Discussion 2 - Oscillators

Problem 1. An apple weighs mg. When you hang it from the end of a long spring of force constant k and negligible mass, it bounces up and down in simple harmonic motion. If you stop the bouncing and let the apple swing from side to side through a small angle, the frequency of this simple pendulum, for the purposes of this problem, assume that it is half the bounce frequency. (Because the angle is small, the back-and-forth swings do not cause any appreciable change in the length of the spring.) What is the unstretched length of the spring with the apple removed?

First we find the stretched length of the spring since we know the bounce frequency is twice the pendulum frequency. We will put the pendulum frequency length in terms if l_0, the unstretched length of the spring, and x_{eq}, the amount that the spring stretches from l_0 due to the weight of the apple

$$\omega_{spring} = \sqrt{\frac{k}{m}} \tag{1}$$

$$\omega_{pendulum} = \sqrt{\frac{g}{l}} \tag{2}$$

$$\sqrt{\frac{g}{l_0 + x_{eq}}} = \frac{1}{2}\sqrt{\frac{k}{m}} \tag{3}$$

$$\tag{4}$$

Next we find how much the spring is stretched due to the weight of the apple by considering the moment when acceleration is zero.

$$l_0 + x_{eq} = \frac{4mg}{k} \tag{5}$$

$$mg - kx_{eq} = 0 \tag{6}$$

$$x_{eq} = \frac{mg}{k} \tag{7}$$

$$\tag{8}$$

The unstretched length of the spring with the apple removed is the difference of these two.

$$l_0 = \frac{3mg}{k} \tag{9}$$

Problem 2. A block of mass m rests in a tray of mass M which is attached to a spring with force constant k. The spring is below the tray, so it can oscillate up and down. Define $y = 0$ as the equilibrium position (where the net force is zero). The force of gravity acts in the $-y$ direction. Assume that at time $t = 0$, the system is set into motion from its equilibrium position by giving the tray and block an initial speed v_0 in the $-y$ direction.

a. Determine $y(t)$ in terms of k, m, M, and v_0. Assume v_0 is small enough that the block remains on the tray at all times.

We start with the standard wave equations for the position and velocity of a spring system with mass $m + M$,

$$y(t) = A\cos(\omega t + \phi) \tag{10}$$

$$\omega = \sqrt{\frac{k}{m+M}} \tag{11}$$

$$v(t) = -A\omega\sin(\omega t + \phi) \tag{12}$$

$$\tag{13}$$

We now apply the boundary condition of the position at $t = 0$

$$y(0) = 0 = A\cos(\omega * 0 + \phi) \tag{14}$$
$$\phi = \pm\frac{\pi}{2} \tag{15}$$
$$\tag{16}$$

Assuming the positive solution, we now apply the boundary condition of the initial velocity at $t = 0$:

$$v(0) = -v_0 = -A\omega\sin\left(\omega * 0 \pm \frac{\pi}{2}\right) \tag{17}$$
$$\phi = \frac{\pi}{2} \tag{18}$$
$$A = \frac{v_0}{\omega} \tag{19}$$
$$y(t) = v_0\sqrt{\frac{m+M}{k}}\cos\left(\sqrt{\frac{k}{m+M}}t + \frac{\pi}{2}\right) \tag{20}$$

We also could have chosen:

$$\phi = -\frac{\pi}{2} \tag{21}$$
$$A = -\frac{v_0}{\omega} \tag{22}$$
$$y(t) = -v_0\sqrt{\frac{m+M}{k}}\cos\left(\sqrt{\frac{k}{m+M}}t - \frac{\pi}{2}\right) \tag{23}$$

b. Determine the condition v_0 such that the block remains on the tray

In order for the block to remain on the tray, the acceleration of the tray should not be greater in magnitude than graviational acceleration, g. Identifying that in general $A = x_{max}$ and that $a_{max} = \omega^2 x$

$$a_{max} \leq -g \tag{24}$$
$$-\omega^2 x_{max} \leq -g \tag{25}$$
$$-\frac{k}{m+M}v_0\sqrt{\frac{m+M}{k}} \leq -g \tag{26}$$
$$v_0 \leq g\sqrt{\frac{m+M}{k}} \tag{27}$$

Problem 3. A block of mass m, moving on the end of a spring with force constant k, is acted on by a damping force $\vec{F} = -b\vec{v}$.

a. If the constant b has the value $\sqrt{mk}$, what is the frequency of oscillation of the block?

$$F = ma = -kx - bv \text{ and } x = Ae^{\frac{-bt}{2m}}\cos(\omega' t + \phi) \tag{28}$$
$$\dot{x} = v = Ae^{\frac{-bt}{2m}}\left(-\frac{b}{2m}\cos(\omega' t + \phi) - \omega'\sin(\omega' t + \phi)\right) \tag{29}$$
$$\ddot{x} = a = Ae^{\frac{-bt}{2m}}\left(\left(\left(\frac{b}{2m}\right)^2 - \omega'^2\right)\cos(\omega' t + \phi) + \frac{b\omega'}{m}\sin(\omega' t + \phi)\right) \tag{30}$$

Consider the moment $t = 0$ in the case that $\phi = 0$. Then

$$e^{\frac{-b\times 0}{2m}} = 1 \tag{31}$$
$$\cos(\omega' \times 0 + 0) = 1 \tag{32}$$
$$\sin(\omega' \times 0 + 0) = 0 \tag{33}$$
$$\tag{34}$$

Plugging x,v and a into F=ma and equating all of the cosine coefficients:

$$\frac{b^2}{4m} - \omega'^2 m = -k + \frac{b^2}{2m} \tag{35}$$
$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{36}$$
$$\omega' = \sqrt{\frac{3k}{4m}} = \frac{\sqrt{3}}{2}\omega_0 \tag{37}$$
$$f = \frac{1}{2\pi}\omega' \tag{38}$$

b. For what value of the constant b will the motion be critically damped?

Critically damping has $b = 2\sqrt{km}$ so that $\omega' = 0$.

c. Is energy conserved?

Energy is not conserved in the mass-spring system. The damping force dissipates energy from the system.

Problem 4. A block with mass M rests on a frictionless surface and is connected to a horizontal spring of force constant k. The other end of the spring is attached to a wall. A second block with mass m rests on top of the first block. The coefficient of static friction between the blocks is μ_s. Find the *maximum* amplitude of oscillation such that the top block will not slip on the bottom block.

See Professor's extra questions solutions.

Problem 5. A compound physical pendulum consists of a disk or radius R fixed at its center to one end of a rod of length L whose other end is attached to a pivot as shown in the figure.

This problem is from the website http://web.mit.edu/8.01t/www/materials/modules/chapter24.pdf on page 10.

a. Find the torque about the pivot due to gravity in terms of the angle θ the pendulum makes with the vertical.

Torque is $\vec{r} \times \vec{F}$ where $\vec{r}$ is a vector from the pivot to the location of the force. Gravity acts on the center of mass of the object. Consider the torque for each object independently and add them. Define $\hat{y}$ to be up and $\hat{z}$ to be out of the page. The angle of the pendulum with respect to the vertical direction is θ.

$$\vec{\tau} = \vec{r_R} \times M_R g(-\hat{y}) + \vec{r_D} \times M_D g(-\hat{y}) \tag{39}$$
$$\vec{\tau} = r_R \sin(\theta) M_R g(-\hat{z}) + r_D \sin(\theta) M_D g(-\hat{z}) \tag{40}$$
$$\vec{\tau} = -\left(\frac{L}{2}M_R + LM_D\right) g \sin(\theta)\hat{z} \tag{41}$$

b. Find the moment of inertia of the compound object about the pivot.

The moment of inertia of a compound object is the sum of the moments of inertia of the pieces. Integrate to find the moment of inertia of a rod where y is the distance from the axis of rotation.

$$I = \int y^2 dm \tag{42}$$

$$dm = \frac{M_R}{L} dy \tag{43}$$

$$I_R = \frac{M_R}{L} \int_0^L y^2 dy \tag{44}$$

$$I_R = \frac{M_R}{L} \left(\frac{L^3}{3} \right) \tag{45}$$

$$I_R = \frac{1}{3} M_R L^2 \tag{46}$$

The thin rod is a 1D object so the moment of inertia only had the integral over length. The thin disk is a 2D object so we should have two integrals, angle and radius. Recall that for a disk $dA = r d\phi dr$.

$$I_D = I_{CM} + M_D L^2 \tag{47}$$

$$dm = \frac{M_D}{A} dA = \frac{M_D}{\pi R^2} r d\phi dr \tag{48}$$

$$I_{CM} = \frac{M_D}{\pi R^2} \int_0^{2\pi} d\phi \int_0^R r^3 dr \tag{49}$$

$$I_{CM} = \frac{1}{2} M_D R^2 \tag{50}$$

$$I_D = \frac{1}{2} M_D R^2 + M_D L^2 \tag{51}$$

$$I_{total} = I_R + I_D = \frac{1}{3} M_R L^2 + \frac{1}{2} M_D R^2 + M_D L^2 \tag{52}$$

c. Find the period and angular frequency of small oscillations.

Write the rotational equation of motion for the physical pendulum.

$$\Sigma \vec{\tau} = I \frac{d^2\theta}{dt^2} \hat{z} \tag{53}$$

$$-\left(\frac{L}{2} M_R + L M_D \right) g \sin(\theta) \hat{z} = \left(\frac{1}{3} M_R L^2 + \frac{1}{2} M_D R^2 + M_D L^2 \right) \frac{d^2\theta}{dt^2} \hat{z} \tag{54}$$

For small angles, $\sin\theta \approx \theta$.

$$-\left(\frac{L}{2} M_R g + L M_D g \right) \theta = \left(\frac{1}{3} M_R L^2 + \frac{1}{2} M_D R^2 + M_D L^2 \right) \frac{d^2\theta}{dt^2} \tag{55}$$

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\frac{L}{2} M_R g + L M_D g}{\frac{1}{3} M_R L^2 + \frac{1}{2} M_D R^2 + M_D L^2} \right) \theta \tag{56}$$

Simple harmonic motion is always of the following form, where the box could be any coordinate.

$$\frac{d^2\Box}{dt^2} = -\omega^2 \Box \tag{57}$$

Therefore, the angular frequency of this physical pendulum is

$$\omega = \sqrt{\frac{\frac{L}{2} M_R g + L M_D g}{\frac{1}{3} M_R L^2 + \frac{1}{2} M_D R^2 + M_D L^2}} \tag{58}$$

and the period is $2\pi/\omega$.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\frac{1}{3}M_R L^2 + \frac{1}{2}M_D R^2 + M_D L^2}{\frac{L}{2}M_R g + L M_D g}} \tag{59}$$

The general formula for the period of a physical pendulum is

$$T = 2\pi\sqrt{\frac{I}{mgd_{cm}}} \tag{60}$$

where I is the total moment of inertia, m is the total mass, and d_{cm} is the distance from the pivot to the center of mass.

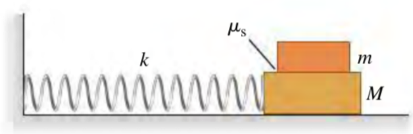

Figure 1: Problem 4

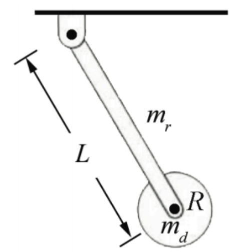

Figure 2: Problem 5