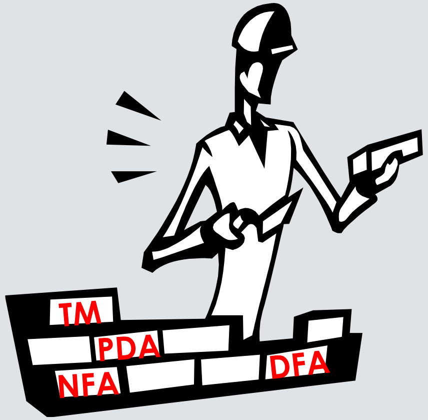
CHAPTER 0

Introduction - Review

What we'll cover this semester



Mathematical models of computation

Decidability theory
(What can be computed, i.e., decided, by models)

Complexity Theory
(Resources required so that problems can be computed)

Review – Mathematical Models

- Sets

Set

S={blue, red, yellow, green} $blue \in S$

Elements (or members)

A={blue, red} $green \notin S$

such that

$A \subseteq S$

All elements in A are also in S

$\forall x \in A \implies x \in S$

$S \subseteq S$

Subsets

$A \subsetneq S$

All elements in A are also in S and
exists at least one element in S that is not in A

$\forall x \in A \implies x \in S \wedge$

$\exists y \in S \implies y \notin A$

Proper subset

A={1, 2, 3, ….} or
A= { n | $n \in \mathbb{N}$}

Infinite set

B={}

The empty set

B={white}

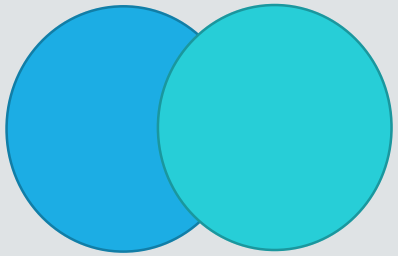
Singleton set

B={red, blue}

Unordered pair

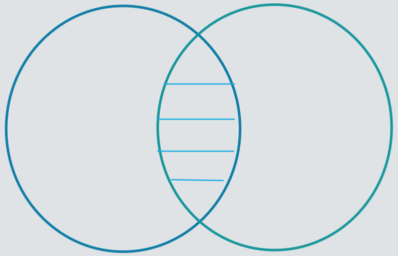
Review – Mathematical Models

- Venn Diagrams of Sets – operations on sets



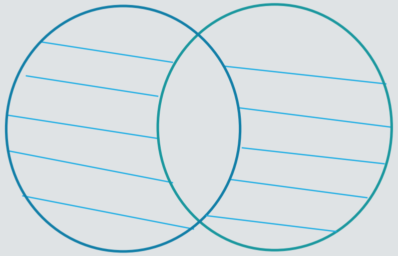
$(A \cup B)$

Union



$(A \cap B)$

Intersection



$(A \cap B)^C \approx \overline{(A \cap B)}$

Complement
(A and B contain all elements)

Review – Mathematical Models

- Sets

$\{1,2,3\} = \{1,1,3,2\}$
$\{1,2,3\} \neq \{1,1,2\}$

Order and repetition of elements does not matter

All elements in set A are also in set B <u>and</u> all elements in Set B are also in set A.

$$\forall x \in A \implies x \in B \land \forall x \in B \implies x \in A$$

Equivalent Sets

$|\{1,2,3\}| = 3$
$|\{1,1,2,3\}| = 3$
$|\{\ \}| = 0$

Cardinality

$$A = \{0, 1\}$$
$$\mathcal{P}(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$$
$$|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$$

Power Set

Review – Mathematical Models

- Sequences and Tuples

$$(a_1, a_2, \ldots, a_k)$$
$$(1, 2, 3) \neq (1, 1, 2, 3)$$
$$(1, 2, 3) \neq (1, 3, 2)$$

Order and repetition of elements do matter

K-tuple (Ordered pair if k=2)

$$|(1, 2, 3)| = 3$$
$$|(1, 2, 2, 3)| = 4$$

Cardinality

$$A = \{0, 1\}$$
$$B = \{x, y\}$$
$$A \times B = \{(0, x), (0, y), (1, x), (1, y)\}$$
$$B \times A = \{(x, 0), (y, 0), (x, 1), (y, 1)\}$$
$$A \times B \neq B \times A$$

Cross (or Cartesian) Product of Sets

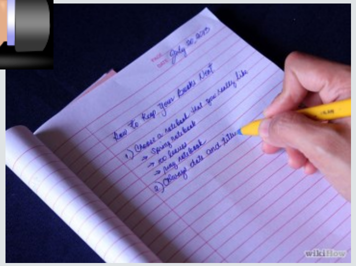
Definitions, Theorems, and Proofs

- Statement: an unambiguous, precise description of some property that an object has.
  - A statement may be true or not
- Theorem: a statement that has been proved to be true
- **Proof**: a convincing logical argument that a statement is true i.e., holds

How to work out a proof

- Understand a statement and all its parts
- Is the statement is true or false? Try a few examples. Find a counterexample
- Can a proof be adapted to a new theorem? Explore differences, what is missing, what is not needed?
- Create an outline of the proof. E.g., First I need to show this, after that I need to show that, etc.
- If you get stuck, try to prove an simplified statement or a special case of it. Try different proof method

If you get frustrated switch to something else, take a step back



Ensure that other people reading your proof (and yourself) can follow your statements

Types of Proofs

- By deduction
  - Sequence of statements whose truth leads us from the initial statements, the hypotheses, to a conclusions statement.
    - Example: All men are mortal. Socrates is a man. Therefore Socrates is mortal.
- By construction
  - Many theorems state that a particular type of object exists. To prove such theorems we can demonstrate how to construct the corresponding object.
    - Example: How to construct a graph with even number of nodes.
- By induction
  - Used to show properties of infinite sets.
  - Show that property holds for case i, which is usually 0 or 1, then assume it holds for the case k where k > i, then show that the property hold for the case k+1 using the results of the case k.
    - Example: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
- By contradiction
  - We assume that the theorem is false and then show that following this assumption leads to a contradiction, i.e., a statement that is both true and false.
    - Example: proof that $\sqrt{2}$ is irrational by starting with the claim that $\sqrt{2}$ is a rational number.

Examples

- Equivalence of sets with repeated elements
  - Consider A = $\{a_1,a_2,a_3,a_4,\ldots\} = \{x \mid x = a_1 \vee x = a_2 \vee x = a_3 \vee x = a_4 \vee \ldots\}$. Show that a set with repeated elements, e.g., {1,1}, is equivalent to a set without repeated elements, e.g., to {1}
    - Thinking about the proof. By the above definition we can rewrite {1,1} as $\{x \mid x = 1 \vee x = 1\}$. However $x = 1 \vee x = 1$ is equivalent to $x = 1$ because v operator is idempotent* for the same terms. Therefore we can rewrite {1,1} as $\{x \mid x = 1\}$ which by the set definition is {1}
    - **Proof**.
      - Assume that a set $\{a_1,a_2,a_3,a_4,\ldots\}$ has some repeated elements.
      - Without loss of generality let it be elements a_i and a_j.
      - Then those elements will appear as terms in disjunctions of the set's definition, i.e., $x = a_i \vee x = a_j$.
      - Because v is idempotent for the same terms one of the terms can be eliminated, i.e., $x = a_j$.
      - The simplified definition will provide us with the set that does not contain repeated elements.

**Element of a set that is unchanged in value when multiplied or otherwise operated on by itself*

Examples – Whiteboard Work

1. Prove that the following statement is true: $\overline{(AUB)} = \bar{A} \cap \bar{B}$
   1. Hint, If X=Y, then that means that Y=X, show that in your proof
2. Let S be a finite subset of some infinite set U. Let T be the complement of S with respect to U. Then T is infinite
   - Hint, think of cardinality of set and proof by contradiction assuming T is finite.

Strings

- An **alphabet** is a set of symbols
  - Examples

$$\Sigma_1 = \{0, 1\}$$

$$\Sigma_2 = \{a, b, c, d, e, f, g, h, \ldots\ldots, z\}$$

- A **string** is a *finite sequence* of symbols over an alphabet
  - Examples

$$(0,0), (1,0,1), (1,1) \ldots.$$

$$00, 101, 11, \ldots$$

Strings over Σ_1

$$(h, i), (h, e, l, l, o), \ldots\ldots$$

$$hi, hello, \ldots\ldots$$

Strings over Σ_2

Simplifications

Strings

- More info on strings
  - Length (number of symbols in a string)

$$w = 101 \text{ then } |w| = 3$$

  - Empty string (string of length zero)

$$\epsilon$$

  - Concatenation (association of strings)

$$x = 101 \ y = 00 \text{ then } \mathbf{xy} = 10100$$

  - Repetition ($\mathbf{w^i}$)

$$w = 10 \text{ then } w^3 = 101010 \text{ and } w^0 = \epsilon$$

Strings Cont.

- Reverse (**w^R**)

$$w = x_1 x_2 \ldots x_n \text{ then } w^R = x_n\, x_{n-1} \ldots x_1$$

$$w = abc \text{ then } w^R = cba$$

$$w = w^R \text{ then } w \text{ is a } \textbf{palindrome}$$

- Substring (string that appears consecutively within another string)

w= abracadabra
y=dab → *y is a substring of w*

Languages

- A **language** is a set of strings over an alphabet
  - Example:

$$\Sigma = \{0, 1\}$$

$$\left.\begin{array}{l}\{00, 11, 0011\} \\ \{01, 10, 1001\}\end{array}\right\} \text{Languages over } \Sigma$$

  - Special cases
    - Empty language $\emptyset$, i.e., a language without strings
      - The size of the empty language is $|\emptyset| = \mathbf{0}$
    - Language $\boldsymbol{\{\epsilon\}}$ is a language with one string that happens to be the empty string
      - The size of the language that includes only the empty string is $|\{\epsilon\}| = \mathbf{1}$
  - Remember!
    - The syntax of a language **constrains** the set of strings that are part of a language to satisfy certain properties

Languages

- We define a language by
  - *Listing* all strings
    - {01,0011,000111,...}
  - *Describing* how to construct a typical string
    - $\{0^k 1^k \mid k > 0\}$
  - Describing how to *test* a string for membership
    - {x over {0,1}| x has the same number of 0's and 1's, has at least one 0, and all 0's precede all 1's }
  - Defining a machine/program that *accepts* all strings in a language
  - Defining a machine/program that *generates* all strings in a language

So Far....

- Review on sets and sequences
- General overview of languages

Symbols → Alphabet → String → Language