UCLouvain

LINMA2171 - Numerical Analysis: Approximation, Interpolation, Integration

Homework 4

Minimax Approximation

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Discrete minimax approximation

1. When $n > m$, prove that the discrete minimax problem admits infinitely many solutions.

Answer

If $n > m$, then $n + 1 > m + 1$. We have more degree of freedom $(n + 1)$ than data points. It is thus possible to interpolate f at $\{x_i\}_{i=0}^{m}$ by a polynomial p of degree n such that the objective function is 0. Furthermore, considering the polynomial:

$$q(x) = \prod_{i=0}^{m} (x - x_i) \in \mathcal{P}_n$$

and any real number $\gamma \in \mathbb{R}$, the polynomial $\overline{p}(x) = p(x) + \gamma q(x)$ also interpolates f at $\{x_i\}_{i=0}^{m}$, so the error is still 0.

We conclude that there is an infinite number of polynomials of degree at most n that satisfies the discrete minimax problem in this case. □

2. When $n = m$, prove that there exists exactly one solution.

Answer

The reasoning is similar to the previous case. However, when $n = m$, the polynomial which interpolates f at $\{x_i\}_{i=0}^{m}$ is unique. To show that this polynomial is indeed the unique solution, let's consider any polynomial $q \in \mathcal{P}_n$ with $q \neq p$. Then, $\exists x_i$ such that $q(x_i) \neq p(x_i)$. This implies that, at this x_i, $|f(x_i) - q(x_i)| > 0$ and thus :

$$\max_{i=0,\ldots,m} |f(x_i) - q(x_i)| > 0$$

which is worse than the solution p.

$\Longrightarrow$ There exists a unique solution p to the problem in this case. □

3. (*Bonus*) When $n < m$, prove that the problem admits at least one solution.

Answer

The discrete minimax problem is equivalent to minimizing:

$$F(a_0, \ldots, a_n) = \max_{i=0,\ldots,m} |f(x_i) - p(x_i)|$$

where $p(x) = \sum_{j=0}^{n} a_j x^j$. Using a vectorial notation:

$$F(\boldsymbol{a}) = \|\mathbf{f} - \boldsymbol{P}(\boldsymbol{a})\|_\infty$$

where $\boldsymbol{P}(\boldsymbol{a}) := (p(x_0), \ldots, p(x_m))^\top$. We want to prove that $F(a)$ is continuous and coercive with respect to the coefficients of the polynomial $\boldsymbol{a} = (a_0, \ldots, a_n)^\top$. This way, using also the fact that the whole set $\mathbb{R}^{n+1}$ is closed, we can conclude that there exists an optimum, which is also global.

- F continuous:
  $\longrightarrow \boldsymbol{P} : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ is a linear function with respect to $\boldsymbol{a}$. We thus have an affine function inside the ∞-norm (which is a continous function) because $\mathbf{f}$ is constant. F, which is a composition of two continous functions, is therefore continuous.

Answer (cont.)

- F coercive:
  $\longrightarrow$ We have to prove that $F \to \infty$ as $\|a\|_2 \to \infty$. Using the triangle inequality, we obtain:
  $$\begin{aligned} F(a) &= \|\mathbf{f} - \boldsymbol{P}(\boldsymbol{a})\|_\infty \\ &\geq \|\mathbf{f}\|_\infty - \|\boldsymbol{P}(\boldsymbol{a})\|_\infty \end{aligned}$$
  It is therefore sufficient to prove that $\|\boldsymbol{P}(\boldsymbol{a})\|_\infty \to \infty$ as $\|\boldsymbol{a}\|_2 \to \infty$.
  If $\|\boldsymbol{a}\|_2 \to \infty$, then at least one coefficient tends (in absolute value) to ∞. Let this coefficient be a_s. Since there is also at least a data point $x_i \neq 0$, the term associated with a_s will also tend (in absolute value) to ∞. We therefore have that $\|\boldsymbol{P}(\boldsymbol{a})\|_\infty \to \infty$ as $\|\boldsymbol{a}\|_2 \to \infty$, which proves that F is coercive.

We have proven that F is a continous and coercive function. Furthermore, we now that $\mathbb{R}^{n+1}$, the feasible set, is closed (because it is the whole space). We conclude that there exists an optimum which is also global. $\square$

Continuous minimax approximation

In this section, I will present the results of my implementation of the *Remez exchange algorithm* to solve the continuous minimax approximation.

1. Implement the Remez exchange algorithm for a given function $f : [a, b] \to \mathbb{R}$, a polynomial degree n, and a tolerance $\varepsilon > 0$. At each iteration, estimate $\|f - p\|_\infty$ by searching for the point where the absolute error is maximal.

Answer

$\longrightarrow$ cfr. `run.py`