Discussion 6 - Gauss' Law

Problem 1. A very long, solid cylinder with radius R has positive charge uniformly distibuted throughout it, with charge per unit volume ρ.

a. Derive the expression for the electric field inside the volume at a distance r from the axis of the cylinder in terms of the charge density ρ.
b. What is the electric field at a point outside the volume in terms of the charge per unit length λ in the cylinder?
c. Compare the answers to parts (a) and (b) for $r = R$.
d. Graph the electric-field magnitude as a fuction of r from $r = 0$ to $r = 3R$.

a. The electric field should point radially outward by the symmetry of the system. Consider a Gaussian surface that is a cylinder of length L and radius $r < R$. The volume is $\pi r^2 L$, the surface area is $2\pi r L$, and $\hat{n} = \hat{r}$. We expect the length L of the Gaussian cylinder to cancel out in our final answer because the field should not depend on your location along a very long cylinder.

$$\oint \vec{E} \cdot \hat{n} dA = \frac{Q_{encl}}{\epsilon_0} \tag{1}$$
$$Q_{enc} = \rho V = \rho \pi r^2 L \tag{2}$$
$$\oint E\hat{r} \cdot \hat{r} dA = \frac{\rho \pi r^2 L}{\epsilon_0} \tag{3}$$
$$E \oint dA = \frac{\rho \pi r^2 L}{\epsilon_0} \tag{4}$$
$$E 2\pi r L = \frac{\rho \pi r^2 L}{\epsilon_0} \tag{5}$$
$$\vec{E} = \frac{\rho r}{2\epsilon_0}\hat{r} \tag{6}$$

b. Now consider a Gaussian cylinder of radius $r > R$. The charge enclosed no longer depends on the radius of the cylinder. For an $r > R$, the Gaussian cylinder will enclose all of the charged surface (within some length L).

$$Q_{enc} = \rho \pi R^2 L \tag{7}$$
$$E 2\pi r L = \frac{\rho \pi R^2 L}{\epsilon_0} \tag{8}$$
$$\vec{E} = \frac{\rho R^2}{2\epsilon_0 r}\hat{r} \tag{9}$$

The charge per unit length is $\lambda = \rho \pi R^2$, so the electric field can be written as

$$\vec{E} = \frac{\lambda}{2\pi r \epsilon_0}\hat{r} \tag{10}$$

c. The answers should agree at $r = R$. $\vec{E}(r) = \rho R/(2\epsilon_0)\hat{r}$.
d. The magnitude of the field is linearly increasing from $r = 0$ to $r = R$ and then decays like $1/r$ from $r = R$ to $r = 3R$.

Problem 2. An insulating sphere with radius a has a uniform charge density ρ. The sphere is not centered at the origin but at $\vec{r} = \vec{b}$. Show that the electric field inside the sphere is given by $\vec{E} = \rho(\vec{r} - \vec{b})/3\epsilon_0$.

See ungraded homework problems solution for 22.57.

Problem 3. An insulating sphere of radius R has a spherical hole of radius a located within its volume and centered a distance b from the center of the sphere, where $a < b < R$. (A cross section of the sphere is shown in Figure 1.) The solid part of the sphere has a uniform volume charge density ρ. Find the magnitude and direction of the electric field $\vec{E}$ inside the hole, and show that $\vec{E}$ is uniform over the entire hole. Use the principle of superposition and the result of Problem 2.

See ungraded homework problems solution for 22.57.

Problem 4. A small conducting spherical shell with inner radius a and outer radius b is concentric with a larger conducting spherical shell with inner radius c and outer radius d as shown in the figure. The inner shell has total charge $+2q$, and the outer shell has charge $+4q$.

a. Calculate the electric field $\vec{E}$ (magnitude and direction) in terms of q and the distance r from the common center of the two shells for every region. Graph the radial component of $\vec{E}$ as a function of r.

b. What is the total charge on the inner surface of the small shell, the outer surface of the small shell, the inner surface of the large shell, and the outer surface of the large shell?

The electric field in the material of a conductor is zero. The configuration is spherically symmetric so we can apply Gauss' Law. The only charges in this problem exist on the surfaces of the conductors. The total charge is the sum of the charge on the inner surface and the outer surface of a conductor.

$$q_a + q_b = 2q \tag{11}$$

$$q_c + q_d = 4q \tag{12}$$

$$\oint \vec{E} \cdot \hat{n} dA = \frac{q_{enc}}{\epsilon_0} \tag{13}$$

$$\vec{E} = E\hat{r} \tag{14}$$

$$E 4\pi r^2 = \frac{q_{enc}}{\epsilon_0} \tag{15}$$

$$\vec{E} = \frac{q_{enc}}{4\pi\epsilon_0 r^2}\hat{r} \tag{16}$$

Consider a Gaussian sphere with radius $r < a$. The charge enclosed is zero; therefore the field is zero.

Consider a Gaussian sphere with radius r such that $a < r < b$. The only charged region enclosed is the inner surface of the small conductor so $q_{enc} = q_a$. The field in this region must be zero since we are inside the conductor. Therefore $q_a = 0$ and then $q_b = 2q$ to maintain the total charge.

Consider a Gaussian sphere with radius r such that $b < r < c$. The charge enclosed is $q_a + q_b = 2q$ so the magnitude of the electric field is $2q/(4\pi\epsilon_0 r^2)$.

A Gaussian sphere with radius $c < r < d$ encloses $q_a + q_b + q_c = 2q + q_c$ but the field should be zero because the region is inside a conductor. So the charge enclosed must add up to zero and $q_c = -2q$. Since $q_c + q_d = 4q$, $q_d = 6q$.

The magnitude of the electric field for $r > d$ is $6q/(4\pi\epsilon_0 r^2)$.

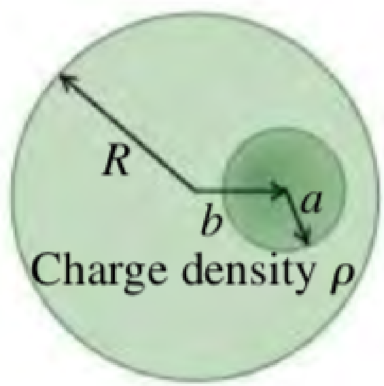

Figure 1: Problem 3

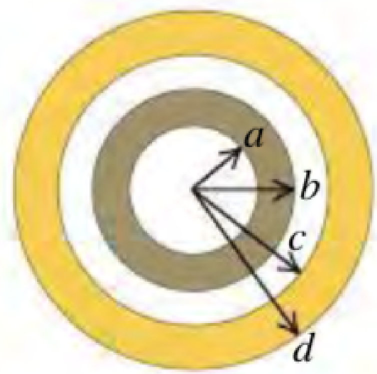

Figure 2: Problem 4