = e^{\sigma T}\left[\cos(\omega T) + J\sin(\omega T)\right]$$

$$\therefore Z = e^{\sigma T}\cos(\omega T) + J\, e^{\sigma T}\sin(\omega T)$$

$$Z = u + JV$$

Where

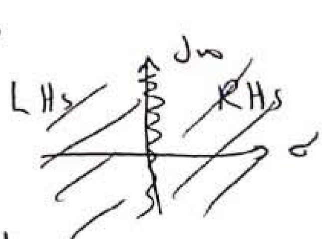
$$u = Re\{Z\} = e^{\sigma T}\cos(\omega T)$$

$$V = Im\{Z\} = e^{\sigma T}\sin(\omega T)$$

$$\sigma = Re\{S\}$$

$$\omega = Im\{S\}$$

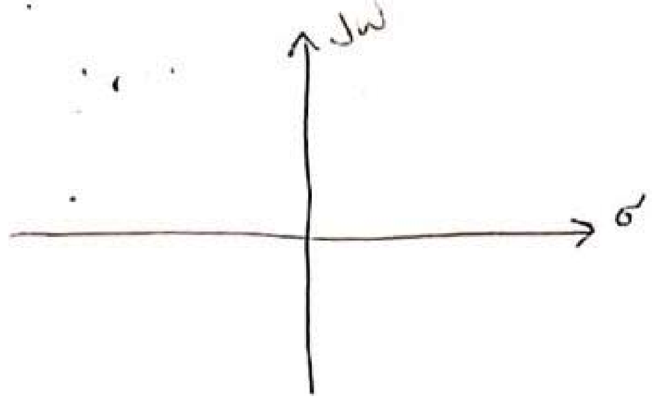
$\boxed{\text{Re Call}}$ S-plane has 3 regions



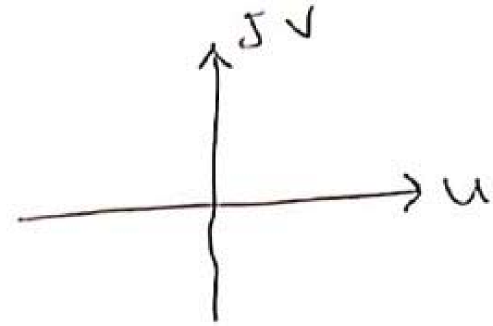
(LHs) Left Hand Side : $\sigma < 0$ Stable

(RHs) Right Hand Side: $\sigma > 0$ unstable

($J\omega$) axis : $\sigma = 0$ marginally stable



S-plane



Z-plane

→ $\boxed{\sigma = 0}$ jω ايه $\Rightarrow$ $u = \sin \omega T$
$v = \cos \omega T$

$\boxed{u^2 + v^2 = 1}$ Circle radius = 1

→ $\boxed{\sigma > 0}$ Right side of S-plane

$u = e^{\sigma T} \sin \omega T$, $v = e^{\sigma T} \cos \omega T$.

$\boxed{u^2 + v^2 = e^{2\sigma T}}$ → circles greater than 1
region outside r=1

→ $\boxed{\sigma < 0}$ Left side of S-plane

$\boxed{u^2 + v^2 = e^{2\sigma T}}$ inside circle r=1

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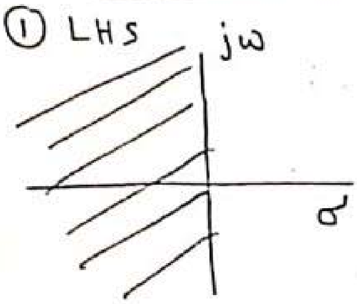
Then

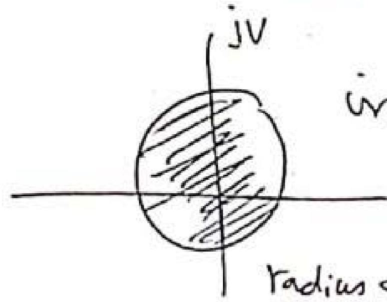
S-plane (for Continuous)		Z-plane (for discrete)
① LHS	يناظر → Stable	inside
② RHS	يناظر → unstable	outside
③ jw axis	يناظر → marginly Stable	

① LHS



يناظر →

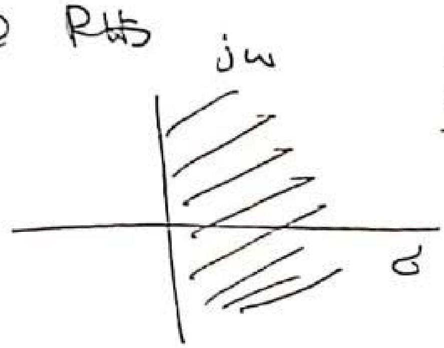
Stable



inside

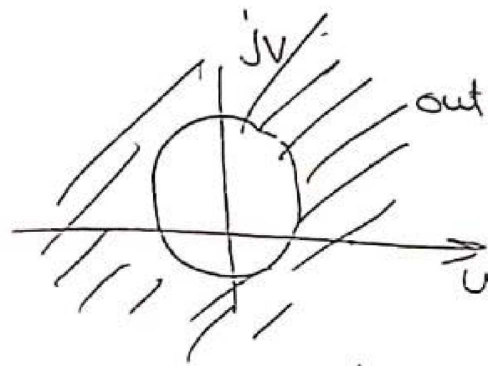
radius < 1

② RHS



يناظر →

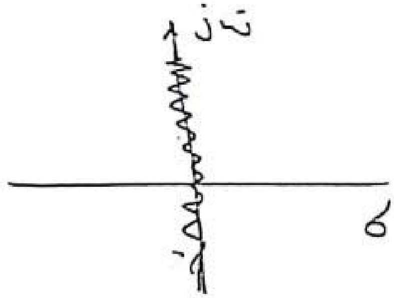
unstable



outside

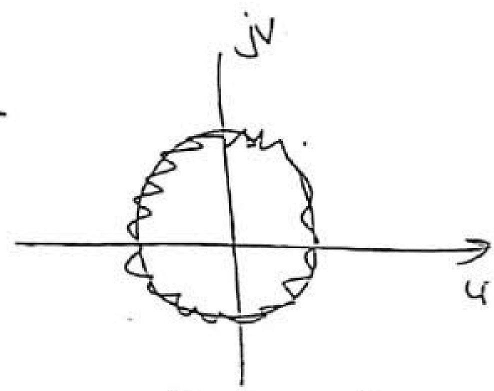
radius > 1

③ jw axis



يناظر →

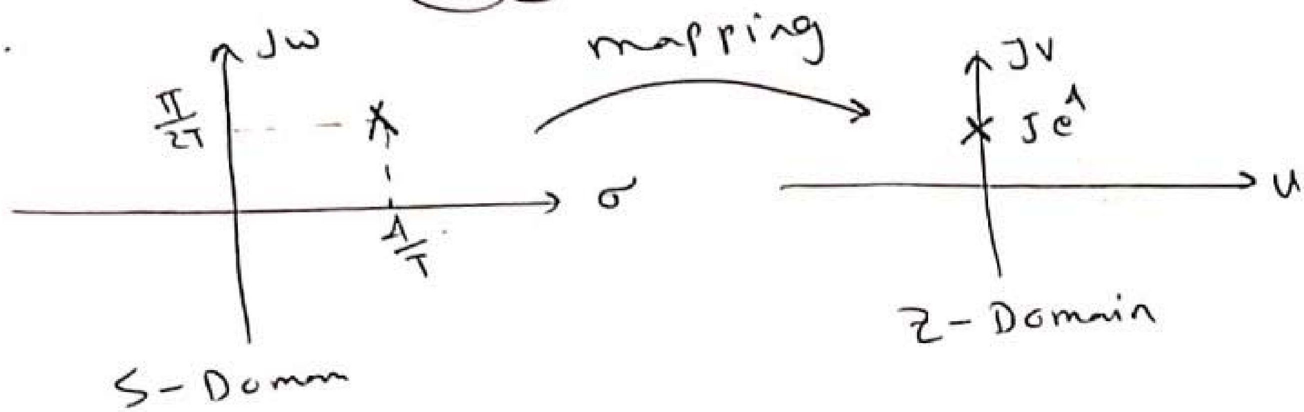
marginly Stable



radius = 1

IF $S = \frac{A}{T} + J\frac{\pi}{2T}$, Find Z_{equiv}?

Sol



given: $\sigma = \frac{A}{T}$, $w = \frac{\pi}{2T}$

$$u = e^{\sigma T}\cos(\omega T) = e^{A}\cos(\tfrac{\pi}{2}) = \boxed{\text{zero}}$$

$$v = e^{\sigma T}\sin(\omega T) = e^{A}\sin(\tfrac{\pi}{2}) = \boxed{e^{A}}$$

$$\therefore\ Z = 0 + J e^{A} = J e^{A}$$

$\boxed{\text{ex}}$ IF $Z = 0.25 + J\,0.26$ Find S_{equiv} ?

Sol

given: $u = 0.25$, $v = 0.26$

$$u = 0.25 = e^{\sigma T}\cos(\omega T)$$

$$v = 0.26 = e^{\sigma T}\sin(\omega T)$$

$$\frac{v}{u} = \frac{0.26}{0.25} = \frac{\cancel{e^{\sigma T}} \sin(\omega T)}{\cancel{e^{\sigma T}} \cos(\omega T)} = \tan(\omega T)$$

$$\tan(\omega T) = \frac{0.26}{0.25} \rightsquigarrow \omega T \approx \frac{\pi}{4}$$

$$\boxed{\omega \simeq \frac{\pi}{4T}}$$ Sub in $u = 0.25 = e^{\sigma T} \cos(\omega T$

$$0.25 = e^{\sigma T} \cos\left(\frac{\pi}{4T} T\right)$$

$$e^{\sigma T} = \frac{0.25}{\cos\left(\frac{\pi}{4}\right)} = \frac{0.25}{\frac{1}{\sqrt{2}}} = 0.35$$

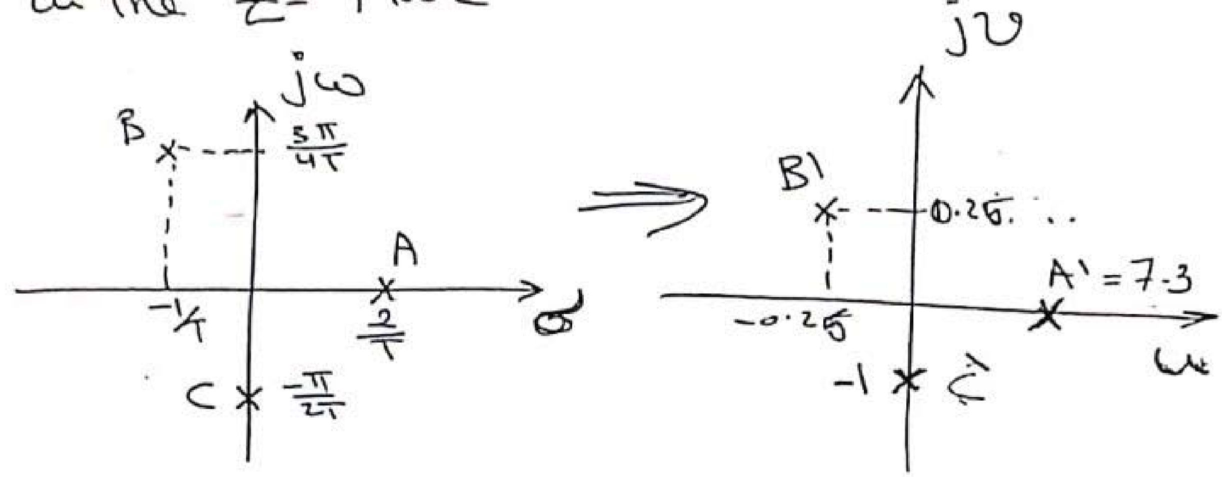
$$e^{\sigma T} = 0.35 \quad (\text{taking } \ln)$$

$$\sigma T = \ln(0.35) \longrightarrow \boxed{\sigma = \frac{-1}{T}}$$

$$\therefore s = \sigma + j\omega = \boxed{\frac{-1}{T} + j\frac{\pi}{4T}}$$

ex(2)

use the relation betn s-plane & Z-plane to find the locations of A, B, C poles in the Z-Plane.



S-plane.

Z-plane

Sol

$\because A = \frac{2}{T} + j0 \longrightarrow \sigma = \frac{2}{T},\ \omega = 0$

$\therefore u = e^{\sigma T}\cos(\omega T) = e^{2}\cdot 1 = 7.3$

$v = e^{\sigma T}\sin(\omega T) = 0$

$\therefore A \xrightarrow[\text{Z-pl}]{\text{in}} Z = e^{2} + j0 = 7.3$

$B = \frac{-1}{T} + j\frac{3\pi}{4T} \longrightarrow \sigma = \frac{-1}{T},\ \omega = \frac{3\pi}{4T}$

$\therefore u = e^{\sigma T}\cos(\omega T) = e^{-1}\cos\left(\frac{3\pi}{4}\right) = -0.26$

$v = e^{\sigma T}\sin(\omega T) = e^{-1}\sin\left(\frac{3\pi}{4}\right) = 0.26$

$\therefore B\backslash \xrightarrow{@} Z = -0.26 + j0.26$

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$$C = 0 - J\frac{\pi}{2T} \longrightarrow \sigma = 0,\ w = \frac{-\pi}{2T}$$

$$u = e^{\sigma T}\cos(wT) = e^{0}\cos\left(\frac{-\pi}{2T}T\right) = \cos\left(-\frac{\pi}{2}\right) = 0$$

$$V = e^{\sigma T}\sin(wT) = \sin\left(\frac{-\pi}{2T}T\right) = -1$$

$$c' \xrightarrow[z\text{-plane}]{@} Z = 0 - J1 = -J1$$

Summary

IF given point in Z-domain

$$Z = \bigcirc + J\triangle$$

$$u = \bigcirc$$

$$V = \triangle$$

1) $$\frac{V}{u} = \frac{e^{\sigma T}\sin(wT)}{e^{\sigma T}\cos(wT)} = \tan(wT)$$

get $w = \_\_$

2) substitute $u = e^{\sigma T}\cos(wT)$

get $\sigma = \_\_$

$\therefore$ $\boxed{S = \sigma + Jw}$

mapping of point in -S- domain

IF given point in S-Domain

$$S = \square + J\triangle$$

$\therefore$ $\sigma = \square$

$w = \triangle$

$$u = e^{\sigma T}\cos(wT) = \_\_$$

$$V = e^{\sigma T}\sin(wT) = \_\_$$

$$\boxed{Z = u + JV}$$

mapping of Point in Z-Domain

Z - Trans Form

$$x(n) \xrightarrow{\text{Z-TrmsForm}} X(z)$$

$$X(z) \xrightarrow{\text{inverse Z-TrmsForm}} x(n)$$

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

because X(Z) is a sum of power series so there will be a Condition of existance of X(z).

$\boxed{ex}$ $x(n) = \begin{cases} 2 & , n=0 \\ 3 & , n=1 \\ 4 & , n=2 \\ 0 & , o.w \end{cases}$ Find X(z)

SoL

$$X(z) = \sum_{n=0}^{2} x(n)\, z^{-n} = x(0)\, z^{-0} + x(1)\, z^{-1} + x(2)\, z^{-2}$$

$$= \boxed{2 + 3z^{-1} + 4z^{-2}}$$

Z - TransForm of Elementary signals

$$\delta(n) \xrightarrow{\text{Z-Trans}} 1$$

$$u(n) \longrightarrow \frac{z}{z-1}, \quad |z|>1$$

$$(a)^n u(n) \longrightarrow \frac{z}{z-a}, \quad |z|> |a|$$

$$e^{-n\alpha} u(n) \longrightarrow \frac{z}{z-e^{-\alpha}}, \quad |z|> e^{-\alpha}$$

$\boxed{\text{Recall}}$ $\sum_{n=0}^{\infty}(a)^n = \begin{cases} \frac{1}{1-a} & , |a|<1 \\ \infty & , \text{o.w} \end{cases}$

→ prove $Z\{\delta(n)\} = 1$

<u>Sol</u>

$$Z\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} \delta(n)\, \cancel{z^{0}}_{1} = 1$$

→ prove: $Z\{u(n)\} = \frac{z}{z-1}$

$\boxed{\text{Sol}}$

$$Z\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n$$

$$Z\{u(n)\} = \frac{1}{1-z^{-1}}, \quad |z^{-1}|<1 \Rightarrow \left|\frac{1}{z}\right| < 1$$

$$\boxed{Z\{u(n)\} = \frac{z}{z-1}, \quad |z| > 1}$$

→ prove: $Z\{a^n u(n)\} = \frac{z}{z-a}, \quad z > a$

$\boxed{\text{Sol}}$

$$Z\{(a)^n u(n)\} = \sum_{n=-\infty}^{\infty} (a)^n u(n)\, z^{-n} = \sum_{n=0}^{\infty} (a)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1-a z^{-1}}, \quad |a z^{-1}|<1$$

$$\left|\frac{a}{z}\right| < 1$$

$$= \boxed{\frac{z}{z-a}, \quad |z| > a}$$

Region of Convergence (Roc):

→ Convergent: (finite) within boundries

→ Divergent: not bounded (tends to infinite)

Roc → The Region of Values that makes any function finite (i.e: Convergent)

example Find Z-Transform

$$x(n) = -a^{n} \, u(-n-1)$$

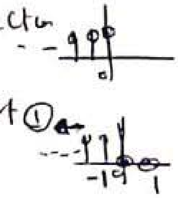
Solution

Note $u(-n-1) = \overrightarrow{u}-[n+1]$

$n \leq -1$

1) reflection

2) shift ①



$$\therefore F(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{-1} -(a^{n})(z^{-n})$$

[Not the Form of Geometric series]

$$F(z) = -\left[\cdots\cdots + a^{-3} z^{3} + a^{-2} z^{2} + a^{-1} z \right]$$

Take $(a^{-1} z)$ common Factor

$$F(z) = -a^{-1} z \left[\cdots + a^{-2} z^{2} + a^{-1} z + 1 \right]$$

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Where

$$1 + a^{-1}z + a^{-2}z^2 \cdots = \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$1 + a^{-1}z + a^{-2}z^2 \cdots = \frac{1}{1 - a^{-1}z}, \quad |a^{-1}z| < 1$$

$$\left|\frac{z}{a}\right| < 1$$

$$|z| < a$$

$$\therefore F(z) = -a^{-1}z \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= -a^{-1}z \cdot \frac{1}{1 - a^{-1}z}, \quad |z| < a$$

$$F(z) = \frac{-z/a}{1 - \frac{z}{a}}, \quad |z| < a$$

$$\therefore F(z) = \frac{-z}{a - z}, \quad |z| < a$$

$$F(z) = \frac{z}{z - a}, \quad |z| < a$$

Note that ROC: $|z| < a$

→ Although the previous problem has the same z-transform of $a^n u(n)$ $\left[\frac{z}{z-a}\right]$ but it has different range of convergence.

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Properties of Z-Transform

[1] Scaling:

$$f(n) \longrightarrow F(z)$$

$$c\,f(n) \longrightarrow c\,F(z)$$

Proof

$$z\{f(n)\} = F(z)$$

$$\therefore z\{c\,f(n)\} = \sum_{n=-\infty}^{\infty} c\,f(n)\,z^{-n}$$

$$z\{c\,f(n)\} = c\sum_{n=-\infty}^{\infty} f(n)\,z^{-n} = c\,F(z)$$

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[2] Linearity:

$$z\{c_1 f_1(n) \pm c_2 f_2(n)\} = c_1 F_1(z) + c_2 F_2(z)$$

Proof

$$z\{c_1 f_1(n) \pm c_2 f_2(n)\} = \sum_{n=-\infty}^{\infty} \left[c_1 f_1(n) \pm c_2 f_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[c_1 f_1(n)\,z^{-n} + c_2 f_2(n)\,z^{-n}\right]$$

$$= c_1 \sum f_1(n)\,z^{-n} + c_2 \sum_{n=-\infty}^{\infty} f_2(n)\,z^{-n}$$

$$= c_1 F_1(z) + c_2 F_2(z)$$

ex:

$$x(n) = \delta(n) + \left(\tfrac{1}{2}\right)^n u(n).$$

Sol

$$X(z) = 1 + \frac{z}{z - \frac{1}{2}} \quad , \quad |z| > \tfrac{1}{2}$$

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ex: $x(n) = \left(\tfrac{1}{2}\right)^n u(n) + (2)^n u(n)$

$$X(z) = \frac{z}{(z - \frac{1}{2})} + \frac{z}{(z-2)}$$

& " $|z| > \frac{1}{2}$ " & " $|z| > 2$ "   نأخذ التقاطع

$$\therefore X(z) = \frac{z}{z - \frac{1}{2}} + \frac{z}{z-2}$$

For $|z| > 2$

ex:

$$x(n) = \delta(n) + u(n) + \left(\frac{-1}{2}\right)^n u(n)$$

$$X(z) = \underset{\text{لايوجد شرط}}{1} + \frac{z}{z-1} + \frac{z}{z+\frac{1}{2}}$$

$$z > 1 \qquad z > \left|-\tfrac{1}{2}\right| \Rightarrow |z| > \tfrac{1}{2}$$

$$X(z) = 1 + \frac{z}{z-1} + \frac{z}{z+\frac{1}{2}}$$

For $|z| > 1$.

$\boxed{\text{ex}}$ Find Z-Transform For $x(n) = \sin(\alpha T n)$, $n \geq 0$

Sol

$$x(n) = \sin(\alpha T n)u(n) = \frac{1}{2J}\left[e^{J\alpha T n} - e^{-J\alpha T n}\right]u(n)$$

$$= \frac{1}{2J}\left[\left(e^{J\alpha T}\right)^n - \left(e^{-J\alpha T}\right)^n\right]u(n)$$

$$X(z) = \frac{1}{2J} Z\left\{\left(e^{J\alpha T}\right)^n u(n) - \left(e^{-J\alpha T}\right)^n u(n)\right\}$$

$|z| > |e^{J\alpha T}| \Rightarrow |z| > 1$ $\qquad$ $|z| > |e^{-J\alpha T}|$, $|z| > 1$

$$X(z) = \frac{1}{2J}\left[\frac{z}{z - e^{J\alpha T}} - \frac{z}{z - e^{-J\alpha T}}\right], |z| > 1$$

[3] Delay shift:

If $x(n) \xrightarrow{Z} X(z)$

$\therefore x(n-m) \xrightarrow{z} Z^{-m} X(z)$

Note

$$\xrightarrow[x(n)]{} \boxed{S} \xrightarrow[x(n-1)]{} \;\equiv\; \xrightarrow[x(n)]{} \boxed{z^{-1}} \xrightarrow[x(n-1)]{}$$

$$\therefore \rightarrow \boxed{S^{m}} \rightarrow \;\equiv\; \rightarrow \boxed{z^{-m}} \rightarrow$$

Shifter Block

example: $Z\{u(n-2)\} = ?$

$$u(n) \longrightarrow \frac{z}{z-2}, \quad |z| > 2$$

$$u(n-2) \longrightarrow z^{-2} \cdot \frac{z}{z-2}, \quad |z| > 2$$

proof $Z\{x(n-m)\} = Z^{-m} X(z)$

proof

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n}$$

let $n-m = u \Rightarrow n = m+u$

at $n = -\infty \Rightarrow u = -\infty$

at $n = \infty \Rightarrow u = \infty$

$$Z\{x(n-m)\} = \sum_{u=-\infty}^{\infty} x(u)\, z^{-(m+u)} = z^{-m} \sum_{u=-\infty}^{\infty} x(u)\, z^{-u}$$

$$Z\{x(n-m)\} = z^{-m} X(z)$$

ex: $x(n) = \delta(n-1) + \delta(n-2) + u(n-1)$

Sol

$\delta(n) \xrightarrow{z} 1 \quad , \quad u(n) \longrightarrow \frac{z}{z-1} \quad , |z|>1$

$X(z) = z^{-1} \cdot 1 + z^{-2} \cdot 1 + z^{-1} \cdot \frac{z}{z-1}$

$X(z) = z^{-1} + z^{-2} + \frac{1}{z-1} \quad , |z|>1$

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ex: $x(n) = (2)^n u(n-2)$

Sol

$x(n) = (2)^n u(n-2) = (2)^{n-2+2} u(n-2)$

$x(n) = (2)^2 \, (2)^{n-2} u(n-2)$

$\downarrow$

$X(z) = 4 \cdot z^{-2} \frac{z}{z-2} \quad , |z|>2$

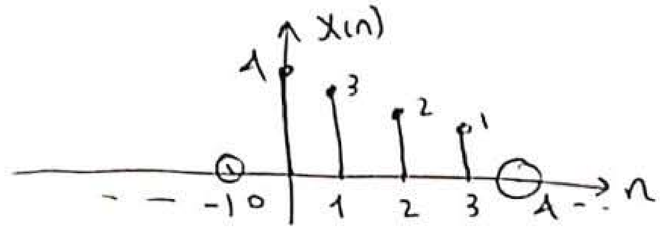
$\boxed{ex}$

$$x(n) = (4-n)\,[u(n) - u(n-4)]$$

Find $X(z)$?

$\boxed{SoL}$

$x(n) = (4-n)$, $n = 0, 1, 2, 3$ [Samples]



$$x(n) = 4\,\delta(n) + 3\,\delta(n-1) + 2\,\delta(n-2) + \delta(n-3)$$

$$\boxed{X(z) = 4 + 3z^{-1} + 2z^{-2} + z^{-3}}$$

<u>ex:</u> $x(n) = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0 & , o.w \end{cases}$ Find $X(z)$

$\boxed{Sol}$

$$x(n) = u(n) - u(n-8)$$

$\downarrow$

$$X(z) = \underset{, |z|>1}{\frac{z}{z-1}} - z^{-8} \cdot \underset{, |z|>1}{\frac{z}{z-1}} = \frac{z}{z-1}\left[1 - z^{-8}\right]$$

$, |z| > 1$

Examples

Find $X(z)$

$$x(n) = \begin{cases} 2 & , 0 \leq n \leq 30 \\ 0 & , \text{else} \end{cases}$$

Solution

$$x(n) = 2\left[u(n) - u(n-31)\right]$$

$$x(n) = 2u(n) - 2u(n-31)$$

$$X(z) = 2 \cdot \frac{z}{z-1} - 2z^{-31} \frac{z}{z-1}$$

$|z| > 1$ $|z| > 1$

$$\therefore X(z) = \frac{2z}{z-1} - 2z^{-31} \cdot \frac{z}{z-1}, \quad |z| > 1$$

→ For long sequence ⇒ Try to make it using unit steps before Z-transform

→ For short sequence ⇒ Make it in terms of deltas before Z-transform

· Example

Find Z-transform of the following

(1) $x(n) = 1 \,, \quad 0 \leq n \leq 15$

(2) $x(n) = n^2 \left[u(n) - u(n-3) \right]$

(3) $x(n) = \sin\left(\frac{\pi}{4} n\right) u(n)$

Solution

(1) $x(n) = 1 \,, \quad 0 \leq n \leq 15$

$$x(n) = u(n) - u(n-16)$$

$$\downarrow$$

$$X(z) = \underbrace{\frac{z}{z-1}}_{|z|>1} - z^{-16} \underbrace{\frac{z}{z-1}}_{|z|>1} = \frac{z\left(1 - z^{-16}\right)}{(z-1)} \,, \quad |z|>1$$

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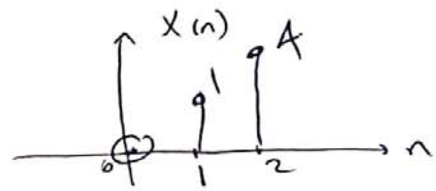
(2) $x(n) = n^2 \left[u(n) - u(n-3) \right]$

$$= n^2 \,, \quad n = 0, 1, 2$$



$$\therefore \; x(n) = \delta(n-1) + 4\,\delta(n-2)$$

$$\downarrow$$

$$X(z) = z^{-1} + 4 z^{-2}$$

(3) $x(n) = \sin\left(\frac{\pi}{4} n\right) u(n)$

$\downarrow$

Recall: $\sin x = \frac{1}{2J}\left[e^{Jx} - e^{-Jx}\right]$

$\cos x = \frac{1}{2}\left[e^{Jx} + e^{-Jx}\right]$

$\therefore\ x(n) = \frac{1}{2J}\left[e^{J\frac{\pi}{4}n} - e^{-J\frac{\pi}{4}n}\right] u(n)$

$$x(n) = \frac{1}{2J}\left[e^{J\frac{\pi}{4}n} u(n) - e^{-J\frac{\pi}{4}n} u(n)\right]$$

$$x(n) = \frac{1}{2J}\left[\left(e^{J\frac{\pi}{4}}\right)^n u(n) - \left(e^{-J\frac{\pi}{4}}\right)^n u(n)\right].$$

$\downarrow$ Z-Transform

$$X(z) = \frac{1}{2J}\left[\frac{z}{z - e^{J\frac{\pi}{4}}} - \frac{z}{z - e^{-J\frac{\pi}{4}}}\right]$$

$|z| > \underbrace{|e^{J\frac{\pi}{4}}|}_{1} \Downarrow |z| > 1$ $\qquad$ $|z| > |e^{-J\frac{\pi}{4}}| \Downarrow |z| > 1$

$$X(z) = \frac{1}{2J}\left[\frac{z}{z - e^{J\frac{\pi}{4}}} - \frac{z}{z - e^{-J\frac{\pi}{4}}}\right], |z| > 1$$

$$X(z) = \frac{1}{2J} \frac{z\left(z - e^{-J\frac{\pi}{4}}\right) - z\left(z - e^{J\frac{\pi}{4}}\right)}{\left(z - e^{J\frac{\pi}{4}}\right)\left(z - e^{-J\frac{\pi}{4}}\right)}, |z| > 1$$

$$X(z) = \frac{z\left[e^{J\frac{\pi}{4}} - e^{-J\frac{\pi}{4}}\right]}{2J\left[z^2 - z\left[e^{J\frac{\pi}{4}} - e^{-J\frac{\pi}{4}}\right] + 1\right]} = \frac{2Jz \sin\frac{\pi}{4}}{2J\left[z^2 - z\, 2J \sin\frac{\pi}{4} + 1\right]}$$

Convolution

$$x(n) * h(n) \longrightarrow X(z) \cdot H(z)$$

[used in LTI system as will be seen later]

$$y(n) = x(n) * h(n)$$

$$\Big\downarrow \text{Z-Transform}$$

$$Y(z) = X(z)\ H(z)$$

where $H(z)$: Transfer Function of LTI system.

Proof: prove $Z\{x(n) * h(n)\} = X(z) \cdot H(z)$

$$Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} [x(n) * h(n)]\, z^{-n}, \quad \text{where } x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} x(k)\, h(n-k)\Big] z^{-n}, \quad \text{let } n-k = u \Rightarrow n = k+u$$

at $n = -\infty \Rightarrow u = -\infty$

at $n = \infty \Rightarrow u = \infty$

$$Z\{x(n) * h(n)\} = \sum_{u=-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} x(k)\, h(u)\Big] z^{-(k+u)},$$

$$Z\{x(n) * h(n)\} = \sum_{u=-\infty}^{\infty} h(u)\, z^{-u} \cdot \sum_{k=-\infty}^{\infty} x(k)\, z^{-k} = \boxed{H(z) \cdot X(z)}$$

the use of Convolution

if given: $x(n) = \checkmark$, $h(n) = \checkmark$

Impulse response

(o/p when i/p $\delta(n)$)

Req: $y(n) = ?$

Z-Transform

1- get $X(z)$, $H(z)$

2- $Y(z) = X(z)\,H(z)$

3- $y(n) = Z^{-1}\{Y(z)\}$.

Convolution

$y(n) = x(n) * h(n)$

Inverse Z-Transform

Inverse Z-transform Can be made using Partial Fractions

Given $X(z) = —$ Find $x(n)$

Steps:

$\boxed{1}$ obtain $\frac{X(z)}{z} = \frac{P(z)}{Q(z)}$

$\boxed{2}$ Factorize Denominator to First degree & Assume Constants

$$(a + zb) \longrightarrow \frac{A}{az+b}$$

$\boxed{3}$ Find Constants using Covering method For First degree brackets

$\boxed{4}$ $\frac{X(z)}{z} = — + —$

Multiply by (z)

$\boxed{5}$ $X(z) = z — + z —$

Apply Inverse Z-transform & get $x(n) = —$

Recall: $\frac{z}{z-a} \xrightarrow{z^{-1}} (a)^n u(n)$, $\frac{z}{z+a} \xrightarrow{z^{-1}} (-a)^n u(n)$

$$\frac{z}{z-1} \xrightarrow{z^{-1}} u(n)$$

ex Find inverse Z-Transform if

$$X(z) = \frac{z^2}{(z-1)(z-0.2)}$$

Sol

1) $$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.2)}$$

2) $$\frac{X(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.2}$$

A, B Covering

$$A = \left.\frac{z}{z-0.2}\right|_{z=1} = \frac{1}{1-0.2} = 1.25$$

$$B = \left.\frac{z}{z-1}\right|_{z=0.2} = \frac{0.2}{0.2-1} = -0.25$$

$$\frac{X(z)}{z} = \frac{1.25}{z-1} - \frac{0.25}{z-0.2}$$

$$X(z) = 1.25\,\frac{z}{z-1} - 0.25\,\frac{z}{z-0.2}$$

$$x(n) = 1.25\,u(n) - 0.25\,(0.2)^n\,u(n)$$

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$\boxed{ex}$ Find inverse Z-Transform using Partial Fraction

$$X(z) = \frac{z^2+2z+1}{z^2-\frac{3}{2}z+\frac{1}{2}}$$

(Sol)

1) $$\frac{X(z)}{z} = \frac{z^2+2z+1}{z(z^2-\frac{3}{2}z+\frac{1}{2})}$$

2). $$\frac{X(z)}{z} = \frac{z^2+2z+1}{z(z-\frac{1}{2})(z-1)}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{z-\frac{1}{2}} + \frac{C}{z-1}$$

$$A = \frac{z^2+2z+1}{(z-\frac{1}{2})(z-1)} = 2$$

$$B = \frac{z^2+2z+1}{z(z-1)} = -9$$

$$C = \frac{z^2+2z+1}{z(z-\frac{1}{2})} = 8$$

3) $X(z) = 2 \cancel{\frac{z}{z}} + (-9) \frac{z}{z-\frac{1}{2}} + 8 \frac{z}{z-1}$

$$X(z) = 2 - 9 \frac{z}{z-\frac{1}{2}} + 8 \frac{z}{z-1}$$

$$x(n) = 2\,\delta(n) - 9 \left(\tfrac{1}{2}\right)^n u(n) + 8 (1)^n u(n)$$

$\boxed{ex}$ Find Inverse Z-Transform

$$X(z) = \frac{z^{-1}}{1 - 0.25 z^{-1} - 0.375 z^{-2}}$$

Sol

← بالضرب في $\frac{z^2}{z^2}$ حتى تكون على الصورة

$$X(z) = \frac{z}{z^2 - 0.25 z - 0.375}$$

Steps

1) $$\frac{X(z)}{z} = \frac{1}{z^2 - 0.75z - 0.375}$$

$$\frac{X(z)}{z} = \frac{1}{(z+\frac{1}{2})(z-\frac{3}{4})}$$

$$\frac{X(z)}{z} = \frac{A}{z+\frac{1}{2}} + \frac{B}{z-3/4}$$

$$A = \left.\frac{1}{z-\frac{3}{4}}\right|_{z=-\frac{1}{2}} = \frac{1}{-\frac{1}{2}-\frac{3}{4}} = \frac{-4}{5}$$

$$B = \left.\frac{1}{z+\frac{1}{2}}\right|_{z=\frac{3}{4}} = \frac{4}{5}$$

$$X(z) = -\frac{4}{5}\,\frac{z}{z+\frac{1}{2}} + \frac{4}{5}\,\frac{z}{z-3/4}$$

$$x(n) = -\frac{4}{5}\cdot\left(-\frac{1}{2}\right)^n u(n) + \frac{4}{5}\left(\frac{3}{4}\right)^n u(n)$$

* Transfere function of LTI system

(1) Transfere function using Impulse response

in T.D.

$$y(n) = x(n) * h(n)$$

$\therefore Y(z) = X(z) \cdot H(z)$, where $H(z) \Rightarrow$ transfer function of the system

$$\therefore H(z) = Z\{h(n)\} = \frac{Y(z)}{X(z)}$$

(2) Transfere function using Difference equation

If the LTI system is described by the following D.E.

$$y(n) + a_1 \cdot y(n-1) + a_2\, y(n-2) + \cdots a_N\, y(n-N) = b_0\, X(n) + \cdots b_M\, X(n-\ldots$$

لازم يبقى (1)

$$\therefore H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}} = \frac{X}{Y}$$

X ← بسط

Y ← مقام

او كان اخدت Z-Transform لل D-E هيبقى $H(z) = \frac{Y(z)}{X(z)}$

③ Transfer function using Poles & zeros

we can write H(z) as

$$H(Z) = K \frac{(Z-Z_1)(Z-Z_2)(Z-Z_3)\cdots(Z-Z_M)}{(Z-P_1)(Z-P_2)\cdots(Z-P_N)}$$

<u>Where</u> K: gain

1. $Z_1, Z_2, Z_3 \cdots Z_M$ are called zero's
   [values of Z which let $H(Z)=0$]
2. $P_1, P_2, P_3, \cdots P_N$ are called poles
   [values of Z which let $H(Z)=\infty$]

<u>for example</u>

the system has two zeros $Z_1=-1, Z_2=2$
& three poles $P_1=0$, $P_2=1/2$, $P_3=\frac{1}{4}$, $K=1$

$$\therefore H(Z) = \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)(Z-P_3)} = \frac{(Z+1)(Z-2)}{Z(Z-\frac{1}{2})(Z-\frac{1}{4})}$$

Summary

- How to get $H(z)$?

1) given: $h(n) \Rightarrow H(z) = Z\{h(n)\}$

2) given: rel between $x(n)$, $y(n)$ [Difference eq]

$$y(n) + a_1 y(n-1) + a_2 y(n-2) \cdots = b_0 x(n) + b_1 x(n-1) \cdots$$

$$H(z) = \frac{x \text{ معاملات}}{y \text{ معاملات}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} \cdots}{1 + a_1 z^{-1} \cdots}$$

3) given: Block diagram! [- Direct Form I or - Direct Form II or any Block Diagram]

$\Downarrow$

Know Difference eq

$\Downarrow$

Know $H(z)$

4) given: zeros & poles, gain $= K$

$z_1, z_2 \cdots \quad p_1, p_2 \cdots$

$$H(z) = K \frac{(z - z_1)(z - z_2) \cdots}{(z - p_1)(z - p_2) \cdots}$$

Stability of LTI system

LTI system may be
- Stable
- marginally Stable
- unstable

1 Stable

Z-Domain H(z)

If all poles of H(z) lie inside the unit circle

$$\left[\, |P_{i's}| < 1 \,\right]$$

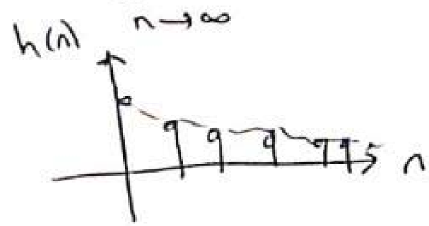
magnitude of each pole < 1

Time Domain h(n)

If h(n) decreases with the increase of n.

$[h(n) \downarrow\downarrow \text{ as } n \uparrow\uparrow]$

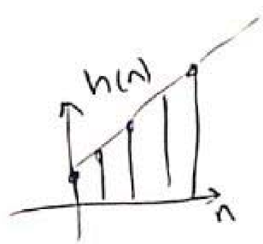
$$\lim_{n\to\infty} h(n) = 0$$



2 UnStable

Z-Domain (H(z))

If $|\text{any pole}| > 1$



Time Domain h(n)

If h(n) increases with the increase of (n)

$[h(n) \uparrow\uparrow \text{ as } n \uparrow]$

$$\lim_{n\to\infty} h(n) = \infty$$

3 Marginally Stable:

Z-Domain (H(z))

If no repeated poles lie on unit circle but one of the poles of H(z) lies on the unit circle its self & others inside unit circle.

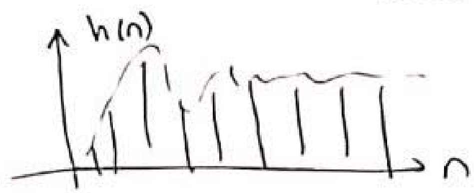
[$\|p_i\| = 1$ & others if exist inside circle]

Time Domain (h(n))

If h(n) approaches a constant level or an oscillation between 2 constant levels as n increases

[h(n) approaches constant value]

$\lim_{n \to \infty} h(n) =$ constant or oscillating



Example: Check the stability of the following LTI systems:

(1) $H(z) = \dfrac{z^{-1}}{(1 - 0.4z^{-1})(1 + 0.2z^{-1})}$

(2) $H(z) = \dfrac{1}{z(z-1)}$.

Sol4

1) $$H(z) = \frac{z^{-1}}{(1-0.4z^{-1})(1+0.2z^{-1})} \times \frac{z^2}{z^2}$$

$$H(z) = \frac{z}{(z-0.4)(z+0.2)}$$

$\therefore$ we have two poles $P_1 = 0.4$, $P_2 = -0.2$

$|P_1| = 0.4$ & $|P_2| = 0.2 < 1$ (stable)

2) $$H(z) = \frac{1}{z(z-1)}$$

we have two poles $P_1 = 0$, $P_2 = 1$

$\therefore$ (critical) marginally stable

Example

Check the stability of the LTI system represented by the following Difference equation:

$$y(n) = x(n) - x(n-1) + x(n-2) + y(n-1) - 0.5\, y(n-2)$$

SoL

Rewrite it as follows:

$$y(n) - y(n-1) + 0.5\, y(n-2) = x(n) - x(n-1) + x(n-2)$$

$$\rightarrow H(z) = \frac{1 - z^{-1} + z^{-2}}{1 - z^{-1} + 0.5\, z^{-2}}$$

multiply by $\frac{z^2}{z^2} \Rightarrow H(z) = \frac{z^2 - z + 1}{z^2 - z + 0.5}$

$$H(z) = \frac{z^2 - z + 1}{\left(z - \left(\frac{1}{2} - J\frac{1}{2}\right)\right)\left(z - \left(\frac{1}{2} + J\frac{1}{2}\right)\right)}$$

roots! $z_{12} = \frac{1 \pm \sqrt{1 - 2}}{2} = \frac{1}{2} - J\frac{1}{2}$ & $\frac{1}{2} + J\frac{1}{2}$

get magnitude of poles:

$$|P_1| = \sqrt{(\text{real})^2 + (\text{Im})^2} = \sqrt{\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^2} < 1$$

$\therefore$ Stable system.

EX(3) a LTI system has

$$y(n) + y(n-1) = x(n).$$

a) find transfer function, check stability

b) find impulse response $h(n)$

c) find the response $y(n)$ of this system when $x(n) = 10\,u(n)$ and it is initially relaxed

Solu $a_1 = 1, \quad b_0 = 1$

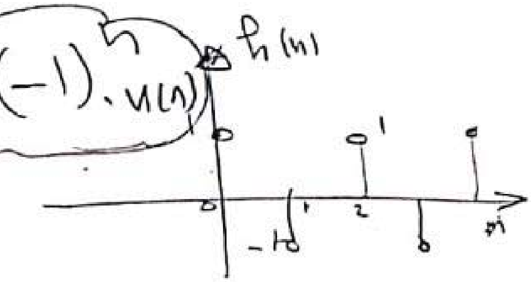
$$H(z) = \frac{1}{1 + z^{-1}}$$

a) $$\boxed{H(z) = \frac{z}{z+1}}$$ → zeros ⇒ $z_1 = 0$; poles → $P_1 = -1$ (magnitude, 1)

it has only one $|pole| = 1$ → marginally stable

b) $$h(n) = Z^{-1}\{H(z)\} = (-1)^n \cdot u(n)$$

$\frac{z}{z+1}$



c) $y(n) = x(n) * h(n)$; $x(n) = 10\,u(n)$

or $Y(z) = X(z) \cdot H(z)$.

$$\therefore Y(z) = \frac{10z}{(z-1)} \cdot \frac{z}{(z+1)}$$

$$Y(z) = \frac{10z^2}{(z+1)(z-1)}$$

$$\therefore \frac{Y(z)}{z} = \frac{10z}{(z+1)(z-1)} = \frac{\overset{5}{\cancel{K_1}}}{(z+1)} + \frac{\overset{5}{\cancel{K_2}}}{(z-1)}$$

$$\therefore Y(z) = \frac{5z}{z+1} + \frac{5z}{z-1}$$

$$\boxed{\therefore y(n) = 5u(n) + 5(-1)^n u(n)}$$

ex(4) LTI system has

$$h(n) = 2\delta(n) + \frac{5}{2}\left(\frac{1}{2}\right)^n - \frac{7}{2}\left(-\frac{1}{4}\right)^n$$

1) Find $H(z)$, Check stability?

2) implement using Direct Form II (Conanic Form)

3) Find the o/p $y(n)$ due to $x(n) = 2u(n)$

4) Find Transfer function of inverse system and Impulse response of inverse system

---

Solu

1] $H(z) = Z\{h(n)\}$

$$= 2 + \frac{5}{2}\left(\frac{z}{z-1/2}\right) - \frac{7}{2}\left(\frac{z}{z+1/4}\right)$$

$$\therefore H(z) = \frac{\left(2z^2 - \frac{1}{2}z - \frac{1}{4}\right) + \frac{5}{2}z\left(z+\frac{1}{4}\right) - \frac{7}{2}z\left(z-1/2\right)}{(z-1/2)(z+1/4)}$$

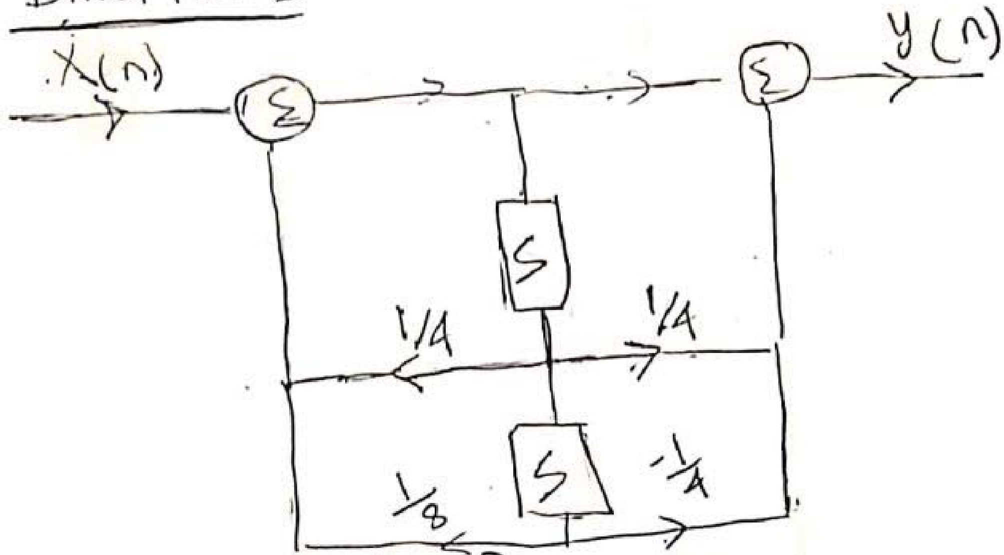
Stable : poles 1/2 and -1/4

$$H(z) = \frac{z^2 + \frac{1}{4}z - \frac{1}{4}}{z^2 - \frac{1}{4}z - \frac{1}{8}} = \frac{1 + \frac{1}{4}z^{-1} - \frac{1}{4}z^{-2}}{1 - \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$\therefore$ D.E. is

$$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + \frac{1}{4}x(n-1) - \frac{1}{4}x(n-2)$$

Direct Form II



(4) Req: $y(n) = ?$

$\therefore x(n) = 2u(n) \longrightarrow X(z) = \frac{2z}{(z-1)}$

$$\therefore Y(z) = X(z)H(z) = \frac{2z(z^2 + \frac{1}{4}z - \frac{1}{4})}{(z-1)(z-\frac{1}{2})(z+\frac{1}{4})}$$

$$\therefore \frac{Y(z)}{z} = \frac{2z^2 + \frac{1}{2}z - \frac{1}{2}}{(z-1)(z-\frac{1}{2})(z+\frac{1}{4})} = \frac{k_1}{z-1} + \frac{k_2}{z-\frac{1}{2}} + \frac{k_3}{z+\frac{1}{4}}$$

K_1, K_2, K_3 by covering.

$$\therefore Y(z) = \frac{k_1 z}{z-1} + \frac{k_2 z}{z-1/2} + \frac{k_3 z}{z+1/4}$$

$\downarrow z^{-1}$

$$\therefore y(n) = k_1 u(n) + k_2 (1/2)^n u(n) + k_3 (-1/4)^n u(n)$$

$$\therefore \boxed{h^{-1}(n) * h(n) = \delta(n)}$$

شرط Invertibility

$\xrightarrow{z}$

$$H^{-1}(z) \cdot H(z) = 1$$

$$\boxed{\therefore H^{-1}(z) = \frac{1}{H(z)} = \frac{z^2 - \frac{1}{4}z - \frac{1}{8}}{z^2 + \frac{1}{4}z - \frac{1}{4}}}$$

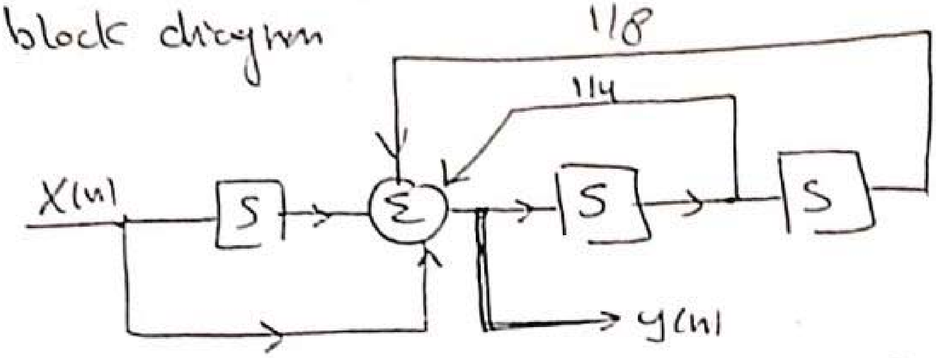
$$H^{-1}(z) = \frac{z^2 - 1/4 z - 1/8}{(z-0.39)(z-0.64)}$$

$$\therefore \frac{H^{-1}(z)}{z} = \frac{z^2 - 1/4 z - 1/8}{z(z-0.39)(z-0.64)} = \frac{A}{z} + \frac{B}{z-0.39} + \frac{C}{z-0.64}$$

$$\therefore H^{-1}(z) = A + \frac{Bz}{z-0.39} + C\frac{z}{z-0.64}$$

$$\therefore h^{-1}(n) = A\,\delta(n) + B(0.39)^n u(n) + C(0.64)^n u(n)$$

CN17 1) Determine D.E. describing the following block diagram



② what's the $H(z) \Longrightarrow$ Transfer function & impulse response of above system

Sol

<u>Sol</u> <u>by tracing</u>

$$y(n) = X(n) + X(n-1) + \frac{1}{4}\, y(n-1) + \frac{1}{8}\, y(n-2)$$

$\therefore$ D.E. is

$$\boxed{y(n) - \frac{1}{4}\, y(n-1) - \frac{1}{8}\, y(n-2) = X(n) + X(n-1)}$$

2nd order D.E.

$$H(z) = \frac{\text{معاملات } X}{\text{معاملات } Y} = \frac{1 + z^{-1}}{1 - \frac{1}{4} z^{-1} - \frac{1}{8} z^{-2}}$$

$$H(z) = \frac{z^2 + z}{z^2 - \frac{1}{4} z - \frac{1}{8}}$$ (Transfer Function)

$\longrightarrow h(n) = ?$ $\quad h(n) = Z^{-1}\{H(z)\}$

Steps

1) $\dfrac{H(z)}{z} = \dfrac{z+1}{z^2 - \frac{1}{4}z - \frac{1}{8}} = \dfrac{z+1}{(z-\frac{1}{2})(z+\frac{1}{4})}$

$\dfrac{H(z)}{z} = \dfrac{A}{z-\frac{1}{2}} + \dfrac{B}{z+\frac{1}{4}}$, $A = \left.\dfrac{z+1}{z+\frac{1}{4}}\right|_{z=\frac{1}{2}} = 2$

$B = -1$

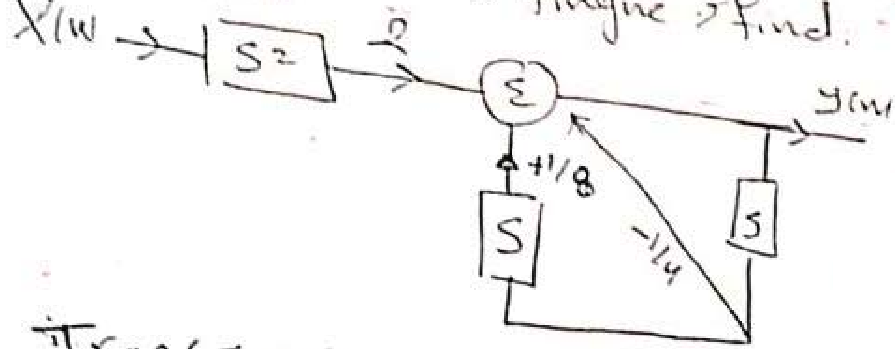
2) $\dfrac{H(z)}{z} = \dfrac{2}{z-\frac{1}{2}} - \dfrac{1}{z+\frac{1}{4}}$

3) $H(z) = \dfrac{2z}{z-\frac{1}{2}} - \dfrac{z}{z+\frac{1}{4}}$

Z^{-1}

$h(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(-\frac{1}{4}\right)^n u(n)$

Ex(7) for the shown fingue, find.



1) Transfer function of that system

2) impulse response

Sol

from block diagram

$$y(n) = 2x(n-2) - \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2)$$

$\therefore$ D.E. is

$$\boxed{y(n) + \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 2x(n-2)}$$

$$H(z) = \frac{X = \text{Yolko}}{Y = \text{Yolko}} = \frac{2z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

$$\therefore \boxed{H(z) = \frac{2}{z^2 + \frac{1}{4}z - \frac{1}{8}}}$$

$h(n) = z^{-1}\{H(z)\} \Rightarrow$ as previous problem

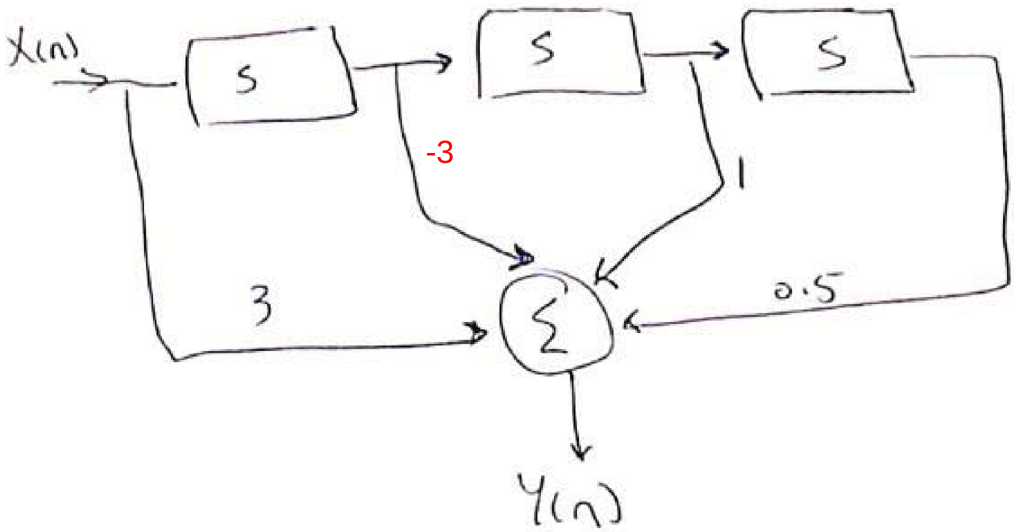
For the discrete time system

Find and sketch.

(i) Impulse response $h(n)$

(ii) the step response [o/p $y(n)$ when $x(n) = u(n)$]

(iii) o/p $y(n)$ when $x(n) = \{\hat{1}, 1, -1, -1\}$



Sol

$$y(n) = 3x(n) - 3x(n-1) + x(n-2) + \tfrac{1}{2}x(n-3)$$

$$\downarrow$$

$$H(z) = \frac{X \text{ معاملات}}{Y \text{ معاملات}} = \frac{3 - 3z^{-1} + z^{-2} + \frac{1}{2}z^{-3}}{1}$$

[non recursive (FIR)]

Scanned by CamScanner

$$H(z) = 3 - 3z^{-1} + z^{-2} + \frac{1}{2} z^{-3}$$

على صوره ، مقلوب Z-Transform لل h(n)

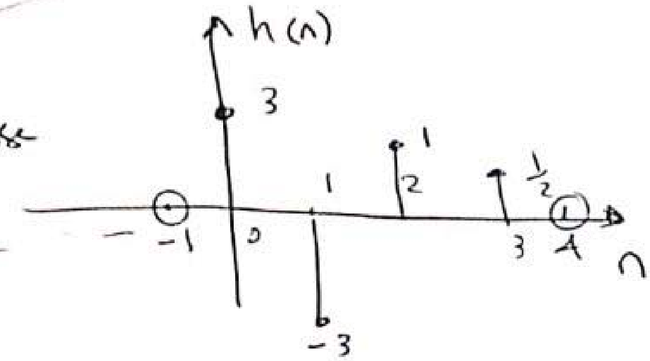
أو

Apply inverse Z-Transform

$$\boxed{h(n) = 3\delta(n) - 3\delta(n-1) + \delta(n-2) + \frac{1}{2}\delta(n-3)}$$

FIR

" Finite Impulse response "



(ii) $y(n) = ?$, $x(n) = u(n)$

Step response

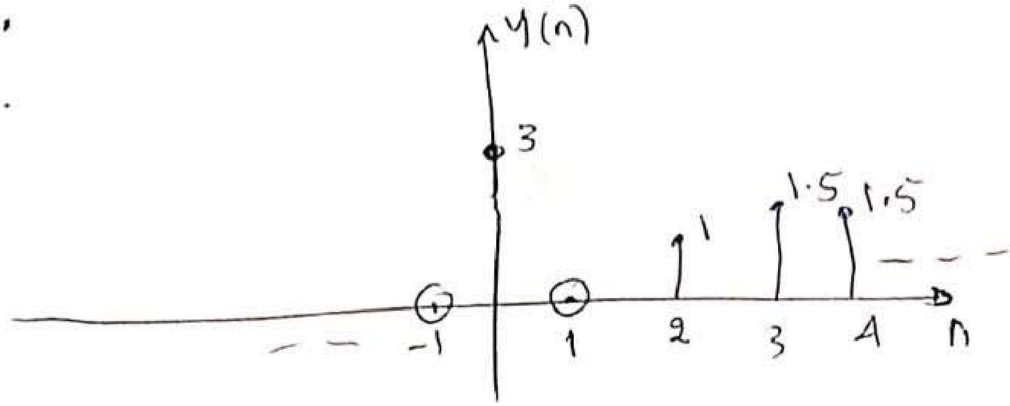
الأسهل ال Convolution

$$y(n) = h(n) * x(n)$$

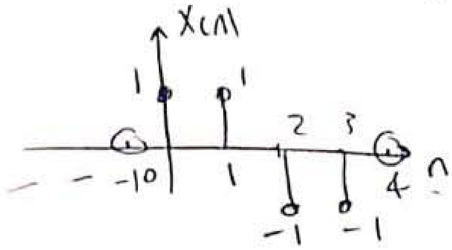
$$y(n) = h(n) * u(n)$$

$$= \left[3\delta(n) - 3\delta(n-1) + \delta(n-2) + \frac{1}{2}\delta(n-3)\right] * u(n)$$

$$= 3u(n) - 3u(n-1) + u(n-2) + \frac{1}{2}u(n-3)$$



(iii) $y(n) = ?$ $\quad x(n) = \{\hat{1}, 1, -1, -1\}$



Using Convolution
(Table)

$$y(n) = x(n) * h(n) = \sum x(k)\, h(n-k)$$

Steps

① $0+0 \leq n \leq 3+3 \rightarrow 0 \leq n \leq 6$

	0	1	2	3	4	5	6	x(n)
h(n) →	3	-3	1	$\frac{1}{2}$				1
		3	-3	1	$\frac{1}{2}$			1
			3	-3	1	$\frac{1}{2}$		-1
				3	-3	1	$\frac{1}{2}$	-1
	3	$\boxed{0}$	-5	$\frac{3}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	y(n)

$$\boxed{y(n) = \{3, 0, -5, \tfrac{3}{2}, \tfrac{3}{2}, -\tfrac{3}{2}, -\tfrac{1}{2}\}}$$

ملاحظة

حساب $y(n)$

نستخدم ال Z-Transform
أسهل لو كان
$x(n)$, $h(n)$
Both expression

① $Y(z) = X(z)H(z)$

② $y(n) = Z^{-1}\{Y(z)\}$

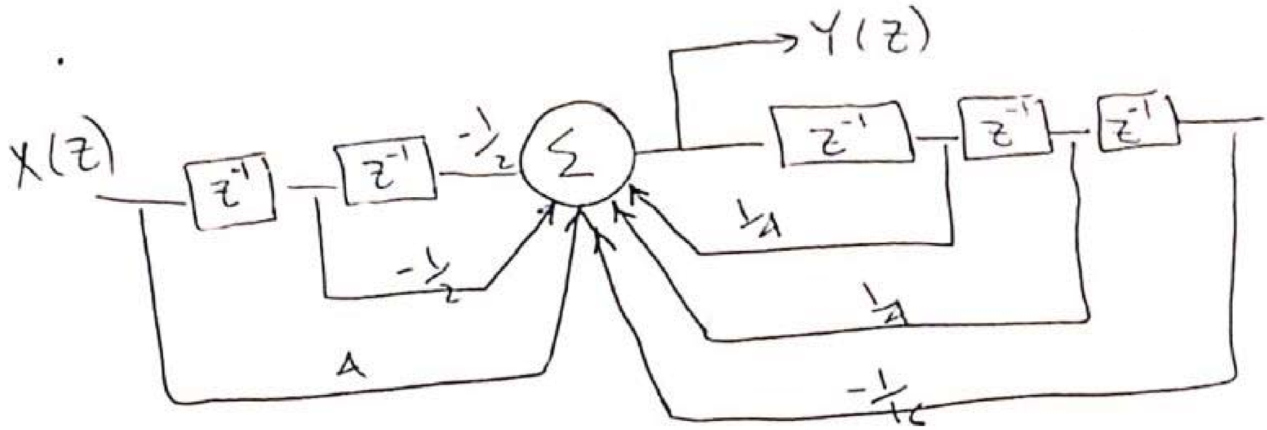
نستخدم Convolution
أسهل في حالة

$h(n)$ or $x(n)$ sequence & other expression

or

$h(n)$ sequence & $x(n)$ sequence

$\boxed{ex}$ Consider the discrete system



(1) Find Transfer Function $H(z)$

(2) Discuss Stability

$\boxed{Sol}$

Note $\boxed{z^{-1}} = \boxed{S}$ & $z^{-2} = S^2$

$$Y(n) = 4X(n) - \frac{1}{2}X(n-1) - \frac{1}{2}X(n-2) + \frac{1}{4}Y(n-1) + \frac{1}{4}Y(n-2) - \frac{1}{16}Y(n-3)$$

$\downarrow$

put in Form of D.E

$$Y(n) - \frac{1}{4}Y(n-1) - \frac{1}{4}Y(n-2) + \frac{1}{16}Y(n-3) = 4X(n) - \frac{1}{2}X(n-1) - \frac{1}{2}X(n-2)$$

$$H(z) = \frac{X}{Y} = \frac{4 - \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}}{1 - \frac{1}{4} z^{-1} - \frac{1}{4} z^{-2} + \frac{1}{16} z^{-3}}$$

$$H(z) = \frac{4z^3 - \frac{1}{2} z^2 - \frac{1}{2} z}{z^3 - \frac{1}{4} z^2 - \frac{1}{4} z + \frac{1}{16}}$$

Poles:

$z^3 - \frac{1}{4} z^2 - \frac{1}{4} z + \frac{1}{16} = 0 \rightarrow$ get poles (calculator)

& check stability