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Calculus and Analytical Geometry

LECTURE #9

Lecture#9(Definite Integrals)

- Unit 9.1: Riemann Sum-Example
- Unit 9.2: Area under a curve as definite Integral-Example
- Unit 9.3: Properties of definite Integral-Example
- Unit 9.4: Theorem of definite integral-Example
- Unit 9.5: Fundamental Theorem of Integral Calculus-1&2
- Unit 9.6: Relation b/w Definite and Indefinite Integral with Example
- Unit 9.7: Mean Value Theorem for Definite Integral-Example

Riemann Sum

The theory of limits of finite approximations was made precise by the German mathematician Bernhard Riemann. We now introduce the notion of a *Riemann sum*, which underlies

We begin with an arbitrary function f defined on a closed interval $[a, b]$.

$$a < x_1 < x_2 < \cdots < x_{n-1} < b.$$

To make the notation consistent, we denote a by x_0 and b by x_n, so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

The set

$$P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$$

is called a **partition** of $[a, b]$.

The partition P divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

The first of these subintervals is $[x_0, x_1]$, the second is $[x_1, x_2]$, and the **kth subinterval of** P is $[x_{k-1}, x_k]$, for k an integer between 1 and n.

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k.$$

The sum S_P is called a **Riemann sum for f on the interval** $[a, b]$.

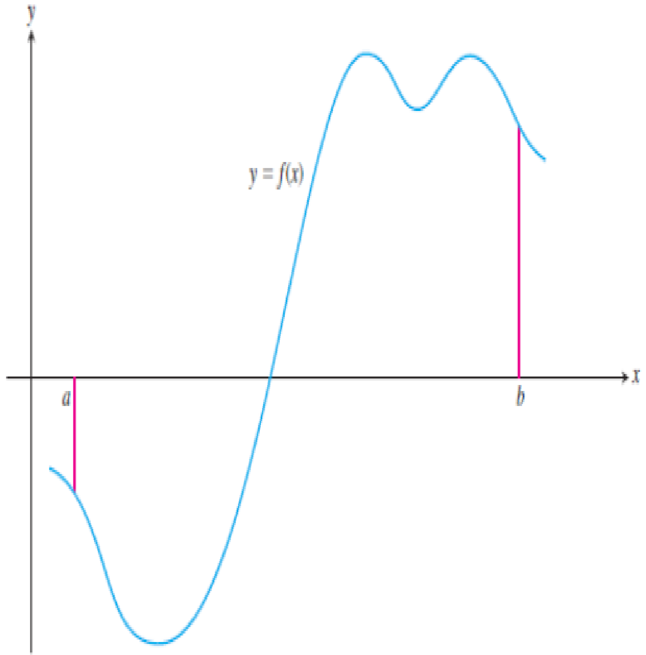
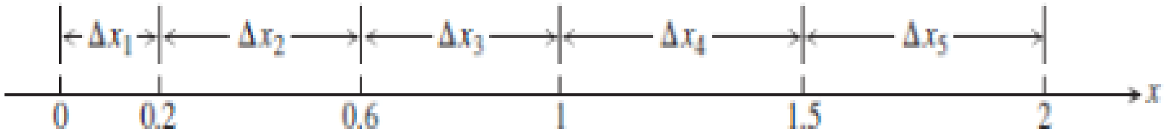


FIGURE 5.8 A typical continuous function $y = f(x)$ over a closed interval $[a, b]$.

Riemann Sum

EXAMPLE 6 Partitioning a Closed Interval

The set $P = \{0, 0.2, 0.6, 1, 1.5, 2\}$ is a partition of $[0, 2]$. There are five subintervals of P: $[0, 0.2]$, $[0.2, 0.6]$, $[0.6, 1]$, $[1, 1.5]$, and $[1.5, 2]$:



The lengths of the subintervals are $\Delta x_1 = 0.2$, $\Delta x_2 = 0.4$, $\Delta x_3 = 0.4$, $\Delta x_4 = 0.5$, and $\Delta x_5 = 0.5$. The longest subinterval length is 0.5, so the norm of the partition is $\|P\| = 0.5$. In this example, there are two subintervals of this length. ■

Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:

Upper limit of integration

The function is the integrand.

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

Area Under a Curve as a Definite Integral

DEFINITION **Area Under a Curve as a Definite Integral**

If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$

► **Example 1** Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

$$\text{(a)} \int_1^4 2\,dx \qquad \text{(b)} \int_{-1}^2 (x+2)\,dx \qquad \text{(c)} \int_0^1 \sqrt{1-x^2}\,dx$$

Solution (a). The graph of the integrand is the horizontal line $y = 2$, so the region is a rectangle of height 2 extending over the interval from 1 to 4 (Figure 4.5.4a). Thus,

$$\int_1^4 2\,dx = (\text{area of rectangle}) = 2(3) = 6$$

Solution (b). The graph of the integrand is the line $y = x + 2$, so the region is a trapezoid whose base extends from $x = -1$ to $x = 2$ (Figure 4.5.4b). Thus,

$$\int_{-1}^2 (x+2)\,dx = (\text{area of trapezoid}) = \frac{1}{2}(1+4)(3) = \frac{15}{2}$$

Area Under a Curve as a Definite Integral

Solution (c). The graph of $y = \sqrt{1 - x^2}$ is the upper semicircle of radius 1, centered at the origin, so the region is the right quarter-circle extending from $x = 0$ to $x = 1$ (Figure 4.5.4*c*). Thus,

$$\int_0^1 \sqrt{1 - x^2}\, dx = (\text{area of quarter-circle}) = \frac{1}{4}\pi(1^2) = \frac{\pi}{4}$$

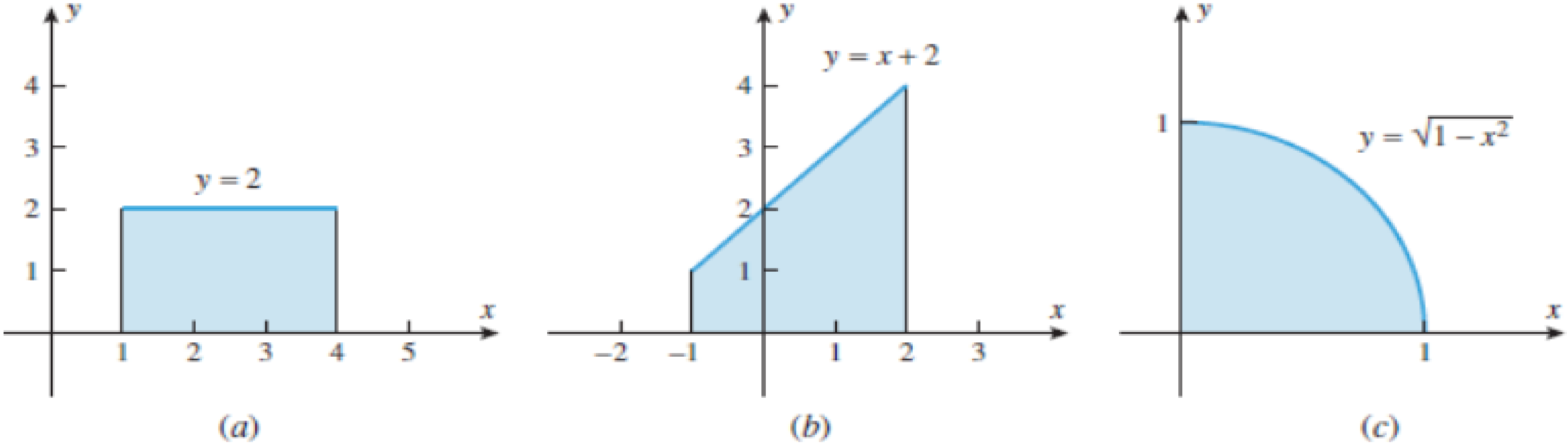


Figure 4.5.4

Area Under a Curve as a Definite Integral

Example 2 Evaluate

$$\text{(a)} \int_0^2 (x-1)\,dx \qquad \text{(b)} \int_0^1 (x-1)\,dx$$

Solution. The graph of $y = x - 1$ is shown in Figure 4.5.5, and we leave it for you to verify that the shaded triangular regions both have area $\frac{1}{2}$. Over the interval $[0, 2]$ the net signed area is $A_1 - A_2 = \frac{1}{2} - \frac{1}{2} = 0$, and over the interval $[0, 1]$ the net signed area is $-A_2 = -\frac{1}{2}$. Thus,

$$\int_0^2 (x-1)\,dx = 0 \quad \text{and} \quad \int_0^1 (x-1)\,dx = -\frac{1}{2}$$

(Recall that in Example 7 of Section 4.4, we used Definition 4.4.5 to show that the net signed area between the graph of $y = x - 1$ and the interval $[0, 2]$ is zero.)

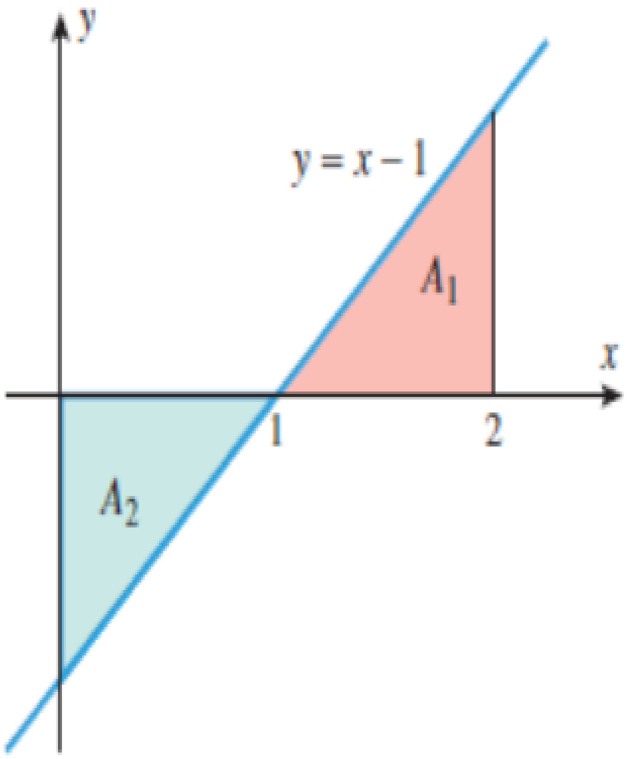


Figure 4.5.5

Total Area between the Curve

If f is a continuous function on the interval $[a, b]$, then we define the *total area* between the curve $y = f(x)$ and the interval $[a, b]$ to be

$$\text{total area} = \int_a^b |f(x)|\,dx \tag{7}$$

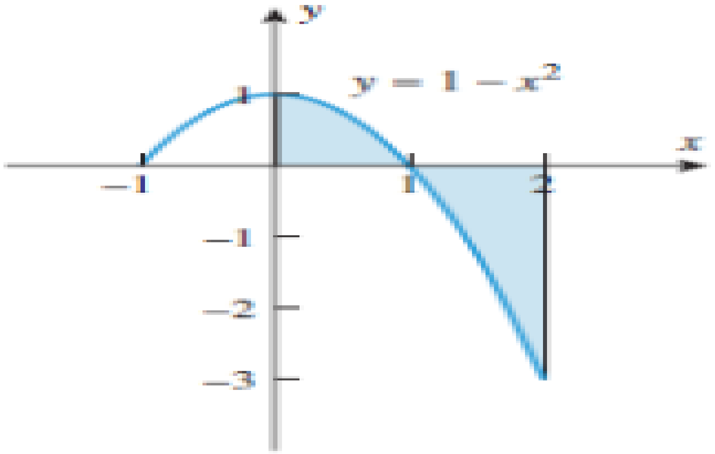
▶ **Example 8** Find the total area between the curve $y = 1 - x^2$ and the x-axis over the interval $[0, 2]$ (Figure 4.6.7).

Solution. The area A is given by

$$A = \int_0^2 |1 - x^2|\,dx = \int_0^1 (1 - x^2)\,dx + \int_1^2 -(1 - x^2)\,dx$$

$$= \left[x - \frac{x^3}{3}\right]_0^1 - \left[x - \frac{x^3}{3}\right]_1^2$$

$$= \frac{2}{3} - \left(-\frac{4}{3}\right) = 2 \quad ◀$$



▲ Figure 4.6.7

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Properties of Definite Integral

4.5.3 DEFINITION

(a) If a is in the domain of f, we define

$$\int_a^a f(x)\,dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

► Example 3

(a) $\int_1^1 x^2\,dx = 0$

(b) $\int_1^0 \sqrt{1-x^2}\,dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\frac{\pi}{4}$ ◄

Example 1(c)

Theorem of Definite Integral

4.5.4 THEOREM *If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and $f-g$ are integrable on $[a, b]$ and*

(a) $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

(b) $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(c) $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

► **Example 4** Evaluate

$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx$$

Solution. From parts (a) and (c) of Theorem 4.5.4 we can write

$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx = \int_0^1 5\,dx - \int_0^1 3\sqrt{1-x^2}\,dx = \int_0^1 5\,dx - 3\int_0^1 \sqrt{1-x^2}\,dx$$

The first integral in this difference can be interpreted as the area of a rectangle of height 5 and base 1, so its value is 5, and from Example 1 the value of the second integral is $\pi/4$. Thus,

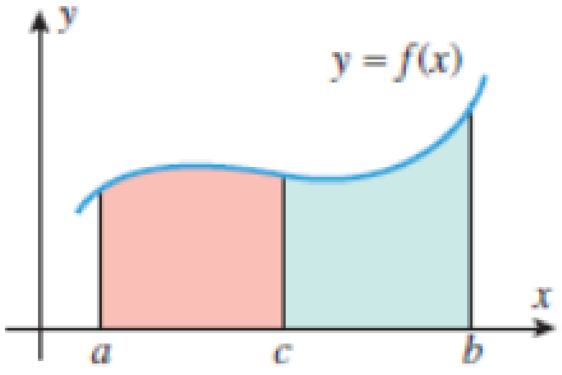
$$\int_0^1 (5 - 3\sqrt{1-x^2})\,dx = 5 - 3\left(\frac{\pi}{4}\right) = 5 - \frac{3\pi}{4}$$ ◄

Theorem of Definite Integral

4.5.5 THEOREM *If f is integrable on a closed interval containing the three points a, b, and c, then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

no matter how the points are ordered.



▲ **Figure 4.5.7**

Fundamental Theorem of Integral Calculus

4.6.1 THEOREM *(The Fundamental Theorem of Calculus, Part 1)* *If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a) \tag{2}$$

Example 1 Evaluate $\int_1^2 x\,dx$.

Solution. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus, from (2)

$$\int_1^2 x\,dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

Example 2 In Example 5 of Section 4.4 we used the definition of area to show that the area under the graph of $y = 9 - x^2$ over the interval $[0, 3]$ is 18 (square units). We can now solve that problem much more easily using the Fundamental Theorem of Calculus:

$$A = \int_0^3 (9 - x^2)\,dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18$$

Fundamental Theorem of Integral Calculus

► Example 3

(a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$ (Figure 4.6.4).
(b) Make a conjecture about the value of the integral

$$\int_0^{\pi} \cos x \, dx$$

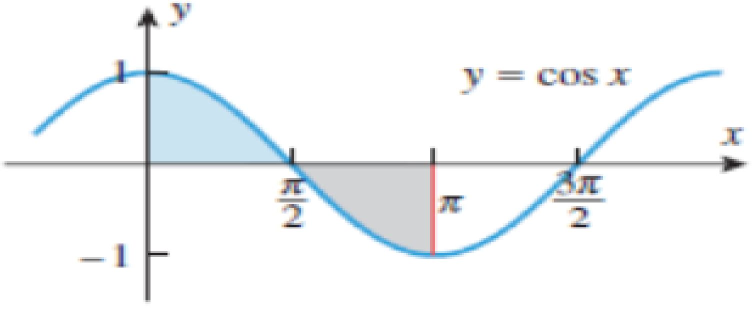
and confirm your conjecture using the Fundamental Theorem of Calculus.

Solution (a). Since $\cos x \geq 0$ over the interval $[0, \pi/2]$, the area A under the curve is

$$A = \int_0^{\pi/2} \cos x \, dx = \sin x \Big]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1$$

Solution (b). The given integral can be interpreted as the signed area between the graph of $y = \cos x$ and the interval $[0, \pi]$. The graph in Figure 4.6.4 suggests that over the interval $[0, \pi]$ the portion of area above the x-axis is the same as the portion of area below the x-axis, so we conjecture that the signed area is zero; this implies that the value of the integral is zero. This is confirmed by the computations

$$\int_0^{\pi} \cos x \, dx = \sin x \Big]_0^{\pi} = \sin \pi - \sin 0 = 0 ◄$$



▲ **Figure 4.6.4**

Fundamental Theorem of Integral Calculus

4.6.3 THEOREM (*The Fundamental Theorem of Calculus, Part 2*) *If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by*

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x) \tag{11}$$

► **Example 10** Find

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right]$$

Solution. The integrand is a continuous function, so from (11)

$$\frac{d}{dx}\left[\int_1^x t^3\,dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3\,dt = \frac{t^4}{4}\bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree. ◄

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Relation b/w Definite and Indefinite Integral

THE RELATIONSHIP BETWEEN DEFINITE AND INDEFINITE INTEGRALS

$$\int_a^b f(x)\,dx = \int f(x)\,dx\bigg]_a^b$$

Example 4

$$\int_1^9 \sqrt{x}\,dx = \int x^{1/2}\,dx\bigg]_1^9 = \frac{2}{3}x^{3/2}\bigg]_1^9 = \frac{2}{3}(27-1) = \frac{52}{3}$$

Example 5 Table 4.2.1 will be helpful for the following computations.

$$\int_4^9 x^2\sqrt{x}\,dx = \int_4^9 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\bigg]_4^9 = \frac{2}{7}(2187-128) = \frac{4118}{7} = 588\frac{2}{7}$$

$$\int_0^{\pi/2} \frac{\sin x}{5}\,dx = -\frac{1}{5}\cos x\bigg]_0^{\pi/2} = -\frac{1}{5}\left[\cos\left(\frac{\pi}{2}\right) - \cos 0\right] = -\frac{1}{5}[0-1] = \frac{1}{5}$$

$$\int_0^{\pi/3} \sec^2 x\,dx = \tan x\bigg]_0^{\pi/3} = \tan\left(\frac{\pi}{3}\right) - \tan 0 = \sqrt{3} - 0 = \sqrt{3}$$

$$\int_{-\pi/4}^{\pi/4} \sec x \tan x\,dx = \sec x\bigg]_{-\pi/4}^{\pi/4} = \sec\left(\frac{\pi}{4}\right) - \sec\left(-\frac{\pi}{4}\right) = \sqrt{2} - \sqrt{2} = 0$$

Relation b/w Definite and Indefinite Integral

► Example 6

$$\int_1^1 x^2\,dx = \frac{x^3}{3}\Big]_1^1 = \frac{1}{3} - \frac{1}{3} = 0$$

$$\int_4^0 x\,dx = \frac{x^2}{2}\Big]_4^0 = \frac{0}{2} - \frac{16}{2} = -8$$

The latter result is consistent with the result that would be obtained by first reversing the limits of integration in accordance with Definition 4.5.3(b):

$$\int_4^0 x\,dx = -\int_0^4 x\,dx = -\frac{x^2}{2}\Big]_0^4 = -\left[\frac{16}{2} - \frac{0}{2}\right] = -8 \blacktriangleleft$$

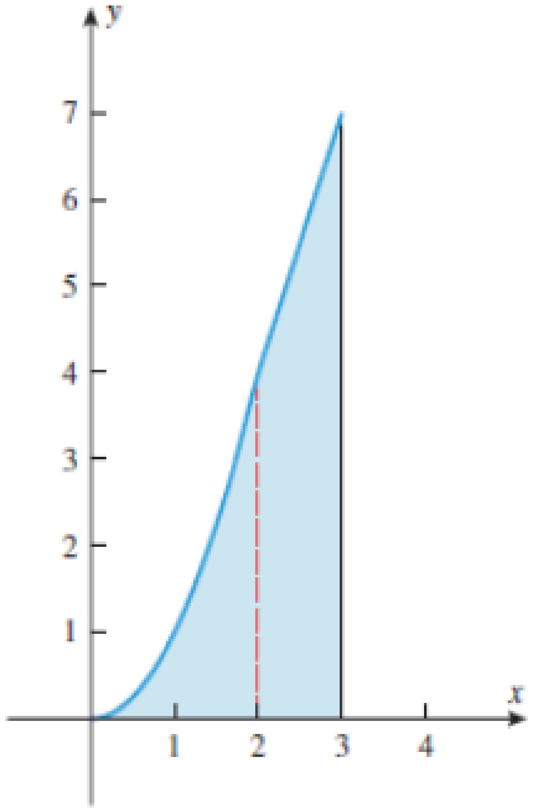
► Example 7 Evaluate $\int_0^3 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2 \end{cases}$$

Solution. See Figure 4.6.5. From Theorem 4.5.5 we can integrate from 0 to 2 and from 2 to 3 separately and add the results. This yields

$$\int_0^3 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^3 (3x - 2)\,dx$$

$$= \frac{x^3}{3}\Big]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \left(\frac{8}{3} - 0\right) + \left(\frac{15}{2} - 2\right) = \frac{49}{6} \blacktriangleleft$$



▲ Figure 4.6.5

Mean Value Theorem for Definite Integral

THEOREM 3 The Mean Value Theorem for Definite Integrals

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

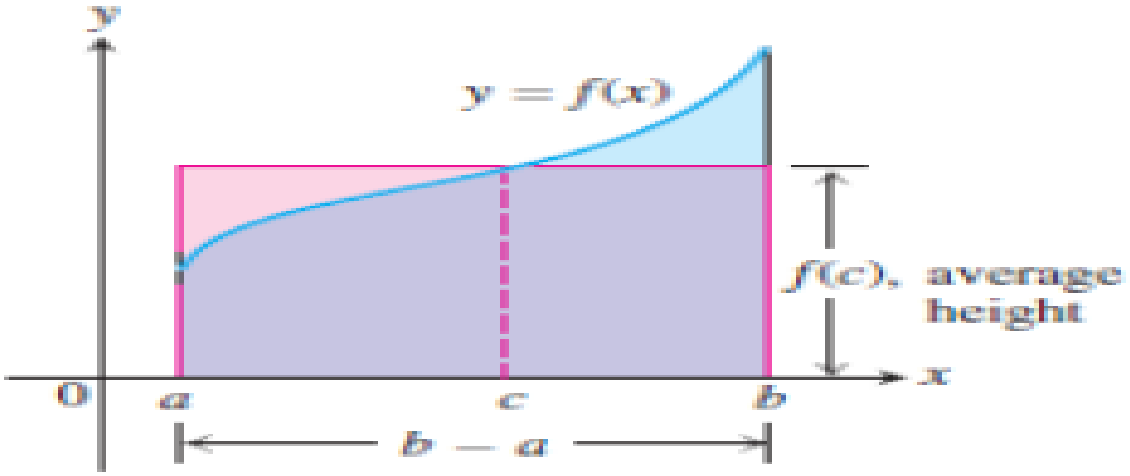


FIGURE 5.16 The value $f(c)$ in the Mean Value Theorem is, in a sense, the average (or *mean*) height of f on $[a, b]$. When $f \geq 0$, the area of the rectangle is the area under the graph of f from a to b,

$$f(c)(b-a) = \int_a^b f(x)\,dx.$$

Mean Value Theorem of Definite Integral

EXAMPLE 1 Applying the Mean Value Theorem for Integrals

Find the average value of $f(x) = 4 - x$ on $[0, 3]$ and where f actually takes on this value at some point in the given domain.

Solution

$$\begin{aligned}
\operatorname{av}(f) &= \frac{1}{b-a}\int_a^b f(x)\,dx \\
&= \frac{1}{3-0}\int_0^3 (4-x)\,dx = \frac{1}{3}\left(\int_0^3 4\,dx - \int_0^3 x\,dx\right) \\
&= \frac{1}{3}\left(4(3-0) - \left(\frac{3^2}{2} - \frac{0^2}{2}\right)\right) \qquad \text{Section 5.3, Eqs. (1) and (2)} \\
&= 4 - \frac{3}{2} = \frac{5}{2}.
\end{aligned}$$

The average value of $f(x) = 4 - x$ over $[0, 3]$ is $5/2$. The function assumes this value when $4 - x = 5/2$ or $x = 3/2$. (Figure 5.18) ■

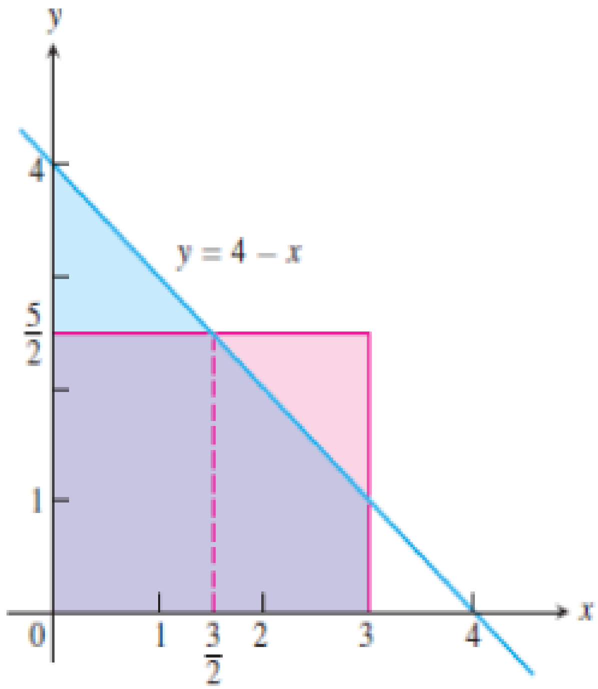


FIGURE 5.18 The area of the rectangle with base $[0, 3]$ and height $5/2$ (the average value of the function $f(x) = 4 - x$) is equal to the area between the graph of f and the x-axis from 0 to 3 (Example 1).

Mean Value Theorem of Definite Integral

EXAMPLE 2 Show that if f is continuous on $[a, b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0$ at least once in $[a, b]$.

Solution The average value of f on $[a, b]$ is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{b-a}\cdot 0 = 0.$$

By the Mean Value Theorem, f assumes this value at some point $c \in [a, b]$.

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Mean Value Theorem of Definite Integral

► **Example 9** Since $f(x) = x^2$ is continuous on the interval $[1, 4]$, the Mean-Value Theorem for Integrals guarantees that there is a point x^* in $[1, 4]$ such that

$$\int_1^4 x^2\,dx = f(x^*)(4-1) = (x^*)^2(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2\,dx = \frac{x^3}{3}\bigg]_1^4 = 21$$

so that

$$3(x^*)^2 = 21 \quad \text{or} \quad (x^*)^2 = 7 \quad \text{or} \quad x^* = \pm\sqrt{7}$$

Thus, $x^* = \sqrt{7} \approx 2.65$ is the point in the interval $[1, 4]$ whose existence is guaranteed by the Mean-Value Theorem for Integrals. ◄

Thank you