UCLouvain

LINMA2171 - Numerical Analysis: Approximation, Interpolation, Integration

Homework 3

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November 12, 2025

Smooth submanifold

Question 1

We know by definition that $J_g(x) \in \mathbb{R}^{m\times n}$ has a rank of m $\forall x \in \mathcal{M}$. By the rank theorem, we have:

$$\underbrace{\operatorname{rank}\big(J_g(x)\big)}_{m} + \dim\left(\underbrace{\operatorname{Ker}\big(J_g(x)\big)}_{T_x\mathcal{M}}\right) = n$$
$$\iff \dim\big(\operatorname{Ker}\big(J_g(x)\big)\big) = n - m = d$$

Question 2

In this case, we have:

$$g : \mathbb{R}^n \to \mathbb{R} : x \mapsto x^\top Bx - 1$$

So $m = 1$. Furthermore, we have that g is smooth because it is a quadratic function in x (with an additional constant). Finally, the jacobian of g is given by:

$$J_g(x) = 2x^\top B$$

which always has a rank of $m = 1$ for any $x \in \mathbb{R}^n$ because it is a single row. Thus $\mathcal{M}$ is a smooth submanifold.

Let's now find an expression for the tangent space $T_x\mathcal{M}$. If $x = 0$, the jacobian is null and we thus have:

$$T_0\mathcal{M} = \mathbb{R}^n$$

Otherwise, because B is positive definite, the only vector that can cancel it is the null vector:

$$T_{x\neq 0}\mathcal{M} = \{0\}$$

Retraction with homotopy continuation

Question 1

Let's first rewrite the projection-like retraction as an optimization problem :

$$\begin{aligned} \min_{x\in\mathcal{M}} f(x) &= \frac{1}{2}\,\|p + v - x\|^2 \\ \text{s.t.}\quad g(x) &= 0 \end{aligned}$$

with $g : \mathbb{R}^m \to \mathbb{R}^m$ smooth, $J_g(x)$ has rank m $\forall x \in \mathcal{M}$ and $\mathcal{M} = \{x \in \mathbb{R}^n : g(x) = 0\}$. To solve a constrained optimization problem, we will use Lagrange multipliers. We define the Lagrangian as:

$$\mathcal{L}(x, \lambda) = \frac{1}{2} \left\| p + v - x \right\|^2 - \lambda^\top g(x)$$

where $\lambda \in \mathbb{R}^m$. To find the optimum, we need to impose the first order optimality condition:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0 \\ \iff -(p + v - x^*) - J_g(x^*)^\top \lambda^* &= 0 \\ \iff \boxed{J_g(x^*)^\top \lambda^* + p + v - x^* = 0} \end{aligned}$$

Of course, the optimum also satisfies $g(x^*) = 0$ thus proving the statement.

Question 2