4. Representing Numbers In Different Bases

In everyday life, numbers are typically represented in base 10. A number written in base 10 as

$$a_n a_{n-1} \cdots a_2 a_1 a_0$$

represents the value

$$a_n 10^n + a_{n-1} 10^{n-1} + \cdots + a_2 10^2 + a_1 10 + a_0$$

In this setting the values $a_n, a_{n-1}, \cdots, a_2, a_1, a_0$ are chosen from 0, 1, ..., 9. For example, in base 10,

$$341 = 3(10)^2 + 4(10) + 1$$

and

$$2437 = 2(10)^3 + 4(10)^2 + 3(10) + 7$$

These numbers can also be written in other bases.

Example 4.1: The number 341 written in base 7 is 665_7 since

$$341 = 6(7)^2 + 6(7) + 5$$

(notice the use of the subscript "7" to denote that this is a base 7 number). Similarly, the number 2437 written in base 3 is 10100021_3 since

$$2437 = 1(3)^7 + 0(3)^6 + 1(3)^5 + 0(3)^4 + 0(3)^3 + 0(3)^2 + 2(3) + 1$$

Example 4.2: The base 6 number 450125_6 can be written in base 10 as follows:

$$\begin{aligned} 450125_6 &= 4(6)^5 + 5(6)^4 + 0(6)^3 + 1(6)^2 + 2(6) + 5 \\ &= 37{,}637 \end{aligned}$$

In general, if $k, m \in \mathbb{N}$ with $m \geq 2$, then the base 10 number *k*, written in base *m* is the expression

$$(b_n b_{n-1} \cdots b_2 b_1 b_0)_m$$

where the values $b_n,\ b_{n-1},\ \cdots,\ b_2,\ b_1,\ b_0 \in \{0, 1, ..., m-1\}$ and

$$k = b_n m^n + b_{n-1} m^{n-1} + \cdots + b_2 m^2 + b_1 m + b_0$$

In the case when $m = 7$, we have the restrictions $b_n,\ b_{n-1},\ \cdots,\ b_2,\ b_1,\ b_0 \in \{0, 1, ..., 6\}$ and

$$k = b_n (7)^n + b_{n-1} (7)^{n-1} + \cdots + b_2 (7)^2 + b_1 (7) + b_0$$

This process is the simplest if the value *m* is no larger than 10. This is because we need single-digit names for values 10 or larger if the base *m* is larger than 10. Otherwise, the expression $(b_n b_{n-1} \cdots b_2 b_1 b_0)_m$ representing a number in base *m* will be difficult to interpret.

Example 4.3: Writing numbers in base 11 requires a single digit to represent the base 10 value 10. Any symbol can be chosen. For example, suppose we chose the symbol Δ to give a single digit representation in base 11 for the base 10 number 10. Then the base 10 number 21 can be written in base 11 as $1\Delta_{11}$ since

$$21 = 1(11) + \Delta_{11} = 11 + 10 = 21$$

Similarly, the base 10 number 5309 can be written in base 11 as $3\Delta 97_{11}$ since

$$\begin{aligned} 5309 &= 3(11)^3 + \Delta_{11}(11)^2 + 9(11) + 7 \\ &= 3(11)^3 + 10(11)^2 + 9(11) + 7 \\ &= 5309 \end{aligned}$$

We can also reverse this process. The base 11 number $4\Delta\Delta 21_{11}$ can be written in base 10 as

$$\begin{aligned} 4(11)^4 + \Delta_{11}(11)^3 + \Delta_{11}(11)^2 + 2(11) + 1 &= 4(11)^4 + 10(11)^3 + 10(11)^2 + 2(11) + 1 \\ &= 73107 \end{aligned}$$

The theorem below tells us that we can write a base $m \geq 2$ representation of any natural number *k*.

Theorem 4.4: Suppose $k, m \in \mathbb{N}$ with $m \geq 2$. Then there is a unique number $n \in \{0, 1, 2, ...\}$ and unique values

$$b_n,\ b_{n-1},\ \cdots,\ b_2,\ b_1,\ b_0 \in \{0, 1, ..., m-1\}$$

so that

$$k = b_n m^n + b_{n-1} m^{n-1} + \cdots + b_2 m^2 + b_1 m + b_0$$

Definition 4.5: Suppose $k, m \in \mathbb{N}$ with $m \geq 2$. The base *m* representation of the number *k* is given by $(b_n b_{n-1} \cdots b_2 b_1 b_0)_m$ where the values $b_n,\ b_{n-1},\ \cdots,\ b_2,\ b_1,\ b_0$ satisfy the equation given in Theorem 4.4.

The process for computing the base $m \geq 2$ representation of a natural number *k* is given below.

Writing A Base 10 Number *k* In Base *m*	
Step 0	If *m* > 10, create symbols for 10, ..., $m-1$.
Step 1	Determine the largest number $n \in \{0, 1, 2, ...\}$ so that $k \geq m^n$.
Step 2	Set $b_n = \text{ipart}\left(k / m^n\right)$ and $r_{n-1} = k - b_n m^n$. If $b_n \geq 10$, use one of the

	symbols from **Step 0**.
Step 3	Set $b_{n-1} = \text{ipart}\left(r_{n-1} / m^{n-1}\right)$ and $r_{n-2} = r_{n-1} - b_{n-1}m^{n-1}$. If $b_{n-1} \geq 10$, use one of the symbols from **Step 0**.
$\vdots$	$\vdots$
Step n+1	Set $b_1 = \text{ipart}\left(r_1 / m\right)$ and $r_0 = r_1 - b_1 m$. If $b_1 \geq 10$, use one of the symbols from **Step 0**.
Step n+2	Set $b_0 = r_0$. If $b_0 \geq 10$, use one of the symbols from **Step 0**.
The base 10 number *k* has the base *m* representation $\left(b_n b_{n-1} \cdots b_2 b_1 b_0\right)_m$	

Example 4.6: Use the algorithm above to write the base 10 number 3147 in base 6.

Solution: There is no need for Step 0 above. The value *n* in Step 1 can be determined by computing some powers of 6. We can see that $6^4 = 1296$ and $6^5 = 7776$. As a result, $n = 4$. The table below gives the values for $b_4,\ r_3,\ b_3,\ r_2,\ b_2,\ r_1,\ b_1,\ r_0,\ b_0$.

$b_4 = \text{ipart}\left(3147 / 6^4\right) = 2$	$r_3 = 3147 - 2(6)^4 = 555$
$b_3 = \text{ipart}\left(555 / 6^3\right) = 2$	$r_2 = 555 - 2(6)^3 = 123$
$b_2 = \text{ipart}\left(123 / 6^2\right) = 3$	$r_1 = 123 - 3(6)^2 = 15$
$b_1 = \text{ipart}\left(15 / 6\right) = 2$	$r_0 = 15 - 2(6) = 3$
$b_0 = 3$	

As a result, the base 10 number 3147 can be written in base 6 as 22323_6.

Example 4.7: Use the algorithm above to write the base 10 number 23469 in base 13.

Solution: Since 13 > 10, we need to do some work in Step 0 to create symbols for single digit representations of 10, 11 and 12. For simplicity, we will use *a*, *b* and *c* to represent 10, 11 and 12 respectively. The value *n* in Step 1 can be determined by computing some powers of 13. Notice that $13^3 = 2197$ and $13^4 = 28561$. So, $n = 3$. The table below shows the values for $b_3,\ r_2,\ b_2,\ r_1,\ b_1,\ r_0,\ b_0$.

$b_3 = \text{ipart}\left(23469 / 13^3\right) = 10 = a$	$r_2 = 23469 - 10(13)^3 = 1499$
$b_2 = \text{ipart}\left(1499 / 13^2\right) = 8$	$r_1 = 1499 - 8(13)^2 = 147$

$b_1 = \text{ipart}(147/13) = 11 = b$	$r_0 = 147 - 11(13) = 4$
$b_4 = 4$	

As a result, the base 10 number 23469 can be written in base 13 as $a8b4_{13}$.

Remark 4.8: Logarithms can be used to determine the value n in Step 1 above. In fact,

$$n = \text{ipart}\left(\log_m(k)\right)$$

Notice that

$$n = \text{ipart}\left(\log_6(3147)\right) = \text{ipart}\left(\ln(3147)/\ln(6)\right) = 4$$

and

$$n = \text{ipart}\left(\log_{13}(23469)\right) = \text{ipart}\left(\ln(23469)/\ln(13)\right) = 3$$

(the values for n in the previous two examples).

Exercises

1. Write the base 10 number 34563 in base 6.
2. Write the base 10 number 14563 in base 2.
3. Write the base 10 number 126067 in base 7.
4. Write the base 10 number 72116 in base 12.
5. Write the base 10 number 72996 in base 13.
6. Write the base 6 number 34542_6 in base 10.
7. Write the base 8 number 57572_8 in base 10.
8. Write the base 2 number 1011011_2 in base 10.
9. Write the base 13 number $c4ab29_{13}$ in base 10, where a, b and c are the single digit representations of 10, 11 and 12 respectively.
10. Develop addition and multiplication tables for base 2.
11. Develop addition and multiplication tables for base 5.
12. Develop addition and multiplication tables for base 7.
13. Develop addition and multiplication tables for base 9.
14. Develop addition and multiplication tables for base 11.
15. Develop addition and multiplication tables for base 13.
16. Develop addition and multiplication tables for base 15.
17. Answer the following questions in base 3.
    a. How old are you?
    b. How many months are there in a year?
    c. How many meters make one kilometer?
    d. How many millimeters make one centimeter?
    e. How many quarters make one dollar?
    f. How many nickels make one quarter?
    g. How many days are there in the month of May?
    h. How many feet are there in one mile?
    i. What is the current year?

A TI-83 Program For Writing Base 10 Numbers In Other Bases

The program **BASE** given below can be used to write a base 10 number in another base. The program prompts for the base 10 representation of the number and the value for the new base. Then it returns the digits for the new base representation in a list. The user must create their own symbols for the single digit representation of values 10, 11, 12, ... as needed.

BASE

```
: ClrList L1
: Disp "BASE 10 NUMBER? "
: Prompt K
: Disp "NEW BASE?"
: Prompt M
: iPart(ln(K)/ln(M)) → N
: For(I,0,N)
: iPart(K/M^(N-I)) → L1 (I+1)
: K – L1 (I+1)*M^(N – I ) → K
: End
: Disp "NEW DIGITS IN L1"
: Disp L1
```

The screen shot below shows the program **BASE** being used to create the base 13 representation of the base 10 number 23469.

```
prgmBASE
BASE 10 NUMBER?
N=?23469
NEW BASE?
M=?13
NEW DIGITS IN L1
                    { 10  8  11  4 }
```

The values shown in the list L_1 agree with the computations in the example above. There we used *a*, *b* and *c* as single digit representations of 10, 11 and 12, and wrote the base 10 number 23469 in base 13 as $a8b4_{13}$.

A Different Method For Converting Base 10 Numbers

Many different methods can be used to write a base 10 number in a different base. We illustrate an additional method with an example. Suppose we want to write the base 10 number 1276 in base 6. To do this, we need to find a number $n \in \{0,1,2,3,...\}$ and values

$$b_n,\ b_{n-1},\ \cdots,\ b_2,\ b_1,\ b_0 \in \{0,1,...,5\}$$

so that

$$1276 = b_n(6)^n + b_{n-1}(6)^{n-1} + \cdots + b_2(6)^2 + b_1(6) + b_0$$

First notice that $6^3 = 216$ and $6^4 = 1296$. So $n = 3$. We can use this to write the equation above in the form

$$1276 = b_3(6)^3 + b_2(6)^2 + b_1(6) + b_0$$

Now we use a subtle trick. Factor 6 from the first three terms on the right hand side.

$$1276 = 6\left(b_3(6)^2 + b_2(6) + b_1\right) + b_0$$

This is the quotient and remainder form that results when 1276 is divided by 6. As a result, b_0 is the remainder when 1276 is divided by 6. That is,

$$b_0 = 1276 - \text{iPart}\left(\frac{1276}{6}\right)6 = 4$$

We place this in the equation above

$$1276 = b_3(6)^3 + b_2(6)^2 + b_1(6) + 4$$

Then we subtract 4 from both sides and divide by 6.

$$\frac{1276-4}{6} = b_3(6)^2 + b_2(6) + b_1$$

Simplifying the left hand side gives

$$212 = b_3(6)^2 + b_2(6) + b_1$$

Now we repeat the process. First, we rewrite the right hand side above by factoring 6 off of the first two terms.

$$212 = 6\left(b_3(6) + b_2\right) + b_1$$

In this form, we can see that b_1 is the remainder when 212 is divided by 6. Consequently,

$$b_1 = 212 - \text{iPart}\left(\frac{212}{6}\right)6 = 2$$

We place this value in the equation above.

$$212 = b_3(6)^2 + b_2(6) + 2$$

Then we subtract 2 from both sides and divide by 6.

$$\frac{212-2}{6} = b_3(6) + b_2$$

Simplifying gives

$$35 = b_3(6) + b_2$$

Continuing this method gives $b_2 = 5$ and $b_3 = 5$. As a result,

$$1276 = 5524_6$$

We can verify our result as follows:

$$5524_6 = 5(6)^3 + 5(6)^2 + 2(6) + 4 = 1276$$

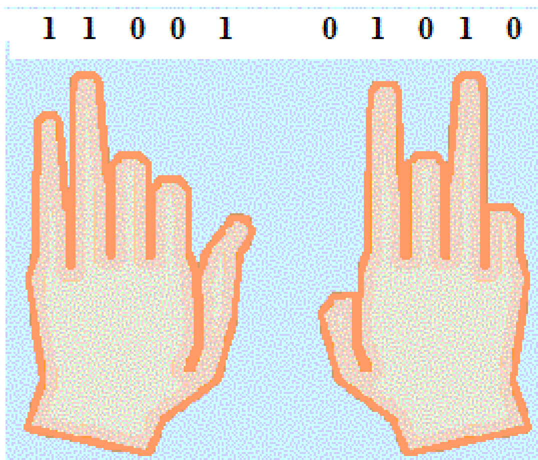
Exercises: Use the method illustrated above to work each of the problems below.

1. Write the base 10 number 34563 in base 6.
2. Write the base 10 number 14563 in base 2.
3. Write the base 10 number 126067 in base 7.
4. Write the base 10 number 72116 in base 12.
5. Write the base 10 number 72996 in base 13.
6. Write the base 6 number 34542_6 in base 10.

7. Write the base 8 number 57572_8 in base 10.

Counting To 1023 On Your Fingers

It is generally accepted that our base 10 numbering system arose as a result of humans having a total of 10 fingers and thumbs on their two hands. Unfortunately, this is not the best base for counting large quantities with your fingers, since it is difficult to get past 10. Interestingly, counting in base 2 is much easier with fingers. In fact, it is possible to use base 2 to count to 1023 on your fingers! For example, the base 10 number 810 has a base 2 representation of 1100101010_2, which can be represented on two hands as shown below.



The graphic above came from an applet demonstrating this process at

http://www.intuitor.com/counting/

Exercises

1. How far can a person count in base 2 if they use their thumbs, fingers and toes?
2. Show that a group of 3 group people can count to 1,073,741,823 in base 2 if they combine the fingers and thumbs on all of their hands. Draw a picture showing how the base 10 number 643,111,231 can be represented.
3. Describe how a person can use their hands to count in base 6, using one hand for the ones digit and the other hand for the six digit. How far can a person count using this method?