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Calculus and Analytical Geometry

LECTURE #11

Lecture#11
(Evaluation of Indefinite Integrals)

- Unit 11.1: Review of Basic integration methods-Example
- Unit 11.2: Integration by Parts-Example
- Unit 11.3: Repeated Integration by Parts-Example
- Unit 11.4: Integration by Parts(Tabular Method)-Example
- Unit 11.5: Integration by Parts For Definite Integral-Example
- Unit 11.6: Integration by Parts Using Reduction Formulae

Basic Integration Formulae

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \frac{du}{u} = \ln|u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln|\cos u| + C$ $= \ln|\sec u| + C$
13. $\int \cot u\,du = \ln|\sin u| + C$ $= -\ln|\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$
19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$
21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

Review of Basic integration methods-Example

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\, dx = \int \frac{du}{\sqrt{u}}$$

$u = x^2 - 9x + 1,$
$du = (2x - 9)\, dx.$

$$= \int u^{-1/2}\, du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C$$

Table 8.1 Formula 4, with $n = -1/2$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16)$$
$$= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 4, u = (x - 4),$
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

Table 8.1, Formula 18

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

Review of Basic integration methods-Example

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2 \, dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned} \int (\sec x + \tan x)^2 \, dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) \, dx \\ &= 2 \int \sec^2 x \, dx + 2 \int \sec x \tan x \, dx - \int 1 \, dx \\ &= 2 \tan x + 2 \sec x - x + C. \end{aligned}$$

EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx &= \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx && \sqrt{u^2} = |u| \\ &= \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx && \text{On } [0, \pi/4], \cos 2x \geq 0, \text{ so } |\cos 2x| = \cos 2x. \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\,dx \\ &= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}. \end{aligned}$$

Review of Basic integration methods-Example

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$\begin{array}{r} x - 3 \\ 3x + 2\overline{)3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ +\ 6 \end{array}$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2}\right) dx = \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$3\int \frac{x\, dx}{\sqrt{1 - x^2}} = 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du$$

$$= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1 - x^2}} = 2\sin^{-1}x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\, dx = -3\sqrt{1 - x^2} + 2\sin^{-1}x + C.$$

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Review of Basic integration methods-Example

EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x \, dx.$$

Solution

$$\int \sec x \, dx = \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{du}{u} \qquad \begin{aligned} &u = \tan x + \sec x, \\ &du = (\sec^2 x + \sec x \tan x)\, dx \end{aligned}$$

$$= \ln |u| + C = \ln |\sec x + \tan x| + C.$$

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Integration By Parts-Example

INTEGRATION BY PARTS

$$\int f(x)g(x)\,dx$$

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx \qquad (2)$$

This formula allows us to integrate $f(x)g(x)$ by integrating $f'(x)G(x)$ instead, and in many cases the net effect is to replace a difficult integration with an easier one. The application of this formula is called *integration by parts*.

In practice, we usually rewrite (2) by letting

$$u = f(x), \quad du = f'(x)\,dx$$

$$v = G(x), \quad dv = G'(x)\,dx = g(x)\,dx$$

This yields the following alternative form for (2):

$$\int u\,dv = uv - \int v\,du \qquad (3)$$

GUIDELINES FOR INTEGRATION BY PARTS

There is another useful strategy for choosing u and dv that can be applied when the integrand is a product of two functions from *different* categories in the list

Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

In this case you will often be successful if you take u to be the function whose category occurs earlier in the list and take dv to be the rest of the integrand. The acronym LIATE will help you to remember the order. The method does not work all the time, but it works

Example 1 Use integration by parts to evaluate $\int x\cos x\,dx$.

Solution. We will apply Formula (3). The first step is to make a choice for u and dv to put the given integral in the form $\int u\,dv$. We will let

$$u = x \quad \text{and} \quad dv = \cos x\,dx$$

(Other possibilities will be considered later.) The second step is to compute du from u and v from dv. This yields

$$du = dx \quad \text{and} \quad v = \int dv = \int \cos x\,dx = \sin x$$

The third step is to apply Formula (3). This yields

$$\int \underbrace{x}_{u}\,\underbrace{\cos x\,dx}_{dv} = \underbrace{x}_{u}\,\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\,\underbrace{dx}_{du}$$

$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C$$

Integration By Parts-Example

Example 2 Evaluate $\int xe^x\,dx$.

Solution. In this case the integrand is the product of the algebraic function x with the exponential function e^x. According to LIATE we should let

$$u = x \quad \text{and} \quad dv = e^x\,dx$$

so that

$$du = dx \quad \text{and} \quad v = \int e^x\,dx = e^x$$

Thus, from (3)

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Example 3 Evaluate $\int \ln x\,dx$.

Solution. One choice is to let $u = 1$ and $dv = \ln x\,dx$. But with this choice finding v is equivalent to evaluating $\int \ln x\,dx$ and we have gained nothing. Therefore, the only reasonable choice is to let

$$u = \ln x \qquad dv = dx$$

$$du = \frac{1}{x}\,dx \qquad v = \int dx = x$$

With this choice it follows from (3) that

$$\int \ln x\,dx = \int u\,dv = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C$$

Integration By Parts-Example

The LIATE method suggests that integrals of the form

$$\int e^{ax} \sin bx\, dx \quad \text{and} \quad \int e^{ax} \cos bx\, dx$$

can be evaluated by letting $u = \sin bx$ or $u = \cos bx$ and $dv = e^{ax}\, dx$. However, this will require a technique that deserves special attention.

► **Example 5** Evaluate $\int e^x \cos x\, dx$.

Solution. Let

$$u = \cos x, \quad dv = e^x\, dx, \quad du = -\sin x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \cos x\, dx = \int u\, dv = uv - \int v\, du = e^x \cos x + \int e^x \sin x\, dx \tag{5}$$

Since the integral $\int e^x \sin x\, dx$ is similar in form to the original integral $\int e^x \cos x\, dx$, it seems that nothing has been accomplished. However, let us integrate this new integral by parts. We let

$$u = \sin x, \quad dv = e^x\, dx, \quad du = \cos x\, dx, \quad v = \int e^x\, dx = e^x$$

Thus,

$$\int e^x \sin x\, dx = \int u\, dv = uv - \int v\, du = e^x \sin x - \int e^x \cos x\, dx$$

Together with Equation (5) this yields

$$\int e^x \cos x\, dx = e^x \cos x + e^x \sin x - \int e^x \cos x\, dx \tag{6}$$

Repeated Integration By Parts-Example

▶ **Example 4** Evaluate $\int x^2 e^{-x}\,dx$.

Solution. Let

$$u = x^2, \quad dv = e^{-x}\,dx, \quad du = 2x\,dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that from (3)

$$\begin{aligned}\int x^2 e^{-x}\,dx &= \int u\,dv = uv - \int v\,du \\ &= x^2(-e^{-x}) - \int -e^{-x}(2x)\,dx \\ &= -x^2 e^{-x} + 2\int x e^{-x}\,dx \qquad (4)\end{aligned}$$

The last integral is similar to the original except that we have replaced x^2 by x. Another integration by parts applied to $\int xe^{-x}\,dx$ will complete the problem. We let

$$u = x, \quad dv = e^{-x}\,dx, \quad du = dx, \quad v = \int e^{-x}\,dx = -e^{-x}$$

so that

$$\int xe^{-x}\,dx = x(-e^{-x}) - \int -e^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

Finally, substituting this into the last line of (4) yields

$$\begin{aligned}\int x^2 e^{-x}\,dx &= -x^2 e^{-x} + 2\int xe^{-x}\,dx = -x^2 e^{-x} + 2(-xe^{-x} - e^{-x}) + C \\ &= -(x^2 + 2x + 2)e^{-x} + C \ ◀\end{aligned}$$

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Integration By Parts(Tabular Method)-Example

Tabular Integration by Parts

Step 1. Differentiate $p(x)$ repeatedly until you obtain 0, and list the results in the first column.

Step 2. Integrate $f(x)$ repeatedly and list the results in the second column.

Step 3. Draw an arrow from each entry in the first column to the entry that is one row down in the second column.

Step 4. Label the arrows with alternating $+$ and $-$ signs, starting with a $+$.

Step 5. For each arrow, form the product of the expressions at its tip and tail and then multiply that product by $+1$ or -1 in accordance with the sign on the arrow. Add the results to obtain the value of the integral.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
$x^2 - x$	$+$	$\cos x$
$2x - 1$	$-$	$\sin x$
2	$+$	$-\cos x$
0		$-\sin x$

$$\begin{aligned}\int (x^2 - x)\cos x\,dx &= (x^2 - x)\sin x + (2x - 1)\cos x - 2\sin x + C\\ &= (x^2 - x - 2)\sin x + (2x - 1)\cos x + C\end{aligned}$$

Integration By Parts(Tabular Method)-Example

Example 6 In Example 9 of Section 4.3 we evaluated $\int x^2\sqrt{x-1}\,dx$ using u-substitution. Evaluate this integral using tabular integration by parts.

Solution.

REPEATED DIFFERENTIATION		REPEATED INTEGRATION
x^2	$+$	$(x-1)^{1/2}$
$2x$	$-$	$\frac{2}{3}(x-1)^{3/2}$
2	$+$	$\frac{4}{15}(x-1)^{5/2}$
0		$\frac{8}{105}(x-1)^{7/2}$

Thus, it follows that

$$\int x^2\sqrt{x-1}\,dx = \tfrac{2}{3}x^2(x-1)^{3/2} - \tfrac{8}{15}x(x-1)^{5/2} + \tfrac{16}{105}(x-1)^{7/2} + C$$

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

Integration By Parts(Tabular Method)-Example

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

Integration By Parts For Definite Integral-Example

INTEGRATION BY PARTS FOR DEFINITE INTEGRALS

For definite integrals the formula corresponding to (3) is

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

Example 7 Evaluate $\int_0^1 \tan^{-1} x\,dx$.

Solution. Let

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{1}{1+x^2}\,dx, \quad v = x$$

Thus,

$$\int_0^1 \tan^{-1} x\,dx = \int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du$$

$$= x\tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

The limits of integration refer to x; that is, $x = 0$ and $x = 1$.

But

$$\int_0^1 \frac{x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx = \frac{1}{2}\ln(1+x^2)\Big]_0^1 = \frac{1}{2}\ln 2$$

so

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big]_0^1 - \frac{1}{2}\ln 2 = \left(\frac{\pi}{4} - 0\right) - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$

Integration by Parts Used Reduction Formulae-Example

REDUCTION FORMULAS

Integration by parts can be used to derive ***reduction formulas*** for integrals. These are formulas that express an integral involving a power of a function in terms of an integral that involves a *lower* power of that function. For example, if n is a positive integer and $n \geq 2$, then integration by parts can be used to obtain the reduction formulas

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx \tag{9}$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \tag{10}$$

▶ **Example 8** Evaluate $\int \cos^4 x\,dx$.

Solution. From (10) with $n = 4$

$$\int \cos^4 x\,dx = \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\int \cos^2 x\,dx$$

Now apply (10) with $n = 2$.

$$= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{4}\left(\tfrac{1}{2}\cos x\sin x + \tfrac{1}{2}\int dx\right)$$

$$= \tfrac{1}{4}\cos^3 x\sin x + \tfrac{3}{8}\cos x\sin x + \tfrac{3}{8}x + C$$ ◀

Integration by Parts Used Reduction Formulae-Example

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Integration by Parts Used Reduction Formulae-Example

EXAMPLE 1 *m* is Odd

Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\begin{aligned}
\int \cos^5 x \, dx &= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad m = 0 \\
&= \int (1 - u^2)^2 \, du \qquad u = \sin x \\
&= \int (1 - 2u^2 + u^4) \, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.
\end{aligned}$$

Integration by Parts Used Reduction Formulae-Example

EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x,$
$du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

Integration by Parts Used Reduction Formulae-Example

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}\int \sin 3x \cos 5x \, dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x] \, dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

Integration by Parts Used Reduction Formulae-Example

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1}x - \int \tan^{n-2}x\,dx \qquad (1)$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \qquad (2)$$

$$\int \sin^n x\cos^m x\,dx = -\frac{\sin^{n-1}x\cos^{m+1}x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2}x\cos^m x\,dx \qquad (n \neq -m). \qquad (3)$$

EXAMPLE 6 Using a Reduction Formula

Find

$$\int \tan^5 x\,dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$

Integration by Parts Used Reduction Formulae-Example

EXAMPLE 7 Deriving a Reduction Formula

Show that for any positive integer n,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Solution We use the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

EXAMPLE 8 Find

$$\int \sin^2 x \cos^3 x\,dx.$$

Solution 1 We apply Equation (3) with $n = 2$ and $m = 3$ to get

$$\int \sin^2 x \cos^3 x\,dx = -\frac{\sin x \cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x \cos^3 x\,dx$$

$$= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\int \cos^3 x\,dx.$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax\,dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax\,dx.$$

Integration by Parts Used Reduction Formulae-Example

EXAMPLE 8 Find

$$\int \sin^2 x \cos^3 x \, dx.$$

Solution 1 We apply Equation (3) with $n = 2$ and $m = 3$ to get

$$\int \sin^2 x \cos^3 x \, dx = -\frac{\sin x \cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x \cos^3 x \, dx$$

$$= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\int \cos^3 x \, dx.$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax \, dx = \frac{\cos^{n-1} ax \sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2} ax \, dx.$$

With $n = 3$ and $a = 1$, we have

$$\int \cos^3 x \, dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x \, dx$$

$$= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.$$

The combined result is

$$\int \sin^2 x \cos^3 x \, dx = -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C\right)$$

$$= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15}\sin x + C'.$$

Solution 2 Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x \, dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x \, dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\int \sin^2 x \cos^3 x \, dx = \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\int \sin^2 x \cos x \, dx$$

$$= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\left(\frac{\sin^3 x}{3}\right) + C$$

$$= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x + C.$$

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Thank you