Example 4 Find the domain and range of each of the following functions.

(a) $f(x) = 5x - 3$

(b) $g(t) = \sqrt{4 - 7t}$

(c) $h(x) = -2x^2 + 12x + 5$

(d) $f(z) = |z - 6| - 3$

(e) $g(x) = 8$

Solution

(a) $f(x) = 5x - 3$

We know that this is a line and that it's not a horizontal line (because the slope is 5 and not zero…). This means that this function can take on any value and so the range is all real numbers. Using "mathematical" notation this is,

$$\text{Range}: \ (-\infty, \infty)$$

This is more generally a polynomial and we know that we can plug any value into a polynomial and so the domain in this case is also all real numbers or,

$$\text{Domain}: \ -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

(b) $g(t) = \sqrt{4 - 7t}$

This is a square root and we know that square roots are always positive or zero and because we can have the square root of zero in this case,

$$g\left(\tfrac{4}{7}\right) = \sqrt{4 - 7\left(\tfrac{4}{7}\right)} = \sqrt{0} = 0$$

We know then that the range will be,

$$\text{Range}: \ [0, \infty)$$

For the domain we have a little bit of work to do, but not much. We need to make sure that we don't take square roots of any negative numbers and so we need to require that,

$$\begin{aligned} 4 - 7t &\geq 0 \\ 4 &\geq 7t \\ \tfrac{4}{7} &\geq t \qquad \Rightarrow \qquad t \leq \tfrac{4}{7} \end{aligned}$$

The domain is then,

$$\text{Domain}: \ t \leq \tfrac{4}{7} \quad \text{or} \quad \left(-\infty, \tfrac{4}{7}\right]$$

(c) $h(x) = -2x^2 + 12x + 5$

Here we have a quadratic which is a polynomial and so we again know that the domain is all real numbers or,

$$\text{Domain}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In this case the range requires a little bit of work. From an Algebra class we know that the graph of this will be a **parabola** that opens down (because the coefficient of the x^2 x^2 is negative) and so the vertex will be the highest point on the graph. If we know the vertex we can then get the range. The vertex is then,

$$x = -\frac{12}{2(-2)} = 3 \qquad y = h(3) = -2(3)^2 + 12(3) + 5 = 23 \qquad \Rightarrow \qquad (3, 23)$$

So, as discussed, we know that this will be the highest point on the graph or the largest value of the function and the parabola will take all values less than this so the range is then,

$$\text{Range}: (-\infty, 23]$$

(d) $f(z) = |z - 6| - 3$

This function contains an absolute value and we know that absolute value will be either positive or zero. In this case the absolute value will be zero if $z = 6$ and so the absolute value portion of this function will always be greater than or equal to zero. We are subtracting 3 from the absolute value portion and so we then know that the range will be,

$$\text{Range}: [-3, \infty)$$

We can plug any value into an absolute value and so the domain is once again all real numbers or,

$$\text{Domain}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

(e) $g(x) = 8$

This function may seem a little tricky at first but is actually the easiest one in this set of examples. This is a constant function and so an value of *x* that we plug into the function will yield a value of 8. This means that the range is a single value or,

$$\text{Range}: 8$$

The domain is all real numbers,

$$\text{Domain}: -\infty < x < \infty \quad \text{or} \quad (-\infty, \infty)$$

In general determining the range of a function can be somewhat difficult. As long as we restrict ourselves down to "simple" functions, some of which we looked at in the previous example, finding the range is not too bad, but for most functions it can be a difficult process.

Because of the difficulty in finding the range for a lot of functions we had to keep those in the previous set somewhat simple, which also meant that we couldn't really look at some of the more complicated domain examples that are liable to be important in a Calculus course. So, let's take a look at another set of functions only this time we'll just look for the domain.

Example 5 Find the domain of each of the following functions.

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

(b) $g(t) = \sqrt{6+t-t^2}$

(c) $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

Solution

(a) $f(x) = \dfrac{x-4}{x^2-2x-15}$

Okay, with this problem we need to avoid division by zero and so we need to determine where the denominator is zero which means solving,

$$x^2-2x-15 = (x-5)(x+3) = 0 \qquad \Rightarrow \qquad x=-3,\ x=5$$

So, these are the only values of *x* that we need to avoid and so the domain is,

$$\text{Domain : All real numbers except } x=-3 \ \&\ x=5$$

(b) $g(t) = \sqrt{6+t-t^2}$

In this case we need to avoid square roots of negative numbers and so need to require that,

$$6+t-t^2 \geq 0 \qquad \Rightarrow \qquad t^2-t-6 \leq 0$$

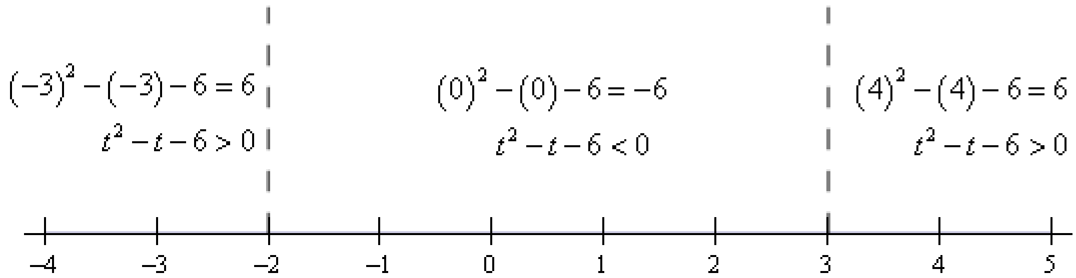
Note that we multiplied the whole inequality by -1 (and remembered to switch the direction of the inequality) to make this easier to deal with. You'll need to be able to solve inequalities like this more than a few times in a Calculus course so let's make sure you can solve these.

The first thing that we need to do is determine where the function is zero and that's not too difficult in this case.

$$t^2-t-6 = (t-3)(t+2) = 0$$

So, the function will be zero at $t=-2$ and $t=3$. Recall that these points will be the only place where the function *may* change sign. It's not required to change sign at these points, but these will be the only points where the function can change sign. This means that all we need to do is break up a number line into the three regions that avoid these two points and test the sign of the function at a single point in each of the regions. If the function is positive at a single point in the region it will be positive at all points in that region because it doesn't contain the any of the points where the function may change sign. We'll have a similar situation if the function is negative for the test point.

So, here is a number line showing these computations.



From this we can see that the only region in which the quadratic (in its modified form) will be negative is in the middle region. Recalling that we got to the modified region by multiplying the quadratic by a -1 this means that the quadratic under the root will only be positive in the middle region and so the domain for this function is then,

$$\text{Domain}: \ -2 \le t \le 3 \qquad \text{or} \qquad [-2,3]$$

(c) $h(x) = \dfrac{x}{\sqrt{x^2-9}}$

In this case we have a mixture of the two previous parts. We have to worry about division by zero and square roots of negative numbers. We can cover both issues by requiring that,

$$x^2 - 9 > 0$$

Note that we need the inequality here to be strictly greater than zero to avoid the division by zero issues. We can either solve this by the method from the previous example or, in this case, it is easy enough to solve by inspection. The domain is this case is,

$$\text{Domain}: \ x < -3 \ \& \ x > 3 \qquad \text{or} \qquad (-\infty,-3) \ \& \ (3,\infty)$$

The next topic that we need to discuss here is that of **function composition**. The composition of *f(x)* and *g(x)* is

$$(f \circ g)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 6 Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(f \circ g)(5)$

(b) $(f \circ g)(x)$

(c) $(g \circ f)(x)$

(d) $(g \circ g)(x)$

Solution

(a) $\left(f \circ g\right)(5)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}\left(f \circ g\right)(5) &= f\left(g(5)\right)\\ &= f(-99) = 29512\end{aligned}$$

(b) $\left(f \circ g\right)(x)$

$$\begin{aligned}\left(f \circ g\right)(x) &= f\left(g(x)\right)\\ &= f(1-20x)\\ &= 3(1-20x)^2 - (1-20x)+10\\ &= 3\left(1-40x+400x^2\right)-1+20x+10\\ &= 1200x^2-100x+12\end{aligned}$$

Compare this answer to the next part and notice that answers are NOT the same. The order in which the functions are listed is important!

(c) $\left(g \circ f\right)(x)$

$$\begin{aligned}\left(g \circ f\right)(x) &= g\left(f(x)\right)\\ &= g\left(3x^2-x+10\right)\\ &= 1-20\left(3x^2-x+10\right)\\ &= -60x^2+20x-199\end{aligned}$$

And just to make the point. This answer is different from the previous part. Order is important in composition.

(d) $\left(g \circ g\right)(x)$

In this case do not get excited about the fact that it's the same function. Composition still works the same way.

$$\begin{aligned}\left(g \circ g\right)(x) &= g\left(g(x)\right)\\ &= g(1-20x)\\ &= 1-20(1-20x)\\ &= 400x-19\end{aligned}$$

Let's work one more example that will lead us into the next section.

Example 7 Given $f(x) = 3x-2$ and $g(x) = \frac{1}{3}x+\frac{2}{3}$ find each of the following.

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

(a)

$$\begin{aligned}(f \circ g)(x) &= f\left(g(x)\right)\\ &= f\left(\frac{1}{3}x+\frac{2}{3}\right)\\ &= 3\left(\frac{1}{3}x+\frac{2}{3}\right)-2\\ &= x+2-2 = x\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g\left(f(x)\right)\\ &= g(3x-2)\\ &= \frac{1}{3}(3x-2)+\frac{2}{3}\\ &= x-\frac{2}{3}+\frac{2}{3} = x\end{aligned}$$

The **difference quotient** of a function $f(x)$ is defined to be,

$$\frac{f(x+h)-f(x)}{h}$$

Compute the difference quotient of the given function.

$f(t) = 2t^2 - 3t + 9$

$$\begin{aligned}f(t+h) &= 2(t+h)^2 - 3(t+h) + 9 = 2\left(t^2+2th+h^2\right) - 3t - 3h\\ &= 2t^2 + 4th + 2h^2 - 3t - 3h + 9\end{aligned}$$

.

$$f(t+h) - f(t) = 2t^2 + 4th + 2h^2 - 3t - 3h + 9 - \left(2t^2 - 3t + 9\right) = 4th + 2$$

$$\frac{f(t+h)-f(t)}{h} = \frac{4th+2h^2-3h}{h} = 4t+2h-3$$

14. Determine all the roots of $W(x) = x^4 + 6x^2 - 27$.

Set the function equal to zero and factor the left side as much as possible.

$$x^4 + 6x^2 - 27 = \left(x^2 - 3\right)\left(x^2 + 9\right) = 0$$

Don't so locked into quadratic equations that the minute you see an equation that is not quadratic you decide you can't deal with it. While this function was not a quadratic it still factored in an obvious manner.

Now, the second term will never be zero (for any real value of *x* anyway and in this class those tend to be the only ones we are interested in) and so we can ignore that term. The first will be zero if,

$$x^2 - 3 = 0 \qquad \Rightarrow \qquad x^2 = 3 \qquad \Rightarrow \qquad x = \pm\sqrt{3}$$

Find the domain and range of the given function.

Find the domain and range of $Y\left(t\right) = 3t^2 - 2t + 1$

This is a polynomial (a 2nd degree polynomial in fact) and so we know that we can plug any value of *t* into the function and so the domain is all real numbers or,

$$\text{Domain} : \ -\infty < t < \infty \ \text{ or } \left(-\infty, \infty\right)$$

The graph of this 2nd degree polynomial (or quadratic) is a **parabola** that opens upwards (because the coefficient of the t^2 t^2 is positive) and so we know that the vertex will be the lowest point on the graph. This also means that the function will take on all values greater than or equal to the *y*-coordinate of the vertex which will in turn give us the range.

So, we need the vertex of the parabola. The *t*-coordinate is,

$$t = -\frac{-2}{2\left(3\right)} = \frac{1}{3}$$

and the *y* coordinate is then, $Y\left(\frac{1}{3}\right) = \frac{2}{3}$.

The range is then,

$$\text{Range} : \left[\frac{2}{3}, \infty\right)$$

Find the domain and range of $f\left(z\right) = 2 + \sqrt{z^2 + 1}$.

We know that when we have square roots that we can't take the square root of a negative number. However, because,

$$z^2 + 1 \geq 1$$

we will never be taking the square root of a negative number in this case and so the domain is all real numbers or,

$$\text{Domain} : -\infty < z < \infty \text{ or } (-\infty, \infty)$$

For the range we need to recall that square roots will only return values that are positive or zero and in fact the only way we can get zero out of a square root will be if we take the square root of zero. For our function, as we've already noted, the quantity that is under the root is always at least 1 and so this root will never be zero. Also recall that we have the following fact about square roots,

$$\text{If } x \geq 1 \text{ then } \sqrt{x} \geq 1$$

So, we now know that,

$$\sqrt{z^2 + 1} \geq 1$$

Finally, we are adding 2 onto the root and so we know that the function must always be greater than or equal to 3 and so the range is,

$$\text{Range} : [3, \infty)$$

22. Find the domain and range of $M(x) = 5 - |x + 8|$.

We're dealing with an absolute value here and the quantity inside is a line, which we can plug all values of *x* into, and so the domain is all real numbers or,

$$\text{Domain} : -\infty < x < \infty \text{ or } (-\infty, \infty)$$

For the range let's again note that the quantity inside the absolute value is a linear function that will take on all real values. We also know that absolute value functions will never be negative and will only be zero if we take the absolute value of zero. So we now know that,

$$|x + 8| \geq 0$$

However, we are subtracting this from 5 and so we'll be subtracting a positive or zero number from 5 and so the range is,

$$\text{Range} : (-\infty, 5]$$

find the domain of the given function.

23. $f(w) = \dfrac{w^3 - 3w + 1}{12w - 7}$

In this case we need to avoid division by zero issues and so we'll need to determine where the denominator is zero. To do this we will solve,

$$12w-7=0 \qquad \Rightarrow \qquad w$$

We can plug all other values of *w* into the function without any problems and so the domain is,

$$\text{Domain : All real numbers except } w=\frac{7}{1:}$$

25. $g(t)=\dfrac{6t-t^3}{7-t-4t^2}$

$$\text{Domain : All real numbers except } t=-\frac{1}{8}\left(1\pm\sqrt{113}\right)$$

29. Find the domain of $f(z)=\sqrt{z-1}+\sqrt{z+6}$.

The domain of this function will be the set of all *z*'s that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

$$z-1\geq 0 \qquad \Rightarrow \qquad z$$

For the second term we need to require,

$$z+6\geq 0 \qquad \Rightarrow \qquad z\geq -6$$

Compute $(f\circ g)(x)$ and $(g\circ f)(x)$ for each of the given pair of functions

$$(f\circ g)(x)=f\left[g(x)\right]=f\left[x^2-14x\right]=5\left(x^2-14x\right)+2=5x^2-70x+2$$

$$(g\circ f)(x)=g\left[f(x)\right]=g\left[5x+2\right]=(5x+2)^2-14(5x+2)=25x^2-50x-24$$