Bonus Assignment

(1) $a_n = 2a_{n-1} + 5$

$a_n - 2a_{n-1} = 5$

$a_n = a_n^{(h)} + a_n^{(p)}$

$r - 2 = 0$

$r_1 = 2$

$a_n^{(h)} = \alpha(2)^n$

$a_n - 2a_{n-1} = 5$

$A - 2A = 5$

$A = -5$

$a_n^{(p)} = -5$

$a_n = a_n^{(h)} + a_n^{(p)}$

$= \alpha(2)^n - 5$

$3 = \alpha(2) - 5$

$8 = \alpha(2)$

$\alpha = 4$

$a_n = 4(2)^n + 5$

(2) $a_n = 2a_{n-1} + a_{n-2} + 5_n$

~~Let~~

$a_n - 2a_{n-1} - a_{n-2} = 5_n$

$a_n = a_n^{(h)} + a_n^{(p)}$

$r^2 - 2r - 1 = 0$

$r = \frac{2 \pm \sqrt{4+4}}{2}$

$= 1 \pm \sqrt{2}$

$\cancel{r_1 = \alpha(1-\sqrt{2})^n}$

$r_1 = (1-\sqrt{2})$

$r_2 = (1+\sqrt{2})$

$a_n^{(p)} = A_1 n + A_0$

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$$A_1 n + A_0 = 2(A_1(n-1) + A_0) + A_1(n-2) + A_0 + 5n$$

$$A_1 n + A_0 = 2(A_1 n - A_1 + A_0) + A_1 n - 2A_1 + A_0 + 5n$$

$$A_1 n + A_0 = 2A_1 n - 2A_1 + 2A_0 + A_1 n - 2A_1 + A_0 + 5n$$

$$A_1 n + A_0 - 2A_1 n + 2A_1 - 2A_0 - A_1 n + 2A_1 - A_0 = 5n$$

$$n(A_1 - 2A_1 - A_1) + (A_0 + 2A_1 - 2A_0 + 2A_1 - A_0) = 5n$$

$$\cancel{A_1} - 2A_1 - \cancel{A_1} = 5$$

$$-2A_1 = 5$$

$$A_1 = -\frac{5}{2}$$

$$\cancel{A_0} + 2 \times -\frac{5}{2} - 2A_0 + 2 \times -\frac{5}{2} - A_0 = 0$$

$$A_0 - 5 - 2A_0 - 5 - A_0 = 0$$

$$-2A_0 - 10 = 0$$

$$-2A_0 = 10$$

$$A_0 = -5$$

$$a_n^{(P)} = -\frac{5}{2} n - 5$$

$(1+\sqrt{2})^n$

Now

$a_1 = 3$, $a_2 = 7$

$3 = \alpha_1(1-\sqrt{2}) + \alpha_2(1+\sqrt{2}) - \frac{5}{2}n - 5$

$7 = \alpha_1(1-\sqrt{2})^2 + \alpha_2(1-\sqrt{2})^2 - \frac{5}{2}\times 2 - 5$

$\alpha_1 = -\left(\frac{28+35\sqrt{2}}{8}\right)$

$\alpha_2 = -\frac{28-35\sqrt{2}}{8}$

$a_n = -\frac{28+35\sqrt{2}}{8}(1-\sqrt{2})^n - \frac{28-35\sqrt{2}}{8}(1+\sqrt{2})^n - \frac{5}{2}n - 5$

③ $a_1 = 3$ and $a_2 = 4$

$a_n - 2a_{n-1} - a_{n-2} = n^2 - 1$

$r^2 - 2r - 1 = 0$

$r_1 = 1 - \sqrt{2}, \; r_2 = 1 + \sqrt{2}$

$a_n = A_2 n^2 + A_1 n + A_0$

$a_{n-1} = A_2 n^2 - 2A_2 n + A_2 + A_1 n - A_1$

$a_{n-2} = A_2 n^2 - 4A_2 n + 4A_2 + A_1 n - 2A_1 + A_0$

$A_2 n^2 + A_1 n + A_0 - 2(A_2 n^2 - 2A_2 n + A_2 + A_1 n - A_1 + A_0)$

$-(A_2 n^2 - 4A_2 n + 4A_2 + A_1 n - 2A + A_0 = n^2 - 1$

$\Rightarrow A_2 n^2 + A_1 n + A_0 - 2A_2 n^2 + 4A_2 n - 2A_2 - 2A_1 n + 2A_1 - 2A_0$

$- A_2 n^2 + 4A_2 n - 4A_2 - A_1 n + 2A_1 - A_0 = n^2 - 1$

$\Rightarrow n^2(A_2 - 2A_2 - A_2) + n(A_1 + 4A_2 - 2A_1 + 4A_2 - A_1)$

$+ (A_0 - 2A_2 + 2A_1 - 2A_0 - 4A_2 + 2A_1 - A_0) = n^2 - 1$

$\cancel{A_2} - 2A_2 - \cancel{A_2} = 1$ | $\cancel{A_1} + 4(-\frac{1}{2}) - 2A_1 + 4(-\frac{1}{2}) - \cancel{A_1} = 0$

$A_2 = -\frac{1}{2}$ | $-4 - 2A_1 = 0$

$A_1 = -2$

$A_0 - 2A_2 + 2A_1 - 2A_0 - 4A_2 + 2A_1 - A_0 = -1$

$A_0 - 2\times(-\frac{1}{2}) + 2(-2) - 2A_0 - 4(-\frac{1}{2}) + 2(-2) - A_0 = -1$

$A_0 = -2$

$a_1 = 3, a_2 = 4$

$\alpha_1(1-\sqrt{2})^1 + \alpha_2(1+\sqrt{2})^1 + \left(-\frac{1}{2}\right)(1)^2 + (-2)(1) + (-2) = 3$

$\alpha_1(1-\sqrt{2}) + \alpha_2(1+\sqrt{2}) = \frac{15}{2}$

$\alpha_1(1-\sqrt{2})^2 + \alpha_2(1+\sqrt{2})^2 + \left(-\frac{1}{2}\right)(2)^2 + (-2)(2) - 2 = 4$

$\alpha_1(3-2\sqrt{2}) + \alpha_2(3+2\sqrt{2}) = 12$

Using calculator, we get

$\alpha_1 = -5.212310601, \alpha_2 = +2.212310601$

$\cancel{a_n = -5.212(1-\sqrt{2})^n + 2.212}$

$\cancel{a_n = -5.212(1-\sqrt{2})^n + 2.2}$

$a_n = -5.212(1-\sqrt{2})^n + 2.212(1+\sqrt{2})^n + \left(-\frac{1}{2}\right)n^2 + (-2)n - 2$

(4) $a_1 = 1, a_2 = 2, b_1 = 0, b_2 = 1$

given, for $n \geq 3$, $a_n = 2a_{n-1} + b_{n-1}$ .. (I)

for $n \geq 2$, $b_n = b_{n-1} + a_{n-1}$ ... (II)

Substracting (I) from (II)

$a_n - b_n = 2a_{n-1} + b_{n-1} - (b_{n-1} + a_{n-1})$

$a_n - b_n = a_{n-1}$ ... (III)

put $n = n+1$ in eqn (I)

$a_{n+1} = 2a_{n+1-1} + b_{n+1-1}$

$a_{n+1} = 2a_n + b_n$ ... (IV)

put b_n from (III) into (IV)

$a_{n+1} = 2a_n + a_n - 2a_{n-1} + a_{n-1}$

$= 3a_n - a_{n-1}$. (V)

now put $n = n-1$ in (V)

$a_n = 3a_{n-1} - a_{n-2}$

$a_n = 3a_{n-1} - a_{n-2}$

$r^2 - 3r + 1 = 0$

$r_1 = \frac{3+\sqrt{5}}{2}$, $r_2 = \frac{3-\sqrt{5}}{2}$

$a_1 = 1 = \left(\frac{3+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{3-\sqrt{5}}{2}\right)\alpha_2$

$a_2 = 2 = \left(\frac{14+6\sqrt{5}}{4}\right)\alpha_1 + \left(\frac{14-6\sqrt{5}}{4}\right)\alpha_2$

Using calculator we get

$\alpha_1 = 0.276393$ and $\alpha_2 = 0.723607$

$a_n = 0.276393\left(\frac{14+6\sqrt{5}}{4}\right) + 0.723607\left(\frac{14-6\sqrt{5}}{4}\right)$

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$$E(x) = \sum P_i x_i$$

$$E(x) = \sum \left[{}^{n}C_{x} \times p^{x} \times q^{n-x} \times x \right]$$

$$= \sum \left[\left[\frac{n!}{(n-x)!\,x!} \times p^{x} \times q^{n-x} \right] \times x \right]$$

$$= \sum \left[\left[\frac{n(n-1)!}{(n-x)!\,x(x-1)!} \times p^{x} \times q^{n-x} \right] \times x \right]$$

$$= \sum \left[\frac{[n(n-1)!]}{(n-x)!\,(x-1)!} \times p^{x-1} \times p \times q^{n-x} \right]$$

$$np = \sum \left[\frac{(n-1)!}{(n-x)!\,(x-1)!} \times p^{(x-1)} \times q^{n-x} \right]$$

$$a = (n-1),\; b = (x-1)$$

$$a-b = (n-1-(x-1))$$

$$a-b = (n-x)$$

$$np \sum \frac{a!}{(a-b)!\,b!} \times p^{b} \times q^{a-b}$$

$\Rightarrow np \times \sum P(x)$; since $\sum P(x) = 1$

$\Rightarrow np$

(b) $E(X) = \sum x\, p(x)$

$$= \sum_{\cancel{x=0}\; x=1}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= \sum_{x=1}^{n} \cancel{x} \frac{n!}{\cancel{x}(x-1)!\,(n-x)!} p^x (1-p)^{n-x}$$

$$\cancel{= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!\,(n-x)!} p^x (1-p)^{n-x}$$

$$= np \sum \frac{x(n-1)!}{(x-1)!\,(n-x)!} p^{x-1} (1-p)^{n-x}$$

$\cancel{= np}$

$\cancel{n - x = (n-1) - (x-1)}$

~~Let~~
Replacing,
$n - x = (n-1) - (x-1)$
$m = n-1$
$y = x-1$

$$= n \cdot p \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y (1-p)^{m-y}$$

$$= np \sum_{y=0}^{m} \cancel{\binom{m}{y}} \binom{m}{y} p^y (1-p)^{m-y}$$

Recall: $\sum_{y=0}^{m} \binom{m}{y} a^y b^{m-y} = (a+b)^m$

So
$$E(X) = np\,(p + (1-p))^m$$
$$= np(1)^m$$
$$= np$$

$$Var(X) = E[(X-\mu)^2]$$
$$= \sum (x-\mu)^2 p(x)$$

$$E[(X-\mu)^2] = E(X^2) - [E(X)]^2$$

$$E(X^2) = \sum_{x=0}^{n} x^2 \frac{n!}{x!\,(n-x)!} p^x (1-p)^{n-x}$$

(5) $E[X(X-1)] = \sum_{x=0}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$

$$= \sum_{x=2}^{n} \frac{x(x-1)\, n!\, p^x (1-p)^{n-x}}{x(x-1)(x-2)!(n-x)!}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!(n-x)!}p^x(1-p)^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!\,p^{x-2}(1-p)^{n-x}}{(x-2)!(n-x)!}$$

Replacing $n-x = (n-2)-(x-2)$

$E[X(X-1)] =$

$$n(n-1)p^2 \times \sum_{x=2}^{n} \frac{(n-2)!\,p^{x-2}(1-p)^{(n-2)-(x-2)}}{(x-2)!((n-2)-(x-2))!}$$

Replacing $m = n-2$, $y = x-2$

$$= n(n-1)p^2 \times \sum_{y=0}^{m} \frac{m!}{y!(m-y)!}p^y(1-p)^{m-y}$$

$E[X(X-1)] = n(n-1)p^2$

$E(X^2) - E(X) = n(n-1)p^2$

Since $E(X) = np$

$E(X^2) = n(n-1)p^2 + np$

$Var(X) = E(X^2) - [E(X)]^2$

$= n(n-1)p^2 + np - (np)^2$

$= np[(n-1)p + 1 - np]$

$= np(\cancel{np} - p + 1 - \cancel{np})$

$= np(1-p)$