ANDO AVRIL 2016

Exercice 1 :

On dispose d'un tableau de données statistiques :

$$\begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 2 & 0 & 4 & 0 \\ 4 & 4 & -2 & 0 \\ 0 & 2 & 0 & -8 \end{pmatrix} \end{array}$$

A, B, C, D des variables sur $\mathbb{R}^3$

x, y, z des individus

1) Les variables A, B, C, D sont-elles linéairement indépendantes ?

2) Donner la dimension de $[A, B, C, D]$?

3) Peut-on expliquer linéairement la variable D par les variables A et C ?

4) Expliquer linéairement D par les variables A, B et C.

5) Expliquer linéairement la variable D par A et B en calculant la projection de D sur l'axe $[A, B]$.

1) $A, B, C, D \in \mathbb{R}^3$ or $\dim \mathbb{R}^3 = 3 \Rightarrow (A, B, C, D)$ est liée

2) $\det(B, C, D) = \begin{vmatrix} 0 & 4 & 0 \\ 4 & -2 & 0 \\ 2 & 0 & -8 \end{vmatrix} = (-8) \times ((-2) \times 0 - 4 \times 4) = -24 \neq 0$

Le rang de la matrice est 3 donc $\dim [A, B, C, D] = 3$

3) $\langle D, A \rangle = 0 \times 2 + 0 \times 4 + (-8) \times 0 = 0$

$\langle D, C \rangle = 0 \times 4 + 0 \times (-2) + (-8) \times 0 = 0$ $\Rightarrow$ C'est impossible

4) $\exists \alpha, \beta, \gamma \in \mathbb{R}$ tel que $D = \alpha A + \beta B + \gamma C$

$$\begin{pmatrix} 0 \\ 0 \\ -8 \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} 2\alpha + 4\gamma = 0 \\ 4\alpha + 4\beta - 2\gamma = 0 \\ 2\beta = -8 \end{cases} \Rightarrow \begin{cases} \beta = -4 \\ \alpha = -2\gamma \\ -8\gamma - 16 - 2\gamma = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \beta = -4 \\ \alpha = -2\gamma \\ -10\gamma - 16 = 0 \end{cases} \Rightarrow \begin{cases} \beta = -4 \\ \gamma = -\frac{8}{5} \\ \alpha = \frac{16}{5} \end{cases}$$

donc $D = \frac{16}{5} A - 4B - \frac{8}{5} C$

5) $\cdot\ \hat{D} = \alpha A + \beta B$, $\hat{D}$ projection de D sur $[A, B]$

$D - \hat{D} \perp [A, B] \Rightarrow \langle D - \hat{D}, A \rangle = 0$ et $\langle D - \hat{D}, B \rangle = 0$

$$\hat{D} = \begin{pmatrix} 2\alpha \\ 4\alpha + 4\beta \\ 2\beta \end{pmatrix} \Rightarrow D - \hat{D} = \begin{pmatrix} -2\alpha \\ -4\alpha - 4\beta \\ -8 - 2\beta \end{pmatrix}$$

$\langle D - \hat{D}, A \rangle = 0 \Rightarrow -4\alpha - 16\alpha - 16\beta = 0 \Rightarrow -20\alpha - 16\beta = 0$

$\Rightarrow -5\alpha - 4\beta = 0$

$\langle D - \hat{D}, B \rangle = 0 \Rightarrow -16\alpha - 16\beta - 16 - 4\beta = 0 \Rightarrow -16\alpha - 20\beta = 16$

$\Rightarrow -4\alpha - 5\beta = 4$

$$\Rightarrow \begin{cases} -5\alpha - 4\beta = 0 & (1) \\ -4\alpha - 5\beta = 4 & (2) \end{cases} \Rightarrow 4\times(1) - 5\times(2) \Rightarrow -16\beta + 25\beta = -20 \Rightarrow 9\beta = -20$$

$$\Rightarrow \beta = -\frac{20}{9}$$

$$\Rightarrow \alpha = \frac{16}{9}$$

$$\hat{D} = \begin{pmatrix} \frac{32}{9} \\ -\frac{16}{9} \\ -\frac{40}{9} \end{pmatrix} \qquad D - \hat{D} = \begin{pmatrix} -\frac{32}{9} \\ \frac{16}{9} \\ -\frac{32}{9} \end{pmatrix}$$

$$\langle D - \hat{D}, \hat{D} \rangle = \frac{32}{9} \times \left(-\frac{32}{9}\right) + \left(-\frac{16}{9}\right) \times \frac{16}{9} + \left(-\frac{40}{9}\right) \times \left(-\frac{32}{9}\right) = 0$$

$$\cos^2(D, \hat{D}) = \frac{\langle D , \hat{D} \rangle}{\|D\|^2 \, \|\hat{D}\|^2} = \frac{0 \times \frac{32}{9} + 0 \times \frac{16}{9} + (-8)\left(-\frac{40}{9}\right)}{[0^2 + 0^2 + (-8)^2]\left[\left(-\frac{32}{9}\right)^2 + \left(\frac{16}{9}\right)^2 + \left(\frac{32}{9}\right)^2\right]} = \frac{5}{256} = 0,0195$$

$$\Rightarrow \cos(D, \hat{D}) = \sqrt{0,0195} = 0,140$$

$$\Rightarrow \cos^{-1}(D, \hat{D}) = 82^\circ$$

Exercice 2 :

$$X = \begin{array}{c} \begin{array}{ccc} X^{(1)} & X^{(2)} & X^{(3)} \end{array} \\ \begin{pmatrix} 4 & \frac{1}{2} & 0 \\ 2 & 3 & \frac{5}{2} \\ 3 & 4 & \frac{7}{2} \\ 5 & 2 & \frac{7}{2} \\ 4 & 1 & \frac{5}{2} \\ 0 & \frac{3}{2} & 3 \end{pmatrix} \end{array}$$

Le poids est $D = \frac{1}{6} I_6 = \frac{1}{6}$ (ou $p_i = \frac{1}{6}$ $\forall i = 1 \dots 6$)

1) Calculer la moyenne $\overline{X^{(1)}}, \overline{X^{(2)}}, \overline{X^{(3)}}$ puis déterminer le centre de gravité.
2) Calculer la matrice des données centrées Y
3) Calculer la matrice de variance-covariance V
4) Diagonaliser V (on montrera que $P_V(\lambda) = -(\lambda-3)(\lambda-2)(\lambda-\frac{1}{2})$)
5) Calculer le pourcentage d'inertie
Déterminer les facteurs principaux associés aux 2 plus grands valeurs propres
6) Calculer les composantes principales et déterminer les coefficients du cercle de corrélation

1) $\overline{X^{(i)}} = \sum_{j=1}^{6} p_i X_j^{(i)}$

$\overline{X^{(1)}} = \frac{1}{6}(4+2+3+5+4+0) = \frac{18}{6} = 3$

$\overline{X^{(2)}} = \frac{1}{6}(\frac{1}{2}+3+4+2+1+\frac{3}{2}) = \frac{12}{6} = 2$

$\overline{X^{(3)}} = \frac{1}{6}(0+\frac{5}{2}+\frac{7}{2}+\frac{7}{2}+\frac{5}{2}+3) = \frac{15}{6} = \frac{5}{2}$

2) $Y^{(i)} = X^{(i)} - \overline{X^{(i)}}$

$$Y = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{5}{2} \\ -1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \\ -3 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

3) $V = {}^tYDY = \frac{1}{6}\,{}^tYY$

$${}^tY = \begin{pmatrix} 1 & -1 & 0 & 2 & 1 & -3 \\ -\frac{3}{2} & 1 & 2 & 0 & -1 & -\frac{1}{2} \\ -\frac{5}{2} & 0 & 1 & 1 & 0 & \frac{1}{2} \end{pmatrix} \Rightarrow V = \frac{1}{6}\,{}^tYY = \frac{1}{6}\begin{pmatrix} 16 & -2 & -2 \\ -2 & \frac{17}{2} & \frac{11}{2} \\ -2 & \frac{11}{2} & \frac{17}{2} \end{pmatrix} = \begin{pmatrix} \frac{8}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{17}{12} & \frac{11}{12} \\ -\frac{1}{3} & \frac{11}{12} & \frac{17}{12} \end{pmatrix}$$

4) $P_V(\lambda) = \det(V - \lambda I)$

$$= \begin{vmatrix} \frac{8}{3}-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{17}{12}-\lambda & \frac{11}{12} \\ -\frac{1}{3} & \frac{11}{12} & \frac{17}{12}-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & -\frac{1}{3} & -\frac{1}{3} \\ 2-\lambda & \frac{17}{12}-\lambda & \frac{11}{12} \\ 2-\lambda & \frac{11}{12} & \frac{17}{12}-\lambda \end{vmatrix} \quad C_1 \leftarrow C_1 + C_2 + C_3$$

$$= (2-\lambda)\begin{vmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & \frac{17}{12}-\lambda & \frac{11}{12} \\ 1 & \frac{11}{12} & \frac{17}{12}-\lambda \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{7}{4}-\lambda & \frac{5}{4} \\ 0 & \frac{5}{4} & \frac{7}{4}-\lambda \end{vmatrix}(2-\lambda) \quad \begin{array}{l} L_2 \leftarrow L_2 - L_1 \\ L_3 \leftarrow L_3 - L_1 \end{array}$$

$= (2-\lambda)\left[(\frac{7}{4}-\lambda)^2 - (\frac{5}{4})^2\right]$

$= (2-\lambda)(\frac{7}{4}-\lambda+\frac{5}{4})(\frac{7}{4}-\lambda-\frac{5}{4})$

$= (2-\lambda)(3-\lambda)(\frac{1}{2}-\lambda)$

Donc $P_V(\lambda) = -(\lambda-3)(\lambda-2)(\lambda-\frac{1}{2})$, d'où $\lambda_1 = 3$, $\lambda_2 = 2$ et $\lambda_3 = \frac{1}{2}$

5). Pourcentage d'inertie : $\dfrac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} = \dfrac{3+2}{3+2+\frac{1}{2}} = \dfrac{10}{11} = 0,91 = 91\%$

· $E_3 = \mathrm{Ker}(V - 3I)$

$$\forall u \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_3 \Rightarrow \begin{cases} (\frac{8}{3}-3)x - \frac{1}{3}y - \frac{1}{3}z = 0 \\ -\frac{1}{3}x + (\frac{17}{12}-3)y + \frac{11}{12}z = 0 \\ -\frac{1}{3}x + \frac{11}{12}y + (\frac{17}{12}-3)z = 0 \end{cases} \Rightarrow \begin{cases} -\frac{1}{3}x - \frac{1}{3}y - \frac{1}{3}z = 0 & (1) \\ -\frac{1}{3}x - \frac{19}{12}y + \frac{11}{12}z = 0 & (2) \\ -\frac{1}{3}x + \frac{11}{12}y - \frac{19}{12}z = 0 & (3) \end{cases}$$

$(2)-(3) \Rightarrow -\frac{30}{12}y + \frac{30}{12}z = 0 \Rightarrow y = z$

(2) et (3) dans $(1) \Rightarrow -\frac{1}{3}x - \frac{2}{3}y = 0 \Rightarrow x = -2y$

$$E_3 = \mathrm{Vect}\left(\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}\right)$$

donc le 1er facteur principal : $u^{(1)} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ → axe 1

· $E_2 = \mathrm{Ker}(V - 2I)$

$$\forall u \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in E_3 \Rightarrow \begin{cases} (\frac{8}{3}-2)x - \frac{1}{3}y - \frac{1}{3}z = 0 \\ -\frac{1}{3}x + (\frac{17}{12}-2)y + \frac{11}{12}z = 0 \\ -\frac{1}{3}x + \frac{11}{12}y + (\frac{17}{12}-2)z = 0 \end{cases} \Rightarrow \begin{cases} \frac{2}{3}x - \frac{1}{3}y - \frac{1}{3}z = 0 & (1) \\ -\frac{1}{3}x - \frac{7}{12}y + \frac{11}{12}z = 0 & (2) \\ -\frac{1}{3}x + \frac{11}{12}y - \frac{7}{12}z = 0 & (3) \end{cases}$$

$(2)-(3) \Rightarrow -\frac{18}{12}y + \frac{18}{12}z = 0 \Rightarrow y = z$

(2) et (3) dans $(1) \Rightarrow \frac{2}{3}x - \frac{2}{3}y = 0 \Rightarrow x = y$

$$E_2 = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$$

donc le 2è facteur principal : $u^{(2)} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ → axe 2

6). · $C^{(i)} = Y u^{(i)}$

$$C^{(1)} = Y u^{(1)} = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{5}{2} \\ -1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \\ -3 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} 6 \\ -3 \\ -3 \\ 3 \\ 3 \\ -6 \end{pmatrix}$$

$$C^{(2)} = Y u^{(2)} = \begin{pmatrix} 1 & -\frac{3}{2} & -\frac{5}{2} \\ -1 & 1 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \\ -3 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} -3 \\ 0 \\ 3 \\ 3 \\ 0 \\ -3 \end{pmatrix}$$

· $\mathrm{Cor}(X^{(i)}, C^{(j)}) = \dfrac{\mathrm{Cov}(X^{(i)}, C^{(j)})}{\sigma_{X^{(i)}} \cdot \sigma_{C^{(j)}}}$

$\mathrm{Cov}(X^{(i)}, C^{(j)}) = \langle Y^{(i)}, C^{(j)} \rangle = \sum_{k=1}^{6} p_k \, Y_k^{(i)} C_k^{(j)}$

$$\sigma_{X^{(1)}} = \sqrt{\frac{8}{3}} \quad , \quad \sigma_{X^{(2)}} = \sigma_{X^{(3)}} = \sqrt{\frac{17}{12}}$$

$$\sigma_{C^{(1)}} = \sqrt{\frac{1}{6} \times \left(\frac{1}{\sqrt{6}}\right)^2 \left[6^2 + (-3)^2 + (-3)^2 + 3^2 + 3^2 + (-6)^2\right]} = \sqrt{\frac{1}{36}(36+9+9+9+9+36)} = \sqrt{3}$$

$$\sigma_{C^{(2)}} = \sqrt{\frac{1}{6} \times \left(\frac{1}{\sqrt{3}}\right)^2 \left[(-3)^2 + 0^2 + 3^2 + 3^2 + 0^2 + (-3)^2\right]} = \sqrt{\frac{1}{18}(9+9+9+9)} = \sqrt{2}$$

$$\mathrm{Cov}(X^{(1)}, C^{(1)}) = \langle Y^{(1)}, C^{(1)} \rangle = \frac{1}{6\sqrt{6}}(6+3+6+3+18) = \frac{36}{6\sqrt{6}} = \frac{6}{\sqrt{6}}$$

$$\Rightarrow \mathrm{Corr}(X^{(1)}, C^{(1)}) = \frac{\frac{6}{\sqrt{6}}}{\sqrt{\frac{8}{3}} \times \sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\mathrm{Cov}(X^{(2)}, C^{(1)}) = \langle Y^{(2)}, C^{(1)} \rangle = \frac{1}{6\sqrt{6}}(-9-3-6-3+3) = \frac{-3}{\sqrt{6}}$$

$$\Rightarrow \mathrm{Corr}(X^{(2)}, C^{(1)}) = \frac{-3/\sqrt{6}}{\sqrt{\frac{17}{12}} \times \sqrt{3}} = -\frac{\sqrt{102}}{17}$$

$$\mathrm{Cov}(X^{(3)}, C^{(1)}) = \langle Y^{(3)}, C^{(1)} \rangle = \frac{1}{6\sqrt{6}}(-15-3+3-3) = \frac{-3}{\sqrt{6}}$$

$$\Rightarrow \mathrm{Corr}(X^{(3)}, C^{(1)}) = \frac{-\sqrt{102}}{17}$$

$$\mathrm{Cov}(X^{(1)}, C^{(2)}) = \langle Y^{(1)}, C^{(2)} \rangle = \frac{1}{6\sqrt{3}}(-3+6+9) = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \mathrm{Corr}(X^{(1)}, C^{(2)}) = \frac{\frac{2}{\sqrt{3}}}{\sqrt{\frac{8}{3}} \times \sqrt{2}} = \frac{1}{2}$$

$$\mathrm{Cov}(X^{(2)}, C^{(2)}) = \langle Y^{(2)}, C^{(2)} \rangle = \frac{1}{6\sqrt{3}}\left(\frac{9}{2}+6+\frac{3}{2}\right) = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \mathrm{Corr}(X^{(2)}, C^{(2)}) = \frac{\frac{2}{\sqrt{3}}}{\sqrt{\frac{17}{12}} \times \sqrt{2}} = \frac{2\sqrt{34}}{17}$$

$$\mathrm{Cov}(X^{(3)}, C^{(2)}) = \langle Y^{(3)}, C^{(2)} \rangle = \frac{1}{6\sqrt{3}}\left(\frac{15}{2}+3+3-\frac{3}{2}\right) = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \mathrm{Corr}(X^{(3)}, C^{(3)}) = \frac{2\sqrt{34}}{17}$$