Midterm 2 Worksheet

Math 251

1. Let $f(x)$ be a differentiable function. Which of the following is true?

A. every critical point of f is also an inflection point. Ex: x^2 at 0

B. every inflection point of f is also a critical point. Ex: $\sin(x)$ at 0

C. every saddle point of f is also an inflection point.

D. every inflection point of f is also a saddle point. Ex: x^4 at 0

2. Find $\dfrac{d}{dx}\left[\dfrac{x^2\sin(x^2)-\log_2(x)}{\tan^{-1}(x)}\right]$.

$$= \frac{\frac{d}{dx}\left[x^2\sin(x^2)-\log_2(x)\right]\cdot\tan^{-1}(x) - \left(x^2\sin(x^2)-\log_2(x)\right)\frac{d}{dx}\left[\tan^{-1}(x)\right]}{\left(\tan^{-1}(x)\right)^2}$$

$$= \frac{\left(2x\sin(x^2)+x^2\cos(x^2)\cdot 2x - \frac{1}{x\ln(2)}\right)\tan^{-1}(x) - \left(x^2\sin(x^2)-\log_2(x)\right)\cdot\frac{1}{1+x^2}}{\left(\tan^{-1}(x)\right)^2}$$

Can simplify, but not required

3. Suppose you're selling concert tickets. You have to pay the band \$20 for every one you print, and you sell them at an increasing price: if you print n tickets, you sell them for $p(n) = n^{1.1}$ each. How much profit do you make by selling the nth ticket?

$C(n) = 20n$

$R(n) = n^{1.1} \cdot n = n^{2.1}$

$P(n) = R(n) - C(n) = n^{2.1} - 20n$

$MP(n) = P'(n) = 2.1n^{1.1} - 20$

this is the profit from selling ticket #n+1.

to get n, we just replace n with n-1.

$MP(n-1) = 2.1(n-1)^{1.1} - 20$

4. Using the Inverse Function Theorem, show that $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$.

IFT tells us about $(f^{-1})'(x)$, so we need

$f^{-1}(x) = \ln(x)$. Then $f(x) = e^x$, so $f'(x) = e^x$.

$$\frac{d}{dx}[\ln(x)] = (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(\ln(x))} = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$$

5. Let $f(x) = x^3 - 3x^2$ on $[-1,3]$. Find and classify all extrema of f, and identify the global maximum and minimum.

First, find the critical points.

$f'(x) = 3x^2 - 6x$

$f'(x) = 0$: $3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ $\quad$ $x = 2$

Also throw in points where $f'(x)$ is undefined, but there are none.

Finally, add endpoints: $x = -1$ and $x = 3$

In total, the critical points are $-1, 0, 2,$ and 3.

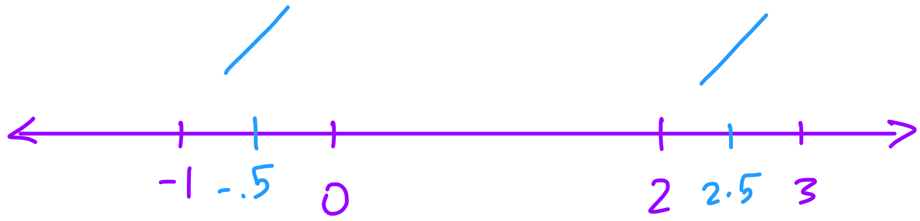
Use the second derivative test on the critical points with $f'(x) = 0$; i.e. 0 and 2

$f''(x) = 6x - 6$

$f''(0) = -6 < 0$: local max

$f''(2) = 12 > 0$: local min

Use the first derivative test to classify the rest:



$f'(-.5) = 3.75 > 0$

$f'(2.5) = 3.75 > 0$

By the diagram above, -1 is a local min and 3 is a local max

Now find the global extrema

Compare maxima: $f(0) = 0$ and $f(3) = 0$, so <ins>either one</ins> can be considered the global max

The same thing actually happens for the minima: $f(-1) = -4$ and $f(2) = -4$.

6. The graph of $x\sin\left(\frac{\pi}{2}y\right) = y$ contains the point $(1,1)$. Find the equation of the tangent line there.

Implicitly differentiate:

$$\sin\left(\frac{\pi}{2}y\right) + x\cos\left(\frac{\pi}{2}y\right)\cdot\frac{\pi}{2}\frac{dy}{dx} = \frac{dy}{dx}$$

$$x\cos\left(\frac{\pi}{2}y\right)\cdot\frac{\pi}{2}\frac{dy}{dx} - \frac{dy}{dx} = -\sin\left(\frac{\pi}{2}y\right)$$

$$\frac{dy}{dx}\left(x\cos\left(\frac{\pi}{2}y\right)\cdot\frac{\pi}{2} - 1\right) = -\sin\left(\frac{\pi}{2}y\right)$$

$$\frac{dy}{dx} = \frac{-\sin\left(\frac{\pi}{2}y\right)}{x\cos\left(\frac{\pi}{2}y\right)\cdot\frac{\pi}{2} - 1}$$

When $x=1$ and $y=1$, $\frac{dy}{dx} = \frac{-\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2} - 1}$

$$= \frac{-1}{0-1} = 1$$

Tangent line: $y = f(a) + f'(a)(x-a) = \underbrace{1}_{y} + \underbrace{1}_{\frac{dy}{dx}}(\underbrace{x-1}_{x})$