Discussion 8 - Capacitance and Dielectrics

Problem 1. Two long, coaxial cylindrical conductors are separated by vacuum. The inner cylinder has radius r_a and linear charge density $+\lambda$. The outer cylinder has inner radius r_b and linear charge density $-\lambda$ as shown in Figure 1.

a. Find the capacitance per unit length for this capacitor. The potential outside a charged conducting cylinder relative to some finite reference point, r_0, is

$$V = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{r_0}{r} \tag{1}$$

b. What is the electric field energy density in the region between the conductors at a distance r from the axis?

c. Integrate the energy density calculated in part (b) over the volume between the conductors in a length L of the capacitor to obtain the total electric-field energy per unit length.

d. Use Eq. 2 and the capcitance per unit length calculated in part (a) to calculate U/L. Does your result agree with that obtained in part (c)?

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2 = \frac{1}{2}QV \tag{2}$$

a. Capacitance is defined by $C = Q/V_{ab}$ so capacitance per length is $C/L = \lambda/V_{ab}$. Find the potential difference between the conductors. The potential is not affected by the presence of the charged outer cylinder. Let's choose our reference point to be $r_0 = r_b$, therefore the potential at the outer cylinder radius is 0.

$$V_a - V_b = \frac{\lambda}{2\pi\epsilon_0}\left(\ln\frac{r_b}{r_a} - 0\right) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_b}{r_a}\right) \tag{3}$$

The capacitance per unit length is

$$\frac{C}{L} = \frac{\lambda}{V_{ab}} = \frac{\lambda}{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{r_b}{r_a}\right)} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_b}{r_a}\right)} \tag{4}$$

The capacitance of coaxial cylinders is determined entirely by their dimensions, just as for parallel-plate and spherical capacitors. Ordinary coaxial cables are made like this but with an insulating material instead of vacuum between the conductors.

b. Use Gauss's law to determine that the electric field in between the cylinders is $\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$. The electric field energy density is

$$u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0\left(\frac{\lambda}{2\pi\epsilon_0 r}\right)^2 = \frac{\lambda^2}{8\pi^2\epsilon_0 r^2} \tag{5}$$

c. Integrate u over the cylindrical volume to find total energy. Divide by the length to find the total electric-field energy per unit length.

$$U = \int u dV \tag{6}$$

$$dV = 2\pi r L dr \tag{7}$$

$$U = 2\pi L \int_{r_a}^{r_b} \frac{\lambda^2}{8\pi^2\epsilon_0 r^2} r dr \tag{8}$$

$$U = \frac{L\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{r_b}{r_a}\right) \tag{9}$$

$$\frac{U}{L} = \frac{\lambda^2}{4\pi\epsilon_0}\ln\left(\frac{r_b}{r_a}\right) \tag{10}$$

d. Use the given formula for energy in a capacitor and see that the result agrees with direct integration of the electric field in part (b).

$$U = \frac{Q^2}{2C} \tag{11}$$
$$Q = \lambda L \tag{12}$$
$$U = (\lambda L)^2 \frac{\ln\left(\frac{r_b}{r_a}\right)}{2\left(2\pi\epsilon_0 L\right)} \tag{13}$$
$$\frac{U}{L} = \frac{\lambda^2}{4\pi\epsilon_0} \ln\left(\frac{r_b}{r_a}\right) \tag{14}$$

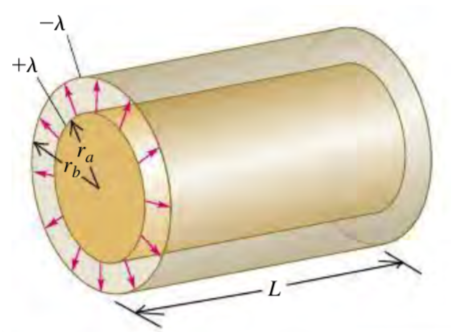

Figure 1: Problem 1

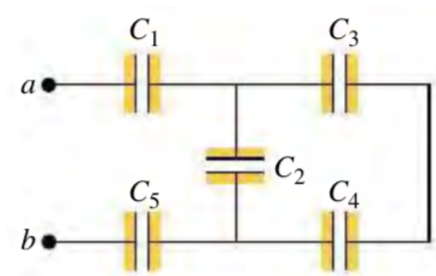

Figure 2: Problem 2

Problem 2. In Figure 2, $C_1 = C_5$ and $C_2 = C_3 = C_4$. The applied potential is V_{ab}.

a. What is the equivalent capacitance of the network between points a and b in terms of C_1 and C_2?

b. Calculate the charge on each capacitor and the potential difference across each capacitor.

a. For capacitors in series the equivalent capacitor is smaller than any of those in series. For capacitors in parallel the equivalent capacitance is larger than any of those in parallel. Start with the two capacitors on the right. They are in series.

$$\frac{1}{C_{34}} = \frac{1}{C_3} + \frac{1}{C_4} \tag{15}$$
$$C_{34} = \frac{C_2}{2} \tag{16}$$

This equivalent capacitor is now in parallel with the middle capacitor.

$$C_{234} = C_2 + C_{34} \tag{17}$$
$$C_{234} = \frac{3}{2} C_2 \tag{18}$$

This is now in series with the two left-most capacitors.

$$\frac{1}{C_{eq}} = \frac{1}{C_{234}} + \frac{1}{C_1} + \frac{1}{C_5} \tag{19}$$
$$\frac{1}{C_{eq}} = \frac{2}{3C_2} + \frac{2}{C_1} \tag{20}$$
$$C_{eq} = \frac{3C_1C_2}{2C_1 + 6C_2} \tag{21}$$

b. Capacitors in series have the same charge. Capacitors in parallel have the same potential. Start with the simplest network and work back to the original circuit.

$$C = \frac{Q}{V} \tag{22}$$
$$Q_{eq} = C_{eq} V_{ab} \tag{23}$$

Every capacitor in the series has this same charge Q_{eq}.

$$Q_{eq} = Q_1 = Q_5 = Q_{234} \tag{24}$$
$$V_1 = \frac{Q_{eq}}{C_1} \tag{25}$$
$$V_5 = \frac{Q_{eq}}{C_5} = \frac{Q_{eq}}{C_1} \tag{26}$$
$$V_{234} = \frac{Q_{eq}}{C_{234}} \tag{27}$$

Every capacitor in parallel has this potential V_{234}.

$$V_{234} = V_2 = V_{34} \tag{28}$$
$$Q_2 = C_2 V_2 \tag{29}$$
$$Q_{34} = C_{34} V_{34} \tag{30}$$
$$\tag{31}$$

This is the same charge on both 3 and 4.

$$Q_{34} = Q_3 = Q_4 \tag{32}$$
$$V_3 = \frac{C_3}{Q_3} \tag{33}$$
$$V_4 = \frac{C_4}{Q_4} \tag{34}$$

Problem 3. A parallel-plate capacitor has capacitance C when the volume between the plates is filled with air. The plates are circular, with radius R. The radius is much bigger than the separation between the plates. The capacitor is connected to a battery, and a charge of magnitude Q goes onto each plate. With the capacitor still connected to the battery, a slab of dielectric is inserted between the plates, completely filling the space between the plates. After the dielectric has been inserted, the charge on each plate has magnitude $1.8Q$.

a. What is the dielectric constant K of the dielectric?

b. What is the potential difference between the plates before and after the dielectric has been inserted?

c. What is the electric field at a point midway between the plates before and after the dielectric has been inserted?

d. With the dielectric inserted, the capacitor is disconnected from the battery. What is the electric field at a point midway between the plates if the dielectric is then removed?

Since the capacitor remains connected to the battery the potential between the plates of the capacitor doesn't change.

a. The capacitance changes by a factor of K when the dielectric is inserted. Since V is unchanged (the battery is still connected), $V_{before} = V_{after}$.

$$V = \frac{Q}{C} \tag{35}$$

$$\frac{Q_{before}}{C_{before}} = \frac{Q_{after}}{C_{after}} \tag{36}$$

$$K = \frac{C_{after}}{C_{before}} = \frac{Q_{after}}{Q_{before}} = 1.80 \tag{37}$$

$$\tag{38}$$

b. The potential difference is the same before and after. $V = Q/C$.

c. The electric field is equal to the potential difference divided by the separation of the plates. The field is uniform between the plates. The capacitance of a parallel-plate capacitor filled with air only depends on the shape.

$$C = \epsilon_0 \frac{A}{d} \tag{39}$$

$$d = \epsilon_0 \frac{\pi R^2}{C} \tag{40}$$

$$E = \frac{V}{d} = \frac{VC}{\epsilon_0 \pi R^2} \tag{41}$$

$$E = \frac{Q}{\epsilon_0 \pi R^2} \tag{42}$$

Alternatively, you could find the electric field using Gauss's law. Because the potential across the capacitor and the distance between the capacitors stays the same, E must be the same after the insertion of a dielectric.

d. When the battery isn't connected, charge on each plate remains the same. Removing the dielectric changes the capacitance from $1.8C$ to C.

$$V = \frac{1.8Q}{C} \tag{43}$$

$$E = \frac{V}{d} = \frac{1.8Q}{Cd} \tag{44}$$

$$E = \frac{1.8Q}{\epsilon_0 \pi R^2} \tag{45}$$