Homework 2

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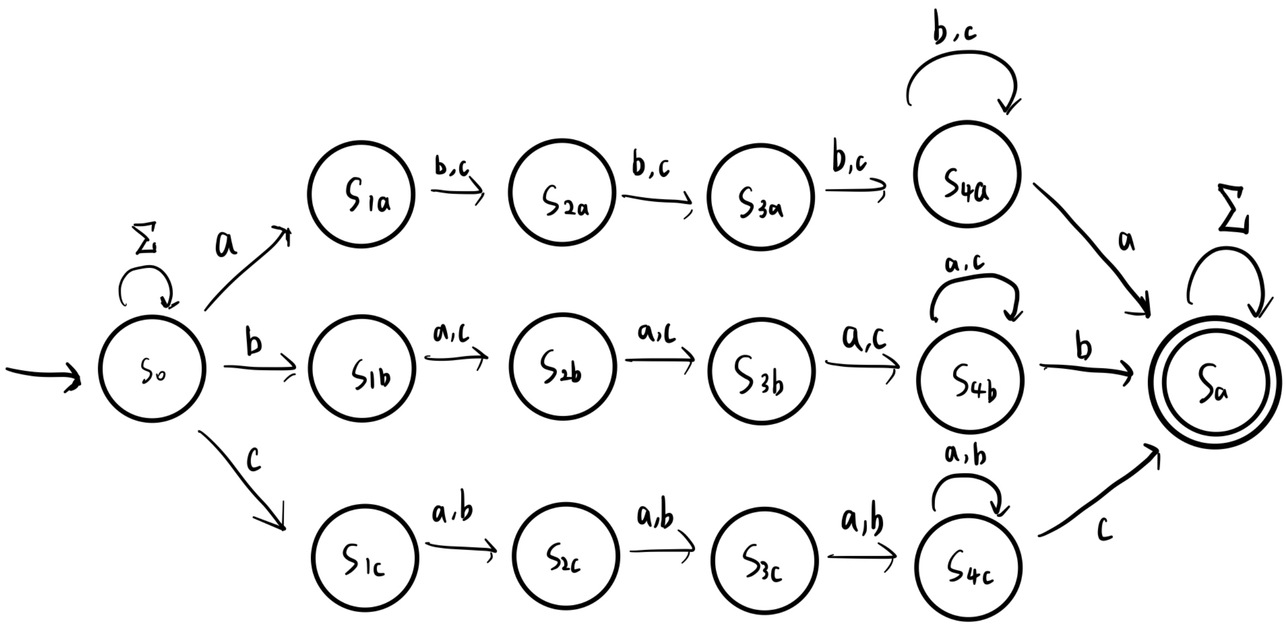

The NFA above recognizes language L_1. There are 3 stages for recognizing an input string w.

The NFA requires that w contains at least one substring s consisting of 2 of the **same symbol** separated by at least 3 occurrences of the other two symbols.

We start with the start state S_0

In the first stage, we try to identify the start symbol of the substring s for some $s \in \Sigma = \{a, b, c\}$. Here, our design uses nondeterminism to loop on Σ before we recognize the start symbol w_s.

In the second stage, after we get the start symbol w_s, we transition into either of the 3 states S_{1a}, S_{1b}, S_{1c} with either a, b, c as the start symbol w_s. Note that the 3 states are on 3 separate branches. Then, we use three transitions to read the followup 3 symbols in $\Sigma \setminus \{w_s\}$. Since there are ≥ 3 occurrences of $\Sigma \setminus \{w_s\}$, we stay in S_{4a}, S_{4b}, S_{4c} after reading the symbols. Here, we continue to loop on $\Sigma \setminus \{w_s\}$

In the 3rd stage, we read the last symbol (same as w_s) and transition into the accept state S_a. Again, our design uses nondeterminism to loop on S_{accept} while reading Σ.

2

We prove that $L_2 = \{a^{2n}b^n | n \geq 0\}$ is not regular by contradiction.

Suppose L_2 is an FSL. Let p be the pumping length. So we can choose $s = a^{2n}b^n \in L$. Assume L_2 is regular. Here, s can be written as $s = xyz$, the concatenation of some substrings x, y, z, where:

1. for each $i \geq 0$, $xy^iz \in A$
2. $|y| = m > 0$
3. $|xy| \leq p$

We consider 3 cases for the formation of y:

2a

The string y consists only of a's. In this case, the number of a's in the string $xyyz$ is more than $2n$, but the number of b's remains the same (n). So $xyyz$ is not a member of L_2, which violates condition 1 of the Pumping Lemma. This case is a contradiction.

2b

The string y consists only of b's. In this case, the number of a's in the string $xyyz$ remains $2n$. However, the number of b's $> n$. So $xyyz$ is still not a member of L_2, which violates condition 1 of the Pumping Lemma.

2c

The string y consists of both a's and b's. In this case, it is possible that within the string $xyyz$, the number of a's is twice the number of b's, specifically, when $y = a^{2m}b^m$ for some $m > 0$). But they will be out of order with some b's before a's. Hence $xyyz$ is still not a member of L_2, which is a contradiction.

From 2a-c, we cannot avoid the contradiction if we assume that L_2 is regular, so L_2 is not regular.

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We prove by contradiction.

$L_3 \cap a^*b^* = L_2 = \{a^{2n}b^n | n \geq 0\}$. Assume L_3 is regular. Since a^*b^* is regular, and regular languages are closed under intersection, the intersection L_2 should also be regular.

However, from **2** we know that $L_2 = \{a^{2n}b^n | n \geq 0\}$ is NOT regular, a contradiction. Hence L_3 is NOT regular

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Sidenote: I am not sure what it means by "at least one 1 before any #). I assume that patterns like 01### is allowed.

A regular expression for L_4

$$(1 \cup \#)^*((0 \cup 1)^*((1)^+\#^*)^*)^*$$

$(1 \cup \#)^*$ means that to the left of any 0's, we can have any number of 1 and $\#$.

The $(0 \cup 1)^*$ specifies that there can be any number of 0 or 1.

The Kleene Plus '$+$' in $(1)^+\#^*$ ensures that there are at least one 1's before the sequence of '$\#$'. Note that the Kleene Star $*$ of $\#^*$ means that, following an 1, the '$\#$' can appear consecutively for any number of times. This makes patterns like $01\#\#\#$ possible. The Kleene Star $*$ in the outer bracket of $((1)^+\#^*)^*$ means that this pattern $(1)^+\#^*$ is optional (it is possible that $\#$ does not appear at all), but can repeat any number of times in $(0, \infty)$.

The Kleene Star * in the entire expression $((0 \cup 1)^*((1)^+\#)^*)^*$ means that this pattern can repeat any number of times in $(0, \infty)$

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I assume the correct statement is: $P(h)$: If T has $k^h + 1$ leaf nodes, then T must have height at least $h + 1$ edges.

To get the maximum number of leaves in a k-ary tree T, we should let every parent possess the maximum number of children (k children per internal node). Thus, for each leaf in the *current level* of a tree, there should be k children in its *next level*. This means that the largest number of leaf nodes for height h edges is k^h.

1) Base case

We try to prove the base case $P(2)$: If T has $k^2 + 1$ leaf nodes, then T must have height at least $2 + 1 = 3$ edges.

We start with a max-leaf tree with height of $h = 1$ edges. This means the leaf nodes are the immediate children of the root nodes. By definition of a full k-ary directed rooted ordered tree, a node can have at most k leaves. So there are at most $k = k^1$ leaves for $h = 1$ (note that such tree contains the most leaf nodes for $h = 1$).

Similarly, for $h = 2$, the root should have exactly k children and each children should have exactly k children. Therefore, the tree can have at most k^2 leaves.

According to Pigeon Hole Principle, to derive a T' with $k = k^h + 1 = k^2 + 1$ leaf nodes, we must spawn some new leaf nodes from existing nodes in T. So there must exist some edges with height 3.

Therefore, for $h = 2$, the base case $P(2)$ is valid.

2) Inductive step

Assume that T_h is a full k-ary tree of height h that has k^h leaf nodes. By definition, it contain the maximum number of leaf nodes among all trees with height h. The inductive hypothesis $P(h)$ states that some T'_h (derived from T_h) with $k^h + 1$ leaf nodes must have height of at least $h + 1$ edges.

To prove $P(h + 1)$, we derive a new k-ary tree T_{h+1} with k^{h+1} nodes from T_h by spawning k nodes from each leaf node of T_h. T_{h+1} should still be a max-leaf tree, but with height $h + 1$. To increase the number of leaf nodes by 1, we must spawn some new leaf nodes from existing nodes in T_{h+1}, which will make the height $h + 2$. So if some T'_{h+1} has $k^{h+1} + 1$ leaf nodes, it must have height of at least $h + 2$ edges since some new leaf nodes must be derived from the max-leaf tree.

Therefore, the inductive hypothesis $P(h + 1)$ is valid for $h + 1$.

From 1-2, we can conclude that the statement is valid.