Final Exam Review Problems

Studying Tips

- Do all of the homework. :p
- Don't memorize formulas. Understand the derivations for each formula you use. Go back and re-derive formulas you need when practicing problems.
- Try problems first without looking at the solutions. Once you've given it full effort, check the solution. Then try again without looking at the solution. Repeat until solved.
- After finishing a problem, go back and think about the process of how you did it. What formulas did you use? What principles did you use?
- Come to OH and practice there!

Test Taking Tips

- Get a full night sleep before the test. Research shows this helps with memory recall. Research also shows that students who get a full night sleep do better than students who stay up all night studying.
- Tell yourself you are going to do well before the test. In your head, say things like "I will remember how to do these problems" and "I am confident in my ability to problem solve". Don't complain with your friends right before about how you are not ready. This has also been scientifically shown to increase your score even though it sounds silly.
- Don't cheat. We have caught people cheating in the past, and we will catch you!
- Write down something for every problem. We are almost always willing to give a point or two for effort.
- Check units! If your answer has the wrong units then you've made an algebra mistake most likely. Even just writing that you have the wrong units shows the grader that you are aware of what you are doing.
- Physically prepare yourself for the test. Eat breakfast and use the bathroom before hand!

Problem 1. As shown in Figure 1, two masses M and m are connected by a very light rigid bar and attached to an ideal massless spring of spring constant k. Assume the mass of the rigid bar is negligible.

a. Using Newton's Second Law, write the differential equation for $x(t)$, the masses' displacement from equilibrium as a function of time, in terms of m, M, and k.

b. Assume that at $t = 0$ the masses are at a positive position $x_0 > 0$ and have a positive velocity $v_{x0} > 0$. Moreover, assume that at $t = 0$ the total kinetic energy of the masses is equal to the potential energy of the spring. Write the soution for $x(t)$ in terms of m, M, k, and x_0.

c. Now consider a physical pendulum consisting of a solid, uniform sphere of radius R suspended on a wire of negligible mass and also of length R, as shown in Figure 2. What must the distance R be so that the period of the pendulum for small oscillations matches the period of the mass-spring system from Figure 1? Give an expression for R in terms of m, M, k, and gravitational acceleration g. (First, find the moment of inertia of a solid, uniform sphere of mass m_s and radius R about an axis through its center.)

a. The rigidly-connected blocks m and M can be treated as a single object of mass $m+M$. Newton's Second Law yields the following.

$$F_{net,x} = -kx(t) \tag{1}$$
$$-kx(t) = (m+M)a_x \tag{2}$$
$$\frac{d^2x(t)}{dt^2} = -\frac{k}{m+M}x(t) \tag{3}$$

b. At $t=0$, the position is $x(0)=x_0$ with $x_0>0$ and the x-velocity is $v_x(0)=v_{x0}$ with $v_{x0}>0$. Moreover, the kinetic and elastic potential energies are equal.

$$K_0 = U_0 \tag{4}$$
$$\frac{1}{2}(m+M)v_{x0}^2 = \frac{1}{2}kx_0^2 \tag{5}$$
$$v_{x0}^2 = \frac{k}{m+M}x_0^2 \tag{6}$$

We assume a solution with frequency ω.

$$x(t) = A\cos(\omega t+\phi) \tag{7}$$
$$\omega = \sqrt{\frac{k}{m+M}} \tag{8}$$
$$v_x(t) = \frac{dx}{dt} = -A\omega\sin(\omega t+\phi) \tag{9}$$
$$x(0) = A\cos(\phi) = x_0 \tag{10}$$
$$v_x(0) = -A\omega\sin(\phi) = v_{x0} \tag{11}$$
$$A^2 = x_0^2 + \frac{v_{x0}^2}{\omega^2} \tag{12}$$
$$A = \sqrt{x_0^2 + \frac{k}{m+M}\frac{m+M}{k}x_0^2} = \sqrt{2}x_0 \tag{13}$$
$$\tan\phi = \frac{-v_{x0}}{\omega x_0} = -\sqrt{\frac{m+M}{k}}\frac{v_{x0}}{x_0} = -1 \tag{14}$$
$$\phi = -\frac{\pi}{4} \tag{15}$$

Check that the choice of ϕ yields $x_0>0$ and $v_{x0}>0$.

$$\cos\phi = \frac{x_0}{A} > 0 \tag{16}$$
$$\sin\phi = \frac{-v_{x0}}{A\omega} < 0 \tag{17}$$

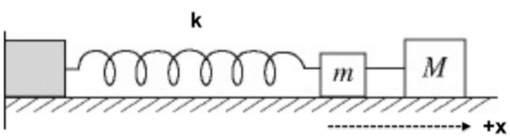


Figure 1: Problem 1

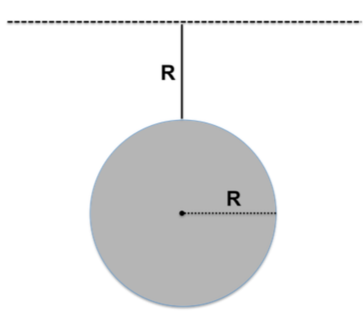


Figure 2: Problem 1

The solution is

$$x(t) = \sqrt{2}x_0 \cos(\sqrt{\frac{k}{m+M}}t - \frac{\pi}{4}) \tag{18}$$

c. We want the period of the physical pendulum to match the period of the oscillator from parts (a) and (b).

$$T_{pendulum} = T_{mass-spring} \tag{19}$$

$$T_{m_s} = 2\pi\sqrt{\frac{m+M}{k}} \tag{20}$$

$$T_{pend} = 2\pi\sqrt{\frac{I_{pivot}}{m_s g d_{cm}}} \tag{21}$$

Since the sphere is uniform, the center of mass is at the center of the sphere, a distance $d_{cm} = 2R$ from the pivot point. The moment of inertia follows from the parallel axis theorem.

$$I_{pivot} = I_{cm} + m_s d_{cm}^2 \tag{22}$$

Find the moment of inertia about a vertical axis through the center of mass by integration.

$$I = \int y^2 dm \tag{23}$$

$$y = r\sin\theta \tag{24}$$

$$dm = \frac{m_s}{V}dV = \frac{m_s}{\frac{4}{3}\pi R^3} r^2 \sin\theta d\theta d\phi dr \tag{25}$$

$$I = \int_0^R \int_0^\pi \int_0^{2\pi} \frac{m_s}{\frac{4}{3}\pi R^3} y^2 r^2 \sin\theta d\theta d\phi dr \tag{26}$$

$$I = \frac{m_s}{\frac{4}{3}\pi R^3} 2\pi \int_0^R r^4 dr \int_0^\pi \sin^2\theta d\theta \tag{27}$$

$$I = \frac{m_s}{\frac{4}{3}\pi R^3} 2\pi \frac{R^5}{5}\frac{4}{3} \tag{28}$$

$$I = \frac{2}{5} m_s R^2 \tag{29}$$

Plug everything into the equation relating the period of the physical pendulum to the period of the mass-spring system and solve for R.

$$T_{pend} = 2\pi\sqrt{\frac{\frac{2}{5}m_s R^2 + m_s(2R)^2}{m_s g(2R)}} = 2\pi\sqrt{\frac{11R}{5g}} \tag{30}$$

$$\sqrt{\frac{11R}{5g}} = \sqrt{\frac{m+M}{k}} \tag{31}$$

$$R = \frac{5g}{11k}(m+M) \tag{32}$$

Problem 2. Consider a musical pipe of length L_P that is open at both ends. Air is blown into the pipe, exciting a longitudinal normal mode. Measurements indicate that there are four pressure fluctuation nodes associated with this normal mode.

a. In the space in Figure 3, draw a representation of this normal mode in terms of the pressure fluctuation, labeling all nodes and anti-nodes. You do not need to label the amplitudes.

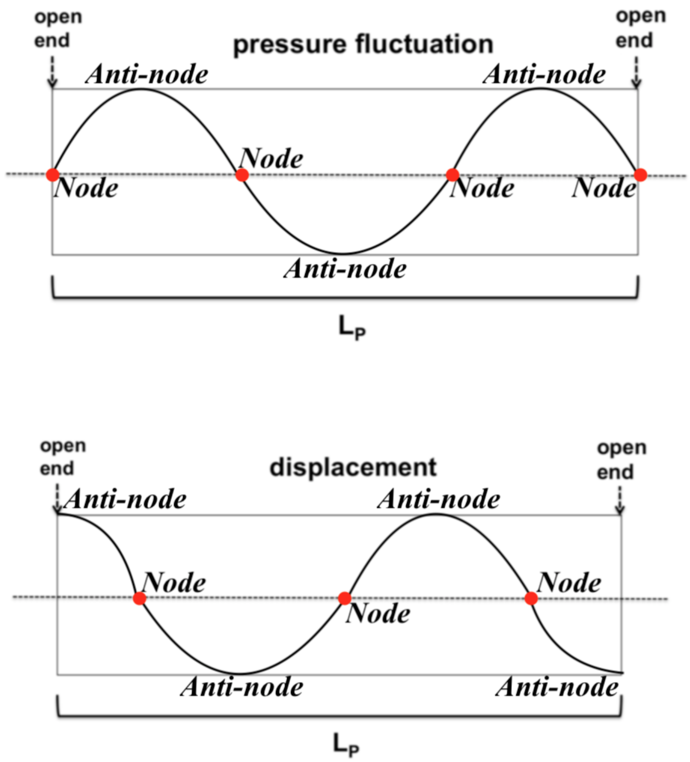


Figure 3: Problem 2

b. Measurements indicate that the pressure fluctuation amplitude is p_{max} and the speed of sound in air is v_s. Determine the expressions for the pressure fluctuation $P_W(x,t)$ as a function of x-position and time t in terms of p_{max}, v_s, and L_P. Assume the ends of the pipe are at $x = 0$ and $x = L_P$, respectively, and that the particles at $x = 0$ have their most positive (i.e. rightmost) displacement at $t = 0$. (Therefore, the pressure at $(x = 0, t = 0)$ is zero.)

c. In a second identical pipe of length L_P, a normal mode corresponding to the next allowed harmonic above the one in Parts (a) and (b) is excited. Both pipes are played simultaneously. At what speed would the original pipe need ot be moved towards a stationary listener such that the listener hears the same frequency from both pipes? Express your answer in terms of v_s.

a. Open ends are pressure nodes and displacement antinodes. Nodes are equally spaced.

b. Start with the general expression for standing waves where the displacement wave and the pressure wave have the same k and ω.

$$P(x,t) = p_{max}\sin(k_n x + \phi_3)\sin(\omega_n t + \phi_4) \tag{33}$$

For the nth mode in a pipe open at both ends, the allowed wavelengths are

$$\lambda_n = \frac{2L_P}{n} \tag{34}$$

Since the wave number is 2π divided by the wavelength and the angular frequency is wavenumber multiplied by the speed of sound,

$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L_P} \tag{35}$$

$$\omega_n = k_n v_{snd} = \frac{n\pi v_{snd}}{L_P} \tag{36}$$

Since there are three half-wavelengths in the pipe, $n = 3$. Find the phase shifts using the initial condition that the displacement is an antinode at $x = 0$. When the slope of the displacement wave is negative, the pressure fluctuation is positive. So the pressure wave looks like a sine wave with no phase shift at $t = 0$. Since $P(x, t)$ is proportional to $-\frac{\partial D}{\partial x}$, it follows that $P(x, t)$ is proportional to $\sin kx \cos \omega t$.

$$P_3(x,t) = p_{max} \sin(\frac{3\pi}{L_P}x) \cos(\frac{3\pi v_{snd}}{L_P}t) \tag{37}$$

c. The mode from Part (a) has frequency

$$f_3 = \frac{\omega_3}{2\pi} = \frac{3v_{snd}}{2L_P} \tag{38}$$

The next allowed harmonic, corresponding to $n = 4$, has frequency

$$f_4 = \frac{1}{2\pi}\frac{4\pi v_{snd}}{L_P} = \frac{2v_{snd}}{L_P} \tag{39}$$

For a pipe moving towards a stationary listener, the Doppler-shifted frequency heard by the listener is

$$f_L = \frac{v_{snd}}{v_{snd} - v_{pipe}} f_{pipe} \tag{40}$$

The original pipe with frequency $f_{pipe} = f_3$ must move towards the listener at a speed such that the listener hears frequency $f_L = f_4$. Solve for v_{pipe}.

$$f_4 = \frac{v_{snd}}{v_{snd} - v_{pipe}} f_3 \tag{41}$$

$$\frac{2v_{snd}}{L_P} = \frac{v_{snd}}{v_{snd} - v_{pipe}} \frac{3v_{snd}}{2L_P} \tag{42}$$

$$\frac{4}{3} = \frac{v_{snd}}{v_{snd} - v_{pipe}} \tag{43}$$

$$\frac{4}{3}(v_{snd} - v_{pipe}) = v_{snd} \tag{44}$$

$$v_{pipe} = \frac{v_{snd}}{4} \tag{45}$$

Problem 3. Figure 4 shows a thin ring of radius a that contains a uniformly distributed negative charge Q (such that $Q < 0$). The ring is placed in the xy-plane, such that the z-axis defines the central axis of the ring.

a. Determine the magnitude and direction of the electric field at point P, which is located at coordinate z along the z-axis, due to the ring of charge. Derive and then solve the necessary integral expression(s) and/or make arguments from symmetry. Express your answer in terms of Q, the given parameters, and fundamental constants.

Consider the ring as a series of point charges and add up the contribution of each point charge. To find the electric field, we can find the potential and use $\vec{E} = -\nabla V$. The potential with reference to infinity due to a point charge is

$$dV = k\frac{dq}{r} \tag{46}$$

where r is the distance from the point charge to the point P. In this case, the distance r is the hypotenuse of the triangle with sides a and z because point P is at height z. Integrate around the whole ring so each piece dq should have charge λds where λ is the total charge divided by total length and ds is the arc length. The radius of the ring is constant, a.

$$dq = \lambda ds = \frac{Q}{2\pi a} a d\theta \tag{47}$$

$$V = \int_{\theta=0}^{\theta=2\pi} k \frac{\frac{Q}{2\pi a} a d\theta}{\sqrt{a^2 + z^2}} \tag{48}$$

$$V = k \frac{Q}{2\pi a} \frac{a 2\pi}{\sqrt{a^2 + z^2}} \tag{49}$$

$$V = k \frac{Q}{\sqrt{a^2 + z^2}} \tag{50}$$

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{z} \tag{51}$$

$$\vec{E} = k \frac{Qz}{(a^2 + z^2)^{3/2}}\hat{z} \tag{52}$$

The electric field has no component in the x or y direction because of the symmetry of the ring and so the potential should only depend on z. The problem tells us $Q < 0$ so the field points toward the ring, as we would expect.

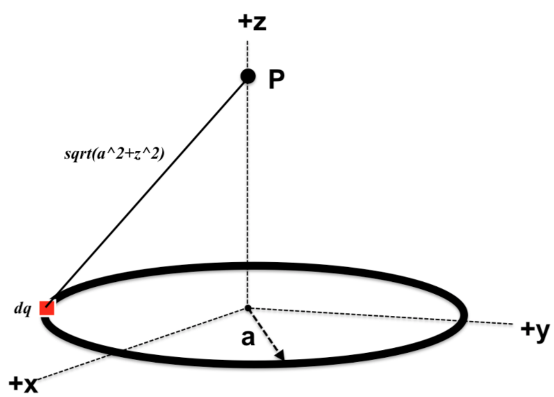


Figure 4: Problem 3

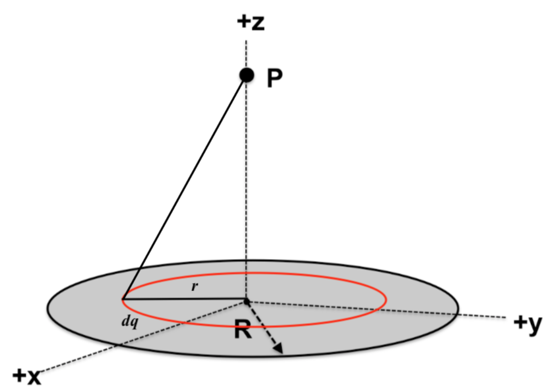


Figure 5: Problem 3

Figure 5 shows a thin, uniformly charged disk of radius R and negative surface charge density σ (such that $\sigma < 0$). The disk is placed in the xy-plane such that the z-axis defines the central axis of the disk.

b. Determine the magnitude and direction of the electric field at point P, which is located at coordinate z along the z-axis, due to the disk of charge. Derive and then solve the necessary integral expression(s) and/or make arguments from symmetry. Express your answer in terms of σ, the given parameters, and fundamental constants. (Make use of the result to Part (a).)

Consider the contribution of a thin ring of charge dq and radius r. Integrate the electric field of a ring over many rings with radii from $r = 0$ to $r = R$ to find the potential of the disk. The charge of each ring is proportional to its area, which is its circumfrence multiplied by dr.

$$dq = \sigma dA = \sigma 2\pi r dr \tag{53}$$

Use the expression for the field of a ring of radius a but replace the radius with the integration variable

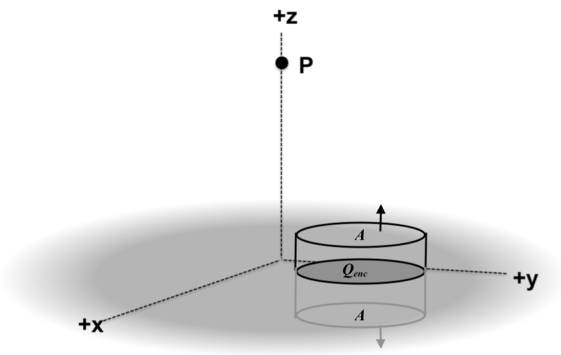


Figure 6: Problem 3

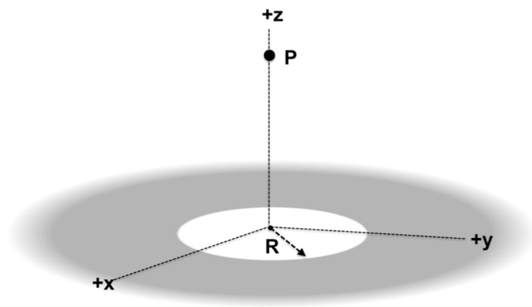


Figure 7: Problem 3

r. The electric field in the x and y directions is zero by symmetry.

$$\vec{E} = \int k \frac{dqz}{(r^2+z^2)^{3/2}} \hat{z} \tag{54}$$

$$\vec{E} = \hat{z} \int_{r=0}^{r=R} k \frac{\sigma 2\pi r dr z}{(r^2+z^2)^{3/2}} \tag{55}$$

$$u = r^2 + z^2 \tag{56}$$

$$du = 2rdr \tag{57}$$

$$\vec{E} = \hat{z} \int_{u=z^2}^{u=R^2+z^2} k\pi\sigma z \frac{du}{u^{3/2}} \tag{58}$$

$$\vec{E} = k2\pi\sigma z \left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right) \hat{z} \tag{59}$$

Figure 6 shows a thin, uniformly charged, infinite sheet of negative surface charge density σ (such that $\sigma < 0$). The sheet is placed in the xy-plane.

c. Using the result from Part (b), take the appropriate limit to determine the magnitude and direction of the electric field at point P due to an infinite sheet of uniform negative surface charge density σ. Express your answer in terms of the given parameters and fundamental constants.

   An infinite sheet is like a disk with a radius at infinity. Take the limit $R \gg z$ of the previous expression.

$$\vec{E} = k2\pi\sigma \left(1 - \frac{z}{\sqrt{R^2+z^2}} \right) \hat{z} \approx k2\pi\sigma \left(1 - \frac{z}{R} \right) \hat{z} \tag{60}$$

$$\vec{E} \approx k2\pi\sigma\hat{z} = \frac{\sigma}{2\epsilon_0}\hat{z} \tag{61}$$

   We recover the expression for the electric field of an intfinite plane with charge σ.

d. Using Gausss Law, determine the magnitude and direction of the electric field at point P due to an infinite sheet of uniform negative surface charge density σ. Demonstrate the application of Gausss Law and clearly state arguments from symmetry. Your answer should match the answer to Part (c).

Consider a Gaussian cylinder through the plane with surface area A. The electric field points away from the plane on both sides by symmetry. The flux through both ends of the cylinder contributes.

$$\oint \vec{E} \cdot \hat{n} dA = \frac{q_{enc}}{\epsilon_0} \tag{62}$$

$$\vec{E} = E\frac{|z|}{z}\hat{z} \tag{63}$$

$$E2A = \frac{A\sigma}{\epsilon_0} \tag{64}$$

$$\vec{E} = \frac{|z|}{z}\frac{\sigma}{2\epsilon_0}\hat{z} \tag{65}$$

Figure 7 shows a thin, uniformly charged infinite sheet of negative surface charge density σ (such that $\sigma < 0$). The sheet is placed in the xy-plane. A circular section of radius R is removed from the center of the sheet such that the z-axis runs through the center of the resulting hole.

e. Determine the magnitude and direction of the electric field at point P, which is located at coordinate z along the z-axis, due to the infinite sheet with a hole. Express your answer in terms of the given parameters and fundamental constants. (Use the principle of superposition and the results to the previous parts of this problem.)

Using the principle of superposition, the electric field due to the infinite sheet with a hole is the electric field due to an infinite sheet plus the electric field due to a disk the size of the hole with opposite charge.

$$\vec{E} = \vec{E}_{sheet,\sigma} + \vec{E}_{hole,-\sigma} \tag{66}$$

Use the results from parts (b) and (c).

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{z} + \frac{-\sigma z}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right)\hat{z} \tag{67}$$

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\frac{1}{\sqrt{\left(\frac{R}{z}\right)^2+1}}\hat{z} \tag{68}$$

f. A positive point charge q_0 that has a mass m_0 is now placed at point P. Assuming that point P is very close to the origin such that $|z| \ll R$, (1) show that the resulting motion of the point charge will be approximately simple harmonic motion and (2) derive an expression for the period of oscillation. Express your answer in terms of the given parameters and fundamental constants.

The acceleration of a test charge of charge q_0 and mass m_0 is given by

$$\vec{a} = \frac{F_{net}}{m_0} \tag{69}$$

$$a_z = \frac{q_0}{m_0}E_z \tag{70}$$

$$\frac{d^2z}{dt^2} = \frac{q_0}{m_0}\frac{-|\sigma|}{2\epsilon_0}\frac{z}{\sqrt{R^2+z^2}} \tag{71}$$

If $|z| \ll R$, $\sqrt{z^2+R^2} \approx R$, such that

$$\frac{d^2z}{dt^2} \approx -\frac{q_0|\sigma|}{2\epsilon_0 m_0 R}z \tag{72}$$

This differential equation describes simple harmonic motion along the z-axis with angular frequency ω. The period oscillation is $T = 2\pi/\omega$.

$$\omega^2 = \frac{q_0|\sigma|}{2\epsilon_0 m_0 R} \tag{73}$$

$$T = 2\pi\sqrt{\frac{2\epsilon_0 m_0 R}{q_0|\sigma|}} \tag{74}$$

Problem 4. Figure 8 shows a diagram for a circuit containing two identical batteries, each of unknown EMF $\mathcal{E}$ and an unknown internal resistance r. The circuit also contains two resistors of resistances R and $3R$. There are two switches S1 and S2, each of which can either be in the open or closed position. An ideal ammeter and an ideal voltmeter are also connected to the circuit as shown. When both switch S1 and switch S2 are open, the voltmeter reads a potential difference V_0. When switch S1 is closed but switch S2 is open, the voltmeter reads a potential difference V_1, where $V_1 < V_0$. Assume that the voltmeter leads are connected in such a way that the measured potential differences are positive ($V_0, V_1 > 0$).

a. Solve for (1) the battery EMF $\mathcal{E}$ and (2) the internal resistance r in terms of R, V_0, and V_1.

b. What is the reading of the ammeter and the direction of the current through switch S1 when (1) both switches are open, (2) switch S1 is closed but switch S2 is open, and (3) both switches are closed? Give the answers in terms of R, V_0, and V_1. State the directions as either left, right, or none.

c. What is the reading of the voltmeter when both switches are closed?

a. With both switches open, no current flows anywhere in the circuit. Since the terminal voltage $V_a - V_b = V_{ab} = \mathcal{E} - Ir$ for a real EMF source and $I = 0$, the voltmeter reading V_0 is just the EMF $\mathcal{E}$ of the battery: $\mathcal{E} = V_0$. With the switch S1 closed, a single loop is formed with current I_1. Kirchoff's loop rule gives

$$+\mathcal{E} - I_1 r - I_1(3R) + \mathcal{E} - I_1 r - I_1 R = 0 \tag{75}$$

Now the terminal voltage is

$$V_{ab} = \mathcal{E} - I_1 r = V_1 \tag{76}$$

where the current I_1 is

$$I_1 = \frac{2\mathcal{E}}{2r + 4R} = \frac{\mathcal{E}}{r + 2R} \tag{77}$$

Solve for the internal resistance, r.

$$V_1 = \mathcal{E} - \left(\frac{\mathcal{E}}{r + 2R}\right) r = \mathcal{E}\frac{2R}{r + 2R} \tag{78}$$

$$\mathcal{E} = V_0 \tag{79}$$

$$r = 2R\left(\frac{V_0}{V_1} - 1\right) \tag{80}$$

b. With both switches open, there is no current anywhere and the ammeter reads zero. With switch S1 closed but S2 open, the current has the value found in part (a).

$$I_1 = \frac{\mathcal{E}}{r + 2R} = \frac{V_0}{2R\left(\frac{V_0}{V_1} - 1\right) + 2R} = \frac{V_1}{2R} \tag{81}$$

The clockwise current moves right through switch S1. With both switches closed, the loop rule gives the following. Define I_1 clockwise on the left branch, I_2 clockwise on the right branch, and I_3 down the middle branch.

$$+\mathcal{E} - I_1 r = 0 \tag{82}$$

$$+\mathcal{E} - I_2 r - I_2(3R) = 0 \tag{83}$$

$$I_2 = \frac{\mathcal{E}}{r + 4R} \tag{84}$$

$$I_1 = \frac{\mathcal{E}}{r} \tag{85}$$

The ammeter reads $\frac{V_0 V_1}{2R(V_0 - V_1)}$ and the clockwise current moves right through switch S1.

c. When both switches are cloes, the voltmeter reads $\mathcal{E} - I_2 r$. Plug in I_2, $\mathcal{E}$, and r from the previous parts and simplify.

$$V_{ab} = \mathcal{E} - I_2 r = \frac{2V_0 V_1}{V_0 + V_1} \tag{86}$$

Problem 5. Three parallel-plate capacitors of capacitances C, $2C$, and $3C$ are connected to an ideal battery of EMF $\mathcal{E}$ and a switch S as shown in Figure 9.

a. With the switch S in the closed position, you perform the following sequence of steps:
   (1) You measure the charge magnitude on the capacitor with capacitance C and record the value as $Q_{initial}$.
   (2) You insert a dielectric material of dielectric constant $K = 2$ between the plates of the capacitor with capacitance $3C$. The dielectric fully fills the volume between the plates.
   (3) You measure the charge magnitude on the capacitor with capacitance C again and record the value as Q_{final}.

   What is the ratio $Q_{final}/Q_{initial}$? Express your answer as a numerical fraction.

b. With the dielectric of dielectric constant $K = 2$ still inserted into the capacitor of capacitance $3C$, you now perform the following sequence of steps:
   (1) You flip switch S into the open position.
   (2) You remove the dielectric.
   (3) You measure the voltage across the capacitor of capacitance $3C$.

   What is the voltage across the capacitor of capacitance $3C$? Express your answer in terms of $\mathcal{E}$.

With the switch closed, the battery is connected, so the voltage is constant. Capacitors in series have the same charge; capacitors in parallel have the same voltage.

a. Capacitor $2C$ and $3C$ are in parallel. Reduce them to one equivalent capacitor $C = 5C$. This $5C$ capacitor is in series with the C capacitor. The equilevent capacitance of all the capacitors together is then $5C/6$.

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{5C} = \frac{6}{5C} \tag{87}$$

The total charge is

$$Q = C_{eq}V = \frac{5C}{6}\mathcal{E} \tag{88}$$

This total charge is equal to the charge on capacitor C because the charge on capacitors in series is equal and capacitor C was in series with capacitor $5C$. Thus,

$$Q_{initial} = \frac{5}{6}C\mathcal{E} \tag{89}$$

Inserting the dielectric into capacitor $3C$ causes the capacitance to increase to

$$C = KC_0 = 2\,(3C) = 6C \tag{90}$$

Adding $2C$ and $6C$ in parallel yields $8C$ which is in series with C.

$$\frac{1}{C_{eq}} = \frac{1}{C} + \frac{1}{8C} \tag{91}$$

The equivalent capacitance is $8C/9$. The total charge on the equivalent capacitor is

$$Q = \frac{8C}{9}\mathcal{E} \tag{92}$$

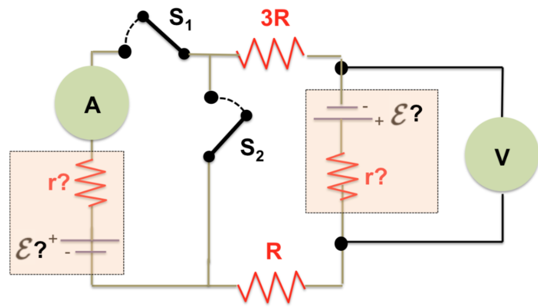


Figure 8: Problem 4

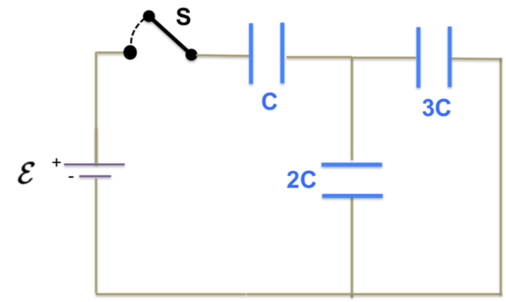


Figure 9: Problem 5

This total charge is equal to the charge on capacitor C. Thus,

$$Q_{final} = \frac{8}{9}C\mathcal{E} \tag{93}$$

So the ratio is

$$\frac{Q_{final}}{Q_{initial}} = \frac{\frac{8}{9}C\mathcal{E}}{\frac{5}{6}C\mathcal{E}} \tag{94}$$

$$\frac{Q_{final}}{Q_{initial}} = \frac{16}{15} \tag{95}$$

b. Once the switch is flipped into the open position, the charge on the capacitor plates is "trapped" such that removing the dielectric does not affect it, $Q = Q_{final}$. The potential difference across capacitor $5C$ is thus

$$V = \frac{Q}{C_{eq}} = \frac{\frac{8}{9}C\mathcal{E}}{5C} = \frac{8}{45}\mathcal{E} \tag{96}$$

Since the potential difference across $5C$ is the same as $2C$ and $3C$, which are in parallel, it follows that the voltage across $3C$ is $8/45\mathcal{E}$.

Problem 6. Two batteries are connected to three conducting wires. Battery 1, which has EMF $2\mathcal{E}$ and negligible internal resistance, has its positive terminal connected to wire 1 and wire 3 and its negative terminal connected to wire 2 and wire 3. Battery 2, which has EMF $3\mathcal{E}$ and negligible internal resistance, has its positive terminal connected to wire 1 and its negative terminal connected to wire 2. Wire 1 and wire 2 have the same resistance, R. Wire 3 has a different resistance R_3.

a. Draw a circuit diagram for the above circuit, clearly labeling the EMF's and resistances. (Model each *real* wire as a resistor connected to *ideal* wires.)

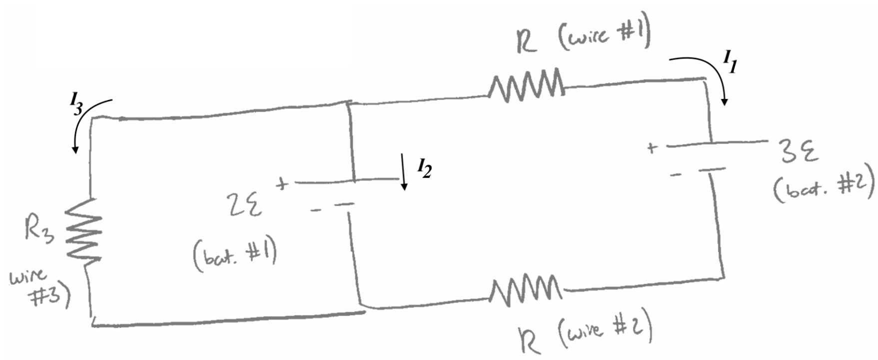
b. Determine the amount of current passing through (1) wire 1, (2) wire 3, and (3) battery 1 in terms of $\mathcal{E}$, R, and R_3. You do not need to state the direction of the current.

c. If you want each wire to dissipate an equal amount of power, what should be the resistance R_3 of wire 3 in terms of R?

a. See figure.

b. To determine the current through each wire, define the current in each branch and apply Kirchoff's rules. The junction rule:

$$I_1 = I_2 + I_3 \tag{97}$$



The loop rule for the left loop and the right loop:

$$+3\mathcal{E} - I_1 R - 2\mathcal{E} - I_1 R = 0 \tag{98}$$
$$+2\mathcal{E} - I_3 R_3 = 0 \tag{99}$$

Therefore

$$I_1 = \frac{\mathcal{E}}{2R} \tag{100}$$
$$I_3 = \frac{2\mathcal{E}}{R_3} \tag{101}$$
$$I_2 = I_1 - I_3 = \frac{\mathcal{E}}{2R} - \frac{2\mathcal{E}}{R_3} \tag{102}$$

c. For each wire to dissipate the same power, we require $P = I^2 R$ to be equal. Plug in the current from the previous part.

$$P = I^2 R \tag{103}$$
$$I_1^2 R = I_3^2 R_3 \tag{104}$$
$$R_3 = \frac{I_1^2}{I_3^2} R \tag{105}$$
$$R_3 = \frac{(\mathcal{E}/2R)^2}{(2\mathcal{E}/R_3)^2} R = \frac{R_3^2}{16R} \tag{106}$$
$$R_3 = 16R \tag{107}$$

Problem 7. The circuit in Figure 10 has been in position a for a long ime. At time $t = 0$ the switch is thrown to position b.

a. What is the current through the resistor just before and just after the switch is thrown?
b. What is the charge on the capacitor just before and just after the switch is thrown?
c. What is the charge on the capacitor at time $t > 0$?
d. After having been in position b for a long time, the switch is thrown back to a. What is the charge on the capacitor at time t after this change? What is the charge on the capacitor just after the switch is thrown?

a. Just before, the current is zero. The voltage across the battery is the same as the voltage across the capacitor. $V_b = V_c$. Just after, $I = V_b/R$.
b. Just before and just after, $Q_0 = CV_b$. Charge does not change instaneously.
c. As the capacitor discharges, $Q(t) = Q_0 e^{-t/RC}$

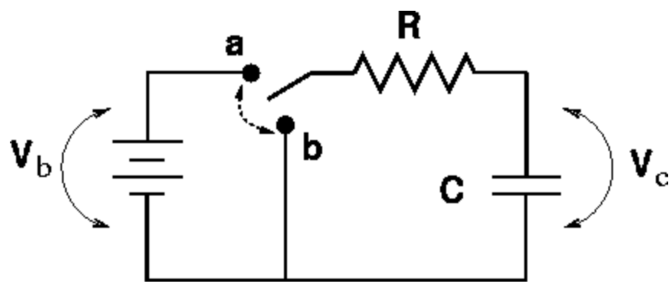


Figure 10: Problem 7

d. The capacitor fully discharges. Then the switch is thrown back so it charges up again, $Q(t) = Q_0(1 - e^{-t/RC})$. Charge does not change instantaneously so it is still zero just after the switch is thrown. We can derive this from the loop equation.

$$V_b - IR - \frac{Q}{C} = 0 \tag{108}$$

$$Q - CV_b = -RC\frac{dQ}{dt} \tag{109}$$

$$\int_0^t \frac{-dt'}{RC} = \int_{Q(0)}^{Q(t)} \frac{dQ'}{Q' - CV_b} \tag{110}$$

$$Q(t = 0) = 0 \tag{111}$$

$$\frac{-t}{RC} = \ln\left[\frac{Q(t) - CV_b}{-CV_b}\right] \tag{112}$$

$$(-CV_b)e^{-t/RC} = Q(t) - CV_b \tag{113}$$

$$Q(t) = CV_b\left(1 - e^{-t/RC}\right) \tag{114}$$

Problem 8. A source of sonic oscillations with frequency f_0 and a receiver are located on the same line normal to a wall. Both the source and the receiver are stationary, and the wall recedes from the source with velocity u. Find the beat frequency registered by the receiver. Denote the velocity of sound as v_{snd}.

First consider the wall as the receiver and find the frequency heard by the wall, f_W. Then consider the wall as a source emiting with frequency f_W. The positive direction is defined from listener to source and the doppler shift formula is

$$f_{obs} = \frac{v_{snd} + v_L}{v_{snd} + v_S} f_0 \tag{115}$$

When the wall is the listener, $v_L = -u$ and $v_S = 0$.

$$f_W = \frac{v_{snd} - u}{v_{snd}} f_0 \tag{116}$$

When the wall is the source, $v_L = 0$ and $v_S = +u$.

$$f_{obs} = \frac{v_{snd}}{v_{snd} + u} f_W \tag{117}$$

Plug in f_W from when the wall was the listener and take the difference with the original frequency to find the beat frequency.

$$f_{beat} = |f_1 - f_2| \tag{118}$$

$$f_{obs} = \frac{v_{snd}}{v_{snd}+u}\left(\frac{v_{snd}-u}{v_{snd}}f_0\right) = \frac{v_{snd}-u}{v_{snd}+u}f_0 \tag{119}$$

$$f_{beat} = f_0 - f_{obs} \tag{120}$$

$$f_{beat} = f_0\left(1 - \frac{v_{snd}-u}{v_{snd}+u}\right) \tag{121}$$

$$f_{beat} = f_0\frac{2u}{v_{snd}+u} \tag{122}$$

Problem 9. For the circuit in Figure 11, the switch is closed at $t = 0$ after having been open for a long time. Find the charge on the capacitor as a function of time. Determine the time constant and the maximum charge on the capacitor in this circuit.

Apply the loop equation twice and the junction equation.

$$I_1 = I_2 + \frac{dQ}{dt} \tag{123}$$

$$0 = V_0 - I_1R_1 - I_2R_2 \tag{124}$$

$$0 = I_2R_2 - \frac{Q}{C} \tag{125}$$

Reduce these to one differential equation for Q.

$$I_2 = \frac{Q}{R_2C} \tag{126}$$

$$I_1 = \frac{Q}{R_2C} + \frac{dQ}{dt} \tag{127}$$

$$0 = V_0 - \left(\frac{Q}{R_2C} + \frac{dQ}{dt}\right)R_1 - \left(\frac{Q}{R_2C}\right)R_2 \tag{128}$$

$$\frac{dQ}{dt} = \frac{V_0}{R_1} - Q\left(\frac{R_1+R_2}{R_1R_2C}\right) \tag{129}$$

Separate the differential equation with Q on the left and t on the right.

$$\int_0^Q \frac{dQ'}{\frac{V_0}{R_1} - Q'\left(\frac{R_1+R_2}{R_1R_2C}\right)} = \int_0^t dt' \tag{130}$$

$$u = \frac{V_0}{R_1} - Q'\left(\frac{R_1+R_2}{R_1R_2C}\right) \tag{131}$$

$$du = -dQ'\left(\frac{R_1+R_2}{R_1R_2C}\right) \tag{132}$$

$$\frac{-1}{\left(\frac{R_1+R_2}{R_1R_2C}\right)}\int_{u_1}^{u_2}\frac{du}{u} = \int_0^t dt' \tag{133}$$

$$u_1 = \frac{V_0}{R_1} \tag{134}$$

$$u_2 = \frac{V_0}{R_1} - Q\left(\frac{R_1+R_2}{R_1R_2C}\right) \tag{135}$$

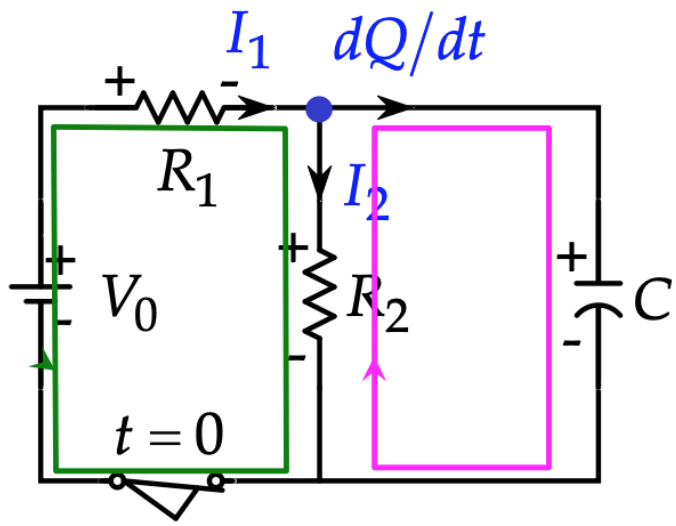


Figure 11: Problem 9

$$\frac{-R_1R_2C}{R_1+R_2}\ln\frac{u_2}{u_1}=t \tag{136}$$

$$u_2=u_1e^{-t\left(\frac{R_1+R_2}{R_1R_2C}\right)} \tag{137}$$

$$\left(\frac{V_0R_2C}{R_1+R_2}\right)-Q=\left(\frac{V_0R_2C}{R_1+R_2}\right)e^{-t\left(\frac{R_1+R_2}{R_1R_2C}\right)} \tag{138}$$

$$Q(t)=\frac{R_2}{R_1+R_2}V_0C\left(1-e^{-t\left(\frac{R_1+R_2}{R_1R_2C}\right)}\right) \tag{139}$$

Note that the time constant depends on the the equivalent resistance of the parallel combination.

$$R_{||}=\frac{R_1R_2}{R_1+R_2} \tag{140}$$

$$Q(t)=\frac{R_2}{R_1+R_2}V_0C\left(1-e^{-t/CR_{||}}\right) \tag{141}$$

The maximum charge occurs when $t\rightarrow\infty$ and the time constant is the inverse of what is multiplying t. The time constant should have units of time and units of RC.

$$Q_{max}=\frac{R_2}{R_1+R_2}V_0C \tag{142}$$

$$\tau=CR_{||}=\frac{R_1R_2C}{R_1+R_2} \tag{143}$$