Computational Methods in Astronomy

2nd Apr 2020

Module D : Gasdynamics (DO)

NS, IUCAA, Pune..

References :

- Feynman Lectures, vol II, chs 40, 41
- D. J. Acheson : Elementary Fluid Dynamics
- A. R. Choudhuri : Physics of fluids & Plasmas.
- * Landau & Lifshitz : Fluid Mechanics (Vol. 6).
- * Mihalas & Mihalas : Foundations of Radiation Hydrodyn.

(* : Advanced level texts).

Spacetime Coordinates : $(x, t) \equiv (x_1, x_2, x_3, t)$

Fixed Orthonormal basis : (e_1, e_2, e_3).

Velocity Field : $v(x, t)$

Ideal Gas (Eq. of State) : $p = \frac{k}{\mu m_p} \rho T$

or $p = (c_p - c_v) \rho T$

(Symbols have usual meanings, p is pressure ...)

What is fluid? ... or a hydrodynamic limit?

$$\lambda_{dB} \ll n^{-1/3} \ll \Delta x \ll L \qquad (1)$$

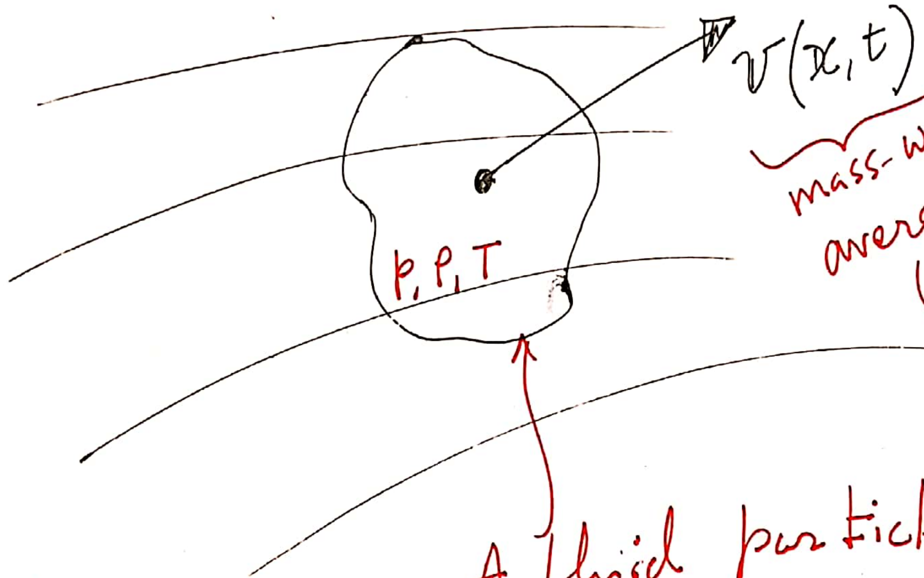
λ_{dB}: de Broglie wavelength; $n^{-1/3}$: inter-particle separation; Δx: Unit cell; L: System scale.

$\underbrace{\lambda_{dB} \ll n^{-1/3}}_{\text{sufficiently dilute}}$, $\underbrace{n^{-1/3} \ll \Delta x \ll L}_{\text{sufficiently Dense.}}$

Such that

$$\lambda_{mfp} \ll \Delta x \ll L \qquad (2)$$

λ_{mfp}: mean free path : Collisions are important.



$v(x,t)$: mass-weighted average vel. (smooth)

A fluid particle. is assumed to be in local thermodynamic equilibrium.

... a continuum limit!

Problem: Convince yourselves that the ISM in a galaxy can be treated as fluid, whereas stars can not be..

- Let us first consider an <u>ideal gas</u> and express fluid equations as "Conservation laws".

$$\frac{\partial}{\partial t}\begin{pmatrix}\text{Density of a}\\ \text{quantity}\end{pmatrix} + \nabla\cdot\begin{pmatrix}\text{Current/flux}\\ \text{Density of the}\\ \text{quantity}\end{pmatrix} = 0 \qquad (3)$$

- Comes from, an integral form. (Take ρ, for example)

$$\frac{\partial}{\partial t}\int \rho \, dV = -\oint \rho v \, d\underbrace{A}_{\text{Area Element}} \quad \cdots$$

Total / Lagrangian / Convective Derivative

- Let a quantity $C(x,t)$, a field variable. A & B two points, s.t., A

$$A : (x, t) \quad ; \quad B : (x + \delta x, t + \delta t)$$

$$C_A \equiv C(x,t) \quad ; \quad C_B \equiv C(x+\delta x, t+\delta t).$$

$$C_B = C(x+\delta x, t+\delta t)$$
$$= \underbrace{C(x,t)}_{\equiv C_A} + v_1 \delta t \frac{\partial C}{\partial x_1} + v_2 \delta t \frac{\partial C}{\partial x_2} + v_3 \delta t \frac{\partial C}{\partial x_3} + \delta t \frac{\partial C}{\partial t}$$

$$\Rightarrow \lim_{\delta t \to 0}\left(\frac{C_B - C_A}{\delta t}\right) = \left(v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3}\right) C + \frac{\partial C}{\partial t}$$

$$\Rightarrow \quad \frac{DC}{Dt} = \Big(\frac{\partial}{\partial t} + \underbrace{v\cdot\nabla}_{\text{Advection.}}\Big) C$$

with

$$\frac{DC}{Dt} \equiv \lim_{\delta t \to 0} \frac{C(x+\delta x, t+\delta t) - C(x,t)}{\delta t}$$

$$\Rightarrow \quad \boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + v\cdot\nabla} \qquad (4)$$

Ideal Fluids : Non-dissipative

- Viscosity and Thermal conductivity are not important, hence ignored

- ~~Viscos~~ Entropy must be conserved.

$$\boxed{\frac{Ds}{Dt} = 0} \qquad (5)$$

s is entropy per unit mass

Usual Thermodynamics :

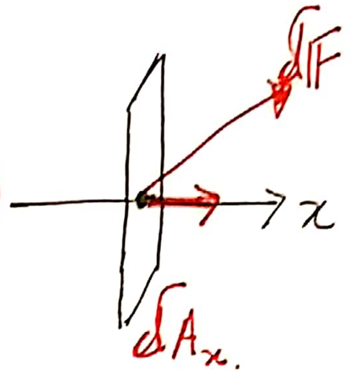
$$T\,ds = du + p\,d\Big(\frac{1}{\rho}\Big) = dh - \frac{1}{\rho}dp$$

Specific enthalpy $h = u + \frac{p}{\rho} \equiv h(T)$

$$u = C_v T \quad ; \quad h = C_p T.$$

Internal Stresses.

$$S_{xx} = \frac{\delta F_x}{\delta A_x} \quad \text{(normal stress)}$$



$$\left.\begin{aligned} S_{yx} &= \frac{\delta F_y}{\delta A_x} \\ S_{zx} &= \frac{\delta F_z}{\partial A_x} \end{aligned}\right\} \text{Tangential stresses.}$$

$\Downarrow$

Cause friction

$\Downarrow$

Viscosity

$\Rightarrow$ Hence ignored for ideal fluids.

- S_{ij} is tensor field

  i^{th} comp. of the force, exerted per unit area, across an area element oriented with its normal in ~~j^{th}~~ j^{th} direct

- For Ideal/Inviscid fluid.

$$S_{ij} = p\,\delta_{ij} \quad \text{(isotropic)}. \qquad (6)$$

- Force per unit volume

$$f_i = -\frac{\partial S_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} \qquad (7)$$

- Mass Conservation :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad (8)$$

$$\Rightarrow \quad \frac{D\rho}{Dt} = - \rho (\nabla \cdot \mathbf{v}) \qquad (9)$$

- Incompressibility $\Rightarrow \nabla \cdot \mathbf{v} = 0 \qquad (10)$

- Momentum Conservation (Euler Eq.).

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = - \nabla p + f_{body} .$$

- f_{body} is a body force, e.g., gravity, ~~pea~~ per unit volume.

- For gravity with φ as its potential
  $\varphi \equiv \varphi(x, t)$ in gnl.

$$f_{body} = - \rho \nabla \varphi .$$

$$\Rightarrow \boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = - \frac{1}{\rho} \nabla p - \nabla \varphi} \qquad (11)$$

In Conservation form:

$$\frac{\partial}{\partial t}\left(\rho v_i\right) + \underbrace{\frac{\partial}{\partial x_j} T_{ij}}_{\text{Divergence of Stress tensor}} = 0 \qquad (12)$$

$$T_{ij} = \rho v_i v_j + p\delta_{ij}$$

- Conservation of Energy:

$$\underbrace{\frac{1}{2}\rho v^2}_{\text{Kinetic Energy}} + \underbrace{\rho u}_{\text{Internal Energy.}}$$

Show that

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho u\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho h\right)v\right] = 0 \qquad (13)$$