Name: ______________________

Midterm 2 **Key**

Math 256

Spring 2023

You have 50 minutes to complete this exam and turn it in. You may use a 3x5 inch two-sided handwritten index card and a scientific calculator, but not a graphing one, and you may not consult the internet or other people. If you have a question, don't hesitate to ask — I just may not be able to answer it. **Enough work should be shown that there is no question about the mathematical process used to obtain your answers.**

You should expect to spend about one minute per question per point it's worth — there are 50 points possible on the exam and 50 minutes total.

Part I (9 points) Multiple choice. You don't need to show your work.

1. (3 points) Only one of the following four matrices is invertible. Which one is it?

A) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

B) $\mathbf{B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$. This one — the others either aren't square or have determinant zero.

C) $\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

D) A 3×3 matrix $\mathbf{D}$ with eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, corresponding to eigenvalues 1, 0, and -6.

2. (3 points) To solve a nonhomogeneous DE, we can use the method of undetermined coefficients or the method of variation of parameters. Which of the following DEs can **only** be solved with variation of parameters?

A) $y'' + 2y = e^t$.

B) $y'' + 4y' = \sin(t) + \cos(2t)$.

C) $y''' - y' + y = t^2 e^{-3t}$.

D) $y' + y = \csc(t)$. Unlike the others, it's not a combination of sin, cos, exponentials, and polynomials.

3. (3 points) Matrix $\mathbf{A}$ has 4 rows and 3 columns, and matrix $\mathbf{C}$ has 4 rows and 2 columns. For the product $\mathbf{AB} = \mathbf{C}$ to be defined, what must be the shape of $\mathbf{B}$?

A) 3×2. We need $\mathbf{B}$ to have 3 rows so that $\mathbf{AB}$ is defined, and for it to equal $\mathbf{C}$, $\mathbf{B}$ needs to have 2 columns.

B) 4×4.

C) 3×3.

D) There is no shape that makes the product defined.

Part II (12 points) Short-answer. Explain your reasoning and show your work for each question.

1. (4 points) One of the eigenvectors of $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ is $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 3$. What is $\mathbf{A}^8\mathbf{v}$ (i.e. the result of multiplying $\mathbf{A}$ by $\mathbf{v}$ eight times)?

Since $\mathbf{A}\mathbf{v} = \lambda\mathbf{v} = 3\mathbf{v}$, $\mathbf{A}^8\mathbf{v} = 3^8\mathbf{v} = \begin{bmatrix} 0 \\ 3^8 \\ 3^8 \end{bmatrix}$.

2. (4 points) Give an example of a differential equation whose general solution is

$$y = c_1 \cos(2t) + c_2 \sin(2t) + c_3 t \cos(2t) + c_4 t \sin(2t).$$

We're looking for a characteristic equation with roots of $\pm 2i$, each of which is repeated. One such polynomial is $(r^2 + 4)^2 = r^4 + 8r^2 + 16$, so the corresponding DE is $y^{(4)} + 8y'' + 16y = 0$.

3. (4 points) Let $\mathbf{B} = \begin{bmatrix} 5 & -4 \\ -6 & 5 \end{bmatrix}$. Find $\mathbf{B}^{-1}$.

Applying the 2×2 inverse formula, $\mathbf{B}^{-1} = \frac{1}{25-24} \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 5 \end{bmatrix}$.

Part III (29 points) More involved questions with multiple parts.

1. (14 points) Let's look at a few variations of a DE.

a) (2 points) Find the general solution to $y'' - 4y = 0$.

The characteristic equation is $r^2 - 4 = 0$, so $r = \pm 2$. The general solution is then $y = c_1 e^{2t} + c_2 e^{-2t}$.

b) (6 points) Find a particular solution to $y'' - 4y = e^{2t}$ using undetermined coefficients.

Our solution is of the form $Y = Ae^{2t}$, but since that's one of the fundamental solutions, we'll change it to $Y = Ate^{2t}$. Then $Y' = A\left(e^{2t} + 2te^{2t}\right)$ and $Y'' = A\left(2e^{2t} + 2e^{2t} + 4te^{2t}\right)$. Plugging it in,

$$\begin{aligned} Y'' - 4Y &= e^{2t} \\ A\left(4e^{2t}\right) &= e^{2t} \\ A &= \frac{1}{4}, \end{aligned}$$

so the general solution is $y = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4}te^{2t}$.

c) (6 points) Find a particular solution to $y'' - 4y = e^{2t}$ (the same as in part b) using variation of parameters.

The Wronskian of the two fundamental solutions is just $W = e^{2t}\frac{d}{dt}\left[e^{-2t}\right] - e^{-2t}\frac{d}{dt}\left[e^{2t}\right] = -4$. Running our fundamental solutions through the variation of parameters formula gives us

$$\begin{aligned} Y &= -y_1 \int \frac{y_2 g(t)}{W[y_1, y_2]}\,dt + y_2 \int \frac{y_1 g(t)}{W[y_1, y_2]}\,dt \\ &= -e^{2t} \int \frac{1}{-4}\,dt + e^{-2t} \int \frac{e^{4t}}{-4}\,dt \\ &= \frac{1}{4} t e^{2t} - \frac{1}{16} e^{2t}. \end{aligned}$$

so the general solution is $y = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4} t e^{2t} - \frac{1}{16} e^{2t}$. We can omit that last term if we like, since it's taken care of by the c_1 term.

2. (15 points) Consider the system of equations

$$3x - 2y = 5$$
$$4x - 3y = 8.$$

a) (2 points) Write this system as an augmented matrix of the form $\left[\begin{array}{c|c} \mathbf{A} & \mathbf{b} \end{array}\right]$.

$$\left[\begin{array}{cc|c} 3 & -2 & 5 \\ 4 & -3 & 8 \end{array}\right].$$

b) (5 points) Solve for $\mathbf{x}$ by row-reducing $\mathbf{A}$. Clearly indicate every row operation.

$$\left[\begin{array}{cc|c} 3 & -2 & 5 \\ 4 & -3 & 8 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 3 & -2 & 5 \\ 1 & -1 & 3 \end{array}\right] \quad \mathbf{r_2} -= \mathbf{r_1}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 3 & -2 & 5 \end{array}\right] \quad \text{swap}\mathbf{r_1},\ \mathbf{r_2}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -4 \end{array}\right] \quad \mathbf{r_2} -= 3\mathbf{r_1}$$

$$\left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -4 \end{array}\right] \quad \mathbf{r_1} += \mathbf{r_2}$$

In total, $x = -1$ and $y = -4$.

c) (3 points) Find the eigenvalues of $\mathbf{A}$. We solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which results in $(3-\lambda)(-3-\lambda)-(-2)(4) = 0$ or just $\lambda^2 - 1 = 0$. The roots are $\lambda = \pm 1$.

d) (5 points) Find the corresponding eigenvectors of $\mathbf{A}$. For $\lambda = 1$, we have

$$\left[\begin{array}{cc|c} 3-1 & -2 & 0 \\ 4 & -3-1 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 4 & -4 & 0 \end{array}\right],$$

which has a solution of $\begin{bmatrix} t \\ t \end{bmatrix}$, so the first eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. For $\lambda = -1$,

$$\left[\begin{array}{cc|c} 3+1 & -2 & 0 \\ 4 & -3+1 & 0 \end{array}\right]$$
$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 4 & -2 & 0 \end{array}\right],$$

which has a solution of $\begin{bmatrix} t \\ 2t \end{bmatrix}$, so the second eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.