



UCLOUVAIN

LINMA2171 - Numerical Analysis: Approximation,
Interpolation, Integration

Homework 3

Student

LUCAS AHOU (35942200)

Teacher

P-A. ABSIL

Smooth submanifold

Question 1

We know by definition that $J_g(x) \in \mathbb{R}^{m \times n}$ has a rank of $m \forall x \in \mathcal{M}$. By the rank theorem, we have:

$$\underbrace{\text{rank}(J_g(x))}_m + \dim\left(\underbrace{\text{Ker}(J_g(x))}_{T_x \mathcal{M}}\right) = n$$

$$\Leftrightarrow \dim(\text{Ker}(J_g(x))) = n - m = d$$

Question 2

In this case, we have:

$$g : \mathbb{R}^n \rightarrow \mathbb{R} : x \mapsto x^\top Bx - 1$$

So $m = 1$. Furthermore, we have that g is smooth because it is a quadratic function in x (with an additional constant). Finally, the jacobian of g is given by:

$$J_g(x) = 2x^\top B$$

which always has a rank of $m = 1$ for any $x \in \mathbb{R}^n$ because it is a single row. Thus \mathcal{M} is a smooth submanifold.

Let's now find an expression for the tangent space $T_x \mathcal{M}$. If $x = 0$, the jacobian is null and we thus have:

$$T_0 \mathcal{M} = \mathbb{R}^n$$

Otherwise:

$$T_{x \neq 0} \mathcal{M} = \{v \in \mathbb{R}^n : x^\top Bv = 0\}$$

Retraction with homotopy continuation

Question 1

Let's first rewrite the projection-like retraction as an optimization problem :

$$\begin{aligned} \min_{x \in \mathcal{M}} f(x) &= \frac{1}{2} \|p + v - x\|^2 \\ \text{s.t. } g(x) &= 0 \end{aligned}$$

with $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ smooth, $J_g(x)$ has rank $m \forall x \in \mathcal{M}$ and $\mathcal{M} = \{x \in \mathbb{R}^n : g(x) = 0\}$. To solve a constrained optimization problem, we will use Lagrange multipliers. We define the Lagrangian as:

$$\mathcal{L}(x, \lambda) = \frac{1}{2} \|p + v - x\|^2 - \lambda^\top g(x)$$

where $\lambda \in \mathbb{R}^m$. To find the optimum, we need to impose the first order optimality condition:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) &= 0 \\ \iff -(p + v - x^*) - J_g(x^*)^\top \lambda^* &= 0 \\ \iff J_g(x^*)^\top \lambda^* + p + v - x^* &= 0 \end{aligned}$$

Of course, the optimum also satisfies $g(x^*) = 0$ thus proving the statement.

Question 2

We are given the system $H(x(t), \lambda(t), t) = 0$ and are asked to give a dynamical system based on that. Let's differentiate the function with respect to t :

$$\frac{d}{dt} H(x(t), \lambda(t), t) = 0$$

Let's denote $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$ where:

$$\begin{cases} H_1(x, \lambda, t) = J_g(x)^\top \lambda - (1-t)J_g(p)^\top \lambda_0 + p + tv - x \\ H_2(x, \lambda, t) = g(x) \end{cases}$$

and where I omitted the time parameter in x and λ for a more pleasant reading experience from the grader point of view.

Let's now compute all the necessary derivatives, starting with H_1 :

- $\frac{\partial H_1}{\partial x}$:

We know that:

$$J_g(x)^\top \lambda = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \dots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^m \lambda_k \frac{\partial g_k}{\partial x_1} \\ \vdots \\ \sum_{k=1}^m \lambda_k \frac{\partial g_k}{\partial x_n} \end{pmatrix}$$

Thus:

$$\frac{\partial}{\partial x} (J_g(x)^\top \lambda) = T(x, \lambda) \in \mathbb{R}^{n \times n}$$

where $[T(x, \lambda)]_{i,j} = \sum_{k=1}^m \lambda_k \frac{\partial^2 g_k}{\partial x_i \partial x_j}$. Finally:

$$\frac{\partial H_1}{\partial x} = \frac{\partial}{\partial x} (J_g(x)^\top \lambda) - \frac{\partial x}{\partial x} = T(x, \lambda) - I_n$$

- $\frac{\partial H_1}{\partial \lambda} = J_g(x)^\top$

- $\frac{\partial H_1}{\partial t} = J_g(p)^\top \lambda_0 + v$

Knowing that $\frac{dH_1}{dt} = \frac{\partial H_1}{\partial x} \dot{x} + \frac{\partial H_1}{\partial \lambda} \dot{\lambda} + \frac{\partial H_1}{\partial t} = 0$, we obtain:

$$(T(x, \lambda) - I_n) \dot{x} + J_g(x)^\top \dot{\lambda} + J_g(p)^\top \lambda_0 + v = 0 \quad (\text{Eq. 1})$$

Differentiating H_2 is straightforward and gives us:

$$\frac{dH_2}{dt} = J_g(x) \dot{x} = 0 \quad (\text{Eq. 2})$$

We can now put everything in matrix form to get:

$$\begin{pmatrix} T(x, \lambda) - I_n & J_g(x)^\top \\ J_g(x) & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} -J_g(p)^\top \lambda_0 - v \\ 0 \end{pmatrix}$$

To find a solution (x^*, λ^*) to the original problem, we notice that $(x(t), \lambda(t))$ is a solution of the original problem in this dynamical system when $t = 1$. We can thus solve the dynamical system for $t \in [0, 1]$ and look at the solution at $t = 1$.

To do that, we need an initial condition. We notice that taking $x(0) = p$ and $\lambda(0) = \lambda_0$ satisfies the system for $t = 0$. We can thus take those to simulate the system.

Implementation

Question 1

Here we have:

$$g(x) = \beta x_1^2 + x_2^2$$

Using the previously derived dynamical model and the parameters given in the statement, we obtain this system to solve:

$$\begin{pmatrix} 2\lambda(t)\beta - 1 & 0 & 2\beta x_1(t) \\ 0 & 2\lambda(t) - 1 & 2x_2(t) \\ 2\beta x_1(t) & 2x_2(t) & 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} -1 \\ -2\lambda_0 \\ 0 \end{pmatrix}$$

We are also asked to plot $p + T_p \mathcal{M}$. We know that:

$$\begin{aligned} J_g(x) &= (2\beta x_1 \ 2x_2) \\ \implies J_g(p) &= (0 \ 2) \end{aligned}$$

$T_p \mathcal{M}$, which is the set of vectors which cancels $J_g(p)$, is thus:

$$T_p \mathcal{M} = \text{span} \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

After solving the dynamical system using the given parameters, we obtain the following plot:

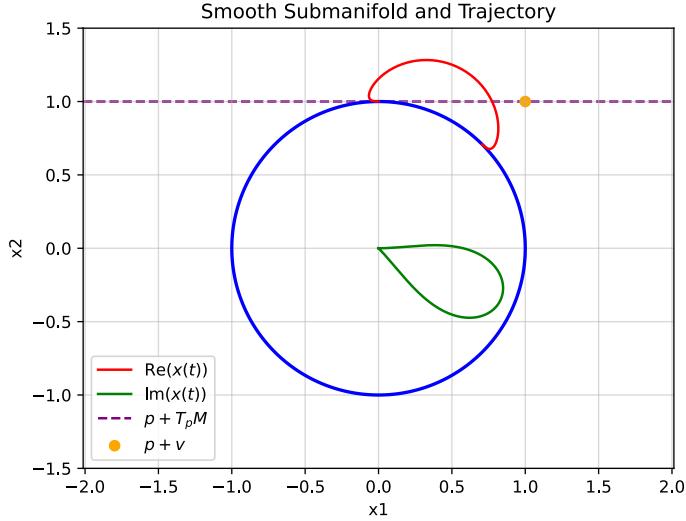


Figure 1 – Trajectory $x(t)$, the smooth submanifold \mathcal{M} and the tangent space $T_p \mathcal{M}$

On Figure 1, we can see that the real part of $x(t)$ converges towards the closest point of \mathcal{M} to the vector $p + v$. We also notice that the imaginary part of $x(t)$ goes back to 0 at the final timestep. This indicates that the retraction was correctly computed.

Question 2

We are now interested at the error of the solution. Let's thus analyze the norm of H :

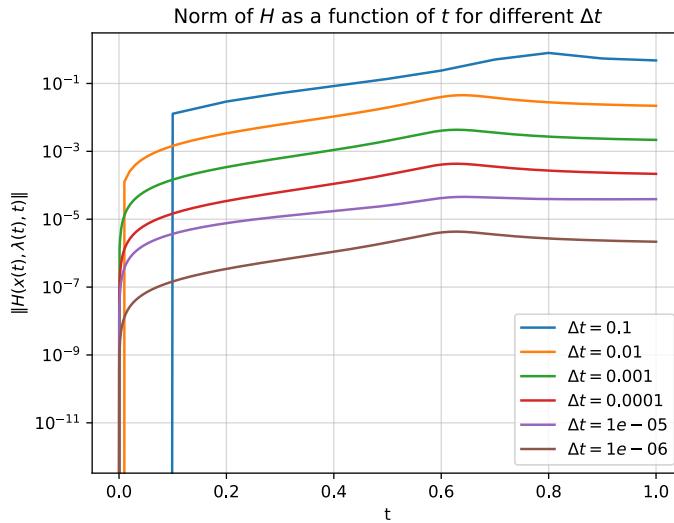


Figure 2 – Errors of the approximation as a function of time for different Δt

On Figure 2, even though the error seems to be very small for small Δt , we actually have to decrease Δt by a considerable amount to achieve a small error. For $\Delta t = 0.1$, we see

that the error is too large to be considered a good approximation. For $\Delta t = 1e-06$, we get a pretty good approximation. However, the cost of computation becomes quite expensive.

As suggested, let's add Newton correction steps into the previous algorithm. To do that, we will keep the Euler explicit method and at each time step, we take the new iterate and perform the Newton-Raphson method until we reach a tolerance of $\epsilon = 1e-10$. We then continue this loop with this new corrected iterate. Here is the plot of the errors for this new method:

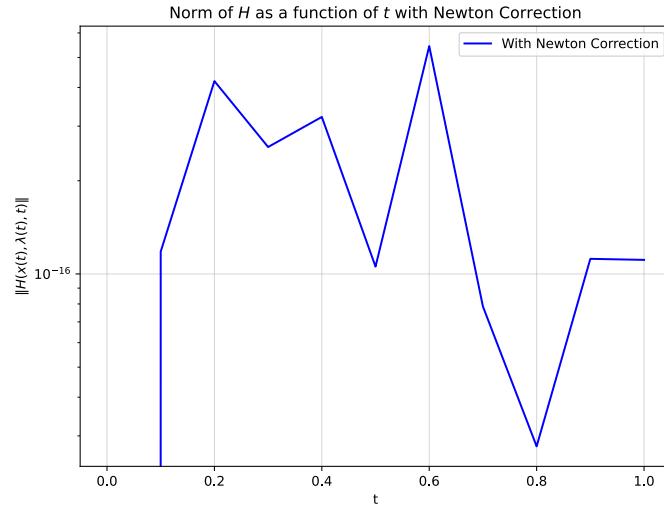


Figure 3 – Error of the approximation for $\Delta t = 0.1$

On Figure 3, we reached a much smaller error at the final time step even though we used a large Δt of 0.1. This allows to converge very quickly to the solution and with greater precision. I chose $\Delta t = 0.1$ because choosing a smaller Δt gave an error of the same order of magnitude and is thus not necessary.

Question 3

Let's now plot the trajectories of $\lambda(t)$ depending on the initial condition λ_0 for different β :

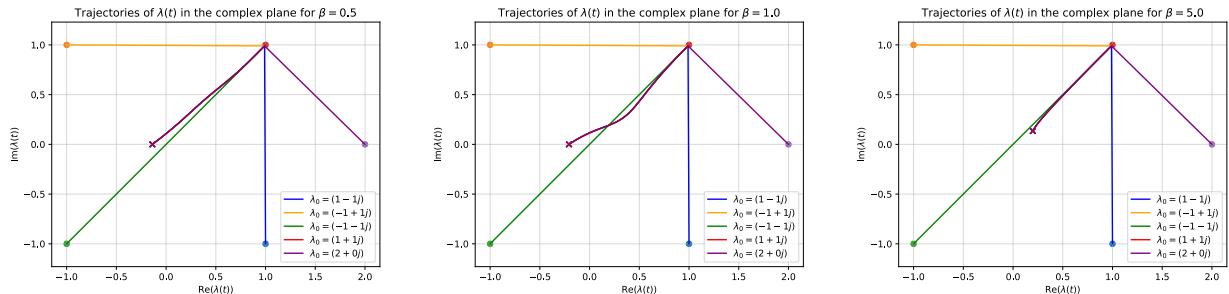


Figure 4 – Trajectories of $\lambda(t)$ in the complex plane depending of the initial condition λ_0 for different values of β

Here, we see that for $\beta = 5$, all λ trajectories converge to the same point that has a non-zero imaginary part. This solution is thus not valid as the optimal Lagrange multiplier λ^* must be real.

Bibliography

- [1] “Newton polynomial.” [Online]. Available: https://en.wikipedia.org/wiki/Newton%27s_method#Multidimensional_formulations