Name: Key

Midterm 2

Math 252

Winter 2022

You have 50 minutes to complete this exam and turn it in. You may use a scientific calculator and a handwritten 3×5 inch index card of notes, but no other resources. When you're finished, first check your work if there is time remaining, then turn it in. If you have a question, don't hesitate to ask — I just may not be able to answer it.

Part I (24 points) Multiple choice. You don't need to show any work.

1. (8 points) Suppose $y = f(x)$, and that the graph of f is rotated about the x-axis. Then

A) the shell method integrates with respect to y and the disk method with respect to x.

B) the shell method integrates with respect to x and the disk method also with respect to x.

C) the shell method integrates with respect to x and the disk method with respect to y.

D) the shell method integrates with respect to y and the disk method also with respect to y.

2. (8 points) It takes 3 J of work to stretch a spring a total of 1 meter from rest. How much work does it take to compress it 2 meters from rest?

A) 3 J.

B) 6 J.

C) 9 J.

D) 12 J.

$$\text{work} = 3 = \int_0^1 kx \, dx$$

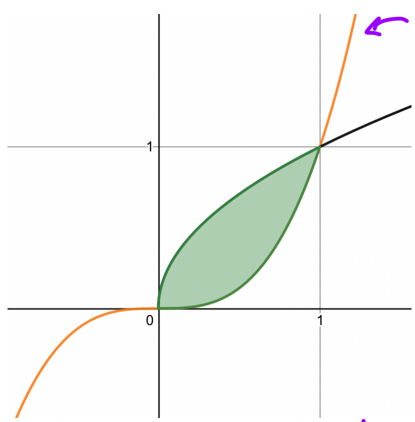
$$3 = \left[k\frac{x^2}{2}\right]\Big|_0^1$$

$$3 = \frac{k}{2}$$

$$k = 6$$

$$\int_0^2 kx \, dx = \int_0^2 6x \, dx = \left[3x^2\right]\Big|_0^2 = \boxed{12 \ J}$$

3. (8 points) Which of the following integrals calculates the area bounded by $f(x) = \sqrt{x}$ and $g(x) = x^3$?



$\leftarrow y = x^3$ or $x = \sqrt[3]{y}$

$\leftarrow y = \sqrt{x}$ or $x = y^2$

A) $\int_0^1 \left(x^3 - \sqrt{x}\right) \, dx$.

B) $\int_0^1 \left(y^2 - \sqrt[3]{y}\right) \, dy$.

C) $\int_0^1 \left(\sqrt{x} + x^3\right) \, dx$.

D) $\int_0^1 \left(\sqrt[3]{y} - y^2\right) \, dy$.

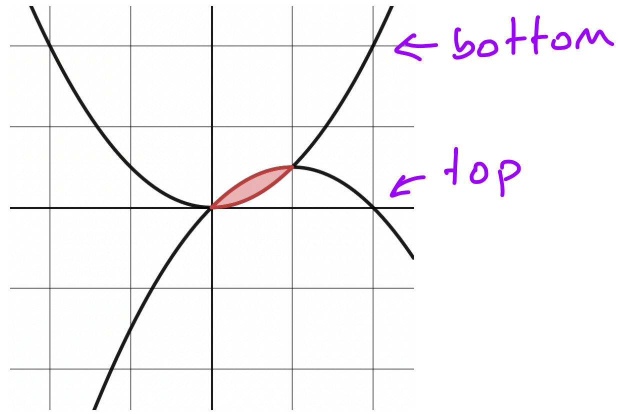
$dx: \int_0^1 (\sqrt{x} - x^3) \, dx$

$dy: \int_0^1 (\sqrt[3]{y} - y^2) \, dy$

$\uparrow$ only this one appears

Part II (32 points) Short answer. Show all your work.

1. (8 points) Find the area between $f(x) = x^2$ and $g(x) = x - x^2$.



intersection: $x^2 = x - x^2$

$2x^2 - x = 0$

$x(2x-1) = 0$

$x = 0$ $\quad$ $x = \frac{1}{2}$

$$\int_0^{1/2} \left((x - x^2) - x^2\right) dx$$

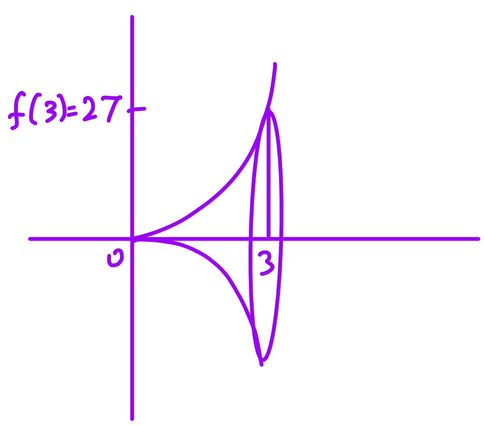
$$= \int_0^{1/2} (x - 2x^2)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2}{3}x^3\right]\Bigg|_0^{1/2}$$

$$= \frac{1}{8} - \frac{1}{12}$$

$$= \frac{1}{24}$$

2. (8 points) Let $f(x) = 3x^2$. Set up the integrals to find the volume of the solid given by rotating the graph of f on $[0, 3]$ about the x-axis, using **both** the disk and shell methods. Don't solve either of the integrals.



Disk method

dx. Top function is $y = 3x^2$ and bottom is $y = 0$.

$$\int_0^3 \pi (3x^2)^2\, dx$$

Shell method

dy. Right function is $x = 3$ and left is $x = \sqrt{\frac{y}{3}}$.

$$\int_0^{27} 2\pi y \left(3 - \sqrt{\frac{y}{3}}\right) dy$$

3. (8 points) The density of a bar is given by $\rho(x) = \ln(x)$ for $x = e$ to $x = e^2$. Find the mass of the bar.

$$\text{mass} = \int_e^{e^2} \rho(x)\,dx$$

$$= \int_e^{e^2} \ln(x)\,dx$$

$$= \left[x\ln(x) - x\right]\Big|_e^{e^2}$$

$$= \left(e^2\ln(e^2) - e^2\right) - \left(e\ln(e) - e\right)$$

$$= \left(2e^2 - e^2\right) - \left(e - e\right)$$

$$= e^2.$$

4. (8 points) Find the surface area of the solid created by revolving the graph of $y = x^3$ on $[0, 2]$ about the x-axis.

$$y' = 3x^2$$

$$(y')^2 = 9x^4$$

$$= \int_0^2 2\pi x^3 \sqrt{1 + 9x^4}\,dx$$

$$u = 1 + 9x^4$$

$$du = 36x^3\,dx \quad \Big| \quad x^3\,dx = \frac{1}{36}\,du$$

$$= \int_0^2 2\pi\sqrt{u} \cdot \frac{1}{36}\,du$$

$$= \frac{\pi}{18}\left[\frac{u^{3/2}}{3/2}\right]\Big|_0^2$$

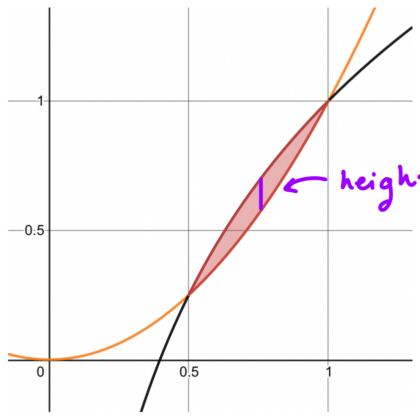
$$= \frac{\pi}{18}\left[\frac{(1 + 9x^4)^{3/2}}{3/2}\right]\Big|_0^2$$

$$= \frac{\pi}{18} \cdot \frac{2}{3}\left(145^{3/2} - 1\right)$$

$$= \frac{\pi}{27}\left(145^{3/2} - 1\right)$$

Part III (32 points) Longer problems that require setting up and solving integrals. Half the credit is for the set-up and half for the solving.

1. (16 points) The functions $f(x) = x^2$ and $g(x) = \dfrac{3}{\ln(16)}\ln(x) + 1$ intersect at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $(1,1)$ and bound a region, as shown below.



height: $\dfrac{3}{\ln(16)}\ln(x) + 1 - x^2$

Find the volume of the solid of revolution given by rotating the region about the y-axis. You may use any method you like. **You may leave your answer in evaluation notation: e.g. $[x^2]|_0^1$. No integrals should be present in your final answer.**

These functions are already in terms of x, so the shell method will require less work to set up.

$$\int_{1/2}^{1} 2\pi x \left(\frac{3}{\ln(16)}\ln(x) + 1 - x^2\right) dx$$

$$= 2\pi \int_{1/2}^{1} \left(\frac{3}{\ln(16)} x\ln(x) + \underbrace{x - x^3}_{\text{can handle these immediately}}\right) dx$$

$$= 2\pi \frac{3}{\ln(16)} \underbrace{\int_{1/2}^{1} x\ln(x)\,dx}_{} + \underbrace{2\pi}_{\text{careful to distribute this}} \left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Bigg|_{1/2}^{1}$$

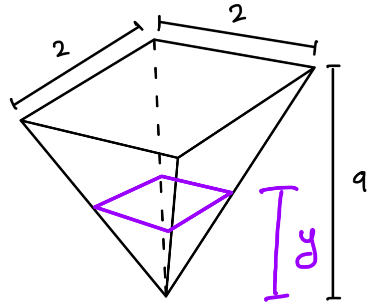
$u = \ln(x)$ $\quad v = \frac{x^2}{2}$

$du = \frac{1}{x}dx$ $\quad dv = x\,dx$

$$= \frac{6\pi}{\ln(16)}\left[\frac{x^2}{2}\ln(x) - \int \frac{x}{2}\,dx\right]\Bigg|_{1/2}^{1} + 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Bigg|_{1/2}^{1}$$

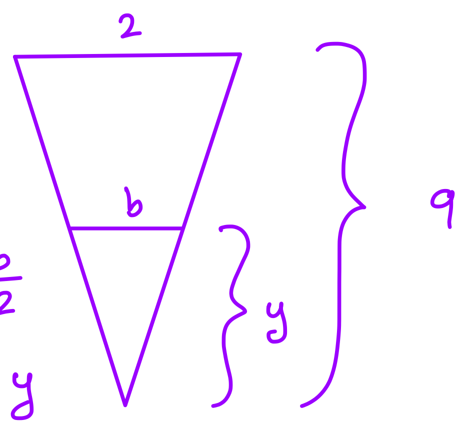
$$= \frac{6\pi}{\ln(16)}\left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right]\Bigg|_{1/2}^{1} + 2\pi\left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Bigg|_{1/2}^{1}.$$

2. (16 points) A tank in the shape of a square pyramid has height 9 meters and a base with side length 2 meters. It's filled up to 5 meters with a liquid that has weight density 2000 $\frac{N}{m^3}$. Find the work done by pumping the liquid out.



Slices are squares — just need to find the side length.

Side view:



Similar triangles: $\frac{y}{9} = \frac{b}{2}$

$b = \frac{2}{9} y$

height of liquid

area

$$\int_0^5 2000 \left(\frac{2}{9} y\right)^2 (9-y)\, dy$$

weight density

distance to move

$$= 2000 \int_0^5 \frac{4}{81} (9y^2 - y^3)\, dy$$

$$= 2000 \left[\frac{4}{27} y^3 - \frac{y^4}{81} \right] \Big|_0^5$$

$$= 2000 \left(\frac{4}{27}(125) - \frac{625}{81} \right).$$