UCLouvain

LINMA2171 - Numerical Analysis: Approximation, Interpolation, Integration

Homework 4

Minimax Approximation

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Discrete minimax approximation

1. When $n > m$, prove that the discrete minimax problem admits infinitely many solutions.

Answer

If $n > m$, then $n+1 > m+1$. We have more degree of freedom $(n+1)$ than data points. It is thus possible to interpolate f at $\{x_i\}_{i=0}^m$ by a polynomial p of degree n such that the objective function is 0. Furthermore, considering the polynomial:

$$q(x) = \prod_{i=0}^{m}(x - x_i) \in \mathcal{P}_n$$

and any real number $\gamma \in \mathbb{R}$, the polynomial $\overline{p}(x) = p(x) + \gamma q(x)$ also interpolates f at $\{x_i\}_{i=0}^m$, so the error is still 0.

We conclude that there is an infinite number of polynomials of degree at most n that satisfies the discrete minimax problem in this case. $\square$

2. When $n = m$, prove that there exists exactly one solution.

Answer

The reasoning is similar to the previous case. However, when $n = m$, the polynomial which interpolates f at $\{x_i\}_{i=0}^m$ is unique. To show that this polynomial is indeed the unique solution, let's consider any polynomial $q \in \mathcal{P}_n$ with $q \neq p$. Then, $\exists x_i$ such that $q(x_i) \neq p(x_i)$. This implies that, at this x_i, $|f(x_i) - q(x_i)| > 0$ and thus :

$$\max_{i=0,\dots,m} |f(x_i) - q(x_i)| > 0$$

which is worse than the solution p.

$\Longrightarrow$ There exists a unique solution p to the problem in this case. $\square$

3. (*Bonus*) When $n < m$, prove that the problem admits at least one solution.

Answer

The discrete minimax problem is equivalent to minimizing:

$$F(a_0, \dots, a_n) = \max_{i=0,\dots,m} |f(x_i) - p(x_i)|$$

where $p(x) = \sum_{j=0}^{n} a_j x^j$. Using a vectorial notation:

$$F(\boldsymbol{a}) = \|\mathbf{f} - \boldsymbol{P}(\boldsymbol{a})\|_\infty$$

where $\boldsymbol{P}(\boldsymbol{a}) := (p(x_0), \dots, p(x_m))^\top$. We want to prove that $F(a)$ is continuous and coercive with respect to the coefficients of the polynomial $\boldsymbol{a} = (a_0, \dots, a_n)^\top$. This way, using also the fact that the whole set $\mathbb{R}^{n+1}$ is closed, we can conclude that there exists an optimum, which is also global.

- F continuous:
  $\longrightarrow$ $\boldsymbol{P} : \mathbb{R}^{n+1} \to \mathbb{R}^{m+1}$ is a linear function with respect to $\boldsymbol{a}$. We thus have an affine function inside the ∞-norm (which is a continous function) because $\mathbf{f}$ is constant. F, which is a composition of two continous functions, is therefore continuous.

Answer (cont.)

- F coercive:
  $\longrightarrow$ We have to prove that $F \to \infty$ as $\|a\|_2 \to \infty$. Using the triangle inequality, we obtain:
  $$\begin{aligned} F(a) &= \||\mathbf{f} - \boldsymbol{P}(\boldsymbol{a})\|_\infty \\ &\geq \|\mathbf{f}\|_\infty - \|\boldsymbol{P}(\boldsymbol{a})\|_\infty \end{aligned}$$
  It is therefore sufficient to prove that $\|\boldsymbol{P}(\boldsymbol{a})\|_\infty \to \infty$ as $\|\boldsymbol{a}\|_2 \to \infty$.
  If $\|\boldsymbol{a}\|_2 \to \infty$, then at least one coefficient tends (in absolute value) to ∞. Let this coefficient be a_s. Since there is also at least a data point $x_i \neq 0$, the term associated with a_s will also tend (in absolute value) to ∞. We therefore have that $\|\boldsymbol{P}(\boldsymbol{a})\|_\infty \to \infty$ as $\|\boldsymbol{a}\|_2 \to \infty$, which proves that F is coercive.

We have proven that F is a continous and coercive function. Furthermore, we now that $\mathbb{R}^{n+1}$, the feasible set, is closed (because it is the whole space). We conclude that there exists an optimum which is also global. $\square$

Continuous minimax approximation

In this section, I will present the results of my implementation of the *Remez exchange algorithm* to solve the continuous minimax approximation.

1. Implement the Remez exchange algorithm for a given function $f : [a, b] \to \mathbb{R}$, a polynomial degree n, and a tolerance $\varepsilon > 0$. At each iteration, estimate $\|f - p\|_\infty$ by searching for the point where the absolute error is maximal.

Answer

$\longrightarrow$ cfr. `run.py`

2.a) For fixed degrees $n \in \{2, 4, 8\}$, plot the resulting minimax polynomial approximation p_n for each function and each initialization strategy, together with the original function.

Answer

As stated in the question, I ran the implementation of the Remez exchange algorithm for $n \in \{2, 4, 8\}$ for both functions to compare both initialization techniques:

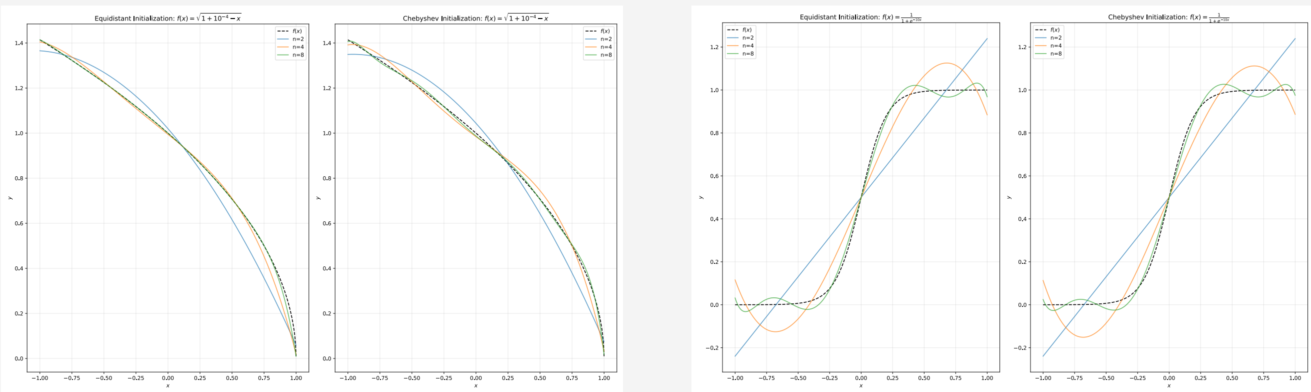


Figure 1: Minimax approximation for f_1 and f_2 using equidistant and Chebyshev points for different degrees

On Figure 1, we notice that, in every case, increasing the degree gives us a better approximation.

Answer (cont.)

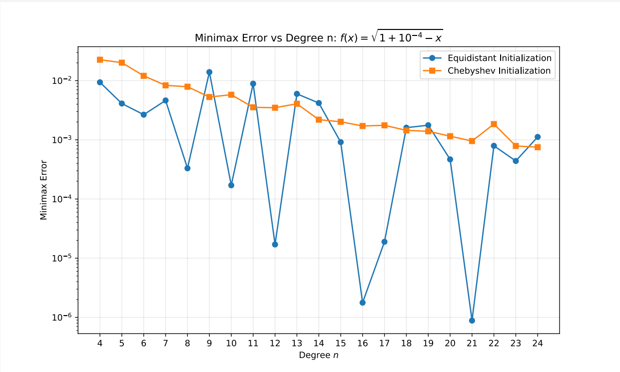
Furthermore, we notice that in the case of the first function, the approximation quickly becomes very close to the original function as the degree of the polynomial increases. This is not the case of the second function where we observe a quite huge approximation error even for $n = 8$. We also notice that, for f_2, the approximation of degree 2 results in a linear function. This is likely due to the fact that the function is anti-symetric around 0.

2.b) Study the evolution of the minimax error as a function of the degree. For n ranging over a larger set (e.g. $n = 4, 5, ..., 24$), compute and plot

$$e(n) = \|f - p\|_\infty$$

for both initialization strategies.

Answer



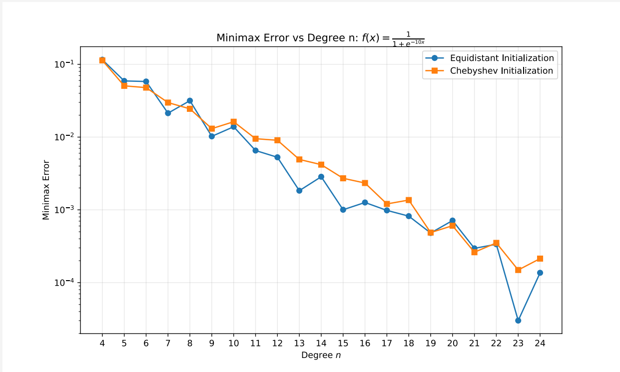


Figure 2: Minimax approximation errors for f_1 and f_2 using equidistant and Chebyshev points as a function of the polynomial degrees

On Figure 2, we notice that for both functions, the equidistant stratgy seems to be slightly better than the Chebyshev one, especially for f_1 where the former sometimes achieves a really small error even for smaller degrees. For f_2, both strategies seem to perform similarly.

2.c) Compare both approaches: which initialization works best, in terms of convergence (numberof Remez iterations) and final minimax error? Justify your claim with theoretical and/or numerical arguments.

Answer

As mentionned above, both initialization strategies share similar performances in terms of the final error. We will thus compare them based on the convergence to the optimal solution. Here is a plot of the number of iterations to obtain a convergence for both functions as a function of the degrees:

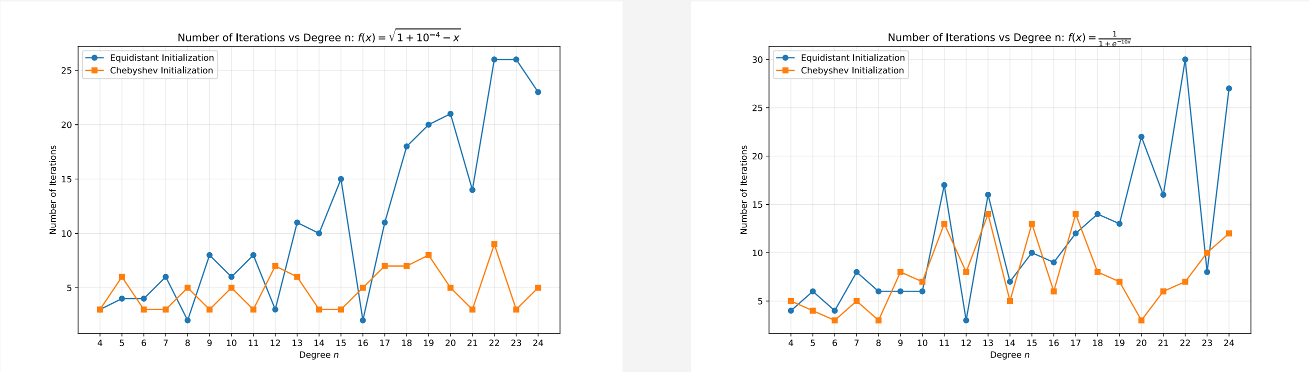


Figure 3: Number of Remez iterations for f_1 and f_2 using equidistant and Chebyshev points as a function of the polynomial degrees

On Figure 3, we notice that the Chebyshev initialization strategy consistently requires less iterations to converge to the optimal solution. This is especially true for f_1 where the difference can be quite significant for higher degrees. This behavior can be explained by the fact that we have a guaranteed bound with the Chebyshev interpolant (Th. 4.2.1 in the course notes), which is not the case with the equidistant points. Therefore, starting from a better initial guess allows the Remez algorithm to converge faster. Chebyshev initialization is thus slightly better in this context.