Properties of DFT

[1] Periodicity :

Both DFT & IDFT are periodic :-

- $x(n) = x(n \pm N)$ for all n
- $X(k) = X(k \pm N)$ for all k.

Where $x(n)$ & $X(k)$ are N-point DFT pair

[2] Circular Convolution

Symbol : $\circledast$ Circular Convolution

It is the Convolution of Two periodic sequences

$$x_1(n) \circledast x_2(n) = y(n)$$

periodic "N" — circular convolution — periodic "N" — periodic "N"

Q: How can we make circular convolution ?

If $x_1(n)$ is periodic with period N

$x_2(n)$ is periodic with period N

$$y(n) = x_1(n) \circledast x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n)_N , \quad m = 0, 1, 2, \ldots, N-1$$

circular convolution

$y(n)$ is also periodic of period N

$$\therefore \underbrace{y(n)}_{\substack{\text{periodic}\\(N)}} = \underbrace{x_1(n)}_{\substack{\text{periodic}\\(N)}} \overset{\text{circular convolution}}{\circledast} \underbrace{x_2(n)}_{\substack{\text{periodic}\\(N)}}$$

$\Big\downarrow$ DFT

$$Y(k) = X_1(k) \cdot X_2(k)$$

__Q:__ How to perform circular convolution?

$\Downarrow$

Using rotating circles

ex: $x_1(n) = \{1, 2, 3, 4\}$, $x_2(n) = \{5, 6, 0, 0\}$

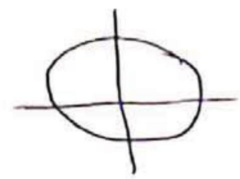
Find the circular Convolution?

Sol

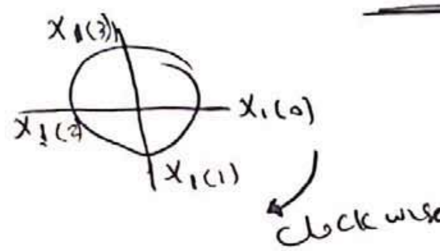
$y(n) = x_1(n) \circledast x_2(n) = ?$

Steps:

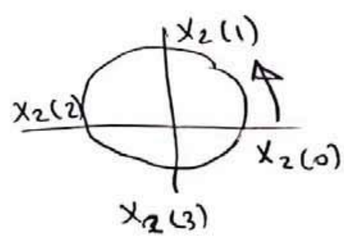
$N = 4$ ⟶ Divide a circle into 4 regions



⟶ Fix one of the 2 sequences and put it clockwise in the outer circle: $x_1(n)$



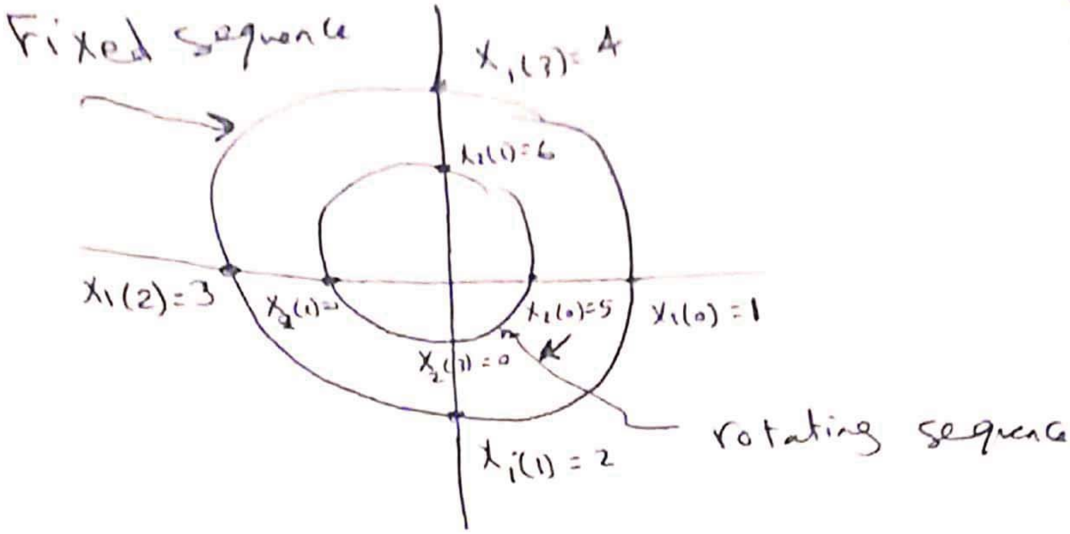
⟶ Put the other sequence $x_2(n)$ in the inner circle Counter clockwise



⟶ To get $y(0)$ ⇒ multiply both sequences

To get $y(1)$ ⇒ rotate $x_2(n)$ one step clockwise & multiply to get $y(1)$

To get $y(2)$ ⇒ rotate $x_2(n)$ 2 steps clockwise & get $y(2)$ (one more step)

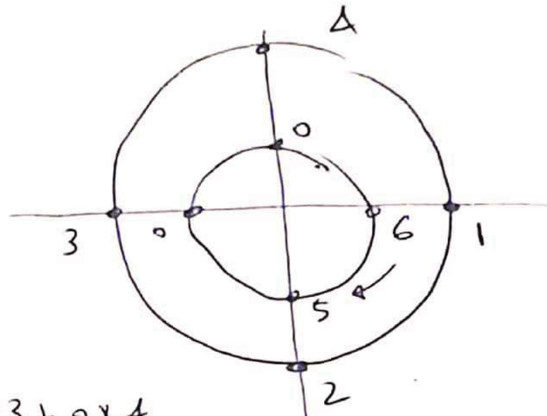


→ $Y(0)$ = product of both circles

$$= 5*1 + 0*2 + 0*3 + 6*4 = 29$$

→ to get $y(1)$

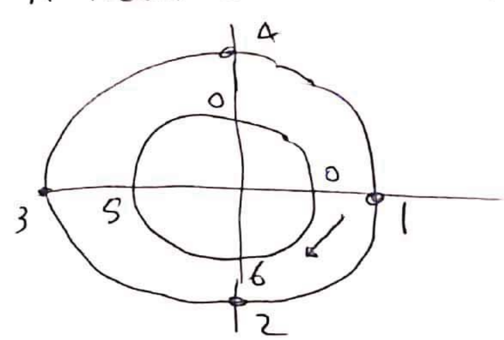
[shift $x_2(n)$ clockwise by one step]



$$Y(1) = 6*1 + 5*2 + 0*3 + 0*4$$
$$= 16$$

→ to get $y(2)$ [shift $x_2(n)$ clockwise by 2 steps]

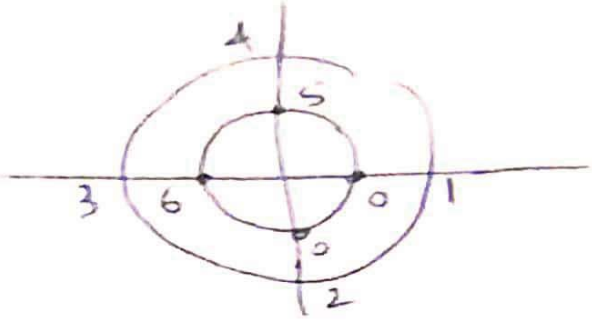
$$Y(2) = 1*0 + 2*6 + 3*5 + 0*4$$
$$= 27$$



→ to get $y(3)$ [shift $x_2(n)$ by 3 steps]

$y(3) = 0*1 + 0*2 + 6*3 + 5*4$

$= 38$



$\therefore y(n) = \{29, 16, 27, 38\}$

output of circular convolution ⇒ periodic of period $N = 4$

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example $x_1(n) = \{1, 2, 3, 4\}$

$x_2(n) = \{1, -1, 0, 1\}$

Find the output of circular convolution

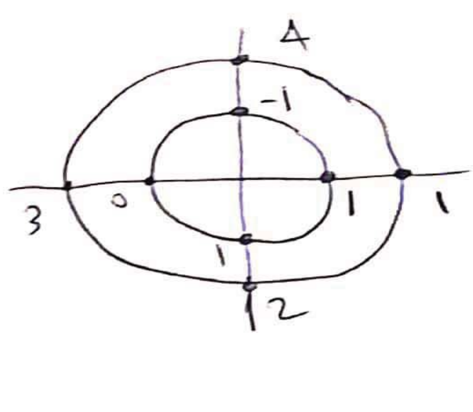
sol

→ Fix $x_1(n)$ clock wise

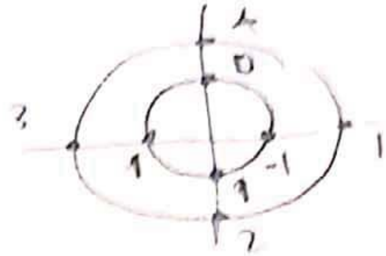
→ arrange $x_2(n)$ counter-clock wise

$y(0) = 1*1 + 2*1 + 3*0 + 4*-1$

$\boxed{y(0) = -1}$



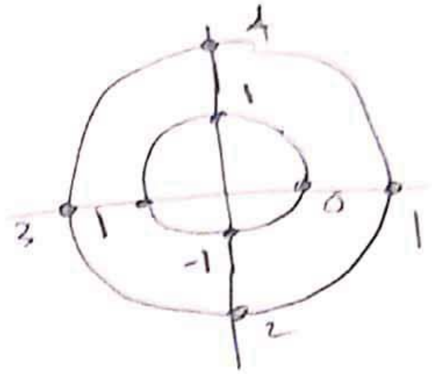
→ shift $x_2(n)$ clockwise one step



$y(1) = -1\times1 + 1\times2 + 1\times3 + 0\times4$

$\boxed{y(1) = 4}$

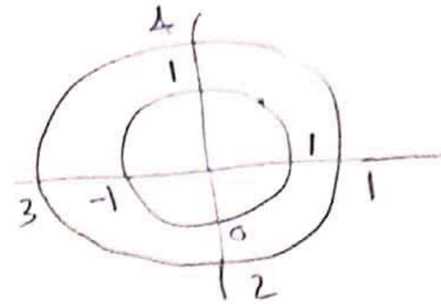
→ shift $x_2(n)$ clockwise one more step



$y(2) = 0\times1 + -1\times2 + 1\times3 + 1\times4$

$\boxed{y(2) = 5}$

→ shift $x_2(n)$ clockwise one more step



$y(3) = 1\times1 + 0\times2 + -1\times3 + 1\times4$

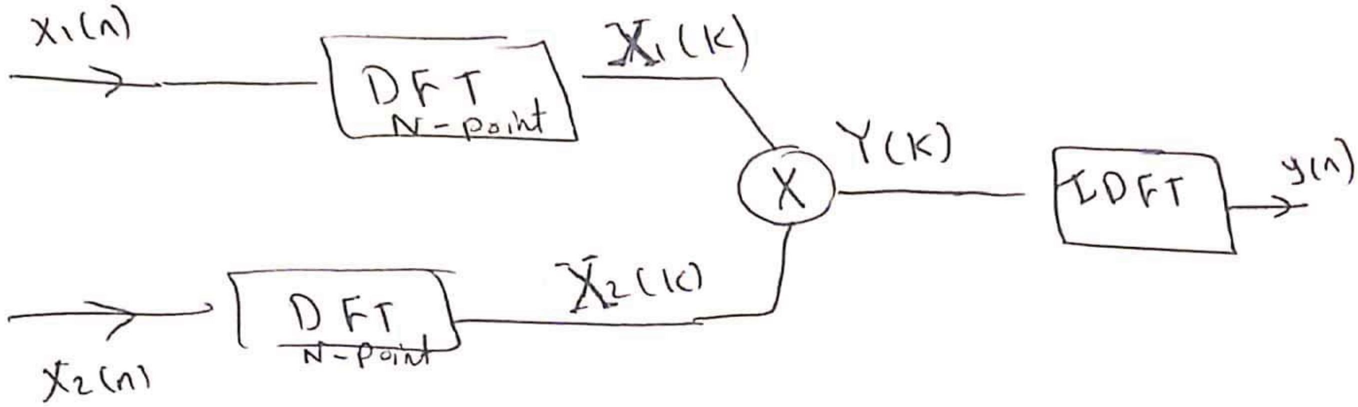
$\boxed{y(3) = 2}$

$\therefore y(n) = \{-1, 4, 5, 2\}$ & repeats

periodic of period $\boxed{4}$

We can make circular convolution using DFT / FFT

Req: $x_1(n) \circledast x_2(n) = y(n)$ (we can use rotating circles if we want) using DFT/FFT.



Steps:

i) get $X_1(k) = DFT\{x_1(n)\}$

ii) get $X_2(k) = DFT\{x_2(n)\}$

iii) get $Y(k) = X_1(k)\, X_2(k)$

iv) get $y(n) = IDFT\{Y(k)\}$.

[This method is to get circular convolution using DFT]. (Note) $x_1(n)$ & $x_2(n)$ must be same length

ex 1

Apply circular Convolution For
$x(n) = \{1, 2, 2, 1\}$, $h(n) = \{1, 2, 3\}$
using DFT and IDFT

Sol

they must be same length. [pad (0) For $h(n)$]

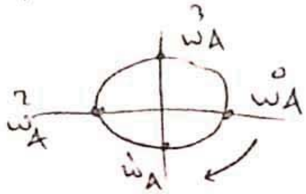
$h(n) = \{1, 2, 3, 0\}$, $x(n) = \{1, 2, 2, 1\}$

steps:

1) get $H(k) = DFT\{h(n)\}$
2) get $X(k) = DFT\{x(n)\}$
3) $Y(k) = X(k)\,H(k) \rightsquigarrow y(n) = IDFT\{Y(k)\}$

$\boxed{1}$ $H(K) = DFT\{h(n)\}$

$$\begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} w_4^0 & w_4^0 & w_4^0 & w_4^0 \\ w_4^0 & w_4^1 & w_4^2 & w_4^3 \\ w_4^0 & w_4^2 & w_4^4 & w_4^6 \\ w_4^0 & w_4^3 & w_4^6 & w_4^9 \end{bmatrix} \end{array} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \\ h(3) \end{bmatrix}$$



$w_4^4 = w_4^0 = 1$, $w_4^6 = w_4^2 = -1$
$w_4^9 = w_4^5 = w_4^1 = -j$,

$$\therefore \begin{bmatrix} H(0) \\ H(1) \\ H(2) \\ H(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -2-2j \\ 2 \\ -2+2j \end{bmatrix}$$

Step 2: $X(k) = DFT\{x(n)\}$

similarly:

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \end{array} \begin{array}{l} \to x(0) \\ \to x(1) \\ \to x(2) \\ \to x(3) \end{array}$$

$$= \begin{bmatrix} 6 \\ -1-j \\ 0 \\ -1+j \end{bmatrix}$$

Step 3: $Y(k) = X(k)H(k)$

$$\therefore Y(k) = \begin{bmatrix} 36 \\ j4 \\ 0 \\ -j4 \end{bmatrix}$$

Step 4: $y(n) = \text{IDFT}\{Y(k)\}$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \underbrace{\frac{1}{4}}_{1/N} \begin{bmatrix} W_4^{0} & W_4^{0} & W_4^{0} & W_4^{0} \\ W_4^{0} & W_4^{-1} & W_4^{-2} & W_4^{-3} \\ W_4^{0} & W_4^{-2} & W_4^{-4} & W_4^{-6} \\ W_4^{0} & W_4^{-3} & W_4^{-6} & W_4^{-9} \end{bmatrix} \begin{bmatrix} 36 \\ j4 \\ 0 \\ -j4 \end{bmatrix}$$

W_N^{-kn}

IDFT [illegible]

$W_4^{-9} = W_4^{-5} = W_4^{-1} = j$

$W_4^{-1} = j \quad W_4^{-2} = -1 \quad W_4^{-3} = -j$

$W_4^{-4} = W_4^{0} = 1 \quad W_4^{-6} = W_4^{-2} = -1$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & J & -1 & -J \\ 1 & -1 & 1 & -1 \\ 1 & -J & -1 & J \end{bmatrix} \begin{bmatrix} 36 \\ J4 \\ 0 \\ -J4 \end{bmatrix}$$

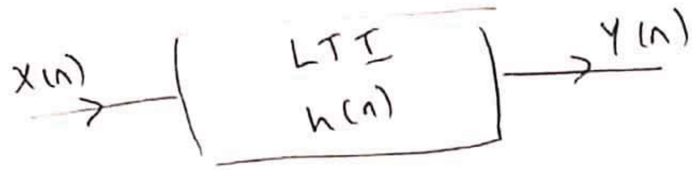
$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 36 \\ 28 \\ 36 \\ 44 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 9 \\ 11 \end{bmatrix}$$

$\therefore y(n) = \{ 9, 7, 9, 11 \}$

$\boxed{\text{Homework}} \longrightarrow$ Apply the circular Convolution using rotating circles and make sure you get the same result.

Performing linear convolution using circular convolution

Recall linear convolution:



$x(n)$: non periodic with length N_x

$h(n)$: non periodic with length N_h

$y(n) = x(n) * h(n)$

linear convolution

$y(n)$: non periodic with length $N_y = N_x + N_h - 1$

Steps to perform linear convolution using circular convolution

1) Find the length of $y(n) = N_x + N_h - 1$
2) zero-pad $x(n)$ & $h(n)$ to have length N_y
3) Now $x(n)$ & $h(n)$ are of the same length N_y

$\Downarrow$

we can make circular convolution using — rotating circles **or** using DFT/FFT example

LTI system with $h(n) = \{2, 2, 1\}$

if the input $x(n) = \{1, 2, 3, 4\}$ Find the output $y(n)$ using circular convolution ?

Sol

Steps!

1) $N_x = 4$, $N_h = 3 \rightarrow N_y = N_x + N_h - 1$

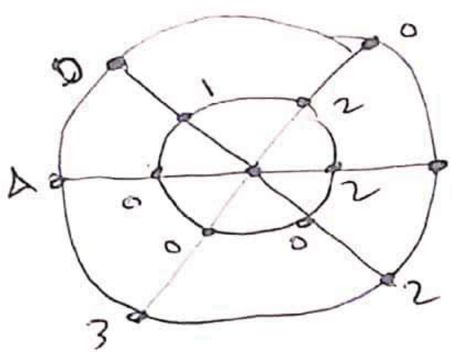
$N_y = 4 + 3 - 1 = 6$

2) Pad 2 zeros for $x(n) \Rightarrow x(n) = \{1, 2, 3, 4, 0, 0\}$

pad 3 zeros for $h(n) \Rightarrow h(n) = \{2, 2, 1, 0, 0, 0\}$

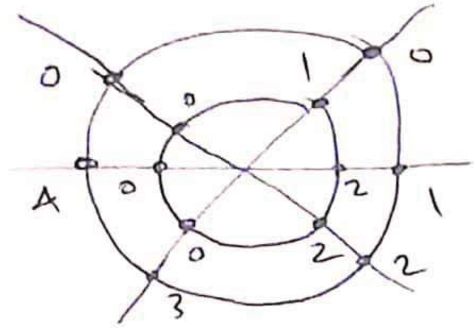
3) Now, we will perform circular convolution using rotating circles

$\longrightarrow$ Fix $x(n)$ clockwise & arrange $h(n)$ Counter-clockwise



$y(0) = 1*2 + 0*2 + 0*1 + 4*0 + 3*0 + 2*0 = 2$

→ rotate $h(n)$ one step clock wise



$y(1) = 2*1 + 0*1 + 0*6 + 0*4 + 0*3 + 2*2 = 6$

Repeat as before to get $y(2)$, $y(3)$, $y(4)$, $y(5)$

$y(n) = \{2, 6, 11, 16, 11, 4\}$

example

LTI system with $h(n) = \{1, 2, 2, 1\}$

if $x(n) = \{1, 2, 3\}$ Find $y(n)$ using

circuler convulution of DFT ?

Sol

Steps:

1) $N_x = 4$, $N_h = 3$

$N_y = N_x + N_h - 1 = 4 + 3 - 1 = 6$

2) pad 2 zeros for $x(n) \Rightarrow x(n) = \{1, 2, 2, 1, 0, 0\}$

pad 3 zers for $h(n) \Rightarrow h(n) = \{1, 2, 3, 0, 0, 0\}$

3) Now, to get $y(n)$, we will use circuler

convolution using DFT.

$\longrightarrow$ get $X(k) = DFT\{x(n)\}$

$\longrightarrow$ get $H(k) = DFT\{h(n)\}$

$\longrightarrow Y(k) = X(k)\,H(k)$

$\longrightarrow y(n) = IDFT\{Y(k)\}$

Result: $y(n) = \{1, 4, 9, 11, 8, 3, 0, 0\}$