(1) No: $\begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix} \notin X$, so X is not closed under addition.

(2) Closure under addition: let $c_1\cos(x)+c_2\sin(x)$, $c_3\cos(x)+c_4\sin(x) \in X$. Then

$c_1\cos(x)+c_2\sin(x)+c_3\cos(x)+c_4\sin(x) =$

$(c_1+c_3)\cos(x) + (c_2+c_4)\sin(x) \in X$.

Closure under scalar multiplication: let $c_1\cos(x)+c_2\sin(x) \in X$ and $c \in \mathbb{R}$. Then

$c(c_1\cos(x)+c_2\sin(x)) = (cc_1)\cos(x)+(cc_2)\sin(x) \in X$.

Zero vector: $0\cos(x) + 0\sin(x) = 0 \in X$.

(3) Closure under addition: let $\begin{bmatrix}a & 0\\0 & b\end{bmatrix}, \begin{bmatrix}c & 0\\0 & d\end{bmatrix} \in X$.

Then $\begin{bmatrix}a & 0\\0 & b\end{bmatrix} + \begin{bmatrix}c & 0\\0 & d\end{bmatrix} = \begin{bmatrix}a+c & 0\\0 & b+d\end{bmatrix} \in X$.

Closure under scalar multiplication: let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in X$ and $c \in \mathbb{R}$. Then $c\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$
$= \begin{bmatrix} ac & 0 \\ 0 & bc \end{bmatrix} \in X$.

Zero vector: with $a = b = 0$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in X$.

④ No: with $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$, $T \in X$,
but $2T \notin X$, since $2T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

⑤ No: the zero transformation $0: \mathbb{R}^3 \to \mathbb{R}^2$
has $\ker 0 = \mathbb{R}^3 \neq \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$.

⑥ Addition: $T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} x_1 + x_2 + y_1 + y_2 + z_1 + z_2 \\ 2x_1 + 2x_2 - y_1 - y_2 - z_1 - z_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + y_1 + z_1 \\ 2x_1 - y_1 - z_1 \end{bmatrix} + \begin{bmatrix} x_2 + y_2 + z_2 \\ 2x_2 - y_2 - z_2 \end{bmatrix}$$

$$= T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$$

Scalar multiplication: $T\left(c\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$

$$= T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix}\right)$$

$$= \begin{bmatrix} cx + cy + cz \\ 2cx - cy - cz \end{bmatrix}$$

$$= c\begin{bmatrix} x + y + z \\ 2x - y - z \end{bmatrix}$$

$$= cT\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$$

Kernel: $\begin{bmatrix} x + y + z \\ 2x - y - z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -3 & -3 & 0 \end{array}\right] \quad \vec{r}_2 -= 2\vec{r}_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \vec{r}_2 \times = -\frac{1}{3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad \vec{r}_1 -= \vec{r}_2$$

$x = 0$, $y = -t$, $z = t$, i.e.

$$\ker T = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

7 Addition: $T(p+q) = \frac{d^2}{dx^2}\left[p(x) + q(x)\right]$

$$= p''(x) + q''(x)$$

$$= T(p) + T(q)$$

Scalar multiplication: $T(c\,p(x)) = \frac{d^2}{dx^2}\left[c\,p(x)\right]$

$$= c\,p''(x)$$

$$= cT(p).$$

Kernel: all polynomials $p(x)$ with $p''(x)=0$, so linear functions $mx+b$. Therefore $\ker T = \operatorname{span}\{1, x\}$.

(8) Addition: $T(f+g) = (f+g)(0) = f(0)+g(0)$
$= T(f)+T(g)$

Scalar multiplication: $T(cf) = (cf)(0) = c\,f(0)$
$= c\,T(f)$.

Kernel: all functions $f: \mathbb{R} \to \mathbb{R}$ with $f(0)=0$.

(9) No: $T\left(2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 4$, but

$2\,T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2$.

(10) Addition: $T(S+R) = (S+R)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$= S\left(\begin{bmatrix}1\\1\end{bmatrix}\right) + R\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= T(S) + T(R)$$

Scalar multiplication: $T(cS) = (cS)\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$

$$= c\,S\left(\begin{bmatrix}1\\1\end{bmatrix}\right)$$

$$= cT(S).$$

Kernel: all $S \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ with

$$S\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix}$$

corresponds to matrices, i.e.

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{matrix}a+b=0\\c+d=0\end{matrix} \implies \begin{matrix}b=-a\\d=-c\end{matrix}$$

so the matrices are of the form

$$\begin{bmatrix} a & -a \\ c & -c \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}.$$

Therefore, $\ker T = \text{span}\{T_1, T_2\}$, where

$$T_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x-y \\ 0 \end{bmatrix} \text{ and } T_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x-y \end{bmatrix}.$$