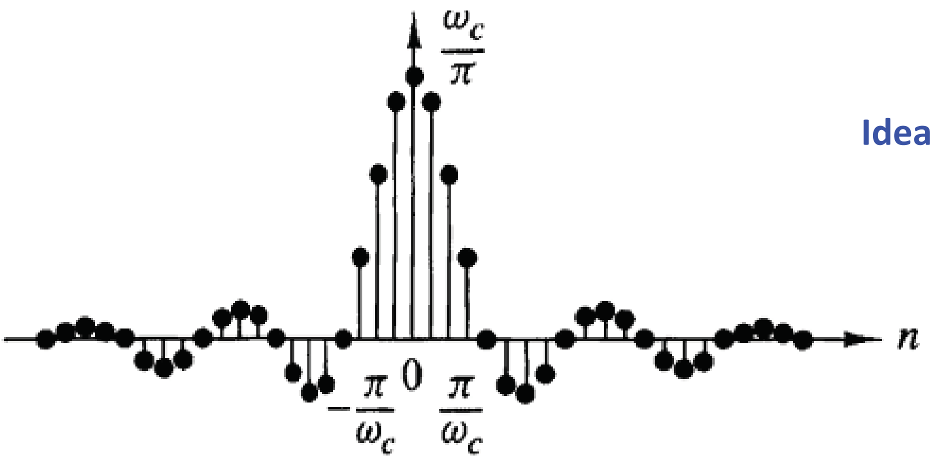
Advantages of FIR filters

- $H(n)$ is of finite length
  - ✓ filter is always stable
- Filter structure is simple
  - ✓ no recursive/feedback elements)
- Can always shift filters to make it causal
- If certain symmetries on $H(n)$ exists
  - ✓ linear phase can be guaranteed

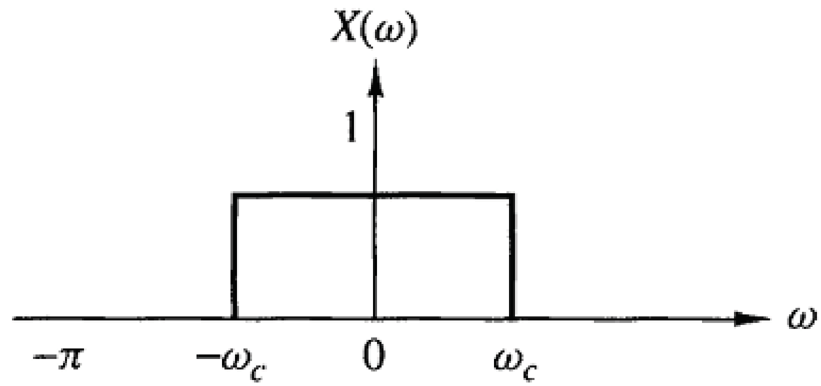
Design FIR filters using windows: Idea

- How to meet a specified frequency response using an FIR filter?
- **Windows Method:** rather than starting with a non-ideal frequency response
  - Start with a non-realizable ideal impulse response
  - Alter it to get a causal, stable, finite length $h(n)$ ➔ Get $H(z)$➔DE

What is the time response of ideal filters?



Ideal LPF



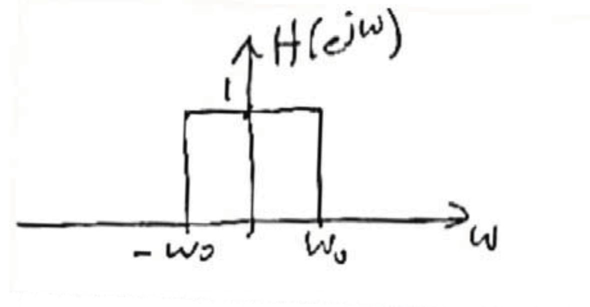
$$X(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \le |\omega| \le \pi \end{cases}$$

$$x(n) = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\sin \omega_c n}{\pi n}, & n \neq 0 \end{cases}$$

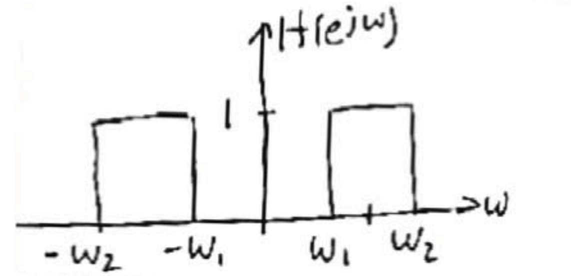
$$x(n) = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

Time response of ideal filters

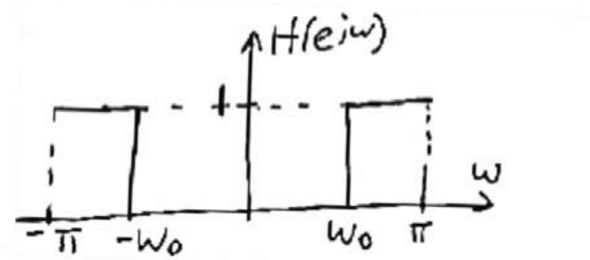
LPF: $h(n) = \dfrac{\sin(w_0 n)}{\pi n}$



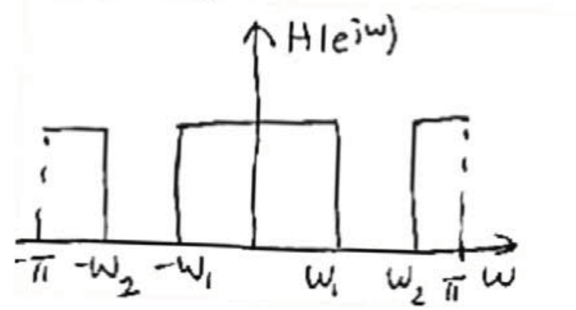
BPF: $h(n) = \dfrac{\sin(w_2 n)}{\pi n} - \dfrac{\sin(w_1 n)}{\pi n}$



HPF: $h(n) = \delta(n) - \dfrac{\sin(w_0 n)}{\pi n}$



BSF: $h(n) = \delta(n) - \dfrac{\sin(w_2 n)}{\pi n} + \dfrac{\sin(w_1 n)}{\pi n}$

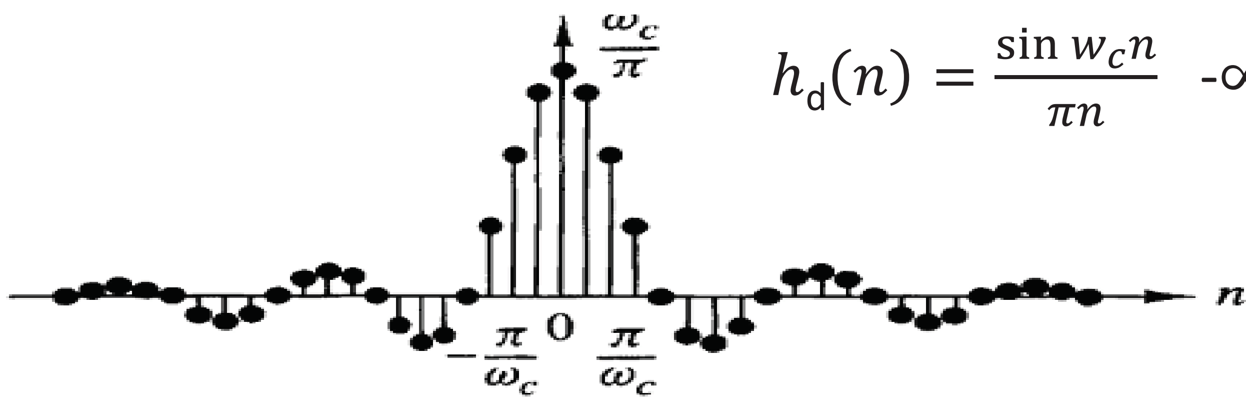


How to get a causal FIR filter from ideal filters (infinite, noncausal)?

To get a causal and finite-length M impulse response from the ideal impulse response $h_{\mathrm{d}}(n)$:

- **Step-1:** Simply truncate and weight (may be with all ones: rectangular window) the desired response $h_{\mathrm{d}}(n)$
- Step-2: Shift right the results by $\frac{M-1}{2}$ to make it causal

$$h_{\mathrm{d}}(n) = \frac{\sin w_c n}{\pi n} \quad -\infty < n < \infty$$

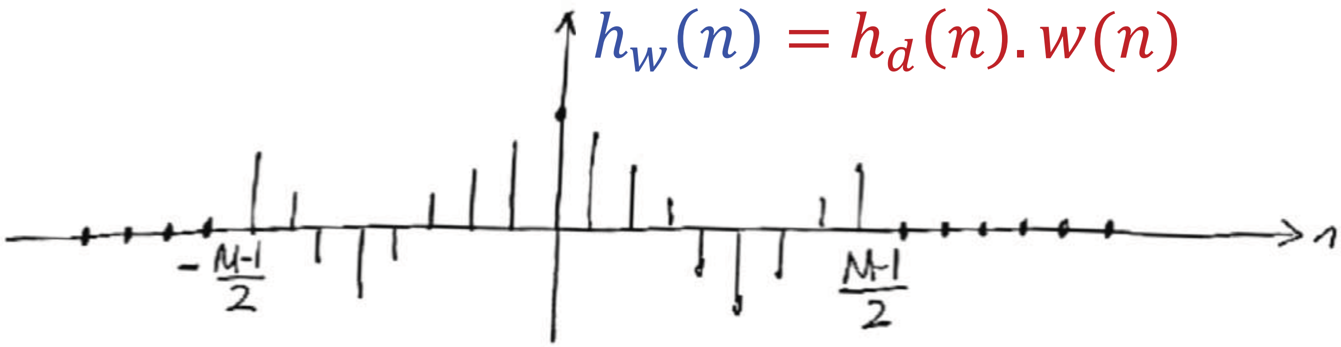


Desired Ideal impulse response of a LPF (infinite and noncausal)

Step-1: Truncate and weight using a window function (may be with all ones: rectangular window) the desired response $h_{\mathrm{d}}(n)$

- Truncate at $[-\frac{M-1}{2}, \frac{M-1}{2}]$
- Weight each term by a window function $w(n)$

$$h_w(n) = h_d(n).w(n)$$



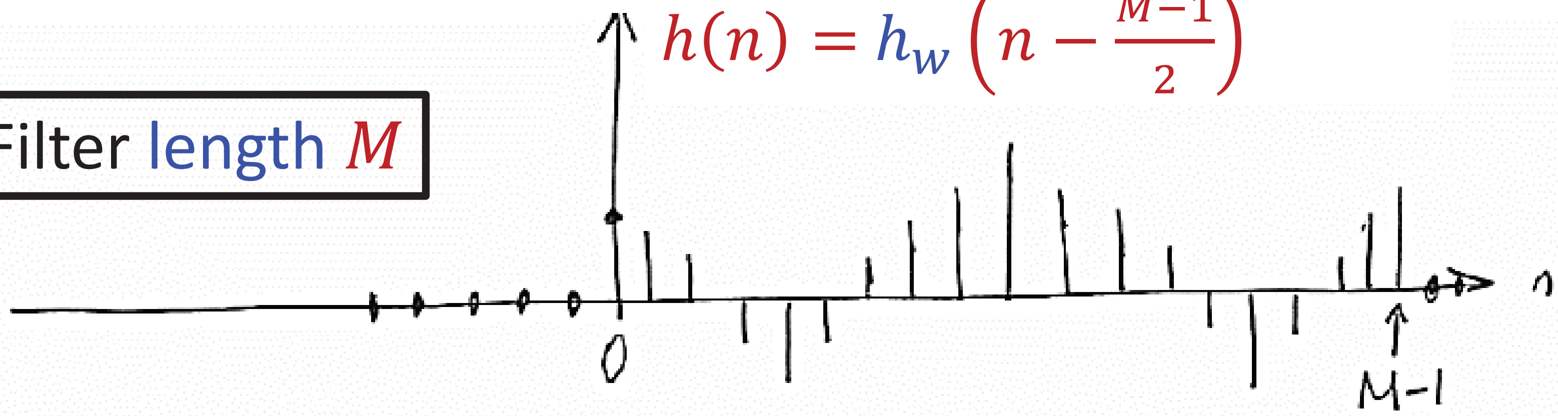
Still non-causal ☹

- Step-2: Shift right the results by $\frac{M-1}{2}$ to make it causal

$$h(n) = h_w\left(n - \frac{M-1}{2}\right)$$

$h(n) = h_w\left(n - \frac{M-1}{2}\right)$

$\boxed{\text{Filter length } M}$



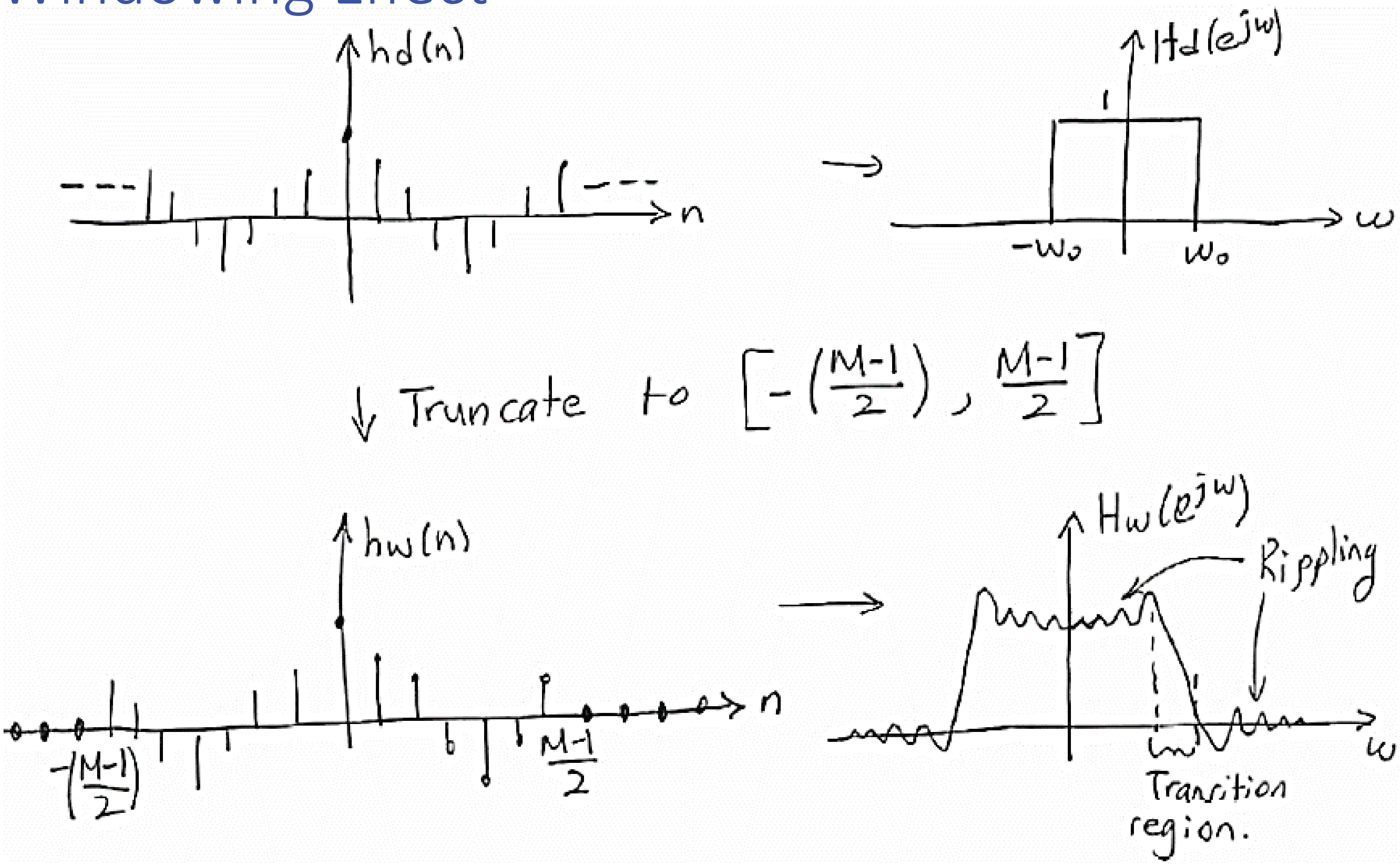
Causal finite length M filter ☺

FIR Filter: $H(z) = h(0) + h(1)z^{-1} + \cdots + h(M-1)z^{-(M-1)}$

$$H(z) = \sum_{0}^{M-1} h(n)\, z^{-n}$$

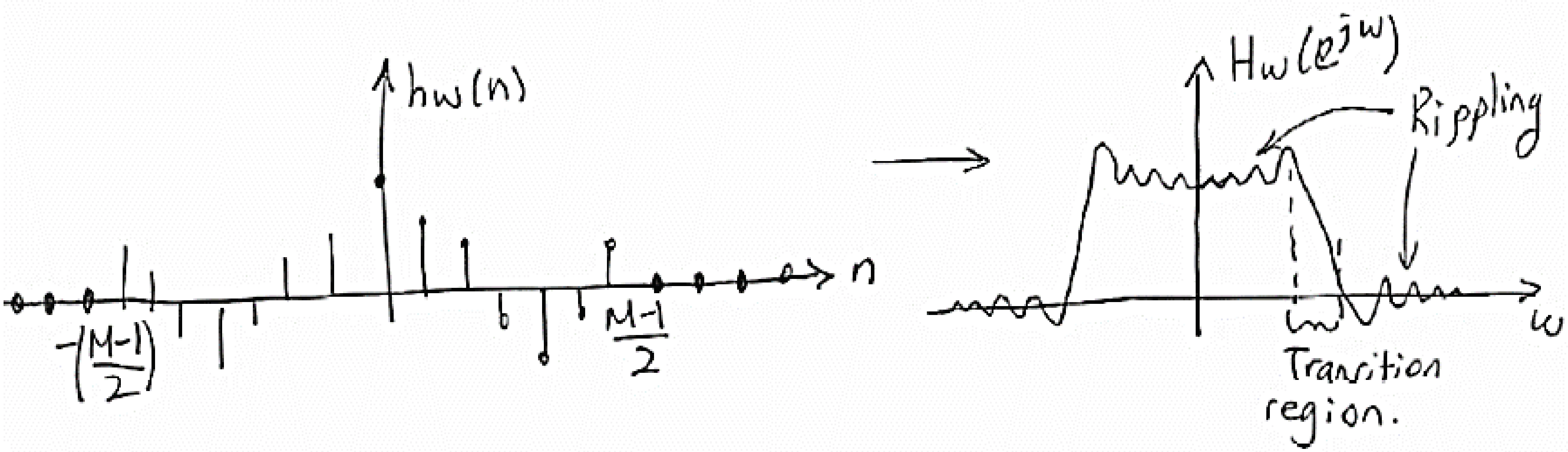
$\boxed{\text{Filter order } M-1}$

Windowing Effect



Windowing Effect (cont'd)

- Smearing of the desired frequency response
  - Transition region
- Rippling of the desired frequency response
  - Nonflat passband/stopband (Gibbs Phenomenon)



How to control $h(n)$ using $w(n)$?

Free control parameters:

- No. of coefficients (filter length) M
- Window Type (Shape)

1. As the length M of the window increases, the width of the main lobe decreases, which results in a linear decrease in the transition width Δw between passbands and stop bands
   - $\Delta w = \frac{c\pi}{M}$; c depends on the window shape,
   - $\text{i.e.}, c = 4 \; and \; \Delta w = \frac{4\pi}{M}$ for a rectangular window)
2. The peak side-lobe amplitude is determine by the shape of the window and it is essentially **independent of the window length** M

How to control $h(n)$ using $w(n)$?

3. If the window shape is changed to decrease side-lobe amplitude (and in turns decreasing the stopband ripples), the **<u>width of the $w(n)$ main lobe ($\sim H(w)$ transition width) will generally increase</u>**

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)
Rectangular	−13	$4\pi/M$	−21
Bartlett	−25	$8\pi/M$	−25
Hanning	−31	$8\pi/M$	−44
Hamming	−41	$8\pi/M$	−53
Blackman	−57	$12\pi/M$	−74

Transition width increases

Stopband ripples decreases

For knowledge, do not keep numbers or rules.

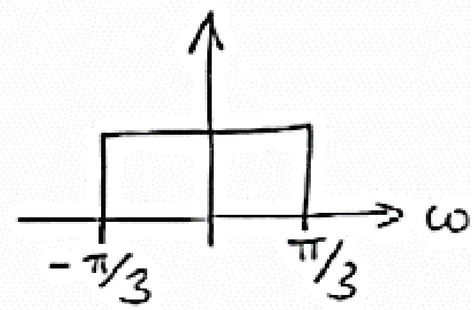
Example:

Apply a rectangular window to design FIR LPF with cut off frequency of π/3. Use an FIR filter of length M =13

Solution

- **Start with an ideal Filter, $h_d(n)$**

$$h_d(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = 0 \\ \dfrac{\sin \omega_c n}{\pi n}, & n \neq 0 \end{cases}$$



Given that the cut off frequency = π/3.

$$h_d(n) = \begin{cases} \dfrac{\sin(\pi/3\, n)}{\pi n} & n \neq 0 \\ 1/3 & n = 0 \end{cases}$$

Given that M = 13.

Step-1: Truncate $h_d(n)$ and weight it using the rectangular window: $h_w(n) = h_d(n).w(n)$

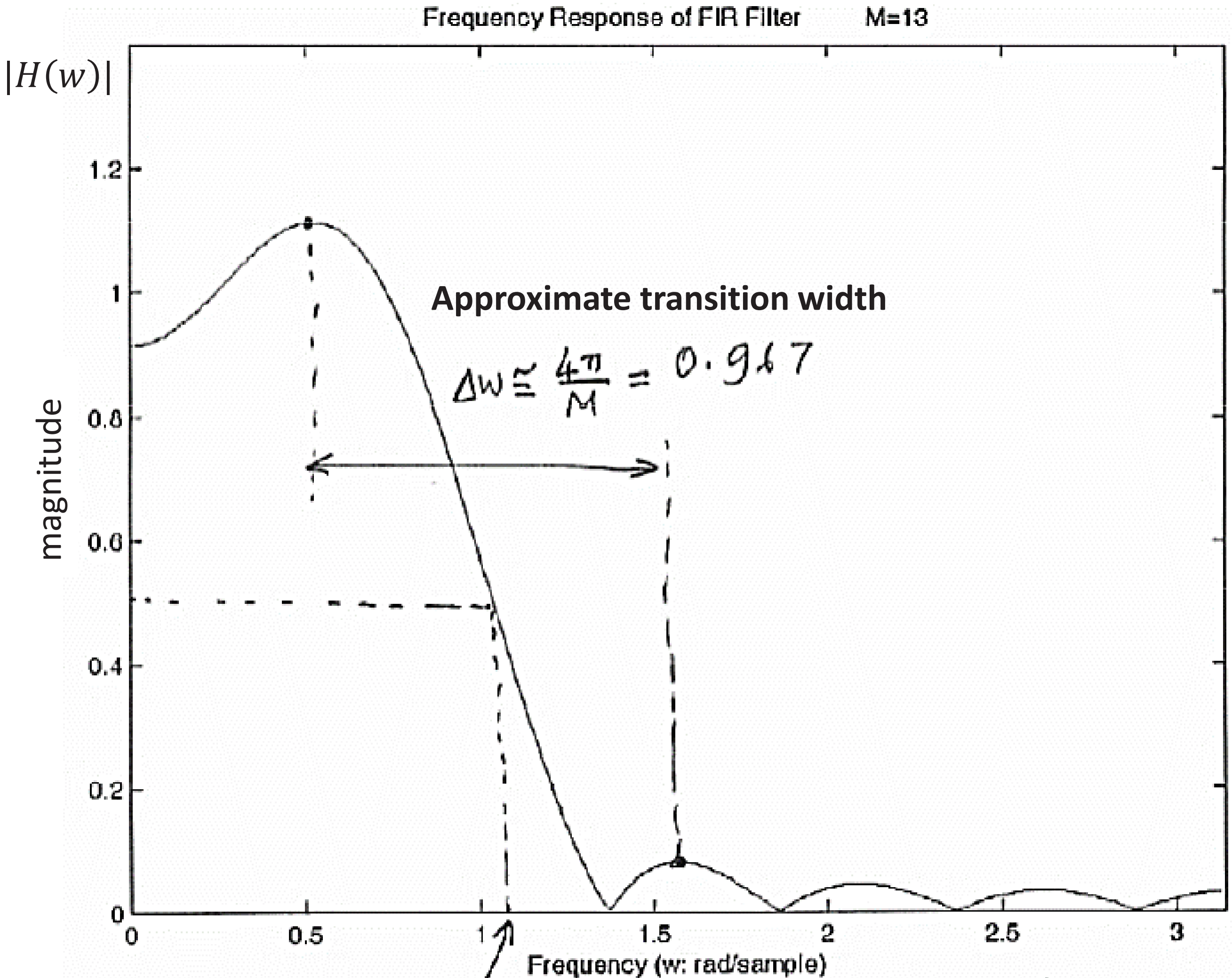
Apply M=13 rectangular window $w(n) = \begin{cases} 1 & -6 \leq n \leq 6 \\ 0 & \text{else} \end{cases}$

Then $h_w(n) = \begin{cases} h_d(n) & |n| \leq 6 \\ 0 & |n| > 6 \end{cases}$

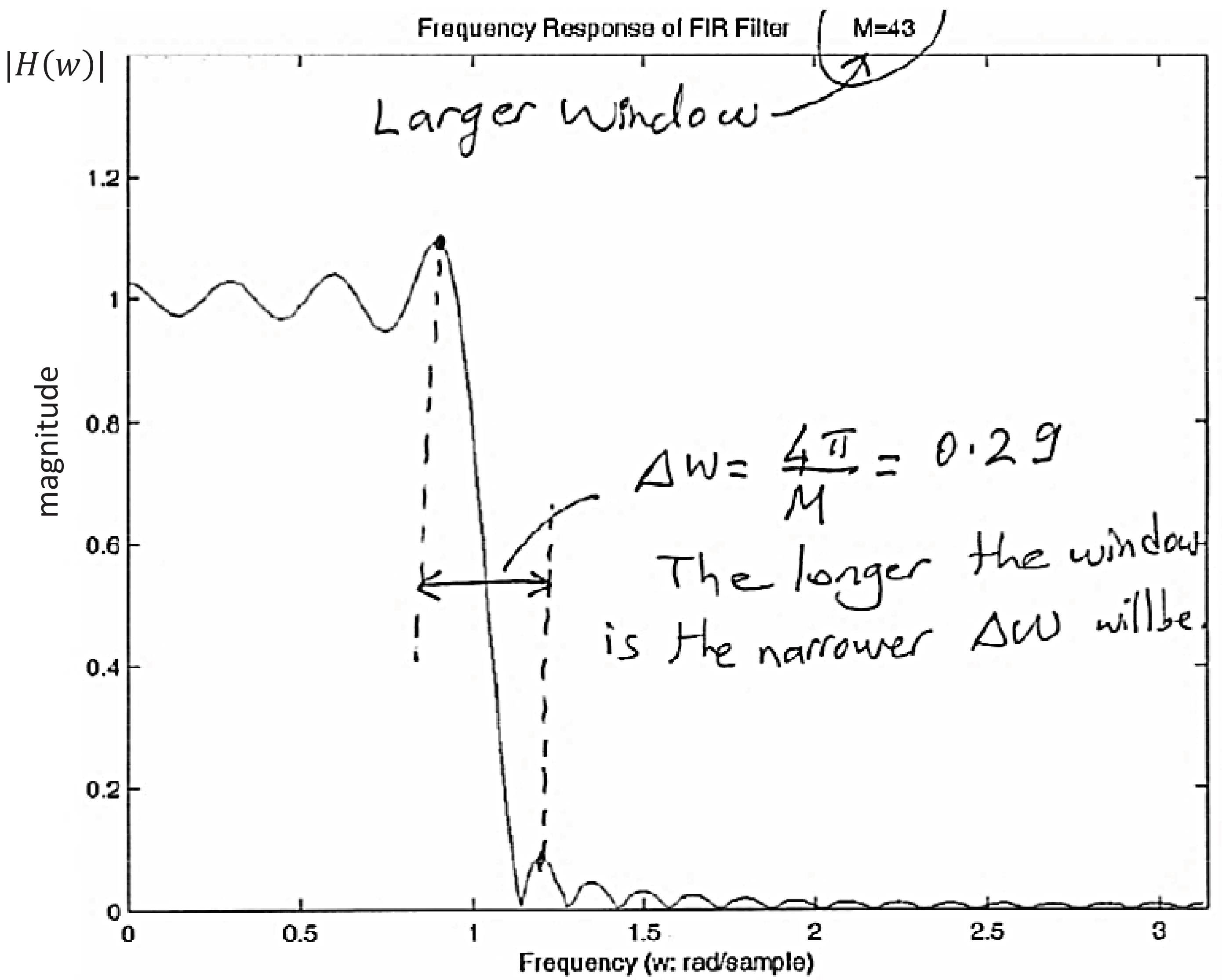
- Step-2: Shift right the results by $\frac{M-1}{2}$ to make it causal

Shift Right by $\frac{M-1}{2} = 6$ to make it causal

$$h(n) = \begin{cases} \dfrac{\sin\left(\frac{\pi}{3}(n-6)\right)}{\pi(n-6)} & 0 \leq n \leq 12, \quad n \neq 6 \\ \dfrac{1}{3} & n = 6 \\ 0 & \text{else} \end{cases}$$



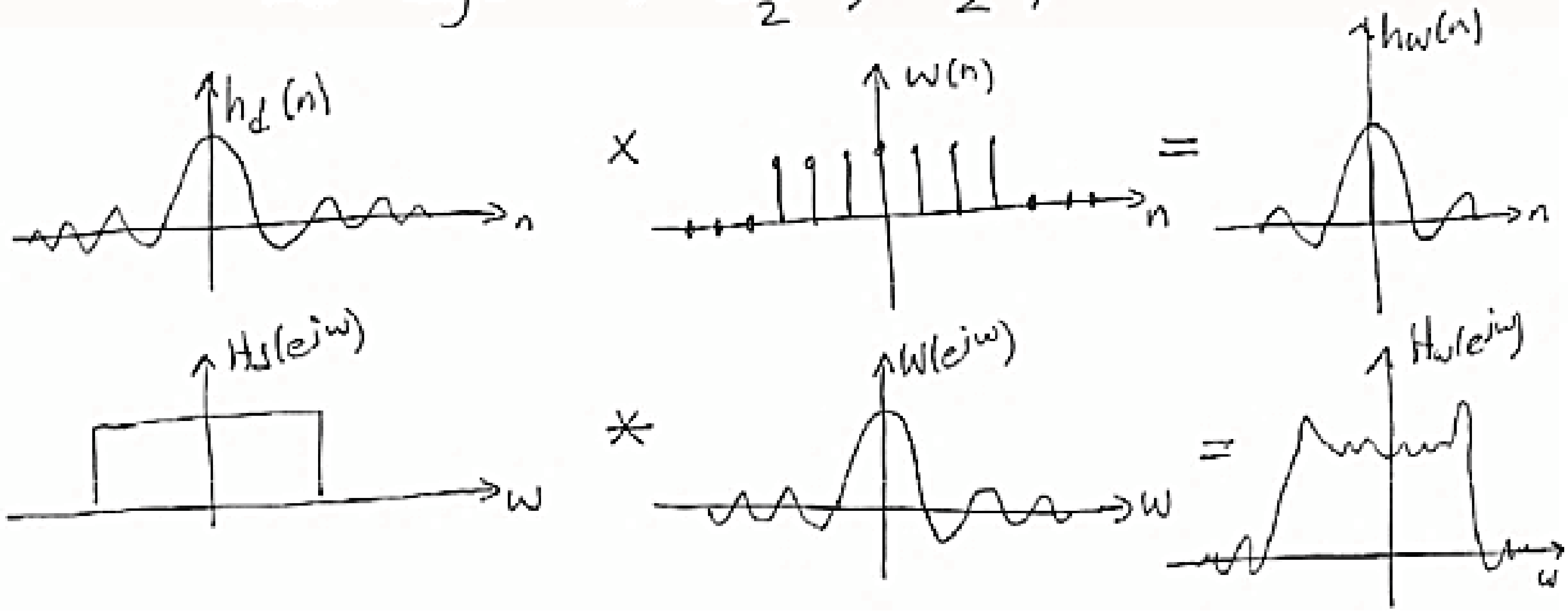
$\pi/3$ Desired W_c ($\pi/3$) is $1/2$ way between peak overshoot and peak undershoot.



Frequency Response of FIR Filter
M=43
$|H(w)|$
Larger Window
magnitude
1.2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3
Frequency (w: rad/sample)
$\Delta W = \frac{4\pi}{M} = 0.29$
The longer the window is the narrower ΔW will be.

FIR Filter Design Using Other Windows

So far: Looked at truncating the infinite length $h(n)$ by simply removing all terms outside the range: $\left(-\frac{(M-1)}{2}, \frac{M-1}{2}\right)$



FIR Filter Design Using Other Windows

Problem: Resulting frequency response is not ideal.

1) transition region (due to mainlobe of window)
2) stopband ripple (due to sidelobes of window)

We can decrease transition region by increasing M but stopband attenuation will always be bad (for rectangular window).

Solution: Use other windows which sacrifice a small transition region (wider mainlobe in window's freq. response) for less ripple (smaller sidelobes in window's freq. response)

For knowledge, do not keep anything.

TABLE 10.1 Window Functions for FIR Filter Design

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Bartlett (triangular)	$1 - \dfrac{2\left\lvert n - \dfrac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5 \cos \dfrac{2\pi n}{M-1} + 0.08 \cos \dfrac{4\pi n}{M-1}$
Hamming	$0.54 - 0.46 \cos \dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos \dfrac{2\pi n}{M-1}\right)$

For knowledge, do not keep anything.

TABLE 10.1 Window Functions for FIR Filter Design

Name of window	Time-domain sequence, $h(n), 0 \le n \le M-1$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2-\left(n-\frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$
Lanczos	$\left\{\dfrac{\sin\left[2\pi\left(n-\frac{M-1}{2}\right)\Big/(M-1)\right]}{2\pi\left(n-\frac{M-1}{2}\right)\Big/\left(\frac{M-1}{2}\right)}\right\}^L, \quad L>0$
	$1, \quad \left\lvert n-\frac{M-1}{2}\right\rvert \le \alpha\frac{M-1}{2}, \qquad 0<\alpha<1$
Tukey	$\frac{1}{2}\left[1+\cos\left(\frac{n-(1+a)(M-1)/2}{(1-\alpha)(M-1)/2}\pi\right)\right]$
	$\alpha(M-1)/2 \le \left\lvert n-\frac{M-1}{2}\right\rvert \le \frac{M-1}{2}$

For knowledge, do not keep anything.

FIR Filter Design Using Other Windows

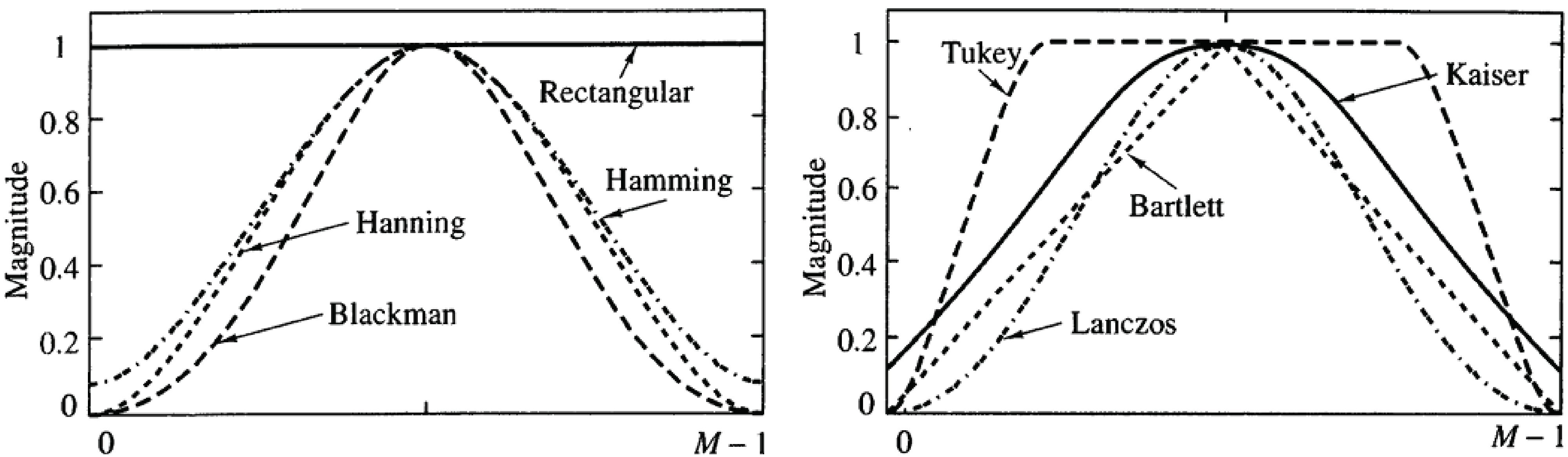


Figure 10.2.3 Shapes of several window functions.

Do not keep any numbers.

FIR Filter Design Using Other Windows

TABLE 7.1 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20 \log_{10} \delta$ (dB)
Rectangular	−13	$4\pi/M$	−21
Bartlett	−25	$8\pi/M$	−25
Hanning	−31	$8\pi/M$	−44
Hamming	−41	$8\pi/M$	−53
Blackman	−57	$12\pi/M$	−74

Transition width increases

Stopband ripples decreases

- To decrease stopband ripples: only way is to change the window
  - main lobe width of increases (wider transition band) ☹

FIR Filter Design Using Other Windows

Summary: FIR Filter Design Via Windowing Method:

Advantages:

1) By using symmetry , we can obtain linear phase filters.
2) Easy to design highpass, bandpass, bandstop as well as lowpass filters
3) Simple procedure
4) Filter is always stable since $h(n)$ is finite length

FIR Filter Design Using Other Windows

Disadvantages:

1) To meet same specs as an IIR filter, FIR filters require much higher orders (40-100 vs. 5-6) (Means more memory and more rounding errors)