Calculus and Analytical Geometry

LECTURE #6

Topics to be Covered

- Unit 6.1: Derivative of Trigonometric and Inverse Trigonometric Functions
- Unit 6.2: Derivative of Logarithmic and Exponential Functions
- Unit 6.3: Chain Rule-Derivatives-Example
- Unit 6.4: Generalized Derivatives-Example
- Unit 6.5: Implicit Differentiation-Example
- Unit 6.6: L'Hospital Rule-Example

Derivative of Trigonometric Functions

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

The derivatives of the remaining trigonometric functions are

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

► **Example 2** Find dy/dx if $y = \dfrac{\sin x}{1+\cos x}$.

Solution. Using the quotient rule together with Formulas (3) and (4) we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{(1+\cos x)\cdot \frac{d}{dx}[\sin x] - \sin x \cdot \frac{d}{dx}[1+\cos x]}{(1+\cos x)^2}\\ &= \frac{(1+\cos x)(\cos x) - (\sin x)(-\sin x)}{(1+\cos x)^2}\\ &= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2} = \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}\end{aligned}$$ ◄

► **Example 1** Find dy/dx if $y = x \sin x$.

Solution. Using Formula (3) and the product rule we obtain

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[x\sin x]\\ &= x\frac{d}{dx}[\sin x] + \sin x \frac{d}{dx}[x]\\ &= x\cos x + \sin x\end{aligned}$$ ◄

Derivative of Inverse Trigonometric Fn

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1 + x^2}$$

EXAMPLE 5 Differentiate (a) $y = \dfrac{1}{\sin^{-1}x}$ and (b) $f(x) = x \arctan\sqrt{x}$.

SOLUTION

(a)
$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x)$$

$$= -\frac{1}{(\sin^{-1}x)^2\sqrt{1 - x^2}}$$

(b)
$$f'(x) = x\frac{1}{1 + (\sqrt{x})^2}\left(\tfrac{1}{2}x^{-1/2}\right) + \arctan\sqrt{x}$$

$$= \frac{\sqrt{x}}{2(1 + x)} + \arctan\sqrt{x}$$

Derivative of Logarithmic Functions

$$\frac{d}{dx}e^x = e^x$$
$$\frac{d}{dx}(\log_e x) = \frac{1}{x}.$$

Differentiate $\log_e(x^2 + 3x + 1)$.

Solution

We solve this by using the chain rule and our knowledge of the derivative of $\log_e x$.

$$\begin{aligned}
\frac{d}{dx}\log_e(x^2+3x+1) &= \frac{d}{dx}(\log_e u) \qquad (\text{where } u = x^2+3x+1) \\
&= \frac{d}{du}(\log_e u) \times \frac{du}{dx} \qquad (\text{by the chain rule}) \\
&= \frac{1}{u} \times \frac{du}{dx} \\
&= \frac{1}{x^2+3x+1} \times \frac{d}{dx}(x^2+3x+1) \\
&= \frac{1}{x^2+3x+1} \times (2x+3) \\
&= \frac{2x+3}{x^2+3x+1}.
\end{aligned}$$

Derivative of Exponential Functions

Example

Find $\frac{d}{dx}(e^{x^3+2x})$.

Solution

Again, we use our knowledge of the derivative of e^x together with the chain rule

$$\begin{aligned}
\frac{d}{dx}(e^{x^3+2x}) &= \frac{de^u}{dx} \qquad \text{(where } u = x^3 + 2x\text{)} \\
&= e^u \times \frac{du}{dx} \qquad \text{(by the chain rule)} \\
&= e^{x^3+2x} \times \frac{d}{dx}(x^3 + 2x) \\
&= (3x^2 + 2) \times e^{x^3+2x}.
\end{aligned}$$

THE CHAIN RULE

2.6.1 THEOREM (The Chain Rule) *If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if*

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{1}$$

► **Example 1** Find dy/dx if $y = \cos(x^3)$.

Solution. Let $u = x^3$ and express y as $y = \cos u$. Applying Formula (1) yields

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \frac{d}{du}[\cos u] \cdot \frac{d}{dx}[x^3] \\ &= (-\sin u) \cdot (3x^2) \\ &= (-\sin(x^3)) \cdot (3x^2) = -3x^2 \sin(x^3)\end{aligned}$$ ◄

Chain Rule-Derivatives

► **Example 2** Find dw/dt if $w = \tan x$ and $x = 4t^3 + t$.

Solution. In this case the chain rule computations take the form

$$\begin{aligned}\frac{dw}{dt} &= \frac{dw}{dx} \cdot \frac{dx}{dt} \\ &= \frac{d}{dx}[\tan x] \cdot \frac{d}{dt}[4t^3 + t] \\ &= (\sec^2 x) \cdot (12t^2 + 1) \\ &= [\sec^2(4t^3 + t)] \cdot (12t^2 + 1) = (12t^2 + 1)\sec^2(4t^3 + t)\end{aligned}$$

Generalized Derivatives

GENERALIZED DERIVATIVE FORMULAS

$$\frac{d}{dx}[u^r] = ru^{r-1}\,\frac{du}{dx}$$

$$\frac{d}{dx}[\sin u] = \cos u\,\frac{du}{dx} \qquad \frac{d}{dx}[\cos u] = -\sin u\,\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u\,\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = -\csc^2 u\,\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u \tan u\,\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = -\csc u \cot u\,\frac{du}{dx}$$

► **Example 5** Find

(a) $\frac{d}{dx}[\sin(2x)]$ (b) $\frac{d}{dx}[\tan(x^2+1)]$ (c) $\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right]$

(d) $\frac{d}{dx}[x^2-x+2]^{3/4}$ (e) $\frac{d}{dx}\left[(1+x^5\cot x)^{-8}\right]$

Solution (a). Taking $u = 2x$ in the generalized derivative formula for $\sin u$ yields

$$\frac{d}{dx}[\sin(2x)] = \frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} = \cos 2x \cdot \frac{d}{dx}[2x] = \cos 2x \cdot 2 = 2\cos 2x$$

Generalized Derivatives

Solution (b). Taking $u = x^2 + 1$ in the generalized derivative formula for $\tan u$ yields

$$\frac{d}{dx}[\tan(x^2+1)] = \frac{d}{dx}[\tan u] = \sec^2 u \frac{du}{dx}$$
$$= \sec^2(x^2+1)\cdot\frac{d}{dx}[x^2+1] = \sec^2(x^2+1)\cdot 2x$$
$$= 2x\sec^2(x^2+1)$$

Solution (c). Taking $u = x^3 + \csc x$ in the generalized derivative formula for $\sqrt{u}$ yields

$$\frac{d}{dx}\left[\sqrt{x^3+\csc x}\right] = \frac{d}{dx}[\sqrt{u}] = \frac{1}{2\sqrt{u}}\frac{du}{dx} = \frac{1}{2\sqrt{x^3+\csc x}}\cdot\frac{d}{dx}[x^3+\csc x]$$
$$= \frac{1}{2\sqrt{x^3+\csc x}}\cdot(3x^2-\csc x\cot x) = \frac{3x^2-\csc x\cot x}{2\sqrt{x^3+\csc x}}$$

Solution (d). Taking $u = x^2 - x + 2$ in the generalized derivative formula for $u^{3/4}$ yields

$$\frac{d}{dx}[x^2-x+2]^{3/4} = \frac{d}{dx}[u^{3/4}] = \frac{3}{4}u^{-1/4}\frac{du}{dx}$$
$$= \frac{3}{4}(x^2-x+2)^{-1/4}\cdot\frac{d}{dx}[x^2-x+2]$$
$$= \frac{3}{4}(x^2-x+2)^{-1/4}(2x-1)$$

Solution (e). Taking $u = 1 + x^5\cot x$ in the generalized derivative formula for u^{-8} yields

$$\frac{d}{dx}[(1+x^5\cot x)^{-8}] = \frac{d}{dx}[u^{-8}] = -8u^{-9}\frac{du}{dx}$$
$$= -8(1+x^5\cot x)^{-9}\cdot\frac{d}{dx}[1+x^5\cot x]$$
$$= -8(1+x^5\cot x)^{-9}\cdot[x^5(-\csc^2 x)+5x^4\cot x]$$
$$= (8x^5\csc^2 x - 40x^4\cot x)(1+x^5\cot x)^{-9} \blacktriangleleft$$

Defined Implicitly and Explicitly

FUNCTIONS DEFINED EXPLICITLY AND IMPLICITLY

An equation of the form $y = f(x)$ is said to define y *explicitly* as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side. However, sometimes functions are defined by equations in which y is not alone on one side; for example, the equation

$$yx + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but it still defines y as a function of x since it can be rewritten as

$$y = \frac{x-1}{x+1}$$

Thus, we say that (1) defines y *implicitly* as a function of x, the function being

$$f(x) = \frac{x-1}{x+1}$$

An equation in x and y can implicitly define more than one function of x. This can occur when the graph of the equation fails the vertical line test, so it is not the graph of a function of x. For example, if we solve the equation of the circle

$$x^2 + y^2 = 1 \tag{2}$$

for y in terms of x, we obtain $y = \pm\sqrt{1-x^2}$, so we have found two functions that are defined implicitly by (2), namely,

$$f_1(x) = \sqrt{1-x^2} \quad \text{and} \quad f_2(x) = -\sqrt{1-x^2} \tag{3}$$

The graphs of these functions are the upper and lower semicircles of the circle $x^2 + y^2 = 1$ (Figure 2.7.1). This leads us to the following definition.

Implicit Differentiation

IMPLICIT DIFFERENTIATION

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation. To illustrate this, let us consider the simple equation

$$xy = 1 \tag{5}$$

One way to find dy/dx is to rewrite this equation as

$$y = \frac{1}{x} \tag{6}$$

from which it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2} \tag{7}$$

this approach we obtain

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

If we now substitute (6) into the last expression, we obtain

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

which agrees with Equation (7). This method of obtaining derivatives is called *implicit differentiation*.

Implicit Differentiation

Example 2 Use implicit differentiation to find dy/dx if $5y^2 + \sin y = x^2$.

$$\frac{d}{dx}[5y^2 + \sin y] = \frac{d}{dx}[x^2]$$

$$5\frac{d}{dx}[y^2] + \frac{d}{dx}[\sin y] = 2x$$

$$5\left(2y\frac{dy}{dx}\right) + (\cos y)\frac{dy}{dx} = 2x$$

The chain rule was used here because y is a function of x.

$$10y\frac{dy}{dx} + (\cos y)\frac{dy}{dx} = 2x$$

Solving for dy/dx we obtain

$$\frac{dy}{dx} = \frac{2x}{10y + \cos y}$$

L' Hospital's Rule

L'HOSPITAL'S RULE Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or ∞/∞.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

L' Hospital's Rule-Examples

EXAMPLE 1 Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

SOLUTION Since

$$\lim_{x \to 1} \ln x = \ln 1 = 0 \quad \text{and} \quad \lim_{x \to 1} (x - 1) = 0$$

we can apply l'Hospital's Rule:

$$\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1/x}{1} = \lim_{x \to 1} \frac{1}{x} = 1$$

EXAMPLE 2 Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$.

SOLUTION We have $\lim_{x \to \infty} e^x = \infty$ and $\lim_{x \to \infty} x^2 = \infty$, so l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$ as $x \to \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's Rule gives

$$\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

L' Hospital's Rule-Examples

EXAMPLE 3 Calculate $\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$.

SOLUTION Since $\ln x \to \infty$ and $\sqrt[3]{x} \to \infty$ as $x \to \infty$, l'Hospital's Rule applies:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}}$$

Notice that the limit on the right side is now indeterminate of type $\frac{0}{0}$. But instead of applying l'Hospital's Rule a second time as we did in Example 2, we simplify the expression and see that a second application is unnecessary:

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x\to\infty} \frac{1/x}{\frac{1}{3}x^{-2/3}} = \lim_{x\to\infty} \frac{3}{\sqrt[3]{x}} = 0$$

L' Hospital's Rule-Examples

EXAMPLE 4 Find $\lim_{x\to 0} \frac{\tan x - x}{x^3}$. (See Exercise 38 in Section 2.2.)

SOLUTION Noting that both $\tan x - x \to 0$ and $x^3 \to 0$ as $x \to 0$, we use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2}$$

Since the limit on the right side is still indeterminate of type $\frac{0}{0}$, we apply l'Hospital's Rule again:

$$\lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

Because $\lim_{x\to 0} \sec^2 x = 1$, we simplify the calculation by writing

$$\lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} = \frac{1}{3} \lim_{x\to 0} \sec^2 x \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x}$$

We can evaluate this last limit either by using l'Hospital's Rule a third time or by writing $\tan x$ as $(\sin x)/(\cos x)$ and making use of our knowledge of trigonometric limits. Putting together all the steps, we get

$$\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x}$$

$$= \frac{1}{3} \lim_{x\to 0} \frac{\tan x}{x} = \frac{1}{3} \lim_{x\to 0} \frac{\sec^2 x}{1} = \frac{1}{3} \qquad \square$$

L' Hospital's Rule-Examples

EXAMPLE 5 Find $\lim_{x \to \pi^-} \dfrac{\sin x}{1 - \cos x}$.

SOLUTION If we blindly attempted to use l'Hospital's Rule, we would get

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \lim_{x \to \pi^-} \frac{\cos x}{\sin x} = -\infty$$

This is **wrong!** Although the numerator $\sin x \to 0$ as $x \to \pi^-$, notice that the denominator $(1 - \cos x)$ does not approach 0, so l'Hospital's Rule can't be applied here.

The required limit is, in fact, easy to find because the function is continuous at π and the denominator is nonzero there:

$$\lim_{x \to \pi^-} \frac{\sin x}{1 - \cos x} = \frac{\sin \pi}{1 - \cos \pi} = \frac{0}{1 - (-1)} = 0 \qquad \square$$

L' Hospital's Rule-Examples

INDETERMINATE PRODUCTS

EXAMPLE 6 Evaluate $\lim_{x \to 0^+} x \ln x$.

SOLUTION The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = 1/(1/x)$, we have $1/x \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0$$

INDETERMINATE DIFFERENCES

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then the limit

$$\lim_{x \to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$. Again

L' Hospital's Rule-Examples

EXAMPLE 7 Compute $\lim_{x \to (\pi/2)^-} (\sec x - \tan x)$.

SOLUTION First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to (\pi/2)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x \to (\pi/2)^-} (\sec x - \tan x) = \lim_{x \to (\pi/2)^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to (\pi/2)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to (\pi/2)^-} \frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to (\pi/2)^-$.

INDETERMINATE POWERS

Several indeterminate forms arise from the limit

$$\lim_{x \to a} [f(x)]^{g(x)}$$

1. $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ type 0^0
2. $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$ type ∞^0
3. $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \pm\infty$ type 1^∞

Each of these three cases can be treated either by taking the natural logarithm:

$$\text{let} \quad y = [f(x)]^{g(x)}, \quad \text{then} \quad \ln y = g(x) \ln f(x)$$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x) \ln f(x)}$$

L' Hospital's Rule-Examples

EXAMPLE 8 Calculate $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$.

SOLUTION First notice that as $x \to 0^+$, we have $1 + \sin 4x \to 1$ and $\cot x \to \infty$, so the given limit is indeterminate. Let

$$y = (1 + \sin 4x)^{\cot x}$$

Then $$\ln y = \ln[(1 + \sin 4x)^{\cot x}] = \cot x \ln(1 + \sin 4x)$$

so l'Hospital's Rule gives

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\dfrac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

EXAMPLE 9 Find $\lim_{x\to 0^+} x^x$.

SOLUTION Notice that this limit is indeterminate since $0^x = 0$ for any $x > 0$ but $x^0 = 1$ for any $x \neq 0$. We could proceed as in Example 8 or by writing the function as an exponential:

$$x^x = (e^{\ln x})^x = e^{x \ln x}$$

In Example 6 we used l'Hospital's Rule to show that

$$\lim_{x\to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x \ln x} = e^0 = 1$$

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Thank you