Discussion 1 - Intro and Math Review

Problem 1. Name 3 systems that undergo periodic motion.

Rocking chair, bouncing ball, tuning fork, swing, grandfather clock, pendulum, spring

Problem 2. Taylor expand the following functions about the given point up to second order.

$$f(x - x_0) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!}f''(x_0)(x - x_0)^2 + \frac{1}{3!}f'''(x_0)(x - x_0)^3 + ... \tag{1}$$

a. $f(x) = \cos(4x)$ about $x = 0$

$$f(x) \approx \cos(4x_0) - 4\sin(4x_0)x - \frac{4^2}{2}\cos(4x_0)x^2 \tag{2}$$

$$f(x) \approx 1 - 8x^2 \tag{3}$$

b. $f(x) = \ln(3 + 4x)$ about $x = 0$

$$f(x) \approx \ln(3 + 4x_0) + \frac{4}{3 + 4x_0}x - \frac{1}{2}\frac{4^2}{(3 + 4x_0)^2}x^2 \tag{4}$$

$$f(x) \approx \ln(3) + \frac{4}{3}x - \frac{8}{9}x^2 \tag{5}$$

c. $f(x) = \frac{1}{x^4}$ about $x = 1$

$$f(x) \approx \frac{1}{x_0^4} - \frac{4}{x_0^5}(x - x_0) + \frac{1}{2}\frac{4 * 5}{x_0^6}(x - x_0)^2 \tag{6}$$

$$f(x) \approx 1 - 4(x - 1) + 10(x^2 - 2x + 1) \tag{7}$$

$$f(x) \approx 15 - 24x + 10x^2 \tag{8}$$

d. $f(x) = (1 + x)^n$ about $x = 0$ for any n

$$f(x) \approx (1 + x_0)^n + n(1 + x_0)^{n-1}x + \frac{1}{2}n(n - 1)(1 + x_0)^{n-2}x^2 \tag{9}$$

$$f(x) \approx 1 + nx + \frac{n(n - 1)}{2}x^2 \tag{10}$$

Problem 3. Calculate the total mass of a sphere with radius R and non-uniform mass density $\rho = Cr$, where C is a constant and r is the distance from the center. [In Eq. 2, dV is the volume of an infinitesimal piece of the object.]

$$M = \int \rho dV \tag{11}$$

Use spherical coordinates where θ is the angle in the r-z plane which ranges from 0 to π and ϕ is the angle in the x-y plane which ranges from 0 to 2π.

$$dV = r^2 \sin(\theta) dr d\theta d\phi \tag{12}$$

$$M = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=R} CrdV \tag{13}$$

$$M = C \int_0^R r^3 dr \int_0^\pi \sin(\theta) d\theta \int_0^{2\pi} d\phi \tag{14}$$

$$M = 4\pi C \frac{R^4}{4} = C\pi R^4 \tag{15}$$

Problem 4. Calculate the moment of inertia of a rod (thin cylinder) with total mass M and length L rotating about an axis perpendicular to the rod, passing through the center of the rod. [In Eq. 3, y is the distance from the axis to a infinitesimal piece of the object with mass dm.]

$$I = \int y^2 dm \tag{16}$$

Assume the rod has uniform linear mass density. Distance from the axis ranges from $-L/2$ to $L/2$.

$$dm = \frac{M}{L} dy \tag{17}$$

$$I = \int_{-L/2}^{L/2} y^2 \frac{M}{L} dy \tag{18}$$

$$I = \frac{M}{L}\frac{1}{3}\left(\left(\frac{L}{2}\right)^3 - \left(\frac{-L}{2}\right)^3\right) \tag{19}$$

$$I = \frac{1}{12} M L^2 \tag{20}$$

Problem 5. The following functions represent the displacement of an object as a function of time. Calculate the velocity, $\frac{dx}{dt}$, and acceleration, $\frac{d^2x}{dt^2}$, for each of the functions.

a. $x(t) = \cos(t)$
b. $x(t) = A\cos(\omega t)$
c. $x(t) = A\cos(t - \frac{\pi}{2})$
d. $x(t) = A\cos(t + \frac{\pi}{4})$

a. $v(t) = -\sin(t)$ and $a(t) = -\cos(t)$
b. $v(t) = -A\omega \sin(\omega t)$ and $a(t) = -A\omega^2 \cos(\omega t)$
c. $v(t) = -A\sin(t - \frac{\pi}{2})$ and $a(t) = -A\cos(t - \frac{\pi}{2})$
d. $v(t) = -A\sin(t + \frac{\pi}{4})$ and $a(t) = -A\cos(t + \frac{\pi}{4})$

Problem 6. Derive the formula for the oscillation frequency for (a) an ideal horizontal spring (spring constant k) with a block of mass m attached to the end (no gravity, no friction), (b) an ideal pendulum with a block of mass m hanging from the end of a string of length l, undergoing small oscillations.

(a) F=ma for an ideal mass spring system is

$$m\ddot{x} = -kx,$$

where the two dots above the x respresent two time derivatives. This can be rewritten as $\ddot{x} = -\frac{k}{m}x$. Without knowing how to solve this differential equation, we can guess at the answer. We know that sine and cosine both come back to themselves with a coefficient out from when we take two derivatives. So let's guess at a solution:

$$x(t) = A\sin{(\omega t + \phi)},$$

where cos could have worked as well. We can now plug this into the differential equation:

$$\ddot{x} = -A\omega^2 \sin{(\omega t + \phi)} \tag{21}$$

$$= -\frac{k}{m} A\sin{(\omega t + \phi)} \tag{22}$$

The sine's and the A's cancel out, and we're left with an equation for the frequency of motion:

$$\omega = \sqrt{\frac{k}{m}}.$$

(b) In the case of a mass on a string, there are two forces to consider: gravity and tension. Start by drawing a force body diagram with these two forces. Then write out a torque equation for the pendulum. We've chosen to use torque here since the periodic motion is in the angle of the pendulum. We will calculate the torque about the pivot point. Gravity is the only contributor to torque (tension is always parallel to $\vec{r}$).

$$\tau = I\ddot{\theta} \tag{23}$$

$$= -mgl\sin\theta \tag{24}$$

The problem stated that we are using the small angle approximation, so $\sin\theta \approx \theta$. We also know the moment of inertia for a point particle l away from the pivot point: $I = ml^2$. Putting it all together, we get the differential equation

$$\ddot{\theta} = -\frac{g}{l}\theta.$$

As in part (a), we guess that the form of the solution is $\theta = A\sin{(\omega t + \phi)}$. Plugging that in and solving for ω gives us

$$\omega = \sqrt{\frac{g}{l}},$$