Since the denominator $\delta(B)$ in Eq. (6.11) determines the structure of the infinitely many coefficients, the stability of $v(B)$ depends on the coefficients in $\delta(B)$. In fact $v(B)$ is said to be stable if all the roots of $m^r - \delta_1 m^{r-1} - \cdots - \delta_r$ are less than 1 in absolute value.

Once the finite number of parameters in $w(B)$ and $\delta(B)$ are estimated, $v(B)$ can be computed recursively from

$$\delta(B)v(B) = w(B)B^b$$

or

$$v_j - \delta_1 v_{j-1} - \delta_2 v_{j-2} - \cdots - \delta_r v_{j-r} = \begin{cases} -w_{j-b}, & j = b+1, \ldots, b+s \\ 0, & j > b+s \end{cases} \tag{6.12}$$

with $v_b = w_0$ and $v_j = 0$ for $j < b$.

The characteristics of the impulse response function are determined by the values of b, r, and s. Recall that in univariate ARIMA modeling, we matched sample autocorrelation and partial autocorrelation functions computed from a time series to theoretical autocorrelation and partial autocorrelation functions of specific ARIMA models to tentatively identify an appropriate model. Thus by knowing the theoretical patterns in the autocorrelation and partial autocorrelation functions for an AR(1) process, for example, we can tentatively identify the AR(1) model when sample autocorrelation and partial autocorrelation functions exhibit the same behavior for an observed time series. The same approach is used in transfer function modeling. However, the primary identification tool is the impulse response function. Consequently, it is necessary that we investigate the nature of the impulse response function and determine what various patterns in the weights imply about the parameters b, r, and s.

Example 6.1 For illustration, we will consider cases for $b = 2$, $r \leq 2$, and $s \leq 2$.

Case 1. $r = 0$ and $s = 2$.

We have

$$y_t = (w_0 - w_1 B - w_2 B^2)x_{t-2}$$

From Eq. (6.12), we have

$$\begin{aligned}
v_0 &= v_1 = 0 \\
v_2 &= w_0 \\
v_3 &= -w_1 \\
v_4 &= -w_2 \\
v_j &= 0, \quad j > 4
\end{aligned}$$

Hence v_t will only be nonzero for $t = 2, 3$, and 4.

Case 2. $r = 1$ and $s = 2$.

We have

$$y_t = \frac{(w_0 - w_1 B - w_2 B^2)}{1 - \delta_1 B} x_{t-2}$$

As in the AR(1) model, the stability of the transfer function is achieved for $|\delta_1| < 1$. Once again from Eq. (6.12), we have

$$\begin{aligned}
v_0 &= v_1 = 0 \\
v_2 &= w_0 \\
v_3 &= \delta_1 w_0 - w_1 \\
v_4 &= \delta_1^2 w_0 - \delta_1 w_1 - w_2 \\
v_j &= \delta_1 v_{j-1}, \quad j > 4
\end{aligned}$$

Since $|\delta_1| < 1$, the impulse response function will approach zero asymptotically.

Case 3. $r = 2$ and $s = 2$.

We have

$$y_t = \frac{(w_0 - w_1 B - w_2 B^2)}{1 - \delta_1 B - \delta_2 B^2} x_{t-2}$$

The stability of the transfer function depends on the roots of the associated polynomial $m^2 - \delta_1 m^1 - \delta_2$. For stability, the roots obtained by

$$m_1, m_2 = \frac{\delta_1 \pm \sqrt{\delta_1^2 + 4\delta_2}}{2}$$

must satisfy $|m_1|, |m_2| < 1$. This also means that

$$\delta_2 - \delta_1 < 1$$
$$\delta_2 + \delta_1 < 1$$
$$-1 < \delta_2 < 1$$

or

$$|\delta_1| < 1 - \delta_2$$
$$-1 < \delta_2 < 1.$$

This set of two equations implies that the stability is achieved with the triangular region given in Figure 6.1. Within that region we might have two real roots or two complex conjugates. For the latter, we need $\delta_1^2 + 4\delta_2 < 0$, which occurs in the area under the curve within the triangle in Figure 6.1. Hence for the values of δ_1 and δ_2 within that curve, the impulse response function would exhibit a damped sinusoid behavior. Everywhere else in the triangle, however, it will have an exponential decay pattern.

Note that when $\delta_2 = 0$ (i.e., $r = 1$), stability is achieved when $|\delta_1| < 1$ as expected.

Table 6.1 summarizes the impulse response functions for the cases we have just discussed with specific values for the parameters.

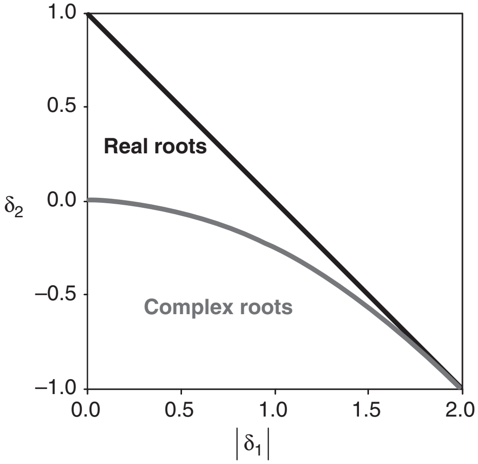
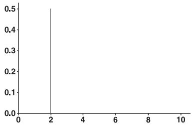
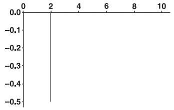
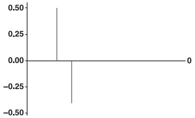
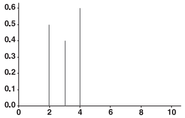
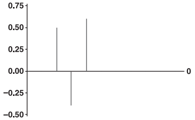
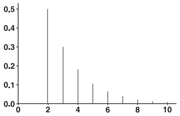
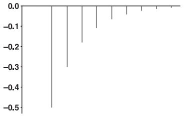
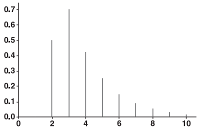
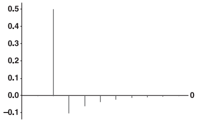
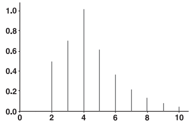
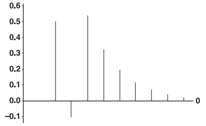
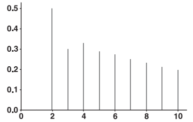
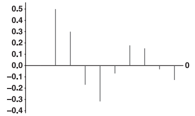
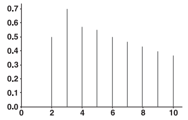
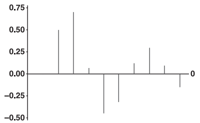
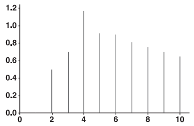
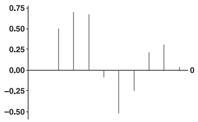


FIGURE 6.1 The stable region for the impulse response function for $r = 2$.

TABLE 6.1 Impulse Response Function with $b = 2, r \leq 2$, and $s \leq 2$

b r s	Model	Impulse response function					
2 0 0	$y_t = w_0 x_{t-2}$	$w_0 = 0.5$			$w_0 = -0.5$		
2 0 1	$y_t = (w_0 - w_1 B)x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$			$w_0 = 0.5$ $w_1 = 0.4$		
2 0 2	$y_t = (w_0 - w_1 B - w_2 B^2)x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$ $w_2 = -0.6$			$w_0 = 0.5$ $w_1 = 0.4$ $w_2 = -0.6$		
2 1 0	$y_t = \frac{w_0}{1 - \delta_1 B} x_{t-2}$	$w_0 = 0.5$		$\delta_1 = 0.6$	$w_0 = -0.5$		$\delta_1 = 0.6$
2 1 1	$y_t = \frac{w_0 - w_1 B}{1 - \delta_1 B} x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$		$\delta_1 = 0.6$	$w_0 = 0.5$ $w_1 = 0.4$		$\delta_1 = 0.6$

2 1 2	$y_t = \dfrac{w_0 - w_1 B - w_2 B^2}{1 - \delta_1 B} x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$ $w_2 = -0.6$		$\delta_1 = 0.6$	$w_0 = 0.5$ $w_1 = -0.4$ $w_2 = -0.6$		$\delta_1 = 0.6$
2 2 0	$y_t = \dfrac{w_0}{1 - \delta_1 B - \delta_2 B^2} x_{t-2}$	$w_0 = 0.5$		$\delta_1 = 0.6$ $\delta_2 = 0.3$	$w_0 = 0.5$		$\delta_1 = 0.6$ $\delta_2 = -0.7$
2 2 1	$y_t = \dfrac{w_0 - w_1 B}{1 - \delta_1 B - \delta_2 B^2} x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$		$\delta_1 = 0.6$ $\delta_2 = 0.3$	$w_0 = 0.5$ $w_1 = -0.4$		$\delta_1 = 0.6$ $\delta_2 = -0.7$
2 2 2	$y_t = \dfrac{w_0 - w_1 B - w_2 B^2}{1 - \delta_1 B - \delta_2 B^2} x_{t-2}$	$w_0 = 0.5$ $w_1 = -0.4$ $w_2 = -0.6$		$\delta_1 = 0.6$ $\delta_2 = 0.3$	$w_0 = 0.5$ $w_1 = -0.4$ $w_2 = -0.6$		$\delta_1 = 0.6$ $\delta_2 = -0.7$

6.3 TRANSFER FUNCTION–NOISE MODELS

As mentioned in the previous section, in the transfer function–noise model in Eq. (6.11) x_t and N_t are assumed to be independent. Moreover, we will assume that the noise N_t can be represented by an ARIMA(p, d, q) model,

$$\underbrace{\phi(B)\,(1-B)^d}_{=\varphi(B)} N_t = \theta(B)\varepsilon_t, \tag{6.13}$$

where $\{\varepsilon_t\}$ is white noise with $E(\varepsilon_t) = 0$. Hence the transfer function–noise model can be written as

$$\begin{aligned} y_t &= v(B)x_t + \psi(B)\varepsilon_t \\ &= \frac{w(B)}{\delta(B)}x_{t-b} + \frac{\theta(B)}{\varphi(B)}\varepsilon_t \end{aligned} \tag{6.14}$$

After rearranging Eq. (6.14), we have

$$\begin{aligned} \underbrace{\delta(B)\varphi(B)}_{=\delta^*(B)}\, y_t &= \underbrace{\varphi(B)w(B)}_{=w^*(B)}\, x_{t-b} + \underbrace{\delta(B)\theta(B)}_{=\theta^*(B)}\, \varepsilon_t \\ \delta^*(B)y_t &= w^*(B)x_{t-b} + \theta^*(B)\varepsilon_t \end{aligned} \tag{6.15}$$

or

$$y_t - \sum_{i=1}^{r^*} \delta_i^* y_{t-i} = w_0^* x_{t-b} - \sum_{i=1}^{s^*} w_i^* x_{t-b-i} + \varepsilon_t - \sum_{i=1}^{q^*} \theta_i^* \varepsilon_{t-i}. \tag{6.16}$$

Ignoring the terms involving x_t, Eq. (6.16) is the ARMA representation of the response y_t. Due to the addition of x_t, the model in Eq. (6.16) is also called an ARMAX model. Hence the transfer function–noise model as given in Eq. (6.16) can be interpreted as an ARMA model for the response with the additional exogenous factor x_t.

6.4 CROSS-CORRELATION FUNCTION

For the bivariate time series (x_t, y_t), we define the **cross-covariance** function as

$$\gamma_{xy}(t, s) = \text{Cov}(x_t, y_s) \tag{6.17}$$

Assuming that (x_t, y_t) is (weakly) stationary, we have

$$\begin{aligned} E(x_t) &= \mu_x, && \text{constant for all } t \\ E(y_t) &= \mu_y, && \text{constant for all } t \\ \mathrm{Cov}(x_t, x_{t+j}) &= \gamma_x(j), && \text{depends only on } j \\ \mathrm{Cov}(y_t, y_{t+j}) &= \gamma_y(j), && \text{depends only on } j \end{aligned}$$

and

$$\mathrm{Cov}(x_t, y_{t+j}) = \gamma_{xy}(j), \quad \text{depends only on } j \text{ for } j = 0, \pm 1, \pm 2, \ldots$$

Hence the **cross-correlation function** (CCF) is defined as

$$\rho_{xy}(j) = \mathrm{corr}(x_t, y_{t+j}) = \frac{\gamma_{xy}(j)}{\sqrt{\gamma_x(0)\gamma_y(0)}} \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots \tag{6.18}$$

It should be noted that $\rho_{xy}(j) \neq \rho_{xy}(-j)$ but $\rho_{xy}(j) = \rho_{yx}(-j)$.

We then define the correlation matrix at lag j as

$$\begin{aligned} \rho(\mathrm{j}) &= \begin{bmatrix} \rho_x(j) & \rho_{xy}(j) \\ \rho_{yx}(j) & \rho_y(j) \end{bmatrix} \\ &= \mathrm{corr}\left[\begin{pmatrix} x_t \\ y_t \end{pmatrix}, (x_{t+j} \quad y_{t+j})\right] \end{aligned} \tag{6.19}$$

For a given sample of N observations, the **sample cross covariance** is estimated from

$$\hat{\gamma}_{xy}(j) = \frac{1}{N}\sum_{t=1}^{N-j}(x_t - \bar{x})(y_{t+j} - \bar{y}) \quad \text{for} \quad j = 0, 1, 2, \ldots \tag{6.20}$$

and

$$\hat{\gamma}_{xy}(j) = \frac{1}{N}\sum_{t=1}^{N+j}(x_{t-j} - \bar{x})(y_t - \bar{y}) \quad \text{for} \quad j = -1, -2, \ldots \tag{6.21}$$

Similarly, the **sample cross correlations** are estimated from

$$r_{xy}(j) = \hat{\rho}_{xy}(j) = \frac{\hat{\gamma}_{xy}(j)}{\sqrt{\hat{\gamma}_x(0)\hat{\gamma}_y(0)}} \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots \tag{6.22}$$

where

$$\hat{\gamma}_x(0) = \frac{1}{N}\sum_{t=1}^{N}(x_t - \bar{x})^2 \quad \text{and} \quad \hat{\gamma}_y(0) = \frac{1}{N}\sum_{t=1}^{N}(y_t - \bar{y})^2$$

Sampling properties such as the mean and variance of the sample CCF are quite complicated. For a few special cases, however, we have the following.

1. For large data sets, $E(r_{xy}(j)) \approx \rho_{xy}(j)$ but the variance is still complicated.
2. If x_t and y_t are autocorrelated but un(cross)correlated at all lags, that is, $\rho_{xy}(j) = 0$, we then have $E(r_{xy}(j)) \approx 0$ and $\text{var}(r_{xy}(j)) \approx (1/N)\sum_{i=-\infty}^{\infty}\rho_x(i)\rho_y(i)$.
3. If $\rho_{xy}(j) = 0$ for all lags j but also x_t is white noise, that is, $\rho_x(j) = 0$ for $j \neq 0$, then we have $\text{var}(r_{xy}(j)) \approx 1/N$ for $j = 0, \pm 1, \pm 2, \ldots$.
4. If $\rho_{xy}(j) = 0$ for all lags j but also both x_t and y_t are white noise, then we have $\text{corr}(r_{xy}(i), r_{xy}(j)) \approx 0$ for $i \neq j$.

6.5 MODEL SPECIFICATION

In this section, we will discuss the issues regarding the specification of the model order in a transfer function–noise model. Further discussion can be found in Bisgaard and Kulahci (2006a,b, 2011).

We will first consider the general form of the transfer function–noise model with time delay given as

$$\begin{aligned} y_t &= v(B)x_t + N_t \\ &= \frac{w(B)}{\delta(B)}x_{t-b} + \frac{\theta(B)}{\varphi(B)}\varepsilon_t. \end{aligned} \tag{6.23}$$

The six-step model specification process is outlined next.

Step 1. Obtaining the preliminary estimates of the coefficients in $v(B)$.

One approach is to assume that the coefficients in $v(B)$ are zero except for the first k lags:

$$y_t \cong \sum_{i=0}^{k} v_i x_{t-i} + N_t.$$

We can then attempt to obtain the initial estimates for $v_1, v_2, \dots, v_k$ through ordinary least squares. However, this approach can lead to highly inaccurate estimates as x_t may have strong autocorrelation. Therefore a method called **prewhitening** of the input is generally preferred.

Method of Prewhitening For the transfer function–noise model in Eq. (6.23), suppose that x_t follows an ARIMA model as

$$\underbrace{\phi_x(B)(1-B)^d}_{=\varphi_x(B)} x_t = \theta_x(B)\alpha_t, \tag{6.24}$$

where α_t is white noise with variance σ_α^2. Equivalently, we have

$$\alpha_t = \theta_x(B)^{-1}\varphi_x(B)x_t. \tag{6.25}$$

In this notation, $\theta_x(B)^{-1}\varphi_x(B)$ can be seen as a filter that when applied to x_t generates a white noise time series, hence the name “prewhitening.”

When we apply this filter to the transfer function–noise model in Eq. (6.23), we obtain

$$\begin{aligned}\underbrace{\theta_x(B)^{-1}\varphi_x(B)y_t}_{=\beta_t} &= \theta_x(B)^{-1}\varphi_x(B)v(B)x_t + \underbrace{\theta_x(B)^{-1}\varphi_x(B)N_t}_{=N_t^*}\\ \beta_t &= v(B)\alpha_t + N_t^*\end{aligned} \tag{6.26}$$

The cross covariance between the filtered series α_t and β_t is given by

$$\begin{aligned}\gamma_{\alpha\beta}(j) &= \mathrm{Cov}(\alpha_t, \beta_{t+j}) = \mathrm{Cov}\left(\alpha_t, v(B)\alpha_{t+j} + N^*_{t+j}\right)\\ &= \mathrm{Cov}\left(\alpha_t, \sum_{i=0}^{\infty} v_i\alpha_{t+j-i} + N^*_{t+j}\right)\\ &= \mathrm{Cov}\left(\alpha_t, \sum_{i=0}^{\infty} v_i\alpha_{t+j-i}\right) + \underbrace{\mathrm{Cov}\left(\alpha_t, N^*_{t+j}\right)}_{=0} \qquad (6.27)\\ &= \sum_{i=0}^{\infty} v_i \,\mathrm{Cov}(\alpha_t, \alpha_{t+j-i})\\ &= v_j \,\mathrm{Var}(\alpha_t).\end{aligned}$$

Note that $\mathrm{Cov}(\alpha_t, N^*_{t+j}) = 0$ since x_t and N_t are assumed to be independent.

From Eq. (6.27), we have $\gamma_{\alpha\beta} = v_j\sigma^2_\alpha$ and hence

$$\begin{aligned}v_j &= \frac{\gamma_{\alpha\beta}(j)}{\sigma^2_\alpha} = \frac{\rho_{\alpha\beta}(j)\sigma_\alpha\sigma_\beta}{\sigma^2_\alpha}\\ &= \rho_{\alpha\beta}(j)\frac{\sigma_\beta}{\sigma_\alpha},\end{aligned} \qquad (6.28)$$

where $\rho_{\alpha\beta}(j) = \mathrm{corr}(\alpha_t, \beta_{t+j})$ is the CCF between α_t and β_t. So through the sample estimates we can obtain the initial estimates for the v_j:

$$\hat{v}_j = r_{\alpha\beta}(j)\frac{\hat{\sigma}_\beta}{\hat{\sigma}_\alpha}. \qquad (6.29)$$

Equation (6.29) implies that there is a simple relationship between the impulse response function, $v(B)$, and the cross-correlation function of the prewhitened response and input series. Hence the estimation of the coefficients in $v(B)$ is possible through this relationship as summarized in Eq. (6.29). A similar relationship exists when the response and the input are not prewhitened (see Box et al., 2008). However, the calculations become fairly complicated when the series are not prewhitened. Therefore we strongly recommend the use of prewhitening in model identification and estimation of transfer function–noise models.

Moreover, since α_t is white noise, the variance of $r_{\alpha\beta}(j)$ is relatively easier to obtain than that of $r_{xy}(j)$. In fact, from the special case 3 in the previous section, we have

$$\mathrm{Var}[r_{\alpha\beta}(j)] \approx \frac{1}{N}, \tag{6.30}$$

if $\rho_{\alpha\beta}(j) = 0$ for all lags j. We can then use $\pm 2/\sqrt{N}$ as the *approximate* 95% confidence interval to judge the significance of $r_{\alpha\beta}(j)$.

Step 2. Specifications of the orders r and s.

Once the initial estimates of the v_j from Eq. (6.29) are obtained, we can use them to specify the orders r and s in

$$\begin{aligned} v(B) &= \frac{w(B)}{\delta(B)} B^b \\ &= \frac{w_0 - w_1 B - \cdots - w_s B^s}{1 - \delta_1 B - \cdots - \delta_r B^r} B^b \end{aligned}$$

The specification of the orders r and s can be accomplished by plotting the v_j. In Figure 6.2, we have an example of the plot of the initial estimates for the v_j in which we can see that $\hat{v}_0 \approx 0$, implying that there might be a

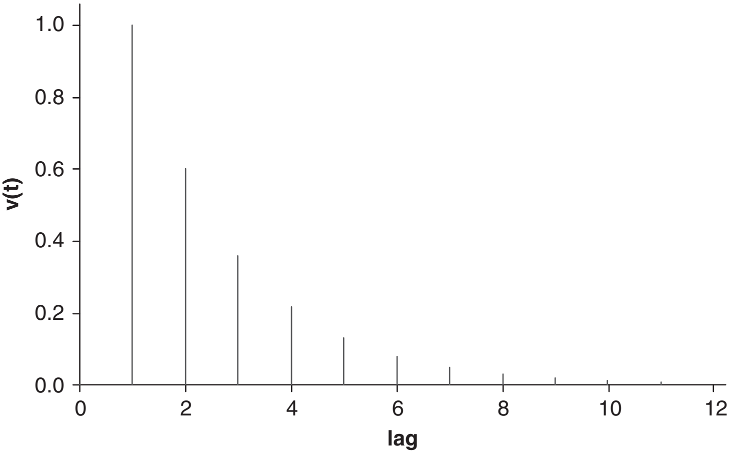


FIGURE 6.2 Example of an impulse response function.

time delay (i.e., $b = 1$). However, for $j > 1$, we have an exponential decay pattern, suggesting that we may have $r = 1$, which implies

$$v_j - \delta v_{j-1} = 0 \quad \text{for} \quad j > 1$$

and

$$s = 0.$$

Hence for this example, our initial attempt in specifying the order of the transfer function noise model will be

$$y_t = \frac{w_0}{1 - \delta B} x_{t-1} + N_t. \tag{6.31}$$

Caution: In model specification, one should be acutely aware of overparameterization as for an arbitrary η, the model in Eq. (6.31) can also be written as

$$\begin{aligned} y_t &= \frac{w_0 (1 - \eta B)}{(1 - \delta B)(1 - \eta B)} x_{t-1} + N_t \\ &= \frac{w_0 - w_1 B}{1 - \delta_1 B - \delta_2 B^2} x_{t-1} + N_t. \end{aligned} \tag{6.32}$$

But the parameters in Eq. (6.32) are not identifiable, since η can arbitrarily take any value.

Step 3. Obtain the estimates of δ_i and w_i.

From $\hat{\delta}(B)\hat{v}(B) = \hat{w}(B)$, we can recursively estimate δ_i and w_i using Eq. (6.12),

$$v_j - \delta_1 v_{j-1} - \delta_2 v_{j-2} - \ldots - \delta_r v_{j-r} = \begin{cases} -w_{j-b}, & j = b+1, \ldots, b+s \\ 0, & j > b+s \end{cases}$$

with $v_b = w_0$ and $v_j = 0$ for $j < b$. Hence for the example in Step 2, we have

$$\begin{aligned} \hat{v}_1 &= \hat{w}_0 \\ \hat{v}_2 - \hat{\delta}\hat{v}_1 &= 0 \\ &\vdots \end{aligned}$$

Step 4. Model the noise.

Once the initial estimates of the model parameters are obtained, the estimated noise can be obtained as

$$\hat{N}_t = y_t - \frac{\hat{w}(B)}{\hat{\delta}(B)} x_{t-\hat{b}}, \tag{6.33}$$

To obtain the estimated noise, we define $\hat{y}_t = (\hat{w}(B)/\hat{\delta}(B))x_{t-\hat{b}}$. We can then calculate $\hat{y}_t$ recursively. To model the estimated noise, we observe its ACF and PACF and determine the orders of the ARIMA model, $\phi(B)\,(1-B)^d N_t = \theta(B)\varepsilon_t$.

Step 5. Fitting the overall model.

Steps 1 through 4 provide us with the model specifications and the initial estimates of the parameters in the transfer function–noise model,

$$y_t = \frac{w(B)}{\delta(B)} x_{t-b} + \frac{\theta(B)}{\phi(B)\,(1-B)^d} \varepsilon_t.$$

The final estimates of the model parameters are then obtained by a nonlinear model fit. Model selection criteria such as AIC and BIC can be used to pick the "best" model among competing models.

Step 6. Model adequacy checks.

At this step, we check the validity of the two assumptions in the fitted model:

1. The assumption that the noise ε_t is white noise requires the examination of the residuals $\hat{\varepsilon}_t$. We perform the usual checks through analysis of the sample ACF and PACF of the residuals.
2. We should also check the independence between ε_t and x_t. For that, we observe the sample cross-correlation function between $\hat{\varepsilon}_t$ and $\hat{x}_t$. Alternatively, we can examine $r_{\hat{\alpha}\hat{\varepsilon}}(j)$, where $\alpha_t = \hat{\theta}_x(B)^{-1}\hat{\varphi}_x(B)x_t$. Under the assumption the model is adequate, $r_{\hat{\alpha}\hat{\varepsilon}}(j)$ will have 0 mean, $1/\sqrt{N}$ standard deviation, and be independent for different lags j. Hence we can use $\pm 2/\sqrt{N}$ as the limit to check the independence assumption.

TABLE 6.2 The viscosity, $y(t)$ and temperature, $x(t)$

$x(t)$	$y(t)$	$x(t)$	$y(t)$	$x(t)$	$y(t)$	$x(t)$	$y(t)$
0.17	0.30	0.08	0.53	0.00	0.34	−0.04	−0.12
0.13	0.18	0.17	0.54	0.02	0.13	0.11	−0.26
0.19	0.09	0.20	0.42	−0.08	0.21	0.19	0.20
0.09	0.06	0.20	0.37	−0.08	0.06	−0.07	0.18
0.03	0.30	0.27	0.34	−0.26	0.04	−0.10	0.32
0.11	0.44	0.23	0.27	−0.06	−0.06	0.13	0.50
0.15	0.46	0.23	0.34	−0.06	−0.16	0.10	0.40
−0.02	0.44	0.20	0.35	−0.09	−0.47	−0.10	0.41
0.07	0.34	0.08	0.43	−0.14	−0.50	−0.05	0.47
0.00	0.23	−0.16	0.63	−0.10	−0.60	−0.12	0.37
−0.08	0.07	−0.08	0.61	−0.25	−0.49	0.00	0.04
−0.15	0.21	0.14	0.52	−0.23	−0.27	0.03	−0.10
−0.15	0.03	0.17	0.06	−0.11	−0.18	−0.06	−0.34
0.04	−0.20	0.27	−0.11	−0.01	−0.37	0.03	−0.41
0.08	−0.39	0.19	−0.01	−0.17	−0.34	0.04	−0.33
0.10	−0.70	0.10	0.02	−0.23	−0.34	0.09	−0.25
0.07	−0.22	0.13	0.34	−0.28	−0.18	−0.25	−0.18
−0.01	−0.08	−0.05	0.21	−0.26	−0.26	−0.25	−0.06
0.06	0.16	0.13	0.18	−0.19	−0.51	−0.40	0.15
0.07	0.13	−0.02	0.19	−0.26	−0.65	−0.30	−0.32
0.17	0.07	0.04	0.05	−0.20	−0.71	−0.18	−0.32
−0.01	0.23	0.00	0.15	−0.08	−0.82	−0.09	−0.81
0.09	0.33	0.08	0.10	0.03	−0.70	−0.05	−0.87
0.22	0.72	0.08	0.28	−0.08	−0.63	0.09	−0.84
0.09	0.45	0.07	0.20	0.01	−0.29	0.18	−0.73

Example 6.2 In a chemical process it is expected that changes in temperature affect viscosity, a key quality characteristic. It is therefore of great importance to learn more about this relationship. The data are collected every 10 seconds and given in Table 6.2 (Note that for each variable the data are centered by subtracting the respective averages). Figure 6.3 shows the time series plots of the two variables.

Since the data are taken in time and at frequent intervals, we expect the variables to exhibit some autocorrelation and decide to fit a transfer function-noise model following the steps provided earlier.

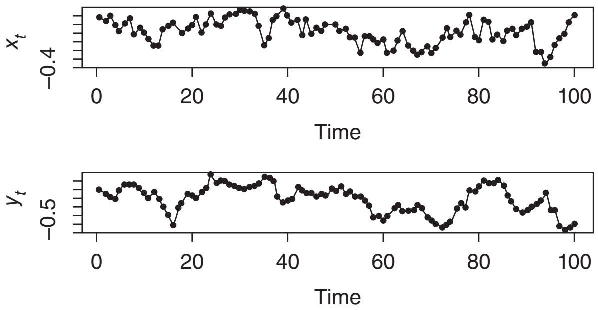


FIGURE 6.3 Time series plots of the viscosity, $y(t)$ and temperature, $x(t)$.

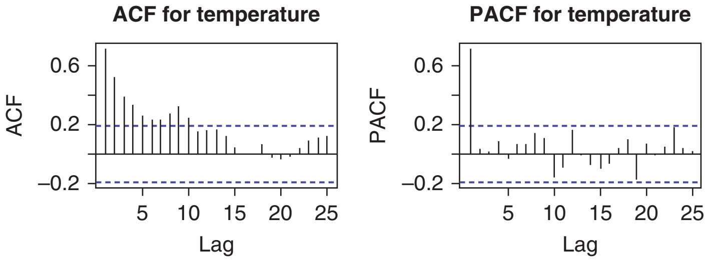


FIGURE 6.4 Sample ACF and PACF of the temperature.

Step 1. Obtaining the preliminary estimates of the coefficients in $v(B)$

In this step we use the prewhitening method. First we fit an ARIMA model to the temperature. Since the time series plot in Figure 6.3 shows that the process is changing around a constant mean and has a constant variance, we will assume that it is stationary.

Sample ACF and PACF plots in Figure 6.4 suggest that an AR(1) model should be used to fit the temperature data. Table 6.3 shows that $\hat{\phi} \cong 0.73$.

TABLE 6.3 AR(1) Model for Temperature, *x*(*t*)

Parameter Estimate of the AR(1) model for $x(t)$

Term	Coef	SE Coef	T	*P*-value
AR 1	0.7292	0.0686	10.63	<0.0001

Number of degrees of freedom: 99
MSE = 0.01009
AIC = −171.08

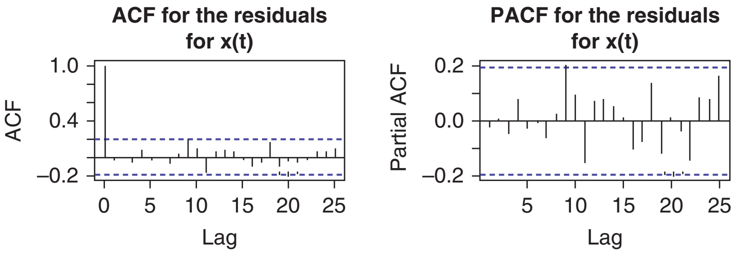


FIGURE 6.5 Sample ACF and PACF of the residuals from the AR(1) model for the Temperature, $x(t)$.

The sample ACF and PACF plots in Figure 6.5 as well as the additional residuals plots in Figure 6.6 reveal that no autocorrelation is left in the data and the model gives a reasonable fit.

Hence we define

$$\alpha_t = (1 - 0.73\,B)x_t$$

and

$$\beta_t = (1 - 0.73\,B)y_t$$

We then compute the sample cross-correlation of α_t and β_t, $r_{\alpha\beta}$ given in Figure 6.7. Since the cross correlation at lags 0, 1 and 2 do not seem to be significant, we conclude that there is a delay of 3 lags (30 seconds) in the system, that is, $b = 3$.

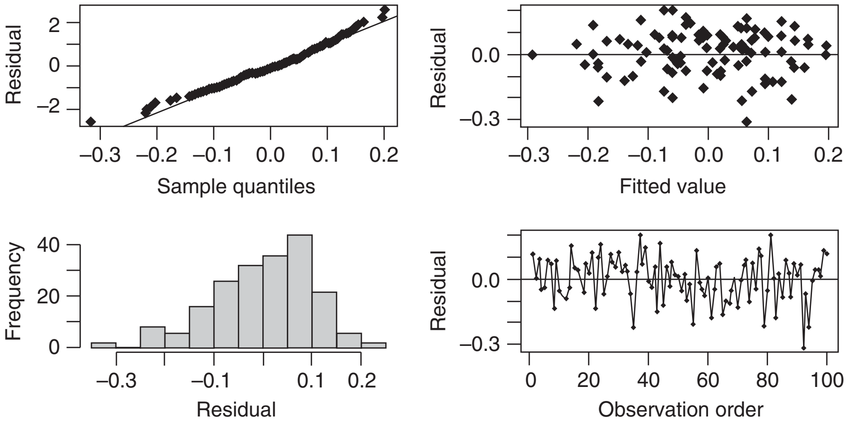


FIGURE 6.6 Residual plots from the AR(1) model for the temperature.

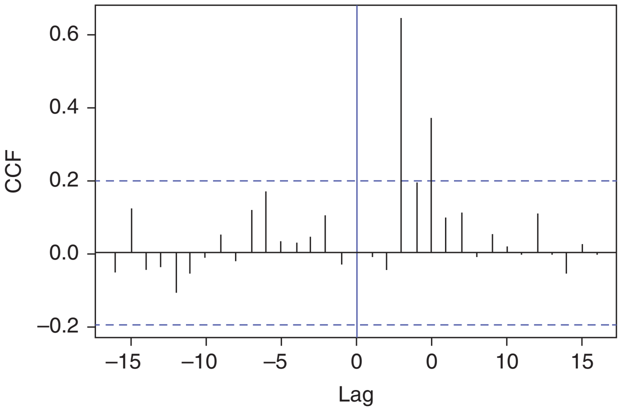


FIGURE 6.7 Sample cross-correlation function between α_t and β_t.

From Eq. (6.34), we have

$$\begin{aligned}\hat{v}_j &= r_{\alpha\beta}(j)\frac{\hat{\sigma}_\beta}{\hat{\sigma}_\alpha}\\ &= r_{\alpha\beta}(j)\frac{0.1881}{0.1008},\end{aligned} \tag{6.34}$$

where $\hat{\sigma}_\alpha$ and $\hat{\sigma}_\beta$ are the sample standard deviations of α_t and β_t. The plot of $\hat{v}_j$ is given in Figure 6.8.

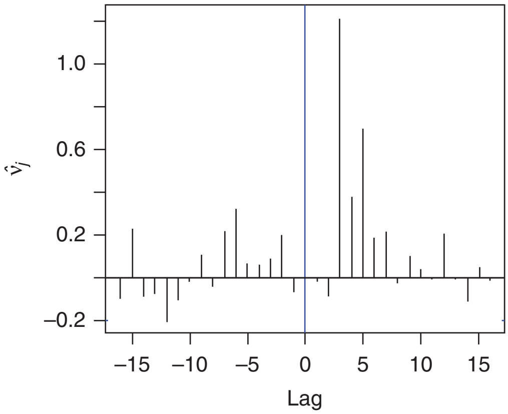


FIGURE 6.8 Plot of the impulse response function for the viscosity data.

Step 2. Specifications of the orders r and s.

To identify the pattern in Figures 6.7 and 6.8, we can refer back to Table 6.1. From the examples of impulse response functions given in that table, we may conclude the denominator of the transfer function is a second order polynomial in B. That is, $r = 2$ so we have $1 - \delta_2 B - \delta_2, B^2$ for the denominator. For the numerator, it seems that $s = 0$ or w_0 would be appropriate. Hence our tentative impulse response function is the following

$$v_t = \frac{w_0}{1 - \delta_1 B - \delta_2 B^2} B^3.$$

Step 3. Obtain the estimates of the δ_i and w$_i$.

To obtain the estimates of δ_i and w$_i$, we refer back to Eq. (6.12) which implies that we have

$$\begin{aligned}
\hat{v}_0 &\approx 0 \\
\hat{v}_1 &\approx 0 \\
\hat{v}_2 &\approx 0 \\
\hat{v}_3 &= 1.21 = \hat{w}_0 \\
\hat{v}_4 &= 0.37 = 1.21\hat{\delta}_1 \\
\hat{v}_5 &= 0.69 = 0.37\hat{\delta}_1 + 1.21\hat{\delta}_2
\end{aligned}$$

The parameter estimates are then

$$\begin{aligned}
\hat{w}_0 &= 1.21 \\
\hat{\delta}_1 &= 0.31 \\
\hat{\delta}_2 &= 0.48
\end{aligned}$$

or

$$\hat{v}_t = \frac{1.21}{1 - 0.31\,B - 0.48\,B^2} B.$$

Step 4. Model the noise.

To model the noise, we first define $\hat{y}_t = \dfrac{\hat{w}(B)}{\hat{\delta}(B)} x_{t-3}$ or

$$\begin{aligned}
\hat{\delta}(B)\hat{y}_t &= \hat{w}(B)x_{t-3} \\
(1 - 0.31\,B - 0.48\,B^2)\hat{y}_t &= 1.21x_{t-3} \\
\hat{y}_t &= 0.31\hat{y}_{t-1} + 0.48\hat{y}_{t-2} + 1.21x_{t-3}.
\end{aligned}$$

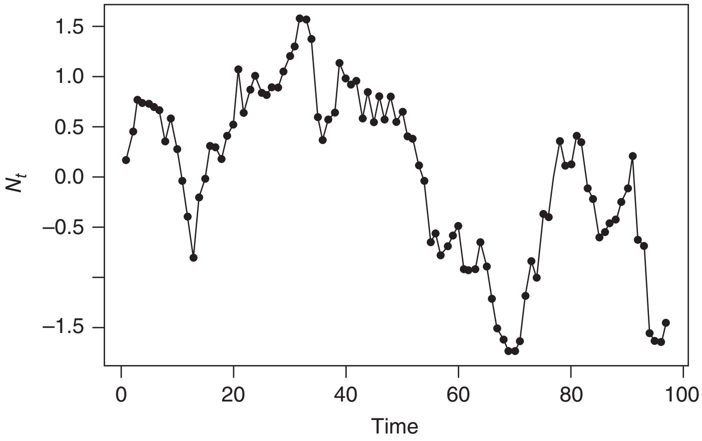


FIGURE 6.9 Time series plot of $\hat{N}_t$.

We then define

$$\hat{N}_t = y_t - \hat{y}_t.$$

Figures 6.9 and 6.10 show the time series plot of $\hat{N}_t$ and its sample ACF/PACF plots respectively which indicate an AR model. Note that partial autocorrelation at lag 3 is borderline significant. However when an AR(3) model is fitted, both ϕ_2 and ϕ_3 are found to be insignificant. Therefore AR(1) model is considered to be the appropriate model.

The parameter estimates for the AR(1) model for $\hat{N}_t$ are given in Table 6.4. Diagnostic checks of the residuals through sample ACF and PACF plots in Figure 6.11 and residuals plots in Figure 6.12 imply that we have a good fit.

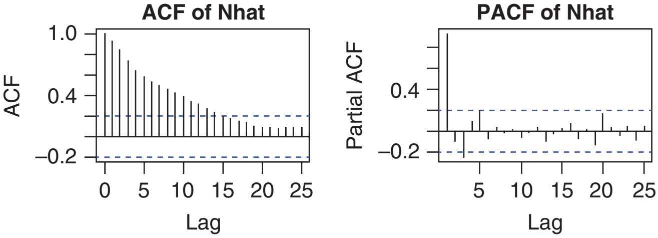


FIGURE 6.10 Sample ACF and PACF of $\hat{N}_t$.

TABLE 6.4 AR(1) Model for N_t

Parameter Estimate of the AR(1) model for $x(t)$

Term	Coef	SE Coef	T	*P*-value
AR 1	0.9426	0.0300	31.42	<0.0001

Number of degrees of freedom: 96
MSE = 0.0141
AIC = −131.9

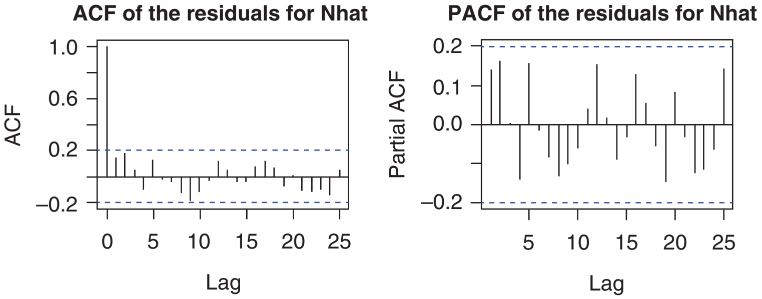


FIGURE 6.11 Sample ACF and PACF of the residuals of the AR(1) model for $\hat{N}_t$.

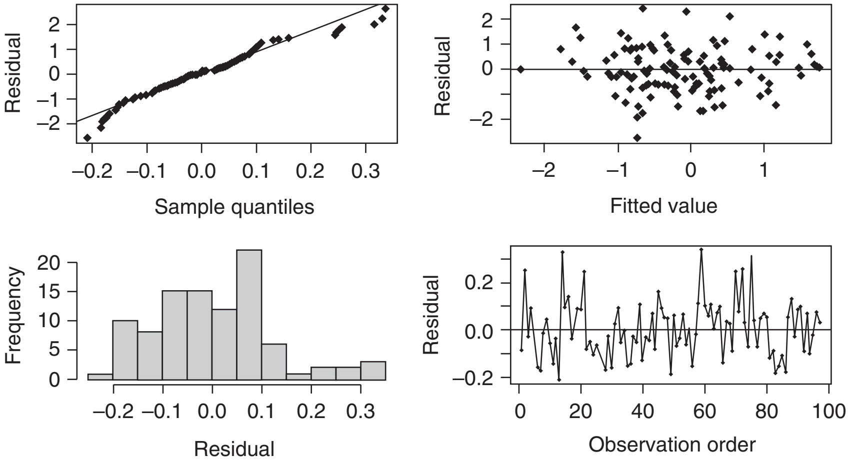


FIGURE 6.12 Residual plots of the AR(l) model for $\hat{N}_t$.

Note that we do not necessarily need the coefficients estimates as they will be re-estimated in the next step. Thus at this step all we need is a sensible model for $\hat{N}_t$ to put into the overall model.

Step 5. Fitting the overall model.

From Step 4, we have the tentative overall model as

$$y_t = \frac{w_0}{1 - \delta_1 B - \delta_2 B^2} x_{t-3} + \frac{1}{1 - \phi_1 B} \varepsilon_t.$$

The calculations that were made so far could have been performed practically in any statistical package. However as we mentioned at the beginning of the Chapter, unfortunately only a few software packages have the capability to fit the overall transfer function-noise model described above. In the following we provide the output from JMP with which such a model can be fitted. At the end of the chapter, we also provide the R code that can be used to fit the transfer function-noise model.

JMP output for the overall transfer function-noise model is provided in Table 6.5. The estimated coefficients are

$$\hat{w}_0 = 1.3276, \quad \hat{\delta}_1 = 0.3414, \quad \hat{\delta}_2 = 0.2667, \quad \hat{\phi}_1 = 0.8295,$$

and they are all significant.

Step 6. Model adequacy checks

The sample ACF and PACF of the residuals provided in Table 6.5 show no indication of leftover autocorrelation. We further check the cross correlation function between $\alpha_t = (1 - 0.73B)x_t$ and the residuals as given in Figure 6.13. There is a borderline significant cross correlation at lag 5. However we believe that it is at this point safe to claim that the current fitted model is adequate.

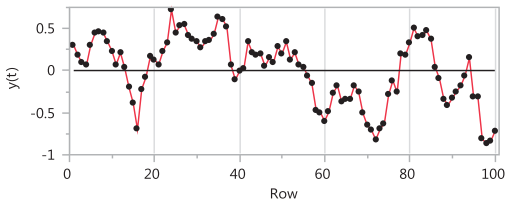
Example 6.2 illustrates transfer function modeling with a single input series where both the input and output time series were stationary. It is often necessary to incorporate **multiple input** time series into the model. A simple generalization of the single-input transfer function is to form an additive model for the inputs, say

$$y_t = \sum_{j=1}^{m} \frac{\omega_j(B)}{\delta_j(B)} x_{j,t-b_j} \tag{6.35}$$

TABLE 6.5 JMP Output for the Viscosity-Temperature Transfer Function-Noise Model

Transfer Function Analysis

Time Series y(t)



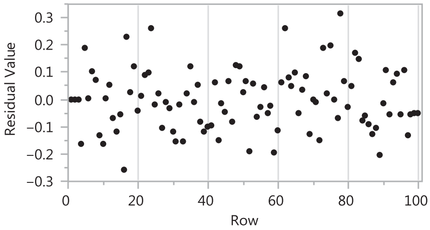
Mean	0.0001
Std	0.3860375
N	100
Zero Mean ADF	-2.038514
Single Mean ADF	-2.020093
Trend ADF	-2.548358

Transfer Function Model (1)

Model Summary

DF	93
Sum of Squared Errors	1.19314545
Variance Estimate	0.01282952
Standard Deviation	0.11326746
Akaike's 'A' Information Criterion	-142.17826
Schwarz's Bayesian Criterion	-131.87942
RSquare	0.91993657
RSquare Adj	0.91743459
MAPE	56.3367497
MAE	0.08684777
-2LogLikelihood	-150.17826

Residuals



Lag	AutoCorr	-.8-.6-.4-.2 0 .2 .4 .6 .8	Ljung-Box Q	p-Value	Lag	Partial	-.8-.6-.4-.2 0 .2 .4 .6 .8
0	1.0000		.	.	0	1.0000	
1	0.0797		0.6542	0.4186	1	0.0797	
2	0.1208		2.1735	0.3373	2	0.1152	
3	-0.0019		2.1739	0.5371	3	-0.0200	
4	-0.1254		3.8440	0.4275	4	-0.1405	
5	0.0867		4.6516	0.4599	5	0.1126	
6	-0.1280		6.4301	0.3768	6	-0.1157	
7	-0.0722		7.0020	0.4287	7	-0.0844	
8	-0.1709		10.2410	0.2485	8	-0.1526	
9	-0.1556		12.9565	0.1646	9	-0.0960	
10	-0.1092		14.3080	0.1594	10	-0.1090	
11	0.0132		14.3280	0.2154	11	0.0585	
12	0.1584		17.2361	0.1409	12	0.1468	
13	0.0907		18.2009	0.1500	13	0.0546	
14	0.0066		18.2061	0.1976	14	-0.0824	
15	-0.0355		18.3577	0.2443	15	-0.0690	
16	0.0755		19.0505	0.2661	16	0.0674	
17	0.1032		20.3586	0.2563	17	0.0600	
18	0.0639		20.8663	0.2862	18	0.0078	
19	-0.1056		22.2705	0.2710	19	-0.1365	
20	0.0298		22.3837	0.3201	20	0.1179	
21	-0.0848		23.3115	0.3276	21	-0.0129	
22	-0.1518		26.3261	0.2379	22	-0.1502	
23	-0.1162		28.1143	0.2115	23	-0.1532	
24	-0.1560		31.3818	0.1431	24	-0.0949	
25	0.0399		31.5985	0.1700	25	0.0526	

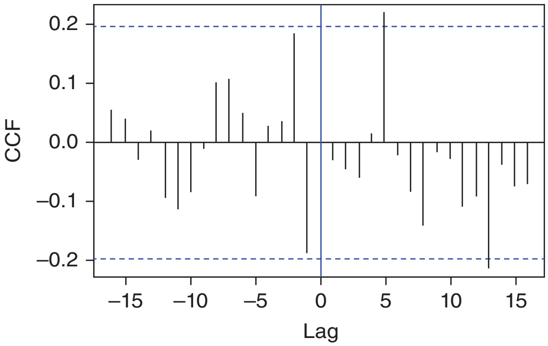


FIGURE 6.13 Sample cross-correlation function between α_1 and the residuals of the transfer function-noise model.

where each input series has a transfer function representation including a potential delay. This is an appropriate approach so long as the input series are uncorrelated with each other. If any of the original series are nonstationary, then differencing may be required. In general, differencing of a higher order than one may be required, and inputs and outputs need not be identically differenced. We now present an example from Montgomery and Weatherby (1980) where two inputs are used in the model.

Example 6.3 Montgomery and Weatherby (1980) present an example of modeling the output viscosity of a chemical process as a function of two inputs, the incoming raw material viscosity $x_{1,t}$ and the reaction temperature $x_{2,t}$. Readings are recorded hourly. Figure 6.14 is a plot of the last 100 readings. All three variables appear to be nonstationary.

Standard univariate ARIMA modeling techniques indicate that the input raw material viscosity can be modeled by an ARIMA(0, 1, 2) or IMA(1,2) process

$$(1 - B)x_{1,t} = (1 + 0.59B + 0.32B^2)\alpha_{1,t},$$

which is then used to prewhiten the output final viscosity. Similarly, an IMA(1,1) model

$$(1 - B)x_{2,t} = (1 + 0.45B)\alpha_{2,t}$$

was used to prewhiten the temperature input. Table 6.6 contains the impulse response functions between the prewhitened inputs and outputs.

Both impulse response functions in Table 6.6 exhibit approximate exponential decay beginning with lag 2 for the initial viscosity input and lag 3

FIGURE 6.14 Hourly readings of final product viscosity y_t, incoming raw material viscosity $x_{r1,t}$, and reaction temperature $x_{2,t}$.

for the temperature input. This is consistent with a tentative model identification of $r = 1, s = 0, b = 2$ for the transfer function relating initial and final viscosities and $r = 1, s = 0, b = 3$ for the transfer function relating temperature and final viscosity. The preliminary parameter estimates for these models are

$$(1 - 0.62B)y_t = 0.34x_{1,t-2}$$
$$(1 - 0.68B)y_t = 0.42x_{2,t-3}.$$

TABLE 6.6 Impulse Response Function for Example 6.3

	Impulse Response Functions	
Lag	Initial and Final Viscosity	Temperature and Final Viscosity
0	−0.422	−0.0358
1	−0.1121	−0.1835
2	0.3446	0.0892
3	0.2127	0.4205
4	0.1327	0.2849
5	0.2418	0.3102
6	0.0851	0.0899
7	0.1491	0.1712
8	0.1402	0.0051

The noise time series is then computed from

$$\hat{N}_t = y_t - \frac{0.34}{1 - 0.62B} x_{1,t-2} - \frac{0.42}{1 - 0.68B} x_{2,t-3}.$$

The sample ACF and PACF of the noise series indicate that it can be modeled as

$$(1 - B)(1 - 0.64B)\hat{N}_t = \varepsilon_t.$$

Therefore, the combined transfer function plus noise model is

$$(1 - B)y_t = \frac{\omega_{10}}{1 - \delta_{11}B}(1 - B)x_{1,t-2} + \frac{\omega_{20}}{1 - \delta_{21}B}(1 - B)x_{2,t-3} + \frac{\varepsilon_t}{1 - \phi_1 B}.$$

The estimates of the parameters in this model are shown in Table 6.7 along with other model summary statistics. The *t*-test statistics indicate that all model parameters are different from zero.

Residual checks did not reveal any problems with model adequacy. The chi-square statistic for the first 25 residual autocorrelations was 10.412. The first 20 cross-correlations between the residuals and the prewhitened input viscosity produced a chi-square test statistic of 11.028 and the first 20 cross-correlations between the residuals and the prewhitened input temperature produced a chi-square test statistic of 15.109. These chi-square

TABLE 6.7 Model Summary Statistics for the Two-Input Transfer Function Model in Example 6.3

Parameter	Estimate	Standard Error	t-statistic for H_0: Parameter = 0	95% Confidence Limits Lower	95% Confidence Limits Upper
δ_{11}	0.789	0.110	7.163	0.573	1.005
ω_{10}	0.328	0.103	3.189	0.127	0.530
δ_{21}	0.677	0.186	3.643	0.313	1.041
ω_{20}	0.455	0.124	3.667	0.212	0.698
ϕ_1	0.637	0.082	7.721	0.475	0.799

test statistics are not significant at the 25% level, so we conclude that the model is adequate. The correlation matrix of the parameter estimates is

	δ_{11}	ω_{10}	δ_{21}	ω_{20}	ϕ_1
δ_{11}	1.00	−0.37	−0.06	0.01	0.02
ω_{10}	−0.37	1.00	−0.22	0.00	0.12
δ_{21}	−0.06	−0.22	1.00	−0.14	−0.07
ω_{20}	−0.01	0.00	−0.14	1.00	−0.08
ϕ_1	−0.02	0.07	−0.07	−0.08	1.00

Notice that a complex relationship between two input variables and one output has been modeled with only five parameters and the small off-diagonal elements in the covariance matrix above imply that these parameter estimates are essentially uncorrelated.

6.6 FORECASTING WITH TRANSFER FUNCTION–NOISE MODELS

In this section we discuss making τ-step-ahead forecasts using the transfer function–noise model in Eq. (6.23). We can rearrange Eq. (6.23) and rewrite it in the difference equation form as

$$\delta(B)\varphi(B)y_t = w(B)\varphi(B)x_{t-b} + \theta(B)\delta(B)\varepsilon_t \tag{6.36}$$

or

$$\delta^*(B)y_t = w^*(B)x_{t-b} + \theta^*(B)\varepsilon_t. \tag{6.37}$$

Then at time $t + \tau$, we will have

$$y_{t+\tau} = \sum_{i=1}^{r+p^*} \delta_i^* y_{t+\tau-i} + w_0^* x_{t+\tau-b} - \sum_{i=1}^{s+p^*} w_i^* x_{t+\tau-b-i} + \varepsilon_{t+\tau} - \sum_{i=1}^{q+r} \theta_i^* \varepsilon_{t+\tau-i}, \tag{6.38}$$

where r is the order of $\delta(B), p^*$ is the order of $\varphi(B)(= \phi(B)(1 - B)^d)$, and s is the order of $\omega(B)$, and q is the order of $\theta(B)$.

The τ-step ahead MSE forecasts are obtained from

$$\begin{aligned}\hat{y}_{t+\tau}(\tau) &= E[y_{t+\tau}|y_t, y_{t-1}, \dots, x_t, x_{t-1}, \dots] \\ &= \sum_{i=1}^{r+p^*} \delta_i^* \hat{y}_{t+\tau-i}(t) + w_0^* \hat{x}_{t+\tau-b}(t) \\ &\quad - \sum_{i=1}^{s+p^*} w_i^* \hat{x}_{t+\tau-b-i}(t) - \sum_{i=1}^{q+r} \theta_i^* \varepsilon_{t+\tau-i} \quad \text{for } \tau = 1, 2, \dots, q.\end{aligned} \tag{6.39}$$

Note that the MA terms will vanish for $\tau > q + r$. We obtain Eq. (6.39) using

$$E(\varepsilon_{t+\tau-i}|y_t, y_{t-1}, \dots, x_t, x_{t-1}, \dots) = \begin{cases} \varepsilon_{t+\tau-i}, & i \geq \tau \\ 0, & i < \tau \end{cases}$$

and

$$\begin{aligned}\hat{x}_t(l) &= E(x_{t+l}|y_t, y_{t-1}, \dots, x_t, x_{t-1}, \dots) \\ &= E(x_{t+l}|x_t, x_{t-1}, \dots).\end{aligned} \tag{6.40}$$

Equation (6.40) implies that the relationship between x_t and y_t is unidirectional and that $\hat{x}_t(l)$ is the forecast from the univariate ARIMA model, $\phi_x(B)(1 - B)^d x_t = \theta_x(B)\alpha_t$.

So forecasts $\hat{y}_{t+1}(t)$, $\hat{y}_{t+2}(t), \dots$ can be computed recursively from Eqs. (6.39) and (6.40).

The variance of the forecast errors can be obtained from the infinite MA representations for x_t and N_t given as

$$\begin{aligned} x_t &= \varphi_x(B)^{-1}\theta_x(B)\alpha_t \\ &= \psi_x(B)\alpha_t \end{aligned} \tag{6.41}$$

and

$$\begin{aligned} N_t &= \varphi(B)^{-1}\theta(B)\varepsilon_t \\ &= \psi(B)\varepsilon_t \\ &= \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}. \end{aligned} \tag{6.42}$$

Hence the infinite MA form of the transfer function–noise model is given as

$$\begin{aligned} y_t &= \underbrace{v(B)\psi_x(B)}_{=v^*(B)} \alpha_{t-b} + \psi(B)\varepsilon_t \\ &= \sum_{i=0}^{\infty} v_i^* \alpha_{t-b-i} + \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}. \end{aligned} \tag{6.43}$$

Thus the minimum MSE forecast can be represented as

$$\hat{y}_{t+\tau}(t) = \sum_{i=\tau-b}^{\infty} v_i^* \alpha_{t+\tau-b-i} + \sum_{i=\tau}^{\infty} \psi_i \varepsilon_{t+\tau-i} \tag{6.44}$$

and the τ-step-ahead forecast error is

$$\begin{aligned} e_t(\tau) &= y_{t+\tau} - \hat{y}_{t+\tau}(t) \\ &= \sum_{i=0}^{\tau-b-1} v_i^* \alpha_{t+\tau-b-i} + \sum_{i=0}^{\tau-1} \psi_i \varepsilon_{t+\tau-i}. \end{aligned} \tag{6.45}$$

As we can see in Eq. (6.45), the forecast error has two components that are assumed to be independent: forecast errors in forecasting x_t, $\sum_{i=0}^{\tau-b-1} v_i^* \alpha_{t+\tau-b-i}$; and forecast errors in forecasting N_t, $\sum_{i=0}^{\tau-1} \psi_i \varepsilon_{t+\tau-i}$. The forecast variance is simply the sum of the two variances:

$$\begin{aligned} \sigma^2(\tau) &= \mathrm{Var}[e_t(\tau)] \\ &= \sigma_\alpha^2 \sum_{i=0}^{\tau-b-1} \left(v_i^*\right)^2 + \sigma_\varepsilon^2 \sum_{i=0}^{\tau-1} \psi_i^2. \end{aligned} \tag{6.46}$$

To check the effect of adding x_t in the model when forecasting, it may be appealing to compare the forecast errors between the transfer function–noise model and the univariate ARIMA model for y_t. Let the forecast error variances for the former and the latter be denoted by $\sigma^2_{\mathrm{TFN}}(\tau)$ and $\sigma^2_{\mathrm{UM}}(\tau)$, respectively. We may then consider

$$\begin{aligned} R^2(\tau) &= 1 - \frac{\sigma^2_{\mathrm{TFN}}(\tau)}{\sigma^2_{\mathrm{UM}}(\tau)} \\ &= \frac{\sigma^2_{\mathrm{UM}}(\tau) - \sigma^2_{\mathrm{TFN}}(\tau)}{\sigma^2_{\mathrm{UM}}(\tau)}. \end{aligned} \tag{6.47}$$

This quantity is expected to go down significantly if the introduction of x_t were indeed appropriate.

Example 6.4 Suppose we need to make forecasts for the next minute (6 observations) for the viscosity data in Example 6.2. We first consider the final model suggested in Example 6.2

$$y_t = \frac{w_0}{1 - \delta_1 B - \delta_2 B^2} x_{t-3} + \frac{1}{1 - \phi_1 B}\varepsilon_t.$$

After some rearrangement, we have

$$\begin{aligned} y_t = {} & (\delta_1 + \phi_1) y_{t-1} + (\delta_2 - \delta_1\phi_1) y_{t-2} - \delta_2\phi_1 y_{t-3} + w_0 x_{t-3} \\ & - w_0\phi_1 x_{t-4} + \varepsilon_t - \delta_1\varepsilon_{t-1} - \delta_2\varepsilon_{t-2}. \end{aligned}$$

From Eq. (6.38), we have the τ-step ahead prediction as

$$\begin{aligned} \hat{y}_{1+\tau}(t) = {} & (\hat{\delta}_1 + \hat{\phi}_1)[y_{t+\tau-1}] + (\hat{\delta}_2 - \hat{\delta}_1\hat{\phi}_1)[y_{t+\tau-2}] - \hat{\delta}_2\hat{\phi}_1[y_{t+\tau-3}] \\ & + \hat{w}_0[x_{t+\tau-3}] - \hat{w}_0\hat{\phi}_1[x_{t+\tau-4}] + [\varepsilon_{t+\tau}] - \hat{\delta}_1[\varepsilon_{t+\tau-1}] - \hat{\delta}_2[\varepsilon_{t+\tau-2}], \end{aligned}$$

where

$$[y_{t+j}] = \begin{cases} y_{t+j}, & j \le 0 \\ \hat{y}_{t+j}(t), & j > 0 \end{cases}$$

$$[x_{t+j}] = \begin{cases} x_{t+j}, & j \le 0 \\ \hat{x}_{t+j}(t), & j > 0 \end{cases}$$

and

$$[\varepsilon_{t+j}] = \begin{cases} \hat{\varepsilon}_{t+j}, & j \leq 0 \\ 0, & j > 0 \end{cases}.$$

Hence for the current and past response and input values, we can use the actual data. For the future response and input values we will instead use their respective forecasts. To forecast the input variable x_t, we will use the AR(1) model, $(1 - 0.73B)x_t = \alpha_t$ from Example 6.2. As for the error estimates, we can use the residuals from the transfer function-noise model or for $b \geq 1$, the one-step-ahead forecast errors for the current and past values of the errors, and set the error estimates equal to zero for future values.

We can obtain the variance of the prediction error from Eq. (6.45). The estimates of σ_α^2 and σ_ε^2 in Eq. (6.45) can be obtained from the univariate AR(1) model for x_t, and the transfer function-noise model from Example 6.2, respectively. Hence for this example we have $\hat{\sigma}_\alpha^2 = 0.0102$ and $\hat{\sigma}_\varepsilon^2 = 0.0128$. The coefficients in $v^*(B)$ and $\psi(B)$ can be calculated from

$$v^*(B) = \sum_{i=0}^{\infty} v_i^* B^i = v(B)\psi_x(B)$$

$$(v_0^* + v_1^* B + v_2^* B^2 + \ldots) = \frac{w_0}{(1 - \delta_1 B - \delta_2 B^2)}(1 - \phi_x B)^{-1}$$

or

$$(v_0^* + v_1^* B + v_2^* B^2 + \ldots)(1 - \delta_1 B - \delta_2 B^2)(1 - \phi_x B) = w_0$$

which means

$$\begin{aligned} v_0^* &= w_0 \\ v_1^* &= (\delta_1 + \phi_x)v_0^* = (\delta_1 + \phi_x)w_0 \\ v_2^* &= (\delta_1 + \phi_x)v_1^* = (\delta_2 - \delta_1\phi_x)v_0^* \\ &= [(\delta_1 + \phi_x)^2 + (\delta_2 - \delta_1\phi_x)]w_0 \\ &\vdots \end{aligned}$$

and

$$\psi_i = \phi_1^i \qquad \text{for } i = 0, 1, 2 \ldots$$

Hence the estimates of the coefficients in $v^*(B)$ and $\psi(B)$ can be obtained by using the estimates of the parameters given in Example 6.2. Note that

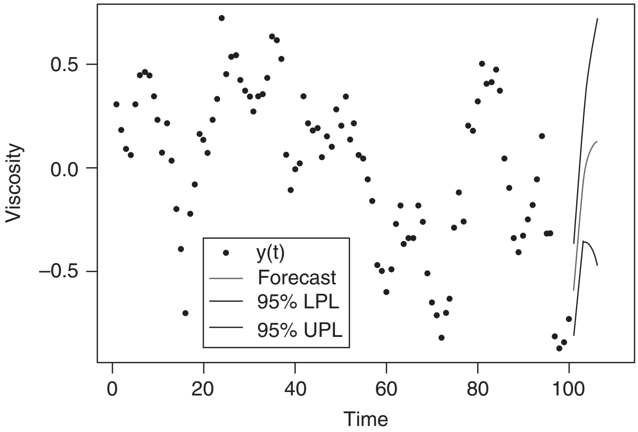


FIGURE 6.15 The time series plots of the actual and 1- to 6-step ahead forecasts for the viscosity data.

for up to 6-step-ahead forecasts, from Eq. (6.45), we will only need to calculate v_0^*, v_1^* and v_2^*.

The time series plot of the forecasts is given in Figure 6.15 together with the approximate 95% prediction limits calculate.by $\pm 2\hat{\sigma}(\tau)$.

For comparison purposes we fit a univariate ARIMA model for y_t. Following the model identification procedure given in Chapter 5, an AR(3) model is deemed a good fit. The estimated standard deviations of the prediction error for the transfer function-noise model and the univariate model are given in Table 6.8. It can be seen that adding the exogenous variable x_t helps to reduce the prediction error standard deviation.

TABLE 6.8 Estimated standard deviations of the prediction error for the transfer function-noise model (TFM) and the univariate model (UM)

	Estimated Standard Deviation of the Prediction Error	
Observation	TFM	UM
101	0.111	0.167
102	0.137	0.234
103	0.149	0.297
104	0.205	0.336
105	0.251	0.362
106	0.296	0.376

6.7 INTERVENTION ANALYSIS

In some cases, the response y_t can be affected by a known event that happens at a specific time such as fiscal policy changes, introduction of new regulatory laws, or switching suppliers. Since these **interventions** do not have to be quantitative variables, we can represent them with indicator variables. Consider, for example, the transfer function–noise model as the following:

$$\begin{aligned} y_t &= \frac{w(B)}{\delta(B)}\xi_t^{(T)} + \frac{\theta(B)}{\varphi(B)}\varepsilon_t \\ &= v(B)\xi_t^{(T)} + N_t, \end{aligned} \tag{6.48}$$

where $\xi_t^{(T)}$ is a deterministic indicator variable, taking only the values 0 and 1 to indicate nonoccurrence and occurrence of some event. The model in Eq. (6.48) is called the **intervention model**. Note that this model has only one intervention event. Generalization of this model with several intervention events is also possible.

The most common indicator variables are the pulse and step variables,

$$P_t^{(T)} = \begin{cases} 0 & \text{if } t \neq T \\ 1 & \text{if } t = T \end{cases} \tag{6.49}$$

and

$$S_t^{(T)} = \begin{cases} 0 & \text{if } t < T \\ 1 & \text{if } t \geq T \end{cases}, \tag{6.50}$$

where T is a specified occurrence time of the intervention event. The transfer function operator $v(B) = w(B)/\delta(B)$ in Eq. (6.48) usually has a fairly simple and intuitive form.

Examples of Responses to Pulse and Step Inputs

1. We will first consider the pulse indicator variable. We will further assume a simple transfer function–noise model as

$$y_t = \frac{w_0}{1 - \delta B} P_t^{(T)}. \tag{6.51}$$

After rearranging Eq. (6.51), we have

$$(1 - \delta B)\, y_t = w_0 P_t^{(T)} = \begin{cases} 0 & \text{if } t \neq T \\ w_0 & \text{if } t = T \end{cases}$$

or

$$y_t = \delta y_{t-1} + w_0 P_t^{(T)}$$

So we have

$$\begin{aligned} y_T &= w_0 \\ y_{T+1} &= \delta y_T = \delta w_0 \\ y_{T+2} &= \delta y_{T+1} = \delta^2 y_T = \delta^2 w_0 \\ &\vdots \\ y_{T+k} &= \cdots = \delta^k y_T = \delta^k w_0, \end{aligned}$$

which means

$$y_t = \begin{cases} 0 & \text{if } t < T \\ w_0 s^{t-T} & \text{if } t \geq T \end{cases}.$$

2. For the step indicator variable with the same transfer function–noise model as in the previous case, we have

$$y_t = \frac{w_0}{1 - \delta B} S_t^{(T)}.$$

Solving the difference equation

$$(1 - \delta B)\, y_t = w_0 S_t^{(T)} = \begin{cases} 0 & \text{if } t < T \\ w_0 & \text{if } t \geq T \end{cases}$$

we have

$$\begin{aligned} y_T &= w_0 \\ y_{T+1} &= \delta y_T + w_0 = w_0(1 + \delta) \\ y_{T+2} &= \delta y_{T+1} + w_0 = w_0(1 + \delta + \delta^2) \\ &\vdots \\ y_{T+k} &= \delta y_{T+k-1} + w_0 = w_0(1 + \delta + \cdots + \delta^k) \end{aligned}$$

or

$$y_t = w_0(1 + \delta + \cdots + \delta^{t-T}) \quad \text{for } t \geq T$$

In intervention analysis, one of the things we could be interested in may be how permanent the effect of the event will be. Generally, for $y_t = (w(B)/\delta(B))\xi_t^{(T)}$ with stable $\delta(B)$, if the intervention event is a pulse, we will then have a transient (short-lived) effect. On the other hand, if the intervention event is a step, we will have a permanent effect. In general, depending on the form of the transfer function, there are many possible responses to the step and pulse inputs. Table 6.9 displays the output

TABLE 6.9 Output responses to step and pulse inputs.

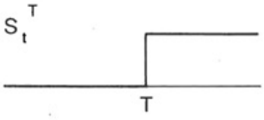
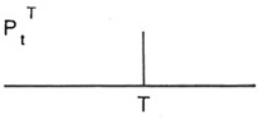
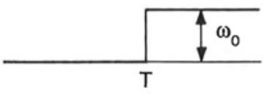
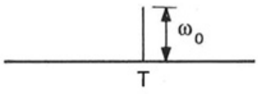
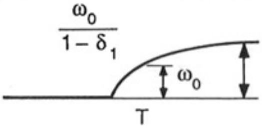
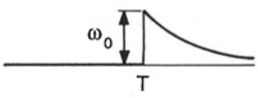
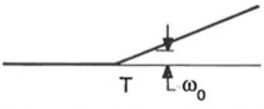
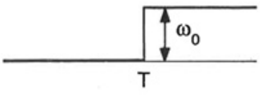
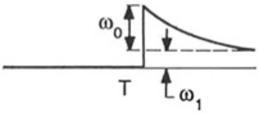
	Step input	Pulse input
$\frac{\omega(B)}{\delta(B)}$		
	Output response	Output response
ω_0		
$\frac{\omega_0}{1-\delta_1 B}$		
$\frac{\omega_0}{1-B}$		
$\frac{\omega_0}{1-\delta_1 B} + \frac{\omega_1}{1-B}$		

TABLE 6.10 Weekly Cereal Sales Data

Week	Sales	Week	Sales	Week	Sales	Week	Sales
1	102,450	27	114,980	53	167,170	79	181,560
2	98,930	28	130,250	54	161,200	80	202,130
3	91,550	29	128,070	55	166,710	81	183,740
4	111,940	30	135,970	56	156,430	82	191,880
5	103,380	31	142,370	57	162,440	83	197,950
6	112,120	32	121,300	58	177,260	84	209,040
7	105,780	33	121,380	59	163,920	85	203,990
8	103,000	34	128,790	60	166,040	86	201,220
9	111,920	35	139,290	61	182,790	87	202,370
10	106,170	36	128,530	62	169,510	88	201,100
11	106,350	37	139,260	63	173,940	89	203,210
12	113,920	38	157,960	64	179,350	90	198,770
13	126,860	39	145,310	65	177,980	91	171,570
14	115,680	40	150,340	66	180,180	92	184,320
15	122,040	41	158,980	67	188,070	93	182,460
16	134,350	42	152,690	68	191,930	94	173,430
17	131,200	43	157,440	69	186,070	95	177,680
18	132,990	44	144,500	70	171,860	96	186,460
19	126,020	45	156,340	71	180,240	97	185,140
20	152,220	46	137,440	72	180,910	98	183,970
21	137,350	47	166,750	73	185,420	99	154,630
22	132,240	48	171,640	74	195,470	100	174,720
23	144,550	49	170,830	75	183,680	101	169,580
24	128,730	50	174,250	76	190,200	102	180,310
25	137,040	51	178,480	77	186,970	103	154,080
26	136,830	52	178,560	78	182,330	104	163,560

responses to the unit step and pulse inputs for several transfer function model structures. This table is helpful in model formulation.

Example 6.5 The weekly sales data of a cereal brand for the last two years are given in Table 6.10. As can be seen from Figure 6.16, the sales were showing a steady increase during most of the two-year period. At the end of the summer of the second year (Week 88), the rival company introduced a similar product into the market. Using intervention analysis, we want to study whether that had an effect on the sales. For that, we will first fit an ARIMA model to the preintervention data from Week 1 to Week 87. The sample ACF and PACF of the data for that time period in Figure 6.17

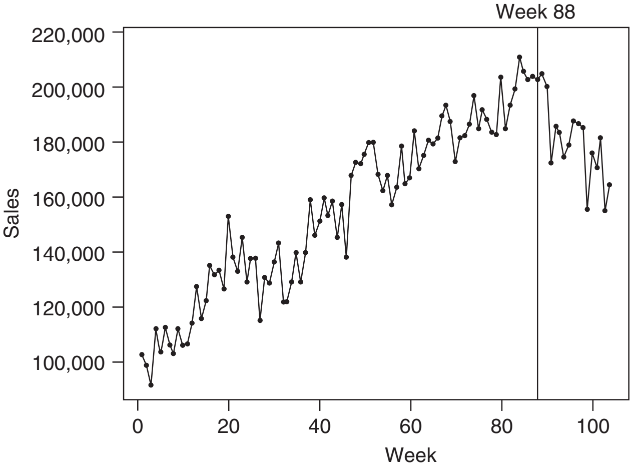


FIGURE 6.16 Time series plot of the weekly sales data.

show that the process is nonstationary. The sample ACF and PACF of the first difference given in Figure 6.18 suggest that an ARIMA(0,1,1) model is appropriate. Then the intervention model has the following form:

$$y_t = w_0 S_t^{(88)} + \frac{1 - \theta B}{1 - B} \varepsilon_t,$$

where

$$S_t^{(88)} = \begin{cases} 0 & \text{if } t < 88 \\ 1 & \text{if } t \geq 88 \end{cases}.$$

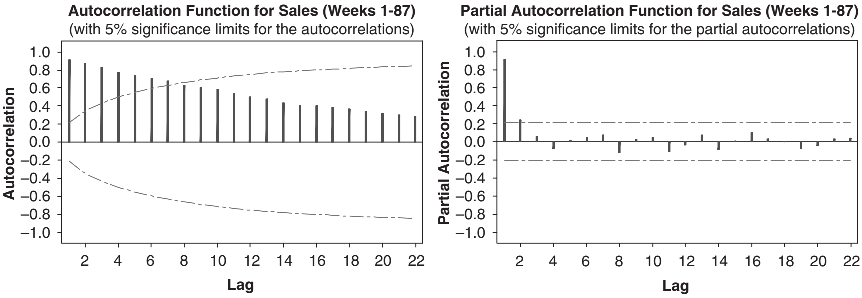


FIGURE 6.17 Sample ACF and PACF pulse of the sales data for weeks 1–87.

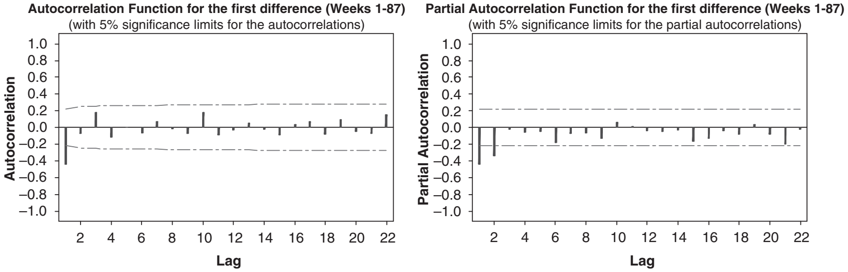


FIGURE 6.18 Sample ACF and PACF plots of the first difference of the sales data for weeks 1–87.

This means that for the intervention analysis we assume that the competition simply slows down (or reverses) the rate of increase in the sales. To fit the model we use the transfer function model option in JMP with $S_t^{(88)}$ as the input. The output in Table 6.11 shows that there was indeed a significant effect on sales due to the introduction of a similar product in the market. The coefficient estimate $\hat{w}_0 = -2369.9$ further suggests that if no appropriate action is taken, the sales will most likely continue to go down.

Example 6.6 Electricity Consumption and the 1973 Arab Oil Embargo The natural logarithm of monthly electric energy consumption in megawatt hours (MWh) for a regional utility from January 1951 to April 1977 is shown in Figure 6.19. The original data exhibited considerable inequality

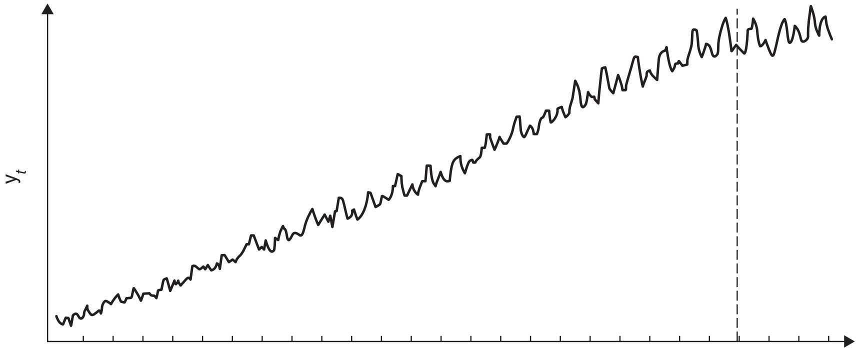
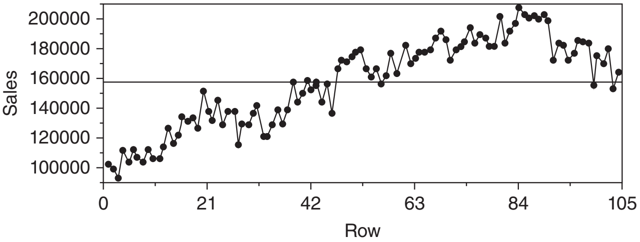


FIGURE 6.19 Natural logarithm of monthly electric energy consumption in megawatt hours (MWh) from January 1951 to April 1977.

TABLE 6.11 JMP Output for the Intervention Analysis in Example 6.5

Transfer Function Analysis
Time Series Sales



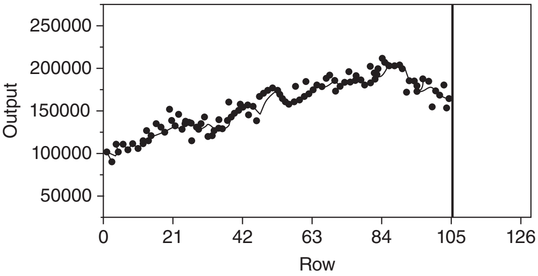
```
Mean            157540.87
Std             29870.833
N                     104
Zero Mean ADF   0.0886817
Single Mean ADF -2.381355
Trend ADF       -3.770646
```

```
Transfer Function Model (1)
 Model Summary
 DF                                             101
 Sum of Squared Errors                   9554575810
 Variance Estimate                       94599755.5
 Standard Deviation                      9726.24056
 Akaike's 'A' Information Criterion      2186.26524
 Schwarz's Bayesian Criterion            2191.5347
 RSquare                                 0.26288323
 RSquare Adj                             0.25558504
 MAPE                                    4.84506071
 MAE                                     7399.35196
 -2LogLikelihood                         2182.26524
```

```
Parameter Estimates
Variable   Term     Factor  Lag   Estimate  Std Error  t Ratio  Prob>|t|
Step       Num0.0        0    0  -2359.900   1104.753    -2.15
Sales      MA1,1         1    1      0.557      0.076     7.36
```

$(1-B)*\mathrm{Sales}_t=- (2369.8995 * \mathrm{Step}_t)+(1 - 0.5571*B)*\mathbf{e}_t$

Interactive Forecasting



of variance over this time period and the natural logarithm stabilizes the variance. In November of 1973 the Arab oil embargo affected the supply of petroleum products to the United States. Following this event, it is hypothesized that the rate of growth of electricity consumption is smaller than in the pre-embargo period. Montgomery and Weatherby (1980) tested this hypothesis using an intervention model.

Although the embargo took effect in November of 1973, we will assume that first impact of this was not felt until the following month, December, 1973. Therefore, the period from January of 1951 until November of 1973 is assumed to have no intervention effect. These 275 months are analyzed to produce a univariate ARIMA model for the noise model. Both regular and seasonal differencing are required to achieve stationary and a multiplicative $(0, 1, 2) \times (0, 1, 1)_{12}$ was fit to the pre-intervention data. The model is

$$(1 - B)(1 - B^{12}) \ln N_t = (1 - 0.40B - 0.28B^2)(1 - 0.64B^{12})\varepsilon_t.$$

To develop the intervention, it is necessary to hypothesize the effect that the oil embargo may have had on electricity consumption. Perhaps the simplest scenario in which the level of the consumption time series is hypothesized to be permanently changed by a constant amount. Logarithms can be thought of as percent change, so this is equivalent to hypothesizing that the intervention effect is a change in the percent growth of electricity consumption. Thus the input series would be a step function

$$S_t = \begin{cases} 0, t = 1, 2, \dots, 275 \\ 1, t = 275, 276, 326 \end{cases}.$$

The intervention model is then

$$(1 - B)(1 - B^{12}) \ln y_t = \omega_0 S_t + (1 - \theta_1 B - \theta_2 B^2)(1 - \Theta_{12} B^{12})\varepsilon_t,$$

where Θ_{12} is the seasonal MA parameter at lag 12. Table 6.12 gives the model parameter estimates and the corresponding 95% confidence intervals. Since the 95% confidence intervals do not contain zero, we conclude that all model parameters are statistically significant. The residual standard error for this model is 0.029195. Since the model was built to the natural logarithm of electricity consumption, the standard error has a simple, direct interpretation: namely, the standard deviation of one-step-ahead forecast errors is 2.9195 percent of the level of the time series.

TABLE 6.12 Model Summary Statistics for Example 6.6

		95% Confidence Limits	
Parameter	Point Estimate	Lower	Upper
ω_0	−0.07303	−0.11605	−0.03000
θ_1	0.40170	0.28964	0.51395
θ_2	0.27739	0.16448	0.39030
Θ_{12}	0.64225	0.54836	0.73604

It is also possible to draw conclusions about the intervention effect. This effect is a level change of magnitude $\hat{\omega}_0 = -0.07303$, expressed in the natural logarithm metric. The estimate of the intervention effect in the original MWh metric is $e^{\hat{\omega}_0} = e^{-0.07303} = 0.9296$. That is, the post-intervention level of electricity consumption is 92.96 percent of the pre-intervention level. The effect of the Arab oil embargo has been to reduce the increase in electricity consumption by 7.04 percent. This is a statistically significant effect.

In this example, there are 275 pre-intervention observations and 41 post-intervention observations. Generally, we would like to have as many observations as possible in the post-intervention period to ensure that the power of the test for the intervention effect is high. However, an extremely long post-intervention period may allow other unidentified factors to affect the output, leading to potential confounding of effects. The ability of the procedure to detect an intervention effect is a function of the number of pre- and post-intervention observations, the size of the intervention effect, the form of the noise model, and the parameter values of the process. In many cases, however, an intervention effect can be identified with relatively short record of post-intervention observations.

It is interesting to consider alternative hypotheses regarding the impact of the Arab oil embargo. For example, it may be more reasonable to suspect that the effect of the oil embargo is not to cause an immediate level change in electricity consumption, but a gradual one. This would suggest a model

$$(1 - B)(1 - B^{12}) \ln y_t = \frac{\omega_0}{1 - \delta_1 B} S_t + (1 - \theta_1 B - \theta_2 B^2)(1 - \Theta_{12} B^{12})\varepsilon_t.$$

The results of fitting this model to the data are shown in Table 6.13. Note that the 95% confidence interval for δ_1 includes zero, implying that we can

TABLE 6.13 Model Summary Statistics for the Alternate Intervention Model for Example 6.6

		95% Confidence Limits	
Parameter	Point Estimate	Lower	Upper
ω_0	−0.06553	−0.11918	−0.01187
δ_1	0.18459	−0.51429	0.88347
θ_1	0.40351	0.29064	0.51637
θ_2	0.27634	0.16002	0.38726
Θ_{12}	0.63659	0.54201	0.73117

drop this parameter from the model. This would leave us with the original intervention model that was fit in Example 6.6. Consequently, we conclude that the Arab oil embargo induced an immediate permanent change in the level of electricity consumption.

In some problems there may be multiple intervention effects. Generally, one indicator variable must be used for each intervention effect. For example, suppose that in the electricity consumption example, we think that the oil embargo had two separate effects: the initial impact beginning in month 276, and a second impact beginning three months later. The intervention model to incorporate these effects is

$$(1 - B)(1 - B^{12}) \ln y_t = \omega_{10} S_{1,t} + \omega_{20} S_{2,t} + (1 - \theta_1 B - \theta_2 B^2)$$
$$(1 - \Theta_{12} B^{12})\varepsilon_t,$$

where

$$S_{1,t} = \begin{cases} 0, & t = 1, 2, 3, \ldots, 275 \\ 1, & t = 276, 277, \ldots, 316 \end{cases}$$

and

$$S_{2,t} = \begin{cases} 0, & t = 1, 2, 3, \ldots, 278 \\ 1, & t = 279, 277, \ldots, 316^{\cdot} \end{cases}$$

In this model the parameters ω_{10} and ω_{20} represent the initial and secondary effects of the oil embargo and $\omega_{10} + \omega_{20}$ represents the long-term total impact.

There have been many other interesting applications of intervention analysis. For some very good examples, see the following references:

- Box and Tiao (1975) investigate the effects on ozone (O_3) concentration in downtown Los Angeles of a new law that restricted the amount of reactive hydrocarbons in locally sold gasoline, regulations that mandated automobile engine design changes, and the diversion of traffic by opening of the Golden State Freeway. They showed that these interventions did indeed lead to reductions in ozone levels.
- Wichern and Jones (1977) analyzed the impact of the endorsement by the American Dental Association of Crest toothpaste as an effective aid in reducing cavities on the market shares of Crest and Colgate toothpaste. The endorsement led to a significant increase in market share for Crest. See Bisgaard and Kulahci (2011) for a detailed analysis of that example.
- Atkins (1979) used intervention analysis to investigate the effect of compulsory automobile insurance, a company strike, and a change in insurance companies' policies on the number of highway accidents on freeways in British Columbia.
- Izenman and Zabell (1981) study the effect of the 9 November, 1965, blackout in New York City that resulted from a widespread power failure, on the birth rate nine months later. An article in *The New York Times* in August 1966 noted that births were up, but subsequent medical and demographic articles appeared with conflicting statements. Using the weekly birth rate from 1961 to 1966, the authors show that there is no statistically significant increase in the birth rate.
- Ledolter and Chan (1996) used intervention analysis to study the effect of a speed change on rural interstate highways in Iowa on the occurrence of traffic accidents.

Another important application of intervention analysis is in the detection of **time series outliers**. Time series observations are often influenced by external disruptive events, such as strikes, social/political events, economic crises, or wars and civil disturbances. The consequences of these events are observations that are not consistent with the other observations in the time series. These inconsistent observations are called **outliers**. In addition to the external events identified above, outliers can also be caused by more mundane forces, such as data recording or transmission errors. Outliers can have a very disruptive effect on model identification, parameter estimation, and forecasting, so it is important to be able to detect their presence so that they can be removed. Intervention analysis can be useful for this.

There are two kinds of time series outliers: additive outliers and innovation outliers. An additive outlier affects only the level of the t^* observation, while an innovation outlier affects all observations $y_{t^*}, y_{t^*+1}, y_{t^*+2}, \cdots$ beyond time t^* where the original outlier effect occurred. An additive outlier can be modeled as

$$z_t = \frac{\theta(B)}{\phi(B)}\varepsilon_t + \omega I_t^{(t^*)},$$

where $I_t^{(t^*)}$is an indicator time series defined as

$$I_t^{(t^*)} = \begin{cases} 1 & \text{if } t = t^* \\ 0 & \text{if } t \neq t^* \end{cases}.$$

An innovation outlier is modeled as

$$z_t = \frac{\theta(B)}{\phi(B)}(\varepsilon_t + \omega I_t^{(t^*)}).$$

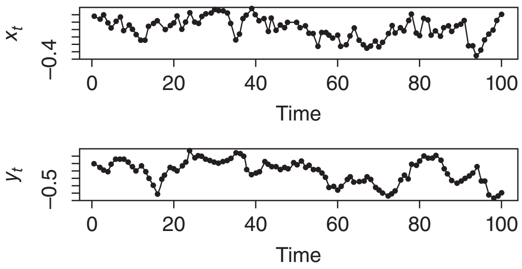
When the timing of the outlier is known, it is relatively straightforward to fit the intervention model. Then the presence of the outlier can be tested by comparing the estimate of the parameter ω, say, $\hat{\omega}$, to its standard error. When the timing of the outlier is not known, an iterative procedure is required. This procedure is described in Box, Jenkins, and Reinsel (1994) and in Wei (2006). The iterative procedure is capable of identifying multiple outliers in the time series.

6.8 R COMMANDS FOR CHAPTER 6

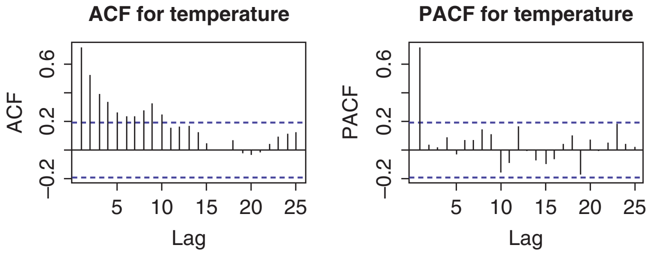
Example 6.2 The data for this example are in the array called vistemp.data of which the two columns represent the viscosity and the temperature respectively.

Below we first start with the prewhitening step.

```
xt<-vistemp.data[,1]
yt<-vistemp.data[,2]

par(mfrow=c(2,1),oma=c(0,0,0,0))
plot(xt,type="o",pch=16,cex=.5,xlab='Time',ylab=expression
    (italic(x[italic(t)])))
plot(yt,type="o",pch=16,cex=.5,xlab='Time',ylab= expression
    (italic(y[italic(t)])))
```



```
#Prewhitening
#Model identification for xt
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(xt,lag.max=25,type="correlation",main="ACF for Temperature")
acf(xt, lag.max=25,type="partial",main="PACF for Temperature",
    ylab="PACF")
```



Fit an AR(1) model to xt.

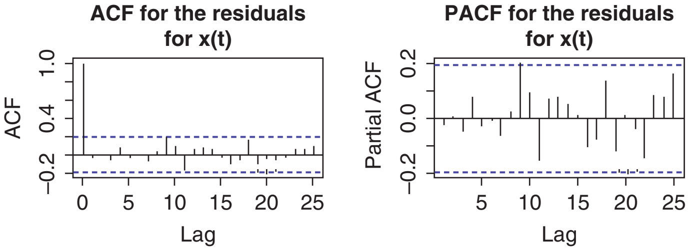
```
xt.ar1<-arima(xt,order=c(1, 0, 0),include.mean=FALSE)
xt.ar1
```

```
        Call:
        arima(x = xt, order = c(1, 0, 0), include.mean = FALSE)

        Coefficients:
                 ar1
              0.7292
        s.e.  0.0686
        sigma^2 estimated as 0.01009:  log likelihood = 87.54,
          aic = -171.08
```

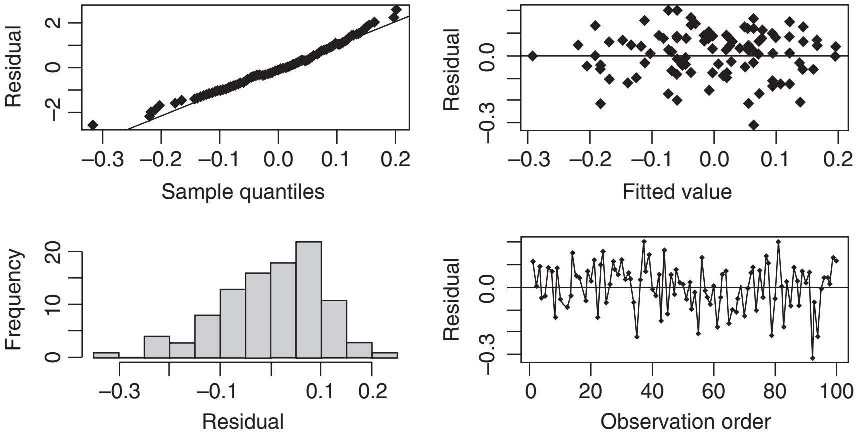
We perform the residual analysis

```
res.xt.ar1(-as.vector(residuals(xt.ar1))
#to obtain the fitted values we use the function fitted() from the
 forecast package
```

```
library(forecast)
fit.xt.ar1<-as.vector(fitted(xt.ar1))
# ACF and PACF of the Residuals
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(res.xt.ar1,lag.max=25,type="correlation",main="ACF of the
    Residuals for x(t)")
acf(res.xt.ar1, lag.max=25,type="partial",main="PACF of the
    Residuals for x(t)")
```

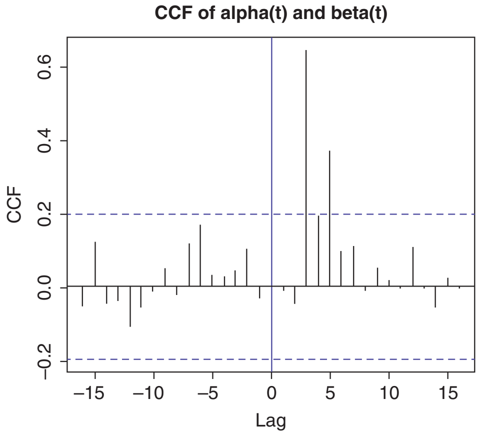


```
# 4-in-1 plot of the residuals
par(mfrow=c(2,2),oma=c(0,0,0,0))
qqnorm(res.xt.ar1,datax=TRUE,pch=16,xlab='Residual',main=")
qqline(res.xt.ar1,datax=TRUE)
plot(fit.xt.ar1,res.xt.ar1,pch=16, xlab='Fitted Value',
     ylab='Residual')
abline(h=0)
hist(res.xt.ar1,col="gray",xlab='Residual',main=")
plot(res.xt.ar1,type="l",xlab='Observation Order',ylab='Residual')
points(res.xt.ar1,pch=16,cex=.5)
abline(h=0)
```

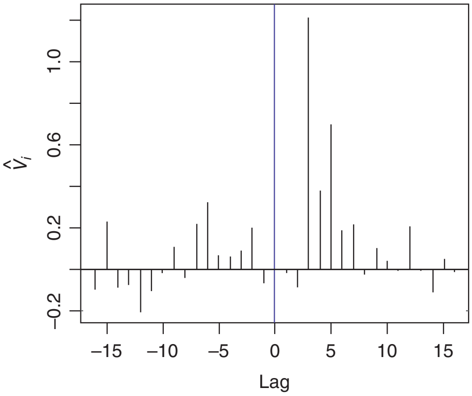


Prewhitening both series using AR(1) coefficients of 0.73.

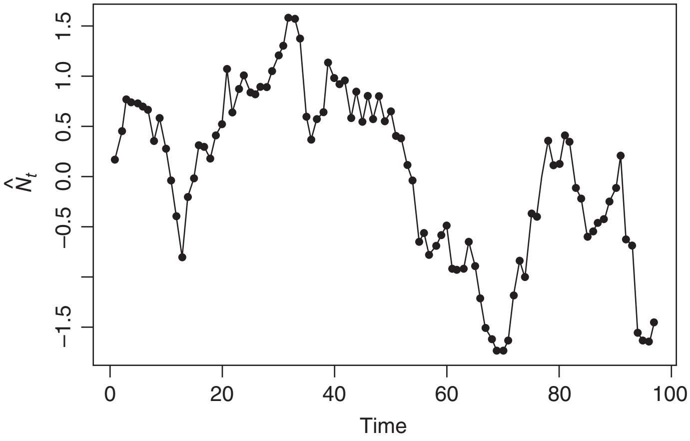
```
T<-length(xt)
alphat<-xt[2:T]-0.73*xt[1:(T-1)]
betat<- yt[2:T]-0.73*yt[1:(T-1)]
ralbe<-ccf(betat,alphat,main='CCF of alpha(t) and beta(t)',
           ylab='CCF')
abline(v=0,col='blue')
```



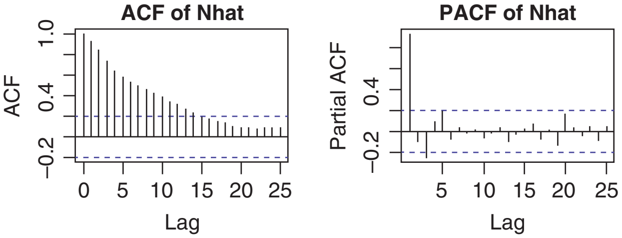
```
#Obtain the estimates of v_t
vhat<-sqrt(var(betat)/var(alphat))*ralbe$acf
nl<-length(vhat)
plot(seq(-(nl-1)/2,(nl-1)/2,1),vhat,
type='h',xlab='Lag',ylab=expression(italic(hat(v)[italic(j)])))
abline(v=0,col='blue')
abline(h=0)
```



```
#Model the noise using the estimates given in the example

Nhat<-array(0,dim=c(1,T))
for (i in 4:T){
Nhat[i]<-yt[i]+0.31*(Nhat[i-1]-yt[i-1])+0.48*(Nhat[i-2]-yt[i-2])
    +1.21*xt[i-3]
         }
Nhat<-Nhat[4:T]
plot(Nhat,type="o",pch=16,cex=.5,xlab='Time',ylab=expression
    (italic(hat(N)[italic(t)])))
```

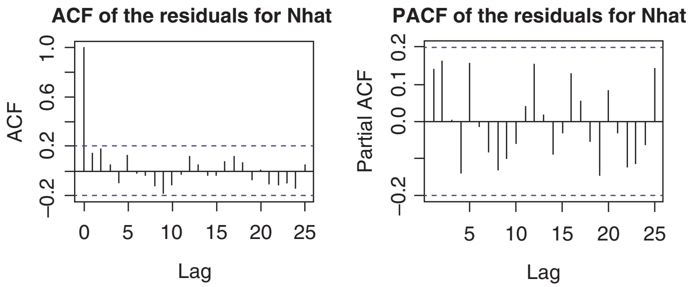


```
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(Nhat,lag.max=25,type="correlation",main="ACF of Nhat")

acf(Nhat, lag.max=25,type="partial",main="PACF of Nhat")
```

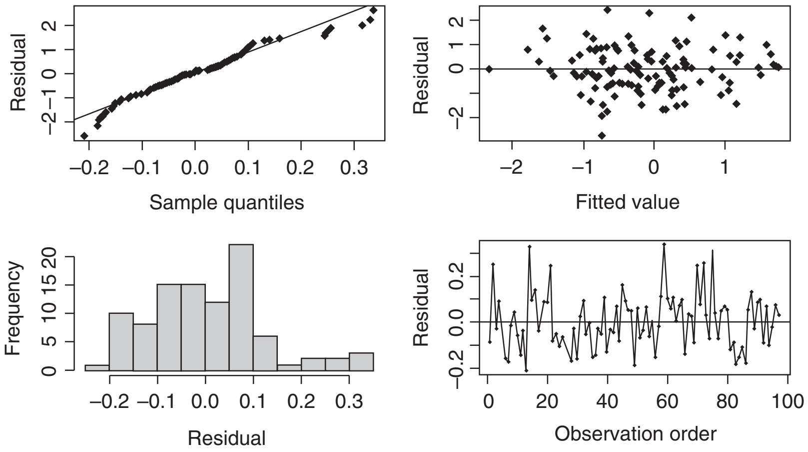


```
#Fit AR(1) and AR(3) models for Nhat

Nhat.ar1<-arima(Nhat,order=c(1, 0, 0),include.mean=FALSE)
Nhat.ar3<-arima(Nhat,order=c(3, 0, 0),include.mean=FALSE)
res.Nhat.ar1<-as.vector(residuals(Nhat.ar1))
```

```
library(forecast)
fit.Nhat.ar1<-as.vector(fitted(Nhat.ar1))
# ACF and PACF of the Residuals
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(res.Nhat.ar1,lag.max=25,type="correlation",main="ACF of the
    Residuals for Nhat")
acf(res.Nhat.ar1, lag.max=25,type="partial",main="PACF of the
    Residuals for Nhat")
```



```
# 4-in-1 plot of the residuals
par(mfrow=c(2,2),oma=c(0,0,0,0))
qqnorm(res.Nhat.ar1,datax=TRUE,pch=16,xlab='Residual',main=")
qqline(res.Nhat.ar1,datax=TRUE)
plot(fit.xt.ar1,res.xt.ar1,pch=16, xlab='Fitted Value',
     ylab='Residual')
abline(h=0)
hist(res.Nhat.ar1,col="gray",xlab='Residual',main=")
plot(res.Nhat.ar1,type="l",xlab='Observation Order',ylab='Residual')
points(res.Nhat.ar1,pch=16,cex=.5)
abline(h=0)
```

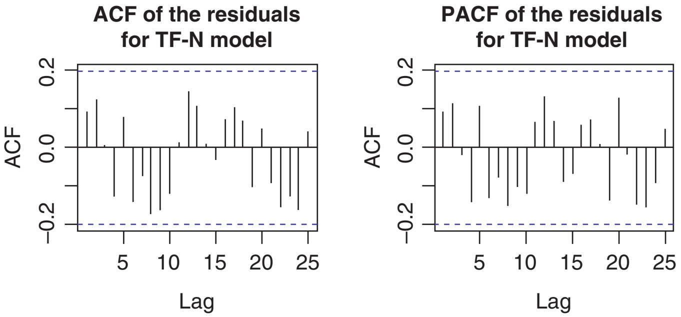


We now fit the following transfer function–noise model

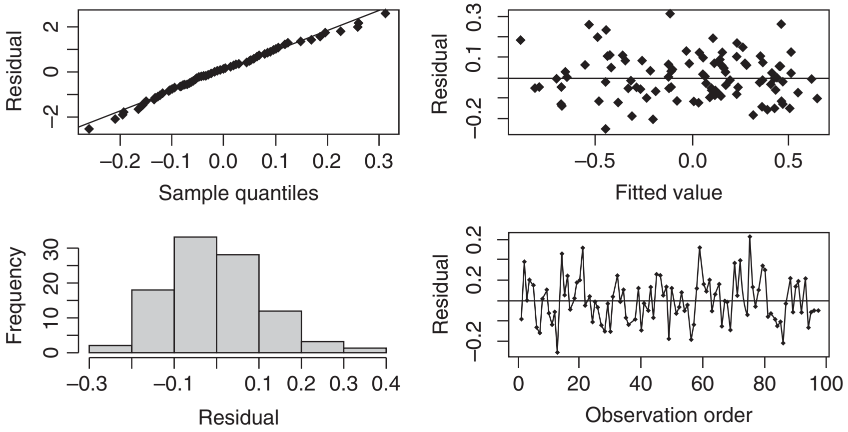
$$y_t = \frac{w_0}{1 - \delta_1 B - \delta_2 B^2} x_{t-3} + \frac{1}{1 - \phi_1 B} \varepsilon_t.$$

For that we will use the “arimax” function in TSA package.

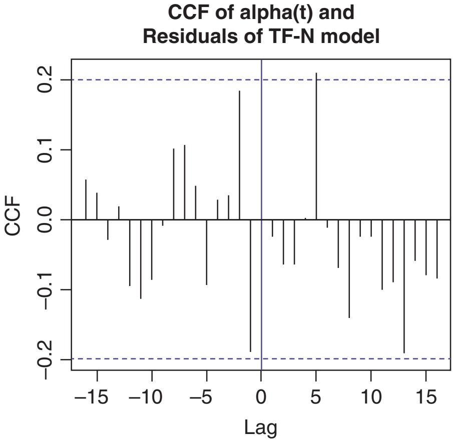
```
library(TSA)
ts.xt<-ts(xt)
lag3.x<-lag(ts.xt,-3)
ts.yt<-ts(yt)
dat3<-cbind(ts.xt,lag3.x,ts.yt)
dimnames(dat3)[[2]]<-c("xt","lag3x","yt")
data2<-na.omit(as.data.frame(dat3))
#Input arguments
#order: determines the model for the error component, i.e. the
#order of the ARIMA model for y(t)
#if there were no x(t)

#xtransf: x(t)
#transfer: the orders (r and s) of the transfer function
visc.tf<-arimax(data2$yt, order=c(1,0,0), xtransf=data.frame(data2$lag3x),
                      transfer=list(c(2,0)), include.mean = FALSE)
visc.tf

Call:
arimax(x = data2$yt, order = c(1, 0, 0), include.mean = FALSE, xtransf =
data.frame(data2$lag3x), transfer = list(c(2, 0)))

Coefficients:
         ar1  data2.lag3x-AR1  data2.lag3x-AR2  data2.lag3x-MA0
      0.8295           0.3414           0.2667           1.3276
s.e.  0.0642           0.0979           0.0934           0.1104

sigma^2 estimated as 0.0123:  log likelihood = 75.09,  aic = -142.18


res.visc.tf<-as.vector(residuals(visc.tf))
library(forecast)
fit.visc.tf <-as.vector(fitted(visc.tf))
# ACF and PACF of the Residuals
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(res.visc.tf,lag.max=25,type="correlation",main="ACF of the
    Residuals \nfor TF-N Model")
acf(res.visc.tf, lag.max=25,type="partial",main="PACF of the
    Residuals \nfor TF-N Model")
```



```
# 4-in-1 plot of the residuals
par(mfrow=c(2,2),oma=c(0,0,0,0))
qqnorm(res.visc.tf,datax=TRUE,pch=16,xlab='Residual',main=")
qqline(res.visc.tf,datax=TRUE)
plot(fit.visc.tf,res.visc.tf,pch=16, xlab='Fitted Value',
     ylab='Residual')
abline(h=0)
hist(res.visc.tf,col="gray",xlab='Residual',main=")
plot(res.visc.tf,type="l",xlab='Observation Order',ylab='Residual')
points(res.visc.tf,pch=16,cex=.5)
abline(h=0)
```



```
T<-length(res.visc.tf)
Ta<-length(alphat)
ccf(res.visc.tf,alphat[(Ta-T+1):Ta],main='CCF of alpha(t) and
    \nResiduals of TF-N Model',ylab='CCF')
abline(v=0,col='blue')
```



Example 6.4 Note that this is a continuation of Example 6.2. Below we assume that the reader followed the modeling efforts required in Example 6.2. For variable and model names, please refer to R-code for Example 6.2.

For forecasting we use the formula given in the example. Before we proceed, we first make forecasts for x(t) based on the AR(1) model. We will only make 6-step-ahead forecasts for x(t) even if not all of them are needed due to delay.

```
tau<-6
xt.ar1.forecast<-forecast(xt.ar1,h=tau)
```

To make the recursive calculations given in the example simpler, we will simply extend the xt, yt and residuals vectors as the following.

```
xt.new<-c(xt,xt.ar1.forecast$mean)
res.tf.new<-c(rep(0,3),res.visc.tf,rep(0,tau))

yt.new<-c(yt,rep(0,tau))

#Note that 3 0's are added to the beginning to compensate for the
#misalignment between xt and the residuals of transfer function
#noise model due to the delay of 3 lags. Last 6 0's are added
#since the future values of the error are assumed to be 0.
```

We now get the parameter estimates for the transfer function–noise model

```
phi1<-visc.tf[[1]][1]
d1<-visc.tf[[1]][2]
d2<-visc.tf[[1]][3]
w0<-visc.tf[[1]][4]
```

The forecasts are then obtained using:

```
T<-length(yt)
for (i in (T+1):(T+tau)){
       yt.new[i]<-(d1+phi1)*yt.new[i-1]+(d2-d1*phi1)*yt.new[i-2]
            -d2*phi1*yt.new[i-3]+w0*xt.new[i-3]
            -w0*phi1*xt.new[i-4]+res.tf.new[i]-d1*res.tf.new[i-1]-
            d2*res.tf.new[i-1]
}
```

To calculate the prediction limits, we need to first calculate the estimate of forecast error variance given in (6.45). As mentioned in the example, since we only need up to six-step ahead forecasts, we need to calculate only v_0^*, v_1^* and v_2^* and ψ_0 through ψ_5.

```
phix<-xt.ar1[[1]][1]
v0star<-w0
v1star<-(d1+phix)*w0
v2star<-(((d1+phix)^2)+(d2-d1*phix))*w0
vstar<-c(v0star,v1star,v2star)
psi<-phix^(0:(tau-1))
sig2.alpha<-xt.ar1$sigma2
sig2.err<-visc.tf$sigma2
sig2.tfn<-rep(0,6)
b<-3
for (i in 1:6) {
       if ((i-b)<=0) {
               sig2.tfn[i]<- sig2.err*sum(psi[1:i]^2)
       }
       else {
               sig2.tfn[i]<- sig2.alpha*sum(vstar[1:(i-b)]^2)
                + sig2.err*sum(psi[1:i]^2)
       }
}
```

For comparison purposes, we also fit a univariate ARIMA model to y(t). An AR(3) model is considered even though the AR coefficient at third lag is borderline significant. The model is given below.

```
yt.ar3<-arima(yt,order=c(3,0,0),include.mean=FALSE)
> yt.ar3

       Series: x
       ARIMA(3,0,0) with zero mean
```

```
Coefficients:
         ar1     ar2      ar3
      0.9852  0.1298  -0.2700
s.e.  0.0954  0.1367   0.0978

sigma^2 estimated as 0.02779:  log likelihood=36.32
AIC=-66.65   AICc=-66.23   BIC=-56.23
```

To calculate the prediction limits, we need to first calculate the estimate of forecast error variance given in (5.83). To estimate ψ_0 through ψ_5 we use the formula given in (5.46).

```
psi.yt<-vector()
psi.yt[1:4]<-c(0,0,0,1)
sig2.yt<-yt.ar3$sigma2

for (i in 5:(4+tau-1)){
       psi.yt[i]<- yt.ar3[[1]][1]*psi.yt[i-1]+yt.ar3[[1]][2]
*psi.yt[i-2]+yt.ar3[[1]][3]*psi.yt[i-3]
}
psi.yt<-psi.yt[4:(4+tau-1)]
psi.yt
sig2.um<-rep(0,6)
b<-3
for (i in 1:6) {
       sig2.um[i]<- sig2.yt*sum(psi.yt[1:i]^2)
}
```

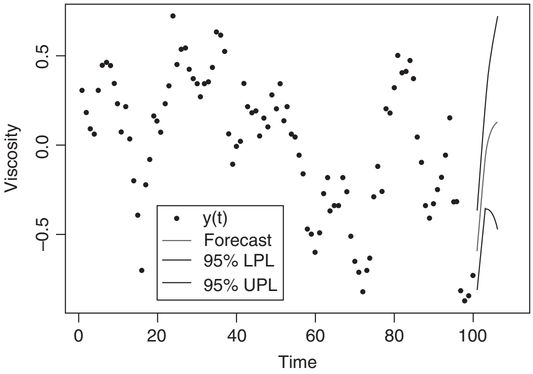
Thus for the transfer function-noise model and the univariate model we have the following prediction error variances.

```
cbind(sig2.tfn,sig2.um
           sig2.tfn    sig2.um
     [1,] 0.01230047 0.02778995
     [2,] 0.01884059 0.05476427
     [3,] 0.02231796 0.08841901
     [4,] 0.04195042 0.11308279
     [5,] 0.06331720 0.13109029
     [6,] 0.08793347 0.14171137
```

We can see that adding the exogenous variable x(t) helps to reduce the prediction error variance by half.

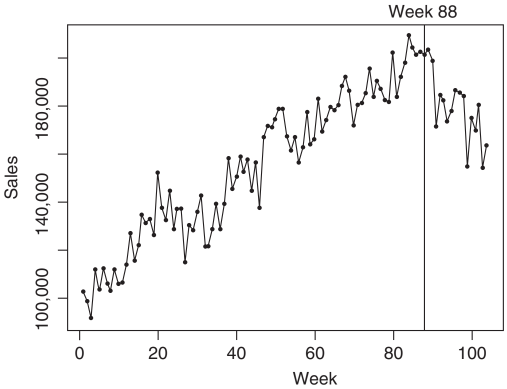
To plot the forecasts and the prediction limits, we have

```
plot(yt.new[1:T],type="p",pch=16,cex=.5,xlab='Time',
     ylab='Viscosity',xlim=c(1,110))
lines(101:106,yt.new[101:106],col="grey40")
lines(101:106, yt.new[101:106]+2*sqrt(sig2.tfn))
lines(101:106, yt.new[101:106]-2*sqrt(sig2.tfn))
legend(20,-.4,c("y(t)","Forecast","95% LPL","95% UPL"), pch=c(16,
NA, NA,NA),lwd=c(NA,.5,.5,.5),cex=.55,col=c("black","grey40",
"black","black"))
```



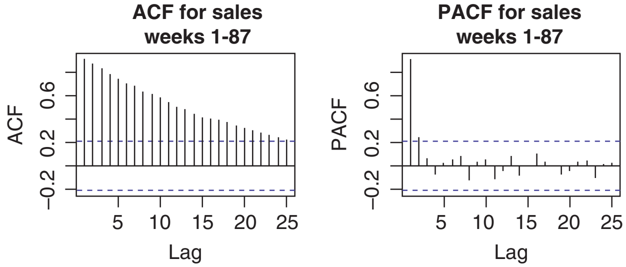
Example 6.5 The data for this example are in the array called cerealsales.data of which the two columns represent the week and the sales respectively. We first start with the plot of the data

```
yt.sales<-cerealsales.data[,2]
plot(yt.sales,type="o",pch=16,cex=.5,xlab='Week',ylab='Sales')
abline(v=88)
mtext("Week 88", side=3, at=88)
```



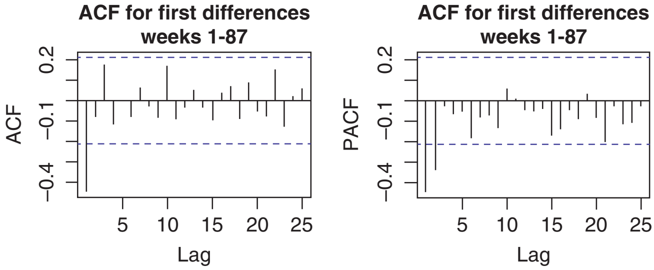
We then try to identify the ARIMA model for the pre-intervention data (up to week 87)

```
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(yt.sales[1:87],lag.max=25,type="correlation",main="ACF
    for Sales \nWeeks 1-87")
acf(yt.sales[1:87], lag.max=25,type="partial",main="PACF for Sales
\nWeeks 1-87",ylab="PACF")
```



The series appears to be non-stationary. We try the first differences

```
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(diff(yt.sales[1:87],1),lag.max=25,type="correlation",main="ACF
    for First Differences \nWeeks 1-87")

acf(diff(yt.sales[1:87],1), lag.max=25,type="partial",main="PACF
    for First Differences \nWeeks 1-87",ylab="PACF")
```



ARIMA(0,0,1) model for the first differences seems to be appropriate. We now fit the transfer function–noise model with the step input. First we define the step indicator variable.

```
library(TSA)
T<-length(yt.sales)
St<-c(rep(0,87),rep(1,(T-87)))
```

```
sales.tf<-arimax(diff(yt.sales), order=c(0,0,1), xtransf= St[2:T],
                      transfer=list(c(0,0)), include.mean = FALSE)
#Note that we adjusted the step function for the differencing we
# did on the yt.sales

sales.tf
      Series: diff(yt.sales)
      ARIMA(0,0,1) with zero mean

      Coefficients:
          ma1       T1-MA0
      -0.5571   -2369.888
      s.e.    0.0757    1104.542

      sigma^2 estimated as 92762871:  log likelihood=-1091.13
      AIC=2186.27   AICc=2186.51   BIC=2194.17
```

EXERCISES

6.1 An input and output time series consists of 300 observations. The prewhitened input series is well modeled by an AR(2) model $y_t = 0.5y_{t-1} + 0.2y_{t-2} + \alpha_t$. We have estimated $\hat{\sigma}_\alpha = 0.2$ and $\hat{\sigma}_\beta = 0.4$. The estimated cross-correlation function between the prewhitened input and output time series is shown below.

Lag, j	0	1	2	3	4	5	6	7	8	9	10
$r_{\alpha\beta}(j)$	0.01	0.03	−0.03	−0.25	−0.35	−0.51	−0.30	−0.15	−0.02	0.09	−0.01

a. Find the approximate standard error of the cross-correlation function. Which spikes on the cross-correlation function appear to be significant?

b. Estimate the impulse response function. Tentatively identify the form of the transfer function model.

6.2 Find initial estimates of the parameters of the transfer function model for the situation in Exercise 6.1.

6.3 An input and output time series consists of 200 observations. The prewhitened input series is well modeled by an MA(1) model $y_t = 0.8\alpha_{t-1} + \alpha_t$. We have estimated $\hat{\sigma}_\alpha = 0.4$ and $\hat{\sigma}_\beta = 0.6$. The estimated cross-correlation function between the prewhitened input and output time series is shown below.

Lag, j	0	1	2	3	4	5	6	7	8	9	10
$r_{\alpha\beta}(j)$	0.01	0.55	0.40	0.28	0.20	0.07	0.02	0.01	−0.02	0.01	−0.01

a. Find the approximate standard error of the cross-correlation function. Which spikes on the cross-correlation function appear to be significant?

b. Estimate the impulse response function. Tentatively identify the form of the transfer function model.

6.4 Find initial estimates of the parameters of the transfer function model for the situation in Exercise 6.3.

6.5 Write the equations that must be solved in order to obtain initial estimates of the parameters in a transfer function model with $b = 2$, $r = 1$, and $s = 0$.

6.6 Write the equations that must be solved in order to obtain initial estimates of the parameters in a transfer function model with $b = 2$, $r = 2$, and $s = 1$.

6.7 Write the equations that must be solved in order to obtain initial estimates of the parameters in a transfer function model with $b = 2$, $r = 1$, and $s = 1$.

6.8 Consider a transfer function model with $b = 2$, $r = 1$, and $s = 0$. Assume that the noise model is AR(1). Find the forecasts in terms of the transfer function and noise model parameters.

6.9 Consider the transfer function model in Exercise 6.8 with $b = 2$, $r = 1$, and $s = 0$. Now assume that the noise model is AR(2). Find the forecasts in terms of the transfer function and noise model parameters. What difference does this noise model make on the forecasts?

6.10 Consider the transfer function model in the Exercise 6.8 with $b = 2$, $r = 1$, and $s = 0$. Now assume that the noise model is MA(1). Find the forecasts in terms of the transfer function and noise model parameters. What difference does this noise model make on the forecasts?

6.11 Consider the transfer function model

$$y_t = \frac{-0.5 - 0.4B - 0.2B^2}{1 - 0.5B} x_{t-2} + \frac{1}{1 - 0.5B} \varepsilon_t.$$

Find the forecasts that are generated from this model.

6.12 Sketch a graph of the impulse response function for the following transfer function:

$$y_t = \frac{2B}{1 - 0.6B} x_t.$$

6.13 Sketch a graph of the impulse response function for the following transfer function:

$$y_t = \frac{1 - 0.2B}{1 - 0.8B} x_t.$$

6.14 Sketch a graph of the impulse response function for the following transfer function:

$$y_t = \frac{1}{1 - 1.2B + 0.4B^2} x_t.$$

6.15 Box, Jenkins, and Reinsel (1994) fit a transfer function model to data from a gas furnace. The input variable is the volume of methane entering the chamber in cubic feet per minute and the output is the concentration of carbon dioxide emitted. The transfer function model is

$$y_t = \frac{-(0.53 + 0.37B + 0.51B^2)}{1 - 0.57B} x_t + \frac{1}{1 - 0.53B + 0.63B^2} \varepsilon_t,$$

where the input and output variables are measured every nine seconds.

a. What are the values of b, s,and r for this model?

b. What is the form of the ARIMA model for the errors?

c. If the methane input was increased, how long would it take before the carbon dioxide concentration in the output is impacted?

6.16 Consider the global mean surface air temperature anomaly and global CO_2 concentration data in Table B.6 in Appendix B. Fit an appropriate transfer function model to this data, assuming that CO_2 concentration is the input variable.

6.17 Consider the chemical process yield and uncontrolled operating temperature data in Table B.12. Fit an appropriate transfer function model to these data, assuming that temperature is the input variable. Does including the temperature data improve your ability to forecast the yield data?

6.18 Consider the U.S. Internal Revenue tax refunds data in Table B.20. Fit an appropriate transfer function model to these data, assuming that population is the input variable. Does including the population data improve your ability to forecast the tax refund data?

6.19 Find time series data of interest to you where a transfer function–noise model would be appropriate.

a. Identify and fit the appropriate transfer function–noise model.

b. Use an ARIMA model to fit only the y_t series.

c. Compare the forecasting performance of the two models from parts a and b.

6.20 Find a time series of interest to you that you think may be impacted by an outlier. Fit an appropriate ARIMA model to the time series and use either the additive outlier or innovation outlier model to see if the potential outlier is statistically significant.

6.21 Table E6.1 provides 100 observations on a time series.

TABLE E6.1 Time Series Data for Exercise 6.21 (100 observations, read down then across)

86.74	83.79	88.42	84.23	82.20
85.32	84.04	89.65	83.58	82.14
84.74	84.10	97.85	84.13	81.80
85.11	84.85	88.50	82.70	82.32
85.15	87.64	87.06	83.55	81.53
84.48	87.24	85.20	86.47	81.73
84.68	87.52	85.08	86.21	82.54
84.68	86.50	84.44	87.02	82.39
86.32	85.61	84.21	86.65	82.42
88.00	86.83	86.00	85.71	82.21
86.26	84.50	85.57	86.15	82.77
85.83	84.18	83.79	85.80	83.12
83.75	85.46	84.37	85.62	83.22
84.46	86.15	83.38	84.23	84.45
84.65	86.41	85.00	83.57	84.91
84.58	86.05	84.35	84.71	85.76
82.25	86.66	85.34	83.82	85.23
83.38	84.73	86.05	82.42	86.73
83.54	85.95	84.88	83.04	87.00
85.16	86.85	85.42	83.70	85.06

a. Plot the data.

b. There is an apparent outlier in the data. Use intervention analysis to investigate the presence of this outlier.

6.22 Table E6.2 provides 100 observations on a time series. These data represent weekly shipments of a product.

a. Plot the data.

b. Note that there is an apparent increase in the level of the time series at about observation 80. Management suspects that this increase in shipments may be due to a strike at a competitor's plant. Build an appropriate intervention model for these data. Do you think that the impact of this intervention is likely to be permanent?

TABLE E6.2 Time Series Data for Exercise 6.22 (100 observations, read down then across)

1551	1556	1613	1552	1838
1548	1557	1595	1558	1838
1554	1564	1601	1543	1834
1557	1592	1587	1552	1840
1552	1588	1568	1581	1832
1555	1591	1567	1578	1834
1556	1581	1561	1587	1842
1574	1572	1558	1583	1840
1591	1584	1576	1573	1840
1575	1561	1572	1578	1838
1571	1558	1554	1574	1844
1551	1571	1560	1573	1848
1558	1578	1550	1559	1849
1561	1580	1566	1552	1861
1560	1577	1560	1563	1865
1537	1583	1570	1555	1874
1549	1564	1577	1541	1869
1551	1576	1565	1547	1884
1567	1585	1571	1553	1886
1553	1601	1559	1538	1867

6.23 Table B.23 contains data on Danish crude oil production. Historically, oil production increased steadily from 1972 up to about 2000, when the Danish government presented an energy strategy containing a number of ambitious goals for national energy policy up through 2025. The aim is to reduce Denmark's dependency on coal, oil and natural gas. The data exhibit a marked downturn in oil production starting in 2005. Fit and analyze an appropriate intervention model to these data.

6.24 Table B.25 contains data on annual US motor vehicle fatalities from 1966 through 2012, along with data on several other factors. Fit a transfer function model to these data using the number of licensed drivers as the input time series. Compare this transfer function model to a univariate ARIMA model for the annual fatalities data.

6.25 Table B.25 contains data on annual US motor vehicle fatalities from 1966 through 2012, along with data on several other factors. Fit a transfer function model to these data using the annual unemployment rate as the input time series. Compare this transfer function model to a univariate ARIMA model for the annual fatalities data. Why do you think that the annual unemployment rate might be a good predictor of fatalities?

6.26 Table B.25 contains data on annual US motor vehicle fatalities from 1966 through 2012, along with data on several other factors. Fit a transfer function model to these data using both the number of licensed drivers and the annual unemployment rate as the input time series. Compare this two-input transfer function model to a univariate ARIMA model for the annual fatalities data, and to the two univariate transfer function models from Exercises 6.24 and 6.25.

CHAPTER 7

SURVEY OF OTHER FORECASTING METHODS

I always avoid prophesying beforehand, because it is a much better policy to prophesy after the event has already taken place.

SIR WINSTON CHURCHILL, *British Prime Minister*

7.1 MULTIVARIATE TIME SERIES MODELS AND FORECASTING

In many forecasting problems, it may be the case that there are more than just one variable to consider. Attempting to model each variable individually may at times work. However, in these situations, it is often the case that these variables are somehow cross-correlated, and that structure can be effectively taken advantage of in forecasting. In the previous chapter we explored this for the "unidirectional" case, where it is assumed that certain inputs have impact on the variable of interest but not the other way around. Multivariate time series models involve several variables that are not only serially but also cross-correlated. As in the univariate case, multivariate or **vector ARIMA** models can often be successfully used in forecasting multivariate time series. Many of the concepts we have seen in Chapter 5

Introduction to Time Series Analysis and Forecasting, Second Edition.
Douglas C. Montgomery, Cheryl L. Jennings and Murat Kulahci.

will be directly applicable in the multivariate case as well. We will first start with the property of stationarity.

7.1.1 Multivariate Stationary Process

Suppose that the vector time series $Y_t = (y_{1t}, y_{2t}, \dots, y_{mt})$ consists of m univariate time series. Then Y_t with finite first and second order moments is said to be **weakly stationary** if

(i) $E(Y_t) = E(Y_{t+s}) = \boldsymbol{\mu}$, constant for all s
(ii) $\mathrm{Cov}(Y_t) = E[(Y_t - \boldsymbol{\mu})(Y_t - \boldsymbol{\mu})'] = \boldsymbol{\Gamma}(0)$
(iii) $\mathrm{Cov}(Y_t, Y_{t+s}) = \boldsymbol{\Gamma}(s)$ depends only on s

Note that the diagonal elements of $\boldsymbol{\Gamma}(s)$ give the autocovariance function of the individual time series, $\gamma_{ii}(s)$. Similarly, the autocorrelation matrix is given by

$$\boldsymbol{\rho}(s) = \begin{bmatrix} \rho_{11}(s) & \rho_{12}(s) & \dots & \rho_{1m}(s) \\ \rho_{21}(s) & \rho_{22}(s) & \dots & \rho_{2m}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{m1}(s) & \rho_{m2}(s) & \dots & \rho_{mm}(s) \end{bmatrix} \tag{7.1}$$

which can also be obtained by defining

$$\begin{aligned} \mathbf{V} &= \mathrm{diag}\{\gamma_{11}(0), \gamma_{22}(0), \dots, \gamma_{mm}(0)\} \\ &= \begin{bmatrix} \gamma_{11}(0) & 0 & \dots & 0 \\ 0 & \gamma_{22}(0) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \gamma_{mm}(0) \end{bmatrix} \end{aligned} \tag{7.2}$$

We then have

$$\boldsymbol{\rho}(s) = \mathbf{V}^{-1/2}\boldsymbol{\Gamma}(s)\mathbf{V}^{-1/2} \tag{7.3}$$

We can further show that $\boldsymbol{\Gamma}(s) = \boldsymbol{\Gamma}(-s)'$ and $\boldsymbol{\rho}(s) = \boldsymbol{\rho}(-s)'$.

7.1.2 Vector ARIMA Models

The stationary vector time series can be represented with a **vector ARMA** model given by

$$\boldsymbol{\Phi}(B)Y_t = \boldsymbol{\delta} + \boldsymbol{\Theta}(B)\boldsymbol{\varepsilon}_t \tag{7.4}$$

where $\mathbf{\Phi}(B) = \mathbf{I} - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \cdots - \mathbf{\Phi}_p B^p$, $\mathbf{\Theta}(B) = \mathbf{I} - \mathbf{\Theta}_1 B - \mathbf{\Theta}_2 B^2 - \cdots - \mathbf{\Theta}_q B^q$, and $\boldsymbol{\varepsilon}_t$ represents the sequence of independent random vectors with $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$ and $\mathrm{Cov}(\boldsymbol{\varepsilon}_t) = \mathbf{\Sigma}$. Since the random vectors are independent, we have $\Gamma_\varepsilon(s) = 0$ for all $s \neq 0$.

The process Y_t in Eq. (7.4) is stationary if the roots of

$$\det[\mathbf{\Phi}(B)] = \det[\mathbf{I} - \mathbf{\Phi}_1 B - \mathbf{\Phi}_2 B^2 - \cdots - \mathbf{\Phi}_p B^p] = 0 \tag{7.5}$$

are all greater than one in absolute value. Then the process Y_t is also said to have infinite MA representation given as

$$\begin{aligned} Y_t &= \boldsymbol{\mu} + \mathbf{\Psi}(B)\boldsymbol{\varepsilon}_t \\ &= \boldsymbol{\mu} + \sum_{i=0}^{\infty} \mathbf{\Psi}_i \boldsymbol{\varepsilon}_{t-i} \end{aligned} \tag{7.6}$$

where $\mathbf{\Psi}(B) = \mathbf{\Phi}(B)^{-1}\mathbf{\Theta}(B)$, $\mu = \mathbf{\Phi}(B)^{-1}\delta$, and $\sum_{i=0}^{\infty} \|\mathbf{\Psi}_i\|^2 < \infty$.

Similarly, if the roots of $\det[\mathbf{\Theta}(B)] = \det[\mathbf{I} - \mathbf{\Theta}_1 B - \mathbf{\Theta}_2 B^2 - \cdots - \mathbf{\Theta}_q B^q] = 0$ are greater than unity in absolute value the process Y_t in Eq. (7.4) is invertible.

To illustrate the vector ARMA model given in Eq. (7.4), consider the bivariate ARMA(1,1) model with

$$\begin{aligned} \mathbf{\Phi}(B) &= \mathbf{I} - \mathbf{\Phi}_1 B \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} B \end{aligned}$$

and

$$\begin{aligned} \mathbf{\Theta}(B) &= \mathbf{I} - \mathbf{\Theta}_1 B \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} B \end{aligned}$$

Hence the model can be written as

$$\left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} B\right] Y_t = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} + \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} B\right] \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$$

or

$$\begin{aligned} y_{1,t} &= \delta_1 + \phi_{11} y_{1,t-1} + \phi_{12} y_{2,t-1} + \varepsilon_{1,t} - \theta_{11}\varepsilon_{1,t-1} - \theta_{12}\varepsilon_{2,t-1} \\ y_{2,t} &= \delta_2 + \phi_{21} y_{1,t-1} + \phi_{22} y_{2,t-1} + \varepsilon_{2,t} - \theta_{21}\varepsilon_{1,t-1} - \theta_{22}\varepsilon_{2,t-1} \end{aligned}$$

As in the univariate case, if nonstationarity is present, through an appropriate degree of differencing a stationary vector time series may be achieved. Hence the vector ARIMA model can be represented as

$$\mathbf{\Phi}(B)\mathbf{D}(B)Y_t = \boldsymbol{\delta} + \mathbf{\Theta}(B)\boldsymbol{\varepsilon}_t$$

where

$$\mathbf{D}(B) = \text{diag}\{(1-B)^{d_1}, (1-B)^{d_2}, \dots, (1-B)^{d_m}\}$$

$$= \begin{bmatrix} (1-B)^{d_1} & 0 & \dots & 0 \\ 0 & (1-B)^{d_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & (1-B)^{d_m} \end{bmatrix}$$

However, the degree of differencing is usually quite complicated and has to be handled with care (Reinsel (1997)).

The identification of the vector ARIMA model can indeed be fairly difficult. Therefore in the next section we will concentrate on the more commonly used and intuitively appealing vector autoregressive models. For a more general discussion see Reinsel (1997), Lütkepohl (2005), Tiao and Box (1981), Tiao and Tsay (1989), Tsay (1989), and Tjostheim and Paulsen (1982).

7.1.3 Vector AR (VAR) Models

The vector AR(p) model is given by

$$\mathbf{\Phi}(B)Y_t = \boldsymbol{\delta} + \boldsymbol{\varepsilon}_t \tag{7.7}$$

or

$$Y_t = \boldsymbol{\delta} + \sum_{i=1}^{p} \mathbf{\Phi}_i Y_{t-i} + \boldsymbol{\varepsilon}_t$$

For a stationary vector AR process, the infinite MA representation is given as

$$Y_t = \boldsymbol{\mu} + \mathbf{\Psi}(B)\boldsymbol{\varepsilon}_t \tag{7.8}$$

where $\mathbf{\Psi}(B) = \mathbf{I} + \mathbf{\Psi}_1 B + \mathbf{\Psi}_2 B^2 + \cdots$ and $\boldsymbol{\mu} = \mathbf{\Phi}(B)^{-1}\delta$. Hence we have $E(Y_t) = \boldsymbol{\mu}$ and $\text{Cov}(\boldsymbol{\varepsilon}_t, Y_{t-s}) = 0$ for any $s > 0$ since Y_{t-s} is only concerned with $\boldsymbol{\varepsilon}_{t-s}$, $\boldsymbol{\varepsilon}_{t-s-1}, \ldots$, which are not correlated with $\boldsymbol{\varepsilon}_t$. Moreover, we also have

$$\begin{aligned}\text{Cov}(\boldsymbol{\varepsilon}_t, Y_t) &= \text{Cov}\left(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t + \boldsymbol{\Psi}_1\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{\Psi}_2\boldsymbol{\varepsilon}_{t-2} + \cdots\right)\\ &= \text{Cov}(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t)\\ &= \boldsymbol{\Sigma}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\Gamma}(s) = \text{Cov}(Y_{t-s}, Y_t) &= \text{Cov}\left(Y_{t-s}, \boldsymbol{\delta} + \sum_{i=1}^{p} \boldsymbol{\Phi}_i Y_{t-i} + \boldsymbol{\varepsilon}_t\right)\\ &= \text{Cov}\left(Y_{t-s}, \sum_{i=1}^{p} \boldsymbol{\Phi}_i Y_{t-i}\right) + \underbrace{\text{Cov}(Y_{t-s}, \boldsymbol{\varepsilon}_t)}_{=0 \text{ for } s>0}\\ &= \sum_{i=1}^{p} \text{Cov}(Y_{t-s}, \boldsymbol{\Phi}_i Y_{t-i})\\ &= \sum_{i=1}^{p} \text{Cov}(Y_{t-s}, Y_{t-i})\boldsymbol{\Phi}_i'\end{aligned} \tag{7.9}$$

Hence we have

$$\boldsymbol{\Gamma}(s) = \sum_{i=1}^{p} \boldsymbol{\Gamma}(s-i)\boldsymbol{\Phi}_i' \tag{7.10}$$

and

$$\boldsymbol{\Gamma}(0) = \sum_{i=1}^{p} \boldsymbol{\Gamma}(-i)\boldsymbol{\Phi}_i' + \boldsymbol{\Sigma} \tag{7.11}$$

As in the univariate case, the Yule–Walker equations can be obtained from the first p equations as

$$\begin{bmatrix}\boldsymbol{\Gamma}(1)\\ \boldsymbol{\Gamma}(2)\\ \vdots\\ \boldsymbol{\Gamma}(p)\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Gamma}(0) & \boldsymbol{\Gamma}(1)' & \ldots & \boldsymbol{\Gamma}(p-1)'\\ \boldsymbol{\Gamma}(1) & \boldsymbol{\Gamma}(0) & \ldots & \boldsymbol{\Gamma}(p-2)'\\ \vdots & \vdots & \ddots & \vdots\\ \boldsymbol{\Gamma}(p-1) & \boldsymbol{\Gamma}(p-2) & \ldots & \boldsymbol{\Gamma}(0)\end{bmatrix}\begin{bmatrix}\boldsymbol{\Phi}_1'\\ \boldsymbol{\Phi}_2'\\ \vdots\\ \boldsymbol{\Phi}_p'\end{bmatrix} \tag{7.12}$$

The model parameters in $\boldsymbol{\Phi}$ and $\boldsymbol{\Sigma}$ can be estimated from Eqs. (7.11) and (7.12).

For the VAR(p), the autocorrelation matrix in Eq. (7.3) will exhibit a decaying behavior following a mixture of exponential decay and damped sinusoid.

Example 7.1 VAR(1) Model The autocovariance matrix for VAR(1) is given as

$$\boldsymbol{\Gamma}(s) = \boldsymbol{\Gamma}(s-1)\boldsymbol{\Phi}' = (\boldsymbol{\Gamma}(s-2)\boldsymbol{\Phi}')\boldsymbol{\Phi}' = \cdots = \boldsymbol{\Gamma}(0)(\boldsymbol{\Phi}')^s \tag{7.13}$$

and

$$\begin{aligned}\boldsymbol{\rho}(s) &= \mathbf{V}^{-1/2}\boldsymbol{\Gamma}(s)\mathbf{V}^{-1/2}\\ &= \mathbf{V}^{-1/2}\boldsymbol{\Gamma}(0)(\boldsymbol{\Phi}')^s\mathbf{V}^{-1/2}\\ &= \mathbf{V}^{-1/2}\boldsymbol{\Gamma}(0)\mathbf{V}^{-1/2}\mathbf{V}^{1/2}(\boldsymbol{\Phi}')^s\mathbf{V}^{-1/2}\\ &= \boldsymbol{\rho}(0)\mathbf{V}^{1/2}(\boldsymbol{\Phi}')^s\mathbf{V}^{-1/2}\end{aligned} \tag{7.14}$$

where $\mathbf{V} = \text{diag}\{\gamma_{11}(0), \gamma_{22}(0), \ldots, \gamma_{mm}(0)\}$. The eigenvalues of $\boldsymbol{\Phi}$ determine the behavior of the autocorrelation matrix. In fact, if the eigenvalues of $\boldsymbol{\Phi}$ are real and/or complex conjugates, the behavior will be a mixture of the exponential decay and damped sinusoid, respectively.

Example 7.2 The pressure readings at two ends of an industrial furnace are taken every 10 minutes and given in Table 7.1. It is expected the individual time series are not only autocorrelated but also cross-correlated. Therefore it is decided to fit a multivariate time series model to this data. The time series plots of the data are given in Figure 7.1. To identify the model we consider the sample ACF plots as well as the cross correlation of the time series given in Figure 7.2. These plots exhibit an exponential decay pattern, suggesting that an autoregressive model may be appropriate. It is further conjectured that a VAR(1) or VAR(2) model may provide a good fit. Another approach to model identification would be to fit ARIMA models to the individual time series and consider the cross correlation of the residuals. For that, we fit an AR(1) model to both time series. The cross-correlation plot of the residuals given in Figure 7.3 further suggests that the VAR(1) model may indeed provide an appropriate fit. Using the SAS ARIMA procedure given in Table 7.2, we fit a VAR(1) model. The SAS output in Table 7.3 confirms that the VAR(1) model provides an appropriate fit for the data. The time series plots of the residuals and the fitted values are given in Figures 7.3, 7.4, and 7.5.

TABLE 7.1 Pressure Readings at Both Ends of the Furnace

	Pressure			Pressure			Pressure			Pressure			Pressure	
Index	Front	Back	Index	Front	Back	Index	Front	Back	Index	Front	Back	Index	Front	Back
1	7.98	20.1	39	10.23	22.2	77	9.23	21.19	115	12.73	19.88	153	5.45	19.46
2	8.64	20.37	40	11.27	18.86	78	10.18	18.52	116	13.86	22.36	154	6.5	18.33
3	10.06	19.99	41	9.57	21.16	79	8.6	21.79	117	12.38	19.04	155	5.23	19.22
4	8.13	19.62	42	10.6	17.89	80	9.66	18.4	118	12.51	23.32	156	4.97	17.7
5	8.84	20.26	43	9.22	20.55	81	8.66	19.17	119	14.32	20.42	157	4.3	18.42
6	10.28	19.46	44	8.91	20.47	82	7.55	18.86	120	13.47	20.88	158	4.47	17.85
7	9.63	20.21	45	10.15	20	83	9.21	19.42	121	12.96	20.25	159	5.48	19.16
8	10.5	19.72	46	11.32	20.07	84	9.45	19.54	122	11.65	20.69	160	5.7	17.91
9	7.91	19.5	47	12.41	20.82	85	10.61	19.79	123	11.99	19.7	161	3.78	19.36
10	9.71	18.97	48	12.41	20.98	86	10.78	18.6	124	9.6	18.51	162	5.86	18.18
11	10.43	22.31	49	10.36	20.91	87	10.68	22.5	125	7.38	18.48	163	6.56	20.82
12	10.99	20.16	50	9.27	20.07	88	14.05	19.1	126	6.98	20.37	164	6.82	18.47
13	10.08	20.73	51	11.77	19.82	89	14.1	23.82	127	8.18	18.21	165	5.18	19.09
14	9.75	20.14	52	11.93	21.81	90	16.11	21.25	128	7.5	20.85	166	6.3	18.91
15	9.37	20.34	53	13.6	20.71	91	13.58	20.46	129	7.04	18.9	167	9.12	20.93
16	11.52	18.83	54	14.26	21.94	92	12.06	22.55	130	9.06	19.84	168	9.3	18.73
17	10.6	24.01	55	14.81	21.75	93	13.76	20.78	131	8.61	19.15	169	10.37	22.17
18	14.31	19.7	56	11.97	18.97	94	13.55	20.94	132	8.93	20.77	170	11.87	19.03
19	13.3	22.53	57	10.99	23.11	95	13.69	21.66	133	9.81	18.95	171	12.36	22.15
20	14.45	20.77	58	10.61	18.92	96	15.07	21.61	134	9.57	20.33	172	14.61	20.67
21	14.8	21.69	59	9.77	20.28	97	15.14	21.69	135	10.31	21.69	173	13.63	22.39

(continued)

TABLE 7.1 (*Continued*)

	Pressure			Pressure			Pressure			Pressure			Pressure	
Index	Front	Back	Index	Front	Back	Index	Front	Back	Index	Front	Back	Index	Front	Back
22	15.09	20.87	60	11.5	21.18	98	14.01	21.85	136	11.88	18.66	174	13.12	19.75
23	12.96	21.42	61	10.52	19.29	99	12.69	20.87	137	12.36	22.35	175	10.07	18.94
24	11.28	18.95	62	12.58	19.9	100	11.6	20.93	138	12.18	19.34	176	10.14	21.47
25	10.78	22.61	63	12.33	19.87	101	12.15	20.57	139	12.94	22.76	177	11.02	19.79
26	10.42	19.93	64	9.77	19.43	102	12.99	21.17	140	14.25	19.6	178	11.37	21.94
27	9.79	21.88	65	10.71	21.32	103	11.89	19.53	141	12.86	23.74	179	10.98	18.73
28	11.66	18.3	66	10.01	17.85	104	10.85	21.14	142	12.14	18.06	180	10.04	21.41
29	10.81	20.76	67	9.48	21.55	105	11.81	20.09	143	10.06	20.11	181	11.3	19.2
30	9.79	17.66	68	9.39	19.04	106	9.46	18.48	144	10.17	19.56	182	10.59	23
31	10.02	23.09	69	9.05	19.04	107	9.25	20.33	145	7.56	19.27	183	11.69	17.47
32	11.09	17.86	70	9.06	21.39	108	9.26	19.82	146	7.77	18.59	184	10.73	21.59
33	10.28	20.9	71	9.87	17.66	109	8.55	20.07	147	9.03	21.85	185	13.64	21.62
34	9.24	19.5	72	7.84	21.61	110	8.86	19.81	148	10.8	19.21	186	12.92	20.23
35	10.32	22.6	73	7.78	18.05	111	10.32	20.64	149	9.41	19.42			
36	10.65	19	74	6.44	19.07	112	11.39	20.04	150	7.81	19.79			
37	8.51	20.39	75	7.67	19.92	113	11.78	21.52	151	7.99	18.81			
38	11.46	19.23	76	8.48	18.3	114	13.13	20.35	152	5.78	18.46			

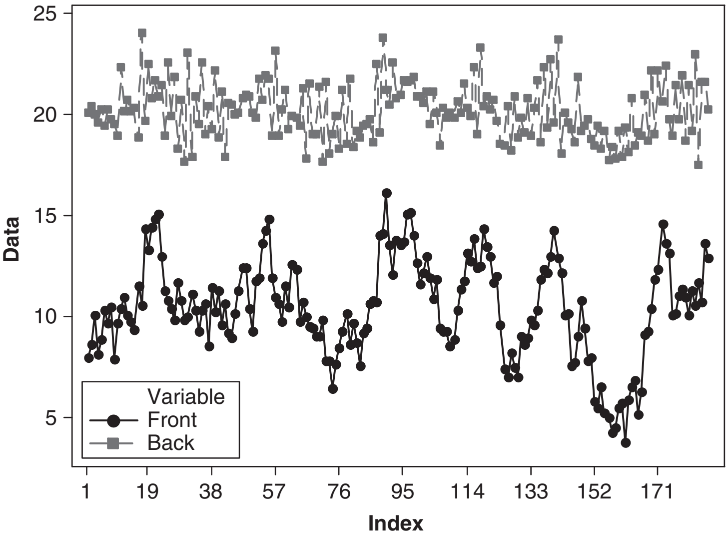


FIGURE 7.1 Time series plots of the pressure readings at both ends of the furnace.

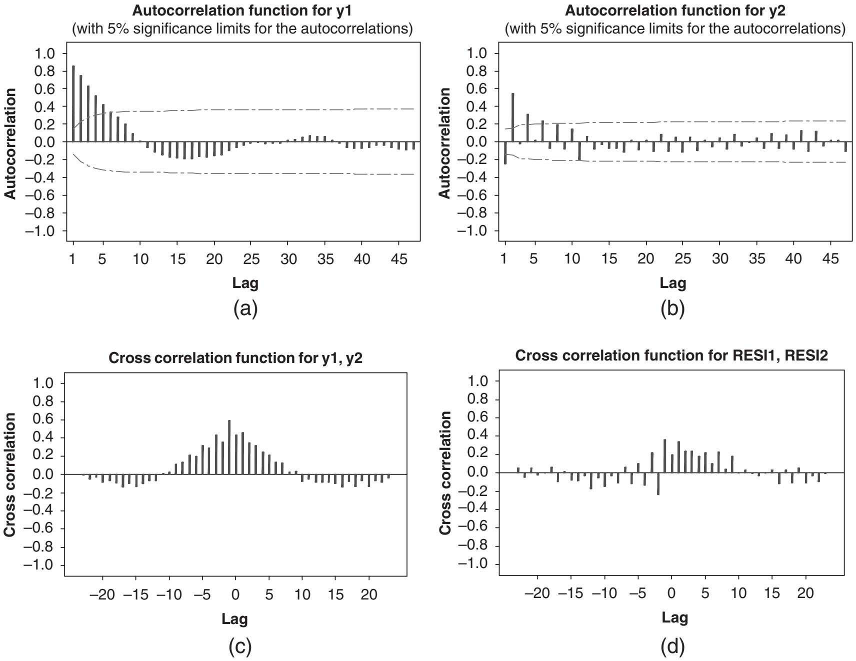


FIGURE 7.2 The sample ACF plot for: (a) the pressure readings at the front end of the furnace, y_1; (b) the pressure readings at the back end of the furnace, y_2; (c) the cross correlation between y_1 and y_2; and (d) the cross correlation between the residuals from the AR(1) model for front pressure and the residuals from the AR(1) model for back pressure.

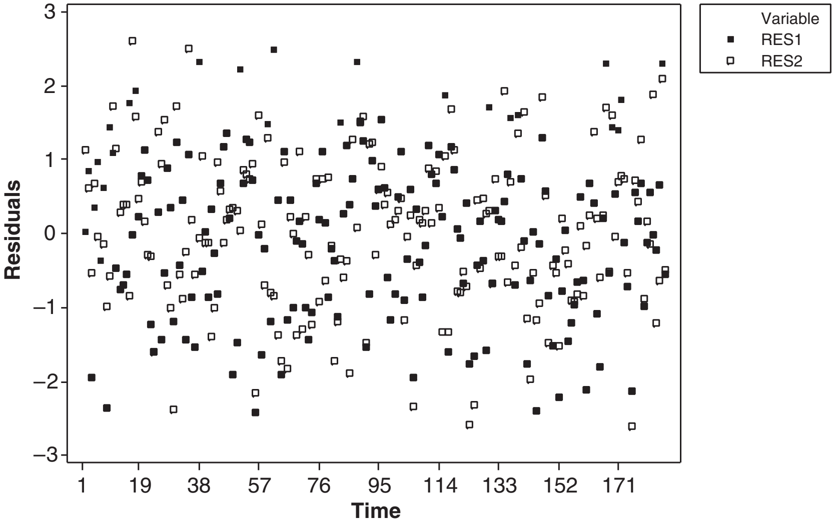


FIGURE 7.3 Time series plots of the residuals from the VAR(1) model.

7.2 STATE SPACE MODELS

In this section we give a brief introduction to an approach to forecasting based on the **state space model**. This is a very general approach that can include regression models and ARIMA models. It can also incorporate a Bayesian approach to forecasting and models with time-varying coefficients. State space models are based on the **Markov property**, which implies the independence of the future of a process from its past, given the present system state. In this type of system, the state of the process at the current time contains all of the past information that is required to predict future process behavior. We will let the system state at time *t* be

TABLE 7.2 SAS Commands to Fit a VAR(1) Model to the Pressure Data

```
proc varmax data=simul4;
       model y1 y2 / p=1 ;
       output out=residuals;
run;

proc print data=residuals;
run;
```

TABLE 7.3 SAS Output for the VAR(1) Model for the Pressure Data

```
                       The VARMAX Procedure

           Type of Model                          VAR(1)
           Estimation Method    Least Squares Estimation

                        Constant Estimates

                      Variable      Constant

                     y1               -6.76331
                     y2               27.23208

                     AR Coefficient Estimates

           Lag   Variable           y1              y2

           1      y1              0.73281        0.47405
                  y2              0.41047       -0.56040

                             Schematic
                         Representation of
                        Parameter Estimates

                             Variable/
                       Lag         C    AR1

                       y1          -    ++
                       y2          +    +-

                        + is > 2*std error,
                       - is < -2*std error,
                           . is between,
                             * is N/A

                     Model Parameter Estimates

                                Standard
Equation  Parameter  Estimate  Error    t Value  Pr > |t|  Variable
    y1     CONST1    -6.76331  1.18977  -5.68    0.0001      1
           AR1_1_1    0.73281  0.03772  19.43    0.0001     y1(t-1)
           AR1_1_2    0.47405  0.06463   7.33    0.0001     y2(t-1)
    y2     CONST2    27.23208  1.11083  24.51    0.0001      1
           AR1_2_1    0.41047  0.03522  11.66    0.0001     y1(t-1)
           AR1_2_2   -0.56040  0.06034  -9.29    0.0001     y2(t-1)
                    Covariances of Innovations

             Variable              y1              y2

             y1               1.25114         0.59716
             y2               0.59716         1.09064

                            Information
                             Criteria

                         AICC    0.041153
                         HQC     0.082413
                         AIC     0.040084
                         SBC     0.144528
                         FPEC    1.040904
```

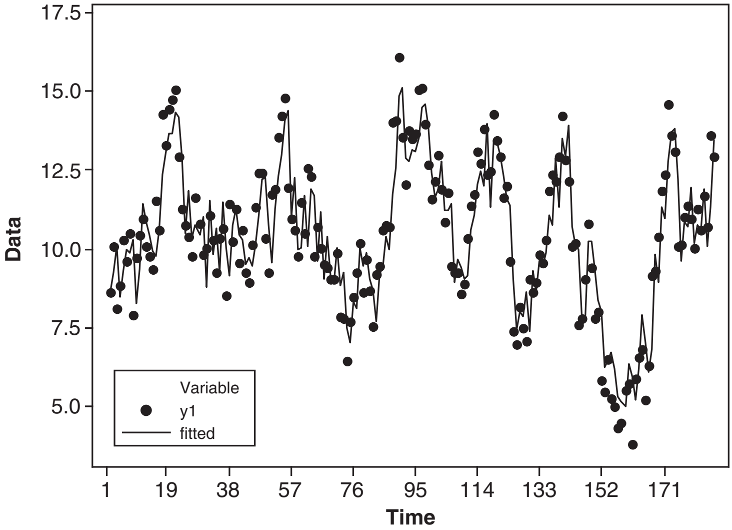


FIGURE 7.4 Actual and fitted values for the pressure readings at the front end of the furnace.

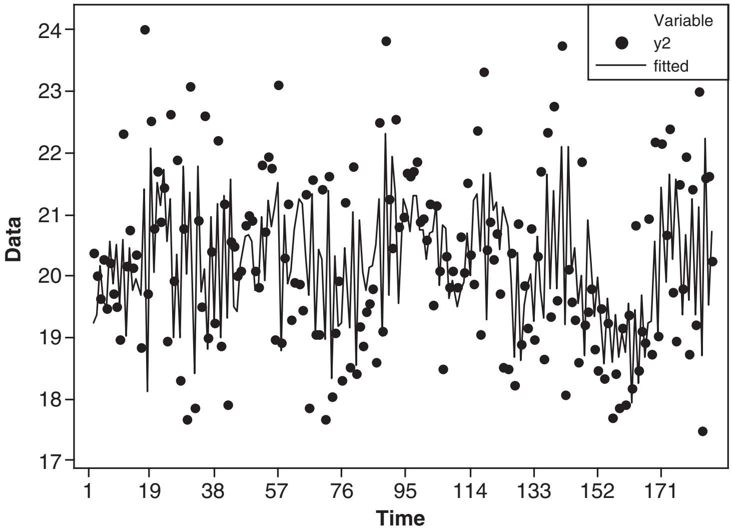


FIGURE 7.5 Actual and fitted values for the pressure readings at the back end of the furnace.

represented by the **state vector** $\mathbf{X}_t$. The elements of this vector are not necessarily observed. A state space model consists of two equations: an **observation** or **measurement equation** that describes how time series observations are produced from the state vector, and a **state** or **system equation** that describes how the state vector evolves through time. We will write these two equations as

$$y_t = \mathbf{h}'_t\mathbf{X}_t + \varepsilon_t \text{ (observation equation)} \tag{7.15}$$

and

$$\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{G}\mathbf{a}_t \text{ (state equation)} \tag{7.16}$$

respectively. In the observation equation $\mathbf{h}_t$ is a known vector of constants and ε_t is the observation error. If the time series is multivariate then y_t and ε_t become vectors $\mathbf{y}_t$ and ε_t, and the vector $\mathbf{h}_t$ becomes a matrix $\mathbf{H}$. In the state equation $\mathbf{A}$ and $\mathbf{G}$ are known matrices and $\mathbf{a}_t$ is the process noise. Note that the state equation resembles a multivariate AR(1) model, except that it represents the state variables rather than an observed time series, and it has an extra matrix $\mathbf{G}$.

The state space model does not look like any of the time series models we have studied previously. However, we can put many of these models in the state space form. This is illustrated in the following two examples.

Example 7.3 Consider an AR(1) model, which we have previously written as

$$y_t = \phi y_{t-1} + \varepsilon_t$$

In this case we let $X_t = y_t$ and $\mathbf{a}_t = \varepsilon_t$ and write the state equation as

$$\begin{gathered}\mathbf{X}_t = \mathbf{A}\mathbf{X}_{t-1} + \mathbf{G}\mathbf{a}_t \\ [y_t] = [\phi][y_{t-1}] + [1]\varepsilon_t\end{gathered}$$

and the observation equation is

$$\begin{aligned}y_t &= \mathbf{h}'_t\mathbf{X}_t + \varepsilon_t \\ y_t &= [1]\mathbf{X}_t + 0 \\ y_t &= \phi y_{t-1} + \varepsilon_t\end{aligned}$$

In the AR(1) model the state vector consists of previous consecutive observations of the time series y_t.

Any ARIMA model can be written in the state space form. Refer to Brockwell and Davis (1991).

Example 7.4 Now let us consider a regression model with one predictor variable and AR(1) errors. We will write this model as

$$\begin{aligned} y_t &= \beta_0 + \beta_1 p_t + \varepsilon_t \\ \varepsilon_t &= \phi\varepsilon_{t-1} + a_t \end{aligned}$$

where p_t is the predictor variable and ε_t is the AR(1) error term. To write this in state space form, define the state vector as

$$\mathbf{X}_t = \begin{bmatrix} \beta_0 \\ \beta_1 \\ p_t - \varepsilon_t \end{bmatrix}$$

The vector $\mathbf{h}_t$ and the matrix $\mathbf{A}$ are

$$\mathbf{h}_t = \begin{bmatrix} 1 \\ p_t \\ 1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi \end{bmatrix}$$

and the state space representation of this model becomes

$$\begin{aligned} y_t &= [1, p_t, 1]\mathbf{X}_t + \varepsilon_t \\ \mathbf{X}_t &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \phi \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ p_{t-1} - \varepsilon_{t-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \phi\varepsilon_{t-1} \end{bmatrix} \end{aligned}$$

Multiplying these equations out will produce the time series regression model with one predictor and AR(1) errors.

The state space formulation does not admit any new forecasting techniques. Consequently, it does not produce better forecasts than any of the other methods. The state space approach does admit a Bayesian formulation of the problem, in which the model parameters have a prior distribution that represents our degree of belief concerning the values of these coefficients. Then after some history of the process (observation) becomes available, this prior distribution is updated into a posterior distribution. Another formulation allows the coefficients in the regression model to vary through time.

The state space formulation does allow a common mathematical framework to be used for model development. It also permits relatively easy generalization of many models. This has some advantages for researchers. It also would allow common computer software to be employed for making forecasts from a variety of techniques. This could have some practical appeal to forecasters.

7.3 ARCH AND GARCH MODELS

In the standard regression and time series models we have covered so far, many diagnostic checks were based on the assumptions that we imposed on the errors: independent, identically distributed with zero mean, and constant variance. Our main concern has mostly been about the **independence** of the errors. The constant variance assumption is often taken as a given. In many practical cases and particularly in finance, it is fairly common to observe the violation of this assumption. Figure 7.6, for example, shows the S&P500 Index (weekly close) from 1995 to 1998. Most of the 1990s enjoyed a bull market up until toward the end when the dot-com bubble burst. The worrisome market resulted in high volatility (i.e., increasing variance). A linear trend model, an exponential smoother, or even an ARIMA model would have failed to capture this phenomenon, as all assume constant variance of the errors. This will in turn result in

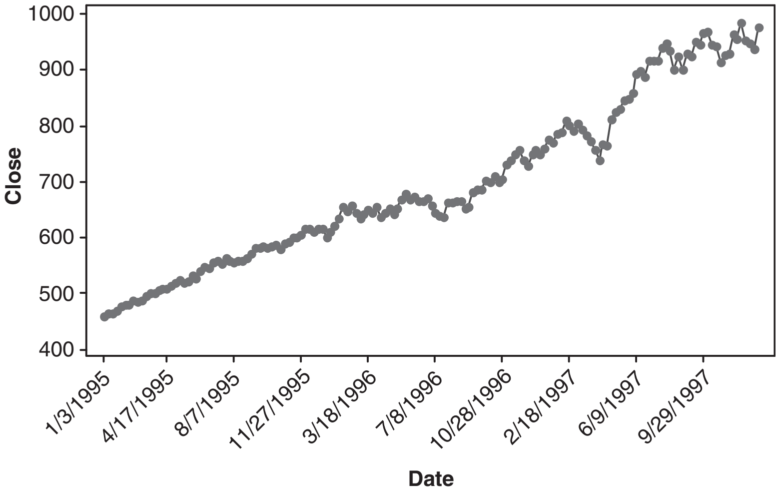


FIGURE 7.6 Time series plot of S&P500 Index weekly close from 1995 to 1998.

the underestimation of the standard errors calculated using OLS and will lead to erroneous conclusions. There are different ways of dealing with this situation. For example, if the changes in the variance at certain time intervals are known, weighted regression can be employed. However, it is often the case that these changes are unknown to the analyst. Moreover, it is usually of great value to the analyst to know why, when, and how these changes in the variance occur. Hence, if possible, modeling these changes (i.e., the variance) can be quite beneficial.

Consider, for example, the simple AR(p) model from Chapter 5 given as

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t \tag{7.17}$$

where e_t is the uncorrelated, zero mean noise with changing variance. Please note that we used e_t to distinguish it from our general white noise error ε_t. Since we let the variance of e_t change in time, one approach is to model e_t^2 as an AR(l) model as

$$e_t^2 = \xi_0 + \xi_1 e_{t-1}^2 + \xi_2 e_{t-2}^2 + \cdots + \xi_l e_{t-l}^2 + a_t \tag{7.18}$$

where a_t is a white noise sequence with zero mean and constant variance σ_a^2. In this notation e_t is said to follow an **autoregressive conditional heteroskedastic** process of order l, ARCH(l).

To check for a need for an ARCH model, once the ARIMA or regression model is fitted, not only the standard residual analysis and diagnostics checks have to be performed but also some serial dependence checks for e_t^2 should be made.

To further generalize the ARCH model, we will consider the alternative representation originally proposed by Engle (1982). Let us assume that the error can be represented as

$$e_t = \sqrt{v_t} w_t \tag{7.19}$$

where w_t is independent and identically distributed with mean 0 and variance 1, and

$$v_t = \zeta_0 + \zeta_1 e_{t-1}^2 + \zeta_2 e_{t-2}^2 + \cdots + \zeta e_{t-l}^2 \tag{7.20}$$

Hence the conditional variance of e_t is

$$\begin{aligned} \mathrm{Var}(e_t|e_{t-1}, \ldots) &= E\left(e_t^2|e_{t-1}^2, \ldots\right) \\ &= v_t \\ &= \zeta_0 + \zeta_1 e_{t-1}^2 + \zeta_2 e_{t-2}^2 + \cdots + l e_{t-l}^2 \end{aligned} \tag{7.21}$$

We can also argue that the current conditional variance should also depend on the previous conditional variances as

$$v_t = \zeta_0 + \varsigma_1 v_{t-1} + \varsigma_2 v_{t-2} + \cdots + \varsigma_k v_{t-k} + \zeta_1 e_{t-1}^2 + \zeta_2 e_{t-2}^2 + \cdots + \zeta_l e_{t-l}^2 \tag{7.22}$$

In this notation, the error term e_t is said to follow a **generalized autoregressive conditional heteroskedastic** process of orders k and l, GARCH(k, l), proposed by Bollerslev (1986). In Eq. (7.22) the model for conditional variance resembles an ARMA model. However, it should be noted that the model in Eq. (7.22) is not a proper ARMA model, as this would have required a white noise error term with a constant variance for the MA part. But none of the terms on the right-hand side of the equation possess this property. For further details, see Hamilton (1994), Bollerslev et al. (1992), and Degiannakis and Xekalaki (2004). Further extensions of ARCH models also exist for various specifications of v_t in Eq. (7.22); for example, Integrated GARCH (I-GARCH) by Engle and Bollerslev (1986), Exponential GARCH (E-GARCH) by Nelson (1991), Nonlinear GARCH by Glosten et al. (1993), and GARCH for multivariate data by Engle and Kroner (1993). But they are beyond the scope of this book. For a brief overview of these models, see Hamilton (1994).

Example 7.5 Consider the weekly closing values for the S&P500 Index from 1995 to 1998 given in Table 7.4. Figure 7.6 shows that the data exhibits nonstationarity. But before taking the first difference of the data, we decided to take the log transformation of the data first. As observed in Chapters 2 and 3, the log transformation is sometimes used for financial data when we are interested, for example, in the rate of change or percentage changes in the price of a stock. For further details, see Granger and Newbold (1986). The time series plot of the first differences of the log of the S&P500 Index is given in Figure 7.7, which shows that while the mean seems to be stable around 0, the changes in the variance are worrisome. The ACF and PACF plots of the first difference given in Figure 7.8 suggest that, except for some borderline significant ACF values at seemingly arbitrary lags, there is no autocorrelation left in the data. As in the case of the Dow Jones Index in Chapter 5, this suggests that the S&P500 Index follows a random walk process. However, the time series plot of the differences does not exhibit a constant variance behavior. For that, we consider the ACF and PACF of the squared differences given in Figure 7.9, which suggests that an AR(3) model can be used. Thus we fit the ARCH(3) model for the variance using the AUTOREG procedure in SAS given in Table 7.5. The SAS output in

TABLE 7.4 Weekly Closing Values for the S&P500 Index from 1995 to 1998

Date	Close	Date	Close	Date	Close	Date	Close	Date	Close
1/3/1995	460.68	8/14/1995	559.21	3/25/1996	645.5	11/4/1996	730.82	6/16/1997	898.7
1/9/1995	465.97	8/21/1995	560.1	4/1/1996	655.86	11/11/1996	737.62	6/23/1997	887.3
1/16/1995	464.78	8/28/1995	563.84	4/8/1996	636.71	11/18/1996	748.73	6/30/1997	916.92
1/23/1995	470.39	9/5/1995	572.68	4/15/1996	645.07	11/25/1996	757.02	7/7/1997	916.68
1/30/1995	478.65	9/11/1995	583.35	4/22/1996	653.46	12/2/1996	739.6	7/14/1997	915.3
2/6/1995	481.46	9/18/1995	581.73	4/29/1996	641.63	12/9/1996	728.64	7/21/1997	938.79
2/13/1995	481.97	9/25/1995	584.41	5/6/1996	652.09	12/16/1996	748.87	7/28/1997	947.14
2/21/1995	488.11	10/2/1995	582.49	5/13/1996	668.91	12/23/1996	756.79	8/4/1997	933.54
2/27/1995	485.42	10/9/1995	584.5	5/20/1996	678.51	12/30/1996	748.03	8/11/1997	900.81
3/6/1995	489.57	10/16/1995	587.46	5/28/1996	669.12	1/6/1997	759.5	8/18/1997	923.54
3/13/1995	495.52	10/23/1995	579.7	6/3/1996	673.31	1/13/1997	776.17	8/25/1997	899.47
3/20/1995	500.97	10/30/1995	590.57	6/10/1996	665.85	1/20/1997	770.52	9/2/1997	929.05
3/27/1995	500.71	11/6/1995	592.72	6/17/1996	666.84	1/27/1997	786.16	9/8/1997	923.91
4/3/1995	506.42	11/13/1995	600.07	6/24/1996	670.63	2/3/1997	789.56	9/15/1997	950.51
4/10/1995	509.23	11/20/1995	599.97	7/1/1996	657.44	2/10/1997	808.48	9/22/1997	945.22
4/17/1995	508.49	11/27/1995	606.98	7/8/1996	646.19	2/18/1997	801.77	9/29/1997	965.03
4/24/1995	514.71	12/4/1995	617.48	7/15/1996	638.73	2/24/1997	790.82	10/6/1997	966.98
5/1/1995	520.12	12/11/1995	616.34	7/22/1996	635.9	3/3/1997	804.97	10/13/1997	944.16
5/8/1995	525.55	12/18/1995	611.95	7/29/1996	662.49	3/10/1997	793.17	10/20/1997	941.64
5/15/1995	519.19	12/26/1995	615.93	8/5/1996	662.1	3/17/1997	784.1	10/27/1997	914.62
5/22/1995	523.65	1/2/1996	616.71	8/12/1996	665.21	3/24/1997	773.88	11/3/1997	927.51
5/30/1995	532.51	1/8/1996	601.81	8/19/1996	667.03	3/31/1997	757.9	11/10/1997	928.35
6/5/1995	527.94	1/15/1996	611.83	8/26/1996	651.99	4/7/1997	737.65	11/17/1997	963.09
6/12/1995	539.83	1/22/1996	621.62	9/3/1996	655.68	4/14/1997	766.34	11/24/1997	955.4
6/19/1995	549.71	1/29/1996	635.84	9/9/1996	680.54	4/21/1997	765.37	12/1/1997	983.79
6/26/1995	544.75	2/5/1996	656.37	9/16/1996	687.03	4/28/1997	812.97	12/8/1997	953.39
7/3/1995	556.37	2/12/1996	647.98	9/23/1996	686.19	5/5/1997	824.78	12/15/1997	946.78
7/10/1995	559.89	2/20/1996	659.08	9/30/1996	701.46	5/12/1997	829.75	12/22/1997	936.46
7/17/1995	553.62	2/26/1996	644.37	10/7/1996	700.66	5/19/1997	847.03	12/29/1997	975.04
7/24/1995	562.93	3/4/1996	633.5	10/14/1996	710.82	5/27/1997	848.28		
7/31/1995	558.94	3/11/1996	641.43	10/21/1996	700.92	6/2/1997	858.01		
8/7/1995	555.11	3/18/1996	650.62	10/28/1996	703.77	6/9/1997	893.27		

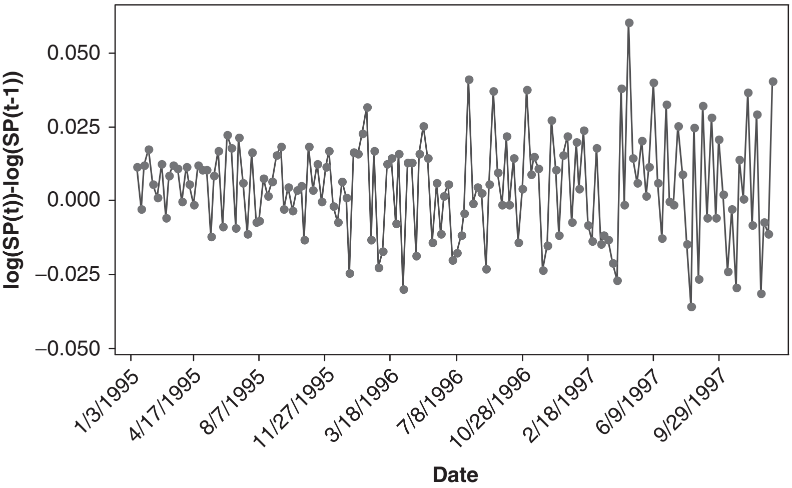


FIGURE 7.7 Time series plot of the first difference of the log transformation of the weekly close for S&P500 Index from 1995 to 1998.

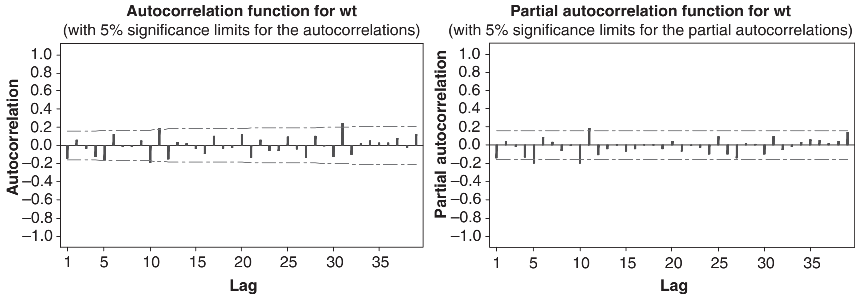


FIGURE 7.8 ACF and PACF plots of the first difference of the log transformation of the weekly close for the S&P500 Index from 1995 to 1998.

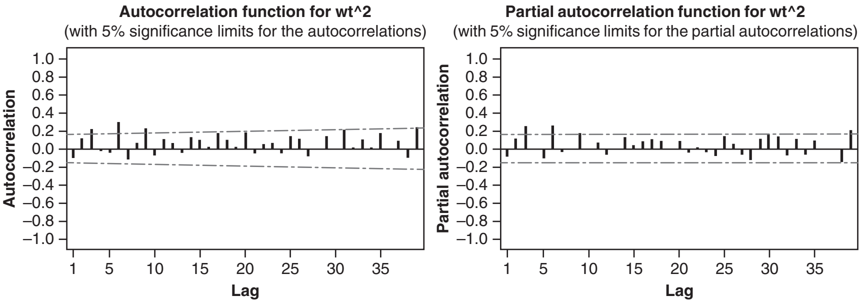


FIGURE 7.9 ACF and PACF plots of the square of the first difference of the log transformation of the weekly close for S&P500 Index from 1995 to 1998.

TABLE 7.5 SAS Commands to Fit the ARCH(3) Model[a]

```
proc autoreg data=sp5003;
   model dlogc = /   garch=( q=3);
run;
```

[a] `dlogc` is the first difference of the log transformed data.

TABLE 7.6 SAS output for the ARCH(3) model

```
                              GARCH Estimates

   SSE                  0.04463228    Observations              156
   MSE                   0.0002861    Uncond Var         0.00030888
   Log Likelihood        422.53308    Total R-Square              .
   SBC                  -824.86674    AIC                -837.06616
   Normality Test           1.6976    Pr > ChiSq             0.4279

                           The AUTOREG Procedure

                          Standard        Approx
Variable     DF           Estimate         Error      t Value      Pr > |t|

Intercept     1           0.004342      0.001254         3.46        0.0005
ARCH0         1           0.000132     0.0000385         3.42        0.0006
ARCH1         1          4.595E-10     3.849E-11        11.94        <.0001
ARCH2         1             0.2377        0.1485         1.60        0.1096
ARCH3         1             0.3361        0.1684         2.00        0.0460
```

Table 7.6 gives the coefficient estimates for the ARCH(3) model for the variance.

There are other studies on financial indices also yielding the ARCH(3) model for the variance, for example, Bodurtha and Mark (1991) and Attanasio (1991). In fact, successful implementations of reasonably simple, low-order ARCH/GARCH models have been reported in various research studies; see, for example, French et al. (1987).

7.4 DIRECT FORECASTING OF PERCENTILES

Throughout this book we have stressed the concept that a forecast should almost always be more than a point estimate of the value of some future event. A prediction interval should accompany most point forecasts, because the PI will give the decision maker some idea about the inherent variability of the forecast and the likely forecast error that could be experienced. Most of the forecasting techniques in this book have been presented showing how both point forecasts and PIs are obtained.

A PI can be thought of as an estimate of the percentiles of the distribution of the forecast variable. Typically, a PI is obtained by forecasting the mean and then adding appropriate multiples of the standard deviation of forecast error to the estimate of the mean. In this section we present and illustrate a different method that directly smoothes the percentiles of the distribution of the forecast variable.

Suppose that the forecast variable y_t has a probability distribution $f(y)$. We will assume that the variable y_t is either stationary or is changing slowly with time. Therefore a model for y_t that is correct at least locally is

$$y_t = \mu + \varepsilon_t$$

Let the observations on y_t be classified into a finite number of bins, where the bins are defined with limits

$$B_0 < B_1 < \cdots < B_n$$

The n bins should be defined so that they do not overlap; that is, each observation can be classified into one and only one bin. The bins do not have to be of equal width. In fact, there may be situations where bins may be defined with unequal width to obtain more information about specific percentiles that are of interest. Typically, $10 \leq n \leq 20$ bins are used.

Let p_k be the probability that the variable y_t falls in the bin defined by the limits B_{k-1} and B_k. That is,

$$p_k = P(B_{k-1} < y_t \leq B_k), \quad k = 1, 2, \ldots, n$$

Assume that $\sum_{k=1}^{n} p_k = 1$. Also, note that $P(y_t \leq B_k) = \sum_{j=1}^{k} p_j$. Now let us consider estimating the probabilities. Write the probabilities as an $n \times 1$ vector $\mathbf{p}$ defined as

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

Let the estimate of the vector $\mathbf{p}$ at time period T be

$$\hat{\mathbf{p}}(T) = \begin{bmatrix} \hat{p}_1(T) \\ \hat{p}_2(T) \\ \vdots \\ \hat{p}_n(T) \end{bmatrix}$$

Note that if we wanted to estimate the percentile of the distribution of y_t corresponding to B_k at time period T we could do this by calculating $\sum_{j=1}^{k} \hat{p}_j(T)$.

We will use an exponential smoothing procedure to compute the estimated probabilities in the vector $\hat{\mathbf{p}}(T)$. Suppose that we are at the end of time period t and the current observation y_T is known. Let $u_k(T)$ be an indicator variable defined as follows:

$$u_k(T) = \begin{cases} 1 & \text{if } B_{k-1} < y_T \leq B_k \\ 0 & \text{otherwise} \end{cases}$$

So the indicator variable $u_k(T)$ is equal to unity if the observation y_T in period T falls in the kth bin. Note that $\sum_{t=1}^{T} u_k(t)$ is the total number of observations that fell in the kth bin during the time periods $t = 1, 2, \ldots, T$. Define the $n \times 1$ observation vector $\mathbf{u}(T)$ as

$$\mathbf{u}(T) = \begin{bmatrix} u_1(T) \\ u_2(T) \\ \vdots \\ u_n(T) \end{bmatrix}$$

This vector will have $n - 1$ elements equal to zero and one element equal to unity. The exponential smoothing procedure for revising the probabilities $\hat{p}_k(T - 1)$ given that we have a new observation y_T is

$$\hat{p}_k(T) = \lambda u_k(T) + (1 - \lambda)\hat{p}_k(T - 1), \quad k = 1, 2, \ldots, n \qquad (7.23)$$

where $0 < \lambda < 1$ is the smoothing constant. In vector form, Eq. (7.23) for updating the probabilities is

$$\hat{\mathbf{p}}_k(T) = \lambda \mathbf{u}_k(T) + (1 - \lambda)\hat{\mathbf{p}}_k(T - 1)$$

This smoothing procedure produces an unbiased estimate of the probabilities p_k. Furthermore, because $u_k(T)$ is a Bernoulli random variable with parameter p_k, the variance of $\hat{p}_k(T)$ is

$$V[\hat{p}_k(T)] = \frac{\lambda}{2 - \lambda} p_k(1 - p_k)$$

Starting estimates or initial values of the probabilities at time $T = 0$ are required. These starting values $\hat{p}_k(0)$, $k = 1, 2, \ldots, n$ could be subjective estimates or they could be obtained from an analysis of historical data.

The estimated probabilities can be used to obtain estimates of specific percentiles of the distribution of the variable y_t. One way to do this would be to estimate the cumulative probability distribution of y_t at time T as follows:

$$F(y) = \begin{cases} 0, & \text{if } y \le B_0 \\ \sum_{j=1}^{k} \hat{p}_j(T), & \text{if } y = B_k, k = 1, 2, \dots, n \\ 1, & \text{if } y \ge B_n \end{cases}$$

The values of the cumulative distribution could be plotted on a graph with $F(y)$ on the vertical axis and y on the horizontal axis and the points connected by a smooth curve. Then to obtain an estimate of any specific percentile, say, $\hat{F}_{1-\gamma} = 1 - \gamma$, all you would need to do is determine the value of y on the horizontal axis corresponding to the desired percentile $1 - \gamma$ on the vertical axis. For example, to find the 95th percentile of the distribution of y, find the value of y on the horizontal axis that corresponds to 0.95 on the vertical axis. This can also be done mathematically. If the desired percentile $1 - \gamma$ exactly matches one of the bin limits so that $F(B_k) = 1 - \gamma$, then the solution is easy and the desired percentile estimate is $\hat{F}_{1-\gamma} = B_k$. However, if the desired percentile $1 - \gamma$ is between two of the bin limits, say, $F(B_{k-1}) < 1 - \gamma < F(B_k)$, then interpolation is required. A linear interpolation formula is

$$\hat{F}_{1-\gamma} = \frac{[F(B_k) - (1 - \gamma)]B_{k-1} + [(1 - \gamma) - F(B_{k-1})]B_k}{F(B_k) - F(B_{k-1})} \tag{7.24}$$

In the extreme tails of the distribution or in cases where the bins are very wide, it may be desirable to use a nonlinear interpolation scheme.

Example 7.6 A financial institution is interested in forecasting the number of new automobile loan applications generated each week by a particular business channel. The information in Table 7.7 is known at the end of week $T - 1$. The next-to-last column of this table is the cumulative distribution of loan applications at the end of week $T - 1$. This cumulative distribution is shown in Figure 7.10.

Suppose that 74 loan applications are received during the current week, T. This number of loan applications fall into the eighth bin ($k = 7$ in Table 7.7). Therefore we can construct the observation vector $\mathbf{u}(T)$

as follows:

$$\mathbf{u}(T) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

TABLE 7.7 Distribution of New Automobile Loan Applications

k	B_{k-1}	B_k	$\hat{p}_k(T-1)$	$F(B_k)$, at the end of week $T-1$	$F(B_k)$, at the end of week $T-1$
0	0	10	0.02	0.02	0.018
1	10	20	0.04	0.06	0.054
2	20	30	0.05	0.11	0.099
3	30	40	0.05	0.16	0.144
4	40	50	0.09	0.25	0.225
5	50	60	0.10	0.35	0.315
6	60	70	0.13	0.48	0.432
7	70	80	0.16	0.64	0.676
8	80	90	0.20	0.84	0.856
9	90	100	0.10	0.94	0.946
10	100	110	0.06	1.00	1.000

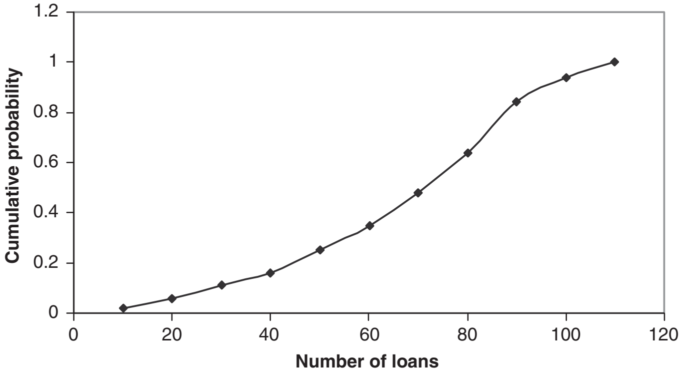


FIGURE 7.10 Cumulative distribution of the number of loan applications, week $T-1$.

Equation (7.23) is now used with $\lambda = 0.10$ to update the probabilities:

$$\hat{\mathbf{p}}_k(T) = \lambda \mathbf{u}_k(T) + (1 - \lambda)\hat{\mathbf{p}}_k(T - 1)$$

$$= 0.1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0.9 \begin{bmatrix} 0.02 \\ 0.04 \\ 0.05 \\ 0.05 \\ 0.09 \\ 0.10 \\ 0.13 \\ 0.16 \\ 0.20 \\ 0.10 \\ 0.06 \end{bmatrix} = \begin{bmatrix} 0.018 \\ 0.036 \\ 0.045 \\ 0.045 \\ 0.081 \\ 0.090 \\ 0.117 \\ 0.244 \\ 0.180 \\ 0.090 \\ 0.054 \end{bmatrix}$$

Therefore the new cumulative distribution of loan applications is found by summing the cumulative probabilities in $\hat{\mathbf{p}}_k(T - 1)$:

$$F(B_k) = \begin{bmatrix} 0.018 \\ 0.054 \\ 0.099 \\ 0.144 \\ 0.225 \\ 0.315 \\ 0.432 \\ 0.676 \\ 0.856 \\ 0.946 \\ 1.000 \end{bmatrix}$$

These cumulative probabilities are also listed in the last column of Table 7.7. The graph of the updated cumulative distribution is shown in Figure 7.11.

Now suppose that we want to find the number of loan applications that corresponds to a particular percentile of this distribution. If this percentile corresponds exactly to one of the cumulative probabilities, such as the 67.6 th percentile, the problem is easy. From the last column of Table 7.7 we would find that

$$\hat{F}_{0.676} = 80$$

That is, in about two of every three weeks we would expect to have 80 or fewer loan applications from this particular channel. However, if the desired

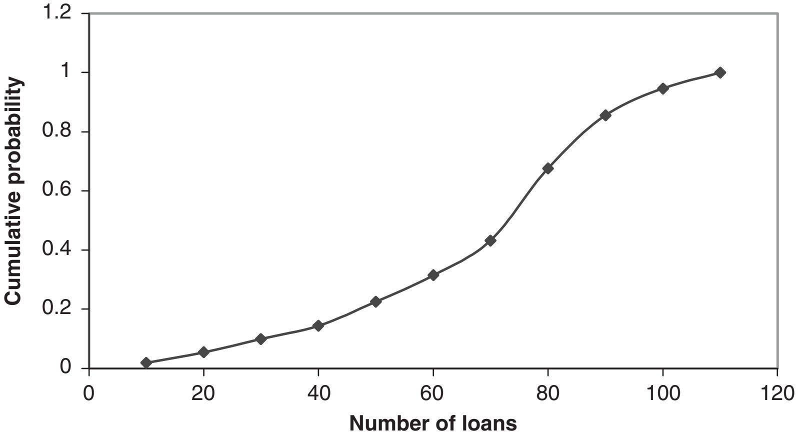


FIGURE 7.11 Cumulative distribution of the number of loan applications, week *T*.

percentile does not correspond to one of the cumulative probabilities in the last column of Table 7.7, we will need to interpolate using Eq. (7.24). For instance, if we want the 75th percentile, we would use Eq. (7.24) as follows:

$$\begin{aligned}\hat{F}_{0.75} &= \frac{[F(B_k) - (0.75)]B_{k-1} + [(0.75) - [F(B_{k-1})]B_k}{F(B_k) - F(B_{k-1})}\\ &= \frac{(0.856 - 0.75)90 + (0.75 - 0.676)80}{0.856 - 0.676}\\ &= 85.89 \approx 86\end{aligned}$$

Therefore, in about three of every four weeks, we would expect to have approximately 86 or fewer loan applications from this loan channel.

7.5 COMBINING FORECASTS TO IMPROVE PREDICTION PERFORMANCE

Readers have been sure to notice that any time series can be modeled and forecast using several methods. For example, it is not at all unusual to find that the time series y_t, $t = 1, 2, \ldots$, which contains a trend (say), can be forecast by both an exponential smoothing approach and an ARIMA model. In such situations, it seems inefficient to use one forecast and ignore all of the information in the other. It turns out that the forecasts from the two methods can be combined to produce a forecast that is superior in terms of forecast error than either forecast alone. For a review paper on the combination of forecasts, see Clemen (1989).

Bates and Granger (1969) suggested using a linear combination of the two forecasts. Let $\hat{y}_{1,T+\tau}(T)$ and $\hat{y}_{2,T+\tau}(T)$ be the forecasts from two different methods at the end of time period T for some future period $T+\tau$ for the time series y_t. The combined forecast is

$$\hat{y}^{\mathrm{c}}_{T+\tau} = k_1\hat{y}_{1,T+\tau}(T) + k_2\hat{y}_{2,T+\tau}(T) \tag{7.25}$$

where k_1 and k_2 are weights. If these weights are chosen properly, the combined forecast $\hat{y}^{\mathrm{c}}_{T+\tau}$ can have some nice properties. Let the two individual forecasts be unbiased. Then we should choose $k_2 = 1 - k_1$ so that the combined forecast will also be unbiased. Let $k = k_1$ so that the combined forecast is

$$\hat{y}^{\mathrm{c}}_{T+\tau} = k\hat{y}_{1,T+\tau}(T) + (1-k)\hat{y}_{2,T+\tau}(T) \tag{7.26}$$

Let the error from the combined forecast be $e^{\mathrm{c}}_{T+\tau}(T) = y_{T+\tau} - \hat{y}^{\mathrm{c}}_{T+\tau}(T)$. The variance of this forecast error is

$$\begin{aligned}\operatorname{Var}\left[e^{\mathrm{c}}_{T+\tau}(T)\right] &= \operatorname{Var}\left[y_{T+\tau} - \hat{y}^{\mathrm{c}}_{T+\tau}(T)\right]\\ &= \operatorname{Var}[ke_{1,T+\tau}(T) + (1-k)e_{2,T+\tau}(T)]\\ &= k^2\sigma_1^2 + (1-k)^2\sigma_2^2 + 2k(1-k)\rho\sigma_1\sigma_2\end{aligned}$$

where $e_{1,T+\tau}(T)$ and $e_{2s,T+\tau}(T)$ are the forecast errors in period $T+\tau$ for the two individual forecasting methods, σ_1^2 and σ_2^2 are the variances of the individual forecast errors for the two forecasting methods, and ρ is the correlation between the two individual forecast errors. A good combined forecast would be one that minimizes the variance of the combined forecast error. If we choose the weight k equal to

$$k^* = \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{7.27}$$

this will minimize the variance of the combined forecast error. By choosing this value for the weight, the minimum variance of the combined forecast error is equal to

$$\text{Min Var}\left[e^{\mathrm{c}}_{T+\tau}(T)\right] = \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{7.28}$$

and this minimum variance of the combined forecast error is less than or equal to the minimum of the variance of the forecast errors of the two individual forecasting methods. That is,

$$\text{Min Var}\left[e^{\text{c}}_{T+\tau}(T)\right] \leq \min\left(\sigma_1^2, \sigma_2^2\right)$$

It turns out that the variance of the combined forecast error depends on the correlation coefficient. Let σ_1^2 be the smaller of the two individual forecast error variances. Then we have the following:

1. If $\rho = \sigma_1/\sigma_2$, then Min Var $[e^{\text{c}}_{T+\tau}(T)] = \sigma_1^2$.
2. If $\rho = 0$, then Var $[e^{\text{c}}_{T+\tau}(T)] = \sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)$.
3. If $\rho \to -1$, then Var $[e^{\text{c}}_{T+\tau}(T)] \to 0$.
4. If $\rho \to 1$, then Var $[e^{\text{c}}_{T+\tau}(T)] \to 0$ if $\sigma_1^2 \neq \sigma_2^2$.

Clearly, we would be happiest if the two forecasting methods have forecast errors with large negative correlation. The best possible case is when the two individual forecasting methods produce forecast errors that are perfectly negatively correlated. However, even if the two individual forecasting methods have forecast errors that are positively correlated, the combined forecast will still be superior to the individual forecasts, provided that $\rho \neq \sigma_1/\sigma_2$.

Example 7.7 Suppose that two forecasting methods can be used for a time series, and that the two variances of the forecast errors are $\sigma_1^2 = 20$ and $\sigma_2^2 = 40$. If the correlation coefficient $\rho = -0.6$, then we can calculate the optimum value of the weight from Eq. (7.27) as follows:

$$\begin{aligned} k^* &= \frac{\sigma_2^2 - \rho\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\ &= \frac{40 - (-0.6)\sqrt{(40)(20)}}{40 + 20 - 2(-0.6)\sqrt{(40)(20)}} \\ &= \frac{56.9706}{93.9411} \\ &= 0.6065 \end{aligned}$$

So the combined forecasting equation is

$$\hat{y}^{\text{c}}_{T+\tau} = 0.6065\hat{y}_{1,T+\tau}(T) + 0.3935\hat{y}_{2,T+\tau}(T)$$

Forecasting method one, which has the smallest individual forecast error variance, receives about 1.5 times the weight of forecasting method two.

The variance of the forecast error for the combined forecast is computed from Eq. (7.28):

$$\begin{aligned}\text{Min Var}\left[e^{\text{c}}_{T+\tau}(T)\right] &= \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\\ &= \frac{(20)(40)[1-(-0.6)^2]}{20+40-2(-0.6)\sqrt{(20)(40)}}\\ &= \frac{512}{93.9411}\\ &= 5.45\end{aligned}$$

This is a considerable reduction in the variance of forecast error. If the correlation had been positive instead of negative, then

$$\begin{aligned}k^* &= \frac{\sigma_2^2-\rho\sigma_1\sigma_2}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\\ &= \frac{40-(0.6)\sqrt{(40)(20)}}{40+20-2(0.6)\sqrt{(40)(20)}}\\ &= \frac{23.0294}{26.0589}\\ &= 0.8837\end{aligned}$$

Now forecasting method one, which has the smallest variance of forecast error, receives much more weight. The variance of the forecast error for the combined forecast is

$$\begin{aligned}\text{Min Var}\left[e^{\text{c}}_{T+\tau}(T)\right] &= \frac{\sigma_1^2\sigma_2^2(1-\rho^2)}{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}\\ &= \frac{(20)(40)[1-(0.6)^2]}{20+40-2(0.6)\sqrt{(20)(40)}}\\ &= \frac{512}{26.0589}\\ &= 19.6478\end{aligned}$$

In this situation, there is very little improvement in the forecast error resulting from the combination of forecasts.

Newbold and Granger (1974) have extended this technique to the combination of n forecasts. Let $\hat{y}_{i,T+\tau}(T)$, $i=1,2,\ldots,n$ be the n unbiased forecasts at the end of period T for some future period $T + \tau$ for the time series y_t. The combined forecast is

$$\hat{y}^{\mathrm{c}}_{T+\tau}(T) = \sum_{i=1}^{n} k_i \hat{y}_{T+\tau}(T)$$
$$= \mathbf{k}'\hat{\mathbf{y}}_{T+\tau}(T)$$

where $\mathbf{k}' = [k_1, k_2, \ldots, k_n]$ is the vector of weights, and $\hat{\mathbf{y}}^{\mathrm{c}}_{T+\tau}(T)$ is a vector of the individual forecasts. We require that all of the weights $0 \leq k_i \leq 1$ and $\sum_{i=1}^{n} k_i = 1$. The variance of the forecast error is minimized if the weights are chosen as

$$\mathbf{k} = \frac{\sum_{T+\tau}^{-1}(T)\mathbf{1}}{\mathbf{1}' \sum_{T+\tau}^{-1}(T)\mathbf{1}}$$

where $\mathbf{\Sigma}_{T+\tau}(T)$ is the covariance matrix of the lead τ forecast errors given by

$$\sum\nolimits_{T+\tau}(T) = E[\mathbf{e}_{T+\tau}(T)\mathbf{e}'{}_{T+\tau}(T)]$$

$\mathbf{1}' = [1, 1, \ldots, 1]$ is a vector of ones, and $\mathbf{e}_{T+\tau}(T) = y_{T+\tau}\mathbf{1} - \hat{\mathbf{y}}_{T+\tau}(T)$ is a vector of the individual forecast errors.

The elements of the covariance matrix are usually unknown and will need to be estimated. This can be done by straightforward methods for estimating variances and covariances (refer to Chapter 2). It may also be desirable to regularly update the estimates of the covariance matrix so that these quantities reflect current forecasting performance. Newbold and Granger (1974) suggested several methods for doing this, and Montgomery, Johnson, and Gardiner (1990) investigate several of these methods. They report that a smoothing approach for updating the elements of the covariance matrix seems to work well in practice.

7.6 AGGREGATION AND DISAGGREGATION OF FORECASTS

Suppose that you wish to forecast the unemployment level of the state in which you live. One way to do this would be to forecast this quantity directly, using the time series of current and previous unemployment data, plus any other predictors that you think are relevant. Another way to do this would be to forecast unemployment at a substate level (say, by county and/or metropolitan area), and then to obtain the state level forecast by summing up the forecasts for each substate region. Thus individual forecasts of a collection of subseries are aggregated to form the forecasts of the quantity of interest. If the substate level forecasts are useful in their own right (as they probably are), this second approach seems very useful. However, there is another way to do this. First, forecast the state level unemployment and then disaggregate this forecast into the individual substate level regional forecasts. This disaggregation could be accomplished by multiplying the state level forecasts by a series of indices that reflect the proportion of total statewide unemployment that is accounted for by each region at the substate level. These indices also evolve with time, so it will be necessary to forecast them as well as part of a complete system.

This problem is sometimes referred to as the **top–down** versus **bottom–up** forecasting problem. In many, if not most, of these problems, we are interested in both forecasts for the top level quantity (the aggregate time series) and forecasts for the bottom level time series that are the components of the aggregate.

This leads to an obvious question: is it better to forecast the aggregate or top level quantity directly and then disaggregate, or to forecast the individual components directly and then aggregate them to form the forecast of the total? In other words, is it better to forecast from the top down or from the bottom up? The literature in statistical forecasting, business forecasting and econometrics, and time series analysis suggests that this question is far from settled at either the theoretical or empirical levels. Sometimes the aggregate quantity is more accurate than the disaggregated components, and sometimes the aggregate quantity is subject to less measurement error. It may be more complete and timely as well, and these aspects of the problem should encourage those who consider forecasting the aggregate quantity and then disaggregating. On the other hand, sometimes the bottom level data is easier to obtain and is at least thought to be more timely and accurate, and this would suggest that a bottom–up approach would be superior to the top–down approach.

In any specific practical application it will be difficult to argue on theoretical grounds what the correct approach should be. Therefore, in most situations, this question will have to be settled empirically by trying both approaches. With modern computer software for time series analysis and forecasting, this is not difficult. However, in conducting such a study it is a good idea to have an adequate amount of data for identifying and fitting the time series models for both the top level series and the bottom level series, and a reasonable amount of data for testing the two approaches. Obviously, **data splitting** should be done here, and the data used for model building should not be used for investigating forecasting model performance. Once an approach is determined, the forecasts should be carefully monitored over time to make sure that the dynamics of the problem have not changed, and that the top–down approach that was found to be optimal in testing (say) is now no longer as effective as the bottom–up approach. The methods for monitoring forecasting model performance presented in Chapter 2 are useful in this regard.

There are some results available about the effect of adding time series together. This is a special case of a more general problem called **temporal aggregation**, in which several time series may be combined as, for instance, when monthly data are aggregated to form quarterly data. For example, suppose that we have a top level time series Y_t that is the sum of two independent time series $y_{1,t}$ and $y_{2,t}$, and let us assume that both of the bottom level time series are moving average (MA) processes of orders q_1 and q_2, respectively. So, using the notation for ARIMA models introduced in Chapter 5, we have

$$Y_t = \theta_1(B)a_t + \theta_2(B)b_t$$

where a_t and b_t are independent white noise processes. Now let q be the maximum of q_1 and q_2. The autocorrelation function for the top level time series Y_t must be zero for all of the lags beyond q. This means that there is a representation of the top level time series as an MA process

$$Y_t = \theta_3(B)u_t$$

where u_t is white noise. This moving average process has the same order as the higher order bottom level time series.

Now consider the general ARIMA(p_1, d, q_1) model

$$\phi_1(B)\nabla^d y_t = \theta_1(B)a_t$$

and suppose that we are interested in the sum of two time series $z_t = y_t + w_t$. A practical situation where this occurs, in addition to the top–down versus bottom–up problem, is when the time series y_t we are interested in cannot be observed directly and w_t represents added noise due to measurement error. We want to know something about the nature of the sum of the two series, z_t. The sum can be written as

$$\phi_1(B)\nabla^d z_t = \theta_1(B)a_t + \phi_1(B)\nabla^d w_t$$

Assume that the time series w_t can be represented as a stationary ARMA$(p_2, 0, q_2)$ model

$$\phi_2(B)w_t = \theta_2(B)b_t$$

where b_t is white noise independent of a_t. Then the top level time series is

$$\phi_1(B)\phi_2(B)\nabla^d z_t = \phi_2(B)\theta_1(B)a_t + \phi_1(B)\theta_2(B)\nabla^d b_t$$

The term on the left-hand side is a polynomial of order $P = p_1 + p_2$, the first term on the right-hand side is a polynomial of order $q_1 + p_2$, and the second term on the right-hand side is a polynomial of order $p_1 + q_2 + d$. Let Q be the maximum of $q_1 + p_2$ and $p_1 + q_2 + d$. Then the top level time series is an ARIMA(P, d, Q) model, say,

$$\phi_3(B)\nabla^d z_t = \theta_3(B)u_t$$

where u_t is white noise.

Example 7.8 Suppose that we have a time series that is represented by an IMA(1, 1) model, and to this time series is added white noise. This could be a situation where measurements on a periodic sample of some characteristic in the output of a chemical process are made with a laboratory procedure, and the laboratory procedure has some built-in measurement error, represented by the white noise. Suppose that the underlying IMA(1, 1) model is

$$y_t = y_{t-1} - 0.6a_{t-1} + a_t$$

Let D_t be the first difference of the observed time series $z_t = y_t + w_t$, where w_t is white noise:

$$\begin{aligned} D_t &= z_t - z_{t-1} \\ &= (1 - \theta B)a_t + (1 - B)w_t \end{aligned}$$

The autocovariances of the differenced series are

$$\begin{aligned} \gamma_0 &= \sigma_a^2(1 + \theta^2) + 2\sigma_w^2 \\ \gamma_1 &= -\sigma_a^2\theta - \sigma_w^2 \\ \gamma_j &= 0, \quad j \geq 2 \end{aligned}$$

Because the autocovariances at and beyond lag 2 are zero, we know that the observed time series will be IMA(1, 1). In general, we could write this as

$$z_t = z_{t-1} - \theta^* u_{t-1} + u_t$$

where the parameter θ^* is unknown. However, we can find θ^* easily. The autocovariances of the first differences of this observed time series are

$$\begin{aligned}\gamma_0 &= \sigma_u^2(1+\theta^{*2})\\ \gamma_1 &= -\sigma_u^2\theta^*\\ \gamma_j &= 0, \quad j \geq 2\end{aligned}$$

Now all we have to do is to equate the autocovariances for this observed series in terms of the parameter θ^* with the autocovariances of the time series D_t and we can solve for θ^* and σ_u^2. This gives the following:

$$\begin{aligned}\frac{\theta^*}{1-\theta^*} &= \frac{0.6}{1-0.6+\sigma_w^2/\sigma_a^2}\\ \sigma_u^2 &= \sigma_a^2\frac{(0.6)^2}{\theta^{*2}}\end{aligned}$$

Suppose that $\sigma_a^2 = 2$ and $\sigma_w^2 = 1$. Then it turns out that the solution is $\theta^* = 0.4$ and $\sigma_u^2 = 4.50$. Adding the measurement error from the laboratory procedure to the original sample property has inflated the variability of the observed value rather considerably over the original variability that was present in the sample property.

7.7 NEURAL NETWORKS AND FORECASTING

Neural networks, or more accurately **artificial neural networks**, have been motivated by the recognition that the human brain processes information in a way that is fundamentally different from the typical digital computer. The neuron is the basic structural element and information-processing module of the brain. A typical human brain has an enormous number of them (approximately 10 billion neurons in the cortex and 60 trillion synapses or connections between them) arranged in a highly complex, nonlinear, and parallel structure. Consequently, the human brain is a very efficient structure for information processing, learning, and reasoning.

An artificial neural network is a structure that is designed to solve certain types of problems by attempting to emulate the way the human brain would solve the problem. The general form of a neural network is a “black-box” type of model that is often used to model high-dimensional, nonlinear data. In the forecasting environment, neural networks are sometimes used to solve prediction problems instead of using a formal model building approach or development of the underlying knowledge of the system that would be required to develop an analytical forecasting procedure. If it was a successful approach that might be satisfactory. For example, a company might want to forecast demand for its products. If a neural network procedure can do this quickly and accurately, the company may have little interest in developing a specific analytical forecasting model to do it. Hill et al. (1994) is a basic reference on artificial neural networks and forecasting.

Multilayer feedforward artificial neural networks are multivariate statistical models used to relate p predictor variables $x_1, x_2, \dots, x_p$ to one or more output or response variables y. In a forecasting application, the inputs could be explanatory variables such as would be used in a regression model, and they could be previous values of the outcome or response variable (lagged variables). The model has several **layers**, each consisting of either the original or some constructed variables. The most common structure involves three layers: the **inputs**, which are the original predictors; the **hidden layer**, comprised of a set of constructed variables; and the output layer, made up of the responses. Each variable in a layer is called a **node**. Figure 7.12 shows a typical three-layer artificial neural network for forecasting the output variable y in terms of several predictors.

A node takes as its input a transformed linear combination of the outputs from the nodes in the layer below it. Then it sends as an output a transformation of itself that becomes one of the inputs to one or more nodes on the next layer. The transformation functions are usually either sigmoidal (S shaped) or linear and are usually called **activation functions** or **transfer**

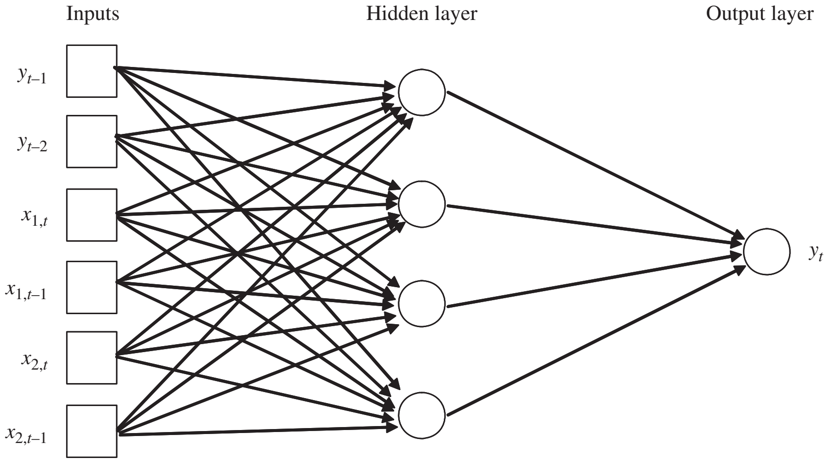


FIGURE 7.12 Artificial neural network with one hidden layer.

functions. Let each of the k hidden layer nodes a_u be a linear combination of the input variables:

$$a_u = \sum_{j=1}^{p} w_{1ju} x_j + \theta_u$$

where the w_{1ju} are unknown parameters that must be estimated (called weights) and θ_u is a parameter that plays the role of an intercept in linear regression (this parameter is sometimes called the bias node).

Each node is transformed by the activation function $g(\ )$. Much of the neural networks literature refers to these activation functions notationally as σ_u because of their S shape (the use of σ is an unfortunate choice of notation so far as statisticians are concerned). Let the output of node a_u be denoted by $z_u = g(a_u)$. Now we form a linear combination of these outputs, say, $b = \sum_{u=1}^{k} w_{uev} z_u$. Finally, the output response or the predicted value for y is a transformation of the b, say, $y = \tilde{g}(b)$, where $\tilde{g}(b)$ is the activation function for the response.

The response variable y is a transformed linear combination of the original predictors. For the hidden layer, the activation function is often chosen to be either a logistic function or a hyperbolic tangent function. The choice of activation function for the output layer often depends on the nature of the response variable. If the response is bounded or dichotomous, the output activation function is usually taken to be sigmoidal, while if it is continuous, an identity function is often used.

The neural network model is a very flexible form containing many parameters, and it is this feature that gives a neural network a nearly universal approximation property. That is, it will fit many historical data sets very well. However, the parameters in the underlying model must be estimated (parameter estimation is called "training" in the neural network literature), and there are a lot of them. The usual approach is to estimate the parameters by minimizing the overall residual sum of squares taken over all responses and all observations. This is a nonlinear least squares problem, and a variety of algorithms can be used to solve it. Often a procedure called **backpropagation** (which is a variation of steepest descent) is used, although derivative-based gradient methods have also been employed. As in any nonlinear estimation procedure, starting values for the parameters must be specified in order to use these algorithms. It is customary to standardize all the input variables, so small essentially random values are chosen for the starting values.

With so many parameters involved in a complex nonlinear function, there is considerable danger of **overfitting**. That is, a neural network will provide a nearly perfect fit to a set of historical or "training" data, but it will often predict new data very poorly. Overfitting is a familiar problem to statisticians trained in empirical model building. The neural network community has developed various methods for dealing with this problem, such as reducing the number of unknown parameters (this is called "optimal brain surgery"), stopping the parameter estimation process before complete convergence and using cross-validation to determine the number of iterations to use, and adding a penalty function to the residual sum of squares that increases as a function of the sum of the squares of the parameter estimates.

There are also many different strategies for choosing the number of layers and number of neurons and the form of the activation functions. This is usually referred to as choosing the **network architecture**. Cross-validation can be used to select the number of nodes in the hidden layer.

Artificial neural networks are an active area of research and application in many fields, particularly for the analysis of large, complex, highly nonlinear problems. The overfitting issue is frequently overlooked by many users and even the advocates of neural networks, and because many members of the neural network community do not have sound training in empirical model building, they often do not appreciate the difficulties overfitting may cause. Furthermore, many computer programs for implementing neural networks do not handle the overfitting problem particularly well. Studies of the ability of neural networks to predict future values of a time series that were not used in parameter estimation (fitting) have been, in many cases, disappointing. Our view is that neural networks are a complement to the familiar statistical tools of forecasting, and they might be one of the approaches you should consider, but they are not a replacement for them.

7.8 SPECTRAL ANALYSIS

This book has been focused on the analysis and modeling of time series in the **time domain**. This is a natural way to develop models, since time series all are observed as a function of time. However, there is another approach to describing and analyzing time series that uses a **frequency domain** approach. This approach consists of using the Fourier representation of a time series, given by

$$y_t = \sum_{k=1}^{T} a_k \sin(2\pi f_k t) + b_k \cos(2\pi f_k t) \tag{7.29}$$

where $f_k = k/T$. This model is named after J.B.J Fourier, an 18th century French mathematician, who claimed that any periodic function could be represented as a series of harmonically related sinusoids. Other contributors to Fourier analysis include Euler, D. Bernoulli, Laplace, Lagrange, and Dirichlet. The original work of Fourier was focused on phenomena in continuous time, such as vibrating strings, and there are still many such applications today from such diverse fields as geophysics, oceanography, atmospheric science, astronomy, and many disciplines of engineering. However, the key ideas carry over to discrete time series. We confine our discussion to **stationary** discrete time series.

Computing the constants a_k and b_k turns out to be quite simple:

$$a_k = \frac{2}{T} \sum_{k=1}^{T} \cos(2\pi f_k t) \tag{7.30}$$

and

$$b_k = \frac{2}{T} \sum_{k=1}^{T} \sin(2\pi f_k t) \tag{7.31}$$

These coefficients are combined to form a periodogram

$$I(f_k) = \frac{T}{2}\left(a_k^2 + b_k^2\right) \tag{7.32}$$

The periodogram is then usually smoothed and scaled to produce the **spectrum** or a **spectral density function**. The spectral density function is just the Fourier transform of the autocorrelation function, so it conveys similar information as is found in the autocorrelations. However, sometimes the spectral density is easier to interpret than the autocorrelation function because adjacent sample autocorrelations can be highly correlated while estimates of the spectrum at adjacent frequencies are approximately independent. Generally, if the frequency k/T is important then $I(f_k)$ will be large, and if the frequency k/T is not important then $I(f_k)$ will be small.

It can be shown that

$$\sum_{k=1}^{T} I(f_k) = \sigma^2 \tag{7.33}$$

where σ^2 is the variance of the time series. Thus the spectrum decomposes the variance of the time series into individual components, each of which is associated with a particular frequency. So we can think of spectral analysis

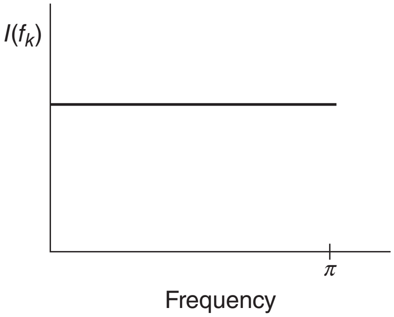


FIGURE 7.13 The spectrum of a white noise process.

as an analysis of variance technique. It decomposes the variability in the time series by frequency.

It is helpful to know what the spectrum looks like for some simple ARMA models. If the time series white noise (uncorrelated observations with constant variance σ^2), it can be shown that the spectrum is a horizontal straight line as shown in Figure 7.13. This means that the contribution to the variance at all frequencies is equal. A logical use for the spectrum is to calculate it for the residuals from a time series model and see if the spectrum is reasonably flat.

Now consider the AR(1) process. The shape of the spectrum depends on the value of the AR(1) parameter ϕ. When $\phi > 0$, which results in a positively autocorrelated time series, the spectrum is dominated by low-frequency components. These low-frequency or long-period components result in a relative smooth time series. When $\phi < 0$, the time series is negatively autocorrelated, and the time series has a more ragged or volatile appearance. This produces a spectrum dominated by high-frequency or short-period components. Examples are shown in Figure 7.14.

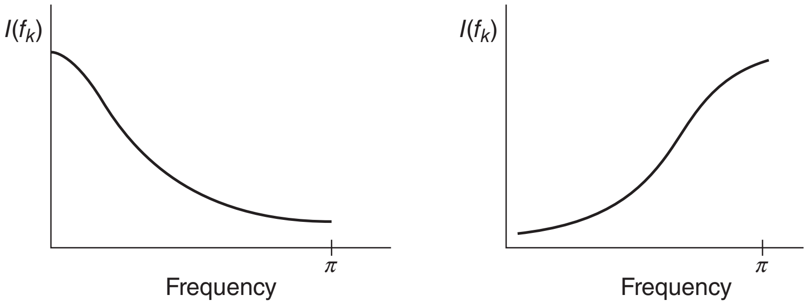


FIGURE 7.14 The spectrum of AR(1) processes.

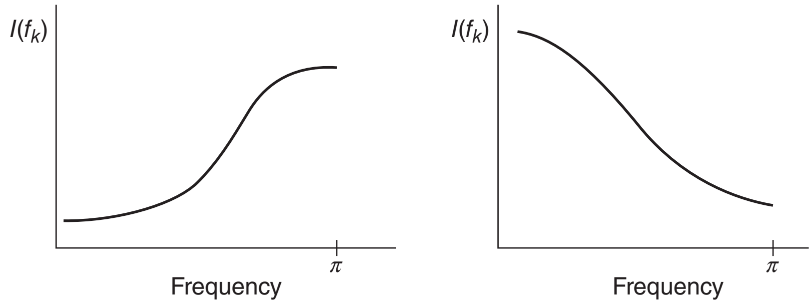


FIGURE 7.15 The spectrum of MA(1) processes.

The spectrum of the MA(1) process is shown in Figure 7.15. When the MA(1) parameter is positive, that is, when $\theta > 0$, the time series is negatively autocorrelated and has a more volatile appearance. Thus the spectrum is dominated by higher frequencies. When the MA(1) parameter is negative ($\theta > 0$), the time series is negatively autocorrelated and has a smoother appearance. This results in a spectrum that is dominated by low frequencies.

The spectrum of seasonal processes will exhibit peaks at the harmonically related seasonal frequencies. For example, consider the simple seasonal model with period 12, as might be used to represent monthly data with an annual seasonal cycle:

$$(1 - \phi^* B^{12}) y_t = \varepsilon_t$$

If ϕ^* is positive, the spectrum will exhibit peaks at frequencies 0 and $2\pi kt/12$, $k = 1, 2, 3, 4, 5, 6$. Figure 7.16 shows the spectrum.

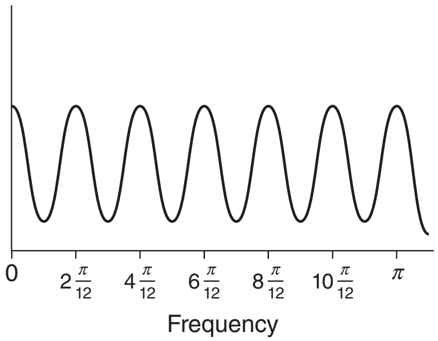


FIGURE 7.16 The spectrum of the seasonal $(1 - \phi^* B^{12}) y_t = \varepsilon_t$ process.

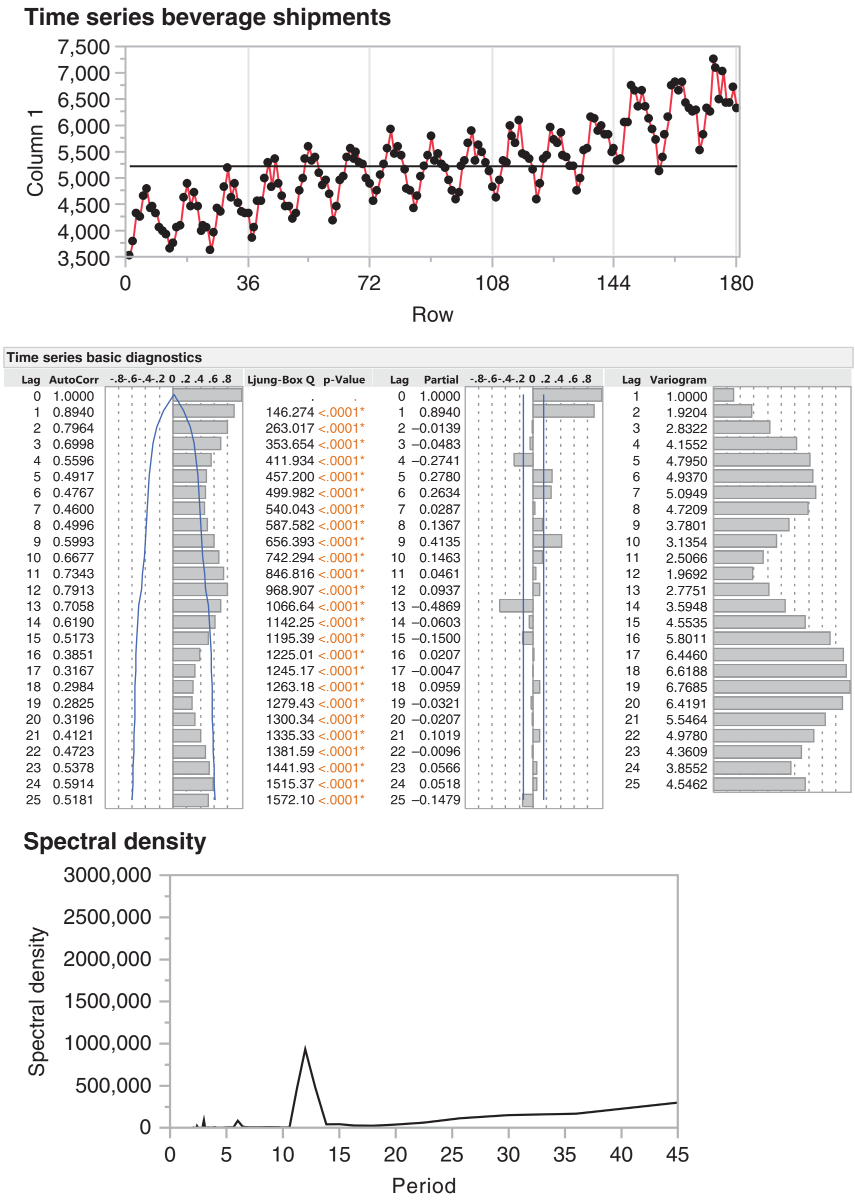
Fisher's Kappa statistic tests the null hypothesis that the values in the series are drawn from a normal distribution with variance 1 against the alternative hypothesis that the series has some periodic component. The test statistic kappa (κ) is the ratio of the maximum value of the periodogram, $I(f_k)$, to its average value. The probability of observing a larger Kappa if the null hypothesis is true is given by

$$P(\kappa > k) = 1 - \sum_{j=0}^{q} (-1)^j \binom{q}{j} \left[\max\left(1 - \frac{jk}{q}, 0\right) \right]^{q-1}$$

where k is the observed value of the kappa statistic, $q = T/2$ if T is even and $q = (T - 1)/2$ if T is odd. The null hypothesis is rejected if the computed probability is less that the desired significance level. There is also a Kolmogorov–Smirnov test due to Bartlett that compares the normalized cumulative periodogram to the cumulative distribution function of the uniform distribution on the interval (0, 1). The test statistic equals the maximum absolute difference of the cumulative periodogram and the uniform CDF. If this quantity exceeds $a/\sqrt{q}$, then we should reject the hypothesis that the series comes from a normal distribution. The values $a = 1.36$ and $a = 1.63$ correspond to significance levels of 0.05 and 0.01, respectively.

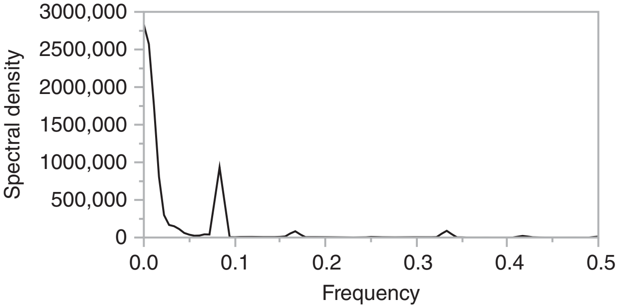
In general, we have found it difficult to determine the exact form of an ARIMA model purely from examination of the spectrum. The autocorrelation and partial autocorrelation functions are almost always more useful and easier to interpret. However, the spectrum is a complimentary tool and should always be considered as a useful supplement to the ACF and PACF. For further reading on spectral analysis and its many applications, see Jenkins and Watts (1969), Percival and Walden (1992), and Priestley (1991).

Example 7.9 JMP can be used to compute and display the spectrum for time series. We will illustrate the JMP output using the monthly US beverage product shipments. These data are shown originally in Figure 1.5 and are in Appendix Table B. These data were also analyzed in Chapter 2 to illustrate decomposition techniques. Figure 7.17 presents the JMP output, including a time series plot, the sample ACF, PACF, and variogram, and the spectral density function. Notice that there is a prominent peak in the spectral density at frequency 0.0833 that corresponds to a seasonal period of 12 months. The JMP output also provides the Fisher kappa statistic and the P-value indicates that there is at least one periodic component. The Bartlett Kolmogorov–Smirnov test statistic is also significant at the 0.01 level indicating that the data do not come from a normal distribution.



Lag	AutoCorr	Ljung-Box Q	p-Value	Lag	Partial	Lag	Variogram
0	1.0000	.	.	0	1.0000	1	1.0000
1	0.8940	146.274	<.0001*	1	0.8940	2	1.9204
2	0.7964	263.017	<.0001*	2	−0.0139	3	2.8322
3	0.6998	353.654	<.0001*	3	−0.0483	4	4.1552
4	0.5596	411.934	<.0001*	4	−0.2741	5	4.7950
5	0.4917	457.200	<.0001*	5	0.2780	6	4.9370
6	0.4767	499.982	<.0001*	6	0.2634	7	5.0949
7	0.4600	540.043	<.0001*	7	0.0287	8	4.7209
8	0.4996	587.582	<.0001*	8	0.1367	9	3.7801
9	0.5993	656.393	<.0001*	9	0.4135	10	3.1354
10	0.6677	742.294	<.0001*	10	0.1463	11	2.5066
11	0.7343	846.816	<.0001*	11	0.0461	12	1.9692
12	0.7913	968.907	<.0001*	12	0.0937	13	2.7751
13	0.7058	1066.64	<.0001*	13	−0.4869	14	3.5948
14	0.6190	1142.25	<.0001*	14	−0.0603	15	4.5535
15	0.5173	1195.39	<.0001*	15	−0.1500	16	5.8011
16	0.3851	1225.01	<.0001*	16	0.0207	17	6.4460
17	0.3167	1245.17	<.0001*	17	−0.0047	18	6.6188
18	0.2984	1263.18	<.0001*	18	0.0959	19	6.7685
19	0.2825	1279.43	<.0001*	19	−0.0321	20	6.4191
20	0.3196	1300.34	<.0001*	20	−0.0207	21	5.5464
21	0.4121	1335.33	<.0001*	21	0.1019	22	4.9780
22	0.4723	1381.59	<.0001*	22	−0.0096	23	4.3609
23	0.5378	1441.93	<.0001*	23	0.0566	24	3.8552
24	0.5914	1515.37	<.0001*	24	0.0518	25	4.5462
25	0.5181	1572.10	<.0001*	25	−0.1479		

FIGURE 7.17 JMP output showing the spectrum, ACF, PACF, and variogram for the beverage shipment data.



White noise test

Fisher's Kappa	28.784553
Prob > Kappa	1.041e-13
Bartlett's Kolmogorov-Smirnov	0.7545515

FIGURE 7.17 (*Continued*)

7.9 BAYESIAN METHODS IN FORECASTING

In many forecasting problems there is little or no historical information available at the time initial forecasts are required. Consequently, the initial forecasts must be based on subjective considerations. As information becomes available, this subjective information can be modified in light of actual data. An example of this is forecasting demand for seasonal clothing, which, because of style obsolescence, has a relatively short life. In this industry a common practice is to, at the start of the season, make a forecast of total sales for a product during the season and then as the season progresses the original forecast can be modified taking into account actual sales.

Bayesian methods can be useful in problems of this general type. The original subjective estimates of the forecast are translated into subjective estimates of the forecasting model parameters. Then Bayesian methods are used to update these parameter estimates when information in the form of time series data becomes available. This section gives a brief overview of the Bayesian approach to parameter estimation and demonstrates the methodology with a simple time series model.

The method of parameter estimation makes use of the Bayes' theorem. Let y be a random variable with probability density function that is characterized by an unknown parameter θ. We write this density as $f(y|\theta)$ to show that the distribution depends on θ. Assume that θ is a random variable with probability distribution $h_0(\theta)$ which is called the **prior distribution** for θ.

The prior distribution summarizes the subjective information that we have about θ, and the treatment of θ as a random variable is the major difference between Bayesian and classical methods of estimation. If we are relatively confident about the value of θ we should choose prior distribution with a small variance and if we are relatively uncertain about the value of θ we should choose prior distribution with a large variance.

In a time series or forecasting scenario, the random variable y is a sequence of observations, say $y_1, y_2, \dots, y_T$. The new estimate of the parameter θ will be in the form of a revised distribution, $h_1(\theta|y)$, called the **posterior distribution** for θ. Using Bayes' theorem the posterior distribution is

$$h_1(\theta|y) = \frac{h_0(\theta)f(y|\theta)}{\int_\theta h_0(\theta)f(y|\theta)d\theta} = \frac{h_0(\theta)f(y|\theta)}{g(y)} \tag{7.34}$$

where $f(y|\theta)$ is usually called the likelihood of y, given the value of θ, and $g(y)$ is the unconditional distribution of y averaged over all θ. If the parameter θ is a discrete random variable then the integral in Eq. (7.34) should be replaced by a summation sign. This equation basically blends the observed information with the prior information to obtain a revised description of the uncertainty in the value of θ in the form of a posterior probability distribution. The **Bayes' estimator of** θ, which we will denote by θ^*, is defined as the expected value of the posterior distribution:

$$\theta^* = \int_\theta \theta h_1(\theta|y)d\theta \tag{7.35}$$

Typically we would use θ^* as the estimate of θ in the forecasting model. In some cases, it turns out that θ^* is optimal in the sense of minimizing the variance of forecast error.

We will illustrate the procedure with a relatively simple example. Suppose that y is normally distributed with mean μ and variance σ_y^2; that is,

$$f(y|\mu) = N\left(\mu, \sigma_y^2\right) = \left(2\pi\sigma_y^2\right)^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma_y}\right)^2\right]$$

We will assume that σ_y^2 is known. The prior distribution for μ is assumed to be normal with mean μ' and variance $\sigma_\mu^{2'}$:

$$h_0(\mu|y) = N\left(\mu', \sigma_\mu^{2'}\right) = \left(2\pi\sigma_\mu^{2'}\right)^{-1/2} \exp\left[-\frac{1}{2}\left(\frac{\mu-\mu'}{\sigma'_\mu}\right)^2\right]$$

The posterior distribution of μ given the observation y is

$$
\begin{aligned}
h_1(\theta|y) &= \frac{2\pi\left(\sigma_\mu^{2'}\sigma_y^2\right)^{-1/2}\exp\left[\frac{1}{2}(\mu-\mu')/\sigma_\mu^{2'}+(y-\mu)\,/\sigma_y^2\right]}{\int_{-\infty}^{\infty}2\pi\left(\sigma_\mu^{2'}\sigma_y^2\right)^{-1/2}\exp\left[\frac{1}{2}(\mu-\mu')/\sigma_\mu^{2'}+(y-\mu)/\sigma_y^2\right]d\mu} \\
&= \left(2\pi\frac{\sigma_\mu^{2'}\sigma_y^2}{\sigma_\mu^{2'}+\sigma_y^2}\right)^{-1/2}\exp\left[-\frac{1}{2}\frac{[(\mu-\left(y\sigma_\mu^{2'}+\mu'\sigma_y^2\right)/\left(\sigma_\mu^{2'}+\sigma_y^2\right)}{\sigma_\mu^{2'}\sigma_y^2/\left(\sigma_\mu^{2'}+\sigma_y^2\right)}\right]
\end{aligned}
$$

which is a normal distribution with mean and variance

$$
\mu'' = E(\mu|y) = \frac{y\sigma_\mu^{2'}+\mu'\sigma_y^2}{\sigma_\mu^{2'}+\sigma_y^2}
$$

and

$$
\sigma_\mu^{2''} = V(\mu|y) = \frac{\sigma_\mu^{2'}\sigma_y^2}{\sigma_\mu^{2'}+\sigma_y^2}
$$

respectively. Refer to Winkler (2003), Raiffa and Schlaifer (1961), Berger (1985), and West and Harrison (1997) for more details of Bayesian statistical inference and decision making and additional examples.

Now let us consider a simple time series model, the constant process, defined in Eq. (4.1) as

$$
y_t = \mu + \varepsilon_t
$$

where μ is the unknown mean and the random component is ε_t, which we will assume to have a normal distribution with mean zero and known variance σ_y^2 Consequently, we are assuming that the observation in any period t has a normal distribution, say

$$
f(y_t|\mu) = N\left(\mu, \sigma_y^2\right)
$$

Since the variance σ_y^2 is known, the problem is to estimate μ.

Suppose that at the start of the forecasting process, time $t = 0$, we estimate the mean demand rate to be μ' and the variance $\sigma_\mu^{2'}$ captures the uncertainty in this estimate. So the prior distribution for μ is the normal prior

$$
h_0(\mu) = N\left(\mu', \sigma_\mu^{2'}\right)
$$

After one period, the observation y_1 is known. The estimate μ' and the variance $\sigma_{\mu}^{2'}$ can now be updated using the results obtained above for a normal sampling process and a normal prior:

$$h_1(\mu|y_1) = N[\mu''(1), \sigma^{2''}(1)]$$

where

$$\mu''(1) = E(\mu|y_1) = \frac{y\sigma_{\mu}^{2'} + \mu'\sigma_y^2}{\sigma_{\mu}^{2'} + \sigma_y^2}$$

and

$$\sigma_{\mu}^{2''}(1) = V(\mu|y_1) = \frac{\sigma_{\mu}^{2'}\sigma_y^2}{\sigma_{\mu}^{2'} + \sigma_y^2}$$

At the end of period 2, when the next observation y_2 becomes available, the Bayesian updating process transforms $h_1(\mu|y_1)$ into $h_2(\mu|y_{1,}y_2)$ in the following way:

$$h_2(\mu|y_1, y_2) = \frac{h_1(\mu|y_1)f(y_2|\mu)}{\int_{\mu} h_1(\mu|y_1)f(y_2|\mu)d\mu}$$

Here the old posterior h_1 is now used as a prior and combined with the likelihood of y_2 to obtain the new posterior distribution of μ at the end of period 2. Using our previous results, we now have

$$h_2(\mu|y_1, y_2) = N[\mu''(2), \sigma^{2''}(2)]$$

and

$$\mu''(2) = E(\mu|y_1, y_2) = \frac{\bar{y}\sigma_{\mu}^{2'} + \mu'\left(\sigma_y^2/2\right)}{\sigma_{\mu}^{2'} + \left(\sigma_y^2/2\right)}$$

$$\sigma_{\mu}^{2''}(2) = V(\mu|y_1, y_2) = \frac{\sigma_{\mu}^{2'}\sigma_y^2}{2\sigma_{\mu}^{2'} + \sigma_y^2}$$

where $\bar{y} = (y_1 + y_2)/2$. It is easy to show that $h_2(\mu|y_1, y_2) = h_2(\mu|\bar{y})$; that is, the same posterior distribution is obtained using the sample average $\bar{y}$ as from using y_1 and y_2 sequentially because the sample average is a sufficient statistic for estimating the mean μ.

In general, we can show that after observing y_T, we can calculate the posterior distribution as

$$h_2(\mu|y_1, y_2, \ldots, y_T) = N[\mu''(T), \sigma^{2''}(T)]$$

where

$$\mu''(T) = \frac{\bar{y}\sigma_\mu^{2'} + \mu'\left(\sigma_y^2/T\right)}{\sigma_\mu^{2'} + \left(\sigma_y^2/T\right)}$$

$$\sigma_\mu^{2''}(T) = \frac{\sigma_\mu^{2'}\sigma_y^2}{T\sigma_\mu^{2'} + \sigma_y^2}$$

where $\bar{y} = (y_1 + y_2 + \cdots + y_T)/T$. The Bayes estimator of μ after T periods is $\mu^*(T) = \mu''(T)$. We can write this as

$$\mu^*(T) = \frac{T}{r+T}\bar{y} + \frac{r}{r+T}\mu' \tag{7.36}$$

where $r = \sigma_y^2/\sigma_\mu^{2'}$. Consequently, the Bayes estimator of μ is just a weighted average of the sample mean and the initial subjective estimate μ'. The Bayes estimator can be written in a recursive form as

$$\mu^*(T) = \lambda(T)y_T + [1 - \lambda(T)]\mu^*(T-1) \tag{7.37}$$

where

$$\lambda(T) = \frac{1}{r+T} = \frac{\sigma_\mu^{2'}}{T\sigma_\mu^{2'} + \sigma_y^2}$$

Equation (7.36) shows that the estimate of the mean in period T is updated at each period by a form that is similar to first-order exponential smoothing. However, notice that the smoothing factor $\lambda(T)$ is a function of T, and it becomes smaller as T increases. Furthermore, since $\sigma_\mu^{2''}(T) = \lambda(T)\sigma_y^2$, the uncertainty in the estimate of the mean decreases to zero as time T becomes large. Also, the weight given to the prior estimate of the mean decreases as T becomes large. Eventually, as more data becomes available, a permanent forecasting procedure could be adopted, perhaps involving exponential smoothing. This estimator is optimal in the sense that it minimizes the variance of forecast error even if the process is not normally distributed.

We assumed that the variance of the demand process was known, or at least a reasonable estimate of it was available. Uncertainty in the value of this parameter could be handled by also treating it as a random variable. Then the prior distribution would be a joint distribution that would reflect the uncertainty in both the mean and the variance. The Bayesian updating process in this case is considerably more complicated than in the known-variance case. Details of the procedure and some useful advice on choosing a prior are in Raiffa and Schlaifer (1961).

Once the prior has been determined, the forecasting process is relatively straightforward. For a constant process, the forecasting equation is

$$\hat{y}_{T+\tau}(T) = \hat{\mu}(T) = \mu^*(T)$$

using the Bayes estimate as the current estimate of the mean. Our uncertainty in the estimate of the mean is just the posterior variance. So the variance of the forecast is

$$V[\hat{y}_{T+\tau}(T)] = \sigma_\mu^{2''}(T)$$

and the variance of forecast error is

$$V[y_{T+\tau} - \hat{y}_{T+\tau}(T)] = V[e_\tau(T+\tau)] = \sigma_y^2 + \sigma_\mu^{2''}(T) \tag{7.38}$$

The variance of forecast error is independent of the lead time in the Bayesian case for a constant process. If we assume that y and μ are normally distributed, then we can use Eq. (3.38) to find a 100(1-α)% prediction interval on the forecast $V[\hat{y}_{T+\tau}(T)]$ as follows:

$$\mu^*(T) \pm Z_{\alpha/2}\sqrt{\sigma_y^2 + \sigma_\mu^{2''}(T)} \tag{7.39}$$

where $Z_{\alpha/2}$ is the usual $\alpha/2$ percentage point of the standard normal distribution.

Example 7.10 Suppose that we are forecasting weekly demand for a new product. We think that demand is normally distributed, and that at least in the short run that a constant model is appropriate. There is no useful historical information, but a reasonable prior distribution for μ is $N(100, 25)$ and σ_y^2 is estimated to be 150. At time period $T = 0$ the forecast for period 1 is

$$\hat{y}_1(0) = 100$$

The variance of forecast error is $150 + 25 = 175$, so a 96% prediction interval for y_1 is

$$100 \pm 1.96\sqrt{175} \quad \text{or} \quad [74.1, 125.9]$$

Suppose the actual demand experienced in period 1 is $y_1 = 86$. We can use Eq. (7.37) to update the estimate of the mean. First, calculate $r = \sigma_y^2/\sigma_\mu^{2'} = 150/25 = 6$ and $\lambda(1) = 1/(6 + 1) = 0.143$, then

$$\begin{aligned}\mu^*(1) = \mu''(1) &= \lambda(1)y_1 + [1 - \lambda(1)]\mu^*(0)\\ &= 0.143(86) + (1 - 0.143)100\\ &= 98.0\end{aligned}$$

and

$$\begin{aligned}\sigma_\mu^{2''}(1) &= \lambda(1)\sigma_y^2\\ &= 0.143(150)\\ &= 21.4\end{aligned}$$

The forecast for period 2 is now

$$\hat{y}_2(1) = 98.0$$

The corresponding 95% prediction interval for y_2 is

$$98.0 \pm 1.96\sqrt{150 + 21.4} \quad \text{or} \quad [72.3, 123.7]$$

In time period the actual demand experienced is 94. Now $\lambda(2) = 1/(6 + 2) = 0.125$, and

$$\begin{aligned}\mu^*(2) = \mu''(2) &= \lambda(2)y_2 + [1 - \lambda(2)]\mu^*(1)\\ &= 0.125(94) + (1 - 0.125)98.0\\ &= 97.5\end{aligned}$$

So the forecast for period 3 is

$$\hat{y}_3(3) = 97.5$$

and the updated variance estimate is

$$\begin{aligned}\sigma_\mu^{2''}(2) &= \lambda(2)\sigma_y^2\\ &= 0.125(150)\\ &= 18.8\end{aligned}$$

Therefore the 96% prediction interval for y_3 is

$$97.5 \pm 1.96\sqrt{150 + 18.8} \quad \text{or} \quad [72.0, 123.0]$$

This procedure would be continued until it seems appropriate to change to a more permanent forecasting procedure. For example, a change to first-order exponential smoothing could be made when $\lambda(T)$ drop to a target level, say $0.05 < \lambda(T) < 0.1$. Then after sufficient data has been observed, an appropriate time series model could be fit to the data.

7.10 SOME COMMENTS ON PRACTICAL IMPLEMENTATION AND USE OF STATISTICAL FORECASTING PROCEDURES

Over the last 35 years there has been considerable information accumulated about forecasting techniques and how these methods are applied in a wide variety of settings. Despite the development of excellent analytical techniques, many business organizations still rely on judgment forecasts by their marketing, sales, and managerial/executive teams. The empirical evidence regarding judgment forecasts is that they are not as successful as statistically based forecasts. There are some fields, such as financial investments, where there is considerable strong evidence that this is so. There are a number of reasons why we would expect judgment forecasts to be inferior to statistical methods.

Inconsistency, or changing one's mind for no compelling or obvious reason, is a significant source of judgment forecast errors. Formalizing the forecasting process through the use of analytical methods is one approach to eliminating inconsistency as a source of error. Formal decision rules that predict the variables of interest using relatively few inputs invariably predict better than humans, because humans are inconsistent over time in their choice of input factors to consider, and how to weight them.

Letting more **recent events** dominate one's thinking, instead of weighting current and previous experience more evenly, is another source of judgment forecast errors. If these recent events are essentially random in nature, they can have undue impact on current forecasts. A good forecasting system will certainly monitor and evaluate recent events and experiences, but will only incorporate them into the forecasts if there is sufficient evidence to indicate that they represent real effects.

Mistaking **correlation** for **causality** can also be a problem. This is the belief that two (or more) variables are related in a causal manner and taking action on this, when the variables exhibit only a correlation between them. It is not difficult to find correlative relationships; any two variables that are monotonically related will exhibit strong correlation. So company sales may appear to be related to some factor that over a short time period is moving synchronously with sales, but relying on this as a causal relationship will lead to problems. The statistical significance of patterns and relationships does not necessarily imply a cause-and-effect relationship.

Judgment forecasts are often dominated by **optimistic thinking**. Most humans are naturally optimistic. An executive wants sales for the product line to increase because his/her bonus may depend on the results. A product manager wants his/her product to be successful. Sometimes bonus payouts are made for exceeding sales goals, and this can lead to unrealistically low forecasts, which in turn are used to set the goals. However, unrealistic forecasts, whether too high or too low, always result in problems downstream in the organization where forecast errors have meaningful impact on efficiency, effectiveness, and bottom-line results.

Humans are notorious for **underestimating variability**. Judgment forecasts rarely incorporate uncertainty in any formal way and, as a result, often underestimate its magnitude and impact. A judgment forecaster often completely fails to express any uncertainty in his/her forecast. Because all forecasts are wrong, one must have some understanding of the magnitude of forecast errors. Furthermore, planning for appropriate actions in the face of likely forecast error should be part of the decision-making process that is driven by the forecast. Statistical forecasting methods can be accompanied by prediction intervals. In our view, every forecast should be accompanied by a PI that adequately expresses for the decision maker how much uncertainty is associated with the point forecast.

In general, both the users of forecasts (decision makers) and the preparers (forecasters) have reasonably good awareness of many of the basic analytical forecasting techniques, such as exponential smoothing and regression-based methods. They are less familiar with time series models such as the ARIMA model, transfer function models, and other more sophisticated methods. Decision makers are often unsatisfied with subjective and judgment methods and want better forecasts. They often feel that analytical methods can be helpful in this regard.

This leads to a discussion of **expectations**. What kind of results can one reasonably expect to obtain from analytical forecasting methods? By results, we mean forecast errors. Obviously, the results that a specific forecaster obtains are going to depend on the specific situation: what variables are being forecast, the availability and quality of data, the methods that can be applied to the problem, and the tools and expertise that are available. However, because there have been many surveys of both forecasters and users of forecasts, as well as forecast competitions (e.g., see Makridakis et al. (1993)) where many different techniques have been applied in head-to-head challenges, some broad conclusions can be drawn.

In general, exponential smoothing type methods, including Winters' method, typically experience mean absolute prediction errors ranging from 10% to 15% for lead-one forecasts. As the lead time increases, the prediction error increases, with mean absolute prediction errors typically in the 17–25% range at lead times of six periods. At 12 period lead times, the mean absolute prediction error can range from 18% to 45%. More sophisticated time series models such as ARIMA models are not usually much better, with the mean absolute prediction error ranging from about 10% for lead-one forecasts, to about 17% for lead-six forecasts, and up to 25% for 12 period lead times. This probably accounts for some of the dissatisfaction that forecasters often express with the more sophisticated techniques; they can be much harder to use, but they do not have substantial payback in terms of reducing forecasting errors. Regression methods often produce mean absolute prediction errors ranging from 12% to 18% for lead-one forecasts. As the lead time increases, the prediction error increases, with mean absolute prediction errors typically in the 17–20% range for six period lead times. At 12 period lead times, the mean absolute prediction error can range from 20% to 25%. Seasonal time series are often easier to predict than nonseasonal ones, because seasonal patterns are relatively stable through time, and relatively simple methods such as Winters' method and seasonal adjustment procedures typically work very well as forecasting techniques. Interestingly, seasonal adjustment techniques are not used nearly as widely as we would expect, given their relatively good performance.

When forecasting is done well in an organization, it is typically done by a group of individuals who have some training and experience in the techniques, have access to the right information, and have an opportunity to see how the forecasts are used. If higher levels of management routinely intervene in the process and use their judgment to modify the forecasts, it is highly desirable if the forecast preparers can interact with these managers to learn why the original forecasts require modification. Unfortunately, in many organizations, forecasting is done in an informal way, and the forecasters are often marketing or sales personnel, or market researchers, for whom forecasting is only a (sometimes small) part of their responsibilities. There is often a great deal of turnover in these positions, and so no long-term experience base or continuity builds up. The lack of a formal, organized process is often a big part of the reason why forecasting is not as successful as it should be.

Any evaluation of a forecasting effort in an organization should consider at least the following questions:

1. What methods are being used? Are the methods appropriate to organizational needs, when planning horizons and other business issues are taken into account? Is there an opportunity to use more than

one forecasting procedure? Could forecasts be combined to improve results?

2. Are the forecasting methods being used correctly?
3. Is an appropriate set of data being used in preparing the forecasts? Is data quality an issue? Are the underlying assumptions of the methods employed satisfied at least well enough for the methods to be successful?
4. Is uncertainty being addressed adequately? Are prediction intervals used as part of the forecast report? Do forecast users understand the PIs?
5. Does the forecasting system take economic/market forces into account? Is there an ability to capitalize on current events, natural forces, and swings in customer preferences and tastes?
6. Is forecasting separate from planning? Very often the forecast is really just a plan or schedule. For example, it may reflect a production plan, not a forecast of what we could realistically expect to sell (i.e., demand). Many individuals do not understand the difference between a forecast and a plan.

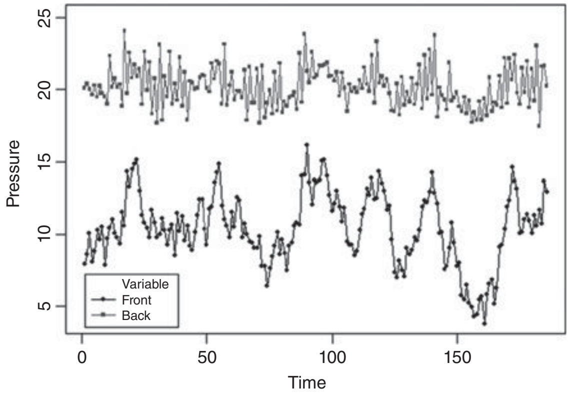
In the short-to-medium term, most businesses can benefit by taking advantage of the relative stability of seasonal patterns and the inertia present in most time series of interest. These are the methods we have focused on in this book.

7.11 R COMMANDS FOR CHAPTER 7

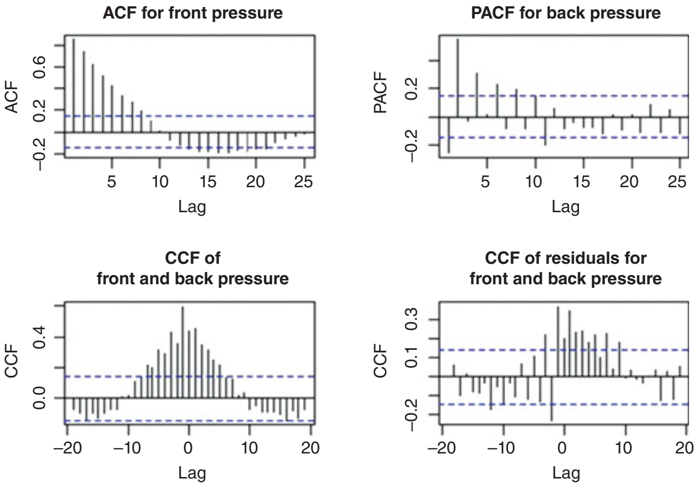
Example 7.11 The data for this example are in the array called pressure .data of which the two columns represent the viscosity and the temperature, respectively. To model the multivariate data we use the “VAR” function in R package “vars.” But we first start with time series, acf, pacf, ccf plots as suggested in the example

```
library(vars)

pf<-pressure.data[,1]
pb<-pressure.data[,2]
plot(pf,type="o",pch=16,cex=.5,xlab='Time', ylab='Pressure',ylim=
  c(4,25))
lines(pb, type="o",pch=15,cex=.5, col="grey40")
legend(1,7,c("Variable","Front","Back"),
```

```
pch=c(NA,16,15),lwd=c(NA,.5,.5),cex=.55,col=c("black","black",
  "grey40"))
```



```
res.pf<- as.vector(residuals(arima(pf,order=c(1,0,0))))
res.pb<- as.vector(residuals(arima(pb,order=c(1,0,0))))
par(mfrow=c(2,2),oma=c(0,0,0,0))
acf(pf,lag.max=25,type="correlation",main="ACF for Front
  Pressure")
acf(pb, lag.max=25, type="correlation",main="PACF for Back
  Pressure",ylab="PACF")
ccf(pb,pf,main='CCF of \nFront and Back Pressures',ylab='CCF')
ccf(res.pb,res.pf,main='CCF of Residuals for \nFront and Back
  Pressures',ylab='CCF')
```



We now fit a VAR(1) model to the data using VAR function:

```
> pressure.var1<-VAR(pressure.data)
> pressure.var1

      VAR Estimation Results:
      =======================

      Estimated coefficients for equation pfront:
      ===========================================
      Call:
      pfront = pfront.l1 + pback.l1 + const

       pfront.l1   pback.l1      const
       0.7329529  0.4735983 -6.7555089

      Estimated coefficients for equation pback:
      ==========================================
      Call:
      pback = pfront.l1 + pback.l1 + const

       pfront.l1   pback.l1      const
       0.4104251 -0.5606308 27.2369791
```

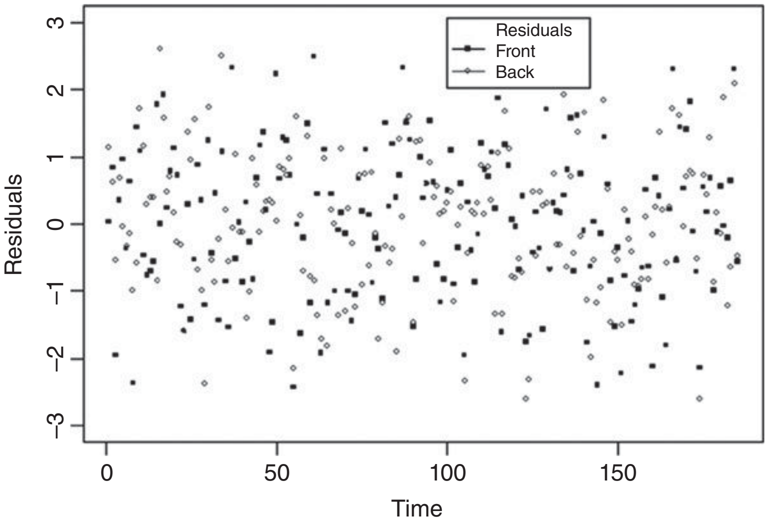
Note that there is also a VARselect function that will automatically select the best p order of the VAR(p) model. In this case we tried p upto 5.

```
> VARselect(pressure.data,lag.max=5)

       $selection
       AIC(n)  HQ(n)  SC(n) FPE(n)
            1      1      1      1

       $criteria
                        1          2         3         4         5
       AIC(n) 0.05227715 0.09275642 0.1241682 0.1639090 0.1882266
       HQ(n)  0.09526298 0.16439946 0.2244684 0.2928665 0.3458413
       SC(n)  0.15830468 0.26946896 0.3715657 0.4819916 0.5769942
       FPE(n) 1.05367413 1.09722530 1.1322937 1.1783005 1.2074691
```

The output shows that VAR(1) was indeed the right choice. We now plot the residuals.

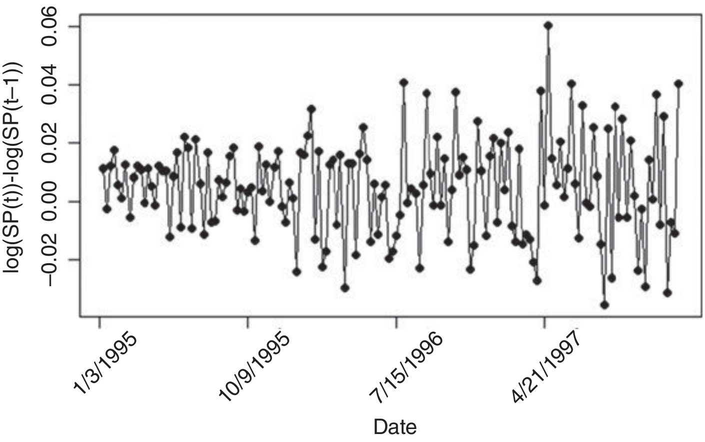
```
plot(residuals(pressure.var1)[,1],type="p",pch=15,cex=.5,xlab=
  'Time', ylab='Residuals',ylim=c(-3,3))
points(residuals(pressure.var1)[,2],pch=1,cex=.5, col="grey40")
```

```
legend(100,3,c("Residuals","Front","Back"),
pch=c(NA,15,1),lwd=c(NA,.5,.5),cex=.55,col=c("black","black",
  "grey40"))
```

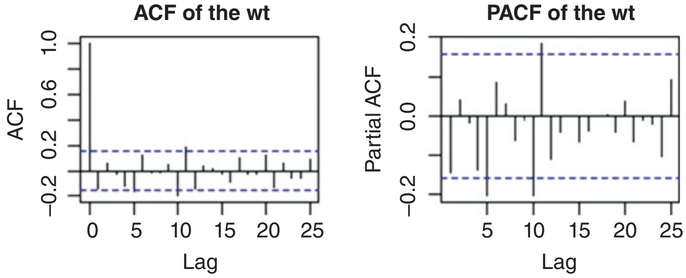


Example 7.12 The data for this example are in the array called sp500.data of which the two columns represent the date and the S&P 500 closing values respectively. To model the multivariate data we use the "garch" function in R package "tseries." But we first start with time series, acf, pacf plots as suggested in the example

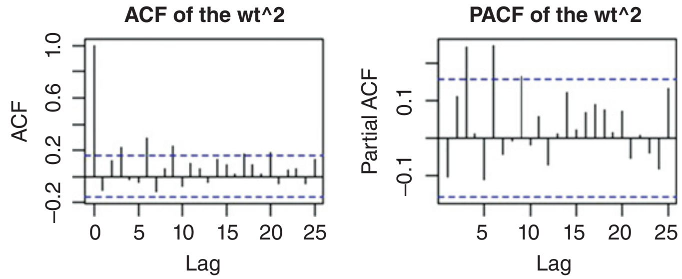
```
library(tseries)

sp<-ts(sp500.data[,2])
logsp.d1<-diff(log(sp))
T<-length(logsp.d1)
plot(logsp.d1,type="o",pch=16,cex=.5,xlab='Date', ylab='log(SP(t))
  -log(SP(t-1))',xaxt='n')
lablist<-as.vector(sp500.data[seq(1,T+1,40),1])
axis(1, seq(1,T+1,40), labels=FALSE)
text(seq(1,T+1,40),par("usr")[3]-.01, ,labels = lablist, srt = 45,
  pos = 1, xpd = TRUE)
```



```
# ACF and PACF of the first difference of the log transformation
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(logsp.d1,lag.max=25,type="correlation",main="ACF of the wt")
acf(logsp.d1, lag.max=25,type="partial",main="PACF of the wt")
```



```
# ACF and PACF of the square of the first difference of the log
# transformation
par(mfrow=c(1,2),oma=c(0,0,0,0))
acf(logsp.d1^2,lag.max=25,type="correlation",main="ACF of the wt^2")
acf(logsp.d1^2, lag.max=25,type="partial",main="PACF of the wt^2")
```



```
# Fit a GARCH(0,3) model
sp.arch3<-garch(logsp.d1, order = c(0,3), trace = FALSE)
summary(sp.arch3)
       Call:
       garch(x = logsp.d1, order = c(0, 3), trace = FALSE)

       Model:
       GARCH(0,3)

       Residuals:
           Min       1Q  Median       3Q      Max
       -2.0351 -0.4486  0.3501  0.8869  2.9320

       Coefficient(s):
           Estimate  Std. Error  t value Pr(>|t|)
       a0 2.376e-04   6.292e-05    3.776 0.000159 ***
       a1 2.869e-15   1.124e-01    0.000 1.000000
       a2 7.756e-02   8.042e-02    0.964 0.334876
       a3 9.941e-02   1.124e-01    0.884 0.376587
       —
       Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 '' 1

       Diagnostic Tests:
               Jarque Bera Test

       data:  Residuals
       X-squared = 0.4316, df = 2, p-value = 0.8059

               Box-Ljung test

       data:  Squared.Residuals
       X-squared = 2.2235, df = 1, p-value = 0.1359
```

EXERCISES

7.1 Show that an AR(2) model can be represented in state space form.

7.2 Show that an MA(1) model can be written in state space form.

7.3 Consider the information on weekly spare part demand shown in Table E7.1. Suppose that 74 requests for 55 parts are received during the current week, T. Find the new cumulative distribution of demand. Use $\lambda = 0.1$. What is your forecast of the 70th percentile of the demand distribution?

7.4 Consider the information on weekly luxury car rentals shown in Table E7.2. Suppose that 37 requests for rentals are received during

TABLE E7.1 Spare Part Demand Information for Exercise 7.3

k	B_{k-1}	B_k	$\hat{p}_k(T-1)$	$F(B_k)$, at the end of week $T-1$
0	0	5	0.02	0.02
1	5	10	0.03	0.05
2	10	15	0.04	0.09
3	15	20	0.05	0.14
4	20	25	0.08	0.22
5	25	30	0.09	0.31
6	30	35	0.12	0.43
7	35	40	0.17	0.60
8	45	50	0.21	0.81
9	50	55	0.11	0.92
10	55	60	0.08	1.00

the current week, T. Find the new cumulative distribution of demand. Use $\lambda = 0.1$. What is your forecast of the 90th percentile of the demand distribution?

7.5 Rework Exercise 7.3 using $\lambda = 0.4$. How much difference does changing the value of the smoothing parameter make in your estimate of the 70th percentile of the demand distribution?

7.6 Rework Exercise 7.4 using $\lambda = 0.4$. How much difference does changing the value of the smoothing parameter make in your estimate of the 70th percentile of the demand distribution?

TABLE E7.2 Luxury Car Rental Demand Information for Exercise 7.4

k	B_{k-1}	B_k	$\hat{p}_k(T-1)$	$F(B_k)$, at the end of week $T-1$
0	0	5	0.06	0.06
1	5	10	0.07	0.13
2	10	15	0.08	0.21
3	15	20	0.09	0.30
4	20	25	0.15	0.45
5	25	30	0.22	0.67
6	30	35	0.24	0.91
7	35	40	0.05	0.96
8	45	50	0.04	1.00

7.7 Suppose that two forecasting methods can be used for a time series, and that the two variances of the forecast errors are $\sigma_1^2 = 10$ and $\sigma_2^2 = 25$. If the correlation coefficient $\rho = -0.75$, calculate the optimum value of the weight used to optimally combine the two individual forecasts. What is the variance of the combined forecast?

7.8 Suppose that two forecasting methods can be used for a time series, and that the two variances of the forecast errors are $\sigma_1^2 = 15$ and $\sigma_2^2 = 20$. If the correlation coefficient $\rho = -0.4$, calculate the optimum value of the weight used to optimally combine the two individual forecasts. What is the variance of the combined forecast?

7.9 Suppose that two forecasting methods can be used for a time series, and that the two variances of the forecast errors are $\sigma_1^2 = 8$ and $\sigma_2^2 = 16$. If the correlation coefficient $\rho = -0.3$, calculate the optimum value of the weight used to optimally combine the two individual forecasts. What is the variance of the combined forecast?

7.10 Suppose that two forecasting methods can be used for a time series, and that the two variances of the forecast errors are $\sigma_1^2 = 1$ and $\sigma_2^2 = 8$. If the correlation coefficient $\rho = -0.65$, calculate the optimum value of the weight used to optimally combine the two individual forecasts. What is the variance of the combined forecast?

7.11 Rework Exercise 7.8 assuming that $\rho = 0.4$. What effect does changing the sign of the correlation coefficient have on the weight used to optimally combine the two forecasts? What is the variance of the combined forecast?

7.12 Rework Exercise 7.9 assuming that $\rho = 0.3$. What effect does changing the sign of the correlation coefficient have on the weight used to optimally combine the two forecasts? What is the variance of the combined forecast?

7.13 Suppose that there are three lead-one forecasts available for a time series, and the covariance matrix of the three forecasts is as follows:

$$\boldsymbol{\Sigma}_{T+1}(T) = \begin{bmatrix} 10 & -4 & -2 \\ -4 & 6 & -3 \\ -2 & -3 & 15 \end{bmatrix}$$

Find the optimum weights for combining these three forecasts. What is the variance of the combined forecast?

7.14 Suppose that there are three lead-one forecasts available for a time series, and the covariance matrix of the three forecasts is as follows:

$$\mathbf{\Sigma}_{T+1}(T) = \begin{bmatrix} 8 & -2 & -1 \\ -2 & 3 & -2 \\ -1 & -2 & 10 \end{bmatrix}$$

Find the optimum weights for combining these three forecasts. What is the variance of the combined forecast?

7.15 Table E7.3 presents 25 forecast errors for two different forecasting techniques applied to the same time series. Is it possible to combine the two forecasts to improve the forecast errors? What is the optimum weight for combining the forecasts? What is the variance of the combined forecast?

TABLE E7.3 Forecast Errors for Exercise 7.15

Time period	Forecast errors, method 1	Forecast errors, method 2
1	−0.78434	6.9668
2	−0.31111	4.5512
3	2.15622	−1.2681
4	−1.81293	6.8967
5	−0.77498	1.6574
6	2.31673	−8.7601
7	−0.94866	0.7472
8	0.81314	−0.7457
9	−2.95718	−0.5355
10	0.08175	−1.3458
11	1.08915	−5.8232
12	−0.20637	1.2722
13	0.57157	−2.4561
14	0.41435	4.3111
15	0.47138	5.9894
16	1.23274	−6.8757
17	−0.66288	1.5996
18	1.71193	10.5031
19	−2.00317	9.8664
20	−2.87901	3.0399
21	−2.87901	14.1992
22	−0.16103	9.0080
23	2.12427	−0.4551
24	0.60598	0.7123
25	0.18259	1.7346

7.16 Show that when combining two forecasts, if the correlation between the two sets of forecast errors is $\rho = \sigma_1/\sigma_2$, then Min Var $[e^{\mathrm{c}}_{T+\tau}(T)] = \sigma_1^2$, where σ_1^2 is the smaller of the two forecast error variances.

7.17 Show that when combining two forecasts, if the correlation between the two sets of forecast errors is $\rho = 0$, then Var $[e^{\mathrm{c}}_{T+\tau}(T)] = \sigma_1^2\sigma_2^2/(\sigma_1^2 + \sigma_2^2)$.

7.18 Let y_t be an IMA(1, 1) time series with parameter $\theta = 0.4$. Suppose that this time series is observed with an additive white noise error.

a. What is the model form of the observed error?

b. Find the parameters of the observed time series, assuming that the variances of the errors in the original time series and the white noise are equal.

7.19 Show that an AR(1) time series that is observed with an additive white noise error is an ARMA(1, 1) process.

7.20 Generate 100 observations of an ARIMA(1, 1, 0) time series. Add 100 observations of white noise to this time series. Calculate the sample ACF and sample PACF of the new time series. Identify the model form and estimate the parameters.

7.21 Generate 100 observations of an ARIMA(1, 1, 0) time series. Generate another 100 observations of an AR(1) time series and add these observations to the original time series. Calculate the sample ACF and sample PACF of the new time series. Identify the model form and estimate the parameters.

7.22 Generate 100 observations of an AR(2) time series. Generate another 100 observations of an AR(1) time series and add these observations to the original time series. Calculate the sample ACF and sample PACF of the new time series. Identify the model form and estimate the parameters.

7.23 Generate 100 observations of an MA(2) time series. Generate another 100 observations of an MA(1) time series and add these observations to the original time series. Calculate the sample ACF and sample PACF of the new time series. Identify the model form and estimate the parameters.

7.24 Table E7.4 presents data on the type of heating fuel used in new single-family houses built in the United States from 1971 through

TABLE E7.4 Data for Exercise 7.24

	Number of Houses (in thousands)				
Year	Total	Gas	Electricity	Oil	Other types or none
1971	1014	605	313	83	15
1972	1143	621	416	93	13
1973	1197	560	497	125	16
1974	940	385	458	85	11
1975	875	347	429	82	18
1976	1034	407	499	110	19
1977	1258	476	635	120	28
1978	1369	511	710	109	40
1979	1301	512	662	86	41
1980	957	394	482	29	52
1981	819	339	407	16	57
1982	632	252	315	17	48
1983	924	400	448	22	53
1984	1025	460	492	24	49
1985	1072	466	528	36	42
1986	1120	527	497	52	45
1987	1123	583	445	58	38
1988	1085	587	402	60	36
1989	1026	596	352	50	28
1990	966	573	318	48	27
1991	838	505	267	37	29
1992	964	623	283	36	22
1993	1039	682	303	34	20
1994	1160	772	333	39	16
1995	1066	708	305	37	16
1996	1129	781	299	37	11
1997	1116	771	296	38	11
1998	1160	809	307	34	10
1999	1270	884	343	35	9
2000	1242	868	329	37	8
2001	1256	875	336	35	9
2002	1325	907	371	38	10
2003	1386	967	377	31	12
2004	1532	1052	440	29	10
2005	1636	1082	514	31	9

2005. Develop an appropriate multivariate time series model for the gas, electricity, and oil time series.

7.25 Reconsider the data on heating fuel in Table E7.4. Suppose that you are interested in forecasting the aggregate series (the Total column in Table E7.4). One way to do this is to forecast the total directly. Another way is to forecast the individual component series and sum the forecasts of the components to obtain a forecast for the total. Investigate these approaches for this data and report on your conclusions.

7.26 Reconsider the data on heating fuel in Table E7.4. Suppose that you are interested in forecasting the four individual components series (the Gas, Electricity, Oil, and Other Types columns in Table E7.4). One way to do this is to forecast the individual time series directly. Another way is to forecast the total and obtain forecasts of the individual component series by decomposing the forecast for the totals into component parts. Investigate these approaches for this data and report on your conclusions.

7.27 Table E7.5 contains data on property crimes reported to the police in the United States. Both the number of property crimes and the crime rate per 100,000 individuals are shown. Using the data on the number of crimes reported, develop an appropriate multivariate time series model for the burglary, larceny-theft, and motor vehicle theft time series.

7.28 Repeat Exercise 7.27 using the property crime rate data. Compare the models obtained using the number of crimes reported versus the crime rate.

7.29 Reconsider the data on property crimes in Table E7.5. Suppose that you are interested in forecasting the aggregate crime rate series. One way to do this is to forecast the total directly. Another way is to forecast the individual component series and sum the forecasts of the components to obtain a forecast for the total. Investigate these approaches for this data and report on your conclusions.

7.30 Reconsider the data on property crimes in Table E7.5. Suppose that you are interested in forecasting the four individual component series (the Burglary, Larceny-Theft, and Motor Vehicle Theft columns in Table E7.5). One way to do this is to forecast the individual time series directly. Another way is to forecast the total

TABLE E7.5 Property Crime Data for Exercise 7.27

	Property Crime (in thousands)			
Year	Total	Burglary	Larceny-theft	Motor vehicle theft
1960	3096	912	1855	328
1961	3199	950	1913	336
1962	3451	994	2090	367
1963	3793	1086	2298	408
1964	4200	1213	2514	473
1965	4352	1283	2573	497
1966	4793	1410	2822	561
1967	5404	1632	3112	660
1968	6125	1859	3483	784
1969	6749	1982	3889	879
1970	7359	2205	4226	928
1971	7772	2399	4424	948
1972	7414	2376	4151	887
1973	7842	2566	4348	929
1974	9279	3039	5263	977
1975	10,253	3265	5978	1010
1976	10,346	3109	6271	966
1977	9955	3072	5906	978
1978	10,123	3128	5991	1004
1979	11,042	3328	6601	1113
1980	12,064	3795	7137	1132
1981	12,062	3780	7194	1088
1982	11,652	3447	7143	1062
1983	10,851	3130	6713	1008
1984	10,608	2984	6592	1032
1985	11,103	3073	6926	1103
1986	11,723	3241	7257	1224
1987	12,025	3236	7500	1289
1988	12,357	3218	7706	1433
1989	12,605	3168	7872	1565
1990	12,655	3074	7946	1636
1991	12,961	3157	8142	1662
1992	12,506	2980	7915	1611
1993	12,219	2835	7821	1563
1994	12,132	2713	7880	1539
1995	12,064	2594	7998	1472
1996	11,805	2506	7905	1394
1997	11,558	2461	7744	1354
1998	10,952	2333	7376	1243
1999	10,208	2102	6956	1152
2000	10,183	2051	6972	1160
2001	10,437	2117	7092	1228
2002	10,451	2152	7053	1246

TABLE E7.5 (*Continued*)

	Crime Rate (per 100,000 population)			
Year	Total	Burglary	Larceny-theft	Motor vehicle theft
1960	1726.3	508.6	1034.7	183.0
1961	1747.9	518.9	1045.4	183.6
1962	1857.5	535.2	1124.8	197.4
1963	2012.1	576.4	1219.1	216.6
1964	2197.5	634.7	1315.5	247.4
1965	2248.8	662.7	1329.3	256.8
1966	2450.9	721.0	1442.9	286.9
1967	2736.5	826.6	1575.8	334.1
1968	3071.8	932.3	1746.6	393.0
1969	3351.3	984.1	1930.9	436.2
1970	3621.0	1084.9	2079.3	456.8
1971	3768.8	1163.5	2145.5	459.8
1972	3560.4	1140.8	1993.6	426.1
1973	3737.0	1222.5	2071.9	442.6
1974	4389.3	1437.7	2489.5	462.2
1975	4810.7	1532.1	2804.8	473.7
1976	4819.5	1448.2	2921.3	450.0
1977	4601.7	1419.8	2729.9	451.9
1978	4642.5	1434.6	2747.4	460.5
1979	5016.6	1511.9	2999.1	505.6
1980	5353.3	1684.1	3167.0	502.2
1981	5263.9	1647.2	3135.3	474.1
1982	5032.5	1488.0	3083.1	458.6
1983	4637.4	1338.7	2871.3	431.1
1984	4492.1	1265.5	2795.2	437.7
1985	4666.4	1291.7	2911.2	463.5
1986	4881.8	1349.8	3022.1	509.8
1987	4963.0	1335.7	3095.4	531.9
1988	5054.0	1316.2	3151.7	586.1
1989	5107.1	1283.6	3189.6	634.0
1990	5073.1	1232.2	3185.1	655.8
1991	5140.2	1252.1	3229.1	659.0
1992	4903.7	1168.4	303.6	631.6
1993	4740.0	1099.7	3033.9	606.3
1994	4660.2	1042.1	3026.9	591.3
1995	4590.5	987.0	3043.2	560.3
1996	4451.0	945.0	2980.3	525.7
1997	4316.3	918.8	2891.8	505.7
1998	4052.5	863.2	2729.5	459.9
1999	3743.6	770.4	2550.7	422.5
2000	3618.3	728.8	2477.3	412.2
2001	3658.1	741.8	2485.7	430.5
2002	3624.1	746.2	2445.8	432.1

and obtain forecasts of the individual component series by decomposing the forecast for the totals into component parts. Investigate these approaches using the crime rate data, and report on your conclusions.

7.31 Table B.1 contains data on market yield of US treasury securities at 10-year constant maturity. Compute the spectrum for this time series. What features of the time series are apparent from examination of the spectrum?

7.32 Table B.3 contains on the viscosity of a chemical product. Compute the spectrum for this time series. What features of the time series are apparent from examination of the spectrum?

7.33 Table B.6 contains the global mean surface air temperature anomaly and the global CO_2 concentration data. Compute the spectrum for both of these time series. What features of the two time series are apparent from examination of the spectrum?

7.34 Table B.11 contains sales data on sales of Champagne. Compute the spectrum for this time series. Is the seasonal nature of the time series apparent from examination of the spectrum?

7.35 Table B.21 contains data on the average monthly retail price of electricity in the residential sector for Arizona from 2001 through 2014. Take the first difference of this timer series and compute the spectrum. Is the seasonal nature of the time series apparent from examination of the spectrum?

7.36 Table B.22 contains data on Danish crude oil production. Compute the spectrum for this time series. What features of the time series are apparent from examination of the spectrum?

7.37 Table B.23 contains data on positive test results for influenza in the US. Compute the spectrum for this time series. Is the seasonal nature of the time series apparent from examination of the spectrum?

7.38 Table B.24 contains data on monthly mean daily solar radiation. Compute the spectrum for this time series. Is the seasonal nature of the time series apparent from examination of the spectrum?

7.39 Table B.27 contains data on airline on-time arrival performance. Compute the spectrum for this time series. Is there any evidence of seasonality in the time series that is apparent from examination of the spectrum?

7.40 Table B.28 contains data on US automobile manufacturing shipments. Compute the spectrum for this time series. Is there any evidence of seasonality in the time series that is apparent from examination of the spectrum?

7.41 Weekly demand for a spare part is assumed to follow a Poisson distribution:

$$f(y|\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0, 1, \ldots$$

The mean λ of the demand distribution is assumed to be a random variable with a gamma distribution

$$h(\lambda) = \frac{b^{a}}{(a-1)!}\lambda^{a-1}e^{-b\lambda}, \quad \lambda > 0$$

where a and b are parameters having subjectively determined values. In the week following the establishment of this prior distribution d parts were demanded. What is the posterior distribution of λ?

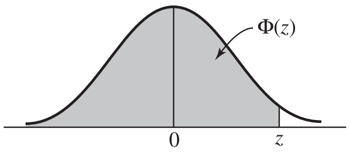
APPENDIX A

STATISTICAL TABLES

Introduction to Time Series Analysis and Forecasting, Second Edition.
Douglas C. Montgomery, Cheryl L. Jennings and Murat Kulahci.

TABLE A.1 Cumulative Standard Normal Distribution

$$\Phi(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$



z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00	z
−3.9	0.000033	0.000034	0.000036	0.000037	0.000039	0.000041	0.000042	0.000044	0.000046	0.000048	−3.9
−3.8	0.000050	0.000052	0.000054	0.000057	0.000059	0.000062	0.000064	0.000067	0.000069	0.000072	−3.8
−3.7	0.000075	0.000078	0.000082	0.000085	0.000088	0.000092	0.000096	0.000100	0.000104	0.000108	−3.7
−3.6	0.000112	0.000117	0.000121	0.000126	0.000131	0.000136	0.000142	0.000147	0.000153	0.000159	−3.6
−3.5	0.000165	0.000172	0.000179	0.000185	0.000193	0.000200	0.000208	0.000216	0.000224	0.000233	−3.5
−3.4	0.000242	0.000251	0.000260	0.000270	0.000280	0.000291	0.000302	0.000313	0.000325	0.000337	−3.4
−3.3	0.000350	0.000362	0.000376	0.000390	0.000404	0.000419	0.000434	0.000450	0.000467	0.000483	−3.3
−3.2	0.000501	0.000519	0.000538	0.000557	0.000577	0.000598	0.000619	0.000641	0.000664	0.000687	−3.2
−3.1	0.000711	0.000736	0.000762	0.000789	0.000816	0.000845	0.000874	0.000904	0.000935	0.000968	−3.1
−3.0	0.001001	0.001035	0.001070	0.001107	0.001144	0.001183	0.001223	0.001264	0.001306	0.001350	−3.0
−2.9	0.001395	0.001441	0.001489	0.001538	0.001589	0.001641	0.001695	0.001750	0.001807	0.001866	−2.9
−2.8	0.001926	0.001988	0.002052	0.002118	0.002186	0.002256	0.002327	0.002401	0.002477	0.002555	−2.8
−2.7	0.002635	0.002718	0.002803	0.002890	0.002980	0.003072	0.003167	0.003264	0.003364	0.003467	−2.7
−2.6	0.003573	0.003681	0.003793	0.003907	0.004025	0.004145	0.004269	0.004396	0.004527	0.004661	−2.6
−2.5	0.004799	0.004940	0.005085	0.005234	0.005386	0.005543	0.005703	0.005868	0.006037	0.006210	−2.5

−2.4	0.006387	0.006569	0.006756	0.006947	0.007143	0.007344	0.007549	0.007760	0.007976	0.008198	−2.4
−2.3	0.008424	0.008656	0.008894	0.009137	0.009387	0.009642	0.009903	0.010170	0.010444	0.010724	−2.3
−2.2	0.011011	0.011304	0.011604	0.011911	0.012224	0.012545	0.012874	0.013209	0.013553	0.013903	−2.2
−2.1	0.014262	0.014629	0.015003	0.015386	0.015778	0.016177	0.016586	0.017003	0.017429	0.017864	−2.1
−2.0	0.018309	0.018763	0.019226	0.019699	0.020182	0.020675	0.021178	0.021692	0.022216	0.022750	−2.0
−1.9	0.023295	0.023852	0.024419	0.024998	0.025588	0.026190	0.026803	0.027429	0.028067	0.028717	−1.9
−1.8	0.029379	0.030054	0.030742	0.031443	0.032157	0.032884	0.033625	0.034379	0.035148	0.035930	−1.8
−1.7	0.036727	0.037538	0.038364	0.039204	0.040059	0.040929	0.041815	0.042716	0.043633	0.044565	−1.7
−1.6	0.045514	0.046479	0.047460	0.048457	0.049471	0.050503	0.051551	0.052616	0.053699	0.054799	−1.6
−1.5	0.055917	0.057053	0.058208	0.059380	0.060571	0.061780	0.063008	0.064256	0.065522	0.066807	−1.5
−1.4	0.068112	0.069437	0.070781	0.072145	0.073529	0.074934	0.076359	0.077804	0.079270	0.080757	−1.4
−1.3	0.082264	0.083793	0.085343	0.086915	0.088508	0.090123	0.091759	0.093418	0.095098	0.096801	−1.3
−1.2	0.098525	0.100273	0.102042	0.103835	0.105650	0.107488	0.109349	0.111233	0.113140	0.115070	−1.2
−1.1	0.117023	0.119000	0.121001	0.123024	0.125072	0.127143	0.129238	0.131357	0.133500	0.135666	−1.1
−1.0	0.137857	0.140071	0.142310	0.144572	0.146859	0.149170	0.151505	0.153864	0.156248	0.158655	−1.0
−0.9	0.161087	0.163543	0.166023	0.168528	0.171056	0.173609	0.176185	0.178786	0.181411	0.184060	−0.9
−0.8	0.186733	0.189430	0.192150	0.194894	0.197662	0.200454	0.203269	0.206108	0.208970	0.211855	−0.8
−0.7	0.214764	0.217695	0.220650	0.223627	0.226627	0.229650	0.232695	0.235762	0.238852	0.241964	−0.7
−0.6	0.245097	0.248252	0.251429	0.254627	0.257846	0.261086	0.264347	0.267629	0.270931	0.274253	−0.6
−0.5	0.277595	0.280957	0.284339	0.287740	0.291160	0.294599	0.298056	0.301532	0.305026	0.308538	−0.5
−0.4	0.312067	0.315614	0.319178	0.322758	0.326355	0.329969	0.333598	0.337243	0.340903	0.344578	−0.4
−0.3	0.348268	0.351973	0.355691	0.359424	0.363169	0.366928	0.370700	0.374484	0.378281	0.382089	−0.3
−0.2	0.385908	0.389739	0.393580	0.397432	0.401294	0.405165	0.409046	0.412936	0.416834	0.420740	−0.2
−0.1	0.424655	0.428576	0.432505	0.436441	0.440382	0.444330	0.448283	0.452242	0.456205	0.460172	−0.1

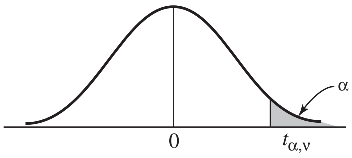
(continued)

TABLE A.1 ***(Continued)***

z	−0.09	−0.08	−0.07	−0.06	−0.05	−0.04	−0.03	−0.02	−0.01	−0.00	*z*
0.0	0.464144	0.468119	0.472097	0.476078	0.480061	0.484047	0.488033	0.492022	0.496011	0.500000	0.0
0.0	0.500000	0.503989	0.507978	0.511967	0.515953	0.519939	0.523922	0.527903	0.531881	0.535856	0.0
0.1	0.539828	0.543795	0.547758	0.551717	0.555760	0.559618	0.563559	0.567495	0.571424	0.575345	0.1
0.2	0.579260	0.583166	0.587064	0.590954	0.594835	0.598706	0.602568	0.606420	0.610261	0.614092	0.2
0.3	0.617911	0.621719	0.625516	0.629300	0.633072	0.636831	0.640576	0.644309	0.648027	0.651732	0.3
0.4	0.655422	0.659097	0.662757	0.666402	0.670031	0.673645	0.677242	0.680822	0.684386	0.687933	0.4
0.5	0.691462	0.694974	0.698468	0.701944	0.705401	0.708840	0.712260	0.715661	0.719043	0.722405	0.5
0.6	0.725747	0.729069	0.732371	0.735653	0.738914	0.742154	0.745373	0.748571	0.751748	0.754903	0.6
0.7	0758036	0.761148	0.764238	0.767305	0.770350	0.773373	0.776373	0.779350	0.782305	0.785236	0.7
0.8	0.788145	0.791030	0.793892	0.796731	0.799546	0.802338	0.805106	0.807850	0.810570	0.813267	0.8
0.9	0.815940	0.818589	0.821214	0.823815	0.826391	0.828944	0.831472	0.833977	0.836457	0.838913	0.9
1.0	0.841345	0.843752	O.846136	0.848495	0.850830	0.853141	0.855428	0.857690	0.859929	0.862143	1.0
1.1	0.864334	0.866500	0.868643	0.870762	0.872857	0.874928	0.876976	0.878999	0.881000	0.882977	1.1
1.2	0.884930	0.886860	0.888767	0.890651	0.892512	0.894350	0.896165	0.897958	0.899727	0.901475	1.2
1.3	0.903199	0.904902	0.906582	0.908241	0.909877	0.911492	0.913085	0.914657	0.916207	0.917736	1.3
1.4	0.919243	0.920730	0.922196	0.923641	0.925066	0.926471	0.927855	0.929219	0.930563	0.931888	1.4
1.5	0.933193	0.934478	0.935744	0936992	0.938220	0.939429	0.940620	0.941792	0.942947	0.944083	1.5
1.6	0.945201	0.946301	0.947384	0.948449	0.949497	0.950529	0.951543	0.952540	0.953521	0.954486	1.6
1.7	0.955435	0.956367	0.957284	0.958185	0.959071	0.959941	0.960796	0.961636	0.962462	0.963273	1.7
1.8	0.964070	0.964852	0.965621	0.966375	0.967116	0.967843	0.968557	0.969258	0.969946	0.970621	1.8
1.9	0.971283	0.971933	0.972571	0.973197	0.973810	0.974412	0.975002	0.975581	0.976148	0.976705	1.9

2.0	0.977250	0.977784	0.978308	0.978822	0.979325	0.979818	0.980301	0.980774	0.981237	0.981691	2.0
2.1	0.982136	0.982571	0.982997	0.983414	0.983823	0.984222	0.984614	0.984997	0.985371	0.985738	2.1
2.2	0.986097	0.986447	0.986791	0.987126	0.987455	0.987776	0.988089	0.988396	0.988696	0.988989	2.2
2.3	0.989276	0.989556	0.989830	0.990097	0.990358	0.990613	0.990863	0.991106	0.991344	0.991576	2.3
2.4	0.991802	0.992024	0.992240	0.992451	0.992656	0.992857	0.993053	0.993244	0.993431	0.993613	2.4
2.5	0.993790	0.993963	0.994132	0.994297	0.994457	0.994614	0.994766	0.994915	0.995060	0.995201	2.5
2.6	0.995339	0.995473	0.995604	0.995731	0.995855	0.995975	0.996093	0.996207	0.996319	0.996427	2.6
2.7	0.996533	0.996636	0.996736	0.996833	0.996928	0.997020	0.997110	0.997197	0.997282	0.997365	2.7
2.8	0.997445	0.997523	0.997599	0.997673	0.997744	0.997814	0.997882	0.997948	0.998012	0.998074	2.8
2.9	0.998134	0.998193	0.998250	0.998305	0.998359	0.998411	0.998462	0.998511	0.998559	0.998605	2.9
3.0	0.998650	0.998694	0.998736	0.998777	0.998817	0.998856	0.998893	0.998930	0.998965	0.998999	3.0
3.1	0.999032	0.999065	0.999096	0.999126	0.999155	0.999184	0.999211	0.999238	0.999264	0.999289	3.1
3.2	0.999313	0.999336	0.999359	0.999381	0.999402	0.999423	0.999443	0.999462	0.999481	0.999499	3.2
3.3	0.999517	0.999533	0.999550	0.999566	0.999581	0.999596	0.999610	0.999624	0.999638	0.999650	3.3
3.4	0.999663	0.999675	0.999687	0.999698	0.999709	0.999720	0.999730	0.999740	0.999749	0.999758	3.4
3.5	0.999767	0.999776	0.999784	0.999792	0.999800	0.999807	0.999815	0.999821	0.999828	0.999835	3.5
3.6	0.999841	0.999847	0.999853	0.999858	0.999864	0.999869	0.999874	0.999879	0.999883	0.999888	3.6
3.7	0.999892	0.999896	0.999900	0.999904	0.999908	0.999912	0.999915	0.999918	0.999922	0.999925	3.7
3.8	0.999928	0.999931	0.999933	0.999936	0.999938	0.999941	0.999943	0.999946	0.999948	0.999950	3.8
3.9	0.999952	0.999954	0.999956	0.999958	0.999959	0.999961	0.999963	0.999964	0.999966	0.999967	3.9

TABLE A.2 Percentage Points $t_{\alpha,\nu}$ of the t Distribution

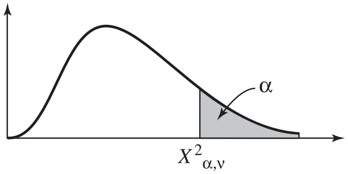


ν \ α	0.40	0.25	0.10	0.05	0.025	0.01	0.005	0.0025	0.001	0.0005
1	0.325	1.000	3.078	6.314	12.706	31.821	63.657	127.32	318.31	636.62
2	0.289	0.816	1.886	2.920	4.303	6.965	9.925	14.089	23.326	31.598
3	0.277	0.765	1.638	2.353	3.182	4.541	5.841	7.453	10.213	12.924
4	0.271	0.741	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.610
5	0.267	0.727	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	4.317	5.208	5.959
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	3.833	4.501	5.041
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	3.690	4.297	4.781
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587
11	0.260	0.697	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437
12	0.259	0.695	1.356	1.782	2.179	2.681	3.055	3.428	3.930	4.318
13	0.259	0.694	1.350	1.771	2.160	2.650	3.012	3.372	3.852	4.221
14	0.258	0.692	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.140
15	0.258	0.691	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073
16	0.258	0.690	1.337	1.746	2.120	2.583	2.921	3.252	3.686	4.015

17	0.257	0.689	1.333	1.740	2.110	2.567	2.898	3.222	3.646	3.965
18	0.257	0.688	1.330	1.734	2.101	2.552	2.878	3.197	3.610	3.922
19	0.257	0.688	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883
20	0.257	0.687	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.850
21	0.257	0.686	1.323	1.721	2.080	2.518	2.831	3.135	3.527	3.819
22	0.256	0.686	1.321	1.717	2.074	2.508	2.819	3.119	3.505	3.792
23	0.256	0.685	1.319	1.714	2.069	2.500	2.807	3.104	3.485	3.767
24	0.256	0.685	1.318	1.711	2.064	2.492	2.797	3.091	3.467	3.745
25	0.256	0.684	1.316	1.708	2.060	2.485	2.787	3.078	3.450	3.725
26	0.256	0.684	1.315	1.706	2.056	2.479	2.779	3.067	3.435	3.707
27	0.256	0.684	1.314	1.703	2.052	2.473	2.771	3.057	3.421	3.690
28	0.256	0.683	1.313	1.701	2.048	2.467	2.763	3.047	3.408	3.674
29	0.256	0.683	1.311	1.699	2.045	2.462	2.756	3.038	3.396	3.659
30	0.256	0.683	1.310	1.697	2.042	2.457	2.750	3.030	3.385	3.646
40	0.255	0.681	1.303	1.684	2.001	2.423	2.704	2.971	3.307	3.551
60	0.254	0.679	1.296	1.671	2.000	2.390	2.660	2.915	3.232	3.460
120	0.254	0.677	1.289	1.658	1.980	2.358	2.617	2.860	3.160	3.373
∞	0.253	0.674	1.282	1.645	1.960	2.326	2.576	2.807	3.090	3.291

ν = degrees of freedom.

TABLE A.3 Percentage Points $\chi^2_{\alpha,\nu}$ of the Chi-Square Distribution

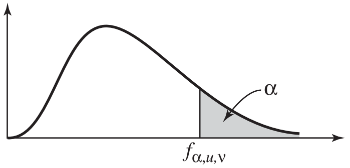


ν \ α	0.995	0.990	0.975	0.950	0.900	0.500	0.100	0.050	0.025	0.010	0.005
1	0.00+	0.00+	0.00+	0.00+	0.02	0.45	2.71	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	0.21	1.39	4.61	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	0.58	2.37	6.25	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	1.06	3.36	7.78	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	1.61	4.35	9.24	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	2.20	5.35	10.65	12.59	14.45	16.81	18.55
7	0.99	1.24	1.69	2.17	2.83	6.35	12.03	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	3.49	7.34	13.36	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	4.17	8.34	14.68	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	4.87	9.34	15.99	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	5.58	10.34	17.28	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	6.30	11.34	18.55	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	7.04	12.34	19.81	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	7.79	13.34	21.06	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	8.55	14.34	22.31	25.00	27.49	30.58	32.80

16	5.14	5.81	6.91	7.96	9.31	15.34	23.54	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	10.09	16.34	24.77	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	10.87	17.34	25.99	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00
21	8.03	8.90	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40
22	8.64	9.54	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80
23	9.26	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18
24	9.89	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56
25	10.52	11.52	13.12	14.61	16.47	24.34	34.28	37.65	40.65	44.31	46.93
26	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29
27	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.65
28	12.46	13.57	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99
29	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34
30	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	55.33	69.33	85.53	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	64.28	79.33	96.58	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	73.29	89.33	107.57	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	82.36	99.33	118.50	124.34	129.56	135.81	140.17

ν = degrees of freedom.

TABLE A.4 Percentage Points $f_{\alpha,\mu,\nu}$ of the F Distribution



$f_{0.25,\mu,\nu}$

		Degrees of Freedom for the Numerator (μ)																		
	ν	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of Freedom for the Denominator (ν)	1	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.41	9.49	9.58	9.63	9.67	9.71	9.76	9.80	9.85
	2	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.41	3.43	3.43	3.44	3.45	3.46	3.47	3.48
	3	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.46	2.46	2.46	2.47	2.47	2.47	2.47	2.47
	4	1.81	2.00	2.05	2.06	2.07	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08	2.08
	5	1.69	1.85	1.88	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.89	1.88	1.88	1.88	1.88	1.87	1.87	1.87
	6	1.62	1.76	1.78	1.79	1.79	1.78	1.78	1.78	1.77	1.77	1.77	1.76	1.76	1.75	1.75	1.75	1.74	1.74	1.74
	7	1.57	1.70	1.72	1.72	1.71	1.71	1.70	1.70	1.70	1.69	1.68	1.68	1.67	1.67	1.66	1.66	1.65	1.65	1.65
	8	1.54	1.66	1.67	1.66	1.66	1.65	1.64	1.64	1.63	1.63	1.62	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.58
	9	1.51	1.62	1.63	1.63	1.62	1.61	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.54	1.54	1.53	1.53
	10	1.49	1.60	1.60	1.59	1.59	1.58	1.57	1.56	1.56	1.55	1.54	1.53	1.52	1.52	1.51	1.51	1.50	1.49	1.48
	11	1.47	1.58	1.58	1.57	1.56	1.55	1.54	1.53	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.47	1.46	1.45
	12	1.46	1.56	1.56	1.55	1.54	1.53	1.52	1.51	1.51	1.50	1.49	1.48	1.47	1.46	1.45	1.45	1.44	1.43	1.42
	13	1.45	1.55	1.55	1.53	1.52	1.51	1.50	1.49	1.49	1.48	1.47	1.46	1.45	1.44	1.43	1.42	1.42	1.41	1.40
	14	1.44	1.53	1.53	1.52	1.51	1.50	1.49	1.48	1.47	1.46	1.45	1.44	1.43	1.42	1.41	1.41	1.40	1.39	1.38

Degrees of Freedom for the Denominator (v)	15	1.43	1.52	1.52	1.51	1.49	1.48	1.47	1.46	1.46	1.45	1.44	1.43	1.41	1.41	1.40	1.39	1.38	1.37	1.36
	16	1.42	1.51	1.51	1.50	1.48	1.47	1.46	1.45	1.44	1.44	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34
	17	1.42	1.51	1.50	1.49	1.47	1.46	1.45	1.44	1.43	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.33
	18	1.41	1.50	1.49	1.48	1.46	1.45	1.44	1.43	1.42	1.42	1.40	1.39	1.38	1.37	1.36	1.35	1.34	1.33	1.32
	19	1.41	1.49	1.49	1.47	1.46	1.44	1.43	1.42	1.41	1.41	1.40	1.38	1.37	1.36	1.35	1.34	1.33	1.32	1.30
	20	1.40	1.49	1.48	1.47	1.45	1.44	1.43	1.42	1.41	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.32	1.31	1.29
	21	1.40	1.48	1.48	1.46	1.44	1.43	1.42	1.41	1.40	1.39	1.38	1.37	1.35	1.34	1.33	1.32	1.31	1.30	1.28
	22	1.40	1.48	1.47	1.45	1.44	1.42	1.41	1.40	1.39	1.39	1.37	1.36	1.34	1.33	1.32	1.31	1.30	1.29	1.28
	23	1.39	1.47	1.47	1.45	1.43	1.42	1.41	1.40	1.39	1.38	1.37	1.35	1.34	1.33	1.32	1.31	1.30	1.28	1.27
	24	1.39	1.47	1.46	1.44	1.43	1.41	1.40	1.39	1.38	1.38	1.36	1.35	1.33	1.32	1.31	1.30	1.29	1.28	1.26
	25	1.39	1.47	1.46	1.44	1.42	1.41	1.40	1.39	1.38	1.37	1.36	1.34	1.33	1.32	1.31	1.29	1.28	1.27	1.25
	26	1.38	1.46	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.37	1.35	1.34	1.32	1.31	1.30	1.29	1.28	1.26	1.25
	27	1.38	1.46	1.45	1.43	1.42	1.40	1.39	1.38	1.37	1.36	1.35	1.33	1.32	1.31	1.30	1.28	1.27	1.26	1.24
	28	1.38	1.46	1.45	1.43	1.41	1.40	1.39	1.38	1.37	1.36	1.34	1.33	1.31	1.30	1.29	1.28	1.27	1.25	1.24
	29	1.38	1.45	1.45	1.43	1.41	1.40	1.38	1.37	1.36	1.35	1.34	1.32	1.31	1.30	1.29	1.27	1.26	1.25	1.23
	30	1.38	1.45	1.44	1.42	1.41	1.39	1.38	1.37	1.36	1.35	1.34	1.32	1.30	1.29	1.28	1.27	1.26	1.24	1.23
	40	1.36	1.44	1.42	1.40	1.39	1.37	1.36	1.35	1.34	1.33	1.31	1.30	1.28	1.26	1.25	1.24	1.22	1.21	1.19
	60	1.35	1.42	1.41	1.38	1.37	1.35	1.33	1.32	1.31	1.30	1.29	1.27	1.25	1.24	1.22	1.21	1.19	1.17	1.15
	120	1.34	1.40	1.39	1.37	1.35	1.33	1.31	1.30	1.29	1.28	1.26	1.24	1.22	1.21	1.19	1.18	1.16	1.13	1.10
	∞	1.32	1.39	1.37	1.35	1.33	1.31	1.29	1.28	1.27	1.25	1.24	1.22	1.19	1.18	1.16	1.14	1.12	1.08	1.00

(*continued*)

TABLE A.4 (*Continued*)

		$f_{0.10,\mu,\nu}$																		
		Degrees of Freedom for the Numerator (μ)																		
	ν	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of Freedom for the Denominator (ν)	1	39.86	49.50	53.59	55.83	57.24	58.20	58.91	59.44	59.86	60.19	60.71	61.22	61.74	62.00	62.26	62.53	62.79	63.06	63.33
	2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48	9.49
	3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14	5.13
	4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78	3.76
	5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12	3.10
	6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74	2.72
	7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49	2.47
	8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32	2.29
	9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18	2.16
	10	3.29	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08	2.06
	11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00	1.97
	12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93	1.90
	13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88	1.85
	14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83	1.80
	15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79	1.76

Degrees of Freedom for the Denominator (v)	16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75	1.72
	17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72	1.69
	18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69	1.66
	19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67	1.63
	20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64	1.61
	21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62	1.59
	22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60	1.57
	23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59	1.55
	24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57	1.53
	25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56	1.52
	26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54	1.50
	27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53	1.49
	28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52	1.48
	29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51	1.47
	30	2.88	2.49	2.28	2.14	2.03	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50	1.46
	40	2.84	2.44	2.23	2.09	2.00	1.93	1.87	1.83	1.79	1.76	1.71	1.66	1.61	1.57	1.54	1.51	1.47	1.42	1.38
	60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35	1.29
	120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26	1.19
	∞	2.71	2.30	2.08	1.94	1.85	1.77	1.72	1.67	1.63	1.60	1.55	1.49	1.42	1.38	1.34	1.30	1.24	1.17	1.00

(*continued*)

TABLE A.4 (*Continued*)

	$f_{0.05,\mu,\nu}$																			
		Degrees of Freedom for the Numerator (μ)																		
	ν	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of Freedom for the Denominator (ν)	1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
	2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
	3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
	4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
	5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
	6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
	7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
	8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
	9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
	10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
	11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
	12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
	13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
	14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
	15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07

Degrees of Freedom for the Denominator (v)	16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
	17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
	18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
	19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
	20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
	21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
	22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
	23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
	24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
	25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
	26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
	27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
	28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
	29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
	30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
	40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
	60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
	120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.55	1.43	1.35	1.25
	∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

(*continued*)

TABLE A.4 (*Continued*)

$f_{0.25,\mu,\nu}$

		Degrees of Freedom for the Numerator (μ)																		
	ν	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of Freedom for the Denominator (ν)	1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010	1014	1018
	2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.49	39.50
	3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95	13.90
	4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31	8.26
	5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07	6.02
	6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90	4.85
	7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.42	4.36	4.31	4.25	4.20	4.14
	8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73	3.67
	9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39	3.33
	10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14	3.08
	11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94	2.88
	12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79	2.72
	13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66	2.60
	14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55	2.49
	15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46	2.40

Degrees of Freedom for the Denominator (v)	16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38	2.32
	17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32	2.25
	18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26	2.19
	19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20	2.13
	20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16	2.09
	21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11	2.04
	22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08	2.00
	23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04	1.97
	24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01	1.94
	25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98	1.91
	26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95	1.88
	27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93	1.85
	28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91	1.83
	29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89	1.81
	30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87	1.79
	40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45	2.39	2.29	2.18	2.07	2.01	1.94	1.88	1.80	1.72	1.64
	60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58	1.48
	120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43	1.31
	∞	5.02	3.69	3.12	2.79	2.57	2.41	2.29	2.19	2.11	2.05	1.94	1.83	1.71	1.64	1.57	1.48	1.39	1.27	1.00

(*continued*)

TABLE A.4 (*Continued*)

	$f_{0.01,\mu,\nu}$																			
		Degrees of Freedom for the Numerator (μ)																		
	ν	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
Degrees of Freedom for the Denominator (ν)	1	4052	4999.5	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339	6366
	2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.42	99.43	99.45	99.46	99.47	99.47	99.48	99.49	99.50
	3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.05	26.87	26.69	26.00	26.50	26.41	26.32	26.22	26.13
	4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.37	14.20	14.02	13.93	13.84	13.75	13.65	13.56	13.46
	5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11	9.02
	6	13.75	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97	6.88
	7	12.25	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74	5.65
	8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95	4.46
	9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40	4.31
	10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00	3.91
	11	9.65	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69	3.60
	12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45	3.36
	13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25	3.17
	14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09	3.00
	15	8.68	6.36	5.42	4.89	4.36	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96	2.87
	16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84	2.75
	17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75	2.65

Degrees of Freedom for the Denominator (v)	18	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66	2.57
	19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58	2.59
	20	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52	2.42
	21	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46	2.36
	22	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40	2.31
	23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35	2.26
	24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31	2.21
	25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27	2.17
	26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23	2.13
	27	7.68	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20	2.10
	28	7.64	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17	2.06
	29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14	2.03
	30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11	2.01
	40	7.31	5.18	4.31	3.83	3.51	3.29	3.12	2.99	2.89	2.80	2.66	2.52	2.37	2.29	2.20	2.11	2.02	1.92	1.80
	60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73	1.60
	120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53	1.38
	∞	6.63	4.61	3.78	3.32	3.02	2.80	2.64	2.51	2.41	2.32	2.18	2.04	1.88	1.79	1.70	1.59	1.47	1.32	1.00

TABLE A.5 Critical Values of the Durbin–Watson Statistic

		k = Number of Regressors (Excluding the Intercept)									
	Probability in Lower Tail (Significance	1		2		3		4		5	
Sample Size	Level = α)	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U	d_L	d_U
15	0.01	0.81	1.07	0.70	1.25	0.59	1.46	0.49	1.70	0.39	1.96
	0.025	0.95	1.23	0.83	1.40	0.71	1.61	0.59	1.84	0.48	2.09
	0.05	1.08	1.36	0.95	1.54	0.82	1.75	0.69	1.97	0.56	2.21
20	0.01	0.95	1.15	0.86	1.27	0.77	1.41	0.63	1.57	0.60	1.74
	0.025	1.08	1.28	0.99	1.41	0.89	1.55	0.79	1.70	0.70	1.87
	0.05	1.20	1.41	1.10	1.54	1.00	1.68	0.90	1.83	0.79	1.99
25	0.01	1.05	1.21	0.98	1.30	0.90	1.41	0.83	1.52	0.75	1.65
	0.025	1.13	1.34	1.10	1.43	1.02	1.54	0.94	1.65	0.86	1.77
	0.05	1.20	1.45	1.21	1.55	1.12	1.66	1.04	1.77	0.95	1.89
30	0.01	1.13	1.26	1.07	1.34	1.01	1.42	0.94	1.51	0.88	1.61
	0.025	1.25	1.38	1.18	1.46	1.12	1.54	1.05	1.63	0.98	1.73
	0.05	1.35	1.49	1.28	1.57	1.21	1.65	1.14	1.74	1.07	1.83
40	0.01	1.25	1.34	1.20	1.40	1.15	1.46	1.10	1.52	1.05	1.58
	0.025	1.35	1.45	1.30	1.51	1.25	1.57	1.20	1.63	1.15	1.69
	0.05	1.44	1.54	1.39	1.60	1.34	1.66	1.29	1.72	1.23	1.79
50	0.01	1.32	1.40	1.28	1.45	1.24	1.49	1.20	1.54	1.16	1.59
	0.025	1.42	1.50	1.38	1.54	1.34	1.59	1.30	1.64	1.26	1.69
	0.05	1.50	1.59	1.46	1.63	1.42	1.67	1.38	1.72	1.34	1.77
60	0.01	1.38	1.45	1.35	1.48	1.32	1.52	1.28	1.56	1.25	1.60
	0.025	1.47	1.54	1.44	1.57	1.40	1.61	1.37	1.65	1.33	1.69
	0.05	1.55	1.62	1.51	1.65	1.48	1.69	1.44	1.73	1.41	1.77
80	0.01	1.47	1.52	1.44	1.54	1.42	1.57	1.39	1.60	1.36	1.62
	0.025	1.54	1.59	1.52	1.62	1.49	1.65	1.47	1.67	1.44	1.70
	0.05	1.61	1.66	1.59	1.69	1.56	1.72	1.53	1.74	1.51	1.77
100	0.01	1.52	1.56	1.50	1.58	1.48	1.60	1.45	1.63	1.44	1.65
	0.025	1.59	1.63	1.57	1.65	1.55	1.67	1.53	1.70	1.51	1.72
	0.05	1.65	1.69	1.63	1.72	1.61	1.74	1.59	1.76	1.57	1.78

Source: Adapted from J. Durbin and G. S. Watson [1951]. Testing for serial correlation in least squares regression II.*Biometrika* **38**, with permission of the publisher.

APPENDIX B

DATA SETS FOR EXERCISES

Introduction to Time Series Analysis and Forecasting, Second Edition.
Douglas C. Montgomery, Cheryl L. Jennings and Murat Kulahci.

TABLE B.1 Market Yield on US Treasury Securities at 10-Year Constant Maturity

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
Apr-1953	2.83	Oct-1966	5.01	Apr-1980	11.47	Oct-1993	5.33
May-1953	3.05	Nov-1966	5.16	May-1980	10.18	Nov-1993	5.72
Jun-1953	3.11	Dec-1966	4.84	Jun-1980	9.78	Dec-1993	5.77
Jul-1953	2.93	Jan-1967	4.58	Jul-1980	10.25	Jan-1994	5.75
Aug-1953	2.95	Feb-1967	4.63	Aug-1980	11.10	Feb-1994	5.97
Sep-1953	2.87	Mar-1967	4.54	Sep-1980	11.51	Mar-1994	6.48
Oct-1953	2.66	Apr-1967	4.59	Oct-1980	11.75	Apr-1994	6.97
Nov-1953	2.68	May-1967	4.85	Nov-1980	12.68	May-1994	7.18
Dec-1953	2.59	Jun-1967	5.02	Dec-1980	12.84	Jun-1994	7.10
Jan-1954	2.48	Jul-1967	5.16	Jan-1981	12.57	Jul-1994	7.30
Feb-1954	2.47	Aug-1967	5.28	Feb-1981	13.19	Aug-1994	7.24
Mar-1954	2.37	Sep-1967	5.30	Mar-1981	13.12	Sep-1994	7.46
Apr-1954	2.29	Oct-1967	5.48	Apr-1981	13.68	Oct-1994	7.74
May-1954	2.37	Nov-1967	5.75	May-1981	14.10	Nov-1994	7.96
Jun-1954	2.38	Dec-1967	5.70	Jun-1981	13.47	Dec-1994	7.81
Jul-1954	230	Jan-1968	5.53	Jul-1981	14.28	Jan-1995	7.78
Aug-1954	2.36	Feb-1968	5.56	Aug-1981	14.94	Feb-1995	7.47
Sep-1954	2.38	Mar-1968	5.74	Sep-1981	15.32	Mar-1995	7.20
Oct-1954	2.43	Apr-1968	564	Oct-1981	15.15	Apr-1995	7.06
Nov-1954	2.48	May-1968	5.87	Nov-1981	13.39	May-1995	6.63
Dec-1954	2.51	Jun-1968	5.72	Dec-1981	13.72	Jun-1995	6.17
Jan-1955	2.61	Jul-1968	5.50	Jan-1982	14.59	Jul-1995	6.28
Feb-1955	2.65	Aug-1968	5.42	Feb-1982	14.43	Aug-1995	6.49
Mar-1955	2.68	Sep-1968	5.46	Mar-1982	13.86	Sep-1995	6.20
Apr-1955	2.75	Oct-1968	5.58	Apr-1982	13.87	Oct-1995	6.04
May-1955	2.76	Nov-1968	5.70	May-1982	13.62	Nov-1995	5.93
Jun-1955	2.78	Dec-1968	6.03	Jun-1982	14.30	Dec-1995	5.71
Jul-1955	2.90	Jan-1969	6.04	Jul-1982	13.95	Jan-1996	5.65
Aug-1955	2.97	Feb-1969	6.19	Aug-1982	13.06	Feb-1996	5.81
Sep-1955	2.97	Mar-1969	6.30	Sep-1982	12.34	Mar-1996	6.27
Oct-1955	2.88	Apr-1969	6.17	Oct-1982	10.91	Apr-1996	6.51
Nov-1955	2.89	May-1969	6.32	Nov-1982	10.55	May-1996	6.74
Dec-1955	2.96	Jun-1969	6.57	Dec-1982	10.54	Jun-1996	6.91
Jan-1956	2.90	Jul-1969	6.72	Jan-1983	10.46	Jul-1996	6.87
Feb-1956	2.84	Aug-1969	6.69	Feb-1983	10.72	Aug-1996	6.64
Mar-1956	2.96	Sep-1969	7.16	Mar-1983	10.51	Sep-1996	6.83
Apr-1956	3.18	Oct-1969	7.10	Apr-1983	10.40	Oct-1996	6.53
May-1956	3.07	Nov-1969	7.14	May-1983	10.38	Nov-1996	6.20
Jun-1956	3.00	Dec-1969	7.65	Jun-1983	10.85	Dec-1996	6.30

(*continued*)

TABLE B.1 (*Continued*)

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
Jul-1956	3.11	Jan-1970	7.79	Jul-1983	11.38	Jan-1997	6.58
Aug-1956	3.33	Feb-1970	7.24	Aug-1983	11.85	Feb-1997	6.42
Sep-1956	3.38	Mar-1970	7.07	Sep-1983	11.65	Mar-1997	6.69
Oct-1956	3.34	Apr-1970	7.39	Oct-1983	11.54	Apr-1997	6.89
Nov-1956	3.49	May-1970	7.91	Nov-1983	11.69	May-1997	6.71
Dec-1956	3.59	Jun-1970	7.84	Dec-1983	11.83	Jun-1997	6.49
Jan-1957	3.46	Jul-1970	7.46	Jan-1984	11.67	Jul-1997	6.22
Feb-1957	3.34	Aug-1970	7.53	Feb-1984	11.84	Aug-1997	6.30
Mar-1957	3.41	Sep-1970	7.39	Mar-1984	12.32	Sep-1997	6.21
Apr-1957	3.48	Oct-1970	7.33	Apr-1984	12.63	Oct-1997	6.03
May-1957	3.60	Nov-1970	6.84	May-1984	13.41	Nov-1997	5.88
Jun-1957	3.80	Dec-1970	6.39	Jun-1984	13.56	Dec-1997	5.81
Jul-1957	3.93	Jan-1971	6.24	Jul-1984	13.36	Jan-1998	5.54
Aug-1957	3.93	Feb-1971	6.11	Aug-1984	12.72	Feb-1998	5.57
Sep-1957	3.92	Mar-1971	5.70	Sep-1984	12.52	Mar-1998	5.65
Oct-1957	3.97	Apr-1971	5.83	Oct-1984	12.16	Apr-1998	5.64
Nov-1957	3.72	May-1971	6.39	Nov-1984	11.57	May-1998	5.65
Dec-1957	3.21	Jun-1971	6.52	Dec-1984	11.50	Jun-1998	5.50
Jan-1958	3.09	Jul-1971	6.73	Jan-1985	11.38	Jul-1998	5.46
Feb-1958	3.05	Aug-1971	6.58	Feb-1985	11.51	Aug-1998	5.34
Mar-1958	2.98	Sep-1971	6.14	Mar-1985	11.86	Sep-1998	4.81
Apr-1958	2.88	Oct-1971	5.93	Apr-1985	11.43	Oct-1998	4.53
May-1958	2.92	Nov-1971	5.81	May-1985	10.85	Nov-1998	4.83
Jun-1958	2.97	Dec-1971	5.93	Jun-1985	10.16	Dec-1998	4.65
Jul-1958	3.20	Jan-1972	5.95	Jul-1985	10.31	Jan-1999	4.72
Aug-1958	3.54	Feb-1972	6.08	Aug-1985	10.33	Feb-1999	5.00
Sep-1958	3.76	Mar-1972	6.07	Sep-1985	10.37	Mar-1999	5.23
Oct-1958	3.80	Apr-1972	6.19	Oct-1985	10.24	Apr-1999	5.18
Nov-1958	3.74	May-1972	6.13	Nov-1985	9.78	May-1999	5.54
Dec-1958	3.86	Jun-1972	6.11	Dec-1985	9.26	Jun-1999	5.90
Jan-1959	4.02	Jul-1972	6.11	Jan-1986	9.19	Jul-1999	5.79
Feb-1959	3.96	Aug-1972	6.21	Feb-1986	8.70	Aug-1999	5.94
Mar-1959	3.99	Sep-1972	6.55	Mar-1986	7.78	Sep-1999	5.92
Apr-1959	4.12	Oct-1972	6.48	Apr-1986	7.30	Oct-1999	6.11
May-1959	4.31	Nov-1972	6.28	May-1986	7.71	Nov-1999	6.03
Jun-1959	4.34	Dec-1972	6.36	Jun-1986	7.80	Dec-1999	6.28
Jul-1959	4.40	Jan-1973	6.46	Jul-1986	7.30	Jan-2000	6.66
Aug-1959	4.43	Feb-1973	6.64	Aug-1986	7.17	Feb-2000	6.52
Sep-1959	4.68	Mar-1973	6.71	Sep-1986	7.45	Mar-2000	6.26
Oct-1959	4.53	Apr-1973	6.67	Oct-1986	7.43	Apr-2000	5.99
Nov-1959	4.53	May-1973	6.85	Nov-1986	7.25	May-2000	6.44

TABLE B.1 (***Continued***)

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
Dec-1959	4.69	Jun-1973	6.90	Dec-1986	7.11	Jun-2000	6.10
Jan-1960	4.72	Jul-1973	7.13	Jan-1987	7.08	Jul-2000	6.05
Feb-1960	4.49	Aug-1973	7.40	Feb-1987	7.25	Aug-2000	5.83
Mar-1960	4.25	Sep-1973	7.09	Mar-1987	7.25	Sep-2000	5.80
Apr-1960	4.28	Oct-1973	6.79	Apr-1987	8.02	Oct-2000	5.74
May-1960	4.35	Nov-1973	6.73	May-1987	8.61	Nov-2000	5.72
Jun-1960	4.15	Dec-1973	6.74	Jun-1987	8.40	Dec-2000	5.24
Jul-1960	3.90	Jan-1974	6.99	Jul-1987	8.45	Jan-2001	5.16
Aug-1960	3.80	Feb-1974	6.96	Aug-1987	8.76	Feb-2001	5.10
Sep-1960	3.80	Mar-1974	7.21	Sep-1987	9.42	Mar-2001	4.89
Oct-1960	3.89	Apr-1974	7.51	Oct-1987	9.52	Apr-2001	5.14
Nov-1960	3.93	May-1974	7.58	Nov-1987	8.86	May-2001	5.39
Dec-1960	3.84	Jun-1974	7.54	Dec-1987	8.99	Jun-2001	5.28
Jan-1961	3.84	Jul-1974	7.81	Jan-1988	8.67	Jul-2001	5.24
Feb-1961	3.78	Aug-1974	8.04	Feb-1988	8.21	Aug-2001	4.97
Mar-1961	3.74	Sep-1974	8.04	Mar-1988	8.37	Sep-2001	4.73
Apr-1961	3.78	Oct-1974	7.90	Apr-1988	8.72	Oct-200l	4.57
May-1961	3.71	Nov-1974	7.68	May-1988	9.09	Nov-2001	4.65
Jun-1961	3.88	Dec-1974	7.43	Jun-1988	8.92	Dec-2001	5.09
Jul-1961	3.92	Jan-1975	7.50	Jul-1988	9.06	Jan-2002	5.04
Aug-1961	4.04	Feb-1975	7.39	Aug-1988	9.26	Feb-2002	4.91
Sep-1961	3.98	Mar-1975	7.73	Sep-1988	8.98	Mar-2002	5.28
Oct-1961	3.92	Apr-1975	8.23	Oct-1988	8.80	Apr-2002	5.21
Nov-1961	3.94	May-1975	8.06	Nov-1988	8.96	May-2002	5.16
Dec-1961	4.06	Jun-1975	7.86	Dec-1988	9.11	Jun-2002	4.93
Jan-1962	4.08	Jul-1975	8.06	Jan-1989	9.09	Jul-2002	4.65
Feb-1962	4.04	Aug-1975	8.40	Feb-1989	9.17	Aug-2002	4.26
Mar-1962	3.93	Sep-1975	8.43	Mar-1989	9.36	Sep-2002	3.87
Apr-1962	3.84	Oct-1975	8.14	Apr-1989	9.18	Oct-2002	3.94
May-1962	3.87	Nov-1975	8.05	May-1989	8.86	Nov-2002	4.05
Jun-1962	3.91	Dec-1975	8.00	Jun-1989	8.28	Dec-2002	4.03
Jul-1962	4.01	Jan-1976	7.74	Jul-1989	8.02	Jan-2003	4.05
Aug-1962	3.98	Feb-1976	7.79	Aug-1989	8.11	Feb-2003	3.90
Sep-1962	3.98	Mar-1976	7.73	Sep-1989	8.19	Mar-2003	3.81
Oct-1962	3.93	Apr-1976	7.56	Oct-1989	8.01	Apr-2003	3.96
Nov-1962	3.92	May-1976	7.90	Nov-1989	7.87	May-2003	3.57
Dec-1962	3.86	Jun-1976	7.86	Dec-1989	7.84	Jun-2003	3.33
Jan-1963	3.83	Jul-1976	7.83	Jan-1990	8.21	Jul-2003	3.98
Feb-1963	3.92	Aug-1976	7.77	Feb-1990	8.47	Aug-2003	4.45
Mar-1963	3.93	Sep-1976	7.59	Mar-1990	8.59	Sep-2003	4.27
Apr-1963	3.97	Oct-1976	7.41	Apr-1990	8.79	Oct-2003	4.29

(*continued*)

TABLE B.1 **(*Continued*)**

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
May-1963	3.93	Nov-1976	7.29	May-1990	8.76	Nov-2003	4.30
Jun-1963	3.99	Dec-1976	6.87	Jun-1990	8.48	Dec-2003	4.27
Jul-1963	4.02	Jan-1977	7.21	Jul-1990	8.47	Jan-2004	4.15
Aug-1963	4.00	Feb-1977	7.39	Aug-1990	8.75	Feb-2004	4.08
Sep-1963	4.08	Mar-1977	7.46	Sep-1990	8.89	Mar-2004	3.83
Ocl-1963	4.11	Apr-1977	7.37	Oct-1990	8.72	Apr-2004	4.35
Nov-1963	4.12	May-1977	7.46	Nov-1990	8.39	May-2004	4.72
Dec-1963	4.13	Jun-1977	7.28	Dec-1990	8.08	Jun-2004	4.73
Jan-1964	4.17	Jul-1977	7.33	Jan-1991	8.09	Jul-2004	4.50
Feb-1964	4.15	Aug-1977	7.40	Feb-1991	7.85	Aug-2004	4.28
Mar-1964	4.22	Sep-1977	7.34	Mar-1991	8.11	Sep-2004	4.13
Apr-1964	4.23	Oct-1977	7.52	Apr-1991	8.04	Oct-2004	4.10
May-1964	4.20	Nov-1977	7.58	May-1991	8.07	Nov-2004	4.19
Jun-1964	4.17	Dec-1977	7.69	Jun-1991	8.28	Dec-2004	4.23
Jul-1964	4.19	Jan-1978	7.96	Jul-1991	8.27	Jan-2005	4.22
Aug-1964	4.19	Feb-1978	8.03	Aug-1991	7.90	Feb-2005	4.17
Sep-1964	4.20	Mar-1978	8.04	Sep-1991	7.65	Mar-2005	4.50
Oct-1964	4.19	Apr-1978	8.15	Oct-1991	7.53	Apr-2005	4.34
Nov-1964	4.15	May-1978	8.35	Nov-1991	7.42	May-2005	4.14
Dec-1964	4.18	Jun-1978	8.46	Dec-1991	7.09	Jun-2005	4.00
Jan-1965	4.19	Jul-1978	8.64	Jan-1992	7.03	Jul-2005	4.18
Feb-1965	4.21	Aug-1978	8.41	Feb-1992	7.34	Aug-2005	4.26
Mar-1965	4.21	Sep-1978	8.42	Mar-1992	7.54	Sep-2005	4.20
Apr-1965	4.20	Oct-1978	8.64	Apr-1992	7.48	Oct-2005	4.46
May-1965	4.21	Nov-1978	8.81	May-1992	7.39	Nov-2005	4.54
Jun-1965	4.21	Dec-1978	9.01	Jun-1992	7.26	Dec-2005	4.47
Jul-1965	4.20	Jan-1979	9.10	Jul-1992	6.84	Jan-2006	4.42
Aug-1965	4.25	Feb-1979	9.10	Aug-1992	6.59	Feb-2006	4.57
Sep-1965	4.29	Mar-1979	9.12	Sep-1992	6.42	Mar-2006	4.72
Oct-1965	4.35	Apr-1979	9.18	Oct-1992	6.59	Apr-2006	4.99
Nov-1965	4.45	May-1979	9.25	Nov-1992	6.87	May-2006	5.11
Dec-1965	4.62	Jun-1979	8.91	Dec-1992	6.77	Jun-2006	5.11
Jan-1966	4.61	Jul-1979	8.95	Jan-1993	6.60	Jul-2006	5.09
Feb-1966	4.83	Aug-1979	9.03	Feb-1993	6.26	Aug-2006	4.88
Mar-1966	4.87	Sep-1979	9.33	Mar-1993	5.98	Sep-2006	4.72
Apr-1966	4.75	Oct-1979	10.30	Apr-1993	5.97	Oct-2006	4.73
May-1966	4.78	Nov-1979	10.65	May-1993	6.04	Nov-2006	4.60
Jun-1966	4.81	Dec-1979	10.39	Jun-1993	5.96	Dec-2006	4.56
Jul-1966	5.02	Jan-1980	10.80	Jul-1993	5.81	Jan-2007	4.76
Aug-1966	5.22	Feb-1980	12.41	Aug-1993	5.68	Feb-2007	4.72
Sep-1966	5.18	Mar-1980	12.75	Sep-1993	5.36		

TABLE B.2 Pharmaceutical Product Sales

Week	Sales (In Thousands)	Week	Sales (In Thousands)	Week	Sales (In Thousands)	Week	Sales (In Thousands)
1	10618.1	31	10334.5	61	10538.2	91	10375.4
2	10537.9	32	10480.1	62	10286.2	92	10123.4
3	10209.3	33	10387.6	63	10171.3	93	10462.7
4	10553.0	34	10202.6	64	10393.1	94	10205.5
5	9934.9	35	10219.3	65	10162.3	95	10522.7
6	10534.5	36	10382.7	66	10164.5	96	10253.2
7	10196.5	37	10820.5	67	10327.0	97	10428.7
8	10511.8	38	10358.7	68	10365.1	98	10615.8
9	10089.6	39	10494.6	69	10755.9	99	10417.3
10	10371.2	40	10497.6	70	10463.6	100	10445.4
11	10239.4	41	10431.5	71	10080.5	101	10690.6
12	10472.4	42	10447.8	72	10479.6	102	10271.8
13	10827.2	43	10684.4	73	9980.9	103	10524.8
14	10640.8	44	10176.5	74	10039.2	104	9815.0
15	10517.8	45	10616.0	75	10246.1	105	10398.5
16	10154.2	46	10627.7	76	10368.0	106	10553.1
17	9969.2	47	10684.0	77	10446.3	107	10655.8
18	10260.4	48	10246.7	78	10535.3	108	10199.1
19	10737.0	49	10265.0	79	10786.9	109	10416.6
20	10430.0	50	10090.4	80	9975.8	110	10391.3
21	10689.0	51	9881.1	81	10160.9	111	10210.1
22	10430.4	52	10449.7	82	10422.1	112	10352.5
23	10002.4	53	10276.3	83	10757.2	113	10423.8
24	10135.7	54	10175.2	84	10463.8	114	10519.3
25	10096.2	55	10212.5	85	10307.0	115	10596.7
26	10288.7	56	10395.5	86	10134.7	116	10650.0
27	10289.1	57	10545.9	87	10207.7	117	10741.6
28	10589.9	58	10635.7	88	10488.0	118	10246.0
29	10551.9	59	10265.2	89	10262.3	119	10354.4
30	10208.3	60	10551.6	90	10785.9	120	10155.4

TABLE B.3 Chemical Process Viscosity

Time Period	Reading	Time Period	Reading	Time Period	Reading	Time Period	Reading
1	86.7418	26	87.2397	51	85.5722	76	84.7052
2	85.3195	27	87.5219	52	83.7935	77	83.8168
3	84.7355	28	86.4992	53	84.3706	78	82.4171
4	85.1113	29	85.6050	54	83.3762	79	83.0420
5	85.1487	30	86.8293	55	84.9975	80	83.6993
6	84.4775	31	84.5004	56	84.3495	81	82.2033
7	84.6827	32	84.1844	57	85.3395	82	82.1413
8	84.6757	33	85.4563	58	86.0503	83	81.7961
9	86.3169	34	86.1511	59	84.8839	84	82.3241
10	88.0006	35	86.4142	60	85.4176	85	81.5316
11	86.2597	36	86.0498	61	84.2309	86	81.7280
12	85.8286	37	86.6642	62	83.5761	87	82.5375
13	83.7500	38	84.7289	63	84.1343	88	82.3877
14	84.4628	39	85.9523	64	82.6974	89	82.4159
15	84.6476	40	86.8473	65	83.5454	90	82.2102
16	84.5751	41	88.4250	66	86.4714	91	82.7673
17	82.2473	42	89.6481	67	86.2143	92	83.1234
18	83.3774	43	87.8566	68	87.0215	93	83.2203
19	83.5385	44	88.4997	69	86.6504	94	84.4510
20	85.1620	45	87.0622	70	85.7082	95	84.9145
21	83.7881	46	85.1973	71	86.1504	96	85.7609
22	84.0421	47	85.0767	72	85.8032	97	85.2302
23	84.1023	48	84.4362	73	85.6197	98	86.7312
24	84.8495	49	84.2112	74	84.2339	99	87.0048
25	87.6416	50	85.9952	75	83.5737	100	85.0572

TABLE B.4 US Production of Blue and Gorgonzola Cheeses

Year	Production (10^3 lb)	Year	Production (10^3 lb)
1950	7,657	1974	28,262
1951	5,451	1975	28,506
1952	10,883	1976	33,885
1953	9,554	1977	34,776
1954	9,519	1978	35,347
1955	10,047	1979	34,628
1956	10,663	1980	33,043
1957	10,864	1981	30,214
1958	11,447	1982	31,013
1959	12,710	1983	31,496
1960	15,169	1984	34,115
1961	16,205	1985	33,433
1962	14,507	1986	34,198
1963	15,400	1987	35,863
1964	16,800	1988	37,789
1965	19,000	1989	34,561
1966	20,198	1990	36,434
1967	18,573	1991	34,371
1968	19,375	1992	33,307
1969	21,032	1993	33,295
1970	23,250	1994	36,514
1971	25,219	1995	36,593
1972	28,549	1996	38,311
1973	29,759	1997	42,773

Source: http://www.nass.usda.gov/QuickStats/.

TABLE B.5 US Beverage Manufacturer Product Shipments, Unadjusted

Month	Dollars (In Millions)	Month	Dollars (In Millions)	Month	Dollars (In Millions)	Month	Dollars (In Millions)
Jan-1992	3519	Oct-1995	4681	Jul-1999	5339	Apr-2003	5576
Feb-1992	3803	Nov-1995	4466	Aug-1999	5474	May-2003	6160
Mar-1992	4332	Dec-1995	4463	Sep-1999	5278	Jun-2003	6121
Apr-1992	4251	Jan-1996	4217	Oct-1999	5184	Jul-2003	5900
May-1992	4661	Feb-1996	4322	Nov-1999	4975	Aug-2003	5994
Jun-1992	4811	Mar-1996	4779	Dec-1999	4751	Sep-2003	5841
Jul-1992	4448	Apr-1996	4988	Jan-2000	4600	Oct-2003	5832
Aug-1992	4451	May-1996	5383	Feb-2000	4718	Nov-2003	5505
Sep-1992	4343	Jun-1996	5591	Mar-2000	5218	Dec-2003	5573
Oct-1992	4067	Jul-1996	5322	Apr-2000	5336	Jan-2004	5331
Nov-1992	4001	Aug-1996	5404	May-2000	5665	Feb-2004	5355
Dec-1992	3934	Sep-1996	5106	Jun-2000	5900	Mar-2004	6057
Jan-1993	3652	Oct-1996	4871	Jul-2000	5330	Apr-2004	6055
Feb-1993	3768	Nov-1996	4977	Aug-2000	5626	May-2004	6771
Mar-1993	4082	Dec-1996	4706	Sep-2000	5512	Jun-2004	6669
Apr-1993	4101	Jan-1997	4193	Oct-2000	5293	Jul-2004	6375
May-1993	4628	Feb-1997	4460	Nov-2000	5143	Aug-2004	6666
Jun-1993	4898	Mar-1997	4956	Dec-2000	4842	Sep-2004	6383
Jul-1993	4476	Apr-1997	5022	Jan-2001	4627	Oct-2004	6118
Aug-1993	4728	May-1997	5408	Feb-2001	4881	Nov-2004	5927
Sep-1993	4458	Jun-1997	5565	Mar-2001	5321	Dec-2004	5750
Oct-1993	4004	Jul-1997	5360	Apr-2001	5290	Jan-2005	5122

Nov-1993	4095	Aug-1997	5490	May-2001	6002	Feb-2005	5398
Dec-1993	4056	Sep-1997	5286	Jun-2001	5811	Mar-2005	5817
Jan-1994	3641	Oct-1997	5257	Jul-2001	5671	Apr-2005	6163
Feb-1994	3966	Nov-1997	5002	Aug-2001	6102	May-2005	6763
Mar-1994	4417	Dec-1997	4897	Sep-2001	5482	Jun-2005	6835
Apr-1994	4367	Jan-1998	4577	Oct-2001	5429	Jul-2005	6678
May-1994	4821	Feb-1998	4764	Nov-2001	5356	Aug-2005	6821
Jun-1994	5190	Mar-1998	5052	Dec-2001	5167	Sep-2005	6421
Jul-1994	4638	Apr-1998	5251	Jan-2002	4608	Oct-2005	6338
Aug-1994	4904	May-1998	5558	Feb-2002	4889	Nov-2005	6265
Sep-1994	4528	Jun-1998	5931	Mar-2002	5352	Dec-2005	6291
Oct-1994	4383	Jul-1998	5476	Apr-2002	5441	Jan-2006	5540
Nov-1994	4339	Aug-1998	5603	May-2002	5970	Feb-2006	5822
Dec-1994	4327	Sep-1998	5425	Jun-2002	5750	Mar-2006	6318
Jan-1995	3856	Oct-1998	5177	Jul-2002	5670	Apr-2006	6268
Feb-1995	4072	Nov-1998	4792	Aug-2002	5860	May-2006	7270
Mar-1995	4563	Dec-1998	4776	Sep-2002	5449	Jun-2006	7096
Apr-1995	4561	Jan-1999	4450	Oct-2002	5401	Jul-2006	6505
May-1995	4984	Feb-1999	4659	Nov-2002	5240	Aug-2006	7039
Jun-1995	5316	Mar-1999	5043	Dec-2002	5229	Sep-2006	6440
Jul-1995	4843	Apr-1999	5233	Jan-2003	4770	Oct-2006	6446
Aug-1995	5383	May-1999	5423	Feb-2003	5006	Nov-2006	6717
Sep-1995	4889	Jun-1999	5814	Mar-2003	5518	Dec-2006	6320

Source: http://www.census.gov/indicator/www/m3/nist/nalcshist2.htm.

TABLE B.6 Global Mean Surface Air Temperature Anomaly and Global CO_2 Concentration

Year	Anomaly (°C)	CO_2 (ppmv)	Year	Anomaly (°C)	CO_2 (ppmv)	Year	Anomaly (°C)	CO_2 (ppmv)
1880	−0.11	290.7	1922	−0.09	303.8	1964	−0.25	319.2
1881	−0.13	291.2	1923	−0.16	304.1	1965	−0.15	320.0
1882	−0.01	291.7	1924	−0.11	304.5	1966	−0.07	321.1
1883	−0.04	292.1	1925	−0.15	305.0	1967	−0.02	322.0
1884	−0.42	292.6	1926	0.04	305.4	1968	−0.09	322.9
1885	−0.23	293.0	1927	−0.05	305.8	1969	0.00	324.2
1886	−0.25	293.3	1928	0.01	306.3	1970	0.04	325.2
1887	−0.45	293.6	1929	−0.22	306.8	1971	−0.10	326.1
1888	−0.23	293.8	1930	−0.03	307.2	1972	−0.05	327.2
1889	0.04	294.0	1931	0.03	307.7	1973	0.18	328.8
1890	−0.22	294.2	1932	0.04	308.2	1974	−0.06	329.7
1891	−0.55	294.3	1933	−0.11	308.6	1975	−0.02	330.7
1892	−0.40	294.5	1934	0.05	309.0	1976	−0.21	331.8
1893	−0.39	294.6	1935	−0.08	309.4	1977	0.16	333.3
1894	−0.32	294.7	1936	0.01	309.8	1978	0.07	334.6
1895	−0.32	294.8	1937	0.12	310.0	1979	0.13	336.9
1896	−0.27	294.9	1938	0.15	310.2	1980	0.27	338.7
1897	−0.15	295.0	1939	−0.02	310.3	1981	0.40	339.9
1898	−0.21	295.2	1940	0.14	310.4	1982	0.10	341.1
1899	−0.25	295.5	1941	0.11	310.4	1983	0.34	342.8

1900	−0.05	295.8	1942	0.10	310.3	1984	0.16	344.4
1901	−0.05	296.1	1943	0.06	310.2	1985	0.13	345.9
1902	−0.30	296.5	1944	0.10	310.1	1986	0.19	347.2
1903	−0.35	296.8	1945	−0.01	310.1	1987	0.35	348.9
1904	−0.42	297.2	1946	0.01	310.1	1988	0.42	351.5
1905	−0.25	297.6	1947	0.12	310.2	1989	0.28	352.9
1906	−0.15	298.1	1948	−0.03	310.3	1990	0.49	354.2
1907	−0.41	298.5	1949	−0.09	310.5	1991	0.44	355.6
1908	−0.30	298.9	1950	−0.17	310.7	1992	0.16	356.4
1909	−0.31	299.3	1951	−0.02	311.1	1993	0.18	357.0
1910	−0.21	299.7	1952	0.03	311.5	1994	0.31	358.9
1911	−0.25	300.1	1953	0.12	311.9	1995	0.47	360.9
1912	−0.33	300.4	1954	−0.09	312.4	1996	0.36	362.6
1913	−0.28	300.8	1955	−0.09	313.0	1997	0.40	363.8
1914	−0.02	301.1	1956	−0.18	313.6	1998	0.71	366.6
1915	0.06	301.4	1957	0.08	314.2	1999	0.43	368.3
1916	−0.20	301.7	1958	0.10	314.9	2000	0.41	369.5
1917	−0.46	302.1	1959	0.05	315.8	2001	0.56	371.0
1918	−0.33	302.4	1960	−0.02	316.6	2002	0.70	373.1
1919	−0.09	302.7	1961	0.10	317.3	2003	0.66	375.6
1920	−0.15	303.0	1962	0.05	318.1	2004	0.60	377.4
1921	−0.04	303.4	1963	0.03	318.7			

Source: http://data.giss.nasa.gov.gistemp/.

TABLE B.7 Whole Foods Market Stock Price, Daily Closing Adjusted for Splits

Date[a]	Dollars	Date	Dollars	Date	Dollars	Date	Dollars	Date	Dollars
1/2/01	28.05	3/15/01	22.01	5/25/01	27.88	8/7/01	32.24	10/23/01	35.20
1/3/01	28.23	3/16/01	22.26	5/29/01	27.78	8/8/01	31.60	10/24/01	35.30
1/4/01	26.25	3/19/01	22.35	5/30/01	28.03	8/9/01	31.78	10/25/01	35.65
1/5/01	25.41	3/20/01	23.06	5/31/01	28.36	8/10/01	32.99	10/26/01	35.96
1/8/01	26.25	3/21/01	22.78	6/1/01	28.31	8/13/01	32.69	10/29/01	35.86
1/9/01	26.03	3/22/01	22.19	6/4/01	27.58	8/14/01	33.31	10/30/01	35.61
1/10/01	26.09	3/23/01	22.19	6/5/01	27.43	8/15/01	32.78	10/31/01	34.42
1/11/01	26.28	3/26/01	22.66	6/6/01	27.16	8/16/01	32.78	11/1/01	34.55
1/12/01	26.00	3/27/01	22.50	6/7/01	27.92	8/17/01	32.82	11/2/01	35.43
1/16/01	25.63	3/28/01	21.36	6/8/01	27.36	8/20/01	33.04	11/5/01	34.92
1/17/01	25.57	3/29/01	20.71	6/11/01	27.17	8/21/01	33.79	11/6/01	35.56
1/18/01	25.57	3/30/01	20.86	6/12/01	27.39	8/22/01	32.69	11/7/01	35.85
1/19/01	25.16	4/2/01	20.95	6/13/01	27.58	8/23/01	32.40	11/8/01	36.89
1/22/01	26.52	4/3/01	20.12	6/14/01	27.55	8/24/01	32.91	11/9/01	37.24
1/23/01	27.18	4/4/01	19.50	6/15/01	27.49	8/27/01	33.38	11/12/01	37.01
1/24/01	26.93	4/5/01	20.30	6/18/01	27.70	8/28/01	34.72	11/13/01	37.52
1/25/01	26.50	4/6/01	20.09	6/19/01	27.19	8/29/01	35.22	11/14/01	37.24
1/26/01	26.50	4/9/01	20.38	6/20/01	26.76	8/30/01	34.77	11/15/01	40.36
1/29/01	27.27	4/10/01	21.13	6/21/01	26.53	8/31/01	34.85	11/16/01	39.42
1/30/01	27.70	4/11/01	20.63	6/22/01	26.45	9/4/01	33.91	11/19/01	40.16
1/31/01	28.17	4/12/01	20.35	6/25/01	25.97	9/5/01	34.39	11/20/01	42.64
2/1/01	28.26	4/16/01	20.39	6/26/01	26.11	9/6/01	34.49	11/21/01	41.86
2/2/01	28.29	4/17/01	20.95	6/27/01	26.50	9/7/01	34.37	11/23/01	42.58
2/5/01	28.23	4/18/01	21.94	6/28/01	26.98	9/10/01	33.44	11/26/01	42.63

2/6/01	28.54	4/19/01	21.43	6/29/01	26.84	9/17/01	33.24	11/27/01	42.14
2/7/01	28.94	4/20/01	21.37	7/2/01	28.03	9/18/01	33.18	11/28/01	41.62
2/8/01	28.51	4/23/01	21.24	7/3/01	28.00	9/19/01	31.26	11/29/01	42.59
2/9/01	27.55	4/24/01	21.13	7/5/01	28.01	9/20/01	31.04	11/30/01	42.50
2/12/01	28.05	4/25/01	22.36	7/6/01	27.20	9/21/01	30.33	12/3/01	42.38
2/13/01	27.98	4/26/01	22.93	7/9/01	27.92	9/24/01	30.69	12/4/01	42.77
2/14/01	23.55	4/27/01	23.26	7/10/01	27.10	9/25/01	30.84	12/5/01	43.80
2/15/01	24.21	4/30/01	24.07	7/11/01	27.15	9/26/01	29.95	12/6/01	45.13
2/16/01	23.92	5/1/01	23.79	7/12/01	27.19	9/27/01	29.22	12/7/01	45.40
2/20/01	23.77	5/2/01	24.56	7/13/01	26.69	9/28/01	31.11	12/10/01	43.81
2/21/01	23.74	5/3/01	24.43	7/16/01	26.79	10/1/01	30.93	12/11/01	42.16
2/22/01	23.55	5/4/01	24.29	7/17/01	27.17	10/2/01	30.98	12/12/01	41.24
2/23/01	23.34	5/7/01	23.33	7/18/01	26.72	10/3/01	32.59	12/13/01	40.91
2/26/01	23.22	5/8/01	25.20	7/19/01	26.33	10/4/01	32.50	12/14/01	41.05
2/27/01	22.87	5/9/01	24.94	7/20/01	26.23	10/5/01	32.12	12/17/01	41.13
2/28/01	21.36	5/10/01	24.95	7/23/01	26.59	10/8/01	32.09	12/18/01	41.55
3/1/01	21.30	5/11/01	25.25	7/24/01	26.82	10/9/01	32.85	12/19/01	41.35
3/2/01	21.51	5/14/01	25.70	7/25/01	27.24	10/10/01	33.44	12/20/01	41.27
3/5/01	21.32	5/15/01	26.33	7/26/01	28.49	10/11/01	32.68	12/21/01	42.46
3/6/01	21.67	5/16/01	27.81	7/27/01	31.65	10/12/01	32.54	12/24/01	42.96
3/7/01	21.48	5/17/01	28.04	7/30/01	34.47	10/15/01	32.07	12/26/01	43.63
3/8/01	21.85	5/18/01	28.75	7/31/01	33.63	10/16/01	33.18	12/27/01	43.63
3/9/01	21.49	5/21/01	28.72	8/1/01	32.58	10/17/01	33.45	12/28/01	43.59
3/12/01	21.48	5/22/01	28.33	8/2/01	32.62	10/18/01	34.35	12/31/01	43.14
3/13/01	22.10	5/23/01	27.61	8/3/01	32.09	10/19/01	33.95		
3/14/01	21.79	5/24/01	27.98	8/6/01	32.41	10/22/01	34.42		

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[a] Date: Month/Day/Year.

TABLE B.8 Unemployment Rate—Full-Time Labor Force, Not Seasonally Adjusted

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
Jan-1963	6.8	Jan-1970	3.8	Jan-1977	7.9	Jan-1984	8.9	Jan-1991	7.0	Jan-1998	5.0
Feb-1963	6.8	Feb-1970	4.3	Feb-1977	8.2	Feb-1984	8.5	Feb-1991	7.4	Feb-1998	4.8
Mar-1963	6.2	Mar-1970	4.2	Mar-1977	7.6	Mar-1984	8.3	Mar-1991	7.1	Mar-1998	4.8
Apr-1963	5.6	Apr-1970	4.1	Apr-1977	6.7	Apr-1984	7.8	Apr-1991	6.6	Apr-1998	4.0
May-1963	5.4	May-1970	4.2	May-1977	6.4	May-1984	7.4	May-1991	6.7	May-1998	4.2
Jun-1963	6.0	Jun-1970	5.5	Jun-1977	7.3	Jun-1984	7.4	Jun-1991	7.0	Jun-1998	4.6
Jul-1963	5.4	Jul-1970	5.1	Jul-1977	6.9	Jul-1984	7.6	Jul-1991	6.9	Jul-1998	4.6
Aug-1963	4.9	Aug-1970	4.7	Aug-1977	6.6	Aug-1984	7.2	Aug-1991	6.5	Aug-1998	4.3
Sep-1963	4.3	Sep-1970	4.5	Sep-1977	6.0	Sep-1984	6.8	Sep-1991	6.3	Sep-1998	4.1
Oct-1963	4.4	Oct-1970	4.5	Oct-1977	5.9	Oct-1994	6.9	Oct-1991	6.3	Oct-1998	3.9
Nov-1963	4.9	Nov-1970	4.9	Nov-1977	5.9	Nov-1984	6.8	Nov-1991	6.6	Nov-1998	3.8
Dec-1963	5.1	Dec-1970	5.2	Dec-1977	5.7	Dec-1984	7.1	Dec-1991	7.0	Dec-1998	3.9
Jan-1964	6.2	Jan-1971	6.1	Jan-1978	6.7	Jan-1985	8.0	Jan-1992	8.1	Jan-1999	4.6
Feb-1964	6.1	Feb-1971	6.2	Feb-1978	6.6	Feb-1985	7.9	Feb-1992	8.3	Feb-1999	4.6
Mar-1964	5.7	Mar-1971	5.9	Mar-1978	6.2	Mar-1985	7.5	Mar-1992	7.9	Mar-1999	4.3
Apr-1964	5.1	Apr-1971	5.4	Apr-1978	5.5	Apr-1985	7.1	Apr-1992	7.4	Apr-1999	4.0
May-1964	4.7	May-1971	5.2	May-1978	5.5	May-1985	7.0	May-1992	7.4	May-1999	3.9
Jun-1964	5.7	Jun-1971	6.4	Jun-1978	6.1	Jun-1985	7.5	Jun-1992	8.0	Jun-1999	4.3
Jul-1964	4.7	Jul-1971	6.0	Jul-1978	6.1	Jul-1985	7.4	Jul-1992	7.8	Jul-1999	4.4
Aug-1964	4.5	Aug-1971	5.6	Aug-1978	5.5	Aug-1985	6.8	Aug-1992	7.3	Aug-1999	4.1
Sep-1964	4.0	Sep-1971	5.1	Sep-1978	5.1	Sep-1985	6.6	Sep-1992	7.0	Sep-1999	3.8
Oct-1964	4.0	Oct-1971	4.8	Oct-1978	4.8	Oct-1985	6.5	Oct-1992	6.7	Oct-1999	3.7
Nov-1964	4.0	Nov-1971	5.1	Nov-1978	5.0	Nov-1986	6.6	Nov-1992	7.0	Nov-1999	3.6

Dec-1964	4.3	Dec-1971	5.2	Dec-1978	5.2	Dec-1985	6.6	Dec-1992	7.1	Dec-1999	3.7
Jan-1965	5.3	Jan-1972	6.1	Jan-1979	5.9	Jan-1986	7.3	Jan-1993	7.9	Jan-2000	4.3
Feb-1965	5.5	Feb-1972	6.0	Feb-1979	6.1	Feb-1986	7.8	Feb-1993	7.9	Feb-2000	4.2
Mar-1965	4.9	Mar-1972	5.8	Mar-1979	5.7	Mar-1986	7.5	Mar-1993	7.5	Mar-2000	4.1
Apr-1965	4.5	Apr-1972	5.2	Apr-1979	5.3	Apr-1986	7.0	Apr-1993	6.9	Apr-2000	3.5
May-1965	4.1	May-1972	5.1	May-1979	5.0	May-1986	7.1	May-1993	6.9	May-2000	3.7
Jun-1965	5.1	Jun-1972	6.0	Jun-1979	5.9	Jun-1986	7.3	Jun-1993	7.2	Jun-2000	4.0
Jul-1965	4.2	Jul-1972	5.7	Jul-1979	5.7	Jul-1986	7.0	Jul-1993	7.1	Jul-2000	4.0
Aug-1965	3.9	Aug-1972	5.2	Aug-1979	5.5	Aug-1986	6.4	Aug-1993	6.5	Aug-2000	3.9
Sep-1965	3.4	Sep-1972	4.6	Sep-1979	5.1	Sep-1986	6.5	Sep-1993	6.2	Sep-2000	3.5
Oct-1965	3.2	Oct-1972	4.5	Oct-1979	5.1	Oct-1986	6.3	Oct-1993	6.1	Oct-2000	3.5
Nov-1965	3.3	Nov-1972	4.2	Nov-1979	5.2	Nov-1986	6.4	Nov-1993	6.0	Nov-2000	3.5
Dec-1965	3.4	Dec-1972	4.2	Dec-1979	5.3	Dec-1986	6.3	Dec-1993	6.2	Dec-2000	3.6
Jan-1966	4.1	Jan-1973	5.1	Jan-1980	6.5	Jan-1987	7.2	Jan-1994	7.5	Jan-2001	4.5
Feb-1966	4.0	Feb-1973	5.2	Feb-1980	6.5	Feb-1987	7.1	Feb-1994	7.4	Feb-2001	4.4
Mar-1966	3.8	Mar-1973	4.9	Mar-1980	6.4	Mar-1987	6.7	Mar-1994	7.0	Mar-2001	4.4
Apr-1966	3.5	Apr-1973	4.4	Apr-1980	6.6	Apr-1987	6.1	Apr-1994	6.3	Apr-2001	4.0
May-1966	3.4	May-1973	4.2	May-1980	7.2	May-1987	6.1	May-1994	5.9	May-2001	4.1
Jun-1966	4.3	Jun-1973	5.1	Jun-1980	8.0	Jun-1987	6.4	Jun-1994	6.3	Jun-2001	4.6
Jul-1966	3.7	Jul-1973	4.7	Jul-1980	8.1	Jul-1987	6.1	Jul-1994	6.4	Jul-2001	4.6
Aug-1966	3.2	Aug-1973	4.3	Aug-1980	7.6	Aug-1987	5.6	Aug-1994	5.8	Aug-2001	4.7
Sep-1966	2.9	Sep-1973	3.9	Sep-1980	6.9	Sep-1987	5.3	Sep-1994	5.5	Sep-2001	4.7
Oct-1966	2.8	Oct-1973	3.6	Oct-1980	6.8	Oct-1987	5.3	Oct-1994	5.3	Oct-2001	4.9
Nov-1966	3.0	Nov-1973	4.0	Nov-1980	7.0	Nov-1987	5.4	Nov-1994	5.2	Nov-2001	5.2
Dec-1966	3.1	Dec-1973	4.1	Dec-1980	7.0	Dec-1987	5.3	Dec-1994	5.0	Dec-2001	5.5
Jan-1967	3.8	Jan-1974	5.2	Jan-1981	8.0	Jan-1988	6.1	Jan-1995	6.1	Jan-2002	6.5

(*continued*)

TABLE B.8 (***Continued***)

Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)	Month	Rate (%)
Feb-1967	3.6	Feb-1974	5.3	Feb-1981	8.0	Feb-1988	6.1	Feb-1995	5.8	Feb-2002	6.3
Mar-1967	3.5	Mar-1974	5.0	Mar-1981	7.6	Mar-1988	5.8	Mar-1995	5.7	Mar-2002	6.2
Apr-1967	3.2	Apr-1974	4.6	Apr-1981	7.0	Apr-1988	5.2	Apr-1995	5.5	Apr-2002	5.9
May-1967	3.0	May-1974	4.5	May-1981	7.2	May-1988	5.4	May-1995	5.4	May-2002	5.7
Jun-1967	4.3	Jurv1974	5.6	Jun-1981	7.8	Jun-1988	5.4	Jun-1995	5.7	Jun-2002	5.2
Jul-1967	3.7	Jul-1974	5.4	Jul-1981	7.4	Jul-1988	5.4	Jul-1995	5.8	Jul-2002	6.0
Aug-1967	3.4	Aug-1974	4.9	Aug-1981	7.0	Aug-1988	5.2	Aug-1995	5.5	Aug-2002	5.6
Sep-1967	3.1	Sep-1974	4.9	Sep-1981	6.9	Sep-1988	4.8	Sep-1995	5.2	Sep-2002	5.3
Oct-1967	3.1	Oct-1974	5.0	Oct-1981	7.3	Oct-1988	4.7	Oct-1995	5.0	Oct-2002	5.4
Nov-1967	3.0	Nov-1974	5.7	Nov-1981	7.8	Nov-1988	4.9	Nov-1995	5.1	Nov-2002	5.7
Dec-1967	3.0	Dec-1974	6.3	Dec-1981	8.5	Dec-1988	5.0	Dec-1995	5.2	Dec-2002	5.9
Jan-1968	3.7	Jan-1975	8.7	Jan-1982	9.5	Jan-1989	5.7	Jan-1996	6.2	Jan-2003	6.6
Feb-1968	3.8	Feb-1975	9.0	Feb-1982	9.6	Feb-1989	5.5	Feb-1996	5.9	Feb-2003	6.5
Mar-1968	3.4	Mar-1975	9.1	Mar-1982	9.7	Mar-1989	5.2	Mar-1996	5.8	Mar-2003	6.3
Apr-1968	2.9	Apr-1975	8.7	Apr-1982	9.4	Apr-1989	5.0	Apr-1996	5.3	Apr-2003	6.0
May-1968	2.7	May-1975	8.6	May-1982	9.5	May-1989	5.0	May-1996	5.3	May-2003	6.0
Jun-1968	4.2	Jun-1975	9.3	Jun-1982	10.3	Jun-1989	5.3	Jun-1996	5.4	Jun-2003	6.6

Jul-1968	3.7	Jul-1975	8.7	Jul-1982	10.1	Jul-1989	5.3	Jul-1996	5.6	Jul-2003	6.4
Aug-1966	3.1	Aug-1975	7.9	Aug-1982	9.8	Aug-1989	4.9	Aug-1996	4.9	Aug-2003	6.1
Sep-1968	2.7	Sep-1975	7.6	Sep-1982	9.7	Sep-1969	4.7	Sep-1996	4.8	Sep-2003	5.7
Oct-1968	2.7	Oct-1975	7.4	Oct-1982	10.1	Oct-1969	4.6	Oct-1996	4.7	Oct-2003	5.6
Nov-1968	2.6	Nov-1975	7.5	Nov-1962	10.6	Nov-1989	4.9	Nov-1996	4.9	Nov-2003	5.7
Dec-1968	2.5	Dec-1975	7.5	Dec-1962	11.0	Dec-1989	4.9	Dec-1996	4.9	Dec-2003	5.6
Jan-1969	3.3	Jan-1976	8.6	Jan-1983	11.9	Jan-1990	5.8	Jan-1997	5.8	Jan-2004	6.3
Feb-1969	3.3	Feb-1976	8.4	Feb-1983	11.9	Feb-1990	5.6	Feb-1997	5.5	Feb-2004	6.1
Mar-1969	3.1	Mar-1976	8.0	Mar-1983	11.4	Mar-1990	5.4	Mar-1997	5.4	Mar-2004	6.2
Apr-1969	2.9	Apr-1976	7.2	Apr-1983	10.6	Apr-1990	5.2	Apr-1997	4.7	Apr-2004	5.4
May-1969	2.7	May-1976	6.8	May-1983	10.3	May-1990	5.1	May-1997	4.7	May-2004	5.5
Jun-1969	4.0	Jun-1976	8.1	Jun-1983	10.6	Jun-1990	5.3	Jun-1997	5.1	Jun-2004	5.8
Jul-1969	3.6	Jul-1976	7.6	Jul-1983	9.8	Jul-1990	5.4	JuL-1997	5.0	Jul-2004	5.8
Aug-1969	3.1	Aug-1976	7.3	Aug-1983	9.4	Aug-1990	5.2	Aug-1997	4.6	Aug-2004	5.3
Sep-1969	3.0	Sep-1976	6.9	Sep-1983	8.8	Sep-1990	5.3	Sep-1997	4.5	Sep-2004	5.1
Oct-1969	2.8	Oct-1976	6.7	Oct-1983	8.4	Oct-1990	5.2	Oct-1997	4.2	Oct-2004	5.0
Nov-1969	2.7	Nov-1976	7.0	Nov-1983	8.2	Nov-1990	5.7	Nov-1997	4.1	Nov-2004	5.1
Dec-1969	2.8	Dec-1976	7.2	Dec-1983	6.2	Dec-1990	6.0	Dec-1997	4.3	Dec-2004	5.2

Source: http://data.bls.gov/cgi-bin/srgate.

TABLE B.9 International Sunspot Numbers

Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number
1700	5.1	1761	86	1622	4.1	1883	63.8	1944	9.7
1701	11.1	1762	61.3	1623	1.9	1884	63.6	1945	33.3
1702	16.1	1763	45.2	1824	8.6	1885	52.3	1946	92.7
1703	23.1	1764	36.5	1825	16.7	1886	25.5	1947	151.7
1704	36.1	1765	21	1826	36.4	1887	13.2	1948	136.4
1705	58.1	1766	11.5	1827	49.7	1888	6.9	1949	134.8
1706	29.1	1767	37.9	1828	64.3	1889	64	1950	84
1707	20.1	1768	69.9	1829	67.1	1890	7.2	1951	69.5
1708	10.1	1769	106.2	1830	71	1891	35.7	1952	31.6
1709	8.1	1770	100.9	1831	47.9	1892	73.1	1953	14
1710	3.1	1771	81.7	1832	27.6	1893	85.2	1954	4.5
1711	0.1	1772	66.6	1833	8.6	1894	78.1	1955	38.1
1712	0.1	1773	34.9	J834	13.3	1895	64.1	1956	141 .8
1713	2.1	1774	30.7	1835	57	1896	41 .9	1957	190.3
1714	11.1	1775	7.1	1836	121.6	1897	26.3	1958	184.9
1715	27.1	1776	19.9	1837	138.4	1898	26.8	1959	159.1
1716	47.1	1777	92.6	1838	103.3	1899	12.2	1960	112.4
1717	63.1	1778	154. 5	1839	85.8	1900	9.6	1961	54
1718	60.1	1779	126	1840	64.7	1901	2.8	1962	37.7
1719	39.1	1780	84.9	1841	36.8	1902	5.1	1963	28
1720	28.1	1781	68.2	1842	24.3	1903	24.5	1964	10.3
1721	26.1	1782	38.6	1843	10.8	1904	42.1	1965	152
1722	22.1	1783	22.9	1844	15.1	1905	63.6	1966	47.1
1723	11.1	1784	10.3	1845	40.2	1906	53.9	1967	93.8
1724	21.1	1785	24.2	1846	61.6	1907	62.1	1966	106
1725	40.1	1786	83	1847	98.6	1908	48.6	1969	105.6
1726	78.1	1787	132.1	1848	124. 8	1909	44	1970	104.6
1727	122.1	1788	131	1849	96.4	1910	18.7	1971	66.7
1728	103.1	1789	118.2	1850	66.7	1911	5.8	1972	69
1729	73.1	1790	90	1851	64.6	1912	3.7	1973	38.1
1730	47.1	1791	66.7	1852	54.2	1913	1.5	1974	34.6
1731	35.1	1792	60.1	1853	39.1	1914	9.7	1975	15.6
1732	11.1	1793	47	1854	20.7	1915	47.5	1976	12.7
1733	5.1	1794	41.1	1855	6.8	1916	57.2	1977	27.6
1734	16.1	1795	21.4	1856	4.4	1917	104	1978	92.6
1735	34.1	1796	16.1	1857	22.8	1918	80.7	1979	155.5
1736	70.1	1797	6.5	1858	54.9	1919	63.7	1980	154.7
1737	81.1	1798	4.2	1859	93.9	1920	37.7	1981	140.6
1738	111.1	1799	6.9	1860	95.9	1921	26.2	1982	116

TABLE B.9 (***Continued***)

Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number	Year	Sunspot Number
1739	101.1	1800	14. 6	1861	77.3	1922	14.3	1983	66.7
1740	73.1	1801	34.1	1862	59.2	1923	59	1984	46
1741	40.1	1802	45.1	1863	44.1	1924	16.8	1985	18
1742	20.1	1803	43.2	1864	47.1	1925	44.4	1986	13.5
1743	16.1	1804	47.6	1865	30.6	1926	64	1987	29.3
1744	5.1	1805	42.3	1866	16.4	1927	69.1	1988	100.3
1745	11.1	1606	28.2	1867	7.4	1928	77.9	1989	157.7
1746	22.1	1807	10.2	1868	37.7	1929	65	1990	142.7
1747	40.1	1808	8.2	1869	74.1	1930	35.8	1991	145.8
1748	60.1	1809	2.6	1870	139.1	1931	21.3	1992	94.4
1749	81	1810	0.1	1871	111.3	1932	11.2	1993	54.7
1750	83.5	1811	1.5	1872	101.7	1933	5.8	1994	30
1751	47.8	1812	5.1	1873	66.3	1934	8.8	1995	17.6
1752	47.9	1813	12.3	1874	44.8	1935	36.2	1996	8.7
1753	30.8	1814	14	1875	17.1	1936	79.8	1997	21.6
1754	12.3	1815	35.5	1876	11.4	1937	114.5	1998	64.4
1755	9.7	1816	45.9	1877	12.5	1938	109.7	1999	93.4
1756	10.3	1817	41.1	1878	3.5	1939	88.9	2000	119.7
1757	32.5	1818	30.2	1879	6.1	1940	67.9	2001	111.1
1758	47.7	1819	24	1880	32.4	1941	47.6	2002	104.1
1759	54.1	1620	15.7	1881	54.4	1942	30.7	2003	63.8
1760	63	1821	6.7	1882	59.8	1943	16.4	2004	40.5

Source: http://sidc.oma.be/html/sunspot.html (yearly sunspot number).

TABLE B.10 United Kingdom Airline Miles Flown

Month	Miles (In Millions)	Month	Miles (In Millions)
Jan-1964	7.269	Jul-1967	12.222
Feb-1964	6.775	Aug-1967	12.246
Mar-1964	7.819	Sep-1967	13.281
Apr-1964	8.371	Oct-1967	10.366
May-1964	9.069	Nov-1967	8.730
Jun-1964	10.248	Dec-1967	9.614
Jul-1964	11.030	Jan-1968	8.639
Aug-1964	10.882	Feb-1968	8.772
Sep-1964	10.333	Mar-1968	10.894
Oct-1964	9.109	Apr-1968	10.455
Nov-1964	7.685	May-1968	11.179
Dec-1964	7.682	Jun-1968	10.588
Jan-1965	8.350	Jul-1968	10.794
Feb-1965	7.829	Aug-1968	12.770
Mar-1965	8.829	Sep-1968	13.812
Apr-1965	9.948	Oct-1968	10.857
May-1965	10.638	Nov-1968	9.290
Jun-1965	11.253	Dec-1968	10.925
Jul-1965	11.424	Jan-1969	9.491
Aug-1965	11.391	Feb-1969	8.919
Sep-1965	10.665	Mar-1969	11.607
Oct-1965	9.396	Apr-1969	8.852
Nov-1965	7.775	May-1969	12.537
Dec-1965	7.933	Jun-1969	14.759
Jan-1966	8.186	Jul-1969	13.667
Feb-1966	7.444	Aug-1969	13.731
Mar-1966	8.484	Sep-1969	15.110
Apr-1966	9.864	Oct-1969	12.185
May-1966	10.252	Nov-1969	10.645
Jun-1966	12.282	Dec-1969	12.161
Jul-1966	11.637	Jan-1970	10.840
Aug-1966	11.577	Feb-1970	10.436
Sep-1966	12.417	Mar-1970	13.589
Oct-1966	9.637	Apr-1970	13.402
Nov-1966	8.094	May-1970	13.103
Dec-1966	9.280	Jun-1970	14.933
Jan-1967	8.334	Jul-1970	14.147
Feb-1967	7.899	Aug-1970	14.057
Mar-1967	9.994	Sep-1970	16.234
Apr-1967	10.078	Oct-1970	12.389
May-1967	10.801	Nov-1970	11.594
Jun-1967	12.953	Dec-1970	12.772

Source: Adapted from Montgomery, Johnson, and Gardner (1990), with permission of the publisher.

TABLE B.11 Champagne Sales

Month	Sales (In Thousands of Bottles)	Month	Sales (In Thousands Bottles)	Month	Sales (In Thousands Bottles)
Jan-1962	2.851	Sep-1964	3.528	May-1967	4.968
Feb-1962	2.672	Oct-1964	5.211	Jun-1967	4.677
Mar-1962	2.755	Nov-1964	7.614	Jul-1967	3.523
Apr-1962	2.721	Dec-1964	9.254	Aug-1967	1.821
May-1962	2.946	Jan-1965	5.375	Sep-1967	5.222
Jun-1962	3.036	Feb-1965	3.088	Oct-1967	6.873
Jul-1962	2.282	Mar-1965	3.718	Nov-1967	10.803
Aug-1962	2.212	Apr-1965	4.514	Dec-1967	13.916
Sep-1962	2.922	May-1965	4.520	Jan-1968	2.639
Oct-1962	4.301	Jun-1965	4.539	Feb-1968	2.899
Nov-1962	5.764	Jul-1965	3.663	Mar-1968	3.370
Dec-1962	7.132	Aug-1965	1.643	Apr-1968	3.740
Jan-1963	2.541	Sep-1965	4.739	May-1968	2.927
Feb-1963	2.475	Oct-1965	5.428	Jun-1968	3.986
Mar-1963	3.031	Nov-1965	8.314	Jul-1968	4.217
Apr-1963	3.266	Dec-1965	10.651	Aug-1968	1.738
May-1963	3.776	Jan-1966	3.633	Sep-1968	5.221
Jun-1963	3.230	Feb-1966	4.292	Oct-1968	6.424
Jul-1963	3.028	Mar-1966	4.154	Nov-1968	9.842
Aug-1963	1.759	Apr-1966	4.121	Dec-1968	13.076
Sep-1963	3.595	May-1966	4.647	Jan-1969	3.934
Oct-1963	4.474	Jun-1966	4.753	Feb-1969	3.162
Nov-1963	6.838	Jul-1966	3.965	Mar-1969	4.286
Dec-1963	8.357	Aug-1966	1.723	Apr-1969	4.676
Jan-1964	3.113	Sep-1966	5.048	May-1969	5.010
Feb-1964	3.006	Oct-1966	6.922	Jun-1969	4.874
Mar-1964	4.047	Nov-1966	9.858	Jul-1969	4.633
Apr-1964	3.523	Dec-1966	11.331	Aug-1969	1.659
May-1964	3.937	Jan-1967	4.016	Sep-1969	5.951
Jun-1964	3.986	Feb-1967	3.957	Oct-1969	6.981
Jul-1964	3.260	Mar-1967	4.510	Nov-1969	9.851
Aug-1964	1.573	Apr-1967	4.276	Dec-1969	12.670

TABLE B.12 Chemical Process Yield, with Operating Temperature (Uncontrolled)

Hour	Yield (%)	Temperature (°F)	Hour	Yield (%)	Temperature (°F)
1	89.0	153	26	99.4	152
2	90.5	152	27	99.6	153
3	91.5	153	28	99.8	153
4	93.2	153	29	98.8	154
5	93.9	154	30	99.9	154
6	94.6	151	31	98.2	153
7	94.7	153	32	98.7	153
8	93.5	152	33	97.5	154
9	91.2	151	34	97.9	152
10	89.3	150	35	98.3	152
11	85.6	150	36	98.8	151
12	80.3	149	37	99.1	150
13	75.9	149	38	99.2	149
14	75.3	147	39	98.6	148
15	78.3	146	40	95.3	147
16	89.1	143	41	94.2	146
17	88.3	148	42	91.3	148
18	89.2	151	43	90.6	145
19	90.1	152	44	91.2	143
20	94.3	153	45	88.3	145
21	97.7	154	46	84.1	150
22	98.6	152	47	86.5	147
23	98.7	153	48	88.2	150
24	98.9	152	49	89.5	151
25	99.2	152	50	89.5	152

TABLE B.13 US Production of Ice Cream and Frozen Yogurt

Year	Ice Cream (10^3 gal)	Frozen Yogurt (10^3 gal)	Year	Ice Cream (10^3 gal)	Frozen Yogurt (10^3 gal)
1950	554,351	–	1975	836,552	–
1951	568,849	–	1976	818,241	–
1952	592,705	–	1977	809,849	–
1953	605,051	–	1978	815,360	–
1954	596,821	–	1979	811,079	–
1955	628,525	–	1980	829,798	–
1956	641,333	–	1981	832,450	–
1957	650,583	–	1982	852,072	–
1958	657,175	–	1983	881,543	–
1959	698,931	–	1984	894,468	–
1960	697,552	–	1985	901,449	–
1961	697,151	–	1986	923,597	–
1962	704,428	–	1987	928,356	–
1963	717,597	–	1988	882,079	–
1964	738,743	–	1989	831,159	82,454
1965	757,000	–	1990	823,610	117,577
1966	751,159	–	1991	862,638	147,137
1967	745,409	–	1992	866,110	134,067
1968	773,207	–	1993	866,248	149,933
1969	765,501	–	1994	876,097	150,565
1970	761,732	–	1995	862,232	152,097
1971	765,843	–	1996	878,572	114,168
1972	767,750	–	1997	913,770	92,167
1973	773,674	–	1998	937,485	87,777
1974	781,971	–	2000	979,645	94,478

Source: USDA–National Agricultural Statistics Service.

TABLE B.14 Atmospheric CO_2 Concentrations at Mauna Loa Observatory

Year	Average CO_2 Concentration (ppmv)	Year	Average CO_2 Concentration (ppmv)
1959	316.00	1982	341.09
1960	316.91	1983	342.75
1961	317.63	1984	344.44
1962	318.46	1985	345.86
1963	319.02	1986	347.14
1964	319.52	1987	348.99
1965	320.09	1988	351.44
1966	321.34	1989	352.94
1967	322.13	1990	354.19
1968	323.11	1991	355.62
1969	324.60	1992	356.36
1970	325.65	1993	357.10
1971	326.32	1994	358.86
1972	327.52	1995	360.90
1973	329.61	1996	362.58
1974	330.29	1997	363.84
1975	331.16	1998	366.58
1976	332.18	1999	368.30
1977	333.88	2000	369.47
1978	335.52	2001	371.03
1979	336.89	2002	373.07
1980	338.67	2003	375.61
1981	339.95		

Source: Adapted from C. D. Keeling, T. P. Whorf, and the Carbon Dioxide Research Group (2004); Scripps Institution of Oceanography (SIO), University of California, La Jolla, California USA 92093-0444, with permission of the publisher.

TABLE B.15 US National Violent Crime Rate

Year	Violent Crime Rate (per 100,000 Inhabitants)
1984	539.9
1985	558.1
1986	620.1
1987	612.5
1988	640.6
1989	666.9
1990	729.6
1991	758.2
1992	757.7
1993	747.1
1994	713.6
1995	684.5
1996	636.6
1997	611.0
1998	567.6
1999	523.0
2000	506.5
2001[a]	504.5
2002	494.4
2003	475.8
2004	463.2
2005	469.2

Source: http://www.census.gov/compendia/statab/hist_stats.html.

[a] The murder and nonnegligent homicides that occurred as a result of the events of September 11, 2001 are not included in the rate for the year 2001.

TABLE B.16 US Gross Domestic Product

Year	GDP, Current Dollars (Billions)	GDP, Real (1996) Dollars (Billions)
1976	1823.9	4311.7
1977	2031.4	4511.8
1978	2295.9	4760.6
1979	2566.4	4912.1
1980	2795.6	4900.9
1981	3131.3	5021.0
1982	3259.2	4919.3
1983	3534.9	5132.3
1984	3932.7	5505.2
1985	4213.0	5717.1
1986	4452.9	5912.4
1987	4742.5	6113.3
1988	5108.3	6368.4
1989	5489.1	6591.8
1990	5803.2	6707.9
1991	5986.2	6676.4
1992	6318.9	6880.0
1993	6642.3	7062.6
1994	7054.3	7347.7
1995	7400.5	7543.8
1996	7813.2	7813.2
1997	8318.4	8159.5
1998	8781.5	8508.9
1999	9274.3	8859.0
2000	9824.6	9191.4
2001	10,082.2	9214.5
2002	10,446.2	9439.9

Source: http://www.census.gov/compendia/statab/hist_stats.html.

TABLE B.17 Total Annual US Energy Consumption

Year	BTUs (Billions)	Year	BTUs (Billions)
1949	31,981,503	1978	79,986,371
1950	34,615,768	1979	80,903,214
1951	36,974,030	1980	78,280,238
1952	36,747,825	1981	76,342,955
1953	37,664,468	1982	73,286,151
1954	36,639,382	1983	73,145,527
1955	40,207,971	1984	76,792,960
1956	41,754,252	1985	76,579,965
1957	41,787,186	1986	76,825,812
1958	41,645,028	1987	79,223,446
1959	43,465,722	1988	82,869,321
1960	45,086,870	1989	84,999,308
1961	45,739,017	1990	84,729,945
1962	47,827,707	1991	84,667,227
1963	49,646,160	1992	86,014,860
1964	51,817,177	1993	87,652,195
1965	54,017,221	1994	89,291,713
1966	57,016,544	1995	91,199,841
1967	58,908,107	1996	94,225,791
1968	62,419,392	1997	94,800,047
1969	65,620,879	1998	95,200,433
1970	67,844,161	1999	96,836,647
1971	69,288,965	2000	98,976,371
1972	72,704,267	2001	96,497,865
1973	75,708,364	2002	97,966,872
1974	73,990,880	2003	98,273,323
1975	71,999,191	2004	100,414,461
1976	76,012,373	2005	99,894,296
1977	77,999,554		

Source: Annual Energy Review—Energy Overview 1949–2005, US Department of Energy–Energy information Center, http://www.eia.doe.gov/aer/overview.html.

TABLE B.18 Annual US Coal Production

Year	Coal Production (10^3 Short Tons)	Year	Coal Production (10^3 Short Tons)
1949	480,570	1978	670,164
1950	560,386	1979	781,134
1951	576,335	1980	829,700
1952	507,424	1981	823,775
1953	488,239	1982	838,112
1954	420,789	1983	782,091
1955	490,838	1984	895,921
1956	529,774	1985	883,638
1957	518,042	1986	890,315
1958	431,617	1987	918,762
1959	432,677	1988	950,265
1960	434,329	1969	980,729
1961	420,423	1990	1,029,076
1962	439,043	1991	995,984
1963	477,195	1992	997,545
1964	504,182	1993	945,424
1965	526,954	1994	1,033,504
1966	546,822	1995	1,032,974
1967	564,882	1996	1,063,856
1968	556,706	1997	1,089,932
1969	570,978	1998	1,117,535
1970	612,661	1999	1,100,431
1971	560,919	2000	1,073,612
1972	602,492	2001	1,127,689
1973	598,568	2002	1,094,283
1974	610,023	2003	1,071,753
1975	654,641	2004	1,112,099
1976	684,913	2005	1,133,253
1977	697,205		

Source: Annual Energy Review—Coal Overview 1949–2005, US Department of Energy–Energy Information Center.

TABLE B.19 Arizona Drowning Rate, Children 1–4 Years Old

Year	Drowning Rate per 100,000 Children 1–4 Years Old	Year	Drowning Rate per 100,000 Children 1–4 Years Old
1970	19.9	1988	9.2
1971	16.1	1989	11.9
1972	19.5	1990	5.8
1973	19.8	1991	8.5
1974	21.3	1992	7.1
1975	15.0	1993	7.9
1976	15.5	1994	8.0
1977	16.4	1995	9.9
1978	18.2	1996	8.5
1979	15.3	1997	9.1
1980	15.6	1998	9.7
1981	19.5	1999	6.2
1982	14.0	2000	7.2
1983	13.1	2001	8.7
1984	10.5	2002	5.8
1985	11.5	2003	5.7
1986	12.9	2004	5.2
1987	8.4		

Source: http://www.azdhs.gov/plan/report/im/dd/drown96/01dro96.htm.

TABLE B.20 US Internal Revenue Tax Refunds

Fiscal Year	Amount Refunded (Millions Dollars)	National Population (Thousands)
1987	96,969	242,289
1988	94,480	244,499
1989	93,613	246,819
1990	99,656	249,464
1991	104,380	252,153
1992	113,108	255,030
1993	93,580	257,783
1994	96,980	260,327
1995	108,035	262,803
1996	132,710	265,229
1997	142,599	267,784
1998	153,828	270,248
1999	185,282	272,691
2000	195,751	282,193
2001	252,787	285,108
2002	257,644	287,985
2003	296,064	290,850
2004	270,893	293,657
2005	255,439	296,410
2006	263,501	299,103

Source: US Department of Energy–Internal Revenue Service, SOI Tax Stats–Individual Time Series Statistical Tables, http://www.irs.gov/taxstats/indtaxstats/article/O,,id=96679,00.html.

TABLE B.21 Arizona Average Retail Price of Residential Electricity (Cents per kWh)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2001	6.99	7.13	7.4	8.09	9.41	9.04	8.84	8.84	8.81	8.95	7.17	7.26
2002	7.01	7.17	7.46	7.69	9.37	8.97	8.65	8.78	8.79	8.99	7.37	7.46
2003	7.06	7.57	7.59	7.82	9.52	9.09	8.78	8.74	8.7	8.83	7.21	7.55
2004	7.27	7.49	7.61	8.05	9.26	9.1	8.88	8.87	8.96	8.79	8.05	7.86
2005	7.75	7.99	8.19	8.67	9.6	9.41	9.3	9.28	9.3	9.23	8.12	7.88
2006	8.05	8.21	8.38	8.92	10.19	10.05	9.9	9.88	9.89	9.88	8.74	8.56
2007	8.33	8.46	8.8	9.19	10.2	9.96	10.37	10.33	10.17	10.16	9.08	8.89
2008	8.85	9.02	9.38	10.02	11.03	11.06	10.95	10.86	10.63	10.46	9.55	9.61
2009	9.51	9.82	9.93	10.65	11.33	11.27	11.3	11.29	11.17	10.97	9.86	9.7
2010	9.57	9.84	9.98	10.24	11.75	11.74	11.78	11.59	11.52	10.96	10.14	10
2011	9.84	9.93	10.25	10.97	11.77	11.77	11.85	11.67	11.53	11.08	10.31	9.98
2012	10.01	10.26	10.44	11.17	11.88	11.9	11.86	11.83	11.66	11.36	10.73	10.41
2013	10.25	10.7	10.87	11.74	12.17	12.18	12.51	12.33	12.22	12.02	11.06	11.01
2014	10.92	11.23	11.32	11.97	–	–	–	–	–	–	–	–

Source: http://www.eia.gov/electricity/data.cfm#sales.

TABLE B.22 Denmark Crude Oil Production (In Thousands of Tons)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2001	1372	1439	1499	1399	1340	1018	1121	1411	1560	1492	1578	1709
2002	1627	1457	1536	1560	1575	1431	1567	1267	1421	1619	1531	1592
2003	1617	1445	1598	1464	1482	1514	1406	1520	1560	1578	1574	1550
2004	1560	1335	1626	1645	1685	1617	1715	1471	1607	1726	1543	1731
2005	1577	1536	1632	1605	1568	1541	1518	1591	1459	1536	1485	1470
2006	1459	1351	1471	1330	1518	1377	1547	1364	1086	1456	1429	1450
2007	1266	1194	1290	1256	1290	1258	1240	1340	1159	1382	1264	1231
2008	1255	1024	1242	1101	1275	1138	1268	1141	1085	1196	1155	1156
2009	1201	1067	1140	1110	1081	1066	1112	1061	1129	1051	925	959
2010	1095	937	1014	1116	1061	906	1110	710	1014	1080	1009	1106
2011	987	791	964	925	1090	872	937	906	861	859	930	818
2012	826	830	854	867	866	860	853	820	724	824	819	838
2013	787	752	808	764	756	682	741	679	635	720	687	671
2014	675	637	691	659	–	–	–	–	–	–	–	–

Source: http://www.ens.dk/en/info/facts-figures/energy-statistics-indicators-energy-efficiency/monthly-statistics.

TABLE B.23 US Influenza Positive Tests (Percentage)

Week	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
1	–	19.30	9.39	24.97	13.37	9.74	5.62	12.43	18.57	11.72	6.97	8.55	8.44	4.01	25.17	3.22	35.42	29.18
2	–	20.92	14.76	21.30	17.80	13.85	8.80	9.33	16.94	13.22	9.24	13.59	10.86	4.76	29.28	4.40	34.26	27.33
3	–	25.61	20.30	16.73	22.11	19.74	11.49	6.01	21.34	14.12	11.58	16.80	16.05	5.63	33.52	5.97	30.65	26.39
4	–	28.07	22.07	12.68	23.22	21.37	16.32	3.98	23.30	15.73	18.19	22.71	20.76	5.11	34.34	8.10	29.80	23.90
5	–	26.72	27.32	10.66	20.93	22.80	20.50	2.92	26.87	16.69	22.38	28.46	23.93	5.16	35.4	10.88	26.65	20.18
6	–	24.37	28.29	10.15	18.99	22.79	26.36	1.84	26.55	17.95	28.18	31.75	25.27	4.56	34.66	12.62	22.83	18.73
7	–	20.13	25.51	7.47	15.43	23.86	25.32	1.79	25.55	19.57	25.80	31.82	24.81	4.78	34.85	14.84	20.68	16.47
8	–	14.24	25.23	6.37	13.94	24.91	22.55	1.48	23.46	20.45	23.80	30.40	24.54	5.42	30.98	19.00	19.76	13.53
9	–	9.69	23.38	3.71	11.46	22.92	22.22	0.68	19.95	22.59	22.57	27.65	24.35	6.14	27.51	23.82	18.81	11.94
10	–	7.60	20.59	3.42	7.20	21.18	18.69	0.59	16.04	22.43	20.25	23.68	22.38	6.61	23.79	28.18	18.01	10.84
11	–	5.42	15.62	2.70	5.91	18.52	17.12	0.58	16.94	19.74	19.53	22.01	20.55	5.71	20.28	30.72	16.47	11.11
12	–	4.10	14.31	1.76	5.78	15.12	14.64	1.33	14.66	18.47	16.87	20.39	16.74	4.42	16.12	27.53	15.63	12.53
13	–	2.79	8.26	1.23	4.38	13.75	9.96	0.92	11.04	17.98	13.76	17.78	13.83	4.35	12.8	21.56	14.48	13.22
14	–	2.22	7.92	1.26	3.27	12.41	8.55	0.51	9.12	13.57	11.03	14.54	10.66	2.76	9.81	22.08	13.91	14.39
15	–	2.87	6.45	1.70	2.23	10.86	5.16	0.96	5.27	11.36	12.14	11.77	7.19	2.05	7.12	20.45	10.79	15.00
16	–	1.48	2.63	1.90	2.58	13.09	5.01	1.66	4.19	9.89	9.67	8.51	7.68	1.26	4.42	20.64	8.62	13.72
17	–	1.29	2.53	1.44	1.28	8.57	4.13	0.80	2.68	7.48	9.75	5.37	12.46	0.71	3.35	17.27	8.11	13.08
18	–	1.40	1.73	0.94	0.59	10.23	5.56	0.69	2.27	7.75	6.75	4.63	15.02	0.80	1.91	15.99	6.81	12.62

19	–	0.87	1.12	0.21	0.52	8.76	2.50	0.16	1.73	7.22	4.98	2.56	18.09	0.62	1.10	15.17	5.82	10.90
20	–	0.96	1.13	0.23	0.68	6.00	2.04	0.36	2.06	5.63	3.93	1.57	26.79	0.52	0.84	13.33	4.87	9.81
21	–	NR	NR	NR	NR	NR	1.83	0.57	1.71	6.32	4.32	1.37	33.49	0.19	0.67	11.16	4.49	8.39
22	–	NR	NR	NR	NR	NR	0.85	0.73	1.37	4.69	1.77	0.99	39.48	0.28	0.53	10.91	4.58	8.77
23	–	NR	NR	NR	NR	NR	1.80	0.38	0.76	3.81	1.20	0.92	41.61	0.28	0.27	9.59	4.17	6.65
24	–	NR	NR	NR	NR	NR	1.02	0.24	0.99	1.97	1.70	0.80	43.05	0.44	0.43	9.97	4.28	6.62
25	–	NR	NR	NR	NR	NR	0.73	0.40	0.90	1.19	0.62	0.48	36.18	0.40	0.25	6.76	4.34	5.12
26	–	NR	NR	NR	NR	NR	0.46	0.41	0.58	1.18	1.17	0.45	31.78	0.22	0.26	6.74	3.56	4.17
27	–	NR	NR	NR	NR	NR	0.16	0.17	0.47	0.88	1.61	0.33	32.04	0.31	0.33	4.72	3.53	–
28	–	NR	NR	NR	NR	NR	0.47	0.27	0.20	0.77	1.92	0.47	27.54	0.68	0.60	4.58	4.13	–
29	–	NR	NR	NR	NR	NR	0.57	0.36	0.31	1.20	1.39	0.33	27.11	0.50	0.67	3.67	3.16	–
30	–	NR	NR	NR	NR	NR	0.63	0.00	0.18	1.97	3.18	0.86	27.56	1.36	0.31	5.13	3.18	–
31	–	NR	NR	NR	NR	NR	1.27	0.32	0.00	2.12	1.97	0.62	23.67	1.05	0.77	9.49	3.81	–
32	–	NR	NR	NR	NR	NR	0.32	0.16	0.41	1.33	2.94	0.64	22.55	1.73	0.91	7.54	3.39	–
33	–	NR	NR	NR	NR	NR	0.39	0.15	0.61	1.39	1.70	0.09	24.36	1.06	1.06	6.72	3.15	–
34	–	NR	NR	NR	NR	NR	0.00	0.49	0.39	1.67	1.97	0.55	22.03	2.03	1.00	4.18	2.69	–
35	–	NR	NR	NR	NR	NR	0.00	0.48	0.57	1.87	2.49	0.39	22.29	1.27	0.78	3.06	4.45	–
36	–	NR	NR	NR	NR	NR	0.39	0.36	0.1	1.48	2.2	0.45	24.01	1.54	0.63	3.05	4.03	–
37	–	NR	NR	NR	NR	NR	0.17	0.77	0.45	2.91	2.44	0.70	25.59	1.85	0.90	2.23	3.82	–
38	–	NR	NR	NR	NR	NR	0.37	0.45	0.16	3.51	3.14	1.06	26.99	1.62	0.80	3.08	3.52	–

(continued)

TABLE B.23 (*Continued*)

Week	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013	2014
39	–	NR	NR	NR	NR	NR	0.00	1.52	0.49	2.99	1.90	0.60	31.98	1.34	1.20	3.59	4.61	–
40	0.00	0.14	1.07	0.07	0.69	0.00	1.54	0.65	0.58	2.43	1.85	0.62	33.54	2.14	1.03	3.71	4.02	–
41	0.73	0.34	1.06	0.27	0.35	0.20	4.58	0.65	0.55	2.68	1.74	0.8	37.58	3.2	0.57	4.12	3.99	–
42	1.10	0.18	2.24	0.24	0.54	0.00	11.63	1.62	1.06	2.76	1.63	1.26	39.38	3.38	0.82	5.95	3.22	–
43	0.42	0.33	2.64	0.83	1.07	0.34	17.83	1.22	0.50	2.59	1.57	1.78	35.58	3.92	1.17	6.90	3.56	–
44	0.53	0.88	3.91	0.81	0.76	0.37	19.43	1.62	0.80	3.80	2.07	1.21	30.93	5.36	0.96	7.95	4.72	–
45	0.28	0.63	5.75	1.99	0.71	0.25	24.26	2.02	1.30	3.64	2.24	1.71	27.61	6.51	1.13	10.03	5.81	–
46	0.36	0.39	8.07	1.97	2.19	0.47	28.10	2.78	1.22	2.94	2.11	1.40	19.62	8.73	0.96	13.32	7.3	–
47	0.91	0.94	10.56	3.50	1.61	1.05	33.12	1.96	1.59	3.55	3.47	1.77	14.56	10.61	1.83	19.15	9.18	–
48	1.65	1.17	13.71	3.91	3.08	0.79	34.74	3.58	2.86	3.70	3.32	2.25	10.74	9.99	1.54	22.89	12.07	–
49	1.53	1.85	19.11	5.67	3.56	1.66	34.06	5.02	5.54	4.55	3.67	2.27	7.64	13.25	2.29	29.86	18.29	–
50	3.18	1.73	23.33	6.90	4.6	3.89	32.93	6.56	8.64	8.09	4.17	3.16	6.44	18.85	2.71	33.04	20.77	–
51	7.13	3.76	30.96	11.69	5.80	6.56	30.80	11.78	12.75	10.29	5.34	4.33	4.95	25.04	3.82	37.87	28.12	–
52	12.60	5.91	29.37	13.87	8.39	6.76	24.73	16.76	13.48	10.39	7.15	4.73	4.87	26.27	3.33	38.43	31.03	–
53	17.95	–	–	–	–	–	20.10	–	–	–	–	5.54	–	–	–	–	–	–

NR: Not Reported

Source: http://gis.cdc.gov/grasp/fluview/fluportaldashboard.html.

TABLE B.24 Mean Daily Solar Radiation in Zion Canyon, Utah (Langleys)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2003	–	–	–	–	–	–	–	–	–	–	184	178
2004	212	229	453	503	619	615	573	535	464	262	208	175
2005	166	216	385	508	529	549	579	474	443	302	224	170
2006	184	272	310	477	572	583	508	509	431	291	211	177
2007	220	263	396	466	590	634	542	511	432	316	233	179
2008	176	270	415	569	542	647	569	499	459	333	208	157
2009	214	240	423	487	593	586	638	617	523	367	283	196
2010	213	282	440	546	672	703	665	595	570	322	244	153
2011	242	294	389	533	584	703	597	625	488	371	248	214
2012	246	321	425	560	713	733	550	517	485	367	241	158
2013	232	333	457	506	541	645	494	463	412	320	207	180
2014	205	265	401	479	549	642	–	–	–	–	–	–

Source: http://www.raws.dri.edu/cgi-bin/rawMAIN.pl?utZIOC.

TABLE B.25 US Motor Vehicle Traffic Fatalities

Year	Fatalities	Resident Population (Thousands)	Licensed Drivers (Thousands)	Registered Motor Vehicles (Thousands)	Vehicle Miles Traveled (Billions)	Annual Unemployment Rate (%)
1966	50,894	196,560	100,998	95,703	926	3.8
1967	50,724	198,712	103,172	98,859	964	3.8
1968	52,725	200,706	105,410	102,987	1016	3.6
1969	53,543	202,677	108,306	107,412	1062	3.5
1970	52,627	205,052	111,543	111,242	1110	4.9
1971	52,542	207,661	114,426	116,330	1179	5.9
1972	54,589	209,896	118,414	122,557	1260	5.6
1973	54,052	211,909	121,546	130,025	1313	4.9
1974	45,196	213,854	125,427	134,900	1281	5.6
1975	44,525	215,973	129,791	126,153	1328	8.5
1976	45,523	218,035	134,036	130,793	1402	7.7
1977	47,878	220,239	138,121	134,514	1467	7.1
1978	50,331	222,585	140,844	140,374	1545	6.1
1979	51,093	225,055	143,284	144,317	1529	5.8
1980	51,091	227,225	145,295	146,845	1527	7.1
1981	49,301	229,466	147,075	149,330	1555	7.6
1982	43,945	231,664	150,234	151,148	1595	9.7
1983	42,589	233,792	154,389	153,830	1653	9.6
1984	44,257	235,825	155,424	158,900	1720	7.5
1985	43,825	237,924	156,868	166,047	1775	7.2
1986	46,087	240,133	159,486	168,545	1835	7
1987	46,390	242,289	161,816	172,750	1921	6.2
1988	47,087	244,499	162,854	177,455	2026	5.5
1989	45,582	246,819	165,554	181,165	2096	5.3
1990	44,599	249,464	167,015	184,275	2144	5.6
1991	41,508	252,153	168,995	186,370	2172	6.8
1992	39,250	255,030	173,125	184,938	2247	7.5
1993	40,150	257,783	173,149	188,350	2296	6.9
1994	40,716	260,327	175,403	192,497	2358	6.1
1995	41,817	262,803	176,628	197,065	2423	5.6
1996	42,065	265,229	179,539	201,631	2484	5.4
1997	42,013	267,784	182,709	203,568	2552	4.9
1998	41,501	270,248	184,861	208,076	2628	4.5
1999	41,717	272,691	187,170	212,685	2690	4.2
2000	41,945	282,162	190,625	217,028	2747	4
2001	42,196	284,969	191,276	221,230	2796	4.7
2002	43,005	287,625	194,602	225,685	2856	5.8
2003	42,884	290,108	196,166	230,633	2890	6
2004	42,836	292,805	198,889	237,949	2965	5.5

TABLE B.25 (*Contionued*)

Year	Fatalities	Resident Population (Thousands)	Licensed Drivers (Thousands)	Registered Motor Vehicles (Thousands)	Vehicle Miles Traveled (Billions)	Annual Unemployment Rate (%)
2005	43,510	295,517	200,549	245,628	2989	5.1
2006	42,708	298,380	202,810	251,415	3014	4.6
2007	41,259	301,231	205,742	257,472	3031	4.6
2008	37,423	304,094	208,321	259,360	2977	5.8
2009	33,883	306,772	209,618	258,958	2957	9.3
2010	32,999	309,326	210,115	257,312	2967	9.6
2011	32,479	311,588	211,875	265,043	2950	8.9
2012	33,561	313,914	211,815	265,647	2969	8.1

Sources: http://www-fars.nhtsa.dot.gov/Main/index.aspx, http://www.bls.gov/data/.

TABLE B.26 Single-Family Residential New Home Sales and Building Permits (In Thousands of Units)

	Jan		Feb		Mar		Apr		May		Jun	
Year	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits
1963	42	43.3	35	43.4	44	63.8	52	82.1	58	78.8	48	71
1964	39	42.7	46	48.5	53	67.6	49	73.1	52	70.9	53	72.3
1965	38	39.2	44	41.4	53	65.6	49	73.2	54	69.3	57	71.3
1966	42	39.1	43	38.1	53	69.6	49	66.2	49	60.5	40	58.8
1967	29	31.5	32	34.2	41	56	44	59.2	49	67.6	47	68.5
1968	35	38.9	43	44.7	46	61.4	46	72	43	70.7	41	60.2
1969	34	39.8	40	42.5	43	58.5	42	69.7	43	64.5	44	61.7
1970	34	27.7	29	34.4	36	50.8	42	63.5	43	58.3	44	62.7
1971	45	44.1	49	48.1	62	78.8	62	89.6	58	90.2	59	93.2
1972	51	61.8	56	64.8	60	94.2	65	95	64	103.2	63	104.1
1973	55	64.9	60	66.2	68	91.6	63	94.9	65	102.2	61	92
1974	37	38.7	44	44.8	55	66.5	53	76.8	58	73.3	50	63.7
1975	29	29.8	34	32.5	44	45.5	54	65.2	57	68.2	51	68.4
1976	41	46.3	53	53.6	55	81.4	62	87.3	55	81.3	56	89.1
1977	57	52.6	68	67.2	84	110.5	81	108.2	78	112.4	74	117.2
1978	57	62.9	63	67.9	75	110.1	85	119.7	80	124.3	77	130.4
1979	53	54.9	58	56.4	73	101.5	72	102.2	68	110	63	100.8
1980	43	45.2	44	47	44	49.1	36	48.4	44	49.6	50	61.2
1981	37	39.6	40	42.9	49	61.2	44	69.3	45	61	38	56.3
1982	28	25.4	29	27.9	36	44	32	46.6	36	45.8	34	50.6

1983	44	48.5	46	50.9	57	79.2	59	81.6	64	93	59	101.2
1984	52	60.2	58	72.6	63	88.6	61	92.3	59	98.2	58	90.3
1985	48	55.8	55	58.1	67	85.9	60	93.9	65	96.8	65	89
1986	55	65.3	59	61.1	89	89.3	84	114.8	75	109.3	66	110.8
1987	53	62.4	59	69.5	73	103.5	72	107.6	62	96.6	58	107.7
1988	43	50.5	55	63	68	98.2	68	93.1	64	98.7	65	105.9
1989	52	60.9	51	58.9	58	84.1	60	87.5	61	93.6	58	92
1990	45	60.7	50	60.7	58	82.8	52	79.2	50	83.1	50	79.7
1991	30	37.6	40	43.5	51	61.2	50	75.7	47	78.1	47	73.9
1992	48	55.2	55	61.1	56	82.4	53	88	52	82.7	53	91.6
1993	44	55.1	50	61.3	60	84.2	66	91.5	58	85.2	59	97
1994	46	63.4	58	69.2	74	104	65	102	65	107.7	55	109.2
1995	47	58.2	47	59.8	60	85.1	58	83.1	63	95.9	64	97.4
1996	54	66	68	74.4	70	95.7	70	109.9	69	109.2	65	100.7
1997	61	65.8	69	70.3	81	88.7	70	104.4	71	101.3	71	100.9
1998	64	70.1	75	78.1	81	105.1	82	113.6	82	107.3	83	115.8
1999	67	74.2	76	86.6	84	118.9	86	119.9	80	115.9	82	128
2000	67	78.3	78	89.1	88	119	78	107.6	77	119.3	71	114.5
2001	72	85.6	85	85.1	94	112.7	84	116.5	80	124.4	79	119.2
2002	66	88.7	84	95.5	90	111	86	125.4	88	127.1	84	118.9
2003	76	98	82	93.9	98	117.7	91	134.2	101	132.1	107	138.3
2004	89	103.4	102	108.4	123	154.8	109	155	115	150.2	105	159.3
2005	92	106.9	109	114.8	127	150.6	116	152.7	120	156	115	166.2

(*continued*)

TABLE B.26 (*Continued*)

	Jan		Feb		Mar		Apr		May		Jun	
Year	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits
2006	89	114.3	88	115.6	108	146.7	100	130.8	102	144.5	98	139.3
2007	66	80.3	68	79.4	80	103.4	83	98.6	79	106.6	73	97.2
2008	44	48	48	48	49	54.1	49	63.4	49	61.9	45	59.4
2009	24	22.1	29	26.3	31	32.7	32	37.8	34	39.5	37	47
2010	24	31.4	27	35.4	36	50.9	41	46.1	26	39.9	28	42.9
2011	21	26.2	22	26.5	28	37.8	30	37.2	28	39.7	28	41.5
2012	23	30.5	30	35.7	34	42.9	34	44.5	35	50.3	34	48.1
2013	32	40.7	36	42.3	41	51.7	43	60.2	40	62.8	43	57.4
2014	33	41.1	35	41.2	39	51.4	40	57.6	49	59.1	-	-
1963	62	72.8	56	68.1	49	65.4	44	69.7	39	51	31	40.8
1964	54	68.2	56	61.1	48	61.9	45	60.8	37	50.8	33	42.2
1965	51	64.9	58	64.6	48	60.3	44	61.5	42	54	37	44.5
1966	40	47.3	36	46.9	29	40.2	31	36.9	26	32.7	23	27.1
1967	46	58.1	47	64.7	43	57.6	45	61.2	34	51.6	31	40.2
1968	44	64.3	47	62.2	41	60.8	40	65	32	51.9	32	42.5
1969	39	55.2	40	50.3	33	51.3	32	53.5	31	40.1	28	38.8
1970	44	59.6	48	58.3	45	60.9	44	62.2	40	51.3	37	57.1
1971	64	86	62	82.7	50	78.1	52	76	50	73.3	44	65.9
1972	63	89.1	72	101.1	61	84.6	65	98.1	51	76.5	47	60.8
1973	54	82.6	52	78	46	61.7	42	60.4	37	49.9	30	37.8
1974	48	61.7	45	56.6	41	46.9	34	48.2	30	36.4	24	30.3

1975	51	69.5	53	63.8	46	65.5	46	68	46	51.5	39	47.6
1976	57	82.5	59	80.6	58	78.5	55	77.4	49	73.8	47	61.8
1977	64	99.7	74	110.1	71	97.2	63	94.9	55	87.6	51	68.4
1978	68	100.9	72	107.3	68	96.6	70	104	53	87.5	50	71
1979	64	93	68	97.1	60	79.4	54	83.3	41	57.7	35	45.1
1980	55	74.6	61	75.3	50	80.2	46	76.3	39	55.3	33	48.2
1981	36	52.6	34	45.2	28	41.8	29	35.6	27	29.2	29	29.6
1982	31	46.8	36	47.6	39	52.4	40	55.5	39	54.8	33	48.9
1983	51	82.3	50	85.9	48	77.8	51	76.5	45	68.3	48	56.5
1984	52	79.1	48	80.7	53	70.8	55	75.8	42	62.7	38	51.2
1985	63	92.1	61	90	54	81.3	52	89.7	51	64.9	47	59.1
1986	57	106.2	52	91.3	60	94.8	54	93.5	48	66.4	49	74.7
1987	55	95.8	56	87.6	52	86.6	52	81.7	43	65.5	37	59.7
1988	57	85.3	59	95.9	54	85.1	57	82.9	43	71.8	42	63.4
1989	62	79	61	89.2	49	78.6	51	81.2	47	69.2	40	57.4
1990	46	70.9	46	72.5	38	57.7	37	62.5	34	47	29	37.1
1991	43	74.9	46	69.9	37	64.2	41	70.4	39	52.6	36	51.7
1992	52	83.3	56	76.6	51	80.1	48	80.3	42	63.8	42	65.8
1993	55	88.2	57	91	57	89.8	56	87.9	53	80.5	51	74.8
1994	52	90.9	59	100.9	54	91.5	57	85.9	45	74.8	40	68.9
1995	64	88.3	63	101.4	54	90.1	54	90.8	46	78.4	45	68.8

(continued)

TABLE B.26 (*Continued*)

	Jan		Feb		Mar		Apr		May		Jun	
Year	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits	Sales	Permits
1996	66	101.9	73	97.6	62	85.9	56	90.8	54	71.5	51	66
1997	69	99.8	72	91.8	67	95.6	62	97.5	61	72.5	51	73.9
1998	75	111.2	75	104.4	68	102.5	69	103.8	70	86.6	61	89
1999	78	114.6	78	112.6	65	103.1	67	97.6	61	90.3	57	84.8
2000	76	98.8	73	111.6	70	95.8	71	102.8	63	87.7	65	73.7
2001	76	110.2	74	116.2	66	92.4	66	104.4	67	89.2	66	79.7
2002	82	122.4	90	119.8	82	110.1	77	123	73	96	70	94.6
2003	99	138.6	105	131	90	130.5	88	138.1	76	99.2	75	109.6
2004	96	145.3	102	145.6	94	134.5	101	128.5	84	114.6	83	113.8
2005	117	145.9	110	161.9	99	151.3	105	139.1	86	124	87	112.5
2006	83	111.6	88	121.5	80	97.7	74	98	71	82.2	71	76
2007	68	89.8	60	87.5	53	66.6	57	70.7	45	54.7	44	45.2
2008	43	55.7	38	48	35	45.9	32	40.4	27	26.2	26	24.6
2009	38	46.9	36	42.9	30	40.7	33	38.6	26	31.9	24	34.7
2010	26	37.5	23	37.2	25	34.3	23	31.5	20	29.6	23	30.6
2011	27	35.9	25	41.6	24	36.3	25	34.4	23	31.6	24	29.8
2012	33	47.3	31	49.8	30	43.3	29	49.6	28	40.4	28	36.3
2013	33	58.7	31	58	31	50.5	36	54.4	32	43.8	31	40.2
2014	–	–	–	–	–	–	–	–	–	–	–	–

Source: http://www.census.gov/housing/hvs/.

TABLE B.27 Best Airline On-Time Arrival Performance (Percentage)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1995	73.83	78.91	79.3	81.3	80.68	75.94	80.08	79.85	85.58	82.09	77.85	67.66
1996	62.67	71.9	75.94	80.18	78.92	74.67	75.29	74.69	78.69	77.19	77.91	66.63
1997	68.41	75.21	78.1	79.85	83.16	76.14	77.5	78.56	85.03	81.55	78.22	73.54
1998	75.05	75.39	75.85	79.12	77.48	70.42	78.86	77.03	78.94	81.75	83.29	73.22
1999	67.66	78.91	78.09	75.73	76.19	70.88	71.07	76.11	79.35	80.06	81.41	78
2000	73.75	74.76	76.99	75.37	74.25	66.34	70.31	69.96	78.1	76.07	72.82	62.75
2001	75.42	72.73	75.22	79.29	81.47	75.18	78.12	76.18	67.66	84.78	84.71	80.22
2002	81.02	84.69	78.59	82.58	82.76	78.64	79.66	82.62	87.95	84.18	85.21	78.34
2003	83.32	76.54	82.58	86.85	84.93	82.36	79.66	79	85.63	86.39	80.2	76.04
2004	74.85	77.48	81.29	83.04	77.62	72.95	75.95	78.29	83.91	81	79.12	71.56
2005	71.39	77.58	76.94	83.44	83.67	75.2	70.92	75.16	82.66	81.26	80.03	71.01
2006	78.76	75.3	76.12	78.42	78.27	72.83	73.7	75.8	76.22	72.91	76.52	70.8
2007	73.11	67.26	73.27	75.71	77.9	68.07	69.78	71.64	81.7	78.21	80.03	64.34
2008	72.36	68.63	71.58	77.67	79.02	70.84	75.7	78.44	84.88	86.04	83.33	65.34
2009	77.02	82.6	78.4	79.14	80.49	76.12	77.6	79.68	86.17	77.27	88.59	71.99
2010	78.69	74.66	79.96	85.31	79.94	76.42	76.69	81.65	85.07	83.77	83.16	72.04
2011	76.3	74.54	79.24	75.5	77.06	76.92	77.84	79.34	83.88	85.54	85.3	84.37
2012	83.75	86.16	82.19	86.26	83.38	80.66	76.01	79.15	83.3	80.21	85.73	76.56
2013	80.98	79.62	–	–	–	–	–	–	–	–	–	–

Source: http://www.rita.dot.gov/bts/subject_areas/airline_information/airline_ontime_tables/2013_02/table_02.

TABLE B.28 US Automobile Manufacturing Shipments (Dollar in Millions)

Year	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
1992	5618	7205	7446	7420	8076	7692	4395	7045	7058	8219	7432	6427
1993	7091	8339	8231	7772	8012	8015	3831	6246	7399	8546	8186	7267
1994	7986	9482	9215	9092	9346	9653	4748	8832	9194	9583	9573	8560
1995	8541	10,048	9796	8781	8851	8877	4207	7898	8999	9167	8333	7236
1996	7379	8852	7817	8688	8993	8521	4900	7837	8895	8739	8213	6,425
1997	7698	8898	8228	8121	7804	8120	5151	7943	8179	9354	8490	7,380
1998	7248	9013	9280	8622	8706	7312	3874	8162	9006	9658	8396	7431
1999	7594	9144	9429	8550	8635	8738	4657	8797	8895	9280	8607	7477
2000	8231	9117	9817	8467	8919	9536	4437	7941	9267	9133	7720	6175
2001	6485	7142	8541	6986	8431	7999	4411	8182	6827	8372	7312	6358
2002	7227	7239	7541	7886	8382	7726	4982	8089	7465	8565	7149	5876
2003	7383	7308	7923	6959	7638	7374	4903	6937	8036	8520	6680	6575
2004	6670	7778	9103	7323	7306	7947	4212	7709	7875	7453	6837	6634
2005	6520	7302	7692	6966	7403	7906	4680	8289	8402	8472	7525	7209
2006	7454	8009	9829	7571	9143	9247	4424	8655	7621	8126	7278	6872
2007	6788	6897	8121	6468	7424	7482	4945	8010	6969	7962	7250	6412
2008	7328	7862	7229	6818	7092	7376	5156	7302	7386	7352	5673	4930
2009	3869	3995	4221	4039	3519	3994	3277	4635	4911	5554	5106	5287
2010	5373	5021	5969	5716	5788	6765	5397	7262	7422	6957	6389	6171
2011	5974	6324	8197	6448	6752	7521	5739	7761	7508	8362	7288	7253
2012	7308	8290	8919	8659	9594	9678	7141	9437	8886	10,081	9,898	9085
2013	9789	10,591	11,387	10,666	10,920	10,992	8076	11,166	10,989	12,195	10,996	8365
2014	8683	9705	9577	8356	9686	–	–	–	–	–	–	–

Source: http://www.census.gov/econ/currentdata/.

APPENDIX C

INTRODUCTION TO R

Throughout the book, we often refer to commercial statistical software packages such as JMP and Minitab when discussing the examples. These software packages indeed provide an effective option particularly for the undergraduate level students and novice statisticians with their pull-down menus and various built-in statistical functions and routines. However there is also a growing community of practitioners and academicians who prefer to use R, an extremely powerful and freely available statistical software package that can be downloaded from http://www.r-project.org/. According to this webpage,

> *R is an integrated suite of software facilities for data manipulation, calculation and graphical display. It includes*
>
> - *an effective data handling and storage facility,*
> - *a suite of operators for calculations on arrays, in particular matrices,*
> - *a large, coherent, integrated collection of intermediate tools for data analysis,*
> - *graphical facilities for data analysis and display either on-screen or on hardcopy, and*

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> - *a well-developed, simple and effective programming language which includes conditionals, loops, user-defined recursive functions and input and output facilities.*
>
> *The term "environment" is intended to characterize it as a fully planned and coherent system, rather than an incremental accretion of very specific and inflexible tools, as is frequently the case with other data analysis software.*
>
> *R, like S, is designed around a true computer language, and it allows users to add additional functionality by defining new functions. Much of the system is itself written in the R dialect of S, which makes it easy for users to follow the algorithmic choices made. For computationally-intensive tasks, C, C++ and Fortran code can be linked and called at run time. Advanced users can write C code to manipulate R objects directly.*
>
> *Many users think of R as a statistics system. We prefer to think of it of an environment within which statistical techniques are implemented. R can be extended (easily) via packages. There are about eight packages supplied with the R distribution and many more are available through the CRAN family of Internet sites covering a very wide range of modern statistics.*

In this second edition of our book, we decided to provide the R-code for most of the examples at the end of the chapters. The codes are generated with the novice R user in mind and we therefore tried to keep them simple and easy to understand, sometimes without taking full advantage of more sophisticated options available in R. We nonetheless believe that they offer readers the possibility to immediately apply the techniques covered in the chapters with the data provided at the end of the book or with their own data. This after all we believe is the best way to learn time series analysis and forecasting.

BASIC CONCEPTS IN R

R can be downloaded from the R project webpage mentioned above. Although there are some generic built-in functions such as mean() to calculate the sample mean or lm() to fit a linear model, R provides the flexibility of writing your own functions as in C++ or Matlab. In fact one of the main advantages of R is its ever-growing user community, who openly shares the new functions they wrote in terms of "packages." Each new package has to be installed and loaded from "Packages" option in order to be able to use its contents. We provide the basic commands in R below.

Data entry can be done manually using c() function such as

```
> temp<-c(75.5,76.3,72.4,75.7,78.6)
```

Now the vector temp contains 5 elements that can displayed using

```
> temp
[1] 75.5 76.3 72.4 75.7 78.6
```

or

```
> print(temp)
[1] 75.5 76.3 72.4 75.7 78.6
```

However for large data sets, importing the data from an ASCII file, for example, a .txt file, is preferred. If, for example, each entry of temp represents the average temperature on a weekday and is stored in a file named temperature.txt, the data can then be imported to R using read.table() function as

```
> temp<-read.table("temperature.txt",header=TRUE,Sep="")
```

This command will assign the contents of temperature.txt file into the data frame "temp." It assumes that the first row of the file contains the names of the individual variables in the file, for example, in this case "Day" and "Temperature" and the data are space delimited. Also note that the command further assumes that the file is in the working directory, which can be changed using the File option. Otherwise the full directory has to be specified, for example, if the file is in C:/Rcoding directory,

```
read.table("C:/Rcoding/temperature.txt",header=T,sep=",")
```

Now we have

```
> temp
  Day Temperature
1   1        75.5
2   2        76.3
3   3        72.4
4   4        75.7
5   5        78.6
```

We can access each column of the temp matrix by one of the two commands

```
> temp$Temperature
[1] 75.5 76.3 72.4 75.7 78.6
> temp[,2]
[1] 75.5 76.3 72.4 75.7 78.6
```

Now that the data are imported, we can start using built-in function such as the sample mean and the log transform of the temperature by

```
> mean(temp$Temperature)
[1] 75.7
> log(temp$Temperature)
[1] 4.324133 4.334673 4.282206 4.326778 4.364372
```

One can also write user-defined functions to analyze the data. As we mentioned earlier, for most basic statistical functions there already exists packages containing the functions that would serve the desired purpose. Some basic examples of functions are provided in the R-code of the examples.

As indicated in the R project's webpage: "One of R's strengths is the ease with which well-designed publication-quality plots can be produced, including mathematical symbols and formulae where needed. Great care has been taken over the defaults for the minor design choices in graphics, but the user retains full control." In order to show the flexibility of plotting options in R, in the examples we provide the code for different plots for time series data and residual analysis with various options to make the plots look very similar to the ones generated by the commercial software packages used in the chapters.

Exporting the output or new data can be done through write.table() function. In order to create, for example, a new data frame by appending to the original data frame the log transform of the temperature and export the new data frame into a .txt file, the following commands can be used

```
> temp.new<-cbind(temp,log(temp$Temperature))
> temp.new
  Day Temperature log(temp$Temperature)
1   1        75.5              4.324133
2   2        76.3              4.334673
3   3        72.4              4.282206
4   4        75.7              4.326778
5   5        78.6              4.364372
> write.table(temp.new," C:/Rcoding/Temperaturenew.txt")
```

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