# Discussion 3 - Mechanical Waves

**Problem 1.** A block with mass $m$ moves in a viscous fluid that provides a frictional force, $F = -bv$. The block is attached on the left to a spring with spring constant $k$. The system is driven from the right with an oscillating force $F = F_0 \cos(\omega_d t)$.

a. Write a differential equation that describes the motion of the block.

b. Find the amplitude of the motion.

c. What driving frequency, $\omega_d$, will maximize the amplitude?

a. Write Newton's second law. Define the equilibrium position where the spring is unstretched as $x = 0$.

$$ma = F = -kx - bv + F_0 \cos(\omega_d t) \tag{1}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{b}{m}\frac{dx}{dt} + \frac{F_0}{m}\cos(\omega_d t) \tag{2}$$

b. We can find the amplitude by coming up with an initial guess, or an ansatz. For a damped driven oscillator, there will be a complicated initial period before the oscillator settles down. Eventually, the oscillator will settle into oscillations at the driving frequency, which can be written as $x = A\cos(\omega_d t - \phi)$. To solve for $A$ and $\phi$, plug this ansatz for $x$ into the differential equation.

$$-A\omega_d^2 \cos(\omega_d t - \phi) = -\frac{k}{m}A\cos(\omega_d t - \phi) + \frac{b}{m}A\omega_d \sin(\omega_d t - \phi) + F_0\cos(\omega_d t) \tag{3}$$

We can use trig identities to split up the sines and cosines.

$$\cos(\omega_d t - \phi) = \cos(\omega_d t)\cos\phi + \sin(\omega_d t)\sin\phi \tag{4}$$

$$\sin(\omega_d t - \phi) = \sin(\omega_d t)\cos\phi - \cos(\omega_d t)\sin\phi \tag{5}$$

After rearranging, we now get the equation

$$\left[A(\omega_0^2 - \omega_d^2)\cos\phi + Ab\omega_d\sin\phi - F_0\right]\cos(\omega_d t) + A\left[(\omega_0^2 - \omega_d^2)\sin\phi - b\omega_d\cos\phi\right]\sin(\omega_d t) = 0 \tag{6}$$

In order for this equation to always equal zero, each coefficient must independently equal zero. We can use the coefficient of sine to solve for $\phi$

$$\phi = \arctan\left(\frac{b\omega_d}{\omega_0^2 - \omega_d^2}\right). \tag{7}$$

Plugging this into the coefficients of cosine and rearranging gets you the amplitude for $A$

$$A = \frac{F_0}{\sqrt{(k - m\omega_d^2)^2 + b^2\omega_d^2}} \tag{8}$$

c. The maximum value of A results from minimizing the denominator in the equation for $A$ as $F_0$ is a constant. We will call this maximizing frequency $\omega_d = \Omega_R$, the resonance frequency. In order to maximize A, we take the derivative of the quantity inside the square root and set it equal to zero.

$$\frac{\partial}{\partial\Omega}\left[(k - m\Omega_R^2)^2 + b^2\Omega_R^2\right] = 0 \tag{9}$$

$$-4m\Omega_R(k - m\Omega_R^2) + 2b^2\Omega_R = 0 \tag{10}$$

Divide by $\Omega_R$ since a solution with 0 frequency would not result in any amplitude

$$k - m\Omega_R^2 = \frac{b^2}{2m} \tag{11}$$

$$\Omega_R = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}} \tag{12}$$

$$\Omega_R = \omega_0\sqrt{1 - 2\left(\frac{b}{2m\omega_0}\right)^2} \tag{13}$$

The last expression is convenient for Taylor expansion to approximate $\Omega_R$ in the case of weak damping where $b \ll m\omega_o$.

We can check our answer by considering the case with no damping. If b = 0 then

$$\Omega_R = \omega_0 \tag{14}$$

and the resulting frequency is the natural frequency of the block and spring.

**Problem 2.** Show that a standing wave is the superposition of an incident wave moving to the right and a reflected wave moving to the left with a $\pi$ phase shift. A wave reflected from a fixed end of a string is inverted. How does the amplitude of the standing wave compare to the amplitude of the incident wave?

You may need to use the identity:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{15}$$

Because wave equations are linear, the resultant function can be written as the sum of the two functions $y_1$, the wave moving to the right and $y_2$ the wave moving to the left.

$$y(x,t) = y_1(x,t) + y_2(x,t) \tag{16}$$

$$\tag{17}$$

The wave moving to the right is

$$y_1(x,t) = A\cos(kx - \omega t) \tag{18}$$

Applying the identity

$$y_2(x,t) = A\cos(kx + \omega t + \pi) \tag{19}$$

$$y_2(x,t) = A\cos(kx + \omega t)\cos\pi - \sin(kx + \omega t)\sin\pi \tag{20}$$

$$y_2(x,t) = -A\cos(kx + \omega t) \tag{21}$$

$$y_2(x,t) = -A[\cos(kx)\cos(\omega t) - \sin(kx)\sin(\omega t)] \tag{22}$$

Summing the two parts

$$y(x,t) = A\left[\cos(kx)\cos(-\omega t) - \sin(kx)\sin(-\omega t)\right] - A\left[\cos(kx)\cos(\omega t) - \sin(kx)\sin(\omega t)\right] \tag{23}$$

$$y(x,t) = A\left[\cos(kx)\cos(\omega t) + \sin(kx)\sin(\omega t)\right] - A\left[\cos(kx)\cos(\omega t) - \sin(kx)\sin(\omega t)\right] \tag{24}$$

$$y(x,t) = 2A\sin(kx)\sin(\omega t) \tag{25}$$

The amplitude of the standing wave is twice the amplitude of the traveling wave.

**Problem 3.** A horizontal string tied at both ends is vibrating in its fundamental mode. The *traveling* waves have speed v, frequency f, amplitude A, and wavelength $\lambda$. Consider the points located at $x = \lambda/2$, $x = \lambda/4$, and $x = \lambda/8$ from the left-hand end of the string.

a. Calculate the maximum transverse velocity and maximum transverse acceleration at these points.

b. What is the amplitude of the motion at these points?

c. How much time does it take the string to go from its largest upward displacement to its largest downward displacement at these points?

$$y(x,t) = A_{SW}\sin(kx)\cos(\omega t) \tag{26}$$
$$v_{max} = A_{SW}\sin(kx)\omega \tag{27}$$
$$a_{max} = A_{SW}\sin(kx)\omega^2 \tag{28}$$
$$A_{SW} = 2A \text{ (This was proven in number 2)} \tag{29}$$

a. $x = \lambda/2$ is a node. There is no motion at this point.
$x = \lambda/4$ is an antinode so $v_{max} = A_{SW}\omega = A_{SW}2\pi f$ and $a_{max} = A_{SW}\omega^2 = A_{SW}4\pi^2 f^2$.
For $x = \lambda/8$

$$v_{max} = A_{SW}\sin(2\pi/8)\omega = A_{SW}2\pi f/\sqrt{2} \tag{30}$$
$$a_{max} = A_{SW}\sin(2\pi/8)\omega^2 = A_{SW}4\pi^2 f^2/\sqrt{2} \tag{31}$$

b. The amplitude of a standing wave is twice the amplitude of the traveling wave. We can think of each point on the string as a vertical oscillator whose equation of motion is $y(t) = A_x\cos(\omega t)$ and the amplitude of motion at any position is given by $A_x = 2A\sin(kx)$.

$$A_x = 2A\sin(kx) \tag{32}$$
$$A(\lambda/2) = 0 \tag{33}$$
$$A(\lambda/4) = 2A \tag{34}$$
$$A(\lambda/8) = 2A\sin(\pi/4) = 2A/\sqrt{2} \tag{35}$$

c. The period is the same at every point and the question asks for half the period, $1/(2f)$.

**Problem 4.** A vibrating string of length L is under a tension of $F_T$. The results from five successive stroboscopic pictures are shown in the figure. The strobe rate is set at N flashes per second, and observations reveal that the maximum displacement occurred at flashes 1 and 5 with no other maxima in between.

a. Find the period, frequency, and wavelength for the traveling waves on this string.
b. In what normal mode (harmonic) is the string vibrating?
c. What is the speed of the traveling waves on the string?
d. How fast is point P moving when the string is in
   i. position 1
   ii. position 3
e. What is the mass of this string?

a. The time between flashes 1 and 5 is T/2. Four flashes occur from position 1 to 5 and the time between each flash is 1/N, so the time elapsed is 4/N. Therefore T is 8/N. The frequency is N/8 and the wavelength is L.
b. This is the second harmonic. The first would have no nodes between the two ends.
c. The speed of the traveling wave is frequency times wavelength, NL/8.
d. At flash 1 the string is maximally displaced so its speed is 0. At flash 3, the string is passing through its equilibrium position and has its maximum speed in the negative direction. We see in the figure that the position of P is $x = L/4$ and we know the time for the system to move from one extreme position to the equilibrium position is a quarter of the period.

$$v_y(x,t) = -A\omega\sin(kx)\sin(\omega t) \tag{36}$$
$$v_y\left(\frac{L}{4},\frac{T}{4}\right) = -A\omega\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) \tag{37}$$
$$v_{max} = -A\omega = -A2\pi f = -A\pi N/4 \tag{38}$$

e.

$$\mu = \frac{m}{L} \tag{39}$$

$$v = \sqrt{\frac{F}{\mu}} = \sqrt{\frac{FL}{m}} \tag{40}$$

$$m = \frac{FL}{v^2} = \frac{FL8^2}{N^2L^2} = \frac{64F_T}{N^2L} \tag{41}$$

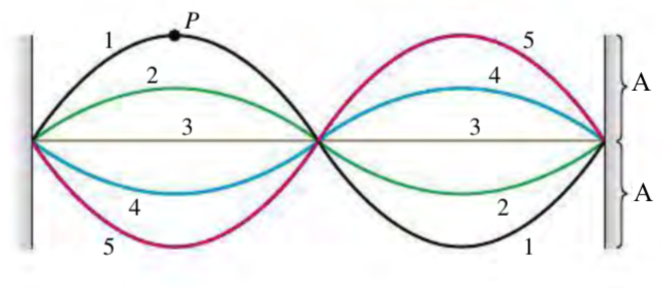


Figure 1: Problem 4