Name: ______________________

Midterm 2

Math 252

Winter 2022

You have 50 minutes to complete this exam and turn it in. **You may use a scientific calculator and a single side of an 8.5×11 inch piece of paper for handwritten notes, but no other resources.** When you're finished, first check your work if there is time remaining, then scan the exam and upload it. If you have a question, don't hesitate to ask — I just may not be able to answer it.

1. (24 points) Multiple choice. You don't need to show any work.

a) (8 points) Suppose $y = f(x)$, and that the graph of f is rotated about the x-axis. Then

A) the shell method integrates with respect to y and the disk method with respect to x.

B) the shell method integrates with respect to x and the disk method also with respect to x.

C) the shell method integrates with respect to x and the disk method with respect to y.

D) the shell method integrates with respect to y and the disk method also with respect to y.

b) (8 points) It takes 3 J of work to stretch a spring a total of 1 meter from rest. How much work does it take to compress it 2 meters from rest?

A) 3 J.

B) 6 J.

C) 9 J.

D) 12 J.

$\int_0^1 kx\,dx = 3$

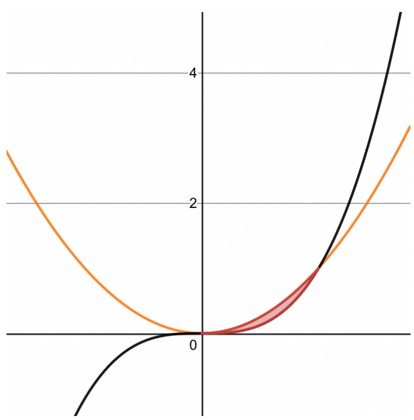
$\left[\frac{kx^2}{2}\right]\Big|_0^1 = 3$

$\frac{k}{2} = 3$

$k = 6$

$\int_0^2 6x\,dx = [3x^2]\Big|_0^2 = 12$

c) (8 points) Which of the following integrals calculates the area bounded by $f(x) = x^2$ and $g(x) = x^3$?



A) $\int_0^1 \cancel{(x^3 - x^2)}\ dx.$

B) $\int_0^1 (\sqrt{y} - \sqrt[3]{y})\ dy.$

C) $\int_0^1 \cancel{(x^2 + x^3)}\ dx.$

D) $\int_0^1 (\sqrt[3]{y} - \sqrt{y})\ dy.$

x^2 on top

$\sqrt[3]{y}$ to the right

2. (32 points) Short answer.

a) (8 points) Find the area between $f(x) = x^2$ and $g(x) = x - x^2$.

$f = g$

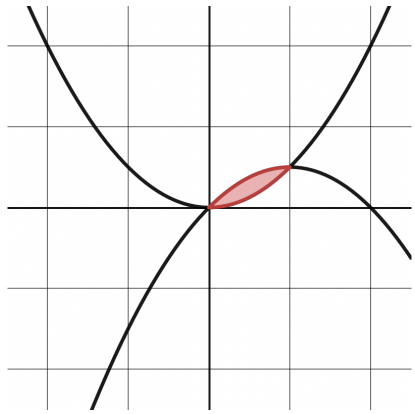
$x^2 = x - x^2$

$2x^2 - x = 0$

$x(2x-1) = 0$

$x = 0,\ x = \frac{1}{2}$

$\int_0^{1/2} x - x^2 - x^2\, dx$



$= \int_0^{1/2} x - 2x^2\, dx$

$= \left[\frac{x^2}{2} - \frac{2x^3}{3}\right]\Big|_0^{1/2}$

$= \frac{1/4}{2} - \frac{1/4}{3}$

$= \frac{1}{24}$

b) (8 points) Let $f(x) = 3x^2$. Set up the integrals to find the volume of the solid given by rotating the graph of f on $[0,3]$ about the x-axis, using **both** the disk and shell methods. Don't solve either of the integrals.

Disk: $\int_0^3 \pi\,(3x^2)^2\, dx$

Shell: $x = \sqrt{\frac{y}{3}}$, so $\int_0^{27} 2\pi y \sqrt{\frac{y}{3}}\, dy$

c) (8 points) The density of a bar is given by $\rho(x) = \ln(x)$ for $x = e$ to $x = e^2$. Find the mass of the bar.

$= \int_e^{e^2} \ln(x)\, dx$

$= \left[x\ln(x) - x\right]\Big|_e^{e^2}$

$= (2e^2 - e^2) - (e - e)$

$= e^2$

d) (8 points) Find the surface area of the solid created by revolving the graph of $y = x^3$ on $[0,2]$ about the x-axis.

$f'(x) = 3x^2$

$f'(x)^2 = 9x^4$

$\int_0^2 2\pi\, x^3 \sqrt{1 + 9x^4}\, dx$

$u = 1 + 9x^4$

$du = 36x^3\, dx$

$= \int_0^2 2\pi\sqrt{u} \cdot \frac{1}{36}\, du$

$= \frac{\pi}{18}\int_0^2 u^{1/2}\, du$

$= \frac{\pi}{18}\left[\frac{2}{3}u^{3/2}\right]\Big|_0^2$

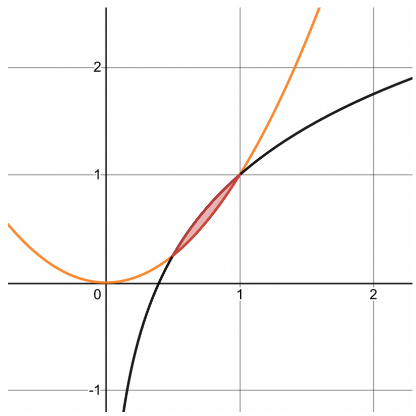
$= \frac{\pi}{18}\left[\frac{2}{3}(1+9x^4)\right]\Big|_0^2$

$= \frac{\pi}{27}(145 - 1)$

$= \frac{16\pi}{3}$

3. (32 points) Longer problems that require setting up and solving integrals. Half the credit is for the set-up and half for the solving.

a) (16 points) The functions $f(x) = x^2$ and $g(x) = \frac{3}{\ln(16)}\ln(x) + 1$ intersect at $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $(1,1)$ and bound a region, as shown below.



Find the volume of the solid of revolution given by rotating the region about the y-axis. You may use any method you like. **You may leave your answer in evaluation notation: e.g. $[x^2]|_0^1$. No integrals should be present in your final answer.**

Disk and shell are both viable, but shell is easier.

$$\int_{1/2}^{1} 2\pi x\left(\frac{3}{\ln(16)}\ln(x) + 1 - x^2\right)dx$$

$$= \int_{1/2}^{1}\left(\frac{6\pi}{\ln(16)}x\ln(x) + 2\pi x - 2\pi x^3\right)dx$$

$$= \underbrace{\int_{1/2}^{1}\frac{6\pi}{\ln(16)}x\ln(x)\,dx}_{} + \int_{1/2}^{1}(2\pi x - 2\pi x^3)\,dx$$

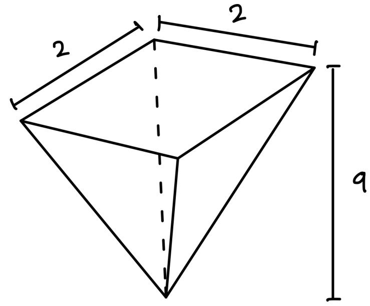
$$u = \ln(x) \qquad v = \frac{3\pi}{\ln(16)}x^2$$

$$du = \frac{1}{x}dx \qquad dv = \frac{6\pi}{\ln(16)}x\,dx$$

$$= \left[\frac{3\pi}{\ln(16)}x^2\ln(x) - \int\frac{3\pi}{\ln(16)}x\,dx + \pi x^2 - \pi\frac{x^4}{2}\right]\Bigg|_{1/2}^{1}$$

$$= \left[\frac{3\pi}{\ln(16)}x^2\ln(x) - \frac{3\pi}{2\ln(16)}x^2 + \pi x^2 - \pi\frac{x^4}{2}\right]\Bigg|_{1/2}^{1}.$$

b) (16 points) A tank in the shape of a square pyramid has height 9 meters and a base with side length 2 meters. It's filled up to 5 meters with a substance that has weight density 2000 $\frac{N}{m^3}$. Find the work done by pumping the liquid out.



A cross-section at height y is a square with side length b. By similar triangles, $\frac{b}{2} = \frac{y}{9}$, so $b = \frac{2}{9}y$. Then the area is $\frac{4}{89}y^2$, and so the integral becomes

$$\int_0^5 (2000)\left(\frac{4}{89}y^2\right)(9-y)\,dy$$

$$= 2000\left(\frac{4}{89}\right)\int_0^5 (9y^2 - y^3)\,dy$$

$$= 2000\left(\frac{4}{89}\right)\left[3y^3 - \frac{y^4}{4}\right]\Big|_0^5$$

$$= 2000\left(\frac{4}{89}\right)\left(375 - \frac{625}{4}\right).$$