Lecture Note

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Chapter 1

Vector Product

1.1 Vector Product (Cross product)

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ are two vectors. We denote the angle θ to be the angle between the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$. The magnitude of the cross product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is defined as to be product of the magnitude of the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with the sine of the angle θ between the two vectors,

$$|\vec{\mathbf{A}} \times \vec{\mathbf{B}}| = |\vec{\mathbf{A}}||\vec{\mathbf{B}}| \sin(\theta) \tag{1.1}$$

The angle θ between the vectors is limited to the values $0 \leq \theta \leq \pi$ ensuring that $\sin(\theta) \geq 0$. Unlike scalar product, vector product (cross product) is a vector quantity.

The direction of the vector product is defined as follows. The vector $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ form

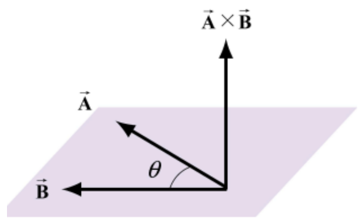

Figure 1.1: Vector product geometry

a plane. Take the perpendicular direction to this plane. Usually We choose the direction of the vector product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ using a the convention called “right-hand rule”, (as shown in Figure 3.1)

1.2 Right-hand Rule for the Direction of Vector Product

For the direction of $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$, redraw the vectors so that the tails are touching. Then draw an arc starting from the vector $\vec{\mathbf{A}}$ and finishing on the vector $\vec{\mathbf{A}}$. Curl your right finger same way as the arc. Your right thumb points in the direction of the vector product (as shown in figure 3.2) You should remember that the direction

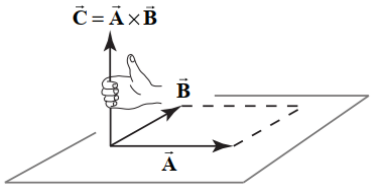

Figure 1.2: Right hand rule

of the vector product $\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

1.3 Properties of the Vector Product

- Vector product is anti-commutative. By changing the order of the vector actually change the direction of the vector product according to the "right hand rule".

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = -\vec{\mathbf{B}} \times \vec{\mathbf{A}} \tag{1.2}$$

- Vector product between a vector $c\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is, where c is a scalar,

$$c\vec{\mathbf{A}} \times \vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \tag{1.3}$$

  Similarly,

$$\vec{\mathbf{A}} \times c\vec{\mathbf{B}} = c(\vec{\mathbf{A}} \times \vec{\mathbf{B}}) \tag{1.4}$$

- The vector product between the sum of two vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ with a vector $\vec{\mathbf{C}}$ is,

$$(\vec{\mathbf{A}} + \vec{\mathbf{B}}) \times \vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{C}} + \vec{\mathbf{B}} \times \vec{\mathbf{C}} \tag{1.5}$$

$$\vec{\mathbf{A}} \times (\vec{\mathbf{B}} + \vec{\mathbf{C}}) = \vec{\mathbf{A}} \times \vec{\mathbf{B}} + \vec{\mathbf{A}} \times \vec{\mathbf{C}} \tag{1.6}$$

1.4 Vector Decomposition and Vector Product

In Cartesian coordinate systems, We can calculate that the magnitude of vector product of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$:

$$|\hat{\imath} \times \hat{\jmath}| = |\hat{\imath}||\hat{\jmath}| \sin(\pi/2) = 1$$

As, $|\hat{\imath}| = |\hat{\jmath}| = 1$, and also $\sin(\pi/2) = 1$. From "right hand rule" the direction of $\hat{\imath} \times \hat{\jmath}$ is along $+\hat{k}$ as shown in figure (3.3). Thus $\hat{\imath} \times \hat{\jmath} = \hat{k}$ Similarly, we can show,

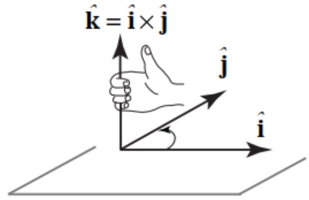

Figure 1.3: Vector product of $\hat{\imath} \times \hat{\jmath}$

$$\hat{j} \times \hat{k} = \hat{i},$$
$$\hat{k} \times \hat{i} = \hat{j}$$

From the anti-commutatively property of the vector product,

$$\hat{j} \times \hat{i} = -\hat{k},$$
$$\hat{i} \times \hat{k} = -\hat{j}$$

The vector product of the unit vector $\hat{i}$ with itself is zero because the two unit vectors are parallel to each other, $(\sin(0°) = 0)$,

$$|\hat{i} \times \hat{i}| = |\hat{i}||\hat{i}| \sin(0°) = 0$$

Similarly,

$$|\hat{j} \times \hat{j}| = 0,$$
$$|\hat{k} \times \hat{k}| = 0$$

With these properties in mind we can now develop an algebraic expression for the vector product in terms of components. We can define the vector product between two vector $\vec{\mathbf{A}} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ and $\vec{\mathbf{B}} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ as,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k} \tag{1.7}$$

We can also write it as matrix notation,

$$\vec{\mathbf{A}} \times \vec{\mathbf{B}} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix}$$

1.5 Exercise

- Derive the expression in equation (3.7).
- Derive the area of a parallelogram formed by two vector $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

1.6 Some Applications in Geometry

1.6.1 Volume of Parallelepiped

We can show that the volume of a Parallelepiped with edges formed by the vectors $\vec{\mathbf{A}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$ is given by $\vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}})$ (known as vector triple product) The volume

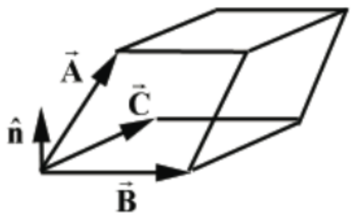

Figure 1.4: Volume of a Parallelepiped

of a parallelepiped is given by area of the base times height. If the base is formed by the vectors $\vec{\mathbf{B}}$ and $\vec{\mathbf{C}}$, then the area of the base is given by the magnitude of $\vec{\mathbf{B}} \times \vec{\mathbf{C}}$. The vector $\vec{\mathbf{B}} \times \vec{\mathbf{C}} = |\vec{\mathbf{B}} \times \vec{\mathbf{C}}|\hat{n}$, where $\hat{n}$ is the unit vector perpendicular to the base. (Figure 3.4)The projection of the vector $\vec{\mathbf{A}}$ along the direction $\hat{n}$ gives the height of the parallelepiped. This projection is given by taking the dot product of $\vec{\mathbf{A}}$ with a unit vector and is equal to $\vec{\mathbf{A}} \cdot \hat{n}\hat{} = height$. Therefore volume of the parallelepiped is,

$$\text{Volume} = \vec{\mathbf{A}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{C}}) = \vec{\mathbf{A}} \cdot |\vec{\mathbf{B}} \times \vec{\mathbf{C}}|\hat{n} = |\vec{\mathbf{B}} \times \vec{\mathbf{C}}|\vec{\mathbf{A}} \cdot \hat{n} = (\text{area})(\text{height}) \quad (1.8)$$

1.7 Exercise

- Justify why $\vec{\mathbf{A}} \cdot (\vec{\mathbf{A}} \times \vec{\mathbf{B}}) = 0$ and $\vec{\mathbf{B}} \cdot (\vec{\mathbf{B}} \times \vec{\mathbf{A}}) = 0$