Data Structures
Hashing

Dictionary & Search ADTs

- Operations
  - create
  - destroy
  - insert
  - find
  - delete

insert

- kohlrabi
  - upscale tuber

find(kreplach)

- kreplach
  - tasty stuffed dough

- kim chi
  - spicy cabbage
- kreplach
  - tasty stuffed dough
- kiwi
  - Australian fruit

- Dictionary: Stores *values* associated with user-specified *keys*
  - keys may be any (homogenous) comparable type
  - values may be any (homogenous) type
  - implementation: data field is a struct with two parts
- Search ADT: keys = values

Implementations So Far

	unsorted list	sorted array	TreesBST – averageAVL – worst casesplay – amortized	Array of size n where keys are 0,…,n-1
insert	find+q(1)	q(n)	q(log n)	
find	q(n)	q(log n)	q(log n)	
delete	find+q(1)	q(n)	q(log n)	

Hash Tables: Basic Idea

- Use a key (arbitrary string or number) to index directly into an array – O(1) time to access records
  - A[“kreplach”] = “tasty stuffed dough”
  - Need a *hash function* to convert the key to an integer

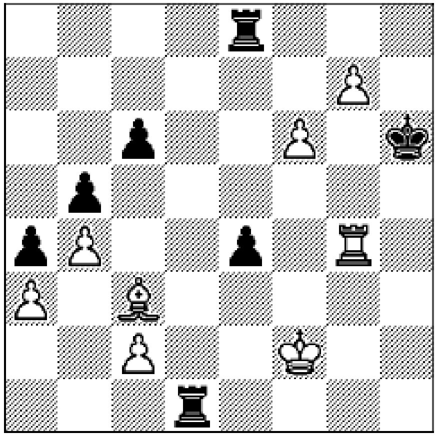
	Key	Data
0	kim chi	spicy cabbage
1	kreplach	tasty stuffed dough
2	kiwi	Australian fruit

Applications

- When log(n) is just too big…
  - Symbol tables in interpreters
  - Real-time databases (in core or on disk)
    - air traffic control
    - packet routing
- When associative memory is needed…
  - Dynamic programming
    - cache results of previous computation

      f(x) ▯if (Find(x)) then Find(x) else f(x)
    - Chess endgames
  - Many text processing applications – e.g. Web

    $Status{$LastURL} = “visited”;



How could you use hash tables to…

- Implement a linked list of unique elements?

- Create an index for a book?

- Convert a document to a Sparse Boolean Vector (where each index represents a different word)?

Properties of Good Hash Functions

- Must return number 0, …, tablesize
- Should be efficiently computable – O(1) time
- Should not waste space unnecessarily
  - For every index, there is at least one key that hashes to it
  - Load factor lambda l = (number of keys / TableSize)
- Should minimize collisions

  = different keys hashing to same index

Integer Keys

- Hash(x) = x % TableSize
- Good idea to make TableSize *prime*. Why?

Integer Keys

- Hash(x) = x % TableSize
- Good idea to make TableSize *prime*. Why?
  - Because keys are typically not randomly distributed, but usually have some *pattern*
    - mostly even
    - mostly multiples of 10
    - in general: mostly multiples of some k
  - If k is a factor of TableSize, then only (TableSize/k) slots will ever be used!
  - Since the only factor of a prime number is itself, this phenomena only hurts in the (rare) case where k=TableSize

Strings as Keys

- If keys are strings, can get an integer by adding up ASCII values of characters in *key*

```
for (i=0;i<key.length();i++)
        hashVal += key.charAt(i);
```

- **Problem 1**: What if *TableSize* is 10,000 and all keys are 8 or less characters long?

- **Problem 2**: What if keys often contain the same characters ("abc", "bca", etc.)?

Hashing Strings

- Basic idea: consider string to be a integer (base 128):
  Hash("abc") = ('a'*128^2 + 'b'*128^1 + 'c') % TableSize
- Range of hash large, anagrams get different values
- **Problem:** although a char can hold 128 values (8 bits), only a subset of these values are commonly used (26 letters plus some special characters)
  - So just use a smaller "base"
  - Hash("abc") = ('a'*32^2 + 'b'*32^1 + 'c') % TableSize

Making the String Hash Easy to Compute

- Horner's Rule

```
int hash(String s) {
  h = 0;
  for (i = s.length() - 1; i >= 0; i--) {
    h = (s.keyAt(i) + h<<5) % tableSize;
  }
  return h;
}
```

What is happening here???

- Advantages:

How Can You Hash…

- A set of values – (name, birthdate) ?
- An arbitrary pointer in C?
- An arbitrary reference to an object in Java?

How Can You Hash…

- A set of values – (name, birthdate) ?

  (Hash(name) ^ Hash(birthdate))% tablesize

  What's this?

- An arbitrary pointer in C?

  ((int)p) % tablesize

- An arbitrary reference to an object in Java?

  Hash(obj.toString())

  or just obj.hashCode() % tablesize

Optimal Hash Function

- The best hash function would distribute keys as evenly as possible in the hash table
- "Simple uniform hashing"
  - Maps each key to a (fixed) random number
  - Idealized gold standard
  - Simple to analyze
  - Can be closely approximated by best hash functions

Collisions and their Resolution

- A collision occurs when two different keys hash to the same value
  - E.g. For *TableSize* = 17, the keys 18 and 35 hash to the same value
  - 18 mod 17 = 1 and 35 mod 17 = 1
- Cannot store both data records in the same slot in array!
- Two different methods for collision resolution:
  - **Separate Chaining**: Use a dictionary data structure (such as a linked list) to store multiple items that hash to the same slot
  - **Closed Hashing (or *probing*):** search for empty slots using a second function and store item in first empty slot that is found

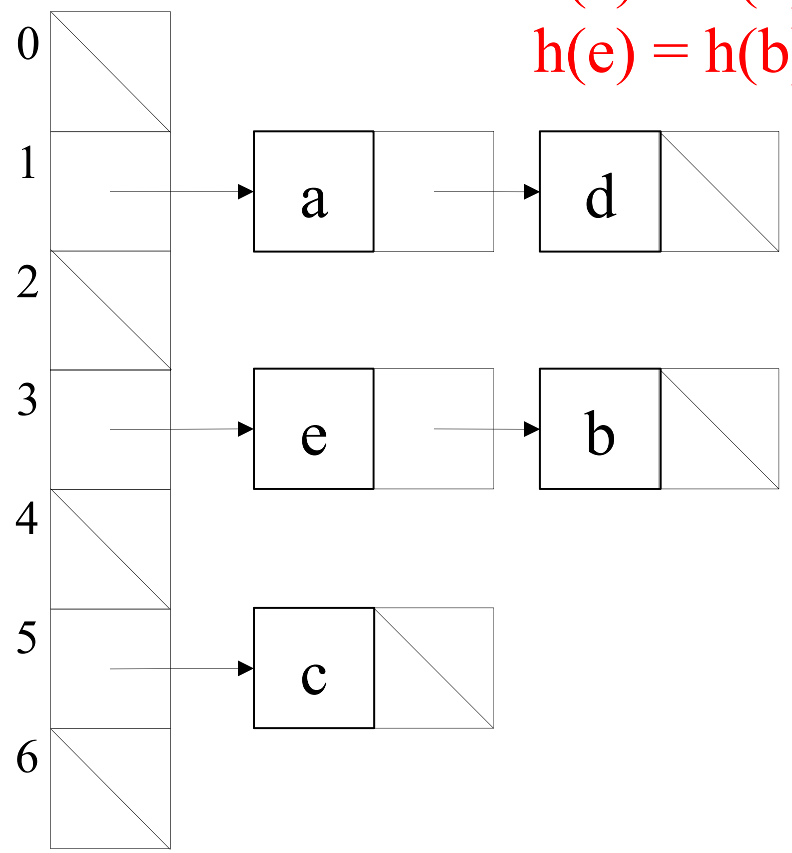
A Rose by Any Other Name…

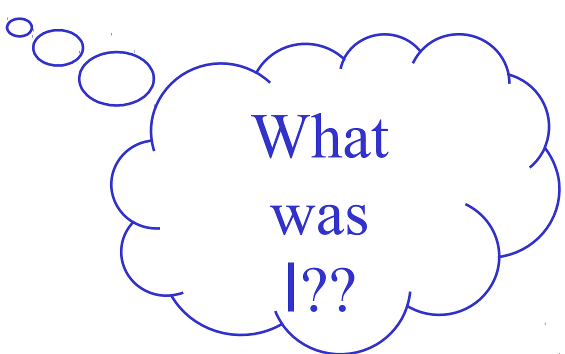
- Separate chaining = Open hashing
- Closed hashing = Open addressing

Hashing with Separate Chaining

- Put a little dictionary at each entry
  - choose type as appropriate
  - common case is unordered linked list (chain)
- Properties
  - performance degrades with length of chains
  - **l can be greater than 1**

h(a) = h(d)
h(e) = h(b)





Load Factor with Separate Chaining

- Search cost
  - unsuccessful search:
  - successful search:
- Optimal load factor:

Load Factor with Separate Chaining

- Search cost (assuming simple uniform hashing)
  - unsuccessful search:

    Whole list – average length l
  - successful search:

    Half the list – average length **l/2+1**

- Optimal load factor:
  - Zero! But between ½ and 1 is fast and makes good use of memory.

Alternative Strategy: Closed Hashing

Problem with separate chaining:

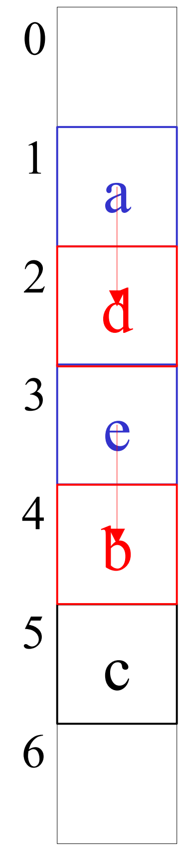
Memory consumed by pointers –
32 (or 64) bits per key!

What if we only allow one Key at each entry?

- two objects that hash to the same spot can't both go there
- first one there gets the spot
- next one must *go in another spot*

- Properties
  - **l £ 1**
  - performance degrades with **difficulty of finding** right spot

h(a) = h(d)
h(e) = h(b)



Collision Resolution by Closed Hashing

- Given an item X, try
  cells $h_0(X)$, $h_1(X)$, $h_2(X)$, …, $h_i(X)$
- $h_i(X) = (Hash(X) + F(i)) \bmod \mathit{TableSize}$
  - Define $F(0) = 0$
- F is the *collision resolution* function. Some possibilities:
  - Linear: $F(i) = i$
  - Quadratic: $F(i) = i^2$
  - Double Hashing: $F(i) = i \cdot Hash_2(X)$

Closed Hashing I: Linear Probing

- Main Idea: When collision occurs, scan down the array one cell at a time looking for an empty cell
  - $h_i(X) = (\text{Hash}(X) + i) \bmod \textit{TableSize} \quad (i = 0, 1, 2, \ldots)$
  - Compute hash value and increment it until a free cell is found

Linear Probing Example

	insert(14) 14%7 = 0	insert(8) 8%7 = 1	insert(21) 21%7 =0	insert(2) 2%7 = 2
0	14	14	14	14
1		8	8	8
2			21	12
3				2
4				
5				
6				
probes:	1	1	3	2

Drawbacks of Linear Probing

- Works until array is full, but as number of items N approaches *TableSize* (l » 1), access time approaches O(N)
- Very prone to cluster formation (as in our example)
  - If a key hashes *anywhere* into a cluster, finding a free cell involves going through the entire cluster – and making it grow!
  - *Primary clustering – clusters grow when keys hash to values close to each other*
- Can have cases where table is empty except for a few clusters
  - Does not satisfy good hash function criterion of *distributing keys uniformly*

Load Factor in Linear Probing

- For *any* l < 1, linear probing will find an empty slot
- Search cost (assuming simple uniform hashing)
  - successful search:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$$

  - unsuccessful search:

$$\frac{1}{2}\left(1+\frac{1}{(1-\lambda)^2}\right)$$

- Performance quickly degrades for l > 1/2

Optimal vs Linear

	successful		unsuccessful	
load factor	optimal	linear	optimal	linear
0.1	1.05	1.06	1.11	1.12
0.2	1.12	1.13	1.25	1.28
0.3	1.19	1.21	1.43	1.52
0.4	1.28	1.33	1.67	1.89
0.5	1.39	1.50	2.00	2.50
0.6	1.53	1.75	2.50	3.63
0.7	1.72	2.17	3.33	6.06
0.8	2.01	3.00	5.00	13.00
0.9	2.56	5.50	10.00	50.50

Closed Hashing II: Quadratic Probing

- Main Idea: Spread out the search for an empty slot – Increment by i^2 instead of i

- $h_i(X) = (\text{Hash}(X) + i^2) \% \; \textit{TableSize}$

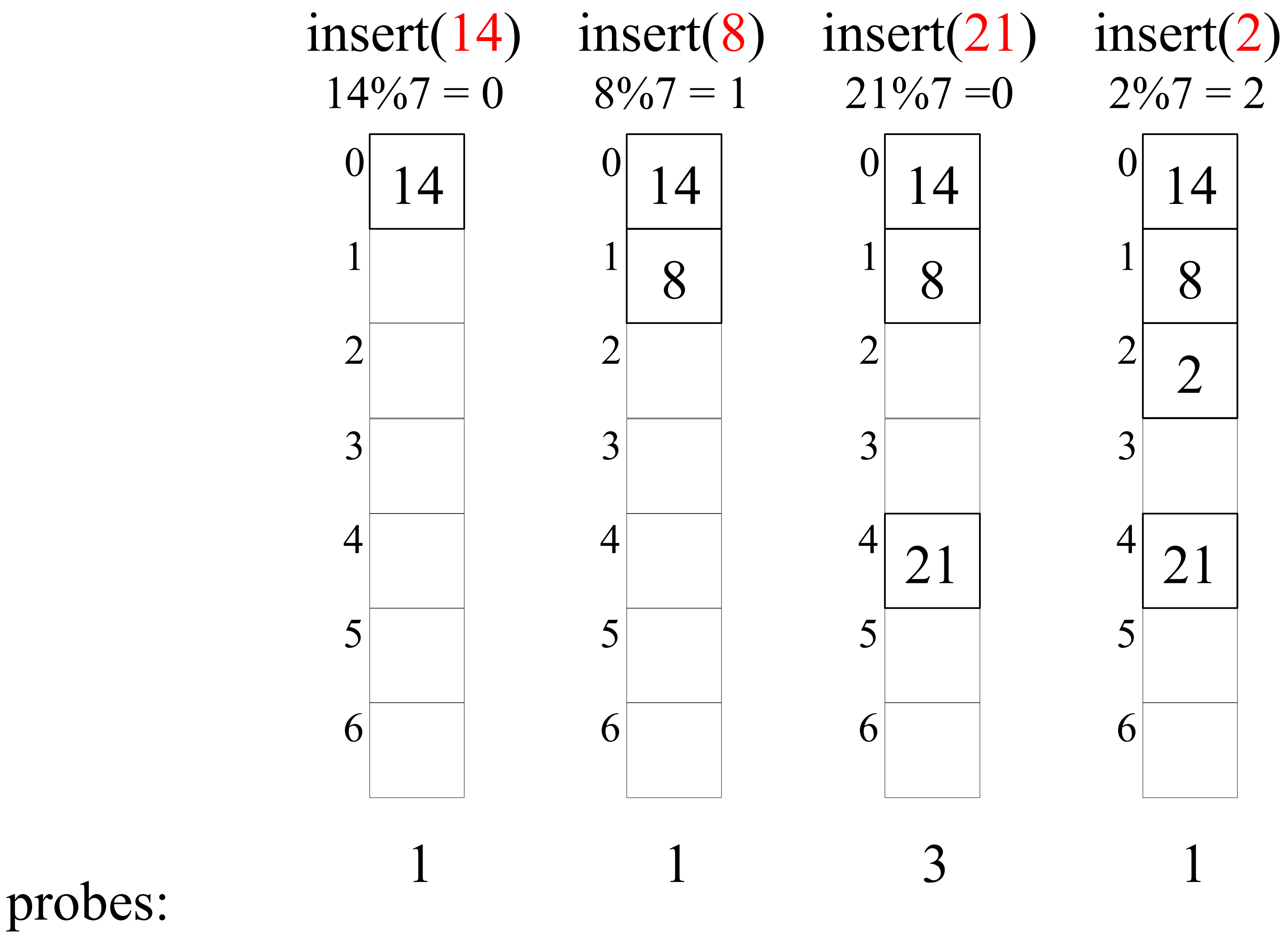
  h0(X) = Hash(X) % TableSize

  h1(X) = Hash(X) + 1 % TableSize

  h2(X) = Hash(X) + 4 % TableSize

  h3(X) = Hash(X) + 9 % TableSize

Quadratic Probing Example



Problem With Quadratic Probing

	insert(14)	insert(8)	insert(21)	insert(2)	insert(7)
	14%7 = 0	8%7 = 1	21%7 =0	2%7 = 2	7%7 = 0
0	14	14	14	14	14
1		8	8	8	8
2				2	2
3					
4			21	21	21
5					
6					
probes:	1	1	3	1	??

Load Factor in Quadratic Probing

- **Theorem**: If TableSize is prime and l £ ½, quadratic probing *will* find an empty slot; for greater l, *might not*
- With load factors near ½ the expected number of probes is empirically near *optimal* – no exact analysis known
- Don't get clustering from *similar* keys (**primary** clustering), still get clustering from *identical* keys (**secondary** clustering)

Closed Hashing III: Double Hashing

- **Idea**: Spread out the search for an empty slot by using a second hash function
  - *No primary or secondary clustering*
- $h_i(X) = (\mathrm{Hash}_1(X) + i \cdot \mathrm{Hash}_2(X)) \bmod \mathit{TableSize}$

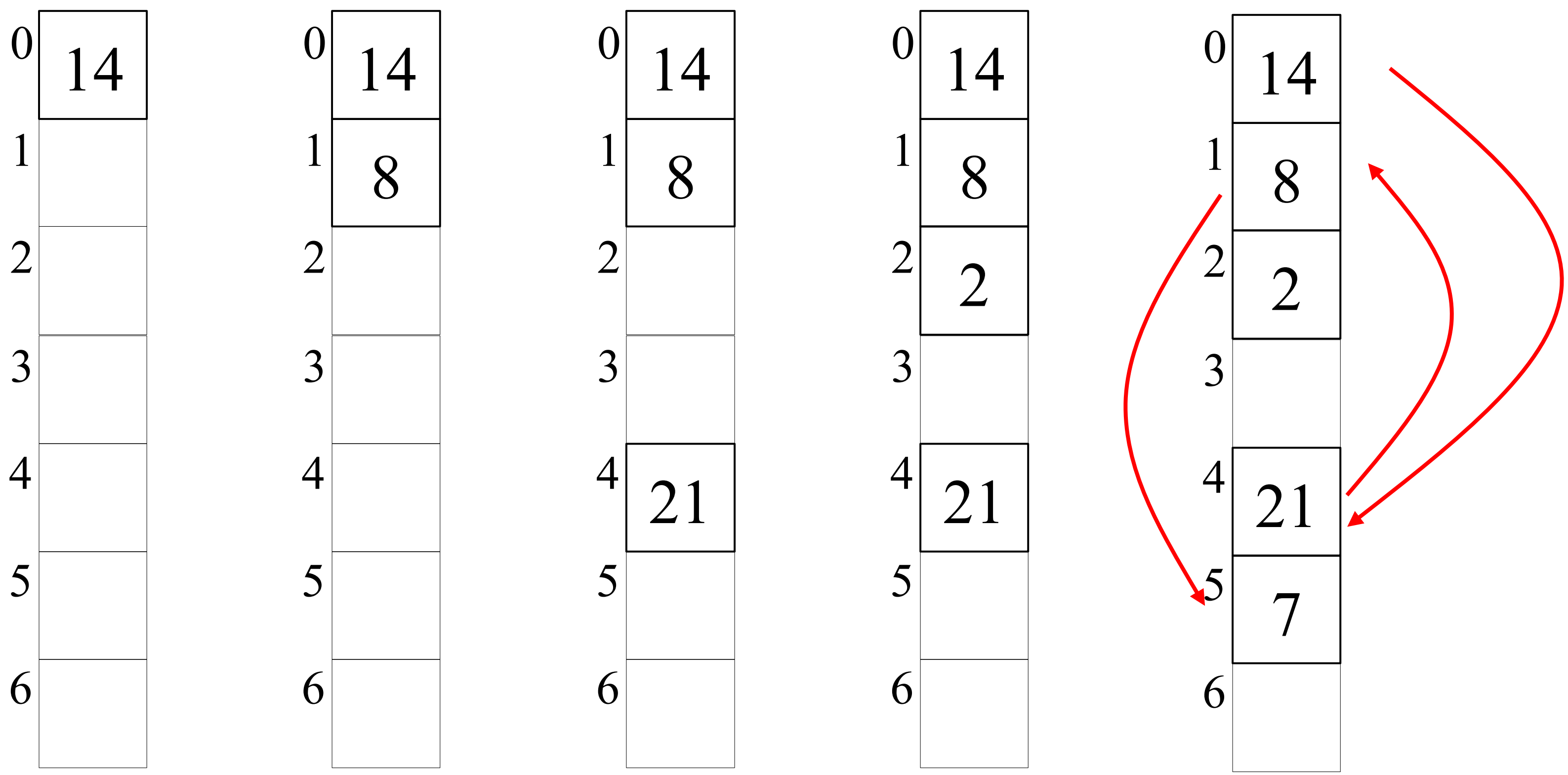
  for $i = 0, 1, 2, \ldots$
- Good choice of $\mathrm{Hash}_2(X)$ can guarantee does not get “stuck” as long as $l < 1$
  - Integer keys:
    $\mathrm{Hash}_2(X) = R - (X \bmod R)$
    where R is a prime smaller than *TableSize*

Double Hashing Example

	insert(14)	insert(8)	insert(21)	insert(2)	insert(7)
	14%7 = 0	8%7 = 1	21%7 =0 5-(21%5)=4	2%7 = 2	7%7 = 0 5-(21%5)=4
0	14	14	14	14	14
1		8	8	8	8
2				2	2
3					
4			21	21	21
5					
6					
probes:	1	1	2	1	??

Double Hashing Example

insert(14)	insert(8)	insert(21)	insert(2)	insert(7)
14%7 = 0	8%7 = 1	21%7 =0	2%7 = 2	7%7 = 0
		5-(21%5)=4		5-(21%5)=4



probes:	1	1	2	1	4

Load Factor in Double Hashing

- For *any* l < 1, double hashing will find an empty slot (given appropriate table size and $hash_2$)
- Search cost approaches optimal (random re-hash):
  - successful search: $$\frac{1}{\lambda} \ln \frac{1}{1-\lambda}$$
  - unsuccessful search: $$\frac{1}{1-\lambda}$$

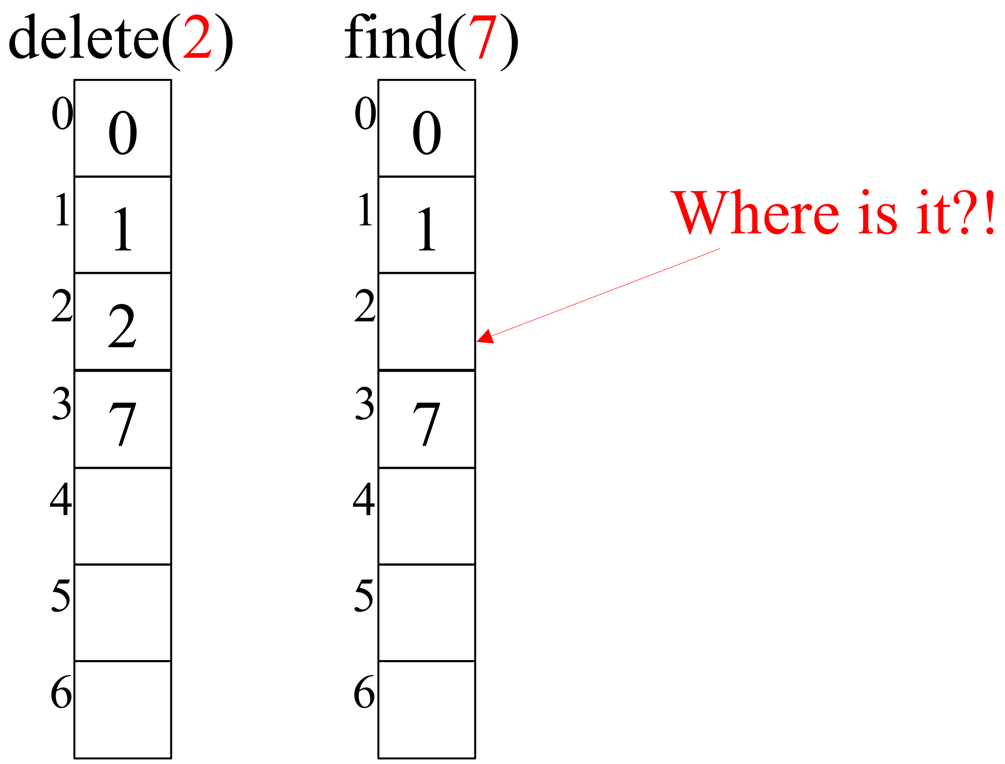
Note natural logarithm!

- No primary clustering and no secondary clustering
- Still becomes costly as l nears 1.

Deletion with Separate Chaining

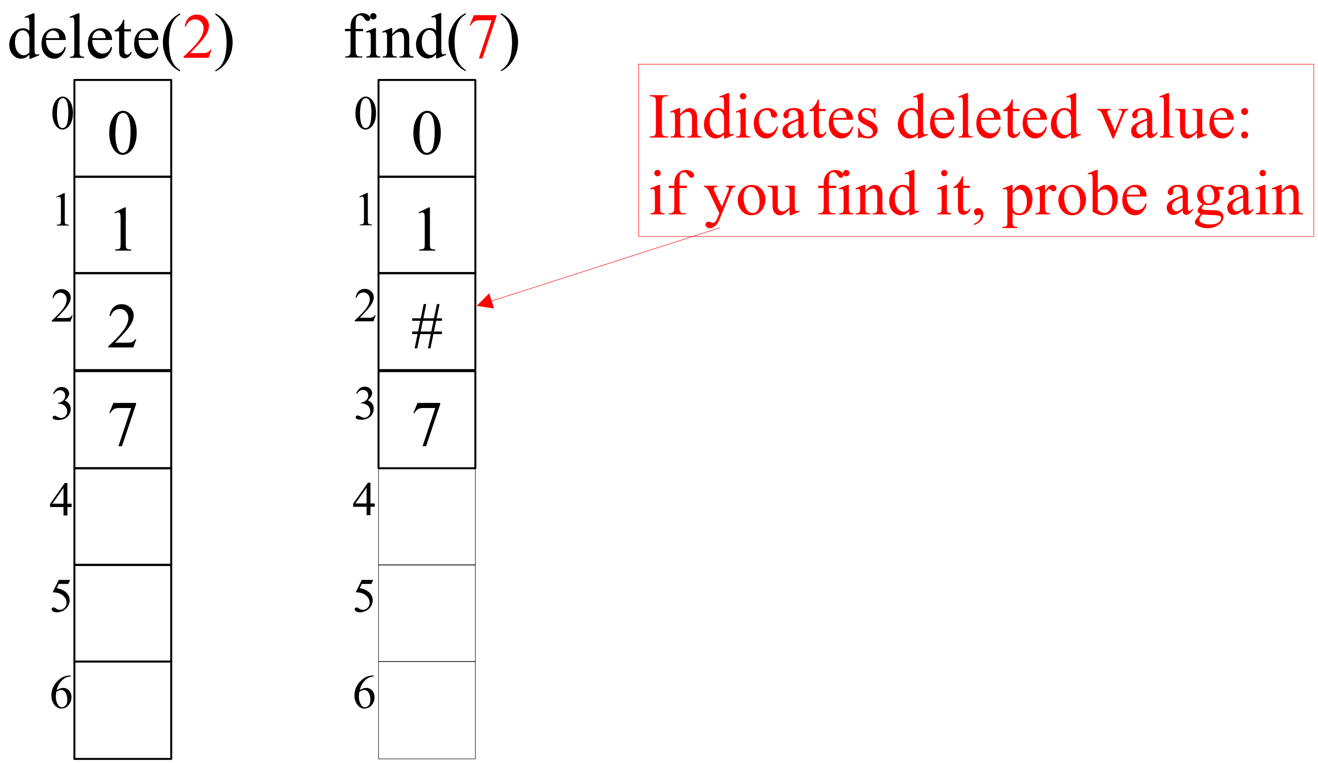
Why is this slide blank?

Deletion in Closed Hashing



What should we do instead?

Lazy Deletion



But *now* what is the problem?

The Squished Pigeon Principle

- An insert using Closed Hashing *cannot* work with a load factor of 1 or more.
  - Quadratic probing can *fail* if l > ½
  - Linear probing and double hashing *slow* if l > ½
  - Lazy deletion never frees space
- Separate chaining becomes slow once l > 1
  - Eventually becomes a linear search of long chains
- How can we relieve the pressure on the pigeons?

REHASH!

Rehashing Example

Separate chaining

$h_1(x) = x \bmod 5$ rehashes to $h_2(x) = x \bmod 11$

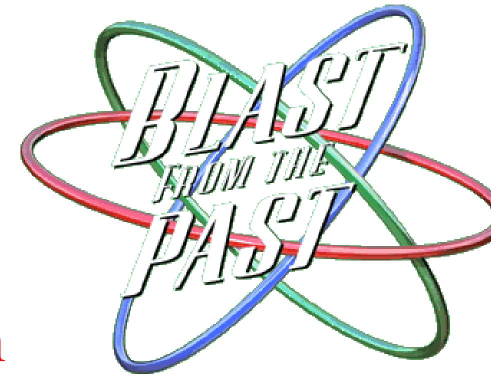
l=1

0	1	2	3	4
25		37	83	
		52	98	

l=5/11

0	1	2	3	4	5	6	7	8	9	10
			25	37		83		52		98

Rehashing Amortized Analysis



- Consider sequence of n operations
  insert(3); insert(19); insert(2); …
- What is the max number of rehashes?

log n

- What is the total time?
  - let's say a regular hash takes time *a*, and rehashing an array contain *k* elements takes time *bk*.

$$an + b(1+2+4+8+\ldots+n) = an + b\sum_{i=o}^{\log n} 2^i$$

$$= an + b(2n-1)$$

- Amortized time = $(an+b(2n-1))/n = O(1)$

Rehashing without Stretching

- Suppose input is a mix of inserts and deletes
  - Never more than TableSize/2 active keys
  - Rehash when $l=1$ (half the table must be deletions)
- Worst-case sequence:
  - T/2 inserts, T/2 deletes, T/2 inserts, Rehash,
    T/2 deletes, T/2 inserts, Rehash, …
- Rehashing at most doubles the amount of work – still O(1)

Case Study

- Spelling dictionary
  - 50,000 words
  - static
  - arbitrary(ish) preprocessing time
- Goals
  - fast spell checking
  - minimal storage

- Practical notes
  - almost all searches are successful
  - words average about 8 characters in length
  - 50,000 words at 8 bytes/word is 400K
  - pointers are 4 bytes
  - there are *many* regularities in the structure of English words

Why?

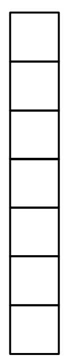
Solutions

- Solutions
  - sorted array + binary search
  - separate chaining
  - open addressing + linear probing

Storage

- Assume words are strings and entries are pointers to strings

Array +
binary search



n pointers

Separate chaining

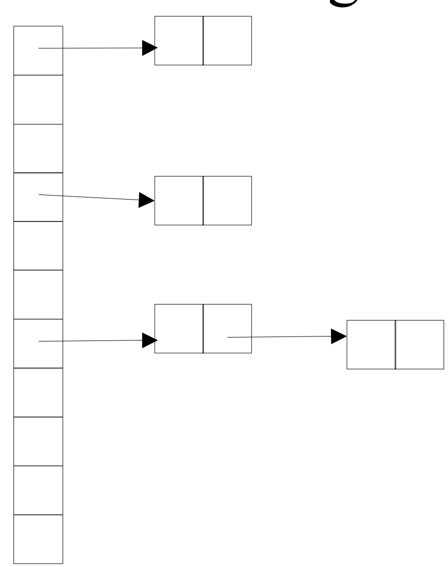
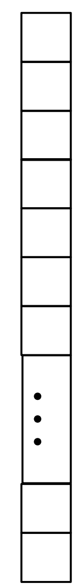


table size + 2n pointers =
n/l + 2n

Closed hashing



n/l pointers

Analysis

50K words, 4 bytes @ pointer

- Binary search
  - storage: n pointers + words = 200K+400K = 600K
  - time: $\log_2 n$ £ 16 probes per access, worst case
- Separate chaining - with l = 1
  - storage: n/l + 2n pointers + words = 200K+400K+400K = 1GB
  - time: 1 + l/2 probes per access on average = 1.5
- Closed hashing - with l = 0.5
  - storage: n/l pointers + words = 400K + 400K = 800K
  - time: $\frac{1}{2}\left(1+\frac{1}{(1-\lambda)}\right)$ probes per access on average = 1.5

Approximate Hashing

- Suppose we want to reduce the space requirements for a spelling checker, by accepting the risk of once in a while overlooking a misspelled word
- Ideas?

Approximate Hashing

Strategy:

- Do not store keys, just a bit indicating cell is in use
- Keep l low so that it is unlikely that a misspelled word hashes to a cell that is in use

Example

- 50,000 English words
- Table of 500,000 cells, each 1 bit
  - 8 bits per byte
- Total memory: 500K/8 = 62.5 K
  - versus 800 K separate chaining, 600 K open addressing
- Correctly spelled words will always hash to a used cell
- What is probability a misspelled word hashes to a used cell?

Rough Error Calculation

- Suppose hash function is optimal - hash is a random number
- Load factor l £ 0.1
  - Lower if several correctly spelled words hash to the same cell
- So probability that a misspelled word hashes to a used cell is £ 10%

Exact Error Calculation

- What is expected load factor?

$$\frac{\text{used cells}}{\text{table size}} = \frac{\text{(Probability a cell is used)(table size)}}{\text{table size}}$$

$= \text{Probability a cell is used} = 1 - \text{(Prob. cell not used)}$

$=1-\text{(Prob. 1st word doesn't use cell)...(Prob. last word doesn't use cell)}$

$=1-((\text{table size} - 1)/\text{table size})^{\text{number words}}$

$$= 1 - \left(\frac{499{,}999}{500{,}000}\right)^{50{,}000} \approx 0.095$$

A Random Hash…

- Extensible hashing
  - Hash tables for disk-based databases – minimizes number disk accesses
- Minimal perfect hash function
  - Hash a given set of *n* keys into a table of size *n* with no collisions
  - Might have to search large space of parameterized hash functions to find
  - Application: compilers
- One way hash functions
  - Used in cryptography
  - Hard (intractable) to *invert*: given just the hash value, recover the key

Puzzler

- Suppose you have a HUGE hash table, that you often need to re-initialize to “empty”. How can you do this in small constant time, *regardless* of the size of the table?

Databases

- A database is a set of records, each a tuple of values
  - E.g.: [name, ss#, dept., salary]
- How can we speed up queries that ask for all employees in a given department?
- How can we speed up queries that ask for all employees whose salary falls in a given range?