LINMA2171 (2025–2026) — Numerical Analysis: Approximation, Interpolation, Integration

Homework 4

Deadline: Tuesday 16 December 2025 at 10:30am

Discrete minimax approximation We first study the minimax approximation problem in a purely discrete setting. We assume that the function f is only known at a finite set of $m+1$ distinct sampling points

$$x_0 < x_1 < \cdots < x_m,$$

and we denote the available data by $f(x_0), \ldots, f(x_m)$. Given a degree $n \geq 0$, we seek a polynomial $p \in \mathcal{P}_n$ that best fits the data in the ℓ^∞ sense. The discrete minimax problem is therefore

$$\min_{p \in \mathcal{P}_n} \max_{i=0,\ldots,m} \left| f(x_i) - p(x_i) \right|.$$

1. When $n > m$, prove that the discrete minimax problem admits infinitely many solutions.
2. When $n = m$, prove that there exists exactly one solution.
3. *(Bonus)* When $n < m$, prove that the problem admits at least one solution.

Continuous minimax approximation We now consider the continuous minimax approximation of a continuously differentiable function $f : [a, b] \to \mathbb{R}$. Given an integer $n \geq 0$, the objective is to find a polynomial $p \in \mathcal{P}_n$ that minimizes the uniform error. The continuous minimax problem is therefore

$$\min_{p \in \mathcal{P}_n} \max_{x \in [a,b]} |f(x) - p(x)|.$$

To approximate such a polynomial, we rely on the *Remez exchange algorithm*, which iteratively enforces an alternating error pattern at $n+2$ reference points

$$a \leq \xi_0 < \xi_1 < \cdots < \xi_{n+1} \leq b.$$

1. Implement the Remez exchange algorithm for a given function $f : [a, b] \to \mathbb{R}$, a polynomial degree n, and a tolerance $\varepsilon > 0$. At each iteration, estimate $\|f - p\|_\infty$ by searching for the point where the absolute error is maximal.
2. Apply your implementation on the interval $[-1, 1]$ to the functions

$$f_1(x) = \sqrt{1 + 10^{-4} - x}, \qquad f_2(x) = \frac{1}{1 + e^{-10x}}.$$

   For the initialization of the reference points $\xi_0, \ldots, \xi_{n+1}$, consider the following two strategies:

   - *Equidistant points:* $\xi_i = -1 + i\Delta x$ with $\Delta x = \frac{2}{n+1}$;
   - *Chebyshev extrema:* $\xi_i = -\cos\left(\pi \dfrac{i}{n+1}\right)$.

   The study consists of two parts:

   (a) For fixed degrees $n \in \{2, 4, 8\}$, plot the resulting minimax polynomial approximation p_n for each function and each initialization strategy, together with the original function.

   (b) Study the evolution of the minimax error as a function of the degree. For n ranging over a larger set (e.g. $n = 4, 5, \ldots, 24$), compute and plot

   $$e(n) = \|f - p_n\|_\infty$$

   for both initialization strategies.

   (c) Compare both approaches: which initialization works best, in terms of convergence (number of Remez iterations) and final minimax error? Justify your claim with theoretical and/or numerical arguments.

Practical information.

Meetings with the TA: by appointment.
Writing : You must do all the writing (report and code) ***individually***. Never share your production, but you are allowed, and even encouraged, to exchange ideas on how to address the homework. If you use an LLM, mention it in the report and explain for which purpose it was used. However, we believe that this assignment will be more useful to you—and the report more pleasant to read for us—if you use LLMs sparingly, if at all.
Questions: Feel free to contact me at antoine.springael@uclouvain.be to ask questions or to set up a meeting.
Submission: Using Moodle. Keep in mind that the deadline is automatically enforced, using the clock of Moodle!
Python/Octave/Matlab/Julia codes: Please don't copy your code to your report and add the `.py`/`.ipynb`/`.m`/`.jl` files as attachments. Add a script named `run` that reproduces all your results.
Language: Reports in French are accepted without penalty. However, English is strongly encouraged. The quality of the English will not impact the grade, provided that the text is intelligible.

Deadline: Tuesday 16 December 2025 at 10:30am