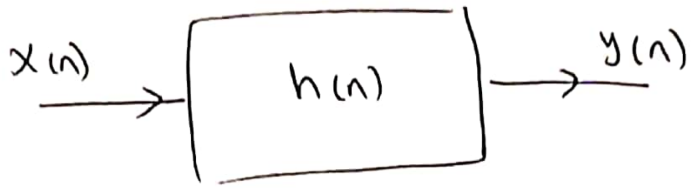
Discrete LTI System

Linear Time Invariant (LTI) Discrete system



$h(n)$: Impulse response of LTI system

[output of the system when the input $= \delta(n)$]

i.e: when $x(n) = \delta(n) \longrightarrow$ output $= h(n)$

$y(n) = x(n) * h(n)$ Discrete Convolution

" proof as in Continuous "

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{or} \quad \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

We usually Fix the small & shift the large.

Notes

(i) $x(n) * \delta(n) = x(n)$

(ii) $x(n) * \delta(n-n_0) = x(n-n_0)$

(iii) If $x(n)$ has length N_x, $h(n)$ has length N_h

$\therefore$ length of $y(n) = N_x + N_h - 1$

How to perform Convolution in LTI Discrete ?

3 cases

1 one of them infinite & the other finite

use the property: $x(n) * \delta(n-n_0) = x(n-n_0)$

example: $x(n) = u(n)$, $h(n) = \begin{cases} 2, & n=0 \\ 1, & n=1 \\ 0, & \text{else} \end{cases}$

SoL

$y(n) = x(n) * h(n) = u(n) * [2\delta(n) + \delta(n-1)]$

$$\boxed{y(n) = 2u(n) + u(n-1)}$$

2 Both $x(n)$ & $h(n)$ are finite samples.

use the Table method " explained in next pages "

ex: $x(n) = \begin{cases} 3, & n=1 \\ 1, & n=2 \\ 0, & \text{else} \end{cases}$ $h(n) = \begin{cases} 1, & n=0 \\ 4, & n=1 \\ 0, & \text{else} \end{cases}$

Both $x(n)$ & $h(n)$ finite $\Rightarrow$ use Table method

[will be solved]

$\boxed{3}$ Long Method : — Both $x(n)$ & $h(n)$ infinite
— Both $x(n)$ & $h(n)$ finite with large number of samples

Use long Method "like the way you used in the Continuous case"

we will need the following relations:

(1) $$\sum_{k=a}^{b} 1 = 1(b-a+1)$$

(2) $$\sum_{k=0}^{\infty} a^k = \begin{cases} \frac{1}{1-a}, & 0<a<1 \\ \infty, & \text{else} \end{cases}$$

(3) $$\sum_{k=m}^{n} (a)^k = a^m \frac{(1-a^{n-m+1})}{(1-a)}$$

Example

Short method "Table"

ex:

$$x(n) = \begin{cases} 2, & n = 0 \\ 3, & n = 1 \\ 2, & n = 2 \\ 0, & \text{o.w} \end{cases} \quad , \quad h(n) = \begin{cases} 1, & n = \pm 1 \\ 2, & n = 0 \\ 0, & \text{o.w.} \end{cases}$$

Find $y(n)$

Sol

Both finite & number of samples ↓↓ [Table].

Steps

1- get range of $y(n)$

بداية ال $x(n)$ + بداية $h(n)$ $\leq n \leq$ نهاية $x(n)$ + نهاية $h(n)$

$$0 + (-1) \leq n \leq 2 + 1$$

$$-1 \leq n \leq 3$$

2) $y(n) = x(n) * h(n)$

$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-1}^{3} x(k)\, h(n-k)$$

3) Construct table by putting «الأقل عينة)) in R.H.S 4

and putting (الأكثر عينة) in 1st row after

time index (n) & shift in each row.

A) multiply 2 functions to get y(n)

n	-1	0	1	2	3	X(K)
put h(n) →	1	2	1			2
		1	2	1		3
			1	2	1	2
y(n) →	2	7	10	7	2	حاصل ضرب كل عمود في X(K)
	2*1	2*2 + 3*1	2*1 + 2*3 + 2*1	3*1 + 2*2	2*1	

$$y(n) = \begin{cases} 2 & , n=-1 \\ 7 & , n=0 \\ 10 & , n=1 \\ 7 & , n=2 \\ 2 & , n=3 \\ 0 & , \text{o.w} \end{cases}$$

length of y(n) $= N_x + N_h - 1$

$= 3 + 3 - 1$

$= 5$

5 samples

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ex 2:

$$x(n) = \begin{cases} 1 & , n = -2 \\ -1 & , n = -1 \\ 2 & , n = 0 \\ 2 & , n = 1 \\ 3 & , n = 2 \end{cases}$$

$$h(n) = \begin{cases} 1 & , n = -2 \\ -3 & , n = -1 \\ 2 & , n = 0 \end{cases}$$, Find $y(n)$

Sol

Steps

1- range of $y(n)$

$$-2 + (-2) \leq n \leq 2 + 0$$

$$-4 \leq n \leq 2$$

2- $y(n) = x(n) * h(n)$ نثبت الصغير $h(n)$ ، نشفت الكبير $x(n)$

$$y(n) = \sum_{K=-4}^{2} h(K)\, x(n-K)$$

3- Construct table

n	-4	-3	-2	-1	0	1	2	$h(k)$
$x(n)$	1	-1	2	2	3			1
		1	-1	2	2	3		-3
			1	-1	2	2	3	2
$y(n)$	1	-4	7	-6	1	5	6	

$$\therefore y(n) = \begin{cases} 1 , & n = -4 \\ -4 , & n = -3 \\ 7 , & n = -2 \\ -6 , & n = -1 \\ 1 , & n = 0 \\ 5 , & n = 1 \\ 6 , & n = 2 \end{cases}$$

Properties of Convolution:

1- $x(n) * \delta(n - n_0) = x(n - n_0)$

2- $[x_1(n) + x_2(n)] * h(n) = x_1(n) * h(n) + x_2(n) * h(n)$

$\boxed{ex}$ $x(n) = \left(\frac{1}{2}\right)^n u(n)$, $h(n) = \delta(n) + \delta(n-1)$

Find $y(n)$

$\left\{ \begin{array}{l} \text{finite} \leftarrow h(n) \\ \text{infinite} \leftarrow x(n) \end{array} \right.$

Sol

Sol

One of them samples & other infinite ⇒ Delta property

$$y(n) = x(n) * h(n) = \left(\tfrac{1}{2}\right)^n u(n) * [\delta(n) + \delta(n-1)]$$

$$= \left(\tfrac{1}{2}\right)^n u(n) + \left(\tfrac{1}{2}\right)^{n-1} u(n-1).$$

$\boxed{\text{II}}$ long Convolution methode

when $x(n), h(n)$ ⇒ large # of Samples or $\boxed{\text{infinite}}$.

ex: Find step response (O/P when $x(n) = u(n)$) For LTI system with

$$h(n) = \left(\tfrac{1}{2}\right)^n u(n)$$

Sol

$x(n) = u(n)$, $h(n) = \left(\tfrac{1}{2}\right)^n u(n)$ Both infinite

Long Convol

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

"long Duration"

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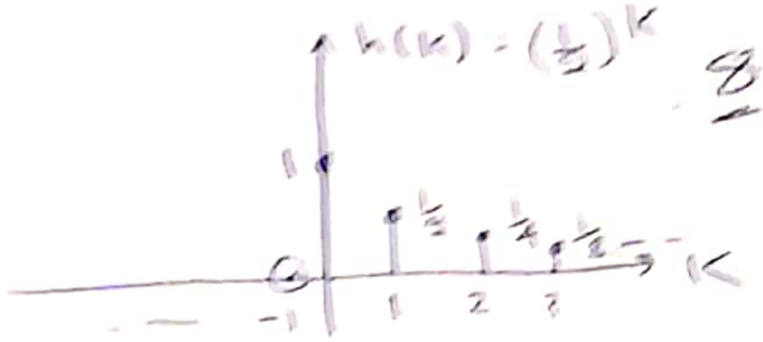
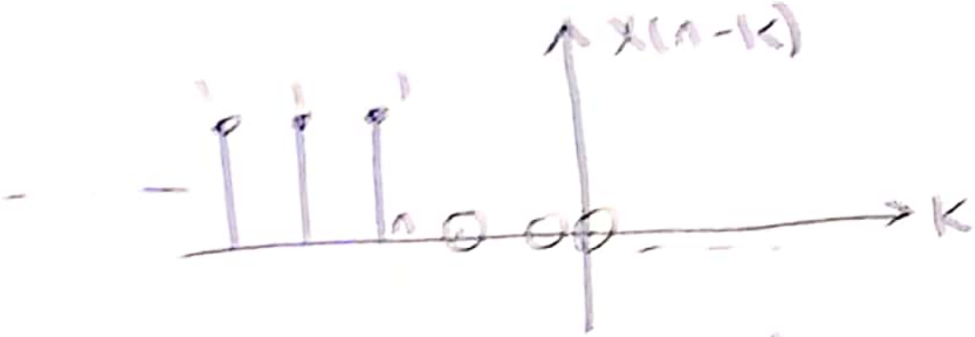


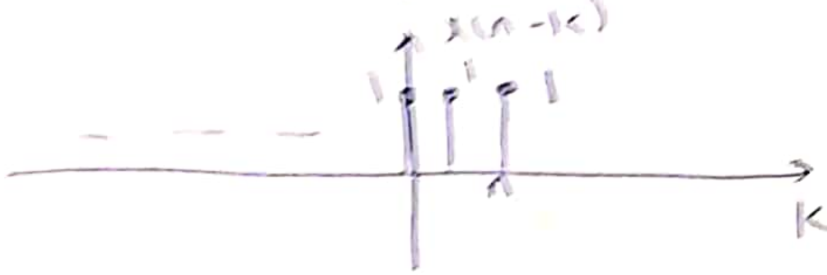
1) $n<0$:

$\boxed{y(n) = 0}$



2) n≥ 0 :

$0 \leq k \leq n$



$$y(n) = \sum_{k=0}^{n} h(k) \cdot 1$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k}$$

$$y(n) = \frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}} = 2\left(1-\left(\frac{1}{2}\right)^{n+1}\right), \ n>0$$

$$\therefore \ y(n) = 2\left(1-\left(\frac{1}{2}\right)^{n+1}\right) u(n)$$

Recall:

$$\sum_{k=0}^{n} a^{k} = \frac{1-a^{n+1}}{1-a}$$

$$\sum_{k=0}^{\infty} a^{k} = \frac{1}{1-k}.$$

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Recall For Long Convolution

1) $$\sum_{k=a}^{b} 1 = b - a + 1$$

2) $$\sum_{k=0}^{\infty} (a)^k = \frac{1}{1-a} \quad , \quad 0 < a < 1$$

3) $$\sum_{k=0}^{n} (a)^k = \frac{1 - a^{n+1}}{1-a}$$

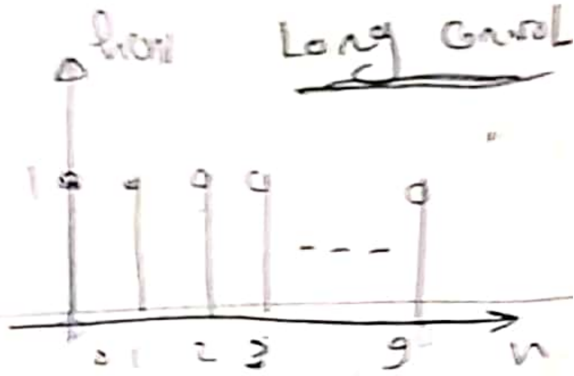
4) $$\sum_{k=m}^{n} (a)^k = a^m \, \frac{1 - a^{n-m+1}}{1-a}$$

Ex(2)

a LTI has $h(n) = u(n) - u(n-10)$, find the Response of this system to $x(n) = 2[u(n-2) - u(n-7)]$

Both finite but number of samples ↑↑ → Long Conv

Solu

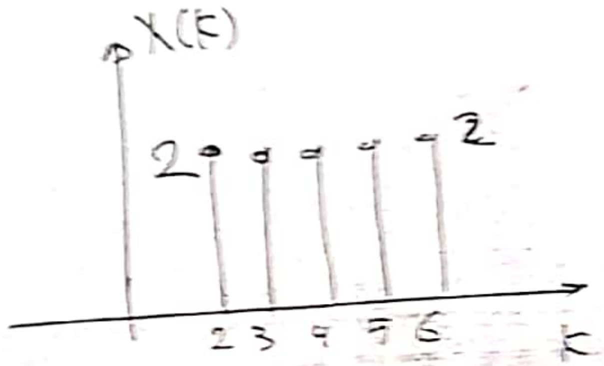


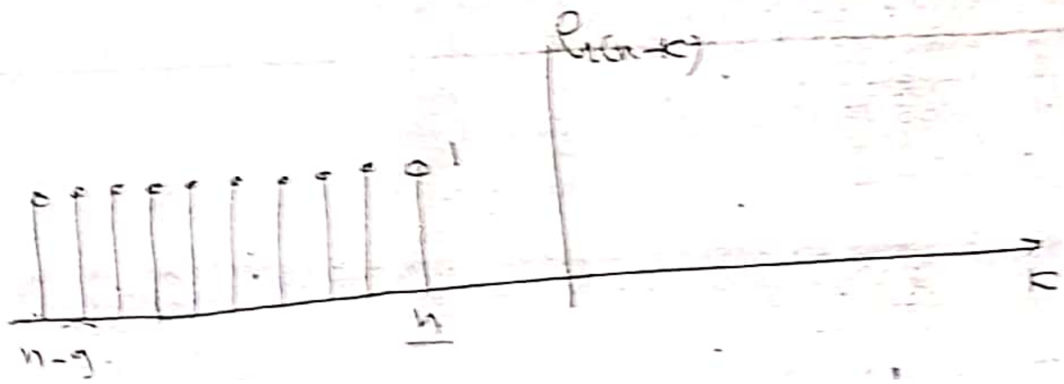
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) = \sum_{k=-\infty}^{\infty} w_n(k)$$

ثبت $x(n)$ المتغير

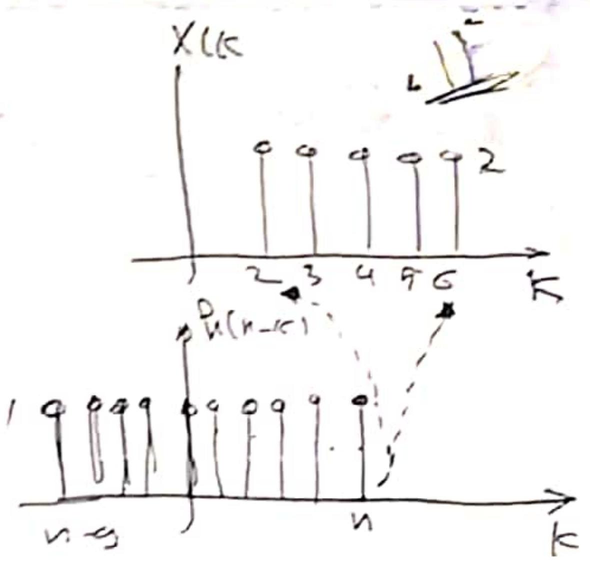
1- $n < 2$

$\therefore y(n) = 0$





2) $2 \le n < 6$

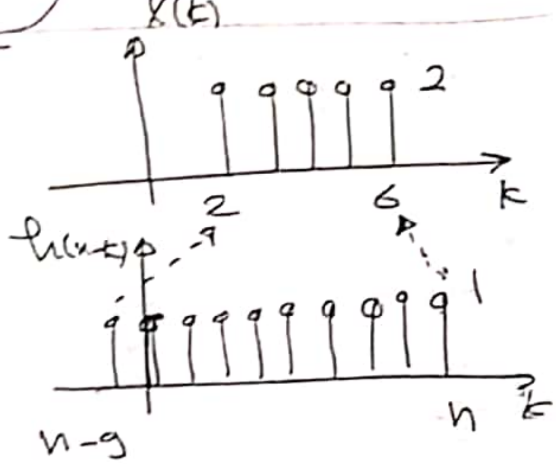


$2 \le k < n$

$$\therefore y(n) = \sum_{k=+2}^{n} 2 = 2[n-(+2)+1]$$

$$= 2n-2$$

3- $n \ge 6$ & $n-9 < 2$ $(6 \le n < 11)$

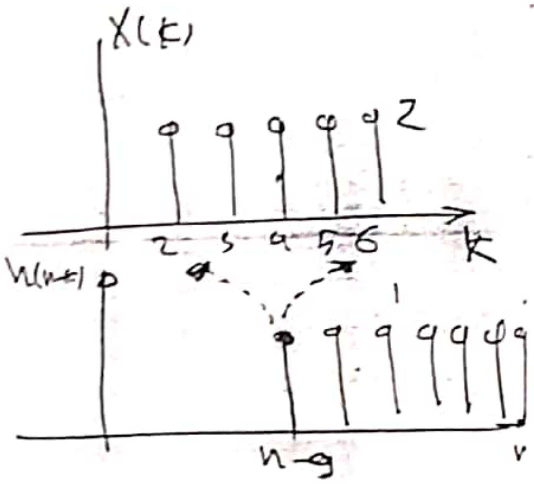


$2 \le k < 6$

$$y(n) = \sum_{k=+2}^{6} 2 = 2[6-2+1]$$

$$= 10$$

4. $2 \le n-9 < 6$ $(11 \le n < 15)$



$n-9 \le k < 6$

$$\therefore y(n) = \sum_{n-9}^{6} 2 = 2[6-(n-9)+1]$$

$$y(n) = 32-2n$$

5- $n-9 > 6 \Rightarrow (n > 15)$

$$y(n) = 0$$

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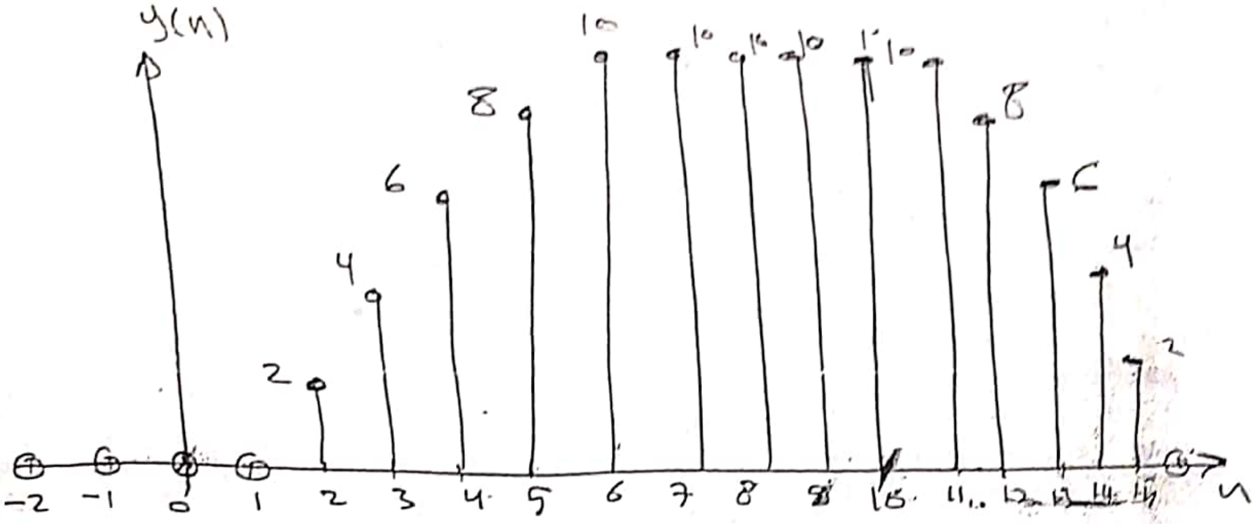
$$\therefore\ y(n) = \begin{cases} 0 & n < 2 \\ 2n-2 & 2 \le n < 6 \\ 10 & 6 \le n < 11 \\ 32-2n & 11 \le n \le 15 \\ 0 & n > 15 \end{cases}$$



Note: You can use the Table method If you want.

⇓

It will give the same result.

Example

$x(n) = 3u(n)$, $h(n) = (2)^{-n}[u(n) - u(n-4)]$

Sketch and Find $y(n)$?

Sol

infinite $x(n)$

finite $h(n)$

← الأول بنستخدم Delta property.

$\left(\frac{1}{2}\right)^n$

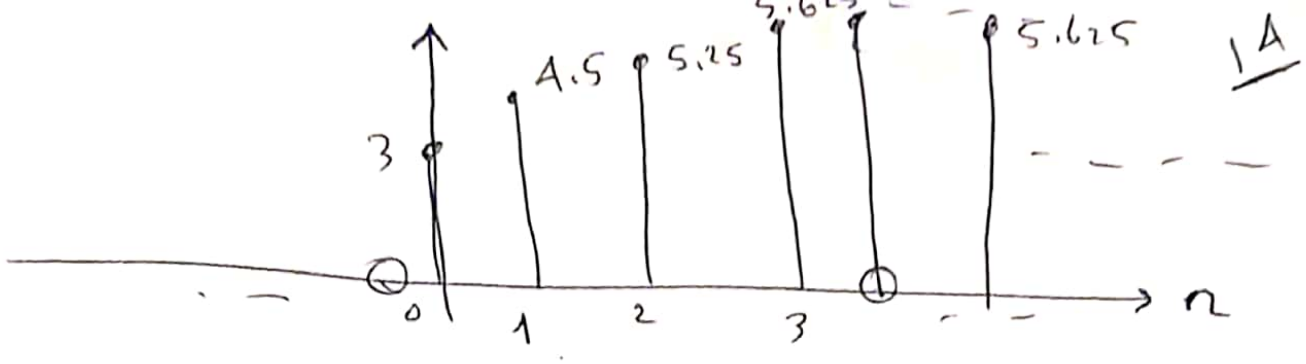
$h(n) = (2)^{-n}$, $n = 0, 1, 2, 3$

$$h(n) = \begin{cases} 1, & n = 0 \\ \frac{1}{2}, & n = 1 \\ \frac{1}{4}, & n = 2 \\ \frac{1}{8}, & n = 3 \\ 0, & o.w \end{cases}$$

$$h(n) = \delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2) + \frac{1}{8}\delta(n-3)$$

$$y(n) = x(n) * h(n) = 3u(n) * \left[\delta(n) + \frac{1}{2}\delta(n-1) + \frac{1}{4}\delta(n-2) + \frac{1}{8}\delta(n-3)\right]$$

$$y(n) = 3u(n) + \frac{3}{2}u(n-1) + \frac{3}{4}u(n-2) + \frac{3}{8}u(n-3)$$

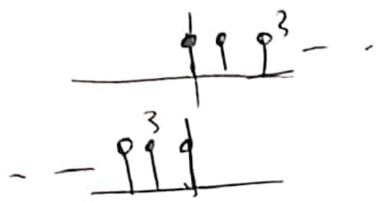


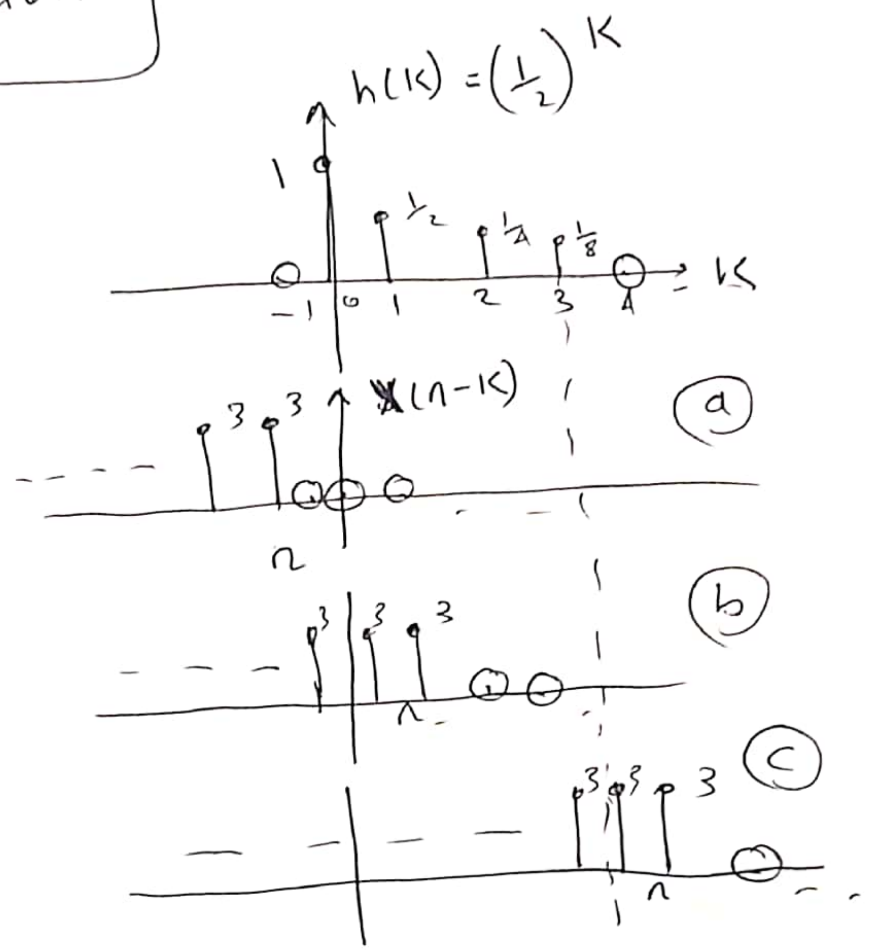
If required to make long convolution

Using long Convolution

1) h(n) نثبت الصغير

و نشقلب الكبير x(n)





a) $\boxed{n < 0}$

$y(n) = 0$

b) $0 \leq n < 3$

$$y(n) = \sum_{K=0}^{n} 3 \left(\frac{1}{2}\right)^K$$

$$y(n) = 3 \cdot \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}$$

$$y(n) = 6\left[1 - \left(\tfrac{1}{2}\right)^{n+1}\right]$$

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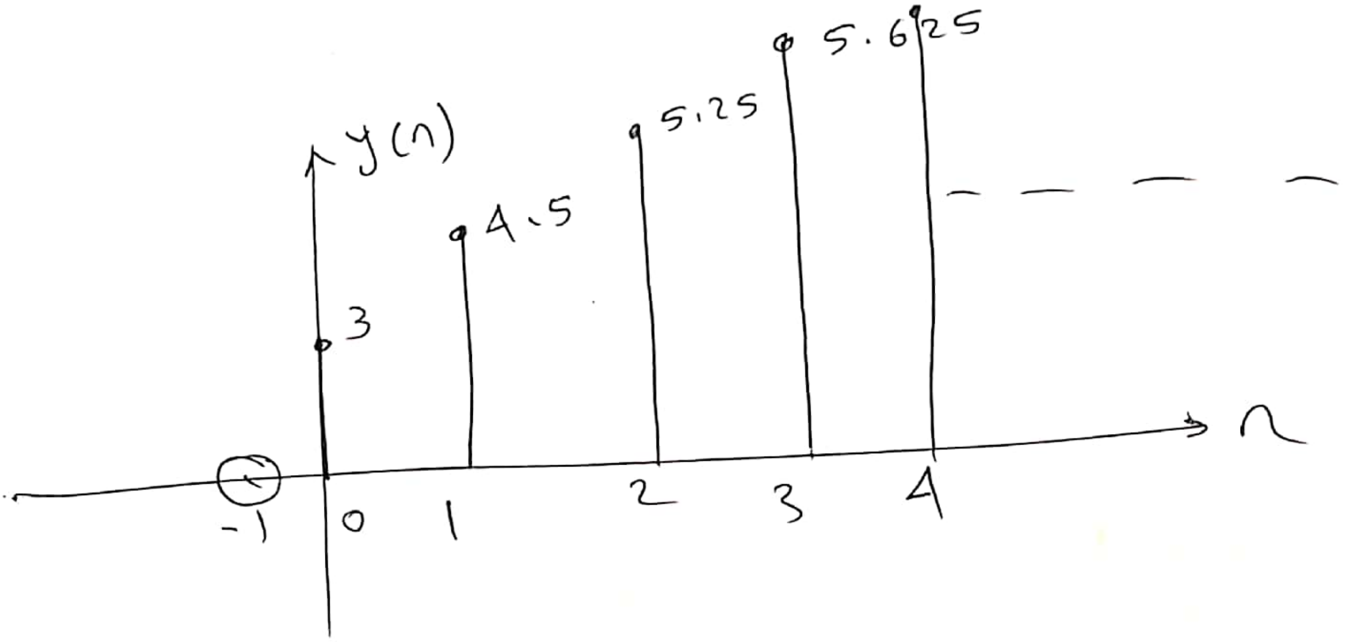
(c) $n \geq 3$

$$y(n) = \sum_{K=0}^{3} 3\left(\tfrac{1}{2}\right)^{K} = 3\,\frac{\left[1 - \left(\tfrac{1}{2}\right)^{3+1}\right]}{\left[1 - \tfrac{1}{2}\right]}$$

$$y(n) = 6\left[1 - \left(\tfrac{1}{2}\right)^{4}\right] = 5.625$$

$$\therefore y(n) = \begin{cases} 0 & , n < 0 \\ 6\left(1 - \left(\tfrac{1}{2}\right)^{n+1}\right) & , 0 \leq n < 3 \\ 5.625 & , n \geq 3 \end{cases}$$



[Same Result as First method].