Propositional Logic

Fall 2014

Learning Outcomes . . .

In this session we will

- Define the elements of propositional logic: statements and operations, including implication, and its converse, inverse, and negation.
- Translate English expressions into logical statements.
- Use both truth tables and derivations to demonstrate equivalence of logical statements.
- Define common tautologies, contradictions, and equivalences.
- Recognize and employ *modus ponens* and *modus tollens* and other forms of valid argumentation.

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- Why is it important?
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- Logic is the skeleton that supports mathematical truth-making, i.e., proof-reading and proof-writing.
- Logic is the glue that holds computers and programs together.

Propositional statements

Definition

A *statement* is a sentence that is either true or false.

Example

Atomic statements:

- It is raining.
- The sun is shining.

Compound statements:

- It is raining *and* the sun is shining.
- It is raining *or* the sun is shining.

Making new statements by connecting propositions

Three fundamental operators:

Operator	**Description**
And (conjunction)	Written $s \wedge t$: true when s *and* t are both true.
Or (disjunction)	Written $s \vee t$: true when either s *or* t (or both) are true.
Not (negation)	Written $\neg s$ (or sometimes $\sim s$): true when s is false.

Keep in mind that s and t may themselves be atomic or compound statements!

Truth tables . . .

The meaning of a logical operation can be expressed as its “truth table.”

- Construct the truth-table for conjunction.
- Construct the truth-table for disjunction.
- Construct the truth-table for negation.

$\boxed{\text{Do in class.}}$

A worked example

Example

Let s be "The sun is shining" and t be "It is raining." Join these into the compound statement:

$$(\neg s \wedge t) \vee \neg t.$$

- Phrase the compound statement in English.
- Construct the truth table.

Do in class.

Truth, falsity, and interpretations

- The truth or falsity of any statement depends upon its *context*.
- "Context" can be visualized as the values that are associated with each variable in a statement.

Example

Is $a \vee b$ true? Well, if *either* $a = true$ or $b = true$, then the statement $a \vee b$ is true.
Ask now if $a \wedge b$ is true, and you will see that it is true, but under fewer interpretations—or, its context is different.

Exclusive or

The word "or" is often used to mean "one or the other," but this is *not* the same meaning of "or" in logic!

Definition

The *exclusive-or* of two statements p and q (written $p \oplus q$), is true when either p is true or q is true, but not both.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Logical equivalences

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Theorem

Let p and q be statement varaiables. Then

$$
\begin{aligned}
& (p \vee q) \wedge \neg(p \wedge q) \;\equiv\; p \oplus q \\
\textit{and} \quad & (p \wedge \neg q) \vee (q \wedge \neg p) \;\equiv\; p \oplus q \,.
\end{aligned}
$$

Prove in class (using Truth Tables).

Laws of Propositional Logic ...

We can do algebra in propositional logic.

Commutative Laws:

$$p \wedge q \equiv q \wedge p$$
$$p \vee q \equiv q \vee p$$

Associative Laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$
$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

Distributive Laws:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

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We can do algebra in propositional logic.

Commutative Laws:

$$p \land q \equiv q \land p$$
$$p \lor q \equiv q \lor p$$

Associative Laws:

$$(p \land q) \land r \equiv p \land (q \land r)$$
$$(p \lor q) \lor r \equiv p \lor (q \lor r)$$

Distributive Laws:

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$
$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

How do we know that these laws are *valid*?

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Prove a Distributive Law in class.

De Morgan's laws . . .

Theorem (De Morgan's laws)

Let p and q be statement varaiables. Then

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\textit{and} \quad \neg(p \wedge q) \equiv \neg p \vee \neg q \ .$$

Examples in English

Example

It is not the case that Alice or Bob went to the store.
$\equiv$ Alice did not go to the store and Bob did not go to the store.

It is not the case that Alice and Bob went to the store.
$\equiv$ Alice did not go to the store or Bob did not go to the store.

Laws of Logic

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold:

1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double Negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. DeMorgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Universal bounds laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

Example of Boolean Algebra

Example from Epp.

$$\neg(\neg p \wedge q) \wedge (p \vee q) \equiv p$$

Prove in class (using Boolean algebra).

Introducing logical implication

Example

If it rains, then Richard brings an umbrella.

We would like a way of assigning truth or falsity to these kinds of statements.

a	**b**	$\mathbf{a} \rightarrow \mathbf{b}$
T	T	T
F	T	T
T	F	F
F	F	T

Note two important qualities of implication:

- There is only one case where the implication is false. WHY IS THIS IMPORTANT?
- Implication is sensitive to direction!. (Compare rows 2 and 3.) WHY IS THIS IMPORTANT?

Implications of implication . . .

- Everything to the left of the implication symbol is called either the antecedent, the hypothesis, or the **sufficient** condition.
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- Everything to the right of the implication symbol is called the consequent, the conclusion, or the **necessary** condition.
- Unlike in common speech, no relationship need exist between the hypothesis and the conclusion.
  - If the sky is blue then 5 is prime.
  - If the sky is blue then 5 is not prime.
  - If the moon is made of cheese, then then 5 is prime.
  - If the moon is made of cheese, then 5 is not prime.

Trying on a few implications . . .

See how well you do with these: Identify the sufficient and the necessary conditions. State under which circumstances these implications are true and under which they are false:

1. If oxygen is present, then rust will form.
2. Only if I study hard will I pass this course. (Read this as p, only if q.)
3. If $f(x) = f(y)$ then $x = y$.
4. If $x \neq y$ then $f(x) \neq f(y)$.

Some important variations of implications

We will define four variations of implication:

- Contrapositive
- Converse
- Inverse
- Negation

The Contrapositive of an Implication

Definition

The *contrapositive* of an implication is obtained by transposing its conclusion with its premise and inverting. So,

Contrapositive of $p \to q$ is $\neg q \to \neg p$.

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Contrapositive: *If I don't live in Maryland, then I don't live in College Park.*

Theorem

The contrapositive of an implication is equivalent to the original statement.

Prove in class.

The converse of an implication

Definition

The *converse* of an implication is obtained by transposing its conclusion with its premise.

Converse of $p \rightarrow q$ is $q \rightarrow p$.

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Converse: *If I live in Maryland, then I live in College Park.*

The inverse of an implication

Definition

The *inverse* of an implication is obtained by negating both its premise and its conclusion.

$$\text{Inverse of } p \to q \text{ is } (\neg p) \to (\neg q).$$

(Parentheses added for emphasis.)

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Inverse: *If I don't live in College Park, then I don't live in Maryland.*

The inverse of an implication is equivalent to the converse!

Why?

Negating an implication

Definition

The *negation* of an implication is obtained by negating it.

Negation of $p \rightarrow q$ is $\neg(p \rightarrow q)$ (which is equivalent to $p \wedge \neg q$).

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Negation: *I live in College Park, and I don't live in Maryland.*

The negation of an implication is not an implication!

Bidirectional implication

Definition

The *biconditional* statement is an implication that is true only when its antecedent and its consequent have the *same* truth values; it is false otherwise. In symbols, $p \leftrightarrow q$ is true *only* when $p \rightarrow q$ **and** $q \rightarrow p$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Experimenting with bi-conditionals . . .

- The bi-conditional, arguably, corresponds to the common use of the word "if" in English sentences.
- The bi-conditional sometimes appears in scientific or technical text as **iff** or as a **necessary and sufficient condition**.
- What do the converse, inverse, and negations of a bi-conditional look like?
- What is the relationship between the exclusive-or (discussed above) and the bi-conditional?

Validity, as a matter of "form."

Definition

An *argument* is a sequence of statements terminating with a conclusion.

- Validity is based upon *formal* properties, not *content*.
- Mastery of logical argumentation translates into a deeper understanding of, and facility for, constructing mathematical proof.

Proofs: Using inference . . .

Here are some common patterns from your text:

- *Modus ponens* (To prove by asserting.) $\dfrac{p \to q,\ p}{q}$.
- *Modus tollens* (To prove by denying.) $\dfrac{p \to q,\ \neg q}{\neg p}$.
- *Disjunctive syllogism* $\dfrac{p \vee q,\ \neg q}{p}$; likewise $\dfrac{p \vee q,\ \neg p}{q}$.
- *Rule of contradiction* Let c be a contradiction: $\dfrac{\neg p \to c}{p}$.

The remaining rules can be derived by systematically applying the rules above with the basic properties of a Boolean algebra, i.e., associativity, commutativity, idempotence, etc.

Valid forms of argumentation

Modus Ponens $p \to q$ p $\therefore q$	Modus Tollens $p \to q$ $\neg q$ $\therefore \neg p$		Disjunctive Syllogism	$p \vee q$ $\neg q$ $\therefore p$	$p \vee q$ $\neg p$ $\therefore q$
Conjunctive Addition	p q $\therefore p \wedge q$		Hypothetical Syllogism	$p \to q$ $q \to r$ $\therefore p \to r$	
Disjunctive Addition	p $\therefore p \vee q$	q $\therefore p \vee q$	Dilemma: Proof by Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$	
Conjunctive Simplification	$p \wedge q$ $\therefore p$	$p \wedge q$ $\therefore q$	Rule of Contradiction	$\neg p \to c$ $\therefore p$	
Closing C.W. without contradiction	$\vert p$ Assumed $\vert q$ derived $\therefore p \to q$		Closing C.W. with contradiction	$\vert p$ Assumed $\vert x \wedge \neg x$ derived $\therefore \neg p$	

Examples of Logical Arguments

$$\begin{array}{ll} & p \vee q \\ & q \rightarrow r \\ & \neg p \\ \therefore & r \end{array} \qquad \begin{array}{ll} & p \wedge q \\ & p \rightarrow s \\ & \neg r \rightarrow \neg q \\ \therefore & s \wedge r \end{array} \qquad \begin{array}{ll} & p \vee q \\ & \neg(q \vee r) \\ & p \rightarrow (m \rightarrow r) \\ \therefore & \neg m \end{array}$$

Prove in class.

Reviewing & moving forward

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- "Arguments" are patterns of reasoning that incorporate these operators over a particular *domain of discourse*, e.g., numbers, objects, etc.
- Most importantly: truth is *formal* in these sytems.

Applications of logic

Logic is everywhere. Shortly, we will examine how logic is used in several areas relevant to computer science.

- Logical primitives & switching circuits.
- Logical statements & sequential circuits. Later in this semester, perhaps
- Logical constructions in combinatorial circuits.