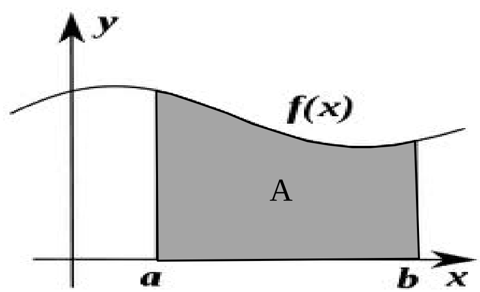
Areas by Integration

1. **Area under a curve – region bounded by the given function, vertical lines and the x –axis.**
2. **Area under a curve – region bounded by the given function, horizontal lines and the y –axis.**
3. **Area between curves defined by two given functions.**

1. Area under a curve – region bounded by the given function, vertical lines and the x –axis.

If *f(x)* is a continuous and nonnegative function of x on the closed interval [a, b], then the area of the region bounded by the graph of *f,* the x-axis and the vertical lines x=a and x=b is given by:

$$Area = \int_a^b f(x)dx$$

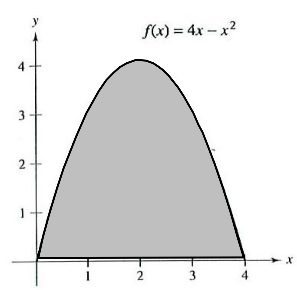


When calculating the area under a curve *f(x),* follow the steps below:

1. Sketch the area.
2. Determine the boundaries a and b,
3. Set up the definite integral,
4. Integrate.

Ex. 1. Find the area in the first quadrant bounded by $f(x) = 4x - x^2$ and the x-axis.

Graph:



To find the boundaries, determine the x-intercepts: $f(x) = 0 \rightarrow 4x - x^2 = 0$

$x(4-x) = 0$

$x = 0$ or $(4-x) = 0$ so $x = 0$ and $x = 4$

Therefore the boundaries are $a = 0$ and $b = 4$

Set up the integral: $A = \int_a^b f(x)dx = \int_0^4 (4x - x^2)dx$

Solve:

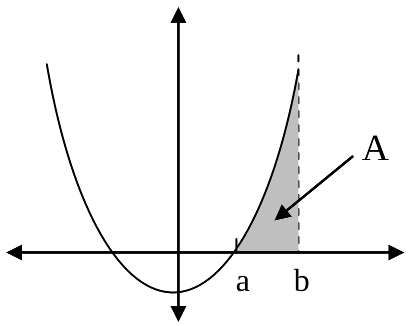
$$\int_0^4 (4x - x^2)dx = \left(4 \cdot \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_0^4 = \left(2x^2 - \frac{1}{3}x^3\right)\Bigg|_0^4 = \left(2 \cdot (4)^2 - \frac{1}{3}(4)^3\right) - \left(2 \cdot (0)^2 - \frac{1}{3}(0)^3\right)$$

$$= \left(2 \cdot 16 - \frac{1}{3} \cdot 64\right) - 0 = \frac{32}{3}$$

The area in the first quadrant under the curve $f(x) = 4x - x^2$ is equal to $\frac{32}{3}$ square units

Ex. 2. Find the area bounded by the following curves: $y = x^2 - 4,\ y = 0,\ x = 4,$

Graph:



Finding the boundaries:

$y = x^2 - 4,$ and $y = 0$ implies $x^2 - 4 = 0$ so $(x-2)(x+2) = 0$

$x = -2$ or $x = 2$

From the graph we see that $x = 2$ is our boundary at a. The value $x = -2$ is a solution to the equation above but it is not bounding the area. (Here's why the graph is an important tool to help us determine correct results. *Don't skip this step*!)

The other boundary value is given by the equation of the vertical line $x = 4,$

Boundaries are: $a = 2,$ and $b = 4,$

Set up the integral:

$$A = \int_a^b f(x)dx = \int_2^4 (x^2 - 4)dx$$

Solve:

$$\int_2^4 (x^2 - 4)dx = \left(\frac{1}{3}x^3 - 4x\right)\Bigg|_2^4 = \left(\frac{1}{3} \cdot (4)^3 - 4 \cdot 4\right) - \left(\frac{1}{3} \cdot (2)^3 - 4 \cdot 2\right)$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) = \frac{64}{3} - 16 - \frac{8}{3} + 8 = \frac{56}{3} - 8 = \frac{32}{3}$$

The area bounded by the curves $y = x^2 - 4,\ y = 0,\ x = 4,$ is equal to $\frac{32}{3}$ square units.