Fourier representation for discrete time signals

Recall:

Time Domain	Frequency Domain.
discrete	periodic
Continous	non periodic
periodic	Discrete
non periodic	Continous

For Discrete Signal

1) $x(n)$: Discrete, periodic of (N), $\omega_0 = \frac{2\pi}{N}$

$\Downarrow$ D.T.F.S (Discrete time Fourier Series)

$X(k)$ discrete, periodic (N), $\omega_0 = \frac{2\pi}{N}$

2) $x(n)$: Discrete, non periodic

$\Downarrow$ D.T.F.T (Discrete time Fourier Transform)

$X(\omega)$ Continous + periodic (2π).

Discrete time Fourier Transform (D.T.F.T)

$$x(n) \text{ non periodic Discrete } \underset{D.T.F.T^{-1}}{\overset{D.T.F.T}{\rightleftarrows}} X(\omega) \text{, periodic } (2\pi) \text{ continous}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad (D.T.F.T)$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega \quad (D.T.F.T^{-1})$$

$x(n)$ non periodic Discrete $\xrightarrow{D.T.F.T}$ $X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-J\omega n}$

periodic (2π) continous.

Recall

1) $\sum_{n=a}^{b} \alpha = \alpha(b-a+1)$

2) $\sum_{n=0}^{\infty} (\alpha)^n = \frac{1}{1-\alpha}$, $0 < \alpha < 1$

3) $\sum_{n=0}^{b} (\alpha)^n = \frac{1-\alpha^{b+1}}{1-\alpha}$.

4) $\sum_{n=a}^{b} (\alpha)^n = (\alpha)^a \frac{1-(\alpha)^{b-a+1}}{1-\alpha}$.

$\boxed{\text{exs}}$ Find spectrum of

1) $x(n) = \delta(n)$ [non periodic, Discrete]

Sol

$X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-J\omega n} = \sum_{n=-\infty}^{\infty} \delta(n)\, \underbrace{\cancel{e^{-J\omega n}}}_{1}\Big|_{n=0}$

$= 1$

$$x(n) = (\alpha)^n u(n)$$

Sol

$x(n)$: non periodic, Discrete $\xrightarrow{DTFT}$ Spectrum $X(\omega)$ Continuous, periodic (2π)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha)^n e^{-j\omega n}$$

$$X(\omega) = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

Complex

If you need $|X(\omega)|$, $\angle X(\omega)$?

$$X(\omega) = \frac{1}{1-\alpha(\cos\omega - j\sin\omega)} = \frac{1}{1-\alpha\cos\omega + j\alpha\sin\omega}$$

$$|X(\omega)| = \frac{1}{\sqrt{(1-\alpha\cos\omega)^2 + (\alpha\sin\omega)^2}}$$

$$\angle X(\omega) = 0 - \tan^{-1}\left[\frac{\alpha\sin\omega}{1-\alpha\cos\omega}\right]$$

ex(4)

Find Spectrum of

$$x(n) = \begin{cases} (2)^n & \boxed{0 \le n < 10} \\ 0 & o.w. \end{cases}$$

take care

Solu

$$x(n) \xrightarrow{DTFT} X(\omega)$$

discrete & nonperiod — Continuous + periodic (2π)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{9} (2)^n e^{-j\omega n} = \sum_{n=0}^{9} \left(2e^{-j\omega}\right)^n$$

$$X(\omega) = \frac{1 - 2e^{-j10\omega}}{1 - 2e^{-j\omega}}$$

Note

1) لو احنا بنحسب الـ spectrum طلع , مطلوب رسم .

$$\Rightarrow \frac{1-e^{-jx}}{1-e^{-jy}} \longrightarrow$$

ناخد $e^{-jx/2}$ بره فوق

ناخد $e^{-jy/2}$ بره تحت

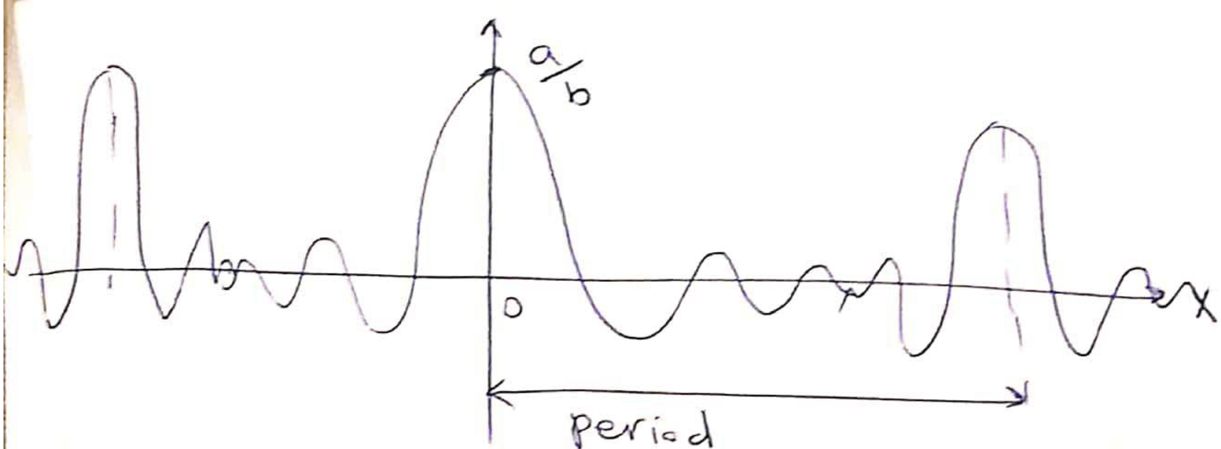
$$\Rightarrow \frac{e^{-jx/2}\left[e^{jx/2}-e^{-jx/2}\right]}{e^{-jy/2}\left[e^{jy/2}-e^{-jy/2}\right]} = \frac{e^{-jx/2}}{e^{-jy/2}} \cdot \frac{\cancel{2j}\sin(x/2)}{\cancel{2j}\sin(y/2)}$$

because $\sin\square = \frac{1}{2j}\left[e^{j\square}-e^{-j\square}\right] \Rightarrow e^{j\square}-e^{-j\square} = 2j\sin\square$

2) Note

$$\frac{\sin(ax)}{\sin(bx)} \longrightarrow \text{Sinc periodic}$$

& peak = $\frac{a}{b}$ at zero



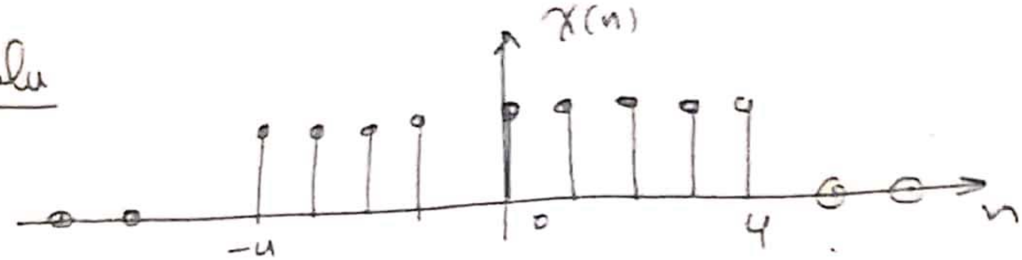
على حسب الـ a

ex(19)

Find the spectrum of

$$x(n) = \begin{cases} 1 & -4 \le n < 5 \\ 0 & o.w \end{cases}$$

Solu



$$x(n) \xrightarrow{\text{DTFT}} X(\omega)$$

- nonp
- discrete

Continuous

periodic (2π)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-4}^{4} e^{-j\omega n}$$

$$= \sum_{n=-4}^{4} \left(e^{-j\omega}\right)^n$$

$$\therefore \quad X(\omega) = \left(e^{-j\omega}\right)^{-4} \frac{1-\left(e^{-j\omega}\right)^9}{1-\left(e^{-j\omega}\right)}$$

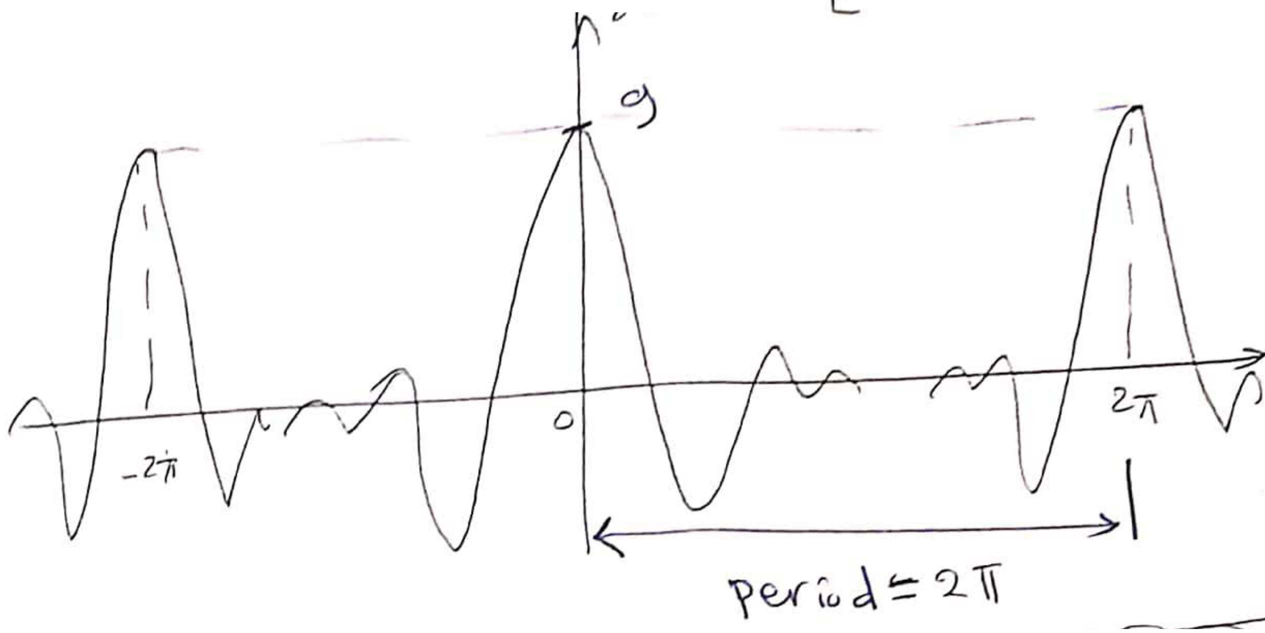
$$= \cancel{e^{j4\omega}}\, \frac{\cancel{e^{-j\frac{9}{2}\omega}}\left[e^{j\frac{9}{2}\omega} - e^{-j\frac{9}{2}\omega}\right]}{\cancel{e^{-j\frac{\omega}{2}}}\left[e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}\right]}$$

$$X(\omega) = \frac{\cancel{2j}\,\sin\left(\frac{9}{2}\cdot\omega\right)}{\cancel{2j}\,\sin\left(\frac{\omega}{2}\right)}$$

$$\therefore \quad X(\omega) = \frac{\sin\left(\frac{9}{2}\,\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \Rightarrow \text{Sinc periodic } 2\pi$$

[Contiuos + periodic (2π)]

Magnitude Spectrum

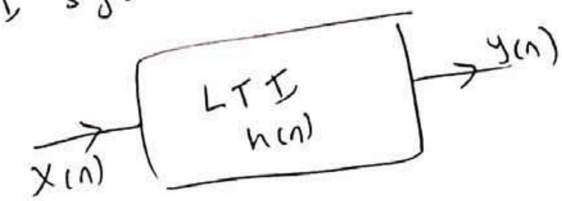


Important Note

→ We can easily get DTFT by getting first Z-transform then put $z = e^{j\omega}$.

Remember the relation between Z-transform and Laplace-Transform $Z = e^{ST}$ or $Z = e^{S}$ for $T = 1$
For $\sigma = 0$, $S = j\omega$ and $Z = e^{j\omega}$

→ For LTI system



$$y(n) = x(n) * h(n)$$ ← impulse response

$$\Downarrow$$

$$Y(z) = X(z) \cdot H(z)$$

$$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$ ← Transfer function

To get the frequency response of LTI system $H(\omega) = ?$

$$H(\omega) = H(z)\Big|_{z = e^{j\omega}}$$

<u>or</u>

$$H(\omega) = DTFT\{h(n)\}$$ using summation rule

Also $Y(\omega) = X(\omega) \cdot H(\omega)$

EX 1

Find Frequency response and impulse response for LTI systems described by

a) output $y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$,

For input $x(n) = \left(\frac{1}{2}\right)^n u(n)$

b) Difference equation:

$$y(n) - 0.25\, y(n-1) - 0.125\, y(n-2) = 3\, x(n) - 0.75\, x(n-1)$$

SoL

(a) $H(z) = \dfrac{Y(z)}{X(z)}$

$$Y(z) = Z\left\{ \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n) \right\} = \frac{1}{4}\,\frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{4}}$$

$$Y(z) = \frac{\frac{1}{4}z^2 - \frac{1}{16}z + z^2 - \frac{1}{2}z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$X(z) = Z\left\{ \left(\frac{1}{2}\right)^n u(n) \right\} = \frac{z}{z-\frac{1}{2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left(\frac{1}{4}z^2 - \frac{1}{16}z + z^2 - \frac{1}{2}z\right)\cancel{\left(z-\frac{1}{2}\right)}}{\cancel{\left(z-\frac{1}{2}\right)}\left(z-\frac{1}{4}\right)z}$$

Transfer function

$$H(z) = \frac{\frac{5}{4} z^2 - \frac{9}{16} z}{z^2 - \frac{1}{4} z}$$

Frequency response: $H(\omega) = H(z)\Big|_{z = e^{j\omega}}$

$$H(\omega) = \frac{\frac{5}{4} e^{j2\omega} - \frac{9}{16} e^{j\omega}}{e^{j2\omega} - \frac{1}{4} e^{j\omega}}$$

↑ Frequency response

(b) given Difference eq:

$$H(z) = \frac{3 - 0.75 z^{-1}}{1 - 0.25 z^{-1} - 0.125 z^{-2}}$$ "Transfer function"

Frequency response: $H(\omega) = H(z)\Big|_{z = e^{j\omega}}$

$$H(\omega) = \frac{3 - 0.75 e^{-j\omega}}{1 - 0.25 e^{-j\omega} - 0.125 e^{-j2\omega}}$$

↑ Frequency response

In order to get h(n), you can easily make take Inverse Z-transform for H(z)

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EX 2

Find Frequency response & Impulse response

a) $y(n) = \frac{1}{6} \sum_{k=0}^{5} x(n-k)$

b) $y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$, $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Sol

ⓐ $y(n) = \frac{1}{6}\left[x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4) + x(n-5)\right]$

$H(z) = \frac{1}{6}\left[1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}\right]$

→ Frequency response: $H(W) = H(z)\Big|_{z = e^{jw}}$

$H(W) = \frac{1}{6}\left[1 + e^{-jw} + e^{-j2w} + e^{-j3w} + e^{-j4w} + e^{-j5w}\right]$

→ Impulse response:

$h(n) = z^{-1}\{H(z)\}$

$h(n) = \frac{1}{6}\left[\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4) + \delta(n-5)\right]$

ⓑ Solved before

Important Note

How to get DTFT & IDTFT ?

Through Z-transform [very easy if possible] or by summation rules

DTFT: $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

IDTFT: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$

$$X(\omega) = X(z)\Big|_{z=e^{j\omega}}$$

* If we have $X(\omega) \xrightarrow{e^{j\omega}=z}$ get $X(z) \xrightarrow{\text{inverse Z-transform}}$ get $x(n)$

* If we have $x(n) \xrightarrow{\text{Z-transform}}$ get $X(z) \xrightarrow{z=e^{j\omega}}$ get $X(\omega)$

Examples

Ex1! Find the DTFT of the causal sequence

$$x(n) = (a)^n \cos(\omega_0 n + \varphi)\, u(n)$$

where a, ω_0, φ are real constants

Solution

Required $X(\omega) = ?$

$$x(n) \xrightarrow[\text{Z-transform}]{\text{get}} X(z) \xrightarrow{z=e^{j\omega}} \text{get } X(\omega)$$

[easier to use the Z-transform instead of the summation Definition]

$$x(n) = (a)^n \cos(\omega_0 n + \varphi)\, u(n)$$

Recall: $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$

$$x(n) = (a)^n \frac{1}{2}\left[e^{j(\omega_0 n + \varphi)} + e^{-j(\omega_0 n + \varphi)}\right] u(n)$$

$$x(n) = \frac{1}{2} e^{j\varphi} (a)^n e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\varphi} (a)^n e^{-j\omega_0 n} u(n)$$

$$x(n) = \frac{1}{2} e^{j\varphi} \left(a e^{j\omega_0}\right)^n u(n) + \frac{1}{2} e^{-j\varphi} \left(a e^{-j\omega_0}\right)^n u(n)$$

Now, we can take Z-transform $\quad (\square)^n \xrightarrow{Z\text{-}tr} \frac{Z}{Z-\square}$

$$X(z) = \frac{1}{2} e^{j\varphi} \frac{z}{z - a e^{j\omega_0}} + \frac{1}{2} e^{-j\varphi} \frac{z}{z - a e^{-j\omega_0}}$$

$$X(\omega) = X(z)\Big|_{z = e^{j\omega}} = \frac{1}{2} e^{j\varphi} \frac{e^{j\omega}}{e^{j\omega} - a e^{j\omega_0}} + \frac{1}{2} e^{-j\varphi} \frac{e^{j\omega}}{e^{j\omega} - a e^{-j\omega_0}}$$

Divide by $e^{j\omega}$

$$X(\omega) = \frac{1}{2} e^{j\varphi} \frac{1}{1 - a e^{-j\omega} e^{j\omega_0}} + \frac{1}{2} e^{-j\varphi} \frac{1}{1 - a e^{-j\omega} e^{-j\omega_0}}$$

#

example

If $x(n) = \left(\frac{1}{2}\right)^n u(n)$, $y(n) = x^2(n)$ Find DTFT of

$y(n)$, $Y(\omega) = ?$

Solution

$$y(n) = x^2(n) = \left(\frac{1}{2}\right)^{2n} [u(n)]^2 = \left(\frac{1}{2}\right)^{2n} u(n)$$

Note $[u(n)]^2 = (1)^2 = 1$, $n = 0, 1, 2 \ldots$

$= u(n)$

$$y(n) = \left[\left(\frac{1}{2}\right)^2\right]^n u(n) = \left[\frac{1}{4}\right]^n u(n)$$

to get $Y(\omega)$ — use DTFT summation rule $\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

or

using z-transform "easier here"

$$Y(z) = Z\left\{ \left(\frac{1}{4}\right)^n u(n) \right\} = \frac{z}{z - \frac{1}{4}}$$

$$Y(\omega) = Y(z)\Big|_{z = e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - \frac{1}{4}} = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

#

Example

Find DTFT of $x(n) = u(n-1) - u(n-4)$

Solution

Required $X(\omega) = ?$ DTFT is easier here using $\rightarrow$ Z-transform

$x(n)$ can be written as

$$x(n) = \delta(n-1) + \delta(n-2) + \delta(n-3)$$

$\downarrow$ Z-transform

$$X(z) = z^{-1} + z^{-2} + z^{-3}$$

$$\boxed{X(\omega) = X(z)\Big|_{z=e^{j\omega}} = e^{-j\omega} + e^{-j1\omega} + e^{-j3\omega}}$$

Example: Find IDFT of $X(\omega) = \cos^2\omega$, $x(n) = ?$

Recall: $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Solution

we can use either $\rightarrow$ IDFT rule: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

or

$\rightarrow$ using Z-transform "easier"

$$X(\omega) = \cos^2\omega = \frac{1}{2}(1+\cos 2\omega)$$

$$X(\omega) = \frac{1}{2}\left[1 + \frac{1}{2}\left(e^{j2\omega} + e^{-j2\omega}\right)\right] = \frac{1}{2} + \frac{1}{4}e^{j2\omega} + \frac{1}{4}e^{-j2\omega}$$

$$X(z) = X(\omega)\Big|_{e^{j\omega}=z} = \frac{1}{2} + \frac{1}{4}z^{2} + 1/4Z^{-2} \xrightarrow[\text{Z-transform}]{\text{inverse}} x(n) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2)$$

Ex2:

Find inverse DTFT $h(n)$ of

$$H(\omega) = [3 + 2\cos\omega + 4\cos 2\omega]\cos(\omega/2)\, e^{-j\omega/2}$$

Z-transform is easier here to get IDFT

Solution

$$H(\omega) \xrightarrow[e^{j\omega} = z]{\text{Convert to } H(z)} H(z) \xrightarrow[\text{z-transform}]{\text{inverse}} \text{get } h(n)$$

$$\left[\begin{aligned} \sin\omega &= \tfrac{1}{2j}(e^{j\omega} - e^{-j\omega}) \\ \cos\omega &= \tfrac{1}{2}[e^{j\omega} + e^{-j\omega}] \end{aligned}\right]$$

$$H(\omega) = [3 + 2\cos\omega + 4\cos 2\omega]\cos(\omega/2)\, e^{-j\omega/2}$$

$$H(\omega) = \left[3 + 2\left[\frac{e^{j\omega} + e^{-j\omega}}{2}\right] + 4\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)\right] \cdot \left(\frac{e^{j\omega/2} + e^{-j\omega/2}}{2}\right) \cdot e^{-j\omega/2}$$

$$H(\omega) = [3 + e^{j\omega} + e^{-j\omega} + 2e^{j2\omega} + 2e^{-j2\omega}]\tfrac{1}{2}[1 + e^{-j\omega}]$$

$$H(\omega) = \frac{1}{2}[3 + e^{j\omega} + e^{-j\omega} + 2e^{j2\omega} + 2e^{-j2\omega} + 3e^{-j\omega} + 1 + e^{-j2\omega} + 2e^{j\omega} + 2e^{-j3\omega}]$$

$$H(\omega) = \frac{1}{2}[2e^{j2\omega} + 3e^{j\omega} + 4 + 4e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega}]$$

$$H(z) = H(\omega)\Big|_{z = e^{j\omega}}$$

$$H(z) = \frac{1}{2}\left[\,2z^{2} + 3z + 4 + 4z^{-1} + 3z^{-2} + 2z^{-3}\right]$$

$$\Big\downarrow \text{ Inverse Z-transform}$$

$$h(n) = \frac{1}{2}\Big[2\,\delta(n+2) + 3\,\delta(n+1) + 4\,\delta(n) + 4\,\delta(n-1)$$
$$+ 3\,\delta(n-2) + 2\,\delta(n-3)\Big]$$

$$h(n) = \{1, 1.5, 2, 2, 1.5, 1\}, \quad -2 \leq n \leq 3$$

$$\downarrow$$

FIR

Example

Consider

$$x(n) = \{3, 0, 1, -2, -3, 4, 1, 0, -1\}, \quad -3 \leq n \leq 5$$

With a DTFT $X(\omega)$. Evaluate the following functions of $X(\omega)$ without computing the transform its self

(a) $X(0)$

(b) $X(\pi)$

(c) $\int_{-\pi}^{\pi} X(\omega)\, d\omega$

(d) $\int_{-\pi}^{\pi} |X(\omega)|^2\, d\omega$

Solution

$\Downarrow$

Solution

Recall: DTFT

$$\text{DTFT: } X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

$$\text{DTFT}^{-1}\text{: } x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$$

(a) Required

$X(0)$ or $X(\omega=0) = ?$

$$\text{DTFT! } X(\omega=0) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\Big|_{\omega=0}$$

$$X(\omega=0) = \sum_{n=-\infty}^{\infty} x(n) = 3+0+1+(-2)+(-3)+4+1+0+(-1)$$

$$= \boxed{3}$$

(b) Required

$X(\pi)$ or $X(\omega=\pi)$

$$\text{DTFT: } X(\omega=\pi) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\Big|_{\omega=\pi} = \sum_{n=-3}^{5} x(n)\, e^{-j\pi n}$$

Where

$$e^{-j\pi n} = \cos(\pi n) - j\sin \pi n$$

$$\sin \pi n = 0, \quad n = 1, 2, 3 \text{ —}$$

$$\cos \pi n = \begin{cases} 1, & n = 0, 2, 4, \dots \\ -1, & n = 1, 3, 5, \dots \end{cases}$$

$$\therefore e^{-j\pi n} = (-1)^n, \quad n = \pm 1, \pm 2, \pm 3 \text{ — —}$$

$$\therefore X(\omega = \pi) = \sum_{n=-3}^{5} x(n)(-1)^n = x(-3) * -1 + x(-2) * 1 + x(-1) * -1 + x(0) * 1 + x(1) * -1 + x(2) * 1 + x(3) * -1 + x(4) * 1 + x(5) * -1$$

$$X(\omega = \pi) = -3 - 1 - 2 + 3 + 4 - 1 + 1 = 1$$

(c) Required: $\int_{-\pi}^{\pi} X(\omega)\, d\omega = ?$

$DTFT^{-1}$: $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\Big|_{n=0}\, d\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, d\omega$$

$$\int_{-\pi}^{\pi} X(\omega)\, d\omega = 2\pi x(0) = 2\pi * -2$$

$$= -4\pi$$

(d) Required: $\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega$

Note → Parserval's Theorem in DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

In this problem

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Required: $\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega = 2\pi \sum_{n=-\infty}^{\infty} |x(n)|^2$

$$= 2\pi \left[(3)^2 + (0)^2 + (1)^2 + (-2)^2 + (-3)^2 + (4)^2 + (1)^2 + (0)^2 + (-1)^2 \right]$$

$$\int_{-\pi}^{\pi} |X(\omega)|^2 \, d\omega = 82\pi$$

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Notes

[1] DTFT is continous and inefficient to be implemented in hardware.

[2] Here, we introduce sampling in Frequency [DFT] as a powerful tool to represent the signal.
" Taking N samples from the DTFT of Discrete time signal "

[3]. In Digital signal processing references,
DTFS is considered the same as DTFT
& only represented by (DFT)

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi n k}{N}} \quad , k=0,1,2,\dots N-1$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi n k}{N}} \quad , n=0,1,2,\dots N-1.$$

" N-Point DFT "

Using N - Point DFT, we can estimate up to N time-samples output sequence.

$\boxed{A}$ For N-Point DFT, $X(k)$ is periodic of period N

Proof

N-point DFT : $$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \quad , k=0,1,2\ldots N-1$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N} n(k+N)}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} \cdot e^{-j2\pi n}$$

But $e^{-j2\pi n} = \underbrace{\cos(2\pi n)}_{1} - j\underbrace{\sin 2\pi n}_{0} = 1$

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n k}{N}} = X(k)$$

$\therefore X(k)$ is periodic of period N

DFT using Weight Matrix

Twiddle Factor

$x(n)$ Discrete, Size $\underline{N}$ points $\xrightarrow{DFT}$ $X(k)$ Discrete, Size $\underline{N}$ points

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-J\frac{2\pi}{N}nk}} \leftarrow DFT$$

$k = 0, 1, 2 \ldots N-1$

we called $e^{-J\frac{2\pi}{N}nk} = W_N^{kn}$ = twiddle Factor.

then

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}}$$

→ example

if we have $\boxed{N = 4}$

$$X(k) = \sum_{n=0}^{3} x(n)\, W_4^{kn} = \sum_{n=0}^{3} x(n)\, e^{-J\frac{2\pi}{4}nk}$$

we have the result For $X(k) = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$

and $x(n) = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$.

as $\boxed{N = 4}$ then we have $x(0), x(1), x(2), x(3)$ T.D (n)

$$\downarrow \text{DFT}$$

$X(0), X(1), X(2), X(3)$ F.D (K)

$\boxed{K=0}$ $X(0) = \sum_{n=0}^{3} x(n)\, W_4^0 = \left[W_4^0 x(0) + W_4^0 x(1) + W_4^0 x(2) + W_4^0 x(3) \right]$

$\boxed{K=1}$ $X(1) = \sum_{n=0}^{3} x(n)\, W_4^n = \left[W_4^0 x(0) + W_4^1 x(1) + W_4^2 x(2) + W_4^3 x(3) \right]$

$\boxed{K=2}$ $X(2) = \sum_{n=0}^{3} x(n)\, W_4^{2n} = \left[W_4^0 x(0) + W_4^2 x(1) + W_4^4 x(2) + W_4^6 x(3) \right]$

$\boxed{K=3}$ $X(3) = \sum_{n=0}^{3} x(n)\, W_4^{3n} = \left[W_4^0 x(0) + W_4^3 x(1) + W_4^6 x(2) + W_4^9 x(3) \right]$

It can be written in matrix Form

$$\underset{4\times1}{X(K)} = \underset{4\times4}{\left[W \right]} \underset{4\times1}{\left[x(n) \right]}$$

[next page]

Generally: For N points $\underset{N\times1}{X(K)} = \underset{N\times N}{\left[W \right]} \underset{N\times1}{\left[x(n) \right]}$

We Can write it in matrix Form

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{array}{c|cccc} & n=0 & n=1 & n=2 & n=3 \\ \hline k=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ k=1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ k=2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ k=3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

So we can get DFT of $x(n)$ $[X(k)]$ i.e, we can get $X(0)$, $X(1)$, $X(2)$, $X(3)$.

Properties of Twiddle Factor

① it is periodic in kn with period N.

$$W_N^{kn+N} = W_N^{kn}$$ → proof is very easy!

ex! → $W_4^6 = W_4^2$ → $W_4^9 = W_4^5 = W_4^1$

[In previous example (4-point), we can write all W_4^{kn} as W_4^0, W_4^1, W_4^2, W_4^3]

② $$W_N^{kn+N/2} = -W_N^{kn}$$ → proof is very easy!

exs: $\rightarrow W_4^3 = -W_4^1 \rightarrow W_4^2 = -W_4^0$.

$\rightarrow W_8^{13} = W_8^5 = -W_8^1$

after knowing properties of twiddle factor,

we return to our example

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}.$$

we can write W matrix in terms of W_4^0, W_4^1

because $W_4^2 = -W_4^0$ (propert 2)

$W_4^3 = -W_4^1$ (// 2)

$W_4^4 = W_4^0$ (// 1)

$\underbrace{W_4^6 = W_4^2}_{\text{prop}(1)} \underbrace{= -W_4^0}_{\text{prop}(2)}$

$W_4^9 = W_4^5 = W_4^1$ (property 1)

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & -w_4^0 & -w_4^1 \\ w_4^0 & -w_4^0 & w_4^0 & -w_4^0 \\ w_4^0 & -w_4 & -w_4^0 & w_4^1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Where $w_4^0 = e^{-J\frac{2\pi}{4}(0)} = e^{-J(0)} = \cos(0) + J\sin(0) = 1$

$w_4^1 = e^{-J\frac{2\pi}{4}(1)} = e^{-J\frac{\pi}{2}} = \cos\frac{\pi}{2} - J\sin\frac{\pi}{2} = -J$

$$\therefore W = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix}$$

Representaton of W in Complex plane

For N = 4

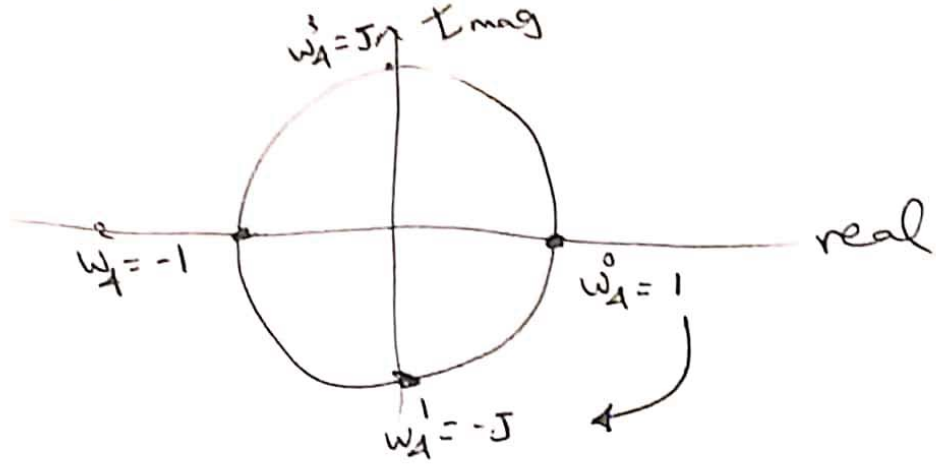
$w_4^0 = e^{-J\frac{2\pi}{4}(0)} = 1$

$w_4^1 = e^{-J(\frac{2\pi}{4})(1)} = e^{-J\frac{\pi}{2}} = -J$

$w_4^2 = e^{-J\frac{2\pi(2)}{4}} = e^{-J\pi} = -1$

$w_4^3 = e^{-J\frac{2\pi(3)}{4}} = e^{-J\frac{3\pi}{2}} = J$

تمثل على ــــــ دائرة نصف قطرها 1



← تقع على دائرة الوحدة

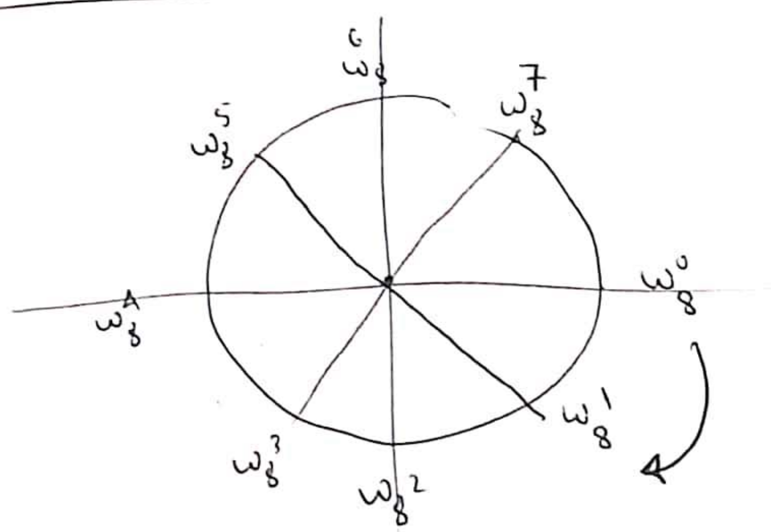
← نستطيع ان نرسم W_N^{kn} بدون حسابات

Steps:

1) Draw unit circle
2) Divide circle by (N) ← # of Points of DFT
3) Start From real axis

ونمشي ع اتجاه عقارب الساعة (مع θ بـ -)

$\boxed{ex}$ $\boxed{\text{For } N=8}$ → angle between 2 points = $\frac{360}{8} = 45°$



• الاتجاهات اللي بينهم زاوية 180° بكونوا متساويين بعكس الاشارة

→ $W_8^7 = -W_8^3$, $W_8^5 = -W_8^1$, - - - - -

IF we want to get $W_8^5 = \cos(135°) + J\sin(135°)$

$$W_8^5 = \frac{-1}{\sqrt{2}} + J\frac{1}{\sqrt{2}}$$

and Directly

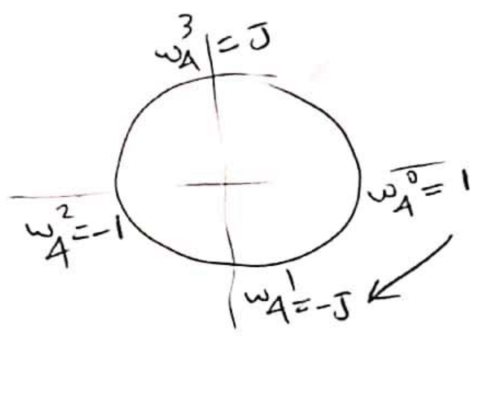
$$W_8^1 = -W_8^5 = \frac{1}{\sqrt{2}} - J\frac{1}{\sqrt{2}}$$

$\boxed{ex}$ Perform 4-point DFT for the following sequence $x(n) = \{1, 0, 0, 1\}$ using W matrix

Sol

$$X(K) = \sum_{n=0}^{3} x(n)\, e^{-J\frac{2\pi}{N}kn} = \sum_{n=0}^{3} x(n)\, W_N^{kn} \quad , \quad N = 4$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{array}{c c} & \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} & \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$



, $W_4^6 = W_4^2 = -1$

$W_4^9 = W_4^5 = W_4^1 = -J$

$W_4^4 = W_4^0 = 1$

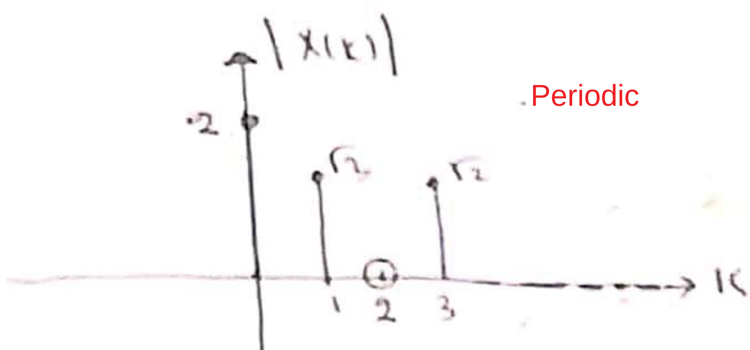
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+j \\ 1-1 \\ 1-j \end{bmatrix} = \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}.$$

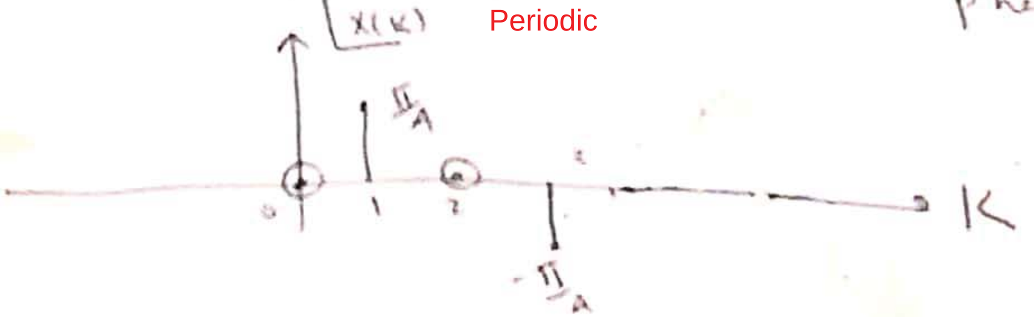
$$X(k) = \begin{cases} 2 = 2\angle 0, & k=0 \\ 1+j = \sqrt{2}\angle \frac{\pi}{4}, & k=1 \\ 0, & k=2 \\ 1-j = \sqrt{2}\angle -\frac{\pi}{4}, & k=3 \end{cases}$$

DC component

if $x(n)$ real samples $\Downarrow$ get complex conjugate



magnitude spectrum



phase spectrum

Find IDFT For $X(k) = \{2, 1+J, 0, 1-J\}$

Sol

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{J\frac{2\pi}{N}kn} \quad [\text{IDFT}]$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)\, W_4^{-kn}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{array}{c} \\ n=0 \\ n=1 \\ n=2 \\ n=3 \end{array} \begin{array}{c} \begin{array}{cccc} k=0 & k=1 & k=2 & k=3 \end{array} \\ \begin{bmatrix} W_4^{-0} & W_4^{-0} & W_4^{-0} & W_4^{-0} \\ W_4^{-0} & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^{-0} & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^{-0} & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \end{array} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

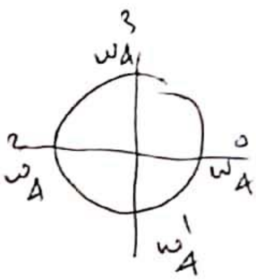
Where $W_4^{-0} = 1$, $W_4^{-1} = W_4^{3} = J$

$W_4^{-2} = W_4^{2} = -1$

$W_4^{-3} = W_4^{1} = -J$

$W_4^{-4} = W_4^{0} = 1$

$W_4^{-9} = W_4^{-5} = W_4^{-1} = W_4^{3} = J$ and so on



$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix} \begin{bmatrix} 2 \\ 1+J \\ 0 \\ 1-J \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} \rightarrow x(n) = \{1, 0, 0, 1\}$$

$\boxed{ex\,2}$ perform DFT To get spectrum of the following sequence. $x(n) = \{ \underset{\uparrow n=0}{1}, 2, 0, 3, 0, 1, 3, 2 \}$

Sol

Excersice

$\boxed{N = 8}$

$$X(k) = \sum_{n=0}^{7} x(n)\, W_N^{kn} = \sum_{n=0}^{7} x(n)\, W_8^{kn}$$

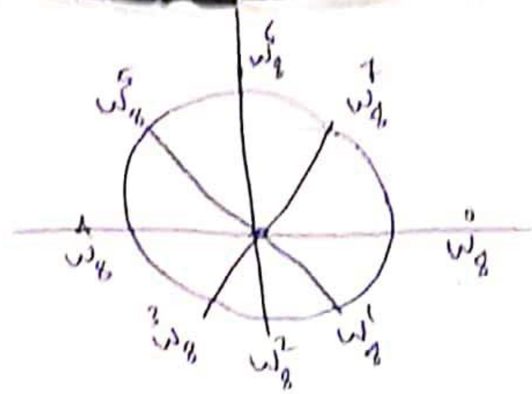
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \\ k=5 \\ k=6 \\ k=7 \end{array} \begin{array}{c} \begin{array}{cccccccc} n=0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & - & - & - \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & - & - & \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & - & - & \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & - & - & \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & - & - & \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & - & - & \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & - & - & \end{bmatrix} \end{array} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

$W_8^0 = 1$

$W_8^4 = -1$

$W_8^2 = -j$

$W_8^6 = j$



$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$, $W_8^5 = \frac{-1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

$W_8^3 = \frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$, $W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$

get $X(k) = \{12, 1+j(3-\sqrt{2}), -2+j2,$

$1-j(3+\sqrt{2}), -4, 1+j(3+\sqrt{2}), -2-j2, 1-j(3-\sqrt{2})\}$

too many calculations

# of multiplications $= N^2 = (8)^2 = 64$

# of additions $= N(N-1) = 8 \times 7 = 56$

We need an efficient algorithm to perform the DFT operation with less calculations $\Longrightarrow$ Called "FFT"

Fast Fourier Transform

Parserval's Theorem

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{K=0}^{N-1} |X(k)|^2$$

Example verify parserval's theorem

$x(n) = \{1, 2, 0, 3\}$

Sol

$\boxed{1}$ Get $\underset{4\times1}{\underline{X}(k)} = \underset{4\times4}{[W]} \; \underset{4\times1}{[x(n)]}$

$$\begin{bmatrix} \underline{X}(0) \\ \underline{X}(1) \\ \underline{X}(2) \\ \underline{X}(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -J & -1 & J \\ 1 & -1 & 1 & -1 \\ 1 & J & -1 & -J \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

From previous examples [draw W_4 & get values]

$$\underline{X}(k) = \begin{bmatrix} 6 \\ 1+J \\ -4 \\ 1-J \end{bmatrix}$$

$$\therefore X(k) = \begin{bmatrix} 6 \\ 1+j \\ -4 \\ 1-j \end{bmatrix}$$

Parserval's Theorem:

$$\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{3} |x(n)|^2 = (1)^2 + (2)^2 + (0)^2 + (3)^2$$

$$= 14$$

$$\frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 = \frac{1}{4} \sum_{k=0}^{3} |X(k)|^2$$

$$= \frac{1}{4}\left[(6)^2 + (\sqrt{2})^2 + (-4)^2 + (\sqrt{2})^2 \right]$$

$$= \frac{1}{4}\left[36 + 2 + 16 + 2 \right]$$

$$= \boxed{14}$$

Example

Consider the sequence $x(n)$ defined for $0 \le n \le 11$

$$x(n) = \{ \underset{\uparrow}{3}, -1, 2, 4, -3, -2, 0, 1, -4, 6, 2, 5 \}$$

with 12-point DFT given by $X(k)$, $0 \le k \le 11$

Evaluate the following functions of $X(k)$ without computing the DFT

(a) $X(0)$ (b) $X(6)$ (c) $\sum_{k=0}^{11} X(k)$

(d) $\sum_{k=0}^{11} |X(k)|^2$

Solution

Recall:

DFT: $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}}$, $k = 0, 1, 2 \ldots N-1$

IDFT: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}}$, $n = 0, 1, 2 \ldots N-1$

(a) $X(0) = ?$ $\Rightarrow$ $X(k=0)$ From DFT

$$X(k=0) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nk}{N}} \Big|_{k=0}, \quad N = 12$$

$$= \sum_{n=0}^{11} x(n) = 3 + (-1) + (2) + \cdots 5$$

$$= \boxed{13}$$

(b) $X(6) = X(k=6) = \sum_{n=0}^{11} x(n)\, e^{-j\frac{2\pi n k}{12}} \Big|_{k=6}$

$$X(6) = \sum_{n=0}^{11} x(n)\, e^{-j\pi n}$$

where $e^{-j\pi n} = \cos \pi n - j \underbrace{\sin \pi n}_{0}$, $n = 0, 1, 2 \dots, 11$

$$e^{-j\pi n} = \cos \pi n = \begin{cases} 1, & n = 0, 2, 4, \dots \\ -1, & n = 1, 3, 5 \dots \end{cases}$$

$\therefore e^{-j\pi} = (-1)^n$, $n = 0, 1, 2, 3, 4 \dots, 11$

$\therefore X(6) = \sum_{n=0}^{11} x(n)\, (-1)^n = 3 * 1 + (-1) * (-1) + 2 * 1 + 4 * -1 + \dots \dots + 5 * -1$

$$\boxed{X(6) = -13}$$

(c) $\sum_{k=0}^{11} X(k) = ? \longrightarrow$ From IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi n k}{N}}$$

$$x(0) = \frac{1}{12} \sum_{k=0}^{11} X(k)$$

$\therefore \sum_{k=0}^{11} X(k) = 12\, x(0) = 12 * 3 = 36$

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(d) $\sum_{k=0}^{11} |X(k)|^2 = ?$ $\longrightarrow$ From Parseval's theorem.

Recall: $\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$

$N = 12$

$\therefore \sum_{n=0}^{11} |x(n)|^2 = \frac{1}{12} \sum_{k=0}^{11} |X(k)|^2$

$\therefore \sum_{k=0}^{11} |X(k)|^2 = 12 * \sum_{n=0}^{11} |x(n)|^2$

$= 12 * [(3)^2 + (-1)^2 + \cdots + (5)^2]$

$\sum_{k=0}^{11} |X(k)|^2 = 1500$

Example:

let $X(k)$, $0 \le k \le 11$ be a 12 point DFT of real $x(n)$ $0 \le n \le 11$

$$X(k) = \{11, 8-j2, 1-j12, 6+j3, -3+j2, 2+j, 15, 2-j, -3-j2, 6-j3, 1+j12, 8+j2\}, \quad 0 \le k \le 11$$

Evaluate the following functions of $x(n)$

ⓐ $x(0)$ ⓑ $x(6)$ ⓒ $\sum_{n=0}^{11} x(n)$ ⓓ $\sum_{n=0}^{11} |x(n)|^2$

Solution

The same way as the previous problem: $\boxed{N = 12}$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j 2\pi n k}{N}}, \quad 0 \le n \le 11$$

$$x(n) = \frac{1}{12} \sum_{k=0}^{11} X(k)\, e^{\frac{j 2\pi n k}{12}}$$

ⓐ $$x(0) = \frac{1}{12} \sum_{k=0}^{11} X(k)\, e^{\frac{j 2\pi n k}{12}} \Big|_{n=0}$$

$x(n=0)$

$$x(n=0) = \frac{1}{12} \sum_{k=0}^{11} X(k) = \frac{1}{12}\left[11 + 8 - j2 + 1 - j12 - - - + 8 + j2\right]$$

$$\boxed{x(n=0) = x(0) = 4.5}$$

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(b) $x(6) = x(n=6) = \frac{1}{12} \sum_{k=0}^{11} X(k)\, e^{\frac{j2\pi nk}{12}} \Big|_{n=6}$

$x(6) = \frac{1}{12} \sum_{k=0}^{11} X(k)\, e^{j\pi k}$,

as shown in previous problem $e^{j\pi k} = (-1)^k$

$x(6) = \frac{1}{12} \sum_{k=0}^{11} X(k) (-1)^k$

$= \frac{1}{12} \left[11 * 1 + (8-j2) * -1 \; - - - - - + (8+j2) * -1 \right]$

$\boxed{x(6) = -0.8333}$

(c) $\sum_{n=0}^{11} x(n) = ?$ $\longrightarrow$ From DFT

$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{n=0}^{11} x(n)\, e^{-j\frac{2\pi nk}{12}}$

let get $X(0) = \sum_{n=0}^{11} x(n)\, \underbrace{e^{0}}_{1}$

$\boxed{\therefore \sum_{n=0}^{11} x(n) = X(0) = 11}$

(d) Req: $\sum_{n=0}^{11} |x(n)|^2$ $\longrightarrow$ From perserval's theorem

$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2 \Rightarrow \sum_{n=0}^{11} |x(n)|^2 = \frac{1}{12} \sum_{k=0}^{11} |X(k)|^2$

$\therefore \sum_{n=0}^{11} |x(n)|^2 = \frac{1}{12} \left[(11)^2 + [(8)^2 + (2)^2] + [(1)^2 + (2)^2] \ldots \right] = \boxed{74.83}$