Calculus and Analytical Geometry

LECTURE #8

Lecture#8(Anti-Derivatives)

- Unit 8.1: Anti-derivatives-Example
- Unit 8.2: Indefinite Integral-Example
- Unit 8.3: Properties of Indefinite Integral-Example
- Unit 8.4: Integration by Substitution-Example

ANTIDERIVATIVES

4.2.1 DEFINITION A function F is called an *antiderivative* of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$
(b) $g(x) = \cos x$
(c) $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$
(b) $G(x) = \sin x$
(c) $H(x) = x^2 + \sin x$

Anti-Derivatives

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives

$$F(0) = -\cos 0 + C = -1 + C.$$

Since $F(0) = 3$, solving for C gives $C = 4$. So

$$F(x) = -\cos x + 4$$

is the antiderivative satisfying $F(0) = 3$. ■

THE INDEFINITE INTEGRAL

The process of finding antiderivatives is called ***antidifferentiation*** or ***integration***. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{1}$$

then ***integrating*** (or ***antidifferentiating***) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (1) is recast using ***integral notation***,

$$\int f(x)\,dx = F(x) + C \tag{2}$$

where C is understood to represent an arbitrary constant. It is important to note that (1) and (2) are just different notations to express the same fact. For example,

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C \quad \text{is equivalent to} \quad \frac{d}{dx}\left[\tfrac{1}{3}x^3\right] = x^2$$

Note that if we differentiate an antiderivative of $f(x)$, we obtain $f(x)$ back again. Thus,

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x) \tag{3}$$

The expression $\int f(x)\,dx$ is called an ***indefinite integral***. The adjective "indefinite" emphasizes that the result of antidifferentiation is a "generic" function, described only up to a constant term. The "elongated s" that appears on the left side of (2) is called an ***integral sign***,* the function $f(x)$ is called the ***integrand***, and the constant C is called the ***constant of integration***. Equation (2) should be read as:

Indefinite Integral

Table 4.2.1

INTEGRATION FORMULAS

DIFFERENTIATION FORMULA	INTEGRATION FORMULA	DIFFERENTIATION FORMULA	INTEGRATION FORMULA
1. $\frac{d}{dx}[x] = 1$	$\int dx = x + C$	5. $\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
2. $\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r \quad (r \neq -1)$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$	6. $\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
3. $\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	7. $\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
4. $\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x\,dx = -\cos x + C$	8. $\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2\,dx = x^3 + C$
$\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + C$
$\frac{d}{du}[u^{3/2}] = \frac{3}{2}u^{1/2}$	$\int \frac{3}{2}u^{1/2}\,du = u^{3/2} + C$

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\frac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\frac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\frac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

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Properties of Indefinite Integral

4.2.3 THEOREM *Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively, and that c is a constant. Then:*

(a) *A constant factor can be moved through an integral sign; that is,*

$$\int cf(x)\,dx = cF(x) + C$$

(b) *An antiderivative of a sum is the sum of the antiderivatives; that is,*

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

(c) *An antiderivative of a difference is the difference of the antiderivatives; that is,*

$$\int [f(x) - g(x)]\,dx = F(x) - G(x) + C$$

Example 2 Evaluate

$$\text{(a)} \int 4\cos x\,dx \qquad \text{(b)} \int (x + x^2)\,dx$$

Solution (a). Since $F(x) = \sin x$ is an antiderivative for $f(x) = \cos x$ (Table 4.2.1), we obtain

$$\int 4\cos x\,dx \underset{(4)}{=} 4\int \cos x\,dx = 4\sin x + C$$

Solution (b). From Table 4.2.1 we obtain

$$\int (x + x^2)\,dx \underset{(5)}{=} \int x\,dx + \int x^2\,dx = \frac{x^2}{2} + \frac{x^3}{3} + C$$

Properties of Indefinite Integral

Example 3

$$\int (3x^6 - 2x^2 + 7x + 1)\,dx = 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + \int 1\,dx$$

$$= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C$$

Example 4 Evaluate

(a) $\int \frac{\cos x}{\sin^2 x}\,dx$ (b) $\int \frac{t^2 - 2t^4}{t^4}\,dt$

Solution (a).

$$\int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin x}\frac{\cos x}{\sin x}\,dx = \int \csc x \cot x\,dx = -\csc x + C$$

Formula 8 in Table 4.2.1

Solution (b).

$$\int \frac{t^2 - 2t^4}{t^4}\,dt = \int \left(\frac{1}{t^2} - 2\right)dt = \int (t^{-2} - 2)\,dt$$

$$= \frac{t^{-1}}{-1} - 2t + C = -\frac{1}{t} - 2t + C$$

EXAMPLE 3 Finding Antiderivatives Using Table 4.2

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \frac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \frac{x}{2}$

Solution

(a) $F(x) = \frac{x^6}{6} + C$ — Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$

Formula 1 with $n = -1/2$

(c) $H(x) = \frac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

(d) $I(x) = \frac{\sin(x/2)}{1/2} + C = 2\sin\frac{x}{2} + C$ — Formula 3 with $k = 1/2$

Integration by Substitution

u-SUBSTITUTION

The method of substitution can be motivated by examining the chain rule from the viewpoint of antidifferentiation. For this purpose, suppose that F is an antiderivative of f and that g is a differentiable function. The chain rule implies that the derivative of $F(g(x))$ can be expressed as

$$\frac{d}{dx}[F(g(x))] = F'(g(x))g'(x)$$

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C \tag{2}$$

For our purposes it will be useful to let $u = g(x)$ and to write $du/dx = g'(x)$ in the differential form $du = g'(x)\,dx$. With this notation (2) can be expressed as

$$\int f(u)\,du = F(u) + C \tag{3}$$

The process of evaluating an integral of form (2) by converting it into form (3) with the substitution

$$u = g(x) \quad \text{and} \quad du = g'(x)\,dx$$

is called the *method of u-substitution*. Here our emphasis is *not* on the interpretation of

Integration by Substitution

► **Example 1** Evaluate $\int (x^2+1)^{50} \cdot 2x\,dx$.

Solution. If we let $u = x^2+1$, then $du/dx = 2x$, which implies that $du = 2x\,dx$. Thus, the given integral can be written as

$$\int (x^2+1)^{50} \cdot 2x\,dx = \int u^{50}\,du = \frac{u^{51}}{51} + C = \frac{(x^2+1)^{51}}{51} + C$$

► **Example 2**

$$\underbrace{\int \sin(x+9)\,dx = \int \sin u\,du}_{\substack{u = x+9 \\ du = 1\cdot dx = dx}} = -\cos u + C = -\cos(x+9) + C$$

$$\underbrace{\int (x-8)^{23}\,dx = \int u^{23}\,du}_{\substack{u = x-8 \\ du = 1\cdot dx = dx}} = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

Integration by Substitution

► Example 3 Evaluate $\int \cos 5x\,dx$.

Solution.

$$\int \cos 5x\,dx = \int (\cos u) \cdot \frac{1}{5}\,du = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \text{ or } dx = \frac{1}{5}\,du \end{array}}$$

Alternative Solution. There is a variation of the preceding method that some people prefer. The substitution $u = 5x$ requires $du = 5\,dx$. If there were a factor of 5 in the integrand, then we could group the 5 and dx together to form the du required by the substitution. Since there is no factor of 5, we will insert one and compensate by putting a factor of $\frac{1}{5}$ in front of the integral. The computations are as follows:

$$\int \cos 5x\,dx = \frac{1}{5}\int \cos 5x \cdot 5\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin 5x + C \blacktriangleleft$$

$$\boxed{\begin{array}{c} u = 5x \\ du = 5\,dx \end{array}}$$

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Integration by Substitution

► Example 4

$$\int \frac{dx}{(\frac{1}{3}x-8)^5} = \int \frac{3\,du}{u^5} = 3\int u^{-5}\,du = -\frac{3}{4}u^{-4} + C = -\frac{3}{4}\left(\frac{1}{3}x-8\right)^{-4} + C$$

$$u = \frac{1}{3}x - 8$$
$$du = \frac{1}{3}\,dx \text{ or } dx = 3\,du$$

► Example 5

$$\int \left(\frac{1}{x^2} + \sec^2 \pi x\right) dx = \int \frac{dx}{x^2} + \int \sec^2 \pi x\,dx$$
$$= -\frac{1}{x} + \int \sec^2 \pi x\,dx$$
$$= -\frac{1}{x} + \frac{1}{\pi}\int \sec^2 u\,du$$

$$u = \pi x$$
$$du = \pi\,dx \text{ or } dx = \frac{1}{\pi}\,du$$

$$= -\frac{1}{x} + \frac{1}{\pi}\tan u + C = -\frac{1}{x} + \frac{1}{\pi}\tan \pi x + C$$

Integration by Substitution

► Example 6 Evaluate $\int \sin^2 x \cos x \, dx$.

Solution. If we let $u = \sin x$, then

$$\frac{du}{dx} = \cos x, \quad \text{so} \quad du = \cos x \, dx$$

Thus,

$$\int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{u^3}{3} + C = \frac{\sin^3 x}{3} + C \blacktriangleleft$$

► Example 7 Evaluate $\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx$.

Solution. If we let $u = \sqrt{x}$, then

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}}, \quad \text{so} \quad du = \frac{1}{2\sqrt{x}} dx \quad \text{or} \quad 2\,du = \frac{1}{\sqrt{x}} dx$$

Thus,

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} \, dx = \int 2 \cos u \, du = 2 \int \cos u \, du = 2 \sin u + C = 2 \sin \sqrt{x} + C$$

Integration by Substitution

Example 8 Evaluate $\int t^4 \sqrt[3]{3-5t^5}\,dt$.

Solution.

$$\int t^4 \sqrt[3]{3-5t^5}\,dt = -\frac{1}{25}\int \sqrt[3]{u}\,du = -\frac{1}{25}\int u^{1/3}\,du$$

$$u = 3-5t^5$$
$$du = -25t^4\,dt \text{ or } -\tfrac{1}{25}\,du = t^4\,dt$$

$$= -\frac{1}{25}\frac{u^{4/3}}{4/3} + C = -\frac{3}{100}(3-5t^5)^{4/3} + C$$

Example 9 Evaluate $\int x^2\sqrt{x-1}\,dx$.

Solution. The composition $\sqrt{x-1}$ suggests the substitution

$$u = x-1 \quad \text{so that} \quad du = dx \tag{4}$$

From the first equality in (4)

$$x^2 = (u+1)^2 = u^2+2u+1$$

so that

$$\int x^2\sqrt{x-1}\,dx = \int (u^2+2u+1)\sqrt{u}\,du = \int (u^{5/2}+2u^{3/2}+u^{1/2})\,du$$

$$= \tfrac{2}{7}u^{7/2} + \tfrac{4}{5}u^{5/2} + \tfrac{2}{3}u^{3/2} + C$$

$$= \tfrac{2}{7}(x-1)^{7/2} + \tfrac{4}{5}(x-1)^{5/2} + \tfrac{2}{3}(x-1)^{3/2} + C$$

Integration by Substitution

► **Example 10** Evaluate $\int \cos^3 x\, dx$.

Solution. The only compositions in the integrand that suggest themselves are

$$\cos^3 x = (\cos x)^3 \quad \text{and} \quad \cos^2 x = (\cos x)^2$$

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx$$

and solve the equation $du = \cos x\, dx$ for $u = \sin x$. Since $\sin^2 x + \cos^2 x = 1$, we then have

$$\int \cos^3 x\, dx = \int \cos^2 x \cos x\, dx = \int (1 - \sin^2 x) \cos x\, dx = \int (1 - u^2)\, du$$

$$= u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C \quad ◄$$

Thank you