Homework 1

Name: Yiqiao Jin
UID: 305107551

1

a

$(B \times B) \times B =$
$\{((0,0),0),((0,1),0),((1,0),0),((1,1),0),((0,0),1),((0,1),1),((1,0),1),((1,1),1)\}$

b

$B \times (B \times B) =$
$\{(0,(0,0)),(0,(0,1)),(0,(1,0)),(0,(1,1)),(1,(0,0)),(1,(0,1)),(1,(1,0)),(1,(1,1))\}$

c

$B \times B \times B = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$

d

$|X| \times |X| = 4 \times 4 = 16$. The power set of $|X| \times |X|$ is the set of all subsets of $|X| \times |X|$. So cardinality $= 2^{16} = 65536$

e

$\{(\epsilon, a),(a,a),(ac,a),(\epsilon,c),(a,c),(ac,c),(\epsilon,aa),(a,aa),(ac,aa)\}$

2

a

$\{\epsilon, a, ac\} \cdot \{a, c, aa\} = \{a, aa, aca, c, ac, acc, aaa, acaa\}$

Note that 'aa' can be formed in 2 ways

b

The result is the empty set $\{\}$

The concatenation of any set with the empty set is the empty set.

c

$\{\epsilon\} \times \Sigma = \{\epsilon\} \times \{a, b, c\} = \{(\epsilon, a), (\epsilon, b), (\epsilon, c)\}$

d

The result is the empty set $\{\}$

Similar to (b), there is no element in $\{\}$ that can form new pairs with elements in L_2^+.

e

$\{\epsilon\} \cdot L_2^+ = L_2^+$

The concatenation of $\{\epsilon\}$ (the set that only contains the empty string) with L_2^+ is the concatention of the empty string with each string in L_2^+. Since $\epsilon x = x$ for $x \in L_2$, the resulting set is L_2^+. Actually, concatenating $\{\epsilon\}$ is just an identity operation for language concatenation.

The result does **NOT** contain ϵ. The resulting set can contain ϵ if and only if L_2^+ contains ϵ. Because it is L_2^+ and not L_2^*, characters in L_2 must appear for a positive number of times. We cannot have zero occurrence of L_2 in this case.

3

This DFA accepts strings that either:

- Start with 1 followed by an even number of symbols in $\Sigma = \{0, 1\}$, or
- Start with 0 followed by an odd number of symbols in $\Sigma = \{0, 1\}$

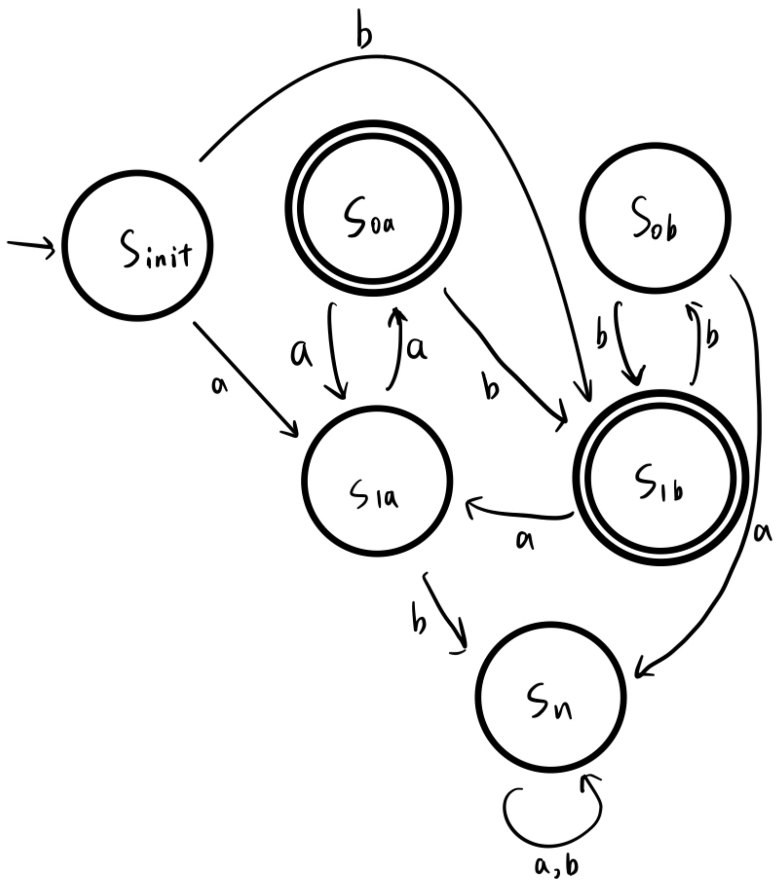
4

a

The language L_{4a} is NOT an FSL. To represent the state of the language, we need to design a DFA that both checks the validity of the grammar and the correctness of the sum. The DFA only has a finite memory. However, there are infinitely many possible combinations of x and y, and there are infinitely many z's that need to be checked against $x + y$ to ensure that the equality is valid. So the language cannot be represented by a **DFA**, thus NOT an FSL

b

The language L_{4b} is an FSL.



	input	
state	a	b
S_{init}	S_{1a}	S_{1b}
S_{0a}	S_{1a}	S_{1b}
S_{1a}	S_{0a}	S_n
S_{0b}	S_n	S_{1b}
S_{1b}	S_{1a}	S_{0b}
S_n	S_n	S_n

Let the DFA $M = (Q, \Sigma, \delta, q_0, F)$, where

- $Q = S_{init}, S_{0a}, S_{1a}, S_{0b}, S_{1b}, S_n$
- $\Sigma = \{a, b\}$
- δ: Transition function is shown as above
- $q_0 = S_{init}$. The start state indicates that we have an even number of a's and b's (zero in this case)
- $F = \{S_{0a}, S_{1b}\}$: set of accept states

The subscript of a state (e.g. S_{1a}) is composed of two parts:

- The number of occurrences of the last word received (0 for even and 1 for odd), and
- The last word we received (either a or b)

Note that S_n is a dead state. We enter this state because we already detect an odd number of a's or an even number of b's in the string, which are unacceptable.

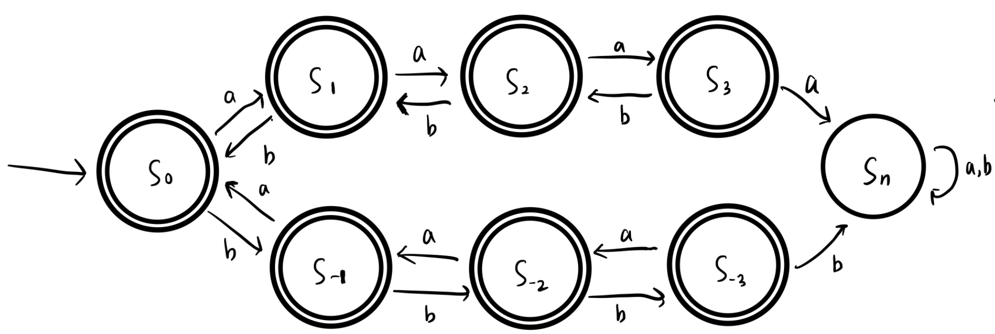
We exclude ϵ because there is no runs of a and b in the empty string ϵ. Any runs of $w \in \Sigma$ must have length ≥ 1. Thus, it is not necessary to have ϵ when we emphasize on the length of runs of a and b.

c

The language L_{4c} is NOT an FSL. The difference between $\#(a, w)$ and $\#(b, w)$ can take an infinite number of values. We need to represent each of those values with a state. Thus, the number of states cannot be finite for this language.

d

The language L_{4d} is an FSL. Since we are constraining the difference between $\#(a, y)$ and $\#(b, y)$ on **ALL** prefixes, once we detect $|\#(a, y) - \#(b, y)| \geq 4$, the string w will NOT be accepted. We only need to record 7 values for $\#(a, y) - \#(b, y)$.



state	input: a	b
S_0	S_1	S_{-1}
S_1	S_2	S_0
S_2	S_3	S_1
S_3	S_n	S_2
S_{-1}	S_0	S_{-2}
S_{-2}	S_{-1}	S_{-3}
S_{-3}	S_{-2}	S_n
S_n	S_n	S_n

Let the DFA $M = (Q, \Sigma, \delta, q_0, F)$, where

- $Q = \{S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, S_3, S_n\}$. The subscript indicates $\#(a, y) - \#(b, y)$. S_n is the state for unacceptable strings
- $\Sigma = \{a, b\}$
- δ: The transition function simply transitions into a new state with subscript+1 when we receive an a, and subscript-1 when we receive a b. If $|(a, y) - \#(b, y)| \geq 4$, we transition into S_n, the state for unacceptable strings
- $q_0 = S_0$ indicates the initial empty string
- $F = \{S_{-3}, S_{-2}, S_{-1}, S_0, S_1, S_2, S_3\}$

S_n is a dead state. We enter this state because at some point we detect that $|\#(a,y) - \#(b,y)| \geq 4$ which makes the string unacceptable.

5

Statement: For all strings $w \in \Sigma^*$, if $w = xy$ for some substrings $x, y \in \Sigma^*$, then $w^R = y^R x^R$

We can let $w = w_1 w_2 ... w_m w_{m+1} ... w_{m+n}$, with $x = w_1 w_2 ... w_m$ and $y = w_{m+1} ... w_{m+n}$. Each $w_i \in \Sigma$

5.1 Basis

Prove that the statement $P(1)$ is true for $|x| = 1$ and $|y| = 1$

Let $w = w^1 w^2$ with $x = w^1$ and $y = w^2$.

Here, $|w| = 2$. Reversing the string, we get:

- $w^R = w^2 w^1$
- $x^R = w^1$
- $y^R = w^2$

$y^R x^R = w^2 w^1 = w^R$. So the base case is proved.

5.2 Inductive Step

Assume the inductive hypothesis $P(n)$ holds for w, x, y, where $|x| = m$ and $|y| = n$. According to $P(n)$, $w = xy$, in which

- $w = w_1 w_2 ... w_m w_{m+1} ... w_{m+n}$
- $x = w_1 w_2 ... w_m$, and
- $y = w_{m+1} ... w_{m+n}$.

Each $w_i \in \Sigma$.

We want to prove that $P(n+1)$ holds for w', x', y'. We prove this by letting $x' = xa, a \in \Sigma$, and $y' = yb, b \in \Sigma$. $|x'| = m + 1$ and $|y'| = n + 1$

Thus we get

- $w' = w_1 w_2 ... w_m a w_{m+1} ... w_{m+n} b$
- $x' = xa = w_1 w_2 ... w_m a$
- $y' = yb = w_{m+1} ... w_{m+n} b$

Reversing the string, we get:

- $w'^R = bw_{m+n}w_{m+1}aw_m...w_2w_1$
- $x'^R = aw_m...w_2w_1$
- $y'^R = bw_{m+n}w_{m+1}$

Indeed, $w'^R = y'^R x'^R$

Thus the inductive hypothesis is true.

From **5.1-2**, we prove that the theorem is valid.

6

Interesting question!

The textbook starts from Chapter 0 Introduction. Each section within chapters is numbered as **Chapter.Section**. The exercises are numbered as **Chapter.Exercise**. The practice problems are attached to the end of each chapters. Figures, examples, and theorems all comprise one sequence within their chapters. So, if there is a figure 1.1, we know that the figure is in chapter one and the next one will be 1.2. Actually I think the syllabus is super helpful when I go through the examples in the book.