Continuity

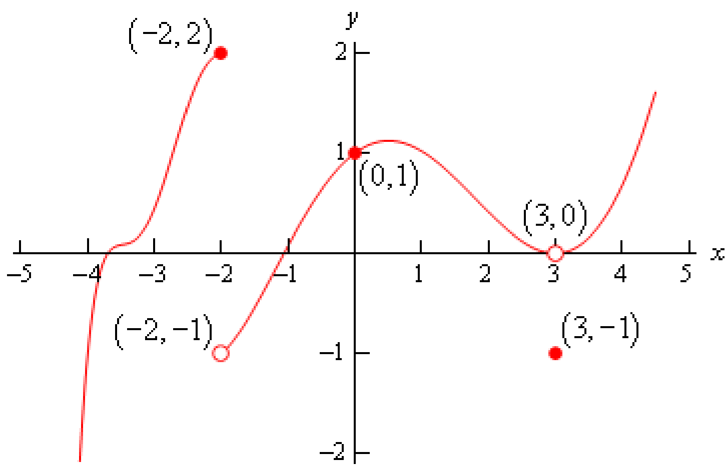
A function $f(x)$ is said to be **continuous** at $x=a$ is $\lim_{x\to a} f(x) = f(a)$. A function is said to be continuous on the interval [*a*, *b*] if it is continuous at each point in the interval.

If $f(x)$ is continuous at $x=a$

then, $\lim_{x\to a} f(x) = f(a)$ $\qquad$ $\lim_{x\to a^-} f(x) = f(a)$ $\qquad$ $\lim_{x\to a^+} f(x) = f(a)$

Q. Given the graph of *f(x)*, shown below, determine if *f(x)* is continuous at $x=-2$

$x=-2$, $x=0$, and $x=3$.



Solution

First $x=-2$.

$$f(-2)=2 \qquad \lim_{x\to -2} f(x) \text{ doesn't exist}$$

Now $x=0$.

$$f(0)=1 \qquad \lim_{x\to 0} f(x)=1$$

The function is continuous at this point since the function and limit have the same value.

Finally $x=3$.

$$f(3)=-1 \qquad \lim_{x\to 3} f(x)=0$$

The function is not continuous at this point. This kind of discontinuity is called a **removable discontinuity**. Removable discontinuities are those where there is a hole in the graph as there is in this case.

Q Determine where the function below is not continuous.

$$h(t) = \frac{4t+10}{t^2 - 2t - 15}$$

Solution

Rational functions are continuous everywhere except where we have division by zero. So all that we need to is determine where the denominator is zero. That's easy enough to determine by setting the denominator equal to zero and solving.

$$t^2 - 2t - 15 = (t-5)(t+3) = 0$$

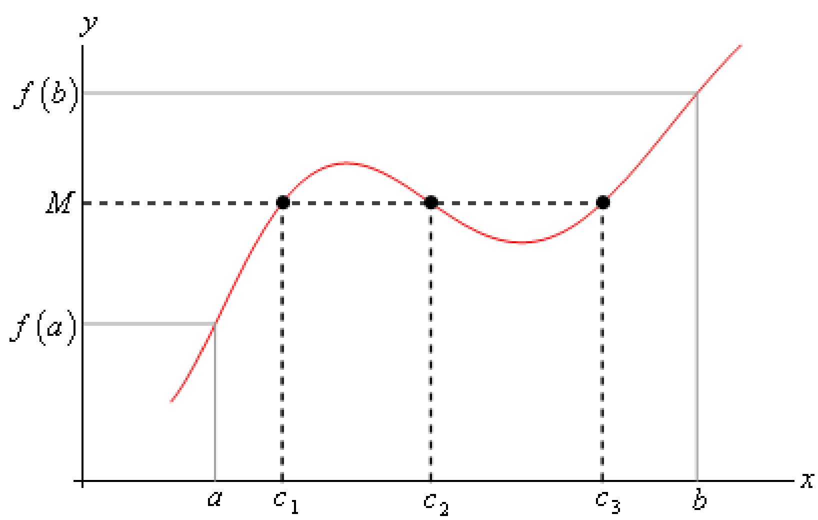
So, the function will not be continuous at *t=-3* and *t=5*.

If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$ then, $\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$

Intermediate Value Theorem

Suppose that *f(x)* is continuous on [*a, b*] and let *M* be any number between *f(a)* and *f(b)*. Then there exists a number *c* such that,

1. $a < c < b$
2. $f(c) = M$



Q. Show that $p(x) = 2x^3 - 5x^2 - 10x + 5$ has a root somewhere in the interval [-1,2].

Solution

What we're really asking here is whether or not the function will take on the value

$$p(x) = 0$$

somewhere between -1 and 2. In other words, we want to show that there is a number *c* such that $-1 < c < 2$ and $p(c) = 0$. However if we define $M = 0$ and acknowledge that $a = -1$ and $b = 2$ we can see that these two condition on *c* are exactly the conclusions of the Intermediate Value Theorem.

So, this problem is set up to use the Intermediate Value Theorem and in fact, all we need to do is to show that the function is continuous and that $M = 0$ is between $p(-1)$ and $p(2)$ (*i.e.* $p(-1) < 0 < p(2)$ or $p(2) < 0 < p(-1)$ and we'll be done.

To do this all we need to do is compute,

$$p(-1) = 8 \qquad\qquad p(2) = -19$$

So we have,

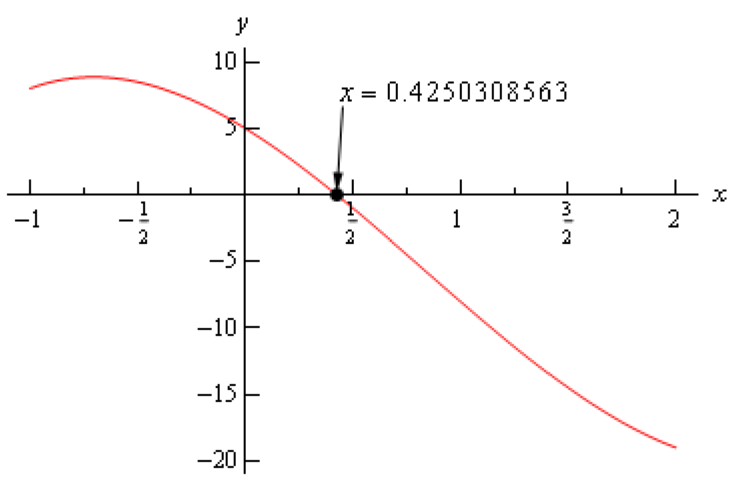
$$-19 = p(2) < 0 < p(-1) = 8$$

Therefore $M = 0$ is between $p(-1)$ and $p(2)$ and since $p(x)$ is a polynomial it's continuous everywhere and so in particular it's continuous on the interval [-1,2]. So by the Intermediate Value Theorem there must be a number $-1 < c < 2$ so that,

$$p(c) = 0$$

Therefore the polynomial does have a root between -1 and 2.

For the sake of completeness here is a graph showing the root that we just proved existed. Note that we used a computer program to actually find the root and that the Intermediate Value Theorem did not tell us what this value was.



Q If possible, determine if $f(x) = 20\sin(x+3)\cos\left(\frac{x^2}{2}\right)$ takes the following values in the interval [0,5].

(a) Does $f(x) = 10$?

(b) Does $f(x) = -10$?

Solution

Now, for each part we will let *M* be the given value for that part and then we'll need to show that *M* lies between $f(0)$ and $f(5)$. So, since we'll need the two function evaluations for each part let's give them here,

$$f(0) = 2.8224 \qquad\qquad f(5) = 19.7436$$

Now, let's take a look at each part.

(a) Okay, in this case we'll define $M = 10$ and we can see that,

$$f(0) = 2.8224 < 10 < 19.7436 = f(5)$$

So, by the Intermediate Value Theorem there must be a number $0 \le c \le 5$ such that

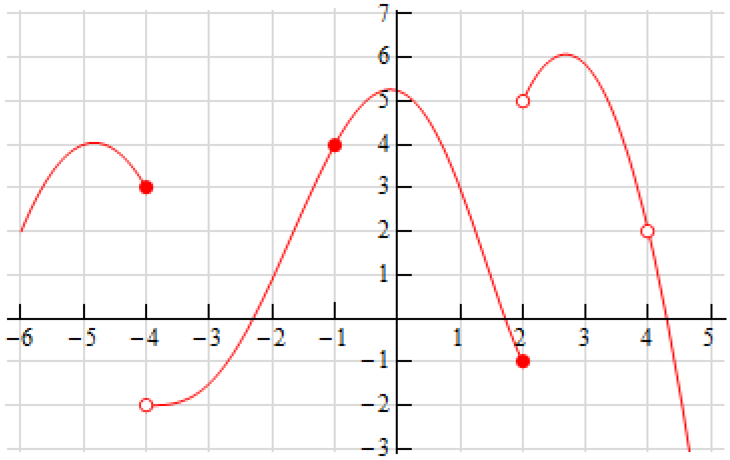
$$f(c) = 10$$

(b) In this part we'll define $M = -10$. We now have a problem. In this part *M* does not live between $f(0)$ and $f(5)$. So, what does this mean for us? Does this mean that $f(x) \neq -10$ in [0,5]?

Unfortunately for us, this doesn't mean anything. It is possible that $f(x) \neq -10$ in [0,5], but is it also possible that $f(x) = -10$ in [0,5]. The Intermediate Value Theorem will only tell us that *c*'s will exist. The theorem will NOT tell us that *c*'s don't exist.

In this case it is not possible to determine if $f(x) = -10$ in [0,5] using the Intermediate Value Theorem.

Q The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Before starting the solution recall that in order for a function to be continuous at $x=a$ both $f(a)$ and $\lim\limits_{x\to a} f(x)$ must exist and we must have,

$$\lim_{x\to a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x=-4$ we have,

$$\lim_{x\to -4^-} f(x) = 3 \neq -2 = \lim_{x\to -4^+} f(x)$$

and therefore we also know that $\lim\limits_{x\to -4} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x=-4$ because the limit does not exist.

Likewise, at $x=2$ we have,

$$\lim_{x\to 2^-} f(x) = -1 \neq 5 = \lim_{x\to 2^+} f(x)$$

and therefore we also know that $\lim\limits_{x\to 2} f(x)$ doesn't exist. So again, because the limit does not exist, we can see that $f(x)$ is **discontinuous** at $x=2$.

Finally let's take a look at $x=4$. Here we can see that,

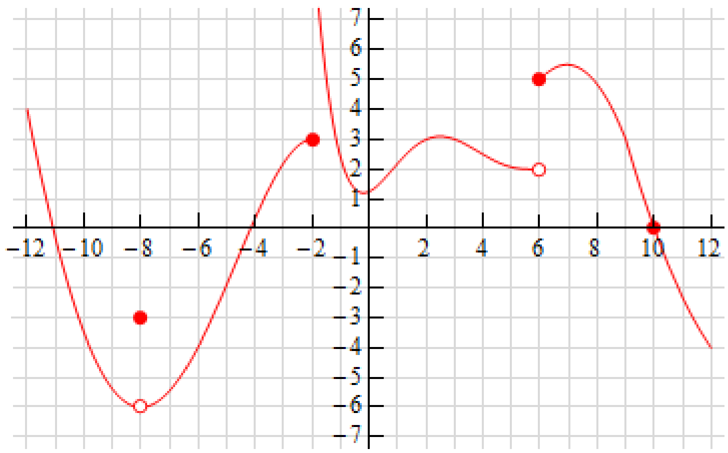
$$\lim_{x\to 4^-} f(x) = 2 = \lim_{x\to 4^+} f(x)$$

and therefore we also know that $\lim\limits_{x\to 4} f(x) = 2$. However, we can also see that $f(4)$ doesn't exist and so once again $f(x)$ is **discontinuous** at $x=4$ because this time the function does not exist at $x=4$.

All other points on this graph will have both the function and limit exist and we'll have $\lim\limits_{x \to a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -4$, $x = 2$ and $x = 4$.

The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



Before starting the solution recall that in order for a function to be continuous at $x = a$ $x = a$ both $f(a)$ and $\lim\limits_{x \to a} f(x)$ must exist and we must have,

$$\lim_{x \to a} f(x) = f(a)$$

Using this idea it should be fairly clear where the function is not continuous.

First notice that at $x = -8$ we have,

$$\lim_{x \to -8^-} f(x) = -6 = \lim_{x \to -8^+} f(x)$$

and therefore we also know that $\lim\limits_{x \to -8} f(x) = -6$. We can also see that $f(-8) = -3$ and so we have,

$$-6 = \lim_{x \to -8} f(x) \neq f(-8) = -3$$

Because the function and limit have different values we can conclude that $f(x)$ is **discontinuous** at $x = -8$.

Next let's take a look at $x = -2$ we have,

$$\lim_{x \to -2^-} f(x) = 3 \neq \infty = \lim_{x \to -2^+} f(x)$$

and therefore we also know that $\lim\limits_{x \to -2} f(x)$ doesn't exist. We can therefore conclude that $f(x)$ is **discontinuous** at $x = -2$ because the limit does not exist.

Finally let's take a look at $x = 6$ $x = 6$. Here we can see we have,

$$\lim_{x \to 6^-} f(x) = 2 \neq 5 = \lim_{x \to 6^+} f(x)$$

and therefore we also know that $\lim\limits_{x \to 6} f(x)$ doesn't exist. So, once again, because the limit does not exist, we can conclude that $f(x)$ is **discontinuous** at $x = 6$.

All other points on this graph will have both the function and limit exist and we'll have $\lim\limits_{x \to a} f(x) = f(a)$ and so will be continuous.

In summary then the points of discontinuity for this graph are : $x = -8$, $x = -2$ and $x = 6$.

Q. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $x = -1$, **(b)** $x = 0$, **(c)** $x = 3$?

$$f(x) = \frac{4x+5}{9-3x}$$

(a) $x = -1$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x \to -1}(4x+5)}{\lim\limits_{x \to -1}(9-3x)} = \frac{4\lim\limits_{x \to -1} x + \lim\limits_{x \to -1} 5}{\lim\limits_{x \to -1} 9 - 3\lim\limits_{x \to -1} x} = \frac{4(-1)+5}{9-3(-1)} = f(-1)$$

So, we can see that $\lim\limits_{x \to -1} f(x) = f(-1)$ and so the function **is continuous** at $x = -1$.

(b) $x = 0$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{4x+5}{9-3x} = \frac{\lim\limits_{x \to 0}(4x+5)}{\lim\limits_{x \to 0}(9-3x)} = \frac{4\lim\limits_{x \to 0} x + \lim\limits_{x \to 0} 5}{\lim\limits_{x \to 0} 9 - 3\lim\limits_{x \to 0} x} = \frac{4(0)+5}{9-3(0)} = f(0)$$

So, we can see that $\lim\limits_{x \to 0} f(x) = f(0)$ and so the function **is continuous** at $x = 0$.

(c) $x = 3$

$$\lim_{x \to 3^-} f(x) = \infty \qquad\qquad \lim_{x \to 3^+} f(x) = -\infty$$

and so $\lim\limits_{x \to 3} f(x)$ also doesn't exist.

Q. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $z=-2$, **(b)** $z=0$, **(c)** $z=5$?

$$g(z)=\frac{6}{z^2-3z-10}$$

(a) $z=-2$

We can see that the function **is not continuous** at $z=-2$.

For practice you might want to verify that,

$$\lim_{z\to-2^-} g(z)=\infty \qquad \lim_{z\to-2^+} g(z)=-\infty$$

and so $\lim_{z\to-2} g(z)$ also doesn't exist.

(b) $z=0$

$$\lim_{z\to0} g(z)=\lim_{z\to0}\frac{6}{z^2-3z-10}=\frac{\lim_{z\to0}6}{\lim_{z\to0}\left(z^2-3z-10\right)}=\frac{}{\lim_{z\to0} z}$$

$$=\frac{}{0^2-3}$$

So, we can see that $\lim_{z\to0} g(x)=g(0)$ and so the function **is continuous** at $z=0$.

(c) $z=5$

we can see that the function **is not continuous** at $z=5$ $z=5$.

$\lim_{z\to5^-} g(z)=-\infty$ $\qquad$ $\lim_{z\to5^+} g(z)=\infty$ and so $\lim_{z\to5} g(z)$ also doesn't exist.

Q. Using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the following function is continuous or discontinuous at **(a)** $t = -2$, **(b)** $t = 10$?

$$h(t) = \begin{cases} t^2 & t < -2 \\ t+6 & t \geq -2 \end{cases}$$

(a) $t = -2$

$$\lim_{t \to -2^-} h(t) = \lim_{t \to -2^-} t^2 = (-2)^2 = 4$$

$$\lim_{t \to -2^+} g(t) = \lim_{t \to -2^+} (t+6) = \lim_{t \to -2^+} t + \lim_{t \to -2^+} 6 = -2 + 6 = 4$$

So we can see that, $\lim_{t \to -2^-} h(t) = \lim_{t \to -2^+} h(t) = 4$ and so $\lim_{t \to -2} h(t) = 4$.

Next, a quick computation shows us that $h(-2) = -2 + 6 = 4$ and so we can see that $\lim_{t \to -2} h(t) = h(-2)$

and so the function **is continuous** at $t = -2$.

(b) $t = 10$

For justification on why we can't just plug in the number here check out the comment at the beginning of the solution to **(a)**.

For this part we can notice that because there are values of *t* on both sides of $t = 10$ in the range $t \geq -2$ we won't need to worry about one-sided limits here. Here is the work for this part.

Here is the work for this part.

$$\lim_{t \to 10} h(t) = \lim_{t \to 10} (t+6) = \lim_{t \to 10} t + \lim_{t \to 10} 6 = 10$$

So, we can see that $\lim_{t \to 10} h(t) = h(10)$ and so the function **is continuous** at $t = 10$.

Q. Determine where the following function is discontinuous.

$$R(t) = \frac{8t}{t^2 - 9t - 1}$$

$$t^2 - 9t - 1 = 0 \qquad \Rightarrow \qquad t = \frac{9 \pm \sqrt{(-9)^2 - 4(1)(-1)}}{2(1)} = \frac{9 \pm \sqrt{85}}{2}$$

The function will therefore be discontinuous at the points : $t = \dfrac{9 \pm \sqrt{85}}{2}$.

Use the Intermediate Value Theorem to show that the given equation has at least one solution in the indicated interval. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval.

Use the Intermediate Value Theorem to show that $25 - 8x^2 - x^3 = 0$ has at least one root in the interval $[-2, 4]$. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval,

Okay, let's start off by defining,

$$f(x) = 25 - 8x^2 - x^3 \qquad \&$$

The problem is then asking us to show that there is a *c* in $[-2, 4]$ so that,

$$f(c) = 0 = M$$

$$f(-2) = 1 \qquad\qquad f(4) = -16$$

Therefore we have,

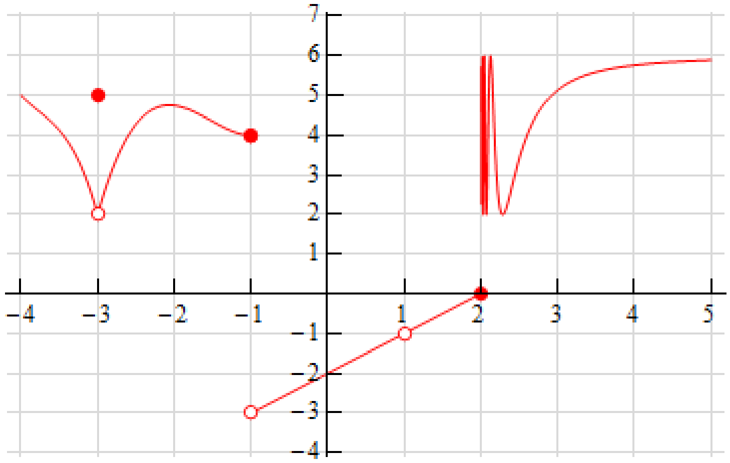
$$f(4) = -167 < 0 < 1 = f(-2)$$

So by the Intermediate Value Theorem there must be a number *c* such that,

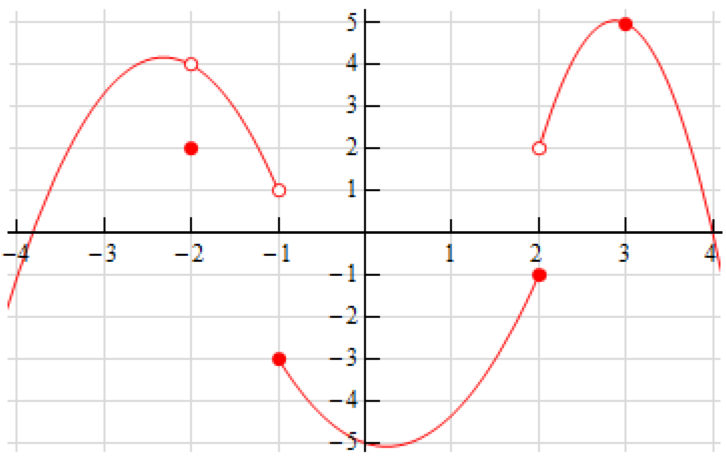
$$-2 < c < 4 \qquad \& \qquad f(c) = 0$$

Assignment

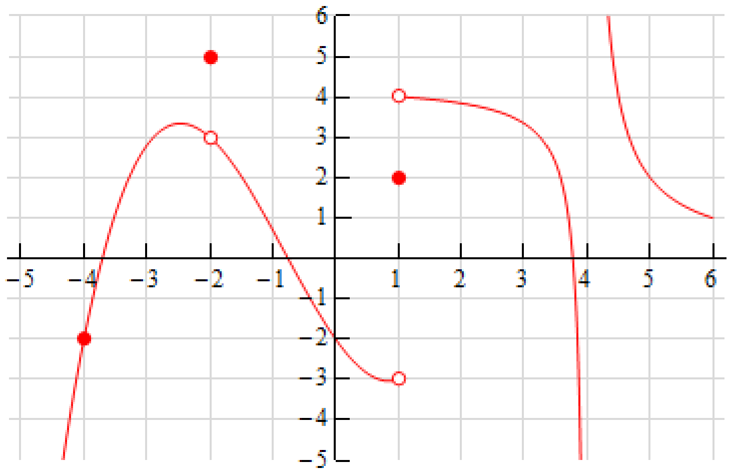
1. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



2. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



3. The graph of $f(x)$ is given below. Based on this graph determine where the function is discontinuous.



For problems 4 _ 13 using only Properties 1- 9 from the **Limit Properties** section, one-sided limit properties (if needed) and the definition of continuity determine if the given function is continuous or discontinuous at the indicated points.

4. $f(x) = \dfrac{6+2x}{7x-14}$

(a) $x = -3$, **(b)** $x = 0$, **(c)** $x = 2$?

5. $R(y) = \dfrac{2y}{y^2 - 25}$

(a) $y = -5$, **(b)** $y = -1$, **(c)** $y = 3$?

6. $g(z) = \dfrac{5z - 20}{z^2 - 12z}$

(a) $z = -1$, **(b)** $z = 0$, **(c)** $z = 4$?

7. $W(x) = \dfrac{2+x}{x^2 + 6x - 7}$

(a) $x = -7$, **(b)** $x = 0$, **(c)** $x = 1$?

8. $h(z) = \begin{cases} 2z^2 & z < -1 \\ 4z + 6 & z \ge -1 \end{cases}$

(a) $z = -6$, **(b)** $z = -1$?

9. $g(x) = \begin{cases} x + \mathbf{e}^x & x < 0 \\ x^2 & x \ge 0 \end{cases}$

(a) $x = 0$, **(b)** $x = 4$?

10. $Z(t)=\begin{cases} 8 & t<5 \\ 1-6t & t\geq 5 \end{cases}$

(a) $t=0$, **(b)** $t=5$?

11. $h(z)=\begin{cases} z+2 & z<-4 \\ 0 & z=-4 \\ 18-z^2 & z>-4 \end{cases}$

(a) $z=-4$, **(b)** $z=2$?

12. $f(x)=\begin{cases} 1-x^2 & x<2 \\ -3 & x=2 \\ 2x-7 & 2<x<7 \\ 0 & x=7 \\ x^2 & x>7 \end{cases}$

(a) $x=2$, **(b)** $x=7$?

13. $g(w)=\begin{cases} 3w & w<0 \\ 0 & w=0 \\ w+6 & 0<w<8 \\ 14 & w=8 \\ 22-w & w>8 \end{cases}$

(a) $w=0$, **(b)** $w=8$?

For problems 14 _ 22 determine where the given function is discontinuous.

14. $f(x)=\dfrac{11-2x}{2x^2-13x-7}$

15. $Q(z)=\dfrac{3}{2z^2+3z-4}$

16. $h(t)=\dfrac{t^2-1}{t^3+6t^2+t}$

17. $f(z)=\dfrac{4z+1}{5\cos\left(\frac{z}{2}\right)+1}$

18. $h(x) = \dfrac{1-x}{x\sin(x-1)}$

19. $f(x) = \dfrac{3}{4\mathbf{e}^{x-7}-1}$

20. $R(w) = \dfrac{\mathbf{e}^{w^2+1}}{\mathbf{e}^{w} - 2\mathbf{e}^{1-w}}$

21. $g(x) = \cot(4x)$

22. $f(t) = \sec\left(\sqrt{t}\right)$

For problems 23 – 27 use the Intermediate Value Theorem to show that the given equation has at least one solution in the indicated interval. Note that you are NOT asked to find the solution only show that at least one must exist in the indicated interval.

23. $1+7x^3-x^4=0$ on $[4,8]$

24. $z^2+11z=3$ on $[-15,-5]$

25. $\dfrac{t^2+t-15}{t-8}=0$ on $[-5,1]$

26. $\ln\left(2t^2+1\right)-\ln\left(t^2-4\right)=0$ on $[-1,2]$

27. $10 = w^3 + w^2\mathbf{e}^{-w} - 5$ on $[0,4]$

For problems 28 – 33 assume that $f(x)$ is continuous everywhere unless otherwise indicated in some way. From the given information is it possible to determine if there is a root of $f(x)$ in the given interval?

If it is possible to determine that there is a root in the given interval clearly explain how you know that a root must exist. If it is not possible to determine if there is a root in the interval sketch a graph of two functions each of which meets the given information and one will have a root in the given interval and the other will not have a root in the given interval.

28. $f(-5)=12$ and $f(0)=-3$ on the interval $[-5,0]$.

29. $f(1)=30$ and $f(9)=6$ on the interval $[1,9]$.

30. $f(20)=-100$ and $f(40)=-100$ on the interval $[20,40]$.

31. $f(-4)=-10$, $f(5)=17$, $\lim_{x\to 1^-} f(x)=-2$, and $\lim_{x\to 1^+} f(x)=4$ on the interval $[-4,5]$.

32. $f(-8)=2$, $f(1)=23$, $\lim_{x\to -4^-} f(x)=35$, and $\lim_{x\to -4^+} f(x)=1$ on the interval $[-8,1]$.

33. $f(0)=-1$, $f(9)=10$, $\lim_{x\to 2^-} f(x)=-12$, and $\lim_{x\to 2^+} f(x)=-3$ on the interval $[0,10]$.