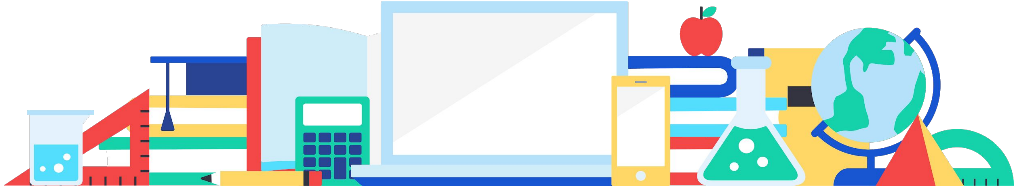
Algebra Concepts & Vocab

Updated March 2020

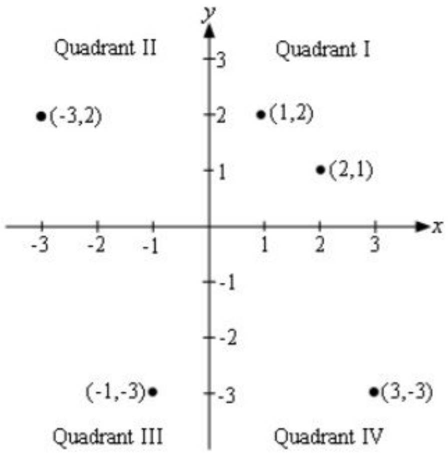


Graphing and Common Graphs

The Cartesian Coordinate System

Key Vocabulary

- **Ordered pair/coordinates** - (x,y); first number is x-coordinate, second number is y-coordinate.
- **Origin** - point where x and y axes cross, at (0,0).
- **Quadrant** - each of four parts of the graph, divided by the x- and y- axes.
  - Quadrant I - x positive, y positive
  - Quadrant II - x negative, y positive
  - Quadrant III - x negative, y negative
  - Quadrant IV - x positive, y negative



The Cartesian Coordinate System

Graphing an Equation

- ➔ A graph is the set of all the ordered pairs whose coordinates satisfy the equation.
- ➔ Pick values of x and plug them into the equation to find the corresponding values of y, then plot the ordered pair given by these values.
- ➔ **x-intercept** - where the graph crosses the x-axis.
  - ◆ To find x-intercepts, set $y = 0$, and solve for x.
- ➔ **y-intercept** - where the graph crosses the y-axis.
  - ◆ To find y-intercepts, set $x = 0$, and solve for y.

Lines

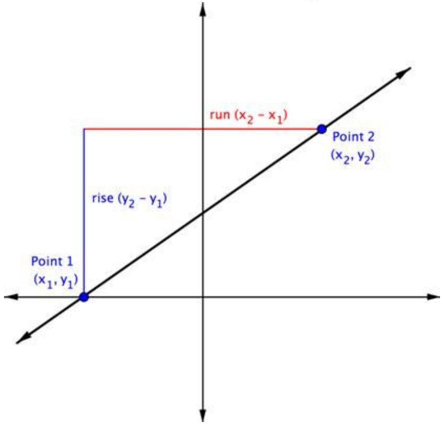
Key Vocabulary

- **Line** - any equation that can be written in the form (standard form) Ax+By=C. A and B cannot both be zero.
  - A line is defined by any two points on the line.
- **Slope** - a measure of the steepness of a line; can also be used to determine whether a line is increasing or decreasing from left to right.
  - Given two points on a line (x_1,y_1) and (x_2,y_2), slope m is given by $m = \frac{y_2-y_1}{x_2-x_1} = \frac{rise}{run}$
  - If slope is positive, then the line is increasing from left to right.
  - If slope is negative, then the line is decreasing from left to right.
    - Rise will be a negative number.
  - The larger the number, the steeper the line.
  - A horizontal line will always have a slope of zero.
    - A horizontal line with y-intercept of b: y = b.
  - A vertical line will always have an undefined slope.
    - A vertical line with x-intercept of a: x = a.

Lines

Key Vocabulary (cont.)

- **Point-slope form** - If a line passes through the point (x_1,y_1) with slope m, then the equation of the line is $y - y_1=m(x - x_1)$. Sometimes written as $y = y_1 + m(x - x_1)$.
- **Slope-intercept form** - If a line has slope m and y-intercept of (0,b), then the equation of the line is $y = mx + b$.
- If there are two lines with slopes m_1 and m_2, then:
  - The lines are **parallel** if $m_1 = m_2$.
  - The lines are **perpendicular** if $m_1m_2 = -1$ or $m_2 = -\frac{1}{m_1}$ or m_2 is the negative reciprocal of m_1.



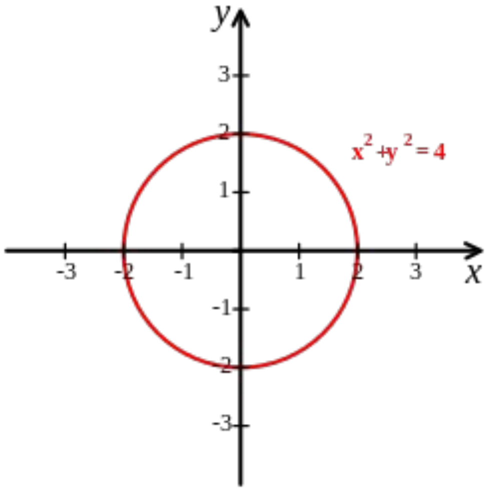
Circles

Graphing Circles

➔ The **standard form of the equation of a circle** with radius r and center:

$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

- rightmost point: (h + r, k)
- leftmost point: (h – r, k)
- topmost point: (h, k + r)
- bottommost point: (h, k – r)



Functions

Functions

Key Vocabulary

- **Relation** - a set of ordered pairs.
- **Function** - a relation for which each value from the set of the first components of the ordered pairs is associated with exactly one value from the set of the second ponents of the ordered pair.
  - Working definition of a function - an equation for which any x that can be plugged into the equation will yield exactly one y out of the equation.
- **Function notation** - f(x) = [insert function here]. Read as “f of x”.
- **Evaluating functions** - find the values of f(x) given a specific value of x.
- **Piecewise functions** - a function that is broken into pieces; the piece you use to evaluate the function depends on the value of x.
- **Domain** - set of all x values that can be plugged into the equation to get a real value for y.
- **Range** - the set of all y values that can result from the equation.
- To graph a function: pick x values, plug into the function to find y values, plot ordered pairs.

Combining Functions

Basic Function Arithmetic

- ➔ Given functions f(x) and g(x):
  $(f + g)(x) = f(x) + g(x)$
  $(f - g)(x) = f(x) - g(x)$
  $(fg)(x) = f(x)g(x)$
  $(f/g)(x) = f(x)/g(x)$

Function Composition

- ➔ Composition of f(x) and g(x):
  $(f \circ g)(x)=f[g(x)]$
- ➔ Composition of g(x) and f(x):
  $(g \circ f)(x)=g[f(x)]$
- ➔ Note: This is not multiplication. It is plugging the second function listed into the first function.

Inverse Functions

Key Vocabulary

- ➔ **One-to-one function** - no two values of x produce the same y.
- ➔ **Inverse function** - Given two one-to-one functions f(x) and g(x), if $(f \circ g)(x)=x$ and $(g \circ f)(x)=x$, then f(x) and g(x) are inverses of each other.
  - ◆ g(x) is the inverse of f(x): $g(x)=f^{-1}(x)$
  - ◆ f(x) is the inverse of g(x): $f(x)=g^{-1}(x)$
- ➔ The graph of the inverse is a reflection of the original function about the line y = x.

Finding the Inverse of a Function

Given f(x), find $f^{-1}(x)$.

1. Replace f(x) with y.
2. Replace every x with y and every y with x.
3. Solve the equation for y.
4. Replace y with $f^{-1}(x)$.
5. Verify answer by checking that $(f \circ f^{-1})(x)=x$ and $(f^{-1} \circ f)(x)=x$.

Common Graphs

Parabolas

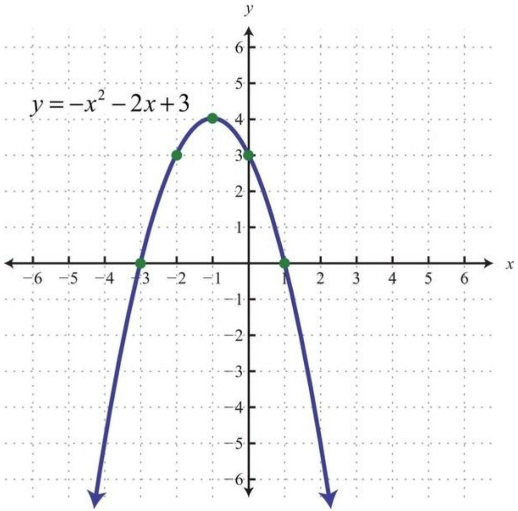
Key Vocabulary

- **Standard form** of a quadratic function - f(x)=ax2+bx+c
- **Parabola** - graph of quadratic function.
- **Vertex** - highest point/lowest point of a parabola.
- **Axis of symmetry** - center line of the parabola.
- **Intercepts** - points where the parabola cross the x- or y-axis.
  - To find y-intercept, set x = 0 and solve for y.
  - To find x-intercept, set y = 0 and solve for x. Note that not all parabolas have x-intercepts.

Parabolas

Sketching Parabolas

1. Find the vertex.
2. Find the y-intercept (0, f(0)).
3. Solve f(x) = 0 to find x-intercepts if they exist. There will be 0, 1, or 2 x-intercepts.
4. Find at least one point on either side of the vertex.
5. Sketch the graph.



Parabolas

Parabolic Tricks

- ➔ $f(x) = a(x - h)^2 + k$
    - If a is positive, then the parabola will open upward. If a is negative, then the parabola will open downward.
    - The vertex of the parabola is the point (h,k).
- ➔ $f(x) = ax^2 + bx + c$
    - If a is positive, then the parabola will open upward. If a is negative, then the parabola will open downward.
    - The vertex of the parabola is given by $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$
    - The y-intercept is $f(0) = a(0)^2 + b(0) + c = c$
- ➔ Convert between two equation forms by completing the square.

Ellipses

- **Standard form of an ellipse** - $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
  - Right side must = 1 to be in standard form
  - center of ellipse: (h, k)
  - rightmost point: (h + a, k)
  - leftmost point: (h − a, k)
  - topmost point: (h, k + b)
  - bottommost point: (h, k − b)
- Equation of an ellipse also $(x - h)^2 + (y - k)^2 = a^2$ if a = b
  - This is standard form of a circle - circles are special cases of ellipses

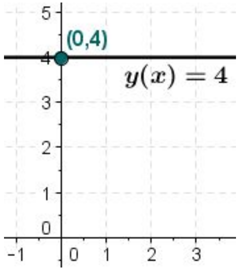
Hyperbolas

Standard Form

Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Center	(h, k)	(h, k)
Opening direction	Left and right	Up and down
Vertices	(h + a, k) and (h − a, k)	(h, k + b) and (h, k − b)
Slope of asymptotes	$\pm ba$	$\pm ba$
Equations of asymptotes	$y=k\pm ba(x-h)$	$y=k\pm ba(x-h)$

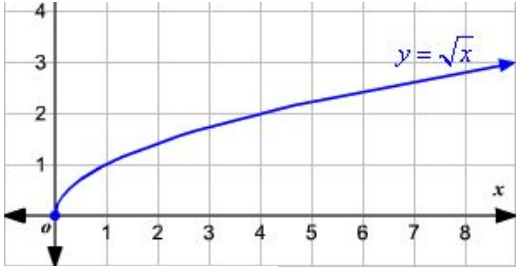
Miscellaneous Functions

→ **Constant function** - f(x) = c, where c = a number

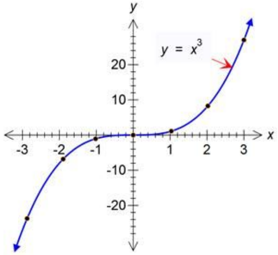


→ **Square root** - $f(x) = \sqrt{x}$, x ≥ 0

- If x < 0, then f(x) is complex, so we restrict x to be ≥0



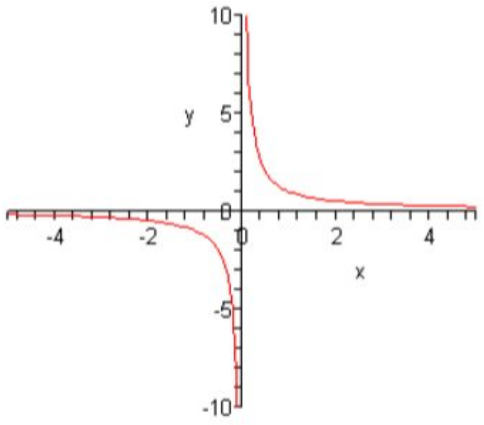
→ **Cubic function** - f(x) = x^3



Rational Functions

Key Concepts

- ➔ $f(x) = \frac{1}{x}$
- ➔ No intercepts, graph is in two pieces.



$(-\frac{b}{2a}, f(-\frac{b}{2a}))$

Rational Functions

Asymptotes

- ➔ The line x = a is a **vertical asymptote** if the graph increases or decreases without bound on one or both sides of the line as x moves in closer and closer to x = a.
- ➔ The line y = b is a **horizontal asymptote** if the graph approaches y = b as x increases or decreases without bound. Note that it doesn't have to approach y = b as x BOTH increases and decreases. It only needs to approach it on one side in order for it to be a horizontal asymptote.
- ➔ **Facts about asymptotes** - Given the rational function
  - ◆ The graph will have a vertical asymptote at x = a $f(x) = \frac{ax^n+...}{bx^m+...}$ minator is zero at x = a and the numerator isn't zero at x = a .
  - ◆ If n < m then the x-axis is the horizontal asymptote.
  - ◆ If n = m then the line y=ab is the horizontal asymptote.
  - ◆ If n > m there will be no horizontal asymptotes.

Rational Functions

How to Graph a Rational Function

1. Find the intercepts, if there are any. Remember that the y-intercept is given by (0, f (0)) and we find the x-intercepts by setting the numerator equal to zero and solving.
2. Find the vertical asymptotes by setting the denominator equal to zero and solving.
3. Find the horizontal asymptote, if it exists, using the fact above.
4. The vertical asymptotes will divide the number line into regions. In each region graph at least one point in each region. This point will tell us whether the graph will be above or below the horizontal asymptote and if we need to we should get several points to determine the general shape of the graph.
5. Sketch the graph.

Manipulating Functions and Symmetry

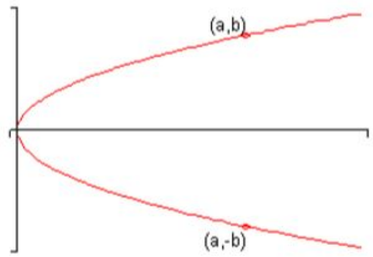
Transformations

Key Vocabulary

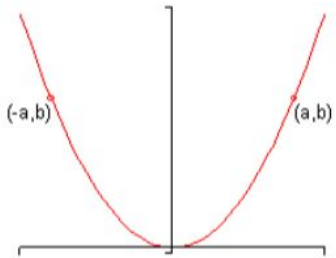
- **Vertical shift** - Given the graph of f(x) the graph of g(x) = f(x) + c will be the graph of f(x) shifted up by c units if c is positive or down by c units if c is negative.
- **Horizontal shift** - Given the graph of f(x) the graph of g(x) = f(x + c) will be the graph of f(x) shifted left by c units if c is positive or right by c units if c is negative.
- **Vertical and horizontal shifts** - If we know the graph of f(x) the graph of g(x) = f(x + c) + k will be the graph of f(x) shifted left or right by c units depending on the sign of c and up or down by k units depending on the sign of k.
- **Reflection about the x-axis** - Given the graph of f(x) then the graph of g(x) = -f(x) is the graph of f(x) reflected about the x-axis. This means that the signs on the all the y coordinates are changed to the opposite sign.
- **Reflection about the y-axis** - Given the graph of f(x) then the graph of g(x) = f(-x) is the graph of f(x) reflected about the y-axis. This means that the signs on the all the x coordinates are changed to the opposite sign.

Symmetry

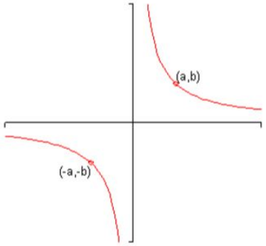
→ A graph is **symmetric about the x-axis** if whenever (a, b) is on the graph then so is (a, -b)



→ A graph is **symmetric about the y-axis** if whenever (a, b) is on the graph then so is (-a, b)



→ A graph is **symmetric about the origin** if whenever (a, b) is on the graph then so is (-a,-b)



Symmetry

Testing for Symmetry

- ➔ A graph will have symmetry about the x-axis if we get an equivalent equation when all the y's are replaced with $-y$.
- ➔ A graph will have symmetry about the y-axis if we get an equivalent equation when all the x's are replaced with $-x$.
- ➔ A graph will have symmetry about the origin if we get an equivalent equation when all the y's are replaced with $-y$ and all the x's are replaced with $-x$.

Exponential and Logarithmic Functions

Exponential Functions

Key Concepts

- **Exponential function** - a function in the form f(x)=bx, where b is the base and x is any real number.
  - b cannot be 0 or 1 for the equation to be a non-constant function. b cannot be negative if the function output is to be real (i.e. not complex).
  - b = 0: $f(x) = 0^x = 0$
  - b = 1: $f(x) = 1^x = 1$
  - b < 0: $f(x) = (-4)^x \rightarrow f(\frac{1}{2}) = (-4)^{\frac{1}{2}} = \sqrt{-4}$
- **e** - a mathematical constant commonly used in applications of exponential equations. e ≈ 2.718281828...

Exponential Functions

Properties of Exponential Functions

- ➔ The graph of f(x) will always contain the point (0,1). Or put another way, f(0)=1 regardless of the value of b.
- ➔ For every possible b, bx > 0 . Note that this implies that bx ≠ 0.
- ➔ If 0 < b < 1 then the graph of bx will decrease as we move from left to right.
- ➔ If b > 1 then the graph of bx will increase as we move from left to right.
- ➔ If bx = by, then x = y.

Logarithmic Functions

Key Vocabulary and Concepts

- Logarithmic function - If b is any number such that b > 0, b ≠ 1, and x > 0, then $y = \log_b x$, which is equivalent to $b^y = x$.
  - Logarithmic form - $y = \log_b x$ (read as "log base b of x)
  - Exponential form - $b^y = x$
- Common logarithm - $\log x = \log_{10} x$
- Natural logarithm - $\ln x = \log_e x$
- Properties of Logarithms
  - $\log_b 1 = 0$. This follows from the fact that b0 = 1.
  - $\log_b b = 1$. This follows from the fact that b1 = b.
  - $\log_b$ bx = x . This can be generalized as logb bf(x) = f(x).
  - b$\log_b$ x = x. This can be generalized as $b^{\text{logb } f(x)} = f(x)$.
  - For the following, assume x > 0 and y > 0.
  - $\log_b (xy) = \log_b x + \log_b y$
  - $\log_b (xy)$ = logb x-$\log_b y$

    Note: $\log_b (x+y) \neq \log_b x + \log_b y$ and $\log_b (x-y) \neq \log_b x - \log_b y$
  - $\log_b (xr) = r \log_b x$
  - If $\log_b x = \log_b y$, then x = y.
- Change of base formula - To change from base a to base b, use $\log_a x = \frac{\log_b x}{\log_b a}$
- To change from base a to common logarithm, use $\log_a x = \frac{\log x}{\log a}$
- To change from base a to natural logarithm, use $\log_a x = \frac{\ln x}{\ln a}$

Solving Equations

Solving Exponential Equations

- Use the fact: "If $b^x = b^y$ then $x = y$" to solve. Note that bases on both sides of the equation must be the same for this property to apply.
- For bases that are not the same, use the logarithm property: $\log_b a' = r \log_b a$.

Solving Logarithmic Equations

- Use the fact: "If $\log_b x = \log_b y$ then $x = y$" to solve. Note that bases on both sides of the equation must be the same for this property to apply.
- For equations where only one term in the equation is a logarithm, use the exponential form of a logarithm: $y = \log_b x \rightarrow b^y = x$.

Applications

Real-World Applications of Exponential and Logarithmic Functions

- **Compound Interest**
  - Starting with P dollars with interest rate of r for t years:
    - If interest is compounded m times per year, then after t years the total amount of money in the account will be $A = P(1 + \frac{r}{m})^{tm}$
    - If interest is compounded continuously, then after t years the total amount of money in the account will be A=Pe $A = Pe^{rt}$
- **Exponential Growth and Decay**
  - $Q=Q_0e^{kt}$, where Q_0 is the positive amount initially present at t = 0, and k is a non-zero constant.
  - If k is positive, then it is exponential growth, and the equation will grow infinitely.
  - If k is negative, then it is exponential decay, and the equation will die down to zero.
- **Earthquake Intensity**
  - Richter scale - used to measure the intensity of an earthquake.
  - $M = \frac{2}{3}log(\frac{E}{E_0})$ where M = earthquake magnitude, E = energy released by the earthquake in joules, $E_0 = 10^{4.4}$ joules.

Solving Linear and Quadratic Equations

Solutions and Solution Sets

Key Vocabulary

- **Solution** - an equation or inequality is any number that, when plugged into the equation/inequality, will satisfy the equation/inequality.
  - Ex: $x = 1$
- **Solution set** - complete set of all solutions of an equation or inequality.
  - Ex: $x = \{-3, 3\}$
- **Set builder notation** - used to express a solution set that is a range of numbers.
  - Ex: $\{z \mid z \geq -5\}$
- **Empty set** - a solution set that contains no solutions, denoted with symbol $\varnothing$.

Linear Equations

- **Linear equation** - any equation that can be written as $ax + b = 0$, where a and b are real numbers and x is a variable.
    - $ax + b = 0$ is also called the **standard form** of a linear equation.

- **Facts about linear equations**
    1. If $a = b$ then $a + c = b + c$ for any c. All this is saying is that we can add a number, c, to both sides of the equation and not change the equation.
    2. If $a = b$ then $a - c = b - c$ for any c. As with the last property we can subtract a number, c, from both sides of an equation.
    3. If $a = b$ then $ac = bc$ for any c. Like addition and subtraction we can multiply both sides of an equation by a number, c, without changing the equation.
    4. If $a = b$ then $a = b$ for any nonzero c. We can divide both sides of an equation by a nonzero number, c, without changing the equation.

Linear Equations

Process for Solving Linear Equations

1. If the equation contains any fractions use the least common denominator to clear the fractions. We will do this by multiplying both sides of the equation by the LCD. Also, if there are variables in the denominators of the fractions identify values of the variable which will give division by zero as we will need to avoid these values in our solution.
2. Simplify both sides of the equation. This means clearing out any parentheses, and combining like terms.
3. Use the first two facts above to get all terms with the variable in them on one side of the equations (combining into a single term of course) and all constants on the other side.
4. If the coefficient of the variable is not equal to 1, use the third or fourth fact above (this will depend on just what the number is) to make the coefficient a one. Note that we usually just divide both sides of the equation by the coefficient if it is an integer or multiply both sides of the equation by the reciprocal of the coefficient if it is a fraction.
5. Verify your answer by plugging the results from the previous steps into the original equation.

Linear Equations

Application of Linear Equations: Word Problems

Process for working through word problems

1. Read the problem carefully, identifying all the given information and what you are being asked to find.
2. Represent one of the unknown quantities with a variable and try to relate all the other unknown quantities (if there are any of course) to this variable.
3. If applicable, sketch a figure illustrating the situation.
4. Form an equation that will relate known quantities to the unknown quantities. To do this, make use of known formulas and the figure sketched in the previous step.
5. Solve the equation formed in the previous step and write down the answer to all the questions. It is important to answer all the questions that you were asked. Often you will be asked for several quantities in the answer and the equation will only give one of them.
6. Check your answer. Do this by plugging into the equation, but also use intuition to make sure that the answer makes sense. Mistakes can often be identified by acknowledging that the answer just doesn't make sense.

Quadratic Functions

Properties of Quadratic Equations

- **Standard form** - $ax^2 + bx + c = 0$, where $a \neq 0$.
- **Zero factor property** - If ab = 0 then either a = 0 and/or b = 0
  - To solve a quadratic equation by factoring, first move all the terms to one side of the equation so the equation can be factored and use the zero factor property (where one side of the equation = 0).
- **Square root property** - If $p^2 = d$, then $p = \pm d$
- **Completing the square** - To get a factorable quadratic equation from $x^2 + bx$, add (b2) $\left(\frac{b}{2}\right)^2$et $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- **Quadratic formula** - $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - Plug a, b, and c into the quadratic formula to get the solutions of the quadratic equation.

Quadratic Functions

Process for Solving Quadratic Equations

1. Is it clearly a square root property problem? In other words, does the equation consist ONLY of something squared and a constant? If this is true, then the square root property is probably the easiest method for use.
2. Does it factor? If so, that is probably the way to go. Note that you shouldn't spend a lot of time trying to determine if the quadratic equation factors. Look at the equation and if you can quickly determine that it factors then go with that. If you can't quickly determine that it factors then don't worry about it.
3. If you've reached this point then you've determined that the equation is not in the correct form for the square root property and that it doesn't factor (or that you can't quickly see that it factors). So, at this point your only real option is the quadratic formula.

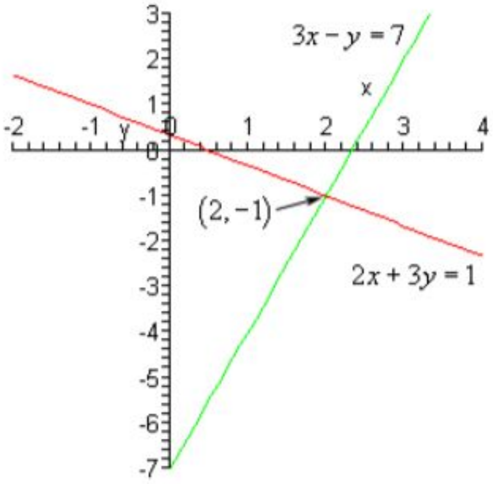
➔ **Discriminant** - in the quadratic formula, the discriminant ($b^2 - 4ac$) will determine which solution set solves the quadratic equation.

➔ **Possible solution sets**

- If $b^2 - 4ac > 0$, then two real distinct (i.e. not equal) solutions.
- If $b^2 - 4ac = 0$, then a double root (recall this arises when we can factor the equation into a perfect square),
- If $b^2 - 4ac < 0$, then two complex solutions.

Systems of Equations

Linear Systems with Two Variables



Key Concepts

- **Linear system of two equations with two variables** - any system that can be written as **ax + by = p and cx + dy = q** where any of the constants (a, b, c, d) can be 0, except that each equation must have at least one variable in it.
  - System is linear if the variables are only to the first power, only in the numerator, and there are no products of variables in any of the equations.
- A **solution to a system of equations** is a value of x and a value of y that, when substituted into the equations, satisfies both equations at the same time.
- Graphically, the solution to a system of linear equations is where the two lines intersect.
- **Method of Substitution** - solve one of the equations for one variable, then substitute this into the other equation and solve, then substitute the solution back into the first equation to find the other variable.
- **Method of Elimination** - multiply one or both of the equations by appropriate numbers so that one of the variables will have the same coefficient with opposite signs, then add equations together. Then solve for one variable, then substitute answer back into the one of the equations to find the other variable.
- If the lines are **parallel**, then the lines never intersect, so there is **no solution** and the system is inconsistent.
- If there are an **infinite number** of solutions, the system is **dependent**.
- For **systems with three variables**, use the same methods as systems with two variables to solve.

Augmented Matrices

Key Concepts

- **Augmented matrix** - a matrix of numbers in which each row represents the constants from one equation (both the coefficients and the constant on the other side of the equal sign) and each column represents all the coefficients for a single variable
- **Elementary Row Operations**
  - Interchange two rows
  - Multiply a row by a constant
  - Add a multiple of a row to another row

Augmented Matrices

Gauss-Jordan Elimination

For the system ax + by = p and cx + dy = q

1. Write the augmented matrix for the system:

$$\left[\begin{array}{cc|c} a & b & p \\ c & d & q \end{array}\right]$$

2. Use elementary row operations to convert it into:

$$\left[\begin{array}{cc|c} 1 & 0 & h \\ 0 & 1 & k \end{array}\right]$$

3. The solution to the system will be x = h and y = k.

If solving for three variables, write the augmented matrix in the same way, then use row operations to convert it to the form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right]$$

Augmented Matrices

Fact: Given any system of equations there are exactly three possibilities for the solution.

1. **There will not be a solution.** If we have a row in which all the entries except for the very last one are zeroes and the last entry is NOT zero then we can stop and the system will have no solution.
2. **There will be exactly one solution.** The matrix is as shown previously, where each row only has one nonzero coefficient in it, and there are no rows that have all zeros.
3. **There will be infinitely many solutions.** If we get a row of all zeros then we will have infinitely many solutions.

Nonlinear Systems

- **Nonlinear system** - a system in which at least one of the variables has an exponent other than 1 and/or there is a product of variables in one of the equations.
- Use either substitution or elimination to solve.
- May get complex solutions in addition to real ones.
- Real solutions represent coordinates of points where graphs intersect.