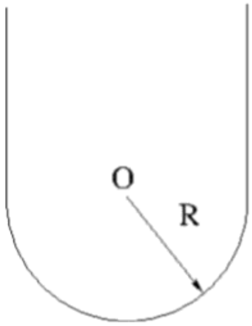
Some Problems from the Kudu Homework

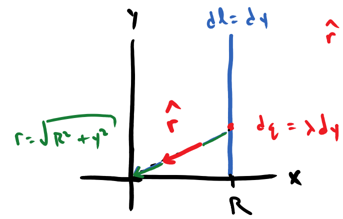
Question 4

A thread carrying a uniform charge λ has two very long straight segments and a half circle. Assuming that the radius R is much smaller than the length of the thread, what is the electric field strength at the center of the circle O?



Line

Use: $$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, dl}{r^2}\,\hat{r}$$



$$\hat{r} = \frac{R(-\hat{i}) + y(-\hat{j})}{\sqrt{R^2+y^2}}$$

Combine:

$$\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\int_0^{L\to\infty} \frac{\lambda\, dy}{R^2+y^2} \cdot \frac{R(-\hat{i}) + y(-\hat{j})}{\sqrt{R^2+y^2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\underbrace{\int_0^{L\to\infty} \frac{R\, dy}{(R^2+y^2)^{3/2}}}_{= \left.\frac{y}{R\sqrt{R^2+y^2}}\right|_0^{L\to\infty}}(-\hat{i}) + \underbrace{\int_0^{L\to\infty} \frac{y\, dy}{(R^2+y^2)^{3/2}}}_{= \left.-\frac{1}{\sqrt{R^2+y^2}}\right|_0^{L\to\infty}}(-\hat{j})\right]$$

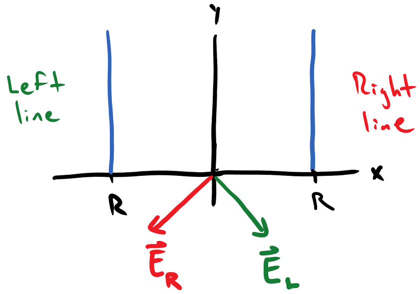
$$\vec{E}(0) = \frac{\lambda}{4\pi\epsilon_0}\left(\frac{L(-\hat{i})}{R\sqrt{\cancel{R^2}+L^2}} - 0 - \cancel{\frac{1(-\hat{j})}{\sqrt{R^2+L^2}}} + \frac{1(-\hat{j})}{\sqrt{R^2+0}}\right)$$

$L \gg R$

Huge denominator

$$= \frac{\lambda}{4\pi\epsilon_0}\cdot\frac{1}{R}(-\hat{i}-\hat{j})$$

COULD do integral for the other line as well. BUT can see that it will not be necessary:



$$\vec{E}_R(0) = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{R}(-\hat{\imath} - \hat{\jmath})$$

Can see:

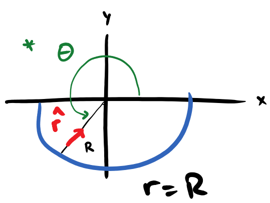
$$\vec{E}_L(0) = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{R}(+\hat{\imath} - \hat{\jmath})$$

x-components cancel, and : $\vec{E}_\ell(0) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{R}(-\hat{\jmath})$

lines

Ring

Again: here we can't use the integral we set up in the Kudu example. We need the component that *cancelled* in that example; will not cancel here.



Use: $\vec{E}(0) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda\, d\ell}{r^2}\hat{r}$

$d\ell = R\, d\theta$

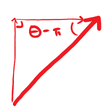
* This is one way to handle ϴ. I am choosing ϴ to be in the range π < ϴ < 2π, so my limits of integration will be from π to 2π.

Can choose a ϴ that is in range 0 < ϴ < π, which will require limits of integration from 0 to π.

$\hat{r}$:

$$\cos(\theta - \pi)\hat{\imath} = (-\cos\theta)\hat{\imath}$$

$$\sin(\theta - \pi)\hat{\jmath} = (-\sin\theta)\hat{\jmath}$$



x-components cancel. Left with y-components only

Will keep integral in terms of ϴ because the integral is simple. Converting to x and y will make the integral complicated.

Combine:

$$\vec{E}_r(0) = \frac{1}{4\pi\epsilon_0}\int_\pi^{2\pi}\frac{\lambda(R\, d\theta)}{R^2}\cdot(-\sin\theta)\hat{\jmath} = \frac{\lambda}{4\pi\epsilon_0 R}\underbrace{\int_\pi^{2\pi}(-\sin\theta)\, d\theta}_{= \cos\theta\big|_\pi^{2\pi} = 2}\hat{\jmath}$$

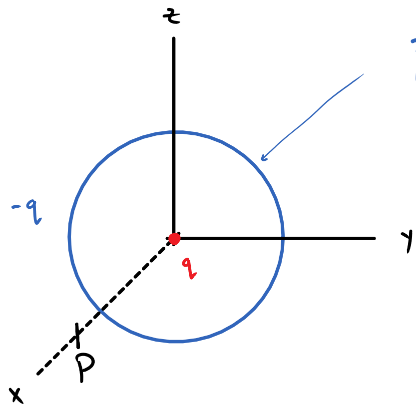
ring

$$\vec{E}_r(0) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{R}\hat{\jmath}$$ ← Same as $\vec{E}_\ell$ but opposite direction!!!

Total $\boxed{\vec{E} = \vec{E}_r + \vec{E}_\ell = 0}$

Question 5

A point charge q is located at the center of a thin ring of radius R with uniformly distributed charge $-q$. What is the electric field at a point lying on the axis of the ring a distance x from its center? Assume $x \gg R$.



$$\vec{E}_-(x) = \frac{1}{4\pi\epsilon_0}\frac{-qx}{(x^2+R^2)^{3/2}}\,\hat{\imath}$$

Same result as the Kudu Example: *Electric Field due to a Ring of Charge*

$$\vec{E}_+(x) = \frac{1}{4\pi\epsilon_0}\frac{q}{x^2}\,\hat{\imath}$$

$$\vec{E}(x) = \vec{E}_+ + \vec{E}_- = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{x^2} + \frac{(-q)x}{(x^2+R^2)^{3/2}}\right]\hat{\imath} = \frac{q}{4\pi\epsilon_0}\left[\frac{(x^2+R^2)^{3/2} - x^3}{x^2(x^2+R^2)^{3/2}}\right]\hat{\imath}$$

The denominator does not vanish:

$$x^2(x^2+R^2)^{3/2} \approx x^2(x^2+0)^{3/2} = x^5$$

The numerator will vanish:

$$(x^2+0)^{3/2} - x^3 = 0$$

We must use a Taylor Expansion in the numerator:

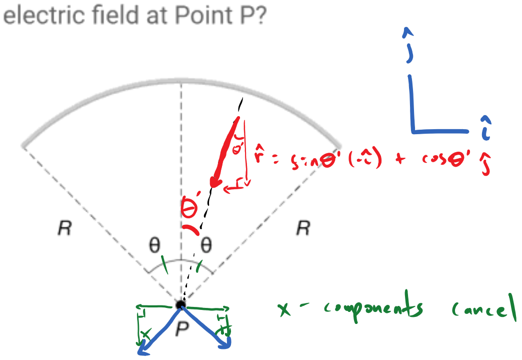
$$(x^2+R^2)^{3/2} = x^3\left[1+\left(\frac{R}{x}\right)^2\right]^{3/2} \qquad \text{Use: } (1+\epsilon)^{3/2} \approx 1 + \frac{3}{2}\epsilon^2$$

$$\vec{E}(x) \approx \frac{q}{4\pi\epsilon_0}\left[\frac{x^3\left[1+\left(\frac{R}{x}\right)^2\right] - x^3}{x^5}\right]\hat{\imath} = \frac{q}{4\pi\epsilon_0}\left(\frac{x^3 + \frac{3}{2}R^2 x - x^3}{x^5}\right)\hat{\imath}$$

$$\boxed{= \frac{3qR^2}{8\pi\epsilon_0 x^4}\,\hat{\imath}}$$

Question 10

The figure shows a charged rod with a constant charge density. If the total charge is Q, what is the electric field at Point P?



Use : $$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda\, dl}{r^2} \hat{r}$$

$$dl = R\, d\theta' \quad , \quad \lambda = \frac{Q}{R(2\theta)} \rightarrow \theta = \frac{s}{R}$$

Combine:

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \int_{-\theta}^{\theta} \frac{Q}{R(2\theta)} \cdot \frac{R\, d\theta'}{R^2} \cos\theta' (-\hat{j})$$

$$= \frac{Q}{4\pi\epsilon_0 R^2 (2\theta)} \int_{-\theta}^{\theta} \cos\theta'\, d\theta' (-\hat{j}) = \frac{Q}{4\pi\epsilon_0 R^2 (2\theta)} \cdot 2\sin\theta \, (-\hat{j})$$

$$\boxed{E(z) = \frac{Q}{4\pi\epsilon_0 R^2} \cdot \frac{\sin\theta}{\theta}}$$