COMPUTATIONAL METHODS IN ASTRONOMY

MODULE D : GASDYNAMICS

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References :

- G. B. Whitham : Linear and Nonlinear Waves.
- D. J. Acheson : Elementary Fluid Dynamics
- Manual of the Pencil Code. http://pencil-code.nordita.org/

- Consider 1D : (x, t)
- Velocity $v(x,t) \equiv \hat{e}_x v_x(x,t) \equiv \hat{e}_x v(x,t)$
- Ideal fluid : $\dfrac{Ds}{Dt} = 0$ (1)

$\Rightarrow$ no heat exchange between different fluid elements

Recall, specific entropy $s \propto \log(p\rho^{-\gamma})$

$\Rightarrow$ $p\rho^{-\gamma}$ remains the same for individual fluid elements at all times.

SOUND WAVES : LINEAR.

Unperturbed State : $p_0,\ \rho_0,\ s_0,\ v_0 = 0$

Perturbed State : $p_1,\ \rho_1,\ s_1 = 0,\ v_1$

Total : $p_0 + p_1,\ \rho_0 + \rho_1,\ v_1,\ s_0$

Eq (1) and the text that follows, implies

$$(p_0 + p_1)(\rho_0 + \rho_1)^{-\gamma} = p_0\, \rho_0^{-\gamma}$$

$$\Rightarrow \quad 1 = \left(1 + \frac{p_1}{p_0}\right)\left(1 + \frac{\rho_1}{\rho_0}\right)^{-\gamma} \qquad (2)$$

(Linear theory) assumes that the perturbations are small : $p_1 \ll p_0,\ \rho_1 \ll \rho_0$ and (check) $v_1 < c_0$

Keep terms that are linear in perturbed quantities, thus ignoring : $p_1 \rho_1$ type terms ~~too~~.

$$(2) \Rightarrow \quad \frac{p_1}{p_0} = \frac{\gamma \rho_1}{\rho_0}$$

Define $$\boxed{c_0 \overset{\text{def}}{=} \sqrt{\frac{\gamma p_0}{\rho_0}}} \qquad (3)$$

$$\Rightarrow \quad \boxed{p_1 = c_0^2\, \rho_1} \qquad (4)$$

Recall Euler Equation :

$$\rho \left[\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right] = - \nabla p \qquad (5)$$

Unperturbed : $\frac{\partial p_0}{\partial x} = 0 \Rightarrow p_0 = \text{const} \qquad (6)$

Total (linearized) : $(\rho_0 + \cancel{\rho_1}) \frac{\partial v_1}{\partial t} = - \frac{\partial p_1}{\partial x}$

$$\Rightarrow \boxed{\rho_0 \frac{\partial v_1}{\partial t} = - \frac{\partial p_1}{\partial x}} \qquad (7)$$

Continuity Equation : $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0 \qquad (8)$

Unperturbed : $\frac{\partial \rho_0}{\partial t} = 0 \Rightarrow \rho_0 = \text{const} \qquad (9)$

Total (linearized) : $\boxed{\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_1}{\partial x} = 0} \qquad (10)$

- Eliminate perturbed quantities using eqns (7) & (10), use (4) to get, e.g.,

$$\boxed{\frac{\partial^2 p_1}{\partial t^2} = c_0^2 \frac{\partial^2 p_1}{\partial x^2}} \qquad (11)$$

In fact, all the perturbed quantities obey the same WAVE EQUATION :

$$\boxed{\frac{\partial^2}{\partial t^2}(p_1, \rho_1, v_1) = c_0^2 \frac{\partial^2}{\partial x^2}(p_1, \rho_1, v_1)} \qquad (12)$$

General Solution to the eqn (11) or (12).

$$v_1 = \underbrace{f(x - c_0 t)}_{\text{wave propagates to the right}} + \underbrace{g(x + c_0 t)}_{\text{... to the left}} \quad (13)$$

$O \longrightarrow x$

Note :- c_0 is the propagation (phase) speed of the waves.

- No change of shape. / NON DISPERSIVE
- Check that the dispersion relation $\omega = \pm c_0 |k|$

- A useful form of the wave eq. (12).

$$\underbrace{\left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x}\right)}_{\text{Rightwards}} \underbrace{\left(\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x}\right)}_{\text{Leftwards}} v_1 = 0 \quad (14)$$

- Retain only of the factors,

$$\frac{\partial v_1}{\partial t} + c_0 \frac{\partial v_1}{\partial x} = 0 \quad (15)$$

with general solution $v_1 = f(x - c_0 t)$

$\longrightarrow$ a simplest wave problem

Linear

$$\frac{\partial \psi}{\partial t} + c_0 \frac{\partial \psi}{\partial x} = 0 \qquad (16)$$

Saw earlier (15) that it admits wave-like solutions :

$$\psi \sim \psi(x - c_0 t) \qquad (17)$$

Nonlinear

$$\frac{\partial \psi}{\partial t} + c(\psi) \frac{\partial \psi}{\partial x} = 0 \qquad (18)$$

- Deceptive simple appearance.
- $c(\psi) = \psi$ (19) is one of the simplest case of nonlinearity.

- With ψ being the (1D) velocity v,

(18), (19) $\Rightarrow$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0 \qquad (20)$$

← Eq (5) without the pressure term.

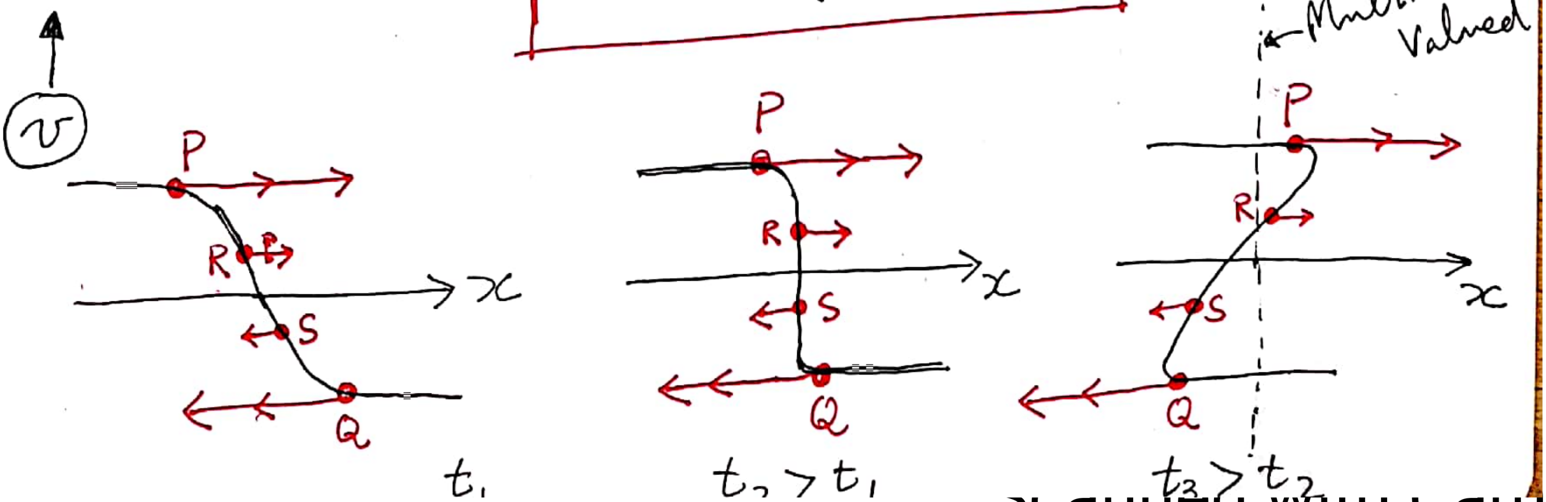
← Inviscid Burgers' Eqn

or $\dfrac{dv}{dt} = 0$ & $\dfrac{dx}{dt} = v$ (21) ⇒ Characteristic curves.

[Here $\dfrac{d}{dt} \equiv \dfrac{D}{Dt}$]

Solution :

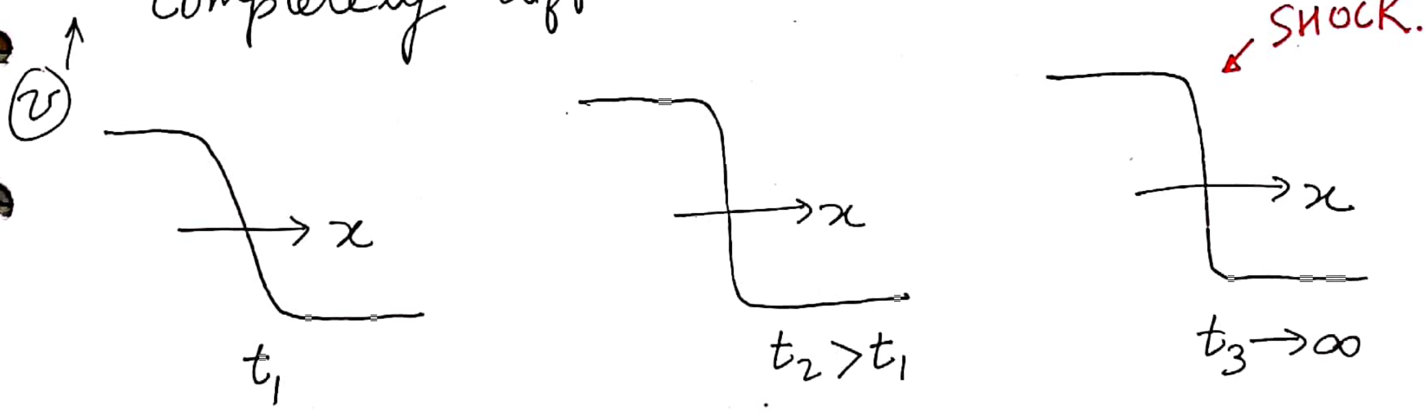
$$\boxed{v = f(x - vt)} \qquad (22)$$



BURGERS' EQUATIONS

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2} \qquad (23)$$

- ν : viscosity (Compare with 20)
- 1D Navier-Stokes Equation without the pressure term
- Non-zero viscosity (ν), however small, inhibits the solutions from becoming multi-valued
- Constitutes a good example where $\nu = 0$ [Eq 20] and $\nu \to 0$ offer completely different solutions.



COLE-HOPF TRANSFORMATION

$$v = -2\nu \frac{1}{\varphi}\frac{\partial \varphi}{\partial x} \qquad (24)$$

Exercise : Solve (23) using (24) and show that the shock thickness $\Delta \propto \nu$