Final Exam Worksheet

Math 251

1. True or false: if $x = a$ is a critical point of $f(x)$ and $f''(a) > 0$, then a is a local minimum. If it's true, explain why, and if it's false, give an example of a function f and a critical point a that are a counterexample.

2. Evaluate $\lim_{x \to 2} (x-1)^{\ln(x-2)}$.

3. Let $g(x) = \dfrac{\sin(x^2)}{\sin^2(x)}$. What is $g'(2)$?

4. Find the equation of the tangent line to $y^2 + \ln(xy) = x$ at $(1,1)$.

5. Define a function f by

$$f(x) = \begin{cases} x^3 \ln(x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

on $[-2,1]$.

a) Show that f is continuous.

b) Find and classify the critical points of f.

c) Find the inflection points of f.

d) Find the global maximum and minimum of f.

6. Let $f(x) = x^3 + x^2 + x + 1$. Find $f'(x)$ **using the limit definition of the derivative**.

① False: for $f(x) = x^2$ on $[-1, 1]$, $x=1$ is a critical point and $f''(1) = 2 > 0$, but $x=1$ is a local max.

② $y = (x-1)^{\ln(x-2)}$: of the form 1^∞

$$\ln(y) = \ln(x-2)\ln(x-1)$$

$$\lim_{x \to 2} \ln(x-2)\ln(x-1) : -\infty \cdot 0$$

$$= \lim_{x \to 2} \frac{\ln(x-2)}{\frac{1}{\ln(x-1)}} : \frac{\infty}{\infty}$$, so L'Hôpital applies

$$= \lim_{x \to 2} \frac{\frac{1}{x-2}}{\frac{0 - \frac{1}{x-1}}{(\ln(x-1))^2}}$$

$$= \lim_{x \to 2} -\frac{1}{x-2} \cdot (\ln(x-1))^2 \cdot (x-1)$$

$$= \underbrace{\left(\lim_{x \to 2} -(x-1) \right)}_{=-1} \underbrace{\left(\lim_{x \to 2} \frac{(\ln(x-1))^2}{x-2} \right)}_{\text{of the form } \frac{0}{0}}$$

$$= -1 \cdot \lim_{x \to 2} \frac{2\ln(x-1) \cdot \frac{1}{x-1}}{1}$$

$$= -1 \cdot \frac{2\ln(1) \cdot \frac{1}{1}}{1}$$

$$= 0$$

Therefore, $\ln(y) = 0$, so $y = e^0 = \boxed{1}$

3

$$g(x) = \frac{\sin(x^2)}{\sin^2(x)}$$

$$g'(x) = \frac{\frac{d}{dx}\left[\sin(x^2)\right] \cdot \sin^2(x) - \sin(x^2) \cdot \frac{d}{dx}\left[\sin^2(x)\right]}{\sin^4(x)}$$

$$= \frac{\cos(x^2) \cdot 2x \sin^2(x) - \sin(x^2) \cdot 2\sin(x) \cdot \cos(x)}{\sin^4(x)}$$

$$g'(2) = \frac{\cos(4) \cdot 4 \cdot \sin^2(2) - \sin(4) \cdot 2\sin(2) \cdot \cos(2)}{\sin^4(2)}$$

4

$$2y\frac{dy}{dx} + \frac{1}{xy}\left(y + x\frac{dy}{dx}\right) = 1$$

$$2y\frac{dy}{dx} + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1$$

$$2y\frac{dy}{dx} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1 - \frac{1}{x}}{2y + \frac{1}{y}}$$

At $(1,1)$: $\dfrac{1-\frac{1}{1}}{2+\frac{1}{1}} = 0$

So the equation of the tangent line is $y = 1 + 0(x-1)$, or just $\boxed{y=1}$

5 Note: there are subtleties regarding derivatives of piecewise functions that we didn't touch on in the course. I will largely be brushing over how we find $f'(0)$, and I don't expect you to have that knowledge prepared for the final.

a) Since $x^3 \ln(x^2)$ is continuous where it's defined, which is everywhere but $x=0$, we just need to show that

$$\lim_{x\to 0} x^3 \ln(x^2) = 0.$$

Since that's of the form $0\cdot(-\infty)$, we use L'Hôpital.

$$\lim_{x\to 0} x^3\ln(x^2) = \lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x^3}} = \lim_{x\to 0} \frac{\frac{1}{x^2}\cdot 2x}{-3x^{-4}}$$

$$= \lim_{x\to 0} -\frac{2}{3}\frac{x^5}{x^2} = \lim_{x\to 0} -\frac{2}{3}x^3 = 0. \checkmark$$

b) $$f'(x) = \begin{cases} 3x^2\ln(x^2) + x^3\cdot\frac{1}{x^2}\cdot 2x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Therefore, one critical point occurs when $x=0$. We also have

$$3x^2\ln(x^2) + x^3\cdot\frac{1}{x^2}\cdot 2x = 0$$

$$3x^2 \ln(x^2) + 2x^2 = 0$$

$$x^2 \left(3 \ln(x^2) + 2\right) = 0$$

$x = 0$

$$\ln(x^2) = -2/3$$

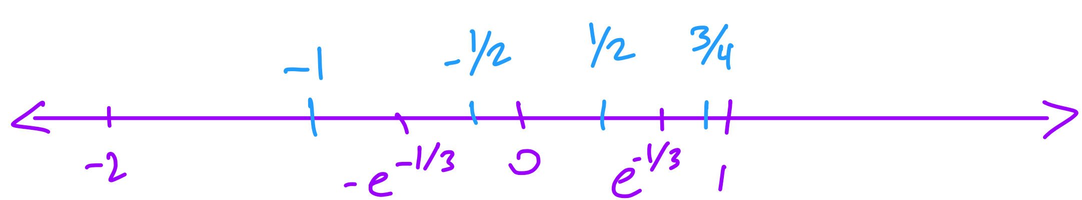
$$x^2 = e^{-2/3}$$

$$x = \pm\sqrt{e^{-2/3}} = \pm e^{-1/3}$$

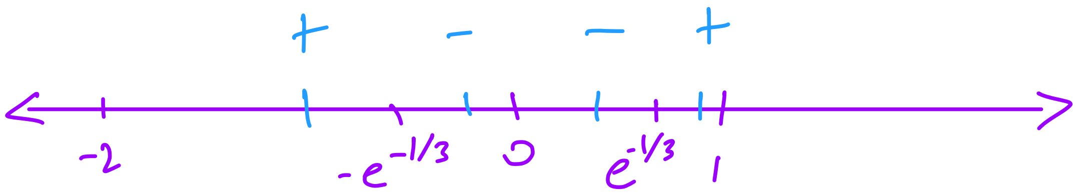
We also have x-values where the derivative is undefined: that's just $x=0$ since $\ln(x^2)$ is only undefined there.

Finally, we have the endpoints: -2 and 1.

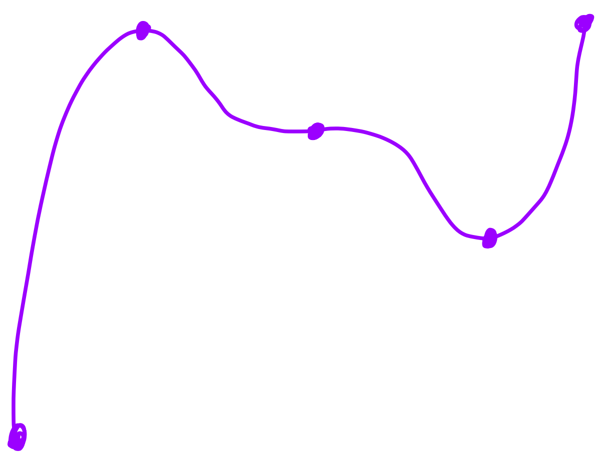
Since the second derivative looks a little gross, let's just use the first derivative test.



Evaluating f' at the blue points, we get the following signs:



In other words, the function looks something like this:



We have that $x = -2$ and $x = e^{-1/3}$ are local minima, while $x = -e^{-1/3}$ and $x = 1$ are local maxima, and $x = 0$ is a saddle point.

c) Now we actually have to find f''.

$$f''(x) = \begin{cases} 6x\ln(x^2) + 3x^2 \cdot \frac{1}{x^2} \cdot 2x + 4x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 6x\ln(x^2) + 10x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

This is zero when $x = 0$ or when

$$6x\ln(x^2) + 10x = 0$$

$$2x(3\ln(x^2) + 5) = 0$$

$$x = 0 \qquad \ln(x^2) = -\frac{5}{3}$$

$$x^2 = e^{-5/3}$$

$$x = \pm\sqrt{e^{-5/3}} = \pm e^{-5/6}$$

d) We just need to compare the local maxima and minima.

minima $\begin{cases} f(-2) \approx -11.09 \\ f(e^{-1/3}) \approx -.245 \end{cases}$ $\leftarrow$ global min

$f(-e^{-1/3}) \approx .245$ $\leftarrow$ global max

$f(1) = 0$

⑥ We have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + (x+h)^2 + (x+h) + 1 - x^3 - x^2 - x - 1}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + 3h^3 + x^2 + 2xh + h^2 + x + h + 1 - x^3 - x^2 - x - 1}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + 3h^3 + 2xh + h^2 + h}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3xh + 3h + 2x + h + 1)$$

$$= 3x^2 + 3x \cdot 0 + 3 \cdot 0 + 2x + 0 + 1$$

$$= 3x^2 + 2x + 1 \quad \checkmark$$