LINMA2171 (2025–2026) — Numerical Analysis: Approximation, Interpolation, Integration

Homework 3

Deadline: Tuesday 25 November 2025 10:30am

Smooth submanifold Let $g : \mathbb{R}^n \to \mathbb{R}^m$ with $n > m$, we consider sets of the form

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid g(x) = 0\}. \tag{1}$$

$\mathcal{M}$ is a smooth submanifold[1] of $\mathbb{R}^n$ with dimension $d = n - m$ if g is smooth[2] and the Jacobian of g at x, denoted $J_g(x) \in \mathbb{R}^{m\times n}$, has rank m for all $x \in \mathcal{M}$. When $\mathcal{M}$ in (1) is a smooth submanifold, we can define the tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$ as

$$T_x\mathcal{M} = \{v \in \mathbb{R}^n \mid J_g(x)v = 0\}$$

1. What is the dimension of the tangent space $T_x\mathcal{M}$?
2. Let $B \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix, show that the ellipsoid

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid x^T Bx = 1\} \tag{2}$$

   is a smooth submanifold of $\mathbb{R}^n$ and give the expression for $T_x\mathcal{M}$.

Retraction with homotopy continuation A retraction on a smooth submanifold $\mathcal{M}$ at $p \in \mathcal{M}$ is a smooth mapping $R_p : T_p\mathcal{M} \to \mathcal{M}$[3] which allows to move on $\mathcal{M}$ from the point p into a direction $v \in T_p\mathcal{M}$. In particular, we focus on the projection-like retraction :

$$R_p(v) = \arg\min_{x\in\mathcal{M}} \|p + v - x\| \tag{3}$$

1. Show that there exists $\lambda^\star \in \mathbb{R}^m$ such that $x^\star = R_p(v)$ satisfies

$$\begin{pmatrix} J_g(x^\star)^T\lambda^\star + p + v - x^\star \\ g(x^\star) \end{pmatrix} = 0 \tag{4}$$

2. We consider a continuum of problems parametrized by $t \in [0, 1]$ given by

$$H(x(t), \lambda(t), t) = \begin{pmatrix} J_g(x(t))^T\lambda(t) - (1-t)J_g(p)^T\lambda_0 + p + tv - x(t) \\ g(x(t)) \end{pmatrix} = 0,$$

   where $\lambda_0 \in \mathbb{C}^m$ is arbitrary. Give the expression of the system describing the dynamics of $(x(t), \lambda(t))$ for $t \in [0, 1]$ and explain how you can use it to find $(x^\star, \lambda^\star)$ in (4).

Implementation Perform the numerical experiments on the ellipsoid (2) with the following data :

$$p = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad B = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{with } \beta > 0.$$

1. Implement an algorithm to compute $R_p(v)$ defined in (3), by discretizing the dynamics of $(x(t), \lambda(t))$ using the Euler explicit method with a time step $\Delta t > 0$. Take[4] $\lambda_0 = 1 + i$ and $\beta = 1$, display $\mathcal{M}$, $p + T_p\mathcal{M}$, $p + v$ and the real and imaginary part of the trajectory $x(t)$.

[1] The general definition of a smooth submanifold can be found in [2].
[2] In this context, smooth means infinitely many times continuously differentiable.
[3] It is also required that each curve $c(t) = R_p(tv)$ satisfies $c(0) = x$ and $c'(0) = v$.
[4] Note that the arrays in your code may not handle complex numbers by default.

2. Keep $\lambda_0 = 1 + i$ and $\beta = 1$, plot[5] the value of $\|H(x(t), \lambda(t), t)\|$ as a function of t, for different values of Δt, what is the issue ? Implement an improved version of the previous algorithm with Newton correction steps, see [3], and show numerically that it works better. Comment on the efficiency of your implementation and motivate your choice for Δt.

3. For several values of β, study the trajectories of $\lambda(t)$ in the complex plane depending on the initial value λ_0. Do they always converge to a value leading to a valid retraction $R_p(v)$? What is a "good" choice for λ_0 ?

You should provide clear graphs and discuss all your numerical results.

Practical information.

Meetings with the TA: By appointment
Writing : You must do all the writing (report and code) ***individually***. Never share your production, but you are allowed, and even encouraged, to exchange ideas on how to address the homework. If you use an LLM, mention it in the report and explain for which purpose it was used. However, we believe that this assignment will be more useful to you—and the report more pleasant to read for us—if you use LLMs sparingly, if at all.
Questions: Feel free to contact me at timothe.taminiau@uclouvain.be to ask questions or to set up a meeting.
Submission: Using Moodle. Keep in mind that the deadline is automatically enforced, using the clock of Moodle!
Python/Octave/Matlab/Julia codes: Please don't copy your code to your report and add the `.py/.ipynb/.m/.jl` files as attachments. Add a script that reproduces all your results.
Language: Reports in French are accepted without penalty. However, English is strongly encouraged. The quality of the English will not impact the grade, provided that the text is intelligible.

Deadline: 25 November, 2025 at 10:30

References

[1] P.-A. Absil and Jérôme Malick. "Projection-like Retractions on Matrix Manifolds". In: *SIAM Journal on Optimization* 22.1 (2012), pp. 135–158. DOI: 10.1137/100802529.

[2] Nicolas Boumal. *An Introduction to Optimization on Smooth Manifolds*. 1st ed. Cambridge University Press, 2023. DOI: 10.1017/9781009166164.

[3] Alexander Heaton and Matthias Himmelmann. *Euclidean distance and maximum likelihood retractions by homotopy continuation*. 2022. DOI: 10.48550/arXiv.2206.14106.

[5] Graphs displaying very small quantities should always be in logarithmic scale for the y-axis.