Final Review

Math 253

Section 1

1. Let $a_n = 2n$, $b_n = n^2 + 1$, and $c_n = (n-1)!$. Write out the first five terms of the sequences (a_n), (b_n), and (c_n).

2. Let $a_1 = 1$ and $a_n = 2a_{n-1}$. Write out the first five terms of (a_n) and find an explicit formula.

3. The Fibonacci numbers are a very famous sequence that begin with $F_1 = 1$ and $F_2 = 1$. After this, every Fibonacci number is the sum of the previous two. Find a recursive formula for the Fibonacci sequence (F_n).

4. Find recursive formulas for each of the three sequences in the previous problem.

5. Let (d_n) be the sequence defined by $d_1 = 0$ and $d_n = d_{n-1} + \frac{1}{2^n}$. Find an explicit formula for (d_n) by writing out terms and finding a pattern, and verify your answer by plugging it back into the recurrence.

6. Let (a_n) be the arithmetic sequence defined by $a_1 = 1$ and $a_n = a_{n-1} - 3$. Find an explicit formula for (a_n) and verify it's correct.

7. Let (b_n) be the geometric sequence defined by $b_1 = 2$ and $b_n = 2b_{n-1}$. Find an explicit formula for (b_n) and verify it's correct.

8. The sequence (a_n) defined by

$$a_n = 1 + \left(-\frac{1}{2}\right)^n$$

converges to 1. Find N such that $|a_n - 1| < \varepsilon$ for $\varepsilon = 1$, $\varepsilon = .1$, and $\varepsilon = .001$.

9. The sequence (b_n) defined by

$$b_n = \frac{1}{n}$$

converges to 0. Find N such that $|b_n| < \varepsilon$ for $\varepsilon = 1$, $\varepsilon = .1$, and $\varepsilon = .001$.

10. Find the limits of the following sequences.

a) $(n!)$.

b) $\left(1 - \frac{1}{n}\right)$.

c) $\left(\sin\left(\frac{n}{2^n}\right)\right)$.

11. Find $\lim\left(\frac{\cos(n)}{n}\right)$.

12. We already understand the behavior of the sequence (r^n) when $r \geq 0$: it converges to zero for $0 \leq r < 1$, to one for $r = 1$, and it diverges for $r > 1$. Use the Squeeze theorem to classify the behavior of (r^n) for negative r.

13. Let (a_n) be a sequence defined recursively by $a_1 = 2$ and

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{2a_{n-1}}$$

for $n \geq 2$.

First, show that (a_n) converges using the MCT. Then find the number it converges to. (Hint: take the limit of both sides of the recurrence.)

Section 2

1. Evaluate the following sums.

a) $\sum_{n=1}^{5} n$.

b) $\sum_{i=-2}^{2} \frac{1}{i^2+1}$.

c) $\sum_{k=0}^{\infty} 2$.

2. For each of the following series, write out the first five terms. Then find the first, third, and fifth partial sums.

a) $\sum_{n=0}^{\infty} n$.

b) $\sum_{m=1}^{\infty} \frac{1}{m}$.

c) $\sum_{k=1}^{\infty} \frac{2}{3^k}$.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{2^n}$ or show it diverges.

4. Evaluate the following series or show they don't converge.

a) $\sum_{i=0}^{\infty} (-1)^i$.

b) $\sum_{m=1}^{\infty} \frac{2}{3^m}$.

c) $\sum_{n=1}^{\infty} \frac{1}{n}$. Note: this is called the **Harmonic series**, and is quite famous. To get a handle on its behavior, try grouping the terms in this manner: put the first and second terms by themselves, then put the third and fourth together, then the fifth through the eighth together, then the 9th through the 16th, and so on, doubling the size of each group. See if you can quantify how large each group is based on its last term.

4. Evaluate $\sum_{n=1}^{\infty} \frac{1}{2^n}$ by using properties of geometric series.

5. Evaluate the following series.

a) $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^{k+1}}$.

b) $\sum_{m=1}^{\infty} e^m$.

6. Evaluate the sum

$$\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right) \right).$$

7. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is actually telescoping, even if it doesn't look like it at first glance. Use partial fractions to split apart $\frac{1}{n(n+1)}$, and then evaluate the sum using properties of telescoping series.

Section 3

1. Apply the divergence test to the following series and clearly state what conclusion you can draw.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

b) $\sum_{j=2}^{\infty} \ln(j)$.

c) $\sum_{i=0}^{\infty} \frac{1}{i+1} \cos(i)$.

2. Determine if the following series converge. Clearly state what conclusions you can draw, if any.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

b) $\sum_{j=1}^{\infty} \sin(j)$.

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$.

3. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within .01 of its actual value.

4. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to within .01 of its actual value.

Section 4

1. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+20}$ converges.

2. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4+n^3}$ converge or diverge?

3. Determine the behavior of $\displaystyle\sum_{m=2}^{\infty} \frac{1}{\ln(m)}$.

4. Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3}$ converge?

5. Determine if $\displaystyle\sum_{k=1}^{\infty} \frac{2^k+3^k}{4^k}$ converges or diverges.

6. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n-n}{3^n}$ converge?

7. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+10}$ converge or diverge?

8. What is the behavior of the series $\sum_{k=1}^{\infty} \frac{k^2}{k(k+3)}$?

9. Determine the behavior of $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$.

Section 5

1. Does $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converge absolutely, conditionally, or not at all?

2. Does $\sum_{i=1}^{\infty} \left(-\frac{1}{i}\right)^i$ converge absolutely, conditionally, or not at all?

3. Does $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sin(n)}$ converge absolutely, converge conditionally, or diverge?

4. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ converges, and then estimate it to within .001 of its true value.

5. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$ converges, and then estimate it to within .2 of its true value.

6. Does the series $\sum_{k=2}^{\infty} \frac{(-1)^k}{\ln(k)}$ converge absolutely, conditionally, or not at all?

7. Does the series $\sum_{k=1}^{\infty} \frac{\sin(k)}{k^2}$ converge absolutely, conditionally, or not at all?

Section 6

1. Does the series $\sum_{n=0}^{\infty} \frac{e^n}{n!}$ converge or diverge?

2. Does the series $\sum_{n=0}^{\infty} \frac{(-1)^n n}{2^n}$ converge absolutely, conditionally, or not at all?

3. Determine the behavior of the series $\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$.

4. Apply the ratio test to the following series and state what conclusion you can draw.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$.

c) $\sum_{n=1}^{\infty} (-1)^n$.

5. Does $\sum_{n=1}^{\infty} \frac{\sin^n(n)}{n^n}$ converge absolutely, conditionally, or not at all?

6. Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge or diverge?

Section 7

1. Find the interval and radius of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n(n+1)}$.

2. Find the interval and radius of convergence of $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$.

3. Write $\frac{x}{1+3x^2}$ as a power series and state the interval and radius of convergence.

4. Write $\frac{x^3}{2+x^2}$ as a power series and state the interval and radius of convergence.

Section 8

1. Express $f(x) = \dfrac{5}{(1-2x)(1+x^2)}$ as a power series in two ways: first with partial fractions and then with power series multiplication. For each, find the interval and radius of convergence, and make sure they match.

2. Express $g(x) = \dfrac{6}{(1-x)(2-x)(3-x)}$ as a power series by using partial fractions and determine the interval and radius of convergence.

3. Express $h(x) = \dfrac{1}{(1-x)^2}$ as a power series by using power series multiplication.

4. Find a power series representation for $\ln(1+x)$ by using calculus on a series representation for $\frac{1}{1+x}$.

5. Express $\dfrac{1}{(1-x)^2}$ as a power series by using calculus on a power series for $\dfrac{1}{1-x}$.

6. Find a power series for e^x by using the fact that it is equal to its own derivative.

Section 9

1. Find a Maclaurin series for $\sin(x)$ and determine its interval of convergence. Find the degree 0, 1, 2, and 3 Maclaurin polynomials. Approximate $\sin(1)$ with the degree 3 one and bound the error.

2. Find a Taylor series for $\ln(x)$ centered at $x = 1$ and determine its interval of convergence. Find the degree 0, 1, 2, and 3 Maclaurin polynomials. Approximate $\ln(1.5)$ with the degree 3 one and bound the error.

3. Find a Taylor series for $\frac{1}{1+x}$ centered at $x = 2$ and determine its interval of convergence. Find the degree 0, 1, 2, and 3 Maclaurin polynomials.

4. Find a Maclaurin series for e^x and determine its interval of convergence. Find the degree 0, 1, 2, and 3 Maclaurin polynomials. Approximate e with an error of no more than .001.

Section 10

1. Approximate $\frac{1}{\sqrt{2}}$ with a degree-4 Maclaurin polynomial and bound the error.

2. Write $(1+x)^{3/2}$ as a Maclaurin series and approximate $1.5^{1.5}$ with a degree-3 polynomial. Bound the error.