Lecture Note

Inspiring Excellence

Department of Mathematics and Natural Sciences

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Chapter 1

One Dimensional Kinematics

Kinematics is the mathematical description of motion. The term is derived from the Greek word "kinema", meaning movement. In order to quantify motion, a mathematical coordinate system, called a reference frame, is used to describe space and time.Once a reference frame has been chosen, we shall introduce the physical concepts of position, velocity and acceleration in a mathematically precise manner.

1.1 Position, Displacement, time interval

Consider a car moving in one dimension. After choosing the origin, we denote the position coordinate of the car by $x(t)$ with respect to the origin. Notice it is a function of time, as the position is varying with time. the position can be positive, negative or zero in value. It has both magnitude and direction. Hence it's a vector quantity. we shall denote as the **position vector** (or simply position) and write as,

$$\vec{r}(t) = x(t)\hat{i} \tag{1.1}$$

Here, the motion is along X-axis. We also denote the position coordinate at $(t = 0)$ by the symbol $x_0 = x(t = 0)$. the SI unit for position is meter $[m]$.

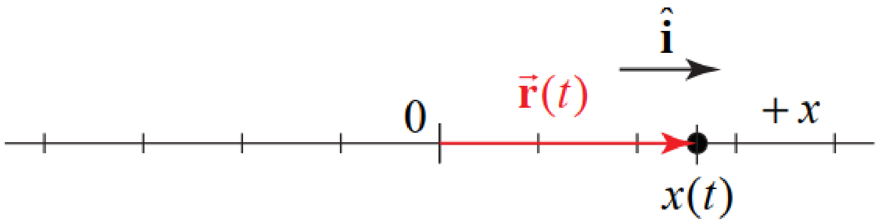

Figure 1.1: The position vector, with reference to a chosen origin.

1.1.1 Time interval

For a closed interval of time $[t_1, t_2]$, we characterize the time interval by the difference of the two end points of the interval,

$$\Delta t = t_2 - t_1 \tag{1.2}$$

The SI unit for time interval is second $[s]$.

1.2 Displacements

The displacement of a body during the time interval $[t_1, t_2]$ is defined as the change of the position of the body.

$$\begin{aligned} \Delta\vec{r} =& \vec{r}(t + \Delta t) - \vec{r}(t) = \vec{r}(t_2) - \vec{r}(t_1) \\ &= x(t_2)\hat{i} - x(t_1)\hat{i} = \Delta x(t)\hat{i} \end{aligned}$$

It's a vector quantity.

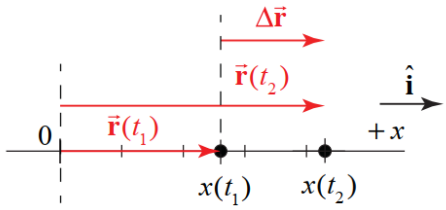

Figure 1.2: The displacement vector of an object over a time interval is the vector difference between the two position vectors.

1.3 Velocity

When describing the motion of objects, words like "speed" and "velocity" are used in natural language; however when introducing a mathematical description of motion, we need to define these terms precisely.

1.3.1 Average Velocity

The average velocity for any time interval Δt is defined as the displacement Δx divided by the time interval Δt,

$$v_{x,avg} = \frac{\Delta x}{\Delta t} \tag{1.3}$$

Because we are describing one-dimensional motion we shall drop the subscript x and denote,

$$v_{avg} = v_{x,avg}$$

The average velocity vector is then,

$$\vec{v_{avg}} = \frac{\Delta x}{\Delta t}\hat{i} \tag{1.4}$$

The SI units for average velocity are meters per second $[m.s^{-1}]$.

Notice that, The average velocity is not necessarily equal to the distance in the time interval Δt traveled divided by the time interval Δt. For example, during a time interval, an object moves in the positive x - direction and then returns to its starting position, the displacement of the object is zero, but the distance traveled is non-zero.

1.4 Instantaneous Velocity

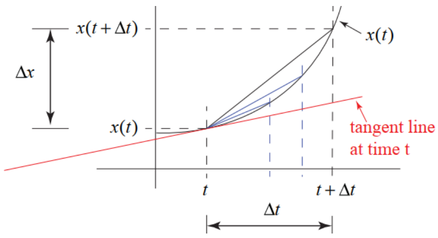

Figure 1.3: Plot of position vs. time showing the tangent line at time t .

Now, lets consider a body is moving in one dimension. At time interval $[t, t{+}\Delta t]$ we can define the average velocity as the slope of the line joining two points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$. As you know, the slope of the line is,

$$v_{avg} = \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} \tag{1.5}$$

Now, if we take the limit $\Delta t \to 0$ then, the slope of the lines connecting the points $(t, x(t))$ and $(t + \Delta t, x(t + \Delta t))$, approach slope of the tangent line to the graph of the function x(t) at the time t in fig (1.3).

Therefore, The instantaneous velocity at time t is given by the slope of the tangent line to the graph of the position function at time t :

$$v(t) = \lim_{\Delta t \to 0} v_{avg} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{(t + \Delta t) - t} = \frac{dx}{dt} \tag{1.6}$$

The instantaneous velocity vector is then,

$$v(\vec{t}) = v(t)\hat{i}$$

The component of the $v(\vec{t})$ can be positive, zero, negative depending whether the body is moving along positive x-axis, standing still or travelling along negative x-axis.

1.5 Acceleration

Let us define Acceleration as the rate of change of velocity with respect to time.

1.5.1 Average Acceleration

Like average velocity, average acceleration is defined as the quantity that measures a change in velocity over a particular time interval. During a time interval Δt a body undergoes a change in velocity,

$$\Delta\vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t)$$

The change in the x -component of the velocity, Δv , for the time interval $[t, t+\Delta t]$ is

$$\Delta v = v(t + \Delta t) - v(t)$$

then the x -component of the average acceleration for the time interval Δt is defined to be,

$$\vec{a}(t) = a_{avg}\hat{i} = \frac{\Delta v}{\Delta t}\hat{i} = \frac{v(t + \Delta t) - v(t)}{\Delta t}\hat{i}$$

The SI units for average acceleration are meters per second squared, $[m.s^{-2}]$.

1.5.2 Instantaneous Acceleration

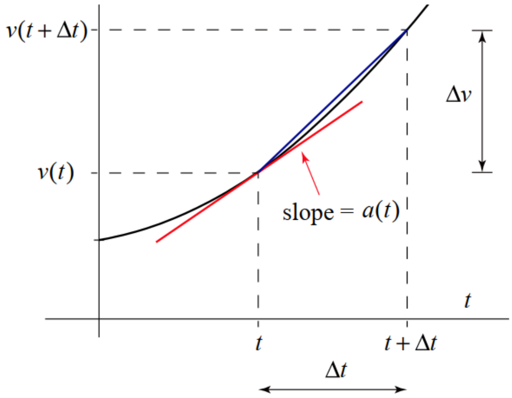

Figure 1.4: Plot of velocity vs. time showing the tangent line at time t .

Lets consider the diagram (Fig-1.4), here as you can see, we can define the average acceleration as a slope of the line joining two points $(t, v(t))$ and $(t + \Delta t, v(t + \Delta t))$. Now, by the same limiting argument like instantaneous velocity, we can define the instantaneous acceleration in terms of the slope of the tangent line.

The x -component of the instantaneous acceleration at time t is the slope of the tangent line at time t of the graph of the x -component of the velocity as a function of time,

$$a(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \frac{dv}{dt} \tag{1.7}$$

The instantaneous acceleration vector at time t is then,

$$\vec{a}(t) = a(t)\,\hat{i} \tag{1.8}$$

Notice that, the velocity is the derivative of position with respect to time, the x

-component of the acceleration is the second derivative of the position function,

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \tag{1.9}$$

1.6 Example 1

Let's consider an example, Say for a moving body, the position is given as $x = x_0 + \frac{1}{2}bt^2$. From the definition of instantaneous velocity we can find it as,

$$\begin{aligned} v(t) &= \frac{dx}{dt} \\ v(t) &= \frac{d}{dt}(x_0 + \frac{1}{2}bt^2) \\ v(t) &= \frac{1}{2}2bt \\ v(t) &= bt \end{aligned}$$

Then the instantaneous acceleration is the first derivative (with respect to time) of the x - component of the velocity:

$$a = \frac{dv}{dt} = \frac{d}{dt}(bt) = b$$

So, the instantaneous acceleration is constant always (Fig-1.5). **Notice:** When you have a position function for a moving body, which is quadratic in time, the acceleration for that moving object must be constant. As you know acceleration is the second derivative of position x with respect to time t. Another thing, notice that $\frac{\Delta v}{\Delta t}$ is also independent of time. From fig(1.5) we see, when the velocity is linear function of time, the average acceleration $\frac{\Delta v}{\Delta t}$ is also constant and exactly equal to the instantaneous acceleration.
Let's consider, a body undergoing constant acceleration for a time interval $[0, t]$, where $\Delta t = t$. Denote the x -component of the velocity at time $t = 0$ by $v_0 v(t = 0)$

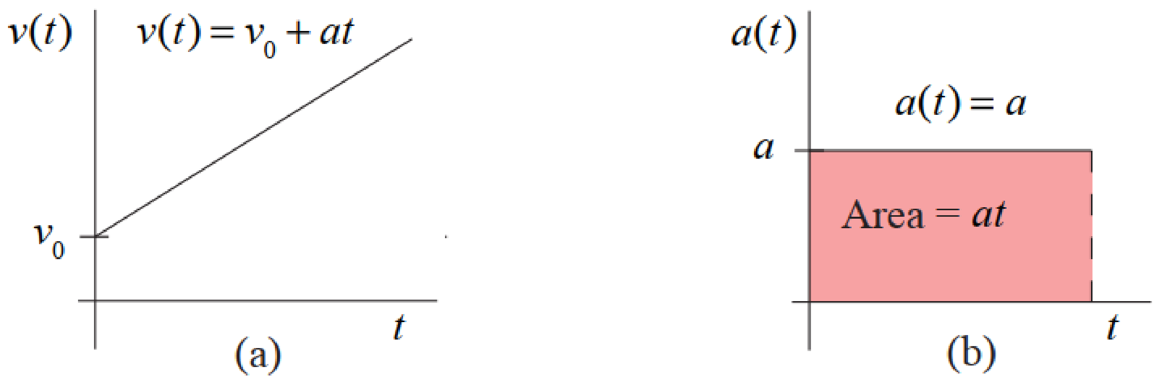

Figure 1.5: Constant acceleration.(a) Velocity, (b) Acceleration

. Therefore the x -component of the acceleration is given by,

$$a(t) = \frac{v(t) - v_0}{t}$$

Thus we get, velocity is linear function of time.

$$v(t) = v_0 + at \tag{1.10}$$

1.7 One Dimensional Kinematics and Integration

When the acceleration a(t) of an object is a non-constant function of time, we would like to determine the time dependence of the position function x(t) and the x -component of the velocity v(t). Because the acceleration is non-constant we no longer can use $v = v_0 + at$, $x = x_0 + v_0 t + \frac{1}{2}at^2$. Instead we shall use integration techniques to determine these functions.

1.7.1 For non-constant acceleration: Change of Velocity as the Indefinite Integral of Acceleration

Let's consider the time interval $t_1 < t < t_2$. From the definition of constant acceleration,

$$a(t) = \frac{dv(t)}{dt} \tag{1.11}$$

Recall the fundamental theorem of calculus, If you have a differentiable function $G(x)$ such that,

$$\frac{d}{dx}(G(x) + constant) = f(x)$$

As the derivative of a constant is zero. then,

$$G(x) + constant = \int f(x)dx \tag{1.12}$$

Integration $\int \cdots dx$ is defined as the inverse operation of differentiation or the 'anti-derivative'. Now, from the equation (1.11) we can write,

$$v(t) + C = \int a(t)dt \tag{1.13}$$

Here, the function $v(t)$ is called the indefinite integral of $a(t)$ with respect to t , and is unique up to an additive constant C. Equivalently we can write the differential $dv(t) = a(t)dt$, called the integrand, and then Eq. (1.1.3) can be written as,

$$v(t) + C = \int dv(t) \tag{1.14}$$

which we interpret by saying that the integral of the differential of function is equal to the function plus a constant. This is called indefinite integral. Now for a given time interval $[t_i, t_f]$, we can determine the change in velocity as definite integral of $a(t)$. By setting $t_f = t$ we can write as,

$$v(t) - v(t_i) = \int_{t'=t_i}^{t'=t} a(t')dt' \tag{1.15}$$

Here, t' is a dummy variable.As the upper limit of the integral, $t_f = t$, is now treated as a variable, we shall use the symbol t' as the integration variable instead of t .

1.7.2 Displacement as the Definite Integral of Velocity

We can repeat the same argument for the definite integral of the x -component of the velocity v(t) vs. time t . Because x(t) is an integral of v(t) the definite integral of v(t) for the time interval $[t_i, t_f]$ is the displacement,

$$x(t_f) - x(t_i) = \int_{t'=t_i}^{t'=t_f} v(t')dt' \tag{1.16}$$

If we set $t_f = t$, then the definite integral gives us the position as a function of time,

$$x(t) = x(t_i) + \int_{t'=t_i}^{t'=t_f} v(t')dt' \tag{1.17}$$

Summarizing the results of these last two sections, for a given acceleration a(t) , we can use integration techniques, to determine the change in velocity and change in position for an interval $[ti, t]$, and given initial conditions (x_i, v_i) , we can determine the position x(t) and the x -component of the velocity v(t) as functions of time.

1.8 Example 2

1.8.1 Non-constant Acceleration

Let's consider a case in which the acceleration, a(t) , is not constant in time and given as,

$$a(t) = b_0 + b_1 t + b_2 t^2$$

The graph of the x -component of the acceleration vs. time is shown in Figure 1.6.

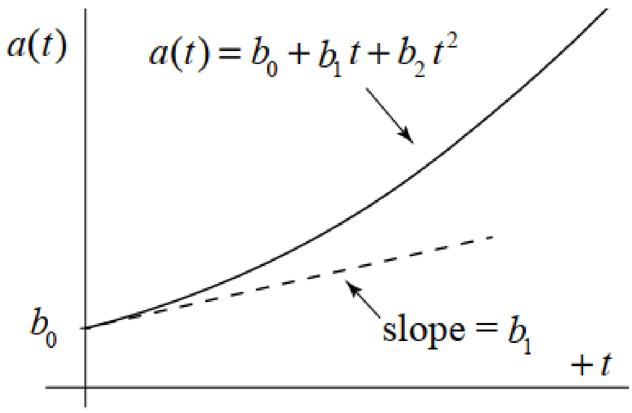

Figure 1.6: Non-constant acceleration vs. time graph

Denote the initial velocity at $t = 0$ by v_0 . Then, the change in the x -component of the velocity as a function of time can be found by integration:

$$v(t) - v_0 = \int_{t'=0}^{t'=t} a(t')dt' = \int_{t'=0}^{t'=t} (b_0 + b_1 t' + b_2 t'^2)dt' = b_0 t + \frac{1}{2}b_1 t^2 + \frac{1}{3}b_2 t^3 \tag{1.18}$$

The x -component of the velocity as a function in time is then,

$$v(t) = v_0 + b_0 t + \frac{1}{2}b_1 t^2 + \frac{1}{3}b_2 t^3 \tag{1.19}$$

1.9 Exercise

For the given acceleration $a(t) = b_0 + b_1 t + b_2 t^2$, find the displacement as a function of time t.