Midterm 1 Review

Math 253

Section 1

1. Let $a_n = 2n$, $b_n = n^2 + 1$, and $c_n = (n-1)!$. Write out the first five terms of the sequences (a_n), (b_n), and (c_n).

2. Let $a_1 = 1$ and $a_n = 2a_{n-1}$. Write out the first five terms of (a_n) and find an explicit formula.

3. The Fibonacci numbers are a very famous sequence that begin with $F_1 = 1$ and $F_2 = 1$. After this, every Fibonacci number is the sum of the previous two. Find a recursive formula for the Fibonacci sequence (F_n).

4. Find recursive formulas for each of the three sequences in the previous problem.

5. Let (d_n) be the sequence defined by $d_1 = 0$ and $d_n = d_{n-1} + \frac{1}{2^n}$. Find an explicit formula for (d_n) by writing out terms and finding a pattern, and verify your answer by plugging it back into the recurrence.

6. Let (a_n) be the arithmetic sequence defined by $a_1 = 1$ and $a_n = a_{n-1} - 3$. Find an explicit formula for (a_n) and verify it's correct.

7. Let (b_n) be the geometric sequence defined by $b_1 = 2$ and $b_n = 2b_{n-1}$. Find an explicit formula for (b_n) and verify it's correct.

8. The sequence (a_n) defined by

$$a_n = 1 + \left(-\frac{1}{2}\right)^n$$

converges to 1. Find N such that $|a_n - 1| < \varepsilon$ for $\varepsilon = 1$, $\varepsilon = .1$, and $\varepsilon = .001$.

9. The sequence (b_n) defined by

$$b_n = \frac{1}{n}$$

converges to 0. Find N such that $|b_n| < \varepsilon$ for $\varepsilon = 1$, $\varepsilon = .1$, and $\varepsilon = .001$.

10. Find the limits of the following sequences.

a) $(n!)$.

b) $\left(1 - \frac{1}{n}\right)$.

c) $\left(\sin\left(\frac{n}{2^n}\right)\right)$.

11. Find $\lim\left(\frac{\cos(n)}{n}\right)$.

12. We already understand the behavior of the sequence (r^n) when $r \geq 0$: it converges to zero for $0 \leq r < 1$, to one for $r = 1$, and it diverges for $r > 1$. Use the Squeeze theorem to classify the behavior of (r^n) for negative r.

13. Let (a_n) be a sequence defined recursively by $a_1 = 2$ and

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{2a_{n-1}}$$

for $n \geq 2$.

First, show that (a_n) converges using the MCT. Then find the number it converges to. (Hint: take the limit of both sides of the recurrence.)

Section 2

1. Evaluate the following sums.

a) $\sum_{n=1}^{5} n$.

b) $\sum_{i=-2}^{2} \frac{1}{i^2+1}$.

c) $\sum_{k=0}^{\infty} 2$.

2. For each of the following series, write out the first five terms. Then find the first, third, and fifth partial sums.

a) $\sum_{n=0}^{\infty} n$.

b) $\sum_{m=1}^{\infty} \frac{1}{m}$.

c) $\sum_{k=1}^{\infty} \frac{2}{3^k}$.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{2^n}$ or show it diverges.

4. Evaluate the following series or show they don't converge.

a) $\sum_{i=0}^{\infty} (-1)^i$.

b) $\sum_{m=1}^{\infty} \frac{2}{3^m}$.

c) $\sum_{n=1}^{\infty} \frac{1}{n}$. Note: this is called the **Harmonic series**, and is quite famous. To get a handle on its behavior, try grouping the terms in this manner: put the first and second terms by themselves, then put the third and fourth together, then the fifth through the eighth together, then the 9th through the 16th, and so on, doubling the size of each group. See if you can quantify how large each group is based on its last term.

4. Evaluate $\sum_{n=1}^{\infty} \frac{1}{2^n}$ by using properties of geometric series.

5. Evaluate the following series.

a) $\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^{k+1}}$.

b) $\sum_{m=1}^{\infty} e^m$.

6. Evaluate the sum

$$\sum_{n=1}^{\infty} \left(\cos\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n+1}\right) \right).$$

7. The series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is actually telescoping, even if it doesn't look like it at first glance. Use partial fractions to split apart $\frac{1}{n(n+1)}$, and then evaluate the sum using properties of telescoping series.

Section 3

1. Apply the divergence test to the following series and clearly state what conclusion you can draw.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

b) $\sum_{j=2}^{\infty} \ln(j)$.

c) $\sum_{i=0}^{\infty} \frac{1}{i+1} \cos(i)$.

2. Determine if the following series converge. Clearly state what conclusions you can draw, if any.

a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

b) $\sum_{j=1}^{\infty} \sin(j)$.

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$.

3. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^3}$ to within .01 of its actual value.

4. Estimate $\sum_{n=1}^{\infty} \frac{1}{n^4}$ to within .01 of its actual value.