Definition

We say that the limit of *f(x)* is *L* as *x* approaches *a* and write this as

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a} f(x) = L$$

provided we can make *f(x)* as close to *L* as we want for all *x* sufficiently close to *a*, from both sides, without actually letting *x* be *a*.

Example 1 Estimate the value of the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

Notice that I did say estimate the value of the limit. Again, we are not going to directly compute limits in this section. The point of this section is to give us a better idea of how limits work and what they can tell us about the function.

So, with that in mind we are going to work this in pretty much the same way that we did in the last section. We will choose values of *x* that get closer and closer to *x=2* and plug these values into the function. Doing this gives the following table of values.

x	*f*(*x*)	*x*	*f*(*x*)
2.5	3.4	1.5	5.0
2.1	3.857142857	1.9	4.157894737
2.01	3.985074627	1.99	4.015075377
2.001	3.998500750	1.999	4.001500750
2.0001	3.999850007	1.9999	4.000150008
2.00001	3.999985000	1.99999	4.000015000

Example 3 Estimate the value of the following limit.

$$\lim_{\theta \to 0} \frac{1 - \cos(\theta)}{\theta}$$

Solution

First don't get excited about the θ in function. It's just a letter, just like *x* is a letter! It's a Greek letter, but it's a letter and you will be asked to deal with Greek letters on occasion so it's a good idea to start getting used to them at this point.

Now, also notice that if we plug in *θ=0* that we will get division by zero and so the function doesn’t exist at this point. Actually, we get 0/0 at this point, but because of the division by zero this function does not exist at *θ=0*.

So, as we did in the first example let’s get a table of values and see what if we can guess what value the function is heading in towards.

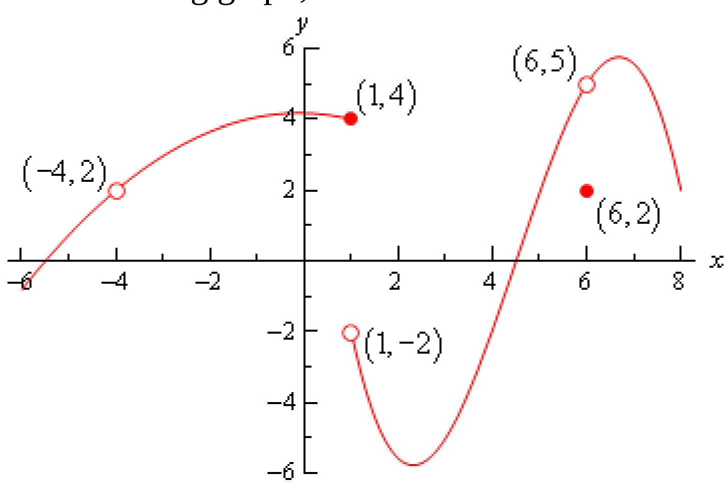
θ	$f(\theta)$	θ	$f(\theta)$
1	0.45969769	-1	-0.45969769
0.1	0.04995835	-0.1	-0.04995835
0.01	0.00499996	-0.01	-0.00499996
0.001	0.00049999	-0.001	-0.00049999

Okay, it looks like the function is moving in towards a value of zero as θ moves in towards 0, from both sides of course.

Therefore, the we will guess that the limit has the value,

$$\lim_{\theta \to 0} \frac{1-\cos(\theta)}{\theta} = 0$$

Example 4 Given the following graph,



compute each of the following.

(a) $f(-4)$ **(b)** $\lim_{x \to -4^-} f(x)$ **(c)** $\lim_{x \to -4^+} f(x)$ **(d)** $\lim_{x \to -4} f(x)$

(e) $f(1)$ **(f)** $\lim_{x \to 1^-} f(x)$ **(g)** $\lim_{x \to 1^+} f(x)$ **(h)** $\lim_{x \to 1} f(x)$

(i) $f(6)$ **(j)** $\lim_{x \to 6^-} f(x)$ **(k)** $\lim_{x \to 6^+} f(x)$ **(l)** $\lim_{x \to 6} f(x)$

Solution

(a) $f(-4)$ doesn't exist. There is no closed dot for this value of *x* and so the function doesn't exist at this point.

(b) $\lim_{x \to -4^-} f(x) = 2$ The function is approaching a value of 2 as *x* moves in towards -4 from the left.

(c) $\lim_{x \to -4^+} f(x) = 2$ The function is approaching a value of 2 as *x* moves in towards -4 from the right.

(d) $\lim_{x \to -4} f(x) = 2$ We can do this one of two ways. Either we can use the fact here and notice that the two one-sided limits are the same and so the normal limit must exist and have the same value as the one-sided limits or just get the answer from the graph.

Also recall that a limit can exist at a point even if the function doesn't exist at that point.

(e) $f(1) = 4$. The function will take on the *y* value where the closed dot is.

(f) $\lim_{x \to 1^-} f(x) = 4$ The function is approaching a value of 4 as *x* moves in towards 1 from the left.

(g) $\lim_{x \to 1^+} f(x) = -2$ The function is approaching a value of -2 as *x* moves in towards 1 from the right. Remember that the limit does NOT care about what the function is actually doing at the point, it only cares about what the function is doing around the point. In this case, always staying to the right of $x = 1$ $x = 1$, the function is approaching a value of -2 and so the limit is -2. The limit is not 4, as that is value of the function at the point and again the limit doesn't care about that!

(h) $\lim_{x \to 1} f(x)$ doesn't exist. The two one-sided limits both exist, however they are different and so the normal limit doesn't exist.

(i) $f(6) = 2$. The function will take on the *y* value where the closed dot is.

(j) $\lim_{x \to 6^-} f(x) = 5$ The function is approaching a value of 5 as *x* moves in towards 6 from the left.

(k) $\lim_{x \to 6^+} f(x) = 5$ The function is approaching a value of 5 as *x* moves in towards 6 from the right.

(l) $\lim_{x \to 6} f(x) = 5$ Again, we can use either the graph or the fact to get this. Also, once more remember that the limit doesn't care what is happening at the point and so it's possible for the limit to have a different value than the function at a point. When dealing with limits we've always got to remember that limits simply do not care about what the function is doing at the point in question. Limits are only concerned with what the function is doing around the point.

Example 2 Evaluate the following limit.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}$$

Solution

First notice that we can use property **4)** to write the limit as,

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} - 2z^4 + 7z^3 + 1}$$

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn't zero. As we will see however, it isn't in this case so we're okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.

$$\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1}$$

$$= \frac{13}{6}$$

xample 1 Evaluate the following limit.

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x}$$

Solution

First let's notice that if we try to plug in $x = 2$ we get,

$$\lim_{x \to 2} \frac{x^2 + 4x - 12}{x^2 - 2x} = \frac{0}{0}$$

So, we can't just plug in $x = 2$ to evaluate the limit. So, we're going to have to do something else.

The first thing that we should always do when evaluating limits is to simplify the function as much as possible. In this case that means factoring both the numerator and denominator. Doing this gives,

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} &= \lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}\\ &= \lim_{x\to 2}\frac{x+6}{x}\end{aligned}$$

So, upon factoring we saw that we could cancel an $x-2$ from both the numerator and the denominator. Upon doing this we now have a new rational expression that we can plug $x = 2$ $x = 2$ into because we lost the division by zero problem. Therefore, the limit is,

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x} = \lim_{x\to 2}\frac{x+6}{x} = \frac{8}{2} = 4$$

Example 2 Evaluate the following limit.

$$\lim_{h\to 0}\frac{2(-3+h)^2-18}{h}$$

Solution

In this case we also get 0/0 and factoring is not really an option. However, there is still some simplification that we can do.

$$\begin{aligned}\lim_{h\to 0}\frac{2(-3+h)^2-18}{h} &= \lim_{h\to 0}\frac{2\left(9-6h+h^2\right)-18}{h}\\ &= \lim_{h\to 0}\frac{18-12h+2h^2-18}{h}\\ &= \lim_{h\to 0}\frac{-12h+2h^2}{h}\end{aligned}$$

So, upon multiplying out the first term we get a little cancellation and now notice that we can factor an *h* out of both terms in the numerator which will cancel against the *h* in the denominator and the division by zero problem goes away and we can then evaluate the limit.

$$\lim_{h\to 0}\frac{2\left(-3+h\right)^2-18}{h}=\lim_{h\to 0}\frac{-12h+2h^2}{h}$$
$$=\lim_{h\to 0}\frac{h\left(-12+2h\right)}{h}$$
$$=\lim_{h\to 0}-12+2h=-12$$

Example 3 Evaluate the following limit.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}$$

Solution

This limit is going to be a little more work than the previous two. Once again however note that we get the indeterminate form 0/0 if we try to just evaluate the limit. Also note that neither of the two examples will be of any help here, at least initially. We can't factor and we can't just multiply something out to get the function to simplify.

When there is a square root in the numerator or denominator we can try to rationalize and see if that helps. Recall that rationalizing makes use of the fact that

$$\left(a+b\right)\left(a-b\right)=a^2-b^2$$

So, if either the first and/or the second term have a square root in them the rationalizing will eliminate the root(s). This *might* help in evaluating the limit.

Let's try rationalizing the numerator in this case.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{\left(t-\sqrt{3t+4}\right)}{\left(4-t\right)}\frac{\left(t+\sqrt{3t+4}\right)}{\left(t+\sqrt{3t+4}\right)}$$

Remember that to rationalize we just take the numerator (since that's what we're rationalizing), change the sign on the second term and multiply the numerator and denominator by this new term.

Next, we multiply the numerator out being careful to watch minus signs.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{t^2-\left(3t+4\right)}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$
$$=\lim_{t\to 4}\frac{t^2-3t-4}{\left(4-t\right)\left(t+\sqrt{3t+4}\right)}$$

Notice that we didn't multiply the denominator out as well. Most students come out of an Algebra class having it beaten into their heads to always multiply this stuff out. However, in this case multiplying out will make the problem very difficult and in the end you'll just end up factoring it back out anyway.

At this stage we are almost done. Notice that we can factor the numerator so let's do that.

$$\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}=\lim_{t\to 4}\frac{(t-4)(t+1)}{(4-t)\left(t+\sqrt{3t+4}\right)}$$

Now all we need to do is notice that if we factor a "-1"out of the first term in the denominator we can do some canceling. At that point the division by zero problem will go away and we can evaluate the limit.

$$\begin{aligned}\lim_{t\to 4}\frac{t-\sqrt{3t+4}}{4-t}&=\lim_{t\to 4}\frac{(t-4)(t+1)}{-(t-4)\left(t+\sqrt{3t+4}\right)}\\&=\lim_{t\to 4}\frac{t+1}{-\left(t+\sqrt{3t+4}\right)}\\&=-\frac{5}{8}\end{aligned}$$

For the function $f(x)=\dfrac{8-x^3}{x^2-4}$ answer each of the following questions.

(a) Evaluate the function the following values of *x* compute (accurate to at least 8 decimal places).

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

(b) Use the information from **(a)** to estimate the value of $\lim_{x\to 2}\dfrac{8-x^3}{x^2-4}$.

(a) Evaluate the function the following values of *x* compute (accurate to at least 8 decimal places).

(i) 2.5	**(ii)** 2.1	**(iii)** 2.01	**(iv)** 2.001	**(v)** 2.0001
(vi) 1.5	**(vii)** 1.9	**(viii)** 1.99	**(ix)** 1.999	**(x)** 1.9999

Here is a table of values of the function at the given points accurate to 8 decimal places.

x	m_{PQ}	x	m_{PQ}
2.5	-3.38888889	1.5	-2.64285714

2.1	-3.07560976	1.9	-2.92564103
2.01	-3.00750623	1.99	-2.99250627
2.001	-3.00075006	1.999	-2.99925006
2.0001	-3.00007500	1.9999	-2.99992500

(b) Use the information from **(a)** to estimate the value of $\lim_{x \to 2} \frac{8 - x^3}{x^2 - 4}$.
From the table of values above it looks like we can estimate that,

$$\lim_{x \to 2} \frac{8 - x^3}{x^2 - 4} = -3$$

For the function $g(\theta) = \frac{\sin(7\theta)}{\theta}$ answer each of the following questions.

(a) Evaluate the function the following values of θ θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

(b) Use the information from **(a)** to estimate the value of $\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta}$.
(a) Evaluate the function the following values of *x* compute (accurate to at least 8 decimal places).

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

Here is a table of values of the function at the given points accurate to 8 decimal places.

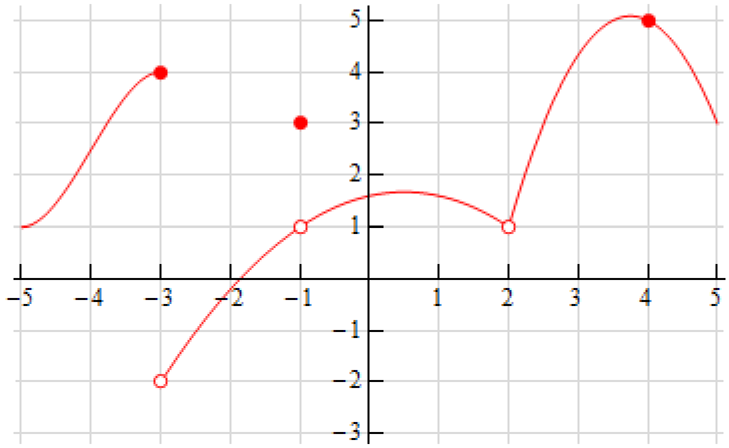
x	m_{PQ}	x	m_{PQ}
0.5	-0.70156646	-0.5	-0.70156646
0.1	6.44217687	-0.1	6.44217687
0.01	6.99428473	-0.01	6.99428473
0.001	6.99994283	-0.001	6.99994283
0.0001	6.99999943	-0.0001	6.99999943

(b) Use the information from **(a)** to estimate the value of $\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta}$.
From the table of values above it looks like we can estimate that,

$$\lim_{\theta \to 0} \frac{\sin(7\theta)}{\theta} = 7$$

Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 4$



(a) $a = -3$
From the graph we can see that,

$$\boxed{f(-3) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = -3$ from both sides the graph is approaching different values (4 from the left and -2 from the right). Because of this we get,

$$\boxed{\lim_{x \to -3} f(x) \text{ does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -1$
From the graph we can see that,

$$\boxed{f(-1) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to -1} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 2$

Because there is no closed dot for $x = 2$ we can see that,

$$\boxed{f(2) \text{ does not exist}}$$

We can also see that as we approach $x = 2$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to 2} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 4$

From the graph we can see that,

$$\boxed{f(4) = 5}$$

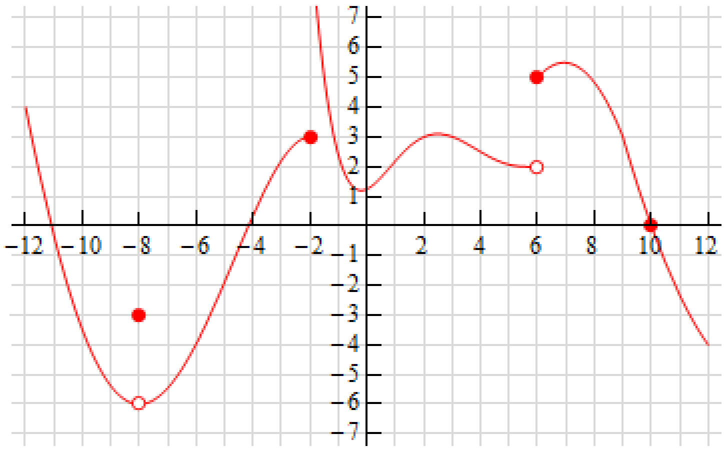
because the closed dot is at the value of $y = 5$.

We can also see that as we approach $x = 4$ from both sides the graph is approaching the same value, 5, and so we get,

$$\boxed{\lim_{x \to 4} f(x) = 5}$$

Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -8$ **(b)** $a = -2$ **(c)** $a = 6$ **(d)** $a = 10$



(a) $a = -8$
From the graph we can see that,

$$\boxed{f(-8) = -3}$$

because the closed dot is at the value of $y = -3$.

We can also see that as we approach $x = -8$ from both sides the graph is approaching the same value, -6, and so we get,

$$\boxed{\lim_{x \to -8} f(x) = -6}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(b) $a = -2$
From the graph we can see that,

$$\boxed{f(-2) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -2$ from both sides the graph is approaching different values (3 from the left and doesn't approach any value from the right). Because of this we get,

$$\boxed{\lim_{x \to -2} f(x) \text{ does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 6$
From the graph we can see that,

$$\boxed{f(6)=5}$$

because the closed dot is at the value of $y=5$.

We can also see that as we approach $x=6$ from both sides the graph is approaching different values (2 from the left and 5 from the right). Because of this we get,

$$\boxed{\lim_{x\to 6} f(x) \text{ does not exist}}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(d) $a=10$
From the graph we can see that,

$$\boxed{f(10)=0}$$

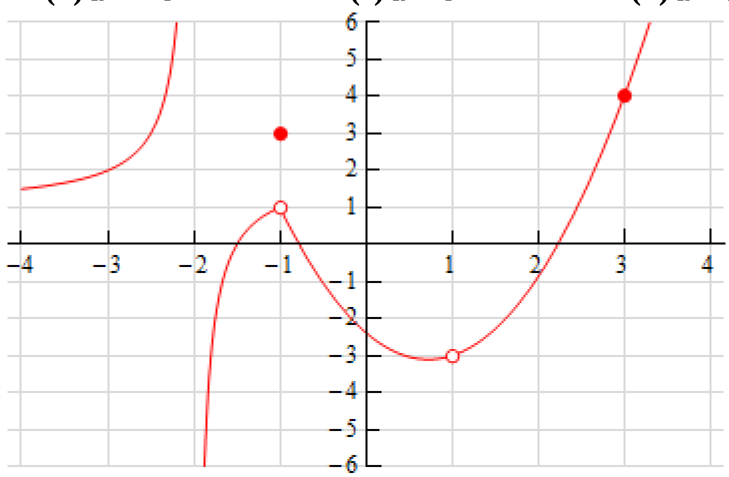
because the closed dot is at the value of $y=0$.

We can also see that as we approach $x=10$ from both sides the graph is approaching the same value, 0, and so we get,

$$\boxed{\lim_{x\to 10} f(x)=0}$$

Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim_{x\to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a=-2$ **(b)** $a=-1$ **(c)** $a=1$ **(d)** $a=3$



(a) $a=-2$

Because there is no closed dot for $x = -2$ we can see that,

$$\boxed{f(-2) \text{ does not exist}}$$

We can also see that as we approach $x = -2$ from both sides the graph is not approaching a value from either side and so we get,

$$\boxed{\lim_{x \to -2} f(x) \text{ does not exist}}$$

(b) $a = -1$
From the graph we can see that,

$$\boxed{f(-1) = 3}$$

because the closed dot is at the value of $y = 3$.

We can also see that as we approach $x = -1$ from both sides the graph is approaching the same value, 1, and so we get,

$$\boxed{\lim_{x \to -1} f(x) = 1}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Often the two will be different.

(c) $a = 1$
Because there is no closed dot for $x = 1$ we can see that,

$$\boxed{f(1) \text{ does not exist}}$$

We can also see that as we approach $x = 1$ from both sides the graph is approaching the same value, -3, and so we get,

$$\boxed{\lim_{x \to 1} f(x) = -3}$$

Always recall that the value of a limit does not actually depend upon the value of the function at the point in question. The value of a limit only depends on the values of the function around the point in question. Therefore, even though the function doesn't exist at this point the limit can still have a value.

(d) $a = 3$
From the graph we can see that,

$$\boxed{f(3) = 4}$$

because the closed dot is at the value of $y = 4$.

We can also see that as we approach $x = 3$ from both sides the graph is approaching the same value, 4, and so we get,

$$\boxed{\lim_{x \to 3} f(x) = 4}$$

For the function $g(x) = \dfrac{x^2 + 6x + 9}{x^2 + 3x}$ answer each of the following questions.

(a) Evaluate the function the following values of *x* compute (accurate to at least 8 decimal places).

(i) -2.5	**(ii)** -2.9	**(iii)** -2.99	**(iv)** -2.999	**(v)** -2.9999
(vi) -3.5	**(vii)** -3.1	**(viii)** -3.01	**(ix)** -3.001	**(x)** -3.0001

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{x \to -3} \frac{x^2 + 6x + 9}{x^2 + 3x}$.

2. For the function $f(z) = \dfrac{10z - 9 - z^2}{z^2 - 1}$ answer each of the following questions.

(a) Evaluate the function the following values of *t* compute (accurate to at least 8 decimal places).

(i) 1.5	**(ii)** 1.1	**(iii)** 1.01	**(iv)** 1.001	**(v)** 1.0001
(vi) 0.5	**(vii)** 0.9	**(viii)** 0.99	**(ix)** 0.999	**(x)** 0.9999

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{z \to 1} \frac{10z - 9 - z^2}{z^2 - 1}$.

3. For the function $h(t) = \dfrac{2 - \sqrt{4 + 2t}}{t}$ answer each of the following questions.

(a) Evaluate the function the following values of θ θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 0.5	**(ii)** 0.1	**(iii)** 0.01	**(iv)** 0.001	**(v)** 0.0001
(vi) -0.5	**(vii)** -0.1	**(viii)** -0.01	**(ix)** -0.001	**(x)** -0.0001

(b) Use the information from **(a)** to estimate the value of $\displaystyle\lim_{t \to 0} \frac{2 - \sqrt{4 + 2t}}{t}$.

4. For the function $g(\theta) = \dfrac{\cos(\theta - 4) - 1}{2\theta - 8}$ answer each of the following questions.

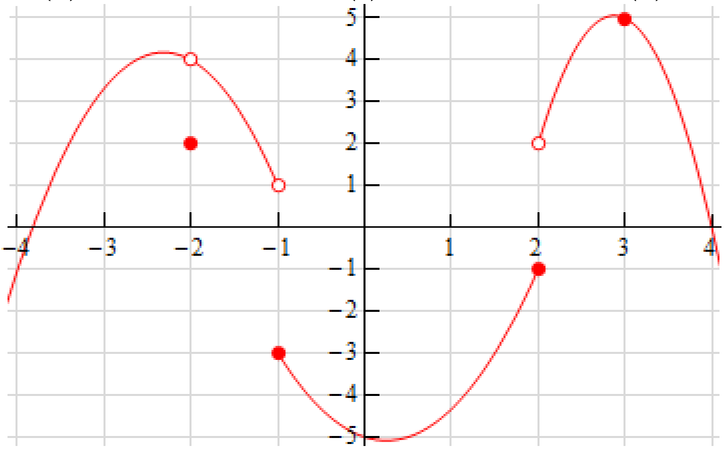
(a) Evaluate the function the following values of θ compute (accurate to at least 8 decimal places). Make sure your calculator is set to radians for the computations.

(i) 4.5	**(ii)** 4.1	**(iii)** 4.01	**(iv)** 4.001	**(v)** 4.0001
(vi) 3.5	**(vii)** 3.9	**(viii)** 3.99	**(ix)** 3.999	**(x)** 3.9999

(b) Use the information from **(a)** to estimate the value of $\lim\limits_{\theta \to 0} \dfrac{\cos(\theta - 4) - 1}{2\theta - 8}$.

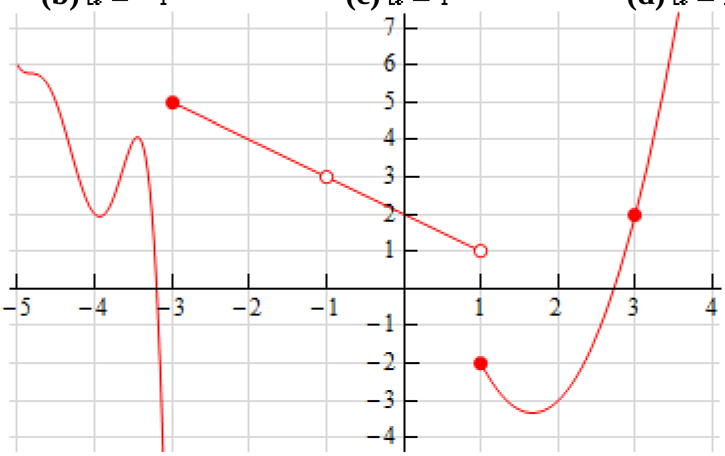
5. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -2$ **(b)** $a = -1$ **(c)** $a = 2$ **(d)** $a = 3$



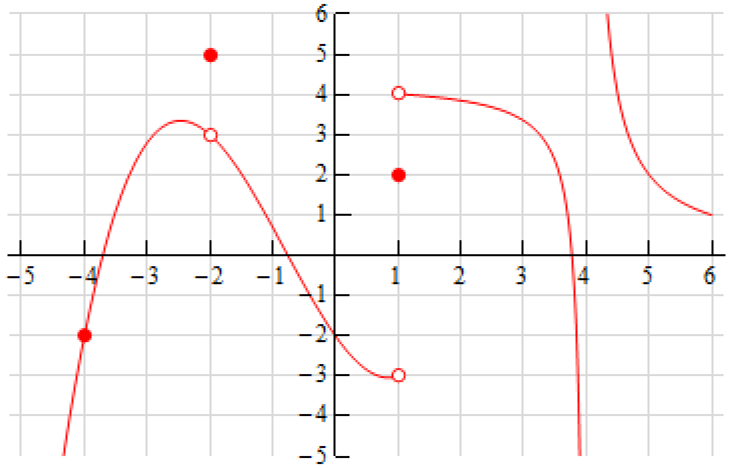
6. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -3$ **(b)** $a = -1$ **(c)** $a = 1$ **(d)** $a = 3$



7. Below is the graph of $f(x)$. For each of the given points determine the value of $f(a)$ and $\lim\limits_{x \to a} f(x)$. If any of the quantities do not exist clearly explain why.

(a) $a = -4$ **(b)** $a = -2$ **(c)** $a = 1$ **(d)** $a = 4$



8. Explain in your own words what the following equation means.

$$\lim_{x \to 12} f(x) = 6$$

9. Suppose we know that $\lim\limits_{x \to -7} f(x) = 18$. If possible, determine the value $f(-7)$. If it is not possible to determine the value explain why not.

10. Is it possible to have $\lim\limits_{x \to 1} f(x) = -23$ and $f(1) = 107$? Explain your answer.