UCLouvain

LINMA2171 - Numerical Analysis: Approximation, Interpolation, Integration

# Homework 3

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# Smooth submanifold

## Question 1

We know by definition that $J_g(x) \in \mathbb{R}^{m \times n}$ has a rank of $m$ $\forall x \in \mathcal{M}$. By the rank theorem, we have:

$$\underbrace{\mathrm{rank}\big(J_g(x)\big)}_{m} + \dim\left(\underbrace{\mathrm{Ker}\big(J_g(x)\big)}_{T_x\mathcal{M}}\right) = n$$
$$\iff \dim\big(\mathrm{Ker}\big(J_g(x)\big)\big) = n - m = d$$

## Question 2

In this case, we have:

$$g : \mathbb{R}^n \to \mathbb{R} : x \mapsto x^\top B x - 1$$

So $m = 1$. Furthermore, we have that $g$ is smooth because it is a quadratic function in $x$ (with an additional constant). Finally, the jacobian of $g$ is given by:

$$J_g(x) = 2x^\top B$$

which always has a rank of $m = 1$ for any $x \in \mathbb{R}^n$ because it is a single row. Thus $\mathcal{M}$ is a smooth submanifold.

Let's now find an expression for the tangent space $T_x\mathcal{M}$. If $x = 0$, the jacobian is null and we thus have:

$$T_0\mathcal{M} = \mathbb{R}^n$$

Otherwise:

$$T_{x \neq 0}\mathcal{M} = \{v \in \mathbb{R}^n : x^\top B v = 0\}$$

# Retraction with homotopy continuation

## Question 1

Let's first rewrite the projection-like retraction as an optimization problem :

$$\min_{x \in \mathcal{M}} f(x) = \frac{1}{2}\,\|p + v - x\|^2$$
$$\text{s.t.} \quad g(x) = 0$$

with $g : \mathbb{R}^m \to \mathbb{R}^m$ smooth, $J_g(x)$ has rank $m$ $\forall x \in \mathcal{M}$ and $\mathcal{M} = \{x \in \mathbb{R}^n : g(x) = 0\}$. To solve a constrained optimization problem, we will use Lagrange multipliers. We define the Lagrangian as:

$$\mathcal{L}(x, \lambda) = \frac{1}{2} \|p + v - x\|^2 - \lambda^\top g(x)$$

where $\lambda \in \mathbb{R}^m$. To find the optimum, we need to impose the first order optimality condition:

$$\begin{aligned} \nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \\ \iff -(p + v - x^*) - J_g(x^*)^\top \lambda^* = 0 \\ \iff \boxed{J_g(x^*)^\top \lambda^* + p + v - x^* = 0} \end{aligned}$$

Of course, the optimum also satisfies $g(x^*) = 0$ thus proving the statement.

## Question 2

We are given the system $H(x(t), \lambda(t), t) = 0$ and are asked to give a dynamical system based on that. Let's differentiate the function with respect to $t$:

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x(t), \lambda(t), t) = 0$$

Let's denote $H = \binom{H_1}{H_2}$ where:

$$\begin{cases} H_1(x, \lambda, t) = J_g(x)^\top \lambda - (1-t) J_g(p)^\top \lambda_0 + p + tv - x \\ H_2(x, \lambda, t) = g(x) \end{cases}$$

and where I omitted the time parameter in $x$ and $\lambda$ for a more pleasant reading experience from the grader point of view.
I will now compute all the necessary derivatives, starting with $H_1$:

- $\frac{\partial H_1}{\partial x}$:

  We know that:

  $$J_g(x)^\top \lambda = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^m \lambda_k \frac{\partial g_k}{\partial x_1} \\ \vdots \\ \sum_{k=1}^m \lambda_k \frac{\partial g_k}{\partial x_n} \end{pmatrix}$$

  Thus:

  $$\frac{\partial}{\partial x} \left( J_g(x)^\top \lambda \right) = T(x, \lambda) \in \mathbb{R}^{n \times n}$$

  where $[T(x, \lambda)]_{i,j} = \sum_{k=1}^m \lambda_k \frac{\partial^2 g_k}{\partial x_i \partial x_j}$. Finally:

  $$\frac{\partial H_1}{\partial x} = \frac{\partial}{\partial x} \left( J_g(x)^\top \lambda \right) - \frac{\partial x}{\partial x} = T(x, \lambda) - I_n$$

- $\frac{\partial H_1}{\partial \lambda} = J_g(x)^\top$

- $\frac{\partial H_1}{\partial t} = J_g(p)^\top \lambda_0 + v$

Knowing that $\frac{\mathrm{d}H_1}{\mathrm{d}t} = \frac{\partial H_1}{\partial x}\dot{x} + \frac{\partial H_1}{\partial \lambda}\dot{\lambda} + \frac{\partial H_1}{\partial t} = 0$, we obtain:

$$(T(x,\lambda) - I_n)\dot{x} + J_g(x)^\top \dot{\lambda} + J_g(p)^\top \lambda_0 + v = 0 \quad (\text{Eq. } 1)$$

Differentiating $H_2$ is straightforward and gives us:

$$\frac{\mathrm{d}H_2}{\mathrm{d}t} = J_g(x)\dot{x} = 0 \quad (\text{Eq. } 2)$$

We can now put everything in matrix form to get:

$$\begin{pmatrix} T(x,\lambda) - I_n & J_g(x)^\top \\ J_g(x) & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} -J_g(p)^\top \lambda_0 - v \\ 0 \end{pmatrix}$$

To find a solution $(x^*, \lambda^*)$ to the original problem, we notice that $(x(t), \lambda(t))$ is a solution of the original problem in this dynamical system when $t = 1$. We can thus solve the dynamical system for $t \in [0, 1]$ and look at the solution at $t = 1$.
To do that, we need an initial condition. We notice that taking $x(0) = p$ and $\lambda(0) = \lambda_0$ satisfies the system for $t = 0$. We can thus take those to simulate the system.

# Implementation