Name: ______________________

Final Exam

Math 253

Fall 2022

You have 2 hours to complete this exam and turn it in. You may use a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

1. (16 points) Multiple choice. You don't need to show your work.

a) (4 points) Which of the following series converges?

A) $\sum_{n=1}^{\infty} \ln(n)$.

B) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n} + 1/4}$.

C) $\sum_{n=1}^{\infty} \frac{1}{n}$.

D) $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n}$.

b) (4 points) Evaluate $\sum_{n=0}^{\infty} (-1)^n \frac{4^n}{(2n)!}$.

A) $\ln(2)$.

B) $\cos(2)$.

C) 1.

D) The sum diverges.

c) (4 points) Which power series has the largest interval of convergence?

A) $\sum_{n=1}^{\infty} n! x^n$.

B) $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

C) $\sum_{n=1}^{\infty} x^n$.

D) $\sum_{n=1}^{\infty} x$.

d) (4 points) The series $\sum_{k=1}^{\infty} \frac{(-2)^k}{3^k + 1}$

A) converges absolutely.

B) converges conditionally.

C) diverges.

2. (48 points) Short-answer. Explain your reasoning and/or show your work for each question.

a) (8 points) Does the series $\sum_{n=0}^{\infty} \frac{1}{n^2 + n + 1}$ converge or diverge?

b) (8 points) The Harmonic series diverges because it is a p-series with $p = 1$. Show that it diverges using another test.

c) (8 points) Estimate $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2+1}$ to within .1 of its actual value.

d) (8 points) Let $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$. Find $f'\left(\frac{1}{2}\right)$.

e) (8 points) Find the Maclaurin series for $x \sin(x^2)$.

f) (8 points) Give an example of a power series with an interval of convergence of exactly $(-2, 2)$. Show that your answer is correct.

3. (32 points) Define a sequence (a_n) by $a_0 = 1$ and $a_n = 3na_{n-1}$.

a) (8 points) Find a_1, a_2, and a_3.

b) (8 points) Find an explicit formula for (a_n). Check your answer by plugging in $n = 0$, $n = 1$, $n = 2$, and $n = 3$, and making sure they match.

c) (8 points) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{a_n}$, where a_n is the same sequence from the previous parts. Determine the interval of convergence of f.

d) (8 points) Find the exact value of $f(-1)$.

4. (32 points) Define a function g by $g(x) = \ln(x)$.

a) (8 points) For $n \geq 1$, find an expression for $g^{(n)}(x)$ (i.e. the nth derivative of g).

b) (12 points) Find the Taylor series for g centered at 1 and determine its interval of convergence.

c) (12 points) Approximate $g(1.1)$ with a degree-3 Taylor polynomial and determine the maximum error.

d) (4 points extra credit) Give an example of a power series with an interval of convergence of **exactly** $[1, 2]$. Hint: try combining the Maclaurin series from this question with another.