# Discussion 5 - Electric Fields

**Problem 1.** A thin disk with a circular hole at its center, called an *annulus*, has inner radius $R_1$ and outer radius $R_2$ as shown in the figure. The disk has a uniform positive surface charge density $\sigma$ on its surface.

a. Determine the total electric charge on the annulus

b. The annulus lies in the yz-plane, with its center at the origion. For an arbitrary point on the x-axis (the axis of the annulus), find the magnitude and direction of the electric field $\vec{E}$. Consider points both above and below the annulus.

c. Show that at points on the x-axis that are sufficiently close to the origin, the magnitude of the electric field is approximately proportional to the distance between the center of the annulus and the point. How close is "sufficiently close"?

d. A point particle with mass m and negative charge -q is free to move along the x-axis (but cannot move off the axis). The particle is originally placed at rest at $x = 0.01R_1$ and released. Find the angular frequency of oscillation of the particle.

a. The worked solution to part a is given on the professor's ungraded homework problems worksheet. The final electric field is

$$\vec{E} = k\sigma 2\pi x \left( \frac{1}{\sqrt{R_1^2 + x^2}} - \frac{1}{\sqrt{R_2^2 + x^2}} \right) \hat{i}$$

b. Small x limit means $x \ll R_1$ (and $x \ll R_2$.)

$$\sqrt{R_1^2 + x^2} \approx \sqrt{R_1^2} = R_1 \tag{1}$$

$$\vec{E} = k\sigma 2\pi x \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \hat{i} \tag{2}$$

$$x\hat{i} = \vec{x} \tag{3}$$

$$\vec{E} \approx k\sigma 2\pi \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \vec{x} \tag{4}$$

c. Write the equation of motion for the particle.

$$\vec{F} = -q\vec{E} = m\vec{a} \tag{5}$$

$$\vec{a} = \frac{-q}{m} k\sigma 2\pi \left( \frac{1}{R_1} - \frac{1}{R_2} \right) \vec{x} \tag{6}$$

Note that $R_2 > R_1$ so the term in parantheses is positive and we have an equation of motion that looks like

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{7}$$

so we can read off the angular frequency as the square root of the constant in front of $x$. The frequency is independent of initial position as long as we are in the limit of $x \ll R_1$.

$$\omega = \sqrt{\frac{q}{m} k\sigma 2\pi \left( \frac{1}{R_1} - \frac{1}{R_2} \right)} \tag{8}$$

**Problem 2.** A hemispherical surface with radius r in a region of uniform electric field $\vec{E}$ has its axis aligned parallel to the direction of the field. Calculate the flux through the surface.

The flux through a hemispherical surface is the same as the flux through the flat bottom circle. Every electric field line that enters the bottom circle exits through the hemisphere. The flux would be the

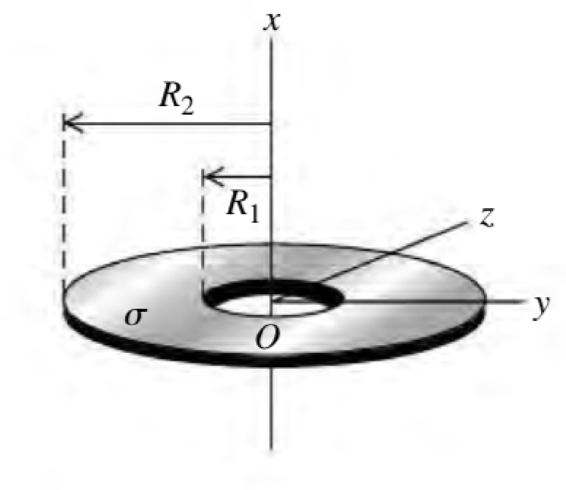


Figure 1: Problem 1

same for any surface bounded by this flat circle. The field lines are perpendicular to the flat circle so the flux is EA.

$$\Phi = E\pi R^2 \tag{9}$$

We could also calculate flux by integration.

$$\Phi = \int \vec{E} \cdot \hat{n} dA \tag{10}$$

$$\vec{E} = E\hat{z} \tag{11}$$

$$\hat{n} = \cos\theta\hat{z} + \sin\theta\hat{\rho} \tag{12}$$

$$dA = R^2 \sin\theta d\theta d\phi \tag{13}$$

$$\Phi = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta E \cos\theta R^2 \sin\theta \tag{14}$$

$$\Phi = ER^2 2\pi \int_0^{\pi/2} d\theta \cos\theta \sin\theta = E\pi R^2 \tag{15}$$

**Problem 3.** An electric dipole with dipole moment $\vec{p}$ is in a uniform external electric field $\vec{E}$. The dipole consists of two particles of equal mass $m$ a distance $d$ form one another.

a. Find the orientations of the dipole for which the torque on the dipole is zero.

b. Which of the orientations in part (a) is stable, and which is unstable?

c. Find the frequency of oscillations for the dipole rotating (small angles) about the stable equilibrium in the electric field. Assume the electric field is constant.

The torque on a dipole is given by $\vec{\tau} = \vec{p} \times \vec{E}$. The cross product is zero when the two vectors are parallel, so the two positions with zero torque are when $\vec{p}$ is parallel and anti-parallel to the electric field.

To determine what is stable, we think about small movement away from the equilibrium point. If a tiny kick sends the dipole spinning, that is not a stable equilibrium point.

See the figure below for the choice of axis and definitions of angles.

We can start looking at the case where the dipole and the electric field are near parallel. The magnitude of the torque will be $\tau = pE\sin\theta$, where $\theta$ is the angle between the two vectors. The sign of the cross product will be in the $-\hat{z}$ direction. Therefore, if the angle between them is positive, the torque is negative, pushing the dipole back towards the x axis. This is a stable equilibrium.

Let's now look at the anti-parallel dipole. We set it up the same way, and take $\vec{p} \to -\vec{p}$. Now the torque is $\vec{\tau} = -pE \sin\theta \hat{z}$, actually the same as before. But if you look at the picture, if $\theta$ decreases, that actually drives the dipole farther away from anti-parallel, causing an unstable equilibrium.

To find the oscillation frequency, we set up a differential equation.

$$\vec{\tau} = I\ddot{\vec{\theta}} = -pE \sin\theta \hat{z}.$$

The moment of inertia about the center of two point particles is $I = 2m(\frac{d}{2})^2$, and we can assume the small angle approximation. Therefore we get

$$m\frac{d^2}{2}\ddot{\vec{\theta}} = -pE\theta\hat{z} \tag{16}$$

$$\ddot{\vec{\theta}} = -\frac{2pE}{md^2}\theta\hat{z} \tag{17}$$

From here we can read off the frequency of oscillations: $\omega = \sqrt{\frac{2pE}{md^2}}$.

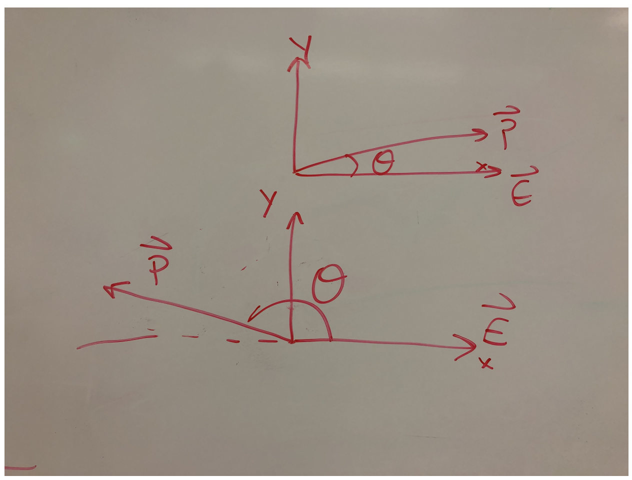


Figure 2: Problem 3