Final Exam

Math 252

Winter 2022

You have 2 hours to complete this exam and turn it in. You may use a scientific calculator and a single two-sided sheet of an 8.5×11 inch piece of paper with handwritten notes, but no other resources. When you're finished, first check your work if there is time remaining, then turn in the exam. If you have a question, don't hesitate to ask — I just may not be able to answer it. Show all your work.

Formulas

- $\sin(\theta) = \dfrac{\text{opp}}{\text{hyp}}$
- $\cos(\theta) = \dfrac{\text{adj}}{\text{hyp}}$
- $\tan(\theta) = \dfrac{\text{opp}}{\text{adj}}$
- $\sec(\theta) = \dfrac{\text{hyp}}{\text{adj}}$
- $\csc(\theta) = \dfrac{\text{hyp}}{\text{opp}}$
- $\cot(\theta) = \dfrac{\text{adj}}{\text{opp}}$
- $\displaystyle\int \tan(\theta)\,d\theta = \ln|\sec(\theta)| + C$
- $\displaystyle\int \sec(\theta)\,d\theta = \ln|\sec(\theta) + \tan(\theta)| + C$
- $\displaystyle\int \sec^2(\theta)\,d\theta = \tan(\theta) + C$
- $\displaystyle\int \sec(\theta)\tan(\theta)\,d\theta = \sec(\theta) + C$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$
- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$
- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$
- $\sin^2(\theta) = 1 - \cos^2(\theta)$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $\sec^2(\theta) - 1 = \tan^2(\theta)$

Part I: Multiple Choice and Short-Answer (32 points possible)

1. (4 points) A differential equation is **separable** if

A) It can be written in the form $y' = xy$.

B) It can be written in the form $y' = f(x)g(y)$. i.e. (x-stuff)(y-stuff)

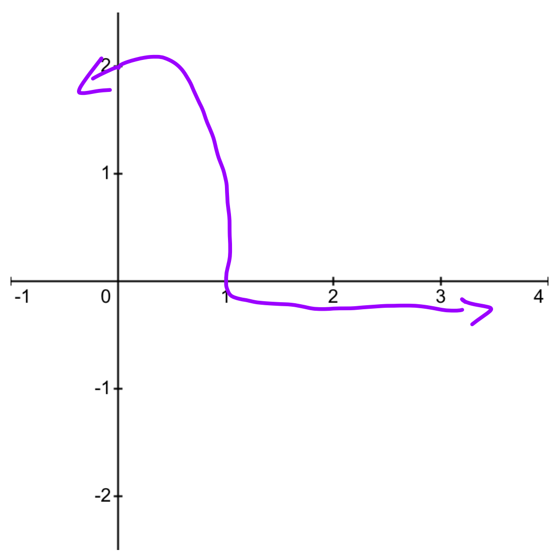
C) It can be solved for y'.

D) $y = x$ is a solution to the equation.

2. (4 points) Draw a continuous function $f(x)$ on $[0,3]$ that has all three of the following properties:

- $f(x) > 0$ when $x < 1$, → forces $f(1) = 0$
- $f(x) < 0$ when $x > 1$, and
- $\int_0^3 f(x)\,dx > 0$.

ex:



3. (4 points) Evaluate $\dfrac{d}{dx}\displaystyle\int_1^{\sqrt{x}} e^{t^2}\,dt$.

A) $\dfrac{e^x}{2\sqrt{x}}$.

B) $\sqrt{x}e^{x^2}$.

C) e^x.

D) $e^x - e$.

FTC I: $F(x) = \int_1^x e^{t^2}\,dt$

$F'(x) = e^{x^2}$

$\frac{d}{dx}\left[F(\sqrt{x})\right] = e^{(\sqrt{x}^2)} \cdot \frac{1}{2\sqrt{x}}$

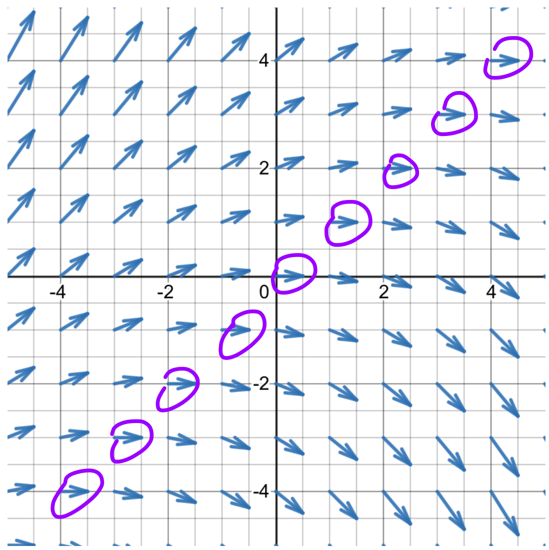
4. (4 points) Give an example of a function $f(x)$ so that $\int_0^1 f(x)\,dx$ diverges.

$f(x) = \frac{1}{x}$ works

$\int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \Big[\ln(x)\Big]\Big|_a^1 = \infty$

$\Rightarrow$ diverges

5. (4 points) Which of the following differential equations could have generated this direction field?



A) $y' = \sin(x)$.

B) $y' = xy$.

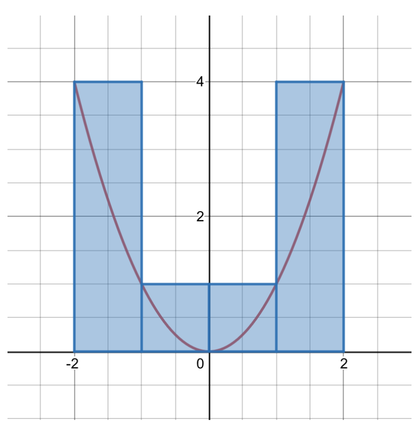
C) $y' = y - x$.

D) $y' = x^2$.

A, B, and D all would have $y'=0$ when $x=0$ but that's not the case

Also, the indicated slopes being 0 imply the answer is C

6. (4 points) The shaded area in the below figure is what kind of Riemann sum of $f(x) = x^2$ on $[-2, 2]$?



A) Left.

B) Right.

C) Upper.

D) Lower.

heights are maxed

7. (4 points) In the following integral, what should we substitute for x and dx?

$$\int \frac{x^2}{\sqrt{4 - x^2}}\, dx$$

trig sub : $x = 2\sin(\theta)$
$dx = 2\cos(\theta)\, d\theta$

8. (4 points) Complete the formula for integration by parts.

$$\int u\, dv = uv - \int v\, du$$

9. (4 points extra credit) Give an example of a function $f(x)$ so that the solid of revolution created by rotating the graph of f on $[1, \infty)$ about the x-axis has finite volume, but $\int_1^\infty f(x)\, dx$ diverges, and verify both.

One possibility: $f(x) = \frac{1}{x}$

volume: $\pi \int_1^\infty \frac{1}{x^2}\, dx = \pi \lim_{b \to \infty} \left[-\frac{1}{x} \right] \Big|_1^b = \pi$

area: $\int_1^\infty \frac{1}{x}\, dx = \lim_{b \to \infty} \left[\ln(x) \right] \Big|_1^b = \infty \Rightarrow$ diverges

Part II: Setting Things Up (24 points possible)

1. (4 points) Let $f(x) = 2x^2 + 1$. Set up, but **do not solve**, the integral to find the volume of the solid of revolution generated by rotating the graph of f on $[1, 2]$ about the x-axis using the **disk** method.

dx

$$\int_1^2 \pi(2x^2+1)^2\,dx$$

2. (4 points) With $f(x) = 2x^2 + 1$ as in the previous question, set up, but **do not solve**, the integral to find the volume of the solid of revolution generated by rotating the graph of f on $[1, 2]$ about the x-axis using the **shell** method.

dy

$$\left.\begin{array}{l} y=2x^2+1 \\ x=\sqrt{\dfrac{y-1}{2}} \end{array}\right\} \quad \begin{array}{l} y=2(1)^2+1=3 \\ y=2(2)^2+1=9 \end{array}$$

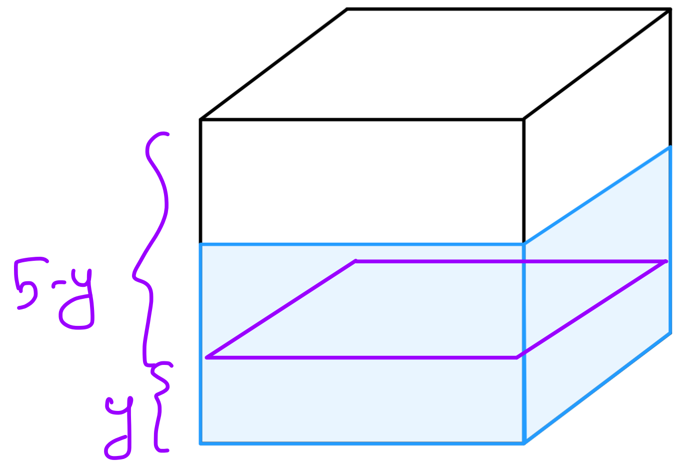
$$\int_3^9 2\pi y\sqrt{\frac{y-1}{2}}\,dy$$

3. (4 points) With $f(x) = 2x^2 + 1$ again, set up, but **do not solve**, the integral to find the arc length of f on $[1, 2]$.

$$f'(x)=4x$$
$$(f'(x))^2=16x^2$$

$$\int_1^2 \sqrt{1+16x^2}\,dx$$

4. (4 points) A tank in the shape of a cube with side length 5 meters is filled to a depth of 3 meters with water, which has weight density $9800 N/m^3$. Set up, but **do not solve**, the integral to calculate the work done by pumping it all out.



slice at height y: square w/ area 25

$$\int_0^3 9800 \cdot 25 \cdot (5-y)\, dy$$

5. (4 points) Set up, but **do not solve**, the integral to find the surface area of the solid given by revolving the graph of $y = \cos(x^2)$ on $[5, 10]$ about the y-axis. need to solve for x

$$x = \sqrt{\arccos(y)}$$

limits: $y = \cos(25)$, $y = \cos(100)$

$$x' = \frac{1}{2}(\arccos(y))^{-1/2}\left(-\frac{1}{\sqrt{1-y^2}}\right)$$

$$(x')^2 = \frac{1}{4}(\arccos(y))^{-1} \cdot \frac{1}{1-y^2}$$

$$\int_{\cos(25)}^{\cos(100)} 2\pi\sqrt{\arccos(y)}\sqrt{\frac{1}{4} \cdot \frac{1}{\arccos(y)} \cdot \frac{1}{1-y^2}}\, dy$$

$$= \pi \int_{\cos(25)}^{\cos(100)} \frac{1}{\sqrt{1-y^2}}\, dy$$

6. (4 points) Write down the partial fraction expansion of $\frac{4x^2+1}{(x^2+x+1)^2(x^2+2x+1)^2}$. Your answer should be in terms of several constants, e.g. A, B, C...

$$\frac{4x^2+1}{(x^2+x+1)^2(x+1)^4}$$

$$= \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2} + \frac{E}{x+1} + \frac{F}{(x+1)^2} + \frac{G}{(x+1)^3} + \frac{H}{(x+1)^4}$$

Part III: Integrals Proper (34 points possible)

1. (8 points) Find the solution to the differential equation $y' = yt\cos(t^2)$, given that $y(0) = e$.

$$\frac{dy}{dt} = y t \cos(t^2)$$

$$\int \frac{1}{y}\, dy = \int t\cos(t^2)\, dt$$

$u = t^2$

$du = 2t\, dt$

$$\ln(y) = \frac{1}{2}\int \cos(u)\, du$$

$$\ln(y) = \frac{1}{2}\sin(u) + C$$

$$\ln(y) = \frac{1}{2}\sin(t^2) + C$$

$y = e^{\frac{1}{2}\sin(t^2) + C}$ and $y = 0$

$\hookrightarrow y(0) \neq e$

$$y(0) = e^{\frac{1}{2}\sin(0) + C} = e$$

$$e^C = e$$

$$C = 1$$

$$\boxed{y = e^{\frac{1}{2}\sin(t^2) + 1}}$$

2. (8 points) Evaluate $\int_1^\infty \frac{\ln(t)}{t^2}\,dt$.

$$= \lim_{b\to\infty} \int_1^b \frac{\ln(t)}{t^2}\,dt$$

$$u = \ln(t) \qquad v = -\frac{1}{t}$$

$$\downarrow \qquad \uparrow$$

$$du = \frac{1}{t}\,dt \qquad dv = \frac{1}{t^2}\,dt$$

$$= \lim_{b\to\infty} \left[-\frac{1}{t}\ln(t) + \int \frac{1}{t^2}\,dt \right]\Bigg|_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{1}{t}\ln(t) - \frac{1}{t} \right]\Bigg|_1^b$$

$$= \lim_{b\to\infty} \left(\Big(\underbrace{-\frac{\ln(b)}{b}}_{\text{L'Hôpital}} \underbrace{- \frac{1}{b}}_{\to 0} \Big) \underbrace{- \Big(-\frac{\ln(1)}{1} - \frac{1}{1} \Big)}_{1} \right)$$

$$= \lim_{b\to\infty} \left(-\frac{1/b}{1} \right) + 1$$

$$= \boxed{1}$$

3. (8 points) Evaluate $\int \frac{4}{x^2\sqrt{x^2-4}}\,dx$.

Trig sub: $x = 2\sec(\theta)$ $\longrightarrow$ $\sec(\theta) = \frac{x}{2}$

$dx = 2\sec(\theta)\tan(\theta)\,d\theta$

$$= \int \frac{8\sec(\theta)\tan(\theta)}{4\sec^2(\theta)\sqrt{4\sec^2(\theta)-4}}\,d\theta$$

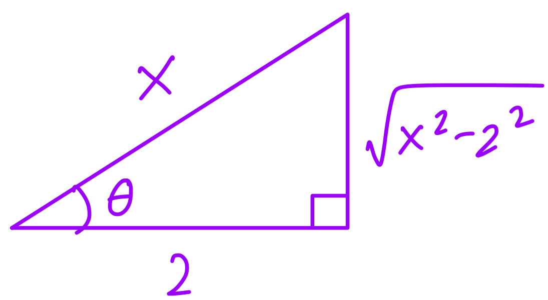
$$= \int \frac{\sec(\theta)\tan(\theta)}{\sec^2(\theta)\tan(\theta)}\,d\theta$$

$$= \int \frac{1}{\sec(\theta)}\,d\theta$$

$$= \int \cos(\theta)\,d\theta$$

$$= \sin(\theta) + C$$

$$= \frac{\sqrt{x^2-4}}{x} + C$$



4. (8 points) Let $f(x) = x$.

a) (2 points) Evaluate $\int_0^2 f(x)\,dx$ using the Fundamental Theorem of Calculus.

$$= \left[\frac{x^2}{2}\right]\Big|_0^2$$

$$= \frac{4}{2} - \frac{0}{2}$$

$$= 2.$$

b) (8 points) Evaluate $\int_0^2 f(x)\,dx$ from the definition of the integral — i.e. taking a limit of Riemann sums.

Right Riemann sum w/ n subintervals:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$x_i^* = 0 + i\Delta x = \frac{2}{n} i$$

$$f(x_i^*) = \frac{2}{n} i$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} i \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{4}{n^2} \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \frac{4}{2} \cdot \frac{n^2+n}{n^2}$$

$$= \frac{4}{2}$$

$$= 2.$$