Protection Against Reconstruction and Its Applications in Private Federated Learning

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Abstract

Federated learning has become an exciting direction for both research and practical training of models with user data. Although data remains decentralized in federated learning, it is common to assume that the model updates are sent in the clear from the devices to the server. Differential privacy has been proposed as a way to ensure the model remains private, but this does not address the issue that model updates can be seen on the server, and lead to leakage of user data. Local differential privacy is one of the strongest forms of privacy protection so that each individual's data is privatized. However, local differential privacy, as it is traditionally used, may prove to be too stringent of a privacy condition in many high dimensional problems, such as in distributed model fitting. We propose a new paradigm for local differential privacy by providing protections against certain adversaries. Specifically, we ensure that adversaries with limited prior information cannot reconstruct, with high probability, the original data within some prescribed tolerance. This interpretation allows us to consider larger privacy parameters. We then design (optimal) DP mechanisms in this large privacy parameter regime. In this work, we combine local privacy protections along with central differential privacy to present a practical approach to do model training privately. Further, we show that these privacy restrictions maintain utility in image classification and language models that is comparable to federated learning without these privacy restrictions.

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1 Introduction

New, more powerful computational hardware and access to substantial amounts of data has made fitting accurate models for image classification, text translation, physical particle prediction, astronomical observation, and other predictive tasks possible with accuracy that was previously completely infeasible [11, 7, 49]. In many modern applications, data comes from measurements on small-scale devices with limited computation and communication ability—remote sensors, fitness monitors—making fitting large scale predictive models both computationally and statistically challenging. Moreover, as more modes of data collection and computing move to peripherals—watches, power-metering, internet-enabled home devices, and even lightbulbs—issues of privacy become ever more salient.

Such large-scale data collection motivates substantial work. Stochastic gradient methods are now the de facto approach to large-scale model-fitting [68, 18, 60, 28], and recent work of McMahan et al. [54] describes systems (which they term federated learning) for aggregating multiple stochastic model-updates from distributed mobile devices. Yet even if only updates to a model are transmitted, leaving all user or participant data on user-owned devices, it is easy to compromise the privacy of users [37, 57]. To see why this issue arises, consider any generalized linear model based on a data vector x, target y, and with loss of the form $\ell(\theta; x, y) = \phi(\langle \theta, x \rangle, y)$. Then trivially one has $\nabla_{\theta} \ell(\theta; x, y) = c \cdot x$ for $c \in \mathbb{R}$, hence, a scalar multiple of the user's clear data x—a clear compromise of privacy. In this paper, we describe an approach to fitting such large-scale models both privately and practically.

A natural approach to addressing the risk of information disclosure in such federated learning scenarios is to use differential privacy [35], which provides strong guarantees on the risk of compromising any user's data. To implement differential privacy, one defines a mechanism M, a randomized mapping from a sample \boldsymbol{x} of data points to some space \mathcal{Z} , which is ε -differentially private if

$$\frac{\mathbb{P}(M(\boldsymbol{x}) \in S)}{\mathbb{P}(M(\boldsymbol{x}') \in S)} \le e^{\varepsilon} \tag{1}$$

for all samples \boldsymbol{x} and \boldsymbol{x}' differing in at most one entry, i.e. if one element is present in one sample and absent in the other. Because of its strength and protection properties, differential privacy (and its variants) are now essentially the standard privacy definition in data analysis and machine learning [22, 32, 23]. Nonetheless, implementing such an algorithm presumes a level of trust between users and a centralized data analyst, which may be undesirable or even untenable, as the data analyst has essentially unfettered access to a user's data. Other approaches to protecting individual updates is to use secure multiparty computation (SMC), sometimes in conjunction with differential privacy protections; see, for example, Bonawitz et al. [17]. Traditional approaches to SMC require substantial communication and computation, making them untenable for large-scale data collection schemes, and Bonawitz et al. [17] address a number of these, though individual user communication and computation still increases with the number of users submitting updates and requires multiple rounds of communication, which may be unrealistic when estimating models from peripheral devices.

An alternative to these approaches is to use *locally private* algorithms [65, 36, 31], in which an individual keeps his or her data private even from the data collector. Such scenarios are natural in distributed (or federated) learning scenarios, where individuals provide data from their devices [53, 8] but wish to maintain privacy. In our learning context, where a user has data $x \in \mathcal{X}$ that he or she wishes to remain private, a randomized mechanism $M: \mathcal{X} \to \mathcal{Z}$ is ε -local differentially private

if for all $x, x' \in \mathcal{X}$ and sets $S \subset \mathcal{Z}$,

$$\frac{\mathbb{P}(M(x) \in S)}{\mathbb{P}(M(x') \in S)} \le e^{\varepsilon}.$$
 (2)

Roughly, a mechanism satisfying inequality (2) guarantees that even if an adversary knows that the initial data is one of x or x', the adversary cannot distinguish them given an outcome Z (the probability of error must be at least $1/(1+e^{\varepsilon})$) [66]. Taking as motivation this testing definition, the "typical" recommendation for the parameter ε is to take ε as a small constant [66, 35, 32].

Local privacy protections provide numerous benefits: they allow easier compliance with regulatory strictures, reduce risks (such as hacking) associated with maintaining private data, and allow more transparent protection of user privacy, because private data never leaves an individual's device in the clear. Yet substantial work in the statistics, machine learning, and computer science communities has shown that local differential privacy and its relaxations cause nontrivial challenges for learning systems. Indeed, Duchi et al. [30, 31] show that in a minimax (worst case population distribution) sense, learning with local differential privacy must suffer a degradation in sample complexity that scales linearly in the dimension of the problem, at least for privacy parameters $\varepsilon = O(1)$. Duchi and Ruan [27] develop this approach further, arguing that a worst-case analysis is too conservative and may not accurately reflect the difficulty of problem instances one actually encounters, so that an instance-specific theory of optimality is necessary. In spite of this instance-specific optimality theory for locally private procedures—that is, fundamental limits on learning that apply to the particular problem at hand—Duchi and Ruan's results suggest that local notions of privacy as currently conceptualized restrict some of the deployments of learning systems.

We consider an alternative conceptualization of privacy protections and the concomitant guarantees from differential privacy and the likelihood ratio bound (2). The testing interpretation of differential privacy suggests that when $\varepsilon \gg 1$, the definition (2) is almost vacuous. We argue that, at least in large-scale learning scenarios, this testing interpretation is unrealistic, and allowing mechanisms with $\varepsilon \gg 1$ may provide meaningful privacy protections. Rather than providing protections against arbitrary inferences, we wish to provide protection against accurate reconstruction of an individual's data x. In the large scale learning scenarios we consider, an adversary given a random observation x likely has little prior information about x, so that protecting only against reconstructing (functions of) x under some assumptions on the adversary's prior knowledge allows substantially improved model-fitting.

The formal setting for our problems is as follows. Given data $X_i \in \mathcal{X}$, i = 1, ..., n, drawn from a distribution P, we seek a parameter vector $\theta \in \mathbb{R}^d$ that will have good future performance when evaluated under loss $\ell : \Theta \times \mathcal{X} \to \mathbb{R}_+$, that is, solve the population risk minimization problem

$$\underset{\theta \in \Theta}{\text{minimize}} \ \{ L(\theta) := \mathbb{E}[\ell(\theta, X)] \}. \tag{3}$$

The standard approach [40] to such problems is to construct the empirical risk minimizer $\widehat{\theta}_n = \arg\min_{\theta} \frac{1}{n} \sum_{i=1}^{n} \ell(\theta, X_i)$. In this paper, however, we consider the stochastic minimization problem (3) while providing both local privacy to individual data X_i and—to maintain the satisfying guarantees of centralized differential privacy (1)—stronger guarantees on the global privacy of the output $\widehat{\theta}_n$ of our procedure. With this as motivation, we describe our contributions at a high level. As above, we argue that large local privacy (2) parameters, $\varepsilon \gg 1$, still provide reasonable privacy protections. We develop new mechanisms and privacy protecting schemes that more carefully reflect the statistical aspects of problem (3), which we demonstrate are (in a sense) theoretically optimal. A substantial portion of this work is devoted to providing practical procedures while providing meaningful local privacy guarantees, which currently do not exist. Consequently, we provide extensive empirical results that demonstrate the tradeoffs between private federated

(distributed) learning scenarios, showing that it is possible to achieve performance comparable to federated learning procedures without privacy safeguards.

1.1 Our approach and results

We propose and investigate a two-pronged approach to model fitting under local privacy. Motivated by the difficulties associated with local differential privacy we discuss in the immediately subsequent section, we reconsider the threat models (or types of disclosure) in locally private learning. Instead of considering an adversary with access to all data, we consider "curious" onlookers, who wish to decode individuals' data but have little prior information on them. Formalizing this (as we discuss in Section 2) allows us to consider substantially relaxed values for the privacy parameter ε , sometimes even scaling with the dimension of the problem, while still providing protection. While this brings us away from the standard guarantees of differential privacy, we can still provide privacy guarantees for the type of onlookers we consider.

This model of privacy is natural in federated learning scenarios [53, 8], where we wish to perform distributed model training. Here, by providing protections against curious onlookers, a company can protect its users against reconstruction of their data by, for example, internal employees. By encapsulating this more relaxed local privacy model within a broader central differential privacy layer, we can still provide satisfactory privacy guarantees to users, protecting them against strong external adversaries as well.

We make several contributions to achieve these goals. In Section 2, we describe a model for curious adversaries, with appropriate privacy guarantees, and demonstrate how (for these curious adversaries) it is still nearly impossible to accurately reconstruct individuals' data. We then detail a prototypical private federated learning system in Section 3. In this direction, we develop new (near-optimal) privacy mechanisms for privatization of high-dimensional vectors in unit balls (Section 4). These mechanisms yield substantial improvements over the minimax optimal schemes Duchi et al. [31, 30] develop, providing order of magnitude improvements over classical noise addition schemes, and we provide a unifying theorem showing the asymptotic behavior of a stochastic-gradient-based private learning scheme in Section 4.4. We conclude our development in Section 5 with several large-scale distributed model-fitting problems, showing how the tradeoffs we make allow for practical procedures. Our approaches allow substantially improved model-fitting and prediction schemes; in situations where local differential privacy with smaller privacy parameter fails to learn a model at all, we can achieve models with performance near non-private schemes.

1.2 Why local privacy makes model fitting challenging

To motivate our approaches, we discuss why local privacy causes some difficulties in a classical learning problem. Duchi and Ruan [27] help to elucidate the precise reasons for the difficulty of estimation under ε -local differential privacy, and we can summarize them heuristically here, focusing on the machine learning applications of interest. To do so, we begin with a brief detour through classical statistical learning and the usual convergence guarantees that are (asymptotically) possible [63].

Consider the population risk minimization problem (3), and let $\theta^* = \operatorname{argmin}_{\theta} L(\theta)$ denote its minimizer. We perform a heuristic Taylor expansion to understand the difference between $\widehat{\theta}_n$ and

 θ^{\star} . Indeed, we have

$$0 = \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\widehat{\theta}_n, X_i) = \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\theta^*, X_i) + \frac{1}{n} \sum_{i=1}^{n} (\nabla^2 \ell(\theta^*, X_i) + \text{error}_i)(\widehat{\theta}_n - \theta^*)$$
$$= \frac{1}{n} \sum_{i=1}^{n} \nabla \ell(\theta^*, X_i) + (\nabla^2 L(\theta^*) + o_P(1))(\widehat{\theta}_n - \theta^*),$$

(for error_i an error term in the Taylor expansion of ℓ), which—when carried out rigorously—implies

$$\widehat{\theta}_n - \theta^* = \frac{1}{n} \sum_{i=1}^n \underbrace{-\nabla^2 L(\theta^*)^{-1} \nabla \ell(\theta^*; X_i)}_{=:\psi(X_i)} + o_P(1/\sqrt{n}). \tag{4}$$

The influence function ψ [63] of the parameter θ measures the effect that changing a single observation X_i has on the resulting estimator $\widehat{\theta}_n$.

All (regular) statistically efficient estimators asymptotically have the form (4) [63, Ch. 8.9], and typically a problem is "easy" when the variance of the function $\psi(X_i)$ is small—thus, individual observations do not change the estimator substantially. In the case of *local* differential privacy, however, as Duchi and Ruan [27] demonstrate, (optimal) locally private estimators typically have the form

$$\widehat{\theta}_n - \theta^* = \frac{1}{n} \sum_{i=1}^n \left[\psi(X_i) + W_i \right] \tag{5}$$

where W_i is a noise term that must be taken so that $\psi(x)$ and $\psi(x')$ are indistinguishable for all x, x'. Essentially, a local differentially private procedure cannot leave $\psi(x)$ small even when it is typically small (i.e. the problem is easy) because it could be large for some value x. In locally private procedures, this means that differentially private tools for typically "insensitive" quantities (cf. [32]) cannot apply, as an individual term $\psi(X_i)$ in the sum (5) is (obviously) sensitive to arbitrary changes in X_i . The consequences of this are striking, and extend even to weakenings of local differential privacy [27]: it makes adaptivity to easy problems essentially impossible for standard ε -locally-differentially private procedures, at least when ε is small, and introduces substantial dimension-dependent penalties in the error $\widehat{\theta}_n - \theta^*$. Thus, to enable high-quality estimates for quantities of interest in machine learning tasks, we explore locally differentially private settings with larger privacy parameter ε .

2 Privacy protections

In developing any statistical machine learning system providing privacy protections, it is important to consider the types of attacks that we wish to protect against. In distributed model fitting and federated learning scenarios, we consider two potential attackers: the first is a curious onlooker who can observe all updates to a model and communication from individual devices, and the second is from a powerful external adversary who can observe the final (shared) model or other information about individuals who may participate in data collection and model-fitting. For the latter adversary, essentially the only effective protection is to use a small privacy parameter in a localized or centralized differentially private scheme [54, 35, 59]. For the curious onlookers, however—for example, internal employees of a company fitting large-scale models—we argue that protecting against reconstruction is reasonable.

2.1 Reconstruction breaches

Abstractly, we work in a setting in which a user or study participant has data X he or she wishes to keep private. Via some process, this data is transformed into a vector W—which may simply be an identity transformation, but W may also be a gradient of the loss ℓ on the datum X or other derived statistic. We then privatize W via a randomized mapping $W \to Z$. An onlooker may then wish to estimate or evaluate some function f on the private data X, $f(X) \in \mathbb{R}^k$. Thus, we have the Markov chain

$$X \to W \to Z$$

and the onlooker, who observes only Z, wishes to estimate f(X). In most scenarios with a curious onlooker, however, if X or f(X) is suitably high dimensional, the onlooker has limited prior information about X, so that relatively little obfuscation is required in the mapping from $W \to Z$.

As a motivating example, consider an image processing scenario. A user has an image X, where $W \in \mathbb{R}^d$ are wavelet coefficients of X (in some prespecified wavelet basis) [52]; without loss of generality, we assume we normalize W to have energy $\|W\|_2 = 1$. Let f(X) be a low-dimensional version of X (say, based on the first 1/8 of wavelet coefficients); then (at least intuitively, and we can make this rigorous) taking Z to be a noisy version of W such that $\|Z - W\|_2 \gtrsim 1$ —that is, noise on the scale of the energy $\|W\|_2$ —should be sufficient to guarantee that the observer is unlikely to be able to reconstruct f(X) to any reasonable accuracy. Moreover, a simple modification of the techniques of Hardt and Talwar [39] shows that for $W \sim \mathsf{Uni}(\mathbb{S}^{d-1})$, and ε -differentially private quantity Z for W satisfies $\mathbb{E}[\|Z - W\|_2] \gtrsim 1$ whenever $\varepsilon \leq d - \log 2$. That is, we might expect that even very large ε provide protections against reconstruction.

With this in mind, let us formalize a reconstruction breach in our scenario. Here, the onlooker (or adversary) has a prior π on $X \in \mathcal{X}$, and there is a known (but randomized) mechanism $M: \mathcal{W} \to \mathcal{Z}, W \mapsto Z = M(W)$. We then have the following definition.

Definition 2.1 (Reconstruction breach). Let π be a prior on \mathcal{X} , and let X, W, Z be generated with Markov structure $X \to W \to Z = M(W)$ for a mechanism M. Let $f: \mathcal{X} \to \mathbb{R}^k$ be the target of reconstruction and $L_{\text{rec}}: \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}_+$ be a loss function. Then an estimator $\hat{v}: \mathcal{Z} \to \mathbb{R}^k$ provides an (α, p, f) -reconstruction breach for the loss L_{rec} if there exists z such that

$$\mathbb{P}\left(L_{\text{rec}}(f(X), \widehat{v}(z)) \le \alpha \mid M(W) = z\right) > p. \tag{6}$$

If for every estimator $\widehat{v}: \mathcal{Z} \to \mathbb{R}^k$,

$$\sup_{z \in \mathcal{Z}} \mathbb{P}\left(L_{\text{rec}}(f(X), \widehat{v}(z)) \le \alpha \mid M(W) = z\right) \le p,$$

then the mechanism M is (α, p, f) -protected against reconstruction for the loss L_{rec} .

Key to Definition 2.1 is that it applies uniformly across all possible observations z of the mechanism M—there are no rare breaches of privacy. This requires somewhat stringent conditions on mechanisms and also disallows recent relaxed privacy definitions [33, 21, 59].

2.2 Separated private mechanisms

We will enforce (analogues of) central differential privacy (1) in our overall model-fitting system; here, McMahan et al. [54] show that certain noise-adding strategies that we discuss in the sequel allow efficient modeling.

¹With that said, we ignore measurability issues here; in our setting, all random variables are mutually absolutely continuous and are generated by regular conditional probability distributions, the conditioning on z in Def. 2.1 has no issues [43].

The more challenging part of our development, then, is to consider mechanisms for providing privacy in the local model. Motivated by the difficulties we outline in Section 1.2 for locally private model fitting—in particular, that estimating the magnitude of a gradient or influence function is challenging—we consider mechanisms that transmit information W by privatizing a pair (U, R), where $Z_1 = M_1(U)$ and $Z_2 = M_2(R)$ are the privatized versions of U and R.

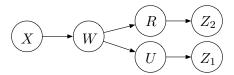


Figure 1: Markovian graphical structure between data X and privatized pair (Z_1, Z_2)

Continuing our running thread of federated learning scenarios, when W is a vector, we split W into its direction $U = W/\|W\|_2$ and magnitude $R = \|W\|_2$. This splitting allows us to develop mechanisms that more carefully reflect the challenges of transmitting vectors that may have varying scales—important, especially, in situations like the one we described in Section 1.2.

More succinctly, we consider mechanisms $M: \mathcal{U} \times \mathcal{R} \to \mathcal{Z}_1 \times \mathcal{Z}_2$ as the pair $M(U,R) = (M_1(U), M_2(R)) = (Z_1, Z_2)$; we would like to guarantee that the pair (Z_1, Z_2) contains little information about W (and so in turn, little about X). As we use this separated scheme repeatedly, we make the following definition.

Definition 2.2 (Separated Differential Privacy). A pair of mechanisms M_1, M_2 mapping from $\mathcal{U} \times \mathcal{R}$ to $\mathcal{Z}_1 \times \mathcal{Z}_2$ (i.e. a channel with the Markovian structure of Fig. 1) is $(\varepsilon_1, \varepsilon_2)$ -separated differentially private if M_1 is ε_1 -locally differentially private and M_2 is ε_2 -locally differentially private (Eq. (2)).

The basic composition properties of differentially private channels [32] guarantee that $M = (M_1, M_2)$ is $(\varepsilon_1 + \varepsilon_2)$ -locally differentially private. Hence, any such mechanism enjoys the typical benefits of differentially private algorithms—group privacy, closure under post-processing, and composition rules [32].

2.3 Protecting against reconstruction

We can now develop guarantees against reconstruction based on our previous definitions. We begin with a simple claim.

Lemma 2.1. Let the pair of mechanisms (M_1, M_2) satisfy $(\varepsilon_1, \varepsilon_2)$ -separated differential privacy over (U, W, R) generated by $U \leftarrow W \rightarrow R$ and $X \rightarrow W$ as in Fig. 1, and let V = f(X) for a measurable function f. Then for any $\pi \in \mathcal{P}$ on \mathcal{X} and measurable sets $A, A' \subset f(\mathcal{X})$, the posterior distribution $\pi_V(\cdot \mid z_1, z_2)$ (for $z_1 = M_1(U)$ and $z_2 = M_2(R)$ satisfies

$$\frac{\pi_V(A \mid z_1, z_2)}{\pi_V(A' \mid z_1, z_2)} \le e^{\varepsilon_1 + \varepsilon_2} \frac{\pi_V(A)}{\pi_V(A')}.$$

The result is immediate by Bayes' rule.

Based on Lemma 2.1, we can show the following result, which guarantees that difficulty of reconstruction of a signal is preserved under private mappings.

Lemma 2.2. Assume that the prior π_0 on X is such that for a tolerance α , probability $p(\alpha)$, target function f, and loss L_{rec} , we have

$$\mathbb{P}(L_{\rm rec}(f(X), v_0) \le \alpha) \le p(\alpha)$$

for all v_0 . Let (M_1, M_2) satisfy $(\varepsilon_1, \varepsilon_2)$ -separated differential privacy. Then the pair (M_1, M_2) is $(\alpha, e^{\varepsilon_1 + \varepsilon_2} \cdot p(\alpha), f)$ -protected against reconstruction for the loss L_{rec} .

Proof Lemma 2.1 immediately implies that for any estimator \hat{v} based on $Z = (M_1(U), M_2(R))$, we have for any realized z and V = f(X)

$$\mathbb{P}(L_{\text{rec}}(V, \widehat{v}(z)) \le \alpha \mid Z = z) = \int \mathbb{1} \left\{ L_{\text{rec}}(v, \widehat{v}(z)) \le \alpha \right\} d\pi_V(v \mid z)$$
$$\le e^{\varepsilon_1 + \varepsilon_2} \int \mathbb{1} \left\{ L_{\text{rec}}(v, \widehat{v}(z)) \le \alpha \right\} d\pi_V(v).$$

The final quantity is $e^{\varepsilon_1+\varepsilon_2}\mathbb{P}(L_{\text{rec}}(f(X),v_0)\leq \alpha)\leq e^{\varepsilon_1+\varepsilon_2}p(\alpha)$ for $v_0=\widehat{v}(z)$, as desired.

Let us now provide a more explicit example of loss and reconstruction that is natural in the distributed learning scenarios we consider. We assume that $W = X \in \mathbb{R}^d$, and for $k \leq d$, consider a matrix $A \in \mathbb{R}^{k \times d}$ with orthonormal rows, so that $AA^T = I$ and $A^TA \in \mathbb{R}^{d \times d}$ is a projection matrix. We consider the problem of reconstructing

$$f_A(x) = \frac{Ax}{\|Ax\|_2} = \frac{Au}{\|Au\|_2} \text{ for } u = x/\|x\|_2.$$
 (7)

Thus, the adversary seeks to reconstruct a k-dimensional projection of the vector x. For example, if x is an image or other signal, A may be the first k rows of the standard Fourier transform matrix, so that we seek low frequency information about x. (In a wavelet scenario, this may be the first level of the wavelet hierarchy.) In these cases, reconstructing Ax is enough for an adversary to get a general sense of the private data, and protecting against reconstruction is more challenging for small k.

Now, for a prior π on X, let π_V be the induced prior on $V = f_A(X) \in \mathbb{R}^k$, and let π_{uni} be the uniform prior π_{uni} on \mathbb{S}^{k-1} . We define

$$\mathcal{P}_A(\rho_0) := \left\{ \pi \text{ over } X \text{ s.t. } \log \frac{d\pi_V(v)}{d\pi_{\text{uni}}(v)} \le \rho_0, \text{ for } v \in \mathbb{S}^{k-1} \right\}.$$
 (8)

We use the ℓ_2 -distance on the sphere as our loss, $L_{\text{rec}}(u,v) = \|u/\|u\|_2 - v/\|v\|_2\|_2$ (when $v \neq 0$, otherwise setting $L_{\text{rec}}(u,v) = \sqrt{2}$). For V uniform and $v_0 \in \mathbb{S}^{k-1}$, we have $\mathbb{E}[\|V - v_0\|_2^2] = 2$, so that thresholds of the form $\alpha = \sqrt{2-2a}$ with a small are the most natural to consider in the reconstruction breaches (6). The following proposition demonstrates that locally differentially private mechanisms protect against reconstruction (and as an immediate consequence, that any separated differentially private scheme, Def. 2.2, does as well).

Proposition 1. Let M be ε -locally differentially private (2) and $k \geq 4$. Let $\pi \in \mathcal{P}_A(\rho_0)$ as in Eq. (8), and let $a \in [0,1]$. Then M is $(\sqrt{2-2a},p(a),f_A)$ -protected against reconstruction for

$$p(a) = \begin{cases} \exp\left(\varepsilon + \rho_0 + \frac{k}{2} \cdot \log(1 - a^2)\right) & \text{if } a \in [0, 1/\sqrt{2}] \\ \exp\left(\varepsilon + \rho_0 + \frac{k-1}{2} \cdot \log(1 - a^2) - \log(2a\sqrt{k})\right) & \text{if } a \in [\sqrt{2/k}, 1]. \end{cases}$$

Simplifying this slightly and rewriting, assuming the reconstruction \hat{v} takes values in \mathbb{S}^{k-1} , we have

$$\mathbb{P}(\left\|f(X) - \widehat{v}(Z)\right\|_2 \leq \sqrt{2 - 2a} \mid Z = z) \leq \exp\left(-\frac{ka^2}{2}\right) \exp(\varepsilon + \rho_0)$$

for any f of the form f(x) = Ax/||Ax|| and $a \le 1/\sqrt{2}$. That is, unless ε or ρ_0 are of the order of k, the probability of obtaining reconstructions better than (nearly) random guessing is extremely low.

Proof Let $Y \sim \text{Uni}(\mathbb{S}^{k-1})$ and $v_0 \in \mathbb{S}^{k-1}$. We then have

$$||Y - v_0||_2^2 = 2 \cdot (1 - \langle Y, v_0 \rangle).$$

We collect a number of more or less standard facts on the uniform distribution on \mathbb{S}^{k-1} in Appendix C, which we reference frequently. Using Lemma C.1 and Eq. (29), we have for all $v_0 \in \mathbb{S}^{k-1}$ and $a \in [0, 1/\sqrt{2}]$ that

$$\mathbb{P}_{\pi_{\text{uni}}}(\|Y - v_0\|_2 < \sqrt{2 - 2a}) = \mathbb{P}(\langle Y, v_0 \rangle > a) \le (1 - a^2)^{k/2}$$

Because $V = f_A(X)$ has prior π_V such that $d\pi_V/d\pi_{\rm uni} \leq e^{\rho_0}$, we obtain

$$\mathbb{P}_{\pi_V} (\|V - v_0\|_2 < \sqrt{2 - 2a}) \le e^{\rho_0} \cdot (1 - a^2)^{k/2}.$$

Then Lemma 2.2 gives the first result of the proposition.

When the desired accuracy is higher (i.e. $a \in [\sqrt{2/k}, 1]$), Lemma C.1 and Eq. (29), with our assumed ratio bound between π_V and $\pi_{\rm uni}$, imply

$$\mathbb{P}_{\pi_V}(\|V - v_0\|_2 \le \sqrt{2 - 2a}) \le e^{\rho_0} \mathbb{P}_{\pi_{\text{uni}}}(\|Y - v_0\|_2 < \sqrt{2 - 2a}) \le e^{\rho_0} \frac{(1 - a^2)^{\frac{k-1}{2}}}{2a\sqrt{k}}.$$

Applying Lemma 2.2 completes the proof.

2.4 Other privacy definitions and existing mechanisms

In our separated mechanisms, we decompose a vector $w \in \mathbb{R}^d$ into its unit direction $u \in \mathbb{S}^{d-1}$ and magnitude r > 0. In the sequel, we design, under appropriate conditions, optimal differentially private mechanisms acting on both u and r, which allow us to provide strong convergence guarantees in different stochastic optimization and learning problems. Given the numerous relaxations of differential privacy [59, 33, 21], however—many of which permit noise addition schemes with simple Gaussian noise addition as well as a number of the benefits of differential privacy—a natural idea is to simply add noise satisfying one of these weaker definitions to u. First, these weakenings can never actually protect against a reconstruction breach for all possible observations z (Definition 2.1)—they can only protect conditional on the observation z lying in some appropriately high probability set (cf. [13, Thm. 1]). Second, as we discuss now, most standard mechanisms add more noise than ours. As we show in the sequel (Section 4), our ε -differentially private mechanisms release Z such that $\mathbb{E}[Z \mid u] = u$ and $\mathbb{E}[\|Z - u\|_2^2 \mid u] \lesssim d \cdot \max\{\varepsilon^{-1}, \varepsilon^{-2}\}$, which we show is optimal.

The first (standard) approach in differential privacy and its weakenings is to add noise via M(u)=u+V, where V is mean zero noise independent of u. For utility, we then consider the mean squared error $\mathbb{E}[\|M(u)-u\|_2^2]=\mathbb{E}[\|V\|_2^2]$. Because two vectors $u,u'\in\mathbb{S}^{d-1}$ can satisfy $\|u-u'\|_1=2\sqrt{d}$, the standard Laplace mechanism [35] adds noise vector V with $V_i\overset{\mathrm{iid}}{\sim} \mathsf{Lap}(\frac{2\sqrt{d}}{\varepsilon})$, yielding $\mathbb{E}[\|M(u)-u\|_2^2]=\frac{4d^2}{\varepsilon^2}$. The ℓ_2 -extension of the Laplace mechanism [35] adds noise V with density $p(v)\propto \exp(\frac{-\varepsilon\|v\|_2}{2})$, so that $\|V\|_2\sim \mathsf{Gamma}(d,\frac{2}{\varepsilon})$ marginally and $\mathbb{E}[\|M(u)-u\|_2^2]>\frac{4d^2}{\varepsilon^2}$. These are evidently of the wrong order of magnitude for $\varepsilon\leq d$.

Duchi et al. [30] provide an alternative sampling strategy that enjoys substantially better dimension dependence. In this case, we sample

$$V = \begin{cases} \text{uniform on } \{v \in \mathbb{S}^{d-1} \mid \langle v, u \rangle \ge 0\} & \text{with probability } \frac{e^{\varepsilon}}{1 + e^{\varepsilon}} \\ \text{uniform on } \{v \in \mathbb{S}^{d-1} \mid \langle v, u \rangle < 0\} & \text{otherwise.} \end{cases}$$
(9)

To debias the vector V, one sets $Z = d\sqrt{\pi} \cdot (\frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}) \cdot \frac{\Gamma(\frac{d-1}{2}+1)}{\Gamma(\frac{d}{2}+1)} \cdot V$; as $\Gamma(\frac{d-1}{2}+1)/\Gamma(\frac{d}{2}+1) \approx 1/\sqrt{d}$, this mechanism satisfies

 $\frac{(e^{\varepsilon}+1)^2}{(e^{\varepsilon}-1)^2}d \lesssim \mathbb{E}[\|Z-u\|_2^2] \lesssim \frac{(e^{\varepsilon}+1)^2}{(e^{\varepsilon}-1)^2}d.$

This linear scaling in dimension is in fact optimal (and better than the noise addition strategies detailed above); unfortunately, it does not converge to zero as ε grows, and relaxing privacy does not give a more accurate estimator beyond O(d) squared ℓ_2 -error.

As we mention above, alternative strategies using weakenings of differential privacy do not provide the strong anti-reconstruction guarantees possible with pure differential privacy. Nonetheless, we discuss three briefly in turn: Rényi [59], concentrated [33, 21], and approximate differential privacy [34]. In the local privacy case where $\varepsilon \lesssim 1$ —the high privacy regime—Duchi and Ruan [27] show that none of these weakenings offers any benefits in terms of statistical power. Each of these is easiest to describe with a small amount of additional notation. Let $Q(\cdot \mid x)$ denote the distribution of the randomized mechanism M(x) conditional on x.

Rényi differential privacy The α -Rényi divergence between distributions P_0 and P_1 is

$$D_{\alpha}\left(P_{0}\|P_{1}\right) := \frac{1}{\alpha - 1}\log \int \left(\frac{dP_{0}}{dP_{1}}\right)^{\alpha}dP_{1},$$

where $\lim_{\alpha\to 1} D_{\alpha}(P_0||P_1) = D_{\rm kl}(P_0||P_1)$. Then a mechanism with distribution $Q(\cdot \mid u)$ is (α, ε) -Rényi-differentially private [59] if $D_{\alpha}(Q(\cdot \mid x)||Q(\cdot \mid x')) \leq \varepsilon$ for all x, x'. For $Z = M(x) \sim Q(\cdot \mid x)$, if π denotes a prior belief on the possible values of x and $\pi(\cdot \mid Z)$ the posterior belief given the observation Z, then Rényi differential privacy is equivalent to the condition that

$$\mathbb{E}_{Z \sim Q(\cdot \mid x)} \left[\left(\frac{\pi(x \mid Z) / \pi(x' \mid Z)}{\pi(x) / \pi(x')} \right)^{\alpha} \right]^{1/\alpha} \le e^{\varepsilon},$$

that is, the prior and posterior odds ratios do not change (on average) much. Clearly ε -differential privacy provides (α, ε) -Rényi privacy for all α . For these weakenings, the basic mechanism for privatizing a single vector in the sphere is the Gaussian mechanism M(u) = u + V for $V \sim \mathsf{N}(0, \sigma^2 I)$. Using that $D_{\alpha}(\mathsf{N}(u, \sigma^2 I) \| \mathsf{N}(u', \sigma^2 I)) = \frac{\alpha}{2\sigma^2} \| u - u' \|_2^2$, we see that the choice $\sigma^2 = \frac{2\alpha}{\varepsilon}$ is sufficient for (α, ε) -Rényi privacy, yielding errors scaling as $\mathbb{E}[\| M(u) - u \|_2^2] = \frac{2\alpha d}{\varepsilon}$. It is not completely clear for which values of α Rényi-differential privacy provides appropriate protections, but for $\alpha \gg 1$, this is again evidently of larger magnitude than our differentially private schemes.

Concentrated differential privacy The concentrated variants [33, 21] of differential privacy are more stringent than Rényi differential privacy—they roughly require $(\alpha, \alpha\rho)$ -Rényi differential privacy (for some ρ) simultaneously for all α , and so they suffer similar drawbacks.

Approximate differential privacy The mechanism with conditional $Q(\cdot \mid x)$ satisfies (ε, δ) -approximate differential privacy if $Q(Z \in A \mid x) \leq e^{\varepsilon}Q(Z \mid x') + \delta$ for all x, x'. (Here one thinks of δ as being a very small value, typically sub-polynomial in the sample size.) In the case that we wish to privatize vectors $u \in \mathbb{S}^{d-1}$, the Gaussian mechanism is (to within constants) optimal [62], adding noise $V \sim \mathsf{N}(0, \frac{C \log \frac{1}{\delta}}{\varepsilon^2}I)$ for a numerical constant C and yielding error $\mathbb{E}[\|M(u) - u\|_2^2] \lesssim \frac{d \log \frac{1}{\delta}}{\varepsilon^2}$. When $\varepsilon \lesssim \log \frac{1}{\delta}$, this error too is of higher order than the mechanisms we develop, while providing weaker privacy guarantees. (See also the discussion of McSherry [56].)

3 Applications in federated learning

Our overall goal is to implement federated learning, where distributed units send private updates to a shared model to a centralized location. Recalling our population risk (3), basic distributed learning procedures (without privacy) iterate as follows [16, 25, 20]:

- 1. A centralized parameter θ is distributed among a batch of b workers, each with a local sample X_i , i = 1, ..., b.
- 2. Each worker computes an update $\Delta_i := \theta_i \theta$ to the model parameters.
- 3. The centralized procedure aggregates $\{\Delta_i\}_{i=1}^b$ into a global update Δ and updates $\theta \leftarrow \theta + \Delta$.

The prototypical situation is to use a stochastic gradient method to implement steps 1–3, so that $\Delta_i = -\eta \nabla \ell(\theta, X_i)$ for some stepsize η in step 2, and $\Delta = \frac{1}{b} \sum_{i=1}^{b} \Delta_i$ is simply the average of the stochastic gradients at each sample X_i in step 3.

In our private distributed learning context, we elaborate steps 2 and 3 so that each provides privacy protections: in the local update step 2, we use locally private mechanisms to protect individual's private data X_i —satisfying Definition 2.1 on protection against reconstruction breaches. Then in the central aggregation step 3, we apply centralized differential privacy mechanisms to guarantee that any models θ communicated to users in the broadcast 1 is globally private. The overall feedback loop then provides meaningful privacy guarantees, as a user's data is never transmitted clearly to the centralized server, and strong centralized privacy guarantees mean that the final and intermediate parameters θ provide no sensitive disclosures.

3.1 A private distributed learning system

Let us more explicitly describe the implementation of a distributed learning system. The outline of our system is similar to the development of Duchi et al. [30, 31, Sec. 5.2] and the system that McMahan et al. [55] outline; we differ in that we allow more general updates and privatize individual users' data before communication, as the centralized data aggregator may not be completely trusted. We decompose the system into five components: (1) transmission of the model to users, (2) computing local updates, (3) transmission of the privatized update, (4) centralized aggregation of the updates, and (5) aggregate model privatization.

Transmission of central model The central aggregator maintains a global model parameter $\theta \in \mathbb{R}^d$ as well as a collection of hyperparameters \mathcal{H} , which govern the behavior of the individual (distributed) updates as well as the aggregation strategies. The parameter θ and \mathcal{H} are distributed to a subset $\mathcal{B} \subset [N]$ of the worker nodes (e.g. a device, user, or individual observation holder). This subset is a random subset of expected size qN, where N is the total number of potential workers and $q \in (0,1)$ is the subsampling rate, where user i is selected independently with probability q.

Local Training On the local workers, we leverage the user data to update the central model parameters θ . We consider a generic method Update that denotes the update rule on each worker. Assuming that worker i has a local sample $\mathbf{x}_i = \{x_{i,1}, \dots, x_{i,m}\}$, each $i \in \mathcal{B}$ performs

$$\theta_i \leftarrow \texttt{Update}\left(\boldsymbol{x}_i, \theta; \mathcal{H}\right)$$
.

To make this abstract description somewhat more concrete, we note that there are many possible updates for solving the problem (3). Perhaps the most popular rule is to apply a gradient update, where for a stepsize $\eta_i \in \mathcal{H}$ we apply

$$ext{Update}\left(oldsymbol{x}_i, heta; \mathcal{H}
ight) := heta - \eta_i rac{1}{m} \sum_{j=1}^m
abla \ell(heta, x_{i,j}).$$

An alternative is to use stochastic proximal-point-type updates [10, 46, 44, 15, 24], which update

$$\mathtt{Update}(\pmb{x}_i,\theta_0;\mathcal{H}) := \operatorname*{argmin}_{\theta} \left\{ \frac{1}{m} \sum_{j=1}^m \ell(\theta,x_{i,j}) + \frac{1}{2\eta_i} \left\|\theta - \theta_0\right\|_2^2 \right\}.$$

Transmission of privatized updates After completing the local update to θ_i , we define the local difference $\Delta_i = \theta_i - \theta$ and transmit this to the aggregator using a separated mechanism (Definition 2.2). In particular, we privatize both the direction $\Delta_i / \|\Delta_i\|$ and magnitude $\|\Delta_i\|$ using the privacy preserving mechanisms we detail in Section 4: letting Z_1 be an unbiased (private) estimate of $\Delta_i / \|\Delta_i\|$ and Z_2 an unbiased estimate of $\|\Delta_i\|$, we return $\widehat{\Delta}_i = Z_1 Z_2$. Our development of mechanisms in the sequel shows the possible norms and privacy levels here.

Centralized aggregation Given the collection of privatized updates $\{\widehat{\Delta}_i\}_{i\in\mathcal{B}}$, we aggregate by projecting each update onto an ℓ_2 -ball of radius S>0. Letting $\phi_S(v)=v\min\{S/\|v\|_2,1\}$ denote this projection, we define the aggregated update

$$\widehat{\Delta} \leftarrow \frac{1}{qN} \sum_{i \in \mathcal{B}} \phi_S(\Delta_i). \tag{10}$$

The presence of the projection is relatively innocuous—for online stochastic gradient settings, where $\Delta_i = \eta \nabla \ell(\theta, x_i)$ for a stepsize η that decreases to zero as the iterative procedure continuous, we eventually have (with probability 1) that $\|\nabla \ell(\theta, x_i)\|_2 \leq r/\eta$ for any r > 0 as $\eta \downarrow 0$. The key is that the truncation to a radius S allows centralized privacy protections:

Aggregate Model Privatization Our local privatization with separated differential privacy provides safeguards against reconstruction breaches from adversaries with diffuse prior knowledge; to provide stronger global protections, we incorporate centralized (approximate) differential privacy at the centralized model. The update (10) has ℓ_2 -sensitivity at most S/(qN) (modifying a single update Δ_i can cause $\widehat{\Delta}$ to change by at most S/(qN) in ℓ_2 -distance). Consequently, addition of appropriate Gaussian noise, as described above, allows us to guarantee (ε , δ)-approximate differential privacy [34]. In particular, at the tth global update we modify the shared parameter via

$$\theta^{(t+1)} = \theta^{(t)} + \widehat{\Delta} + Z \quad \text{for} \quad Z \sim \mathsf{N}\left(0, \zeta^2 \frac{S^2}{q^2 N^2} I_d\right),\tag{11}$$

²A formal argument using Borel-Cantelli is straightforward but tedious; we omit this.

so that $\frac{S^2}{q^2N^2}$ reflects the ℓ_2 -sensitivity. One must choose the parameter ζ to enforce the desired privacy guarantees after a sufficient number of rounds; in our experimental work, we use the "moments accountant" analysis of Abadi et al. [6] to guarantee centralized privacy.

3.2 Asymptotic Analysis

To provide a fuller story and demonstrate the general consequences of our development, we now turn to an asymptotic analysis of our distributed statistical learning scheme for solving problem (3) under locally privatized updates. We ignore the central privatization term (11), as it is asymptotically negligible in our setting. To set the stage, we consider minimizing the population risk $L(\theta) := \mathbb{E}[\ell(\theta, X)]$ using an i.i.d. sample $X_1, \ldots, X_N \stackrel{\text{iid}}{\sim} P$ for some population P.

The simplest strategy in this case is to use the stochastic gradient method, which (using a

The simplest strategy in this case is to use the stochastic gradient method, which (using a stepsize sequence η_t) performs updates

$$\theta^{(t+1)} \leftarrow \theta^{(t)} - \eta_t \nabla \ell(\theta^{(t)}; X_t),$$

where $X_t \stackrel{\text{iid}}{\sim} P$. In this case, under the assumptions that L is \mathcal{C}^2 in a neighborhood of $\theta^* = \operatorname{argmin}_{\theta} L(\theta)$ with $\nabla^2 L(\theta^*) \succ 0$ and that $\mathbb{E}[\|\nabla \ell(\theta; X)\|^2] \leq C(1 + \|\theta - \theta^*\|^2)$ for some $C < \infty$ and all θ , Polyak and Juditsky [61] provide the following result.

Corollary 3.1 (Theorem 2 [61]). Let $\bar{\theta}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} \theta^{(t)}$. Assume the stepsizes $\eta_t \propto t^{-\beta}$ for some $\beta \in (1/2, 1)$. Then under the preceding conditions,

$$\sqrt{T} \left(\bar{\theta}^{(T)} - \theta^{\star} \right) \stackrel{d}{\to} \mathsf{N} \left(0, \nabla^2 L(\theta^{\star})^{-1} \Sigma_{\star} \nabla^2 L(\theta^{\star})^{-1} \right),$$

where $\Sigma_{\star} = \text{Cov}(\nabla \ell(\theta^{\star}; X))$.

We now consider the impact that local privacy has on this result. Let M be a local privatizing mechanism, and define $Z(\theta;x) = M(\nabla \ell(\theta;x))$. We assume that each application of the mechanism M is (conditional on the pair (θ,x)) independent of all others. In this case, the stochastic gradient update becomes

$$\theta^{(t+1)} \leftarrow \theta^{(t)} - \eta_t \cdot Z(\theta^{(t)}; X_t).$$

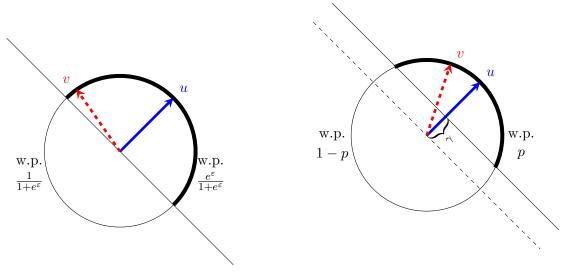
As a consequence of Corollary 3.1, we have

Corollary 3.2. Let the conditions of Corollary 3.1 hold. Assume that $\sup_{\theta \in \Theta} \sup_{x \in \mathcal{X}} \|\nabla \ell(\theta, x)\|_2 \le r_{\max} < \infty$. Assume additionally that the privatization mechanism is (conditional on θ, x) unbiased $\mathbb{E}[Z(\theta; x)] = \nabla \ell(\theta; x)$ and that there exists $C < \infty$ such that $\mathbb{E}[\|Z(\theta; x)\|^2] \le C(1 + \|\theta - \theta^*\|^2)$. Then

$$\sqrt{T} \left(\bar{\theta}^{(T)} - \theta^\star \right) \overset{d}{\to} \mathsf{N} \left(0, \nabla^2 L(\theta^\star)^{-1} \Sigma^{\mathrm{priv}} \nabla^2 L(\theta^\star)^{-1} \right),$$

where $\Sigma^{\text{priv}} := \text{Cov}(Z(\theta^{\star}; X))$.

Key to Corollary 3.2 is that—as we describe in the next section—we can design privatization schemes for which $\Sigma^{\text{priv}} = O(d/\min\{\varepsilon, \varepsilon^2\})\Sigma_{\star}$. This is (in a sense) the "correct" scaling of the problem with dimension and local privacy level ε . This scaling is in contrast to previous work in local privacy (e.g. that by Duchi et al. [31]). In this work, the best such asymptotics (see Section 5.2.1 of [31]) have asymptotic mean-square error of $\|\nabla^2 L(\theta^{\star})^{-1}\|_{\text{op}}^2 r_{\text{max}}^2 d/\min\{\varepsilon^2,1\}$. Already $r_{\text{max}}^2 \geq \text{tr}(\Sigma_{\star})$, and the given operator norms are potentially much worse than the asymptotics Corollary 3.2 reveals. Thus, our ability to develop mechanisms that are (near) optimal in high ε regimes allows model fitting that was impossible with previous mechanisms and guarantees of local privacy.



- (a) Sampling strategy from Duchi et al. [29].
- (b) Sampling strategy for $PrivUnit_2(u; \gamma, \varepsilon')$.

Figure 2. Comparison between sampling strategy from Duchi et al. [30] and PrivUnit₂ given in Algorithm 1 for d=2. The solid-black circle has unit radius. The input unit vector u is then given in solid blue and the returned privatized vector v is given in dashed red. In Plot (a), we sample uniformly from the bold portion of the unit circle with probability $e^{\varepsilon}/(1+e^{\varepsilon})$ and otherwise we sample uniformly from the rest of the unit circle. In Plot (b), we sample uniformly from the bold portion of the unit circle, which is parameterized by $\gamma \in [0,1]$, with probability $p \geq \frac{1}{2}$ and otherwise we sample uniformly from the rest of the unit circle.

4 Private Mechanisms for Releasing High Dimensional Vectors

The main application of the privacy mechanisms we develop is to private (distributed) learning scenarios, where we wish to perform stochastic gradient-like updates to a shared parameter. The key to efficiency in all of these applications is to have accurate estimates of the actual update Δ —frequently simply the stochastic gradient $\nabla \ell(\theta; x)$ —so in this section, we develop new (and optimal) mechanisms for privatizing d-dimensional vectors. We consider two regimes of the highest interest: Euclidean settings [61, 60] (where we wish to privatize vectors belonging to ℓ_2 balls) and the highly non-Euclidean scenarios that arise in high-dimensional estimation and optimization, e.g. mirror descent settings [60, 14] (where we wish to privatize vectors belonging to ℓ_{∞} balls). In each application, we consider mechanisms that release an estimate of the magnitude ||w|| of the vector w and the direction $w/\|w\|$. We thus describe mechanisms for releasing unit vectors in ℓ_2 - and ℓ_{∞} -balls, after which we show how to release the scalar magnitude; the combination allows us to release (optimally accurate) unbiased vector estimates, which we can employ in distributed and online statistical learning problems. We conclude the section with an asymptotic normality result on the convergence of stochastic gradient procedures that unifies our entire development, providing a convergence guarantee that is better than any available for previous locally differentially private learning procedures.

4.1 Privatizing unit ℓ_2 vectors with high accuracy

We begin with the Euclidean case, which arises in most classical applications of stochastic gradient-like methods [69, 61, 60]. In this case, we have a vector $u \in \mathbb{S}^{d-1}$ (i.e. $||u||_2 = 1$), and we wish to

Algorithm 1 Privatized Unit Vector: PrivUnit2

Require: $u \in \mathbb{S}^{d-1}, \ \gamma \in [0,1], \ p \ge \frac{1}{2}$.

Draw random vector V according to the following distribution,

$$V = \begin{cases} \text{uniform on } \{v \in \mathbb{S}^{d-1} \mid \langle v, u \rangle \ge \gamma\} & \text{with probability } p \\ \text{uniform on } \{v \in \mathbb{S}^{d-1} \mid \langle v, u \rangle < \gamma\} & \text{otherwise.} \end{cases}$$
 (12)

Set $\alpha = \frac{d-1}{2}$, $\tau = \frac{1+\gamma}{2}$, and

$$m = \frac{(1 - \gamma^2)^{\alpha}}{2^{d-2}(d-1)} \left[\frac{p}{B(\alpha, \alpha) - B(\tau; \alpha, \alpha)} - \frac{1 - p}{B(\tau; \alpha, \alpha)} \right]$$
(13)

return $Z = \frac{1}{m} \cdot V$

generate an ε -differentially private vector Z with the property that

$$\mathbb{E}[Z \mid u] = u \quad \text{for all } u \in \mathbb{S}^{d-1},$$

where the size $||Z||_2$ is as small as possible to maximize the efficiency in Corollary 3.2.

We modify Duchi et al.'s sampling strategy (9) to develop an optimal mechanism. Our insight is that instead of sampling from the hemispheres $\{v \in \mathbb{S}^{d-1} \mid \langle v,u \rangle \leqslant 0\}$, we can instead sample from spherical caps. That is, we draw our vector V from a cap $\{v \in \mathbb{S}^{d-1} \mid \langle v,u \rangle \geq \gamma\}$ with some probability $p \geq \frac{1}{2}$ or from its complement $\{v \in \mathbb{S}^{d-1} \mid \langle v,u \rangle < \gamma\}$ with probability 1-p, where $\gamma \in [0,1]$ and p are constants we shift to trade accuracy and privacy more precisely. In Figure 2, we give a visual representation comparing the approach of Duchi et al. [30] and our mechanism, which we term $\texttt{PrivUnit}_2$ (see Algorithm 1); in the next subsection we demonstrate the choices of γ and scaling factors to make the scheme differentially private and unbiased. Algorithm 1 takes as input $u \in \mathbb{S}^{d-1}$, $\gamma \in [0,1]$, and $p \geq \frac{1}{2}$ and returns Z satisfying $\mathbb{E}[Z \mid u] = u$. In the algorithm, we require the incomplete beta function

$$B(x; \alpha, \beta) := \int_0^x t^{\alpha - 1} (1 - t)^{\beta - 1} dt \text{ where } B(\alpha, \beta) := B(1; \alpha, \beta) = \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)}.$$

For completeness, in Appendix D we show how to approximately sample conditional vectors on the surface of a hypersphere for PrivUnit₂.

In the remainder of this subsection, we describe the privacy preservation and utility (unbiasedness and small variance) properties of Algorithm 1.

4.1.1 Privacy analysis

Most importantly, Algorithm 1 protects privacy for appropriate choices of the spherical cap level γ . Indeed, the next result shows that $\gamma_{\varepsilon} \approx \sqrt{\varepsilon/d}$ is sufficient to guarantee ε -differential privacy.

Theorem 1. Let $\gamma \in [0,1]$ and $p_0 = \frac{e^{\varepsilon_0}}{1+e^{\varepsilon_0}}$. Then algorithm $PrivUnit_2(\cdot, \gamma, p_0)$ is $(\varepsilon + \varepsilon_0)$ -differentially private whenever $\gamma \geq 0$ is such that

$$\varepsilon \ge \log \frac{1 + \gamma \cdot \sqrt{2(d-1)/\pi}}{\left(1 - \gamma \cdot \sqrt{2(d-1)/\pi}\right)_{\perp}}, \quad i.e. \quad \gamma \le \frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1} \sqrt{\frac{\pi}{2(d-1)}}, \tag{14a}$$

or

$$\varepsilon \ge \frac{1}{2}\log(d) + \log 6 - \frac{d-1}{2}\log(1-\gamma^2) + \log \gamma \quad and \quad \gamma \ge \sqrt{\frac{2}{d}}.$$
 (14b)

Proof We collect a number of results in Appendix C that make the proof more or less straightforward. The random vector $V \in \mathbb{S}^{d-1}$ in Alg. 1 has density (conditional on $u \in \mathbb{S}^{d-1}$)

$$p(v \mid u) \propto \begin{cases} p_0/\mathbb{P}(\langle U, u \rangle \ge \gamma) & \text{if } \langle v, u \rangle \ge \gamma \\ (1 - p_0)/\mathbb{P}(\langle U, u \rangle < \gamma) & \text{if } \langle v, u \rangle < \gamma. \end{cases}$$

By definition of p_0 , for any $u, u' \in \mathbb{S}^{d-1}$, we use that $\gamma \mapsto \mathbb{P}(\langle U, u \rangle < \gamma)$ is increasing in γ to obtain

$$\frac{p(v \mid u)}{p(v \mid u')} \le e^{\varepsilon_0} \cdot \frac{\mathbb{P}(\langle U, u' \rangle < \gamma)}{\mathbb{P}(\langle U, u \rangle \ge \gamma)}.$$
 (15)

It is thus sufficient to prove that the last fraction has upper bound e^{ε} .

We consider two cases in inequality (15). In the first, suppose that $\gamma \geq \sqrt{2/d}$. Then Lemma C.1 implies

$$\frac{\mathbb{P}(\langle U, u' \rangle < \gamma)}{\mathbb{P}(\langle U, u \rangle \geq \gamma)} \leq \frac{6\gamma \sqrt{d}}{(1 - \gamma^2)^{\frac{d-1}{2}}},$$

which is bounded by e^{ε} when $\log 6 + \frac{1}{2} \log d + \log \gamma - \frac{d-1}{2} \log (1-\gamma^2) \le \varepsilon$. In the second case, Lemma C.3 implies

$$\frac{\mathbb{P}(\langle U, u' \rangle < \gamma)}{\mathbb{P}(\langle U, u \rangle \geq \gamma)} \leq \frac{1 + \gamma \sqrt{2(d-1)/\pi}}{1 - \gamma \sqrt{2(d-1)/\pi}},$$

which is bounded by e^{ε} if and only if $\gamma \leq \frac{e^{\varepsilon}-1}{e^{\varepsilon}+1} \sqrt{\pi/(2(d-1))}$.

4.1.2 Utility analysis

We now turn to optimality and utility properties of Algorithm 1. Our first result is an optimality result, which shows that differentially private algorithms must introduce inaccuracy in the vector sampled. To state the strongest version of the lower bound, recall that a set \mathcal{X} is an r-packing in the norm $\|\cdot\|$ if for each $x, x' \in \mathcal{X}$ with $x \neq x'$, then $\|x - x'\| \geq r$. By volume arguments, there are $\frac{1}{2}$ -packings of \mathbb{S}^{d-1} (in ℓ_2) of cardinality at least 2^d (cf. [64]).

Proposition 2. Assume that the mechanism $M: \mathbb{S}^{d-1} \to \mathbb{R}^d$ is ε -differentially private for input $x \in \mathbb{S}^{d-1}$ and releases a vector $M(x) \in \mathbb{R}^d$ such that $\mathbb{E}[M(x)] = x$. Let $\mathcal{X} \subset \mathbb{S}^{d-1}$, $|\mathcal{X}| \geq 2^d$, be a $\frac{1}{2}$ -packing of \mathbb{S}^{d-1} . Then there exists a numerical constant c > 0 such that for all $\varepsilon \in [0, d \log 2 - \log \frac{4}{3}]$,

$$\frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} \mathbb{E}[\|M(x) - x\|_2^2] \ge c \cdot \left(\frac{d}{\varepsilon} \vee \frac{d}{\varepsilon^2}\right).$$

See Appendix B.1 for the proof of this result.

As a consequence of Proposition 2, we can show that Algorithm 1 is order optimal for all privacy levels $\varepsilon \leq d \log 2 - \log \frac{4}{3}$, improving on all previously known mechanisms for (locally) differentially private vector release. To see this, we show that $PrivUnit_2$ indeed produces an unbiased estimator with small norm.

Lemma 4.1. Let $Z = PrivUnit_2(u, \gamma, p)$ for some $u \in \mathbb{S}^{d-1}$, $\gamma \in [0, 1]$, and $p \in [\frac{1}{2}, 1]$. Then $\mathbb{E}[Z] = u$.

See Appendix B.2 for a proof of the lemma.

Letting γ satisfy either of the sufficient conditions (14) in PrivUnit₂(·, γ , p_0), where $p_0 = e^{\varepsilon_0}/(1+e^{\varepsilon_0})$, thus ensures that it is $(\varepsilon + \varepsilon_0)$ -differentially private. With these choices of γ , we then have the following utility guarantee for the privatized vector Z.

Proposition 3. Assume that $0 \le \varepsilon \le d$. Let $u \in \mathbb{S}^{d-1}$ and $p \ge \frac{1}{2}$. Then there exists a numerical constant $c < \infty$ such that if γ saturates either of the two inequalities (14), then $\gamma \gtrsim \min\{\varepsilon/\sqrt{d}, \sqrt{\varepsilon/d}\}$, and the output $Z = PrivUnit_2(u, \gamma, p)$ satisfies

$$||Z||_2 \le c \cdot \sqrt{\frac{d}{\varepsilon} \vee \frac{d}{(e^{\varepsilon} - 1)^2}}.$$

Additionally, $\mathbb{E}[\|Z - u\|_2^2] \lesssim \frac{d}{\varepsilon} \vee \frac{d}{(e^{\varepsilon} - 1)^2}$.

See Appendix B.3 for a proof.

The salient point here is that, evidently, the mechanism of Alg. 1 is order optimal—achieving unimprovable dependence on the dimension d and privacy level ε —and substantially improving the earlier results of Duchi et al. [31], who provide a different mechanism that achieves order-optimal guarantees only when $\varepsilon \lesssim 1$.

4.2 Privatizing unit ℓ_{∞} vectors with high accuracy

We now consider privatization of vectors on the surface of the unit ℓ_{∞} box, $\mathbb{H}^d := [-1,1]^d$, constructing an ε -differentially private vector Z with the property that $\mathbb{E}[Z \mid u] = u$ for all $u \in \mathbb{H}^d$. The importance of this setting arises in very high-dimensional estimation and statistical learning problems, specifically those in which the dimension d dominates the sample size n. In these cases, mirror-descent-based methods [60, 14] have convergence rates for stochastic optimization problems that scale as $\frac{M_{\infty}R_1\sqrt{\log d}}{\sqrt{T}}$, where M_{∞} denotes the ℓ_{∞} -radius of the gradients $\nabla \ell$ and R_1 the ℓ_1 -radius of the constraint set Θ in the problem (3). With the ℓ_2 -based mechanisms in the previous section, we thus address the two most important scenarios for online and stochastic optimization.

Our procedure parallels that for the ℓ_2 case, except that we now use "caps" of the hypercube rather than the sphere. Given $u \in \mathbb{H}^d$, we first round each coordinate randomly to ± 1 to generate $\widehat{u} \in \{-1,1\}^d$ with $\mathbb{E}[\widehat{u} \mid u] = u$. We then sample a privatized vector $V \in \{-1,+1\}^d$ such that with probability $p \geq \frac{1}{2}$ we have $V \in \{v \mid \langle v, \widehat{u} \rangle > \kappa\}$, while with the remaining probability $V \in \{v \mid \langle v, \widehat{u} \rangle \leq \kappa\}$, where $\kappa \in \{0, \dots, d-1\}$. We debias the resulting vector to construct Z satisfying $\mathbb{E}[Z \mid u] = u$. See Algorithm 2.

As in Section 4.1, we divide our analysis into a proof that Algorithm 2 provides privacy and an argument for its utility.

4.2.1 Privacy analysis

We follow a similar analysis to Theorem 1 to give the precise quantity that we need to bound to ensure (local) differential privacy. We defer the proof to Appendix A.1.

Theorem 2. Let
$$\kappa \in \{0, \dots, d-1\}$$
, $p_0 = \frac{e^{\varepsilon_0}}{1+e^{\varepsilon_0}}$ for some $\varepsilon_0 \ge 0$, and $\tau := \frac{\lceil \frac{d+\kappa+1}{2} \rceil}{d}$. If

$$\log\left(\sum_{\ell=0}^{d\tau-1} \binom{d}{\ell}\right) - \log\left(\sum_{\ell=d\tau}^{d} \binom{d}{\ell}\right) \le \varepsilon \tag{17}$$

Algorithm 2 Privatized Unit Vector: PrivUnit_∞

Require: $u \in [-1,1]^d$, $\kappa \in \{0, \dots, d-1\}, p \ge \frac{1}{2}$. Round each coordinate of $u \in [-1,1]^d$ to a corner of \mathbb{H}^d :

$$\widehat{U}_j = \begin{cases} 1 & \text{w.p. } \frac{1+u_j}{2} \\ -1 & \text{otherwise} \end{cases}$$
 for $j \in [d]$.

Draw random vector V via

$$V = \begin{cases} \text{uniform on } \{v \in \{-1, +1\}^d \mid \langle v, \widehat{U} \rangle > \kappa\} & \text{with probability } p \\ \text{uniform on } \{v \in \{-1, +1\}^d \mid \langle v, \widehat{U} \rangle \le \kappa\} & \text{otherwise.} \end{cases}$$
 (16)

Set $\tau = \frac{\left\lceil \frac{d+\kappa+1}{2} \right\rceil}{d}$ and

$$m = p \frac{\binom{d-1}{d\tau - 1}}{\sum_{\ell=\tau \cdot d}^{d} \binom{d}{\ell}} - (1-p) \frac{\binom{d-1}{d\tau - 1}}{\sum_{\ell=0}^{d\tau - 1} \binom{d}{\ell}}$$

return $Z = \frac{1}{m} \cdot V$

then $PrivUnit_{\infty}(\cdot, \kappa, p_0)$ is $(\varepsilon + \varepsilon_0)$ -differentially private.

By approximating (17), we can understand the scaling for κ on the dimension and the privacy parameter ϵ . Specifically, we show that when $\epsilon = \Omega(\log(d))$, setting $\kappa \approx \sqrt{\varepsilon}d$ guarantees ε differential privacy; similarly, for any $\varepsilon = O(1)$, setting $\kappa \approx \varepsilon \sqrt{d}$ gives ε -differential privacy.

Corollary 4.1. Assume that $d, \kappa \in \mathbb{Z}$ are both even and let $p_0 = e^{\varepsilon_0}/(1+e^{\varepsilon_0})$. If $0 \le \kappa < \sqrt{3/2d+1}$ and

$$\varepsilon \ge \log\left(1 + \kappa \cdot \sqrt{\frac{2}{3d+2}}\right) - \log\left(1 - \kappa \cdot \sqrt{\frac{2}{3d+2}}\right),$$
 (18)

then $PrivUnit_{\infty}(\cdot, \kappa, p_0)$ is $(\varepsilon + \varepsilon_0)$ -DP. Let $\kappa_2 = \kappa + 2$. Then if

$$\varepsilon \ge \frac{1}{2}\log(2) + \frac{1}{2}\log\left(d - \frac{\kappa_2^2}{d}\right) + \frac{d}{2}\left[\left(1 + \frac{\kappa_2}{d}\right)\log\left(1 + \frac{\kappa_2}{d}\right) + \left(1 - \frac{\kappa_2}{d}\right)\log\left(1 - \frac{\kappa_2}{d}\right)\right],\tag{19}$$

 $PrivUnit_{\infty}(\cdot, \kappa, p_0)$ is $(\varepsilon + \varepsilon_0)$ -DP.

4.2.2Utility analysis

Paralleling our analysis of the ℓ_2 -case, we now analyze the utility of our ℓ_∞ -privatization mechanism. We first prove that $PrivUnit_{\infty}$ indeed produces an unbiased estimator.

Lemma 4.2. Let $Z = PrivUnit_{\infty}(u, \kappa)$ for some $\kappa \in \{0, \dots, d\}$ and $u \in [-1, 1]$. Then $\mathbb{E}[Z] = u$.

See Appendix B.4 for a proof.

The results of Duchi et al. [31] imply that for $u \in \{-1,1\}^d$ the output $Z = \mathtt{PrivUnit}_{\infty}(u, \kappa = 1)$ (0,p) has magnitude $\|Z\|_{\infty} \lesssim \sqrt{d} \frac{p}{1-p}$, which is $\sqrt{d} \frac{e^{\varepsilon}+1}{e^{\varepsilon}-1}$ for $p=e^{\varepsilon}/(1+e^{\varepsilon})$. We can, however, provide stronger guarantees. Letting κ satisfy the sufficient condition (17) in PrivUnit $_{\infty}(\cdot,\kappa,p_0)$ for $p_0 = \frac{e^{\varepsilon_0}}{e^{\varepsilon_0} + 1}$ ensures that Z is $(\varepsilon + \varepsilon_0)$ -differentially private, and we have the utility bound

Proposition 4. Let $u \in \{-1,1\}^d$, $p \geq \frac{1}{2}$, and $Z = PrivUnit_{\infty}(u,\kappa,p)$. Then $\mathbb{E}[Z] = u$, and there exist numerical constants $0 < c_0, c_1 < \infty$ such that the following holds.

(i) Assume that $\varepsilon \geq \log d$. If κ saturates the bound (19), then $\kappa \geq c_0 \sqrt{\varepsilon d}$ and

$$||Z||_{\infty} \le c_1 \sqrt{\frac{d}{\varepsilon}}.$$

(ii) Assume that $\varepsilon < \log d$. If κ saturates the bound (18), then $\kappa \geq c_0 \min\{\sqrt{d}, \varepsilon\sqrt{d}\}$, and

$$||Z||_{\infty} \le c_1 \frac{\sqrt{d}}{\min\{1, \varepsilon\}}.$$

See Appendix B.5 for a proof.

Thus, comparing to the earlier guarantees of Duchi et al. [31], we see that this hypercube-capbased method we present in Algorithm 2 obtains no worse error in all cases of ε , and when $\varepsilon \geq \log d$, the dependence on ε is substantially better. An argument paralleling that for Proposition 2 shows that the bounds on the ℓ_{∞} -norm of Z are unimprovable except for $\varepsilon \in [1, \log d]$; we believe a slightly more careful probabilistic argument should show that case (i) holds for $\varepsilon \geq 1$.

4.3 Privatizing the magnitude

The final component of our mechanisms for releasing unbiased vectors is to privately release single values $r \in [0, r_{\text{max}}]$ for some $r_{\text{max}} < \infty$. The first (Sec. 4.3.1) provides a randomized-response-based mechanism achieving order optimal scaling for the mean-squared error $\mathbb{E}[(Z-r)^2]$, which is $r_{\text{max}}^2 e^{-2\varepsilon/3}$ for $\varepsilon \geq 1$ (see Corollary 8 in [38]). In the second (Sec. 4.3.2), we provide a mechanism that achieves better relative error guarantees—important for statistical applications in which we wish to adapt to the ease of a problem (recall the introduction), so that "easy" (small magnitude update) examples indeed remain easy.

4.3.1 Absolute error

We first discuss a generalized randomized-response-based scheme for differentially private release of values $r \in [0, r_{\text{max}}]$, where r_{max} is some a priori upper bound on r. We fix a value $k \in \mathbb{N}$ and then follow a three-phase procedure: first, we randomly round r to an index value J taking values in $\{0, 1, 2, \ldots, k\}$ so that

$$\mathbb{E}[r_{\max}J/k \mid r] = r \text{ and } \lfloor kr/r_{\max} \rfloor \le J \le \lceil kr/r_{\max} \rceil.$$

In the second step, we employ randomized response [65] over k outcomes. The third step debiases this randomized quantity to obtain the estimator Z for r. We formalize the procedure in Algorithm 3, ScalarDP.

Importantly, the mechanism ScalarDP is ε -differentially private, and we can control its accuracy via the next lemma, whose proof we defer to Appendix A.3.

Lemma 4.3. Let $\varepsilon > 0$, $k \in \mathbb{N}$, and $0 \le r_{\max} < \infty$. Then the mechanism $\mathit{ScalarDP}(\cdot, \varepsilon; k, r_{\max})$ is ε -differentially private and for $Z = \mathit{ScalarDP}(r, \varepsilon; k, r_{\max})$, if $0 \le r \le r_{\max}$, then $\mathbb{E}[Z] = r$ and

$$\mathbb{E}[(Z-r)^2] = \frac{r_{\max}^2 \cdot (k+1)}{(e^{\varepsilon}-1)^2} \cdot \left(\frac{(2k+1)(e^{\varepsilon}+k)}{6k} - \frac{k+1}{4}\right)$$
$$-r_{\max} \cdot \left(\frac{r(k+1)}{e^{\varepsilon}-1}\right) + \frac{r_{\max}^2}{k^2} \mathbb{E}[J^2] \cdot \left(\frac{e^{\varepsilon}+k}{e^{\varepsilon}-1}\right) - r^2$$

Algorithm 3 Privatize the magnitude with absolute error: ScalarDP

Require: Magnitude r, privacy parameter $\varepsilon > 0$, $k \in \mathbb{N}$, bound r_{max}

 $r \leftarrow \min\{r, r_{\max}\}$

Sample $J \in \{0, 1, \dots, k\}$ such that

$$J = \begin{cases} \lfloor kr/r_{\text{max}} \rfloor & \text{w.p. } (\lceil kr/r_{\text{max}} \rceil - kr/r_{\text{max}}) \\ \lceil kr/r_{\text{max}} \rceil & \text{otherwise.} \end{cases}$$

Use randomized response to obtain

$$\widehat{J} \mid (J=i) = \begin{cases} i & \text{w.p. } \frac{e^{\varepsilon}}{e^{\varepsilon} + k} \\ \text{uniform in } \{0, \dots, k\} \setminus i & \text{w.p. } \frac{k}{e^{\varepsilon} + k}. \end{cases}$$

Debias \widehat{J} , by setting

$$Z = a\left(\widehat{J} - b\right)$$
 for $a = \left(\frac{e^{\varepsilon} + k}{e^{\varepsilon} - 1}\right) \frac{r_{\max}}{k}$ and $b = \frac{k(k+1)}{2(e^{\varepsilon} + k)}$.

return Z

By choosing k appropriately, we can achieve optimal [38] mean-squared error as ε grows:

Lemma 4.4. Let $k = \lceil e^{\varepsilon/3} \rceil$. Then for $Z = ScalarDP(r, \varepsilon; k, r_{max})$,

$$\sup_{r \in [0, r_{\text{max}}]} \mathbb{E}[(Z - r)^2 \mid r] \le C \cdot r_{\text{max}}^2 e^{-2\varepsilon/3}$$

for a universal (numerical) constant C independent of r_{max} and ε .

Proof We use the result of Lemma 4.3, which gives us the bound

$$\begin{split} \mathbb{E}[(Z-r)^2] & \leq \frac{r_{\max}^2 \cdot (k+1)}{(e^{\varepsilon}-1)^2} \cdot \left(\frac{(2k+1)(e^{\varepsilon}+k))}{6k}\right) + \frac{r_{\max}^2}{k^2} \mathbb{E}[J^2] \cdot \left(\frac{e^{\varepsilon}+k}{e^{\varepsilon}-1}\right) - r^2 \\ & = \frac{r_{\max}^2 \cdot (k+1)}{(e^{\varepsilon}-1)^2} \cdot \left(\frac{(2k+1)(e^{\varepsilon}+k))}{6k}\right) + \frac{r_{\max}^2}{k^2} \cdot \left(\frac{k+1}{e^{\varepsilon}-1}\right) \cdot \mathbb{E}[J^2] + \frac{r_{\max}^2}{k^2} \cdot \operatorname{Var}[J] \end{split}$$

Note that Var[J] = O(1) and $\mathbb{E}[J^2] = O(k^2)$. When $k = O(e^{\varepsilon})$, we thus obtain that

$$\mathbb{E}[(Z-r)^2] = r_{\max}^2 \cdot O\left(\frac{k}{e^{\varepsilon}} + \frac{1}{k^2}\right)$$

Taking $k = \lceil e^{\varepsilon/3} \rceil$ gives the result.

4.3.2 Relative error

As our discussion in the introductory Section 1.2 shows, to develop optimal learning procedures it is frequently important to know when the problem is easy—observations are low variance—and for this, releasing scalars with relative error can be important. Consequently, we consider an alternative

mechanism that first breaks the range $[0, r_{\text{max}}]$ into intervals of increasing length based on a fixed accuracy $\alpha > 0$, $k \in \mathbb{N}$, and $\nu > 1$, where we define the intervals

$$E_0 = [0, \nu \alpha], \quad E_i = [\nu^i \alpha, \nu^{i+1} \alpha] \text{ for } i = 1, \dots, k-1.$$
 (20)

The resulting mechanism works as follows: we determine the interval that r belongs to, we randomly round r to an endpoint of the interval (in an unbiased way), then use randomized response to obtain a differentially private quantity, which we then debias. We formalize the algorithm in Algorithm 4.

Algorithm 4 Privatize the magnitude with relative error: ScalarRelDP

Require: Magnitude r, privacy parameter $\varepsilon > 0$, integer k, accuracy $\alpha > 0$, $\nu > 1$, bound r_{max} . $r \leftarrow \min\{r, r_{\text{max}}\}$

Form the intervals $\{E_0, E_1, \dots, E_{k-1}\}$ given in (20) and let i^* be the index such that $r \in E_{i^*}$. Sample $J \in \{0, 1, \dots, k\}$ such that

$$J = \begin{cases} 0 & \text{w.p. } \frac{\nu\alpha - r}{\nu\alpha} \\ 1 & \text{w.p. } \frac{r}{\nu\alpha} \end{cases} \text{ if } i^* = 0 \quad \text{and} \quad J = \begin{cases} i^* & \text{w.p. } \frac{\nu^{i^* + 1}\alpha - r}{\nu^{i^*}(\nu - 1)\alpha} \\ i^* + 1 & \text{w.p. } \frac{r - \nu^{i^*}\alpha}{\nu^{i^*}(\nu - 1)\alpha} \end{cases} \text{ if } i^* \ge 1.$$

Use randomized response to obtain \widehat{J}

$$\widehat{J} \mid (J = i) = \begin{cases} i & \text{w.p. } \frac{e^{\varepsilon}}{e^{\varepsilon} + k} \\ \text{uniform in } \{0, \dots, k\} \setminus i & \text{w.p. } \frac{k}{e^{\varepsilon} + k}. \end{cases}$$

Set
$$\tilde{J} = \nu^{\hat{J}} \cdot \mathbb{1} \left\{ \hat{J} \ge 1 \right\}$$

Debias \tilde{J} , by setting

$$Z = a\left(\tilde{J} - b\right) \text{ for } a = \alpha \cdot \left(\frac{e^{\varepsilon} + k}{e^{\varepsilon} - 1}\right) \text{ and } b = \frac{1}{e^{\varepsilon} + k} \cdot \sum_{i=1}^{k} \nu^{j}.$$

return Z

As in the absolute error case, we can provide an upper bound on the error of the mechanism ScalarRelDP, though in this case—up to the small accuracy α —our error guarantee is relative.

Lemma 4.5. Fix $\alpha > 0$, $k \in \mathbb{N}$, and $\nu > 1$. Then $Z = \textit{ScalarRelDP}(\cdot, \varepsilon; k, \alpha, \nu, r_{\max})$ is ε -differentially private and for $r < r_{\max}$, we have $\mathbb{E}[Z \mid r] = r$ and

$$\frac{\mathbb{E}[(Z-r)^2 \mid r]}{(r \vee \alpha)^2} \le \frac{(k+1)}{(e^{\varepsilon}-1)}\nu^2 + \left(\frac{\nu^{2k} \cdot (e^{\varepsilon}+k)}{(e^{\varepsilon}-1)^2}\right) \left(\frac{1-\nu^{-2k}}{1-\nu^{-2}}\right) + (\nu-1)^2.$$

See Appendix A.4 for a proof of Lemma 4.5.

We perform a few calculations with Lemma 4.5 when $k \leq e^{\varepsilon}$. Let $\nu = 1 + \Delta$ for a $\Delta \in (0,1)$ to be chosen, and note that the choice

$$k = \left\lceil \frac{\log \frac{r_{\text{max}}}{\alpha}}{\log \nu} \right\rceil \approx \frac{\log \frac{r_{\text{max}}}{\alpha}}{\Delta}$$

is the smallest value of k that gives $\nu^k \alpha \geq r_{\text{max}}$. Let $\alpha = r_{\text{max}} \alpha_0$, so that $\nu^{2k} \approx \frac{1}{\alpha_0^2}$. Hence, from Lemma 4.5 we have the truncated relative error bound

$$\frac{\mathbb{E}[(Z-r)^2\mid r]}{(r\vee r_{\max}\alpha_0)^2}\lesssim \frac{k}{e^{\varepsilon}}+\frac{e^{-\varepsilon}}{\alpha_0^2\cdot\Delta}+\Delta^2=e^{-\varepsilon}\left(\log\frac{1}{\alpha_0}+\frac{1}{\alpha_0^2}\right)\cdot\left(\frac{1}{\Delta}\right)+\Delta^2\lesssim \frac{1}{e^{\varepsilon}\alpha_0^2\Delta}+\Delta^2.$$

We assume the relative accuracy threshold $\alpha/r_{\rm max}=\alpha_0\gtrsim e^{-\varepsilon/2}$. Then setting $\Delta=\alpha_0^{-2/3}e^{-\varepsilon/3}$ gives

$$\frac{\mathbb{E}[(Z-r)^2 \mid r]}{(r \vee r_{\text{max}}\alpha_0)^2} \lesssim \alpha_0^{-4/3} e^{-2\varepsilon/3}.$$
 (21)

Let us compare with Lemma 4.4, which yields

$$\mathbb{E}[(Z-r)^2 \mid r] = O\left(r_{\max}^2 e^{-2\varepsilon/3}\right).$$

The choice $\alpha_0 = 1$ in inequality (21) shows that the geometrically-binned mechanism can recover this bound. For $r \leq r_{\text{max}}\alpha_0$, the former inequality (21) is stronger; for example, the choice $\alpha_0 = e^{-\varepsilon/4}$ yields that $\mathbb{E}[(Z-r)^2 \mid r] = O(r_{\text{max}}^2 e^{-5\varepsilon/6})$ for $r \leq r_{\text{max}} e^{-\varepsilon/4}$, and $\mathbb{E}[(Z-r)^2 \mid r] = O(r^2 e^{-\varepsilon/3})$ otherwise.

4.4 Asymptotic analysis with local privacy

Finally, with our development of (optimal) private vector sampling mechanisms complete, we revisit the statistical risk minimization problem (3) and our development of asymptotics in Section 3.2. Recall our setting, where we wish to minimize $L(\theta) = \mathbb{E}_P[\ell(\theta, X)]$ using a sample $X_t \stackrel{\text{iid}}{\sim} P$, $t = 1, \ldots, T$. We consider a stochastic gradient procedure, where we privatize each stochastic gradient $\nabla \ell(\theta, X)$ using a separated mechanism that obfuscates both the direction $\nabla \ell / \|\nabla \ell\|_2$ and magnitude $\|\nabla \ell\|_2$. Our scheme ε -differentially private, and for the privacy level ε , we let $\varepsilon_1 + \varepsilon_2 = \varepsilon$, where we use ε_1 as the privacy level for the direction and ε_2 as the privacy level for the magnitude. For fixed ε_1 , we let $\gamma(\varepsilon_1)$ be the largest value of γ satisfying one of the inequalities (14) so that Algorithm 1 (PrivUnit₂) is ε_2 -differentially private and $\gamma(\varepsilon) \gtrsim \min\{\varepsilon, \sqrt{\varepsilon}\}/\sqrt{d}$ (recall Proposition 3). We use Alg. 3 to privatize the magnitude (with a maximum scalar value r_{max} to be chosen), and thus we define the ε -differentially private mechanism for privatizing a vector w by

$$M(w) := \mathtt{PrivUnit}_2\left(\frac{w}{\|w\|_2}; \gamma(\varepsilon_1), p = \frac{1}{2}\right) \cdot \mathtt{ScalarDP}(\|w\|_2, \varepsilon_2; k = \lceil e^{\varepsilon_2/3} \rceil, r_{\max}). \tag{22}$$

Using this mechanism, we define $Z(\theta;x) := M(\nabla \ell(\theta;x))$, and the private stochastic gradient method iterates

$$\theta^{(t+1)} \leftarrow \theta^{(t)} - \eta_t \cdot Z(\theta^{(t)}; X_t)$$

for t = 1, 2, ... and $X_t \stackrel{\text{iid}}{\sim} P$, where η_t is a stepsize sequence.

To see the asymptotic behavior of the average $\overline{\theta}^{(T)} = \frac{1}{T} \sum_{t=1}^{T} \theta^{(t)}$, we will use Corollary 3.2. We begin by computing the asymptotic variance $\Sigma^{\text{priv}} = \mathbb{E}[Z(\theta^{\star}; X) Z(\theta^{\star}; X)^{\top}]$.

Lemma 4.6. Assume that $0 < \varepsilon_1, \varepsilon_2 \le d$ and let Z be defined as above. Define the unit vector $U = \nabla \ell(\theta^*; X) / \|\nabla \ell(\theta^*; X)\|_2$ and let $\Sigma = \text{Cov}(\nabla \ell(\theta^*; X))$ and $\Sigma_{\text{norm}} = \text{Cov}(\nabla \ell(\theta^*; X) / \|\nabla \ell(\theta^*; X)\|_2)$. Assume additionally that $\|\nabla \ell(\theta^*; X)\|_2 \le r_{\text{max}}$ with probability 1. Then

$$\Sigma^{\text{priv}} \preceq O\left(\frac{dr_{\text{max}}^2 e^{-2\varepsilon_2/3}}{\varepsilon_1 \wedge \varepsilon_1^2}\right) \left(\Sigma_{\text{norm}} + \frac{\text{tr}(\Sigma_{\text{norm}})}{d}I_d\right) + O\left(\frac{d}{\varepsilon_1 \wedge \varepsilon_1^2}\right) \cdot \left(\Sigma + \frac{\text{tr}(\Sigma)}{d}I_d\right).$$

See Appendix A.5 for the proof.

Lemma 4.6 is the key result out of which our main convergence theorem builds. Combining this result with Corollary 3.2, we obtain the following theorem, which highlights the asymptotic convergence results possible when we use somewhat larger privacy parameters ε . For simplicity in the theorem, we assume that the asymptotic covariance of the loss gradients is full rank.

Theorem 3. Let the conditions of Lemma 4.6 hold. Define the optimal asymptotic covariance $\Sigma_{\star} := \text{Cov}(\nabla \ell(\theta^{\star}; X))$, and assume that $\lambda_{\min}(\Sigma_{\star}) = \lambda_{\min} > 0$. Let the privacy levels $0 < \varepsilon_1, \varepsilon_2$ satisfy $\varepsilon_2 \geq \frac{3}{2} \log \frac{d}{\varepsilon_1 \lambda_{\min}}$ and $1 \leq \varepsilon_1 \leq d$. Assume that the stepsizes $\eta_t \propto t^{-\beta}$ for some $\beta \in (1/2, 1)$, and let $\theta^{(t)}$ be generated by the private stochastic gradient method (4.4). Then

$$\sqrt{T} \left(\overline{\theta}^{(T)} - \theta^\star \right) \overset{d}{\to} \mathsf{N} \left(0, \Sigma^{\mathrm{priv}} \right) \quad \text{where} \quad \Sigma^{\mathrm{priv}} \preceq O(1) \frac{d}{\varepsilon_1} \nabla^2 L(\theta^\star)^{-1} \left(\Sigma_\star + \frac{\mathrm{tr}(\Sigma_\star)}{d} I_d \right) \nabla^2 L(\theta^\star)^{-1}.$$

We provide some commentary on Theorem 3. We may compare it to the (minimax optimal) asymptotic results that Duchi et al. [31] achieve for generalized linear model estimation [31, Sec. 5.2]. In their case, in the identical scenario, they achieve an asymptotic variance Σ^{minimax} satisfying

$$\Sigma^{\text{minimax}} \succeq \Omega(1) \left(\frac{e^{\varepsilon} + 1}{e^{\varepsilon} - 1} \right)^{2} \nabla^{2} L(\theta^{\star})^{-1} \left(\Sigma_{\star} + \sup_{x, \theta} \left\| \nabla \ell(\theta; x) \right\|_{2}^{2} I_{d} \right) \nabla^{2} L(\theta^{\star})^{-1}.$$

As $\sup_{x,\theta} \|\nabla \ell(\theta;x)\|_2^2 \ge \operatorname{tr}(\Sigma_{\star})$, this is always worse than the bound of Theorem 3. Even as $\varepsilon \uparrow d$, the minimax remains larger than that in Theorem 3: when $\varepsilon, \varepsilon_1 \gtrsim d$, even in the most favorable case for the minimax bound, we have

$$\operatorname{tr}(\Sigma^{\operatorname{minimax}}) \gtrsim \operatorname{tr}(\nabla^2 L(\theta^{\star})^{-1} \Sigma_{\star} \nabla^2 L(\theta^{\star})^{-1}) + \operatorname{tr}(\Sigma_{\star}) \operatorname{tr}(\nabla^2 L(\theta^{\star})^{-2}) \text{ and}$$
$$\operatorname{tr}(\Sigma^{\operatorname{priv}}) \lesssim \operatorname{tr}(\nabla^2 L(\theta^{\star})^{-1} \Sigma_{\star} \nabla^2 L(\theta^{\star})^{-1}) + \frac{1}{d} \operatorname{tr}(\Sigma_{\star}) \operatorname{tr}(\nabla^2 L(\theta^{\star})^{-2}).$$

As $\operatorname{tr}(AB) \leq \|A\|_{\operatorname{op}} \operatorname{tr}(B)$ for $A, B \succeq 0$, we see that if $\kappa(\Sigma_{\star})$ is the condition number of Σ_{\star} , then we find that

$$\operatorname{tr}(\Sigma^{\operatorname{priv}}) \lesssim \min \left\{ \frac{\kappa(\Sigma_{\star})}{d}, 1 \right\} \operatorname{tr}(\Sigma^{\operatorname{minimax}})$$

for large ε . That is, when the gradient matrix is well-conditioned, our procedure is at least d-times more efficient than prior locally private learning schemes.

It is also instructive to compare the asymptotic covariance Σ^{priv} Theorem 3 to the optimal asymptotic covariance without privacy, which is $\nabla^2 \ell(\theta^*)^{-1} \Sigma_* \nabla^2 \ell(\theta^*)^{-1}$ (cf. [26, 47, 63]). When the privacy level ε_1 scales with the dimension, our asymptotic covariance is within a numerical constant of this optimal value. We can of course never quite achieve optimal covariance, because the privacy channel forces some loss of efficiency, but this loss of efficiency is now bounded.

5 Empirical Results

We now present empirical results of our private learning system, which uses locally differentially private (separated) mechanisms to protect against reconstruction from adversaries with limited prior information while incorporating central differential privacy to provide strong global privacy guarantees. In the settings we consider—with a large dataset distributed across multiple devices or units—the non-private alternative is to communicate and aggregate model updates without local or centralized privacy. In all of our experiments, we permute the data, then assigning a number of

examples to each "user" for local training at each round. After aggregating individual updates, we re-permute the data and continue.

We consider applications in image classification and language modeling. In each experiment, we use $(\varepsilon_1, \varepsilon_2)$ -separated differentially private mechanisms as in the mechanism (22). Letting $\gamma(\varepsilon)$ be the largest value of γ satisfying the privacy condition (14) in our ℓ_2 mechanisms and $p(\varepsilon) = \frac{e^{\varepsilon}}{1+e^{\varepsilon}}$, for any vector $w \in \mathbb{R}^d$ we use

$$M(w) := \mathtt{PrivUnit}_2\left(\frac{w}{\|w\|_2}; \gamma(.99\varepsilon_1), p(.01\varepsilon_1)\right) \cdot \mathtt{ScalarDP}\left(\|w\|_2, \varepsilon_2, k = \left\lceil e^{\varepsilon_2/3} \right\rceil, r_{\max}\right). \tag{23}$$

In our experiments, we set $\varepsilon_2 = 10$, which is large enough (recall Theorem 3) so that its contribution to the final error is negligible relative to the sampling error in PrivUnit₂, but of much smaller order than ε_1 . In each experiment, we vary ε_1 , which is the dominant term in the asymptotic convergence of Theorem 3.

Let us describe the reconstruction protections we provide. In each experiment, we assume as in Eq. (8) that for a matrix $A \in \mathbb{R}^{k \times d}$, k = .01d, the prior $\pi_0 \in \mathcal{P}_A(\rho)$ for $\rho = \exp(\sqrt{d})$, that is, for input random vectors $X \in \mathbb{R}^d$, the induced prior on $v = f_A(x) = Ax/\|Ax\|_2$ satisfies $d\pi(v)/d\pi_{\mathrm{uni}}(v) \leq \rho$. We set the parameters of our mechanisms in Proposition 1 to ensure that no estimator can cause a $(\alpha = \frac{1}{2}, p, f_A)$ -reconstruction breach with probability $p = 10^{-500}$. (Reducing the local privacy parameters $(\varepsilon_1, \varepsilon_2)$ only ensures stronger protections against reconstruction.)

In the centralized aggregation steps (10)–(11), we project the updates onto a ball of radius S and add Gaussian noise with identity covariance and standard deviation $\sigma = \zeta \frac{S}{qN}$, where q is the fraction of users we subsample, N is the total population size, and ζ is a parameter we choose. In our experiments, we report the resulting centralized privacy levels for each experiment. We remark in passing that, because of the randomized subsampling, individuals may have their data sampled more than once (though this is unlikely because of the scale of the data); in this case, the privacy and reconstruction protections degrade gracefully via the standard composition rules for differential privacy, though we do not belabor this [21, 33].

We make a concession to computational feasibility, slightly reducing the value σ that we actually use in our experiments beyond the theoretical recommendations. In particular, we use batch sizes of at most 200 and use $\sigma \in \{.001, .002, .005, .01\}$, depending on our experiment, which of course requires either small ζ above or larger subsampling rate q than our effective rate. As McMahan et al. [55] note, increasing this batch size has negligible effect on the accuracy of the centralized model, so that we report results (following [55]) that use this inflated batch size estimate from a population of size N = 10,000,000.

Experiments							
Task	Dataset	d					
Image Classification over 10 Classes	MNIST	3,274,634					
Image Classification over 10 Classes	CIFAR10	1,756,426					
Image Classification over 100 Classes	Flickr	1,255,524					
Next Word Prediction	REDDIT	13,352,875					

Table 1: Private Federated Learning Experiments.

Throughout our experiments, our goal is to show that for a particular set of hyperparameters our private federated learning system—which includes separated differentially private mechanisms (providing local privacy protections against reconstruction) and central differential privacy—performs nearly as well as models fit using the same methods without *no* privacy. Given the prevalence of

large-scale models that researchers have already published and fit on multiple datasets, we are able to present results both for models trained *tabula rasa* (from scratch, with random initialization) as well as those pre-trained on other data. We present the task, dataset, and dimension of each model we train in each experiment in Table 1.

5.1 Training tabula rasa

We first present results for our private federated learning system on image classification tasks when the model starts from a random initialization. Recall our mechanism (23), so that we allocate $\gamma = \gamma(.99\varepsilon_1)$ for the spherical cap threshold and $p = p(.01 \cdot \varepsilon_1)$ for the probability with which we choose a particular spherical cap in the randomization $PrivUnit_2(\cdot, \gamma, p)$, which ensures ε_1 -differential privacy.

$\texttt{PrivUnit}_2(\cdot,\gamma,p)$			$oxed{ t ScalarDP(\cdot, arepsilon_2, k, r_{ ext{max}})}$			Central DP	
ε_1	$\gamma(.99\varepsilon_1)$	$p = \frac{e^{.01\varepsilon_1}}{1 + e^{.01\varepsilon_1}}$	ε_2	$k = \lceil e^{\varepsilon_2/3} \rceil$	$r_{\rm max}$	S	σ
500	0.01729	1.0	10	29	5	100	0.005
250	0.01217	0.924	10	29	5	100	0.005
100	0.00760	0.731	10	29	5	100	0.005
50	0.00526	0.622	10	29	5	100	0.005
						1	

Table 2. Parameters in MNIST Experiments over 100 rounds with corresponding plots in Figure 3a.

MNIST. We first present results on the classic MNIST dataset [48]. We use the same CNN model architecture as in the TensorFlow tutorial [5] with an Adam optimizer. For this setting, we train the full network from scratch so that d=3,274,634. In each round we randomly shuffle the training images and put disjoint sets of 100 of the images on an expected batch size of 200 devices. Each device does 5 local epochs over their training data where the local batch size is set to be the full set of 100 images. For sampling the magnitude of local updates, we use ScalarDP(\cdot , $\varepsilon_2 = 10, k, r_{\text{max}} = 5$), and for the unit vector direction privatization we use PrivUnit₂ and vary ε_1 . We present the privacy parameters in Table 2 and we present the results in Figure 3a for mechanisms that satisfy ($\varepsilon_1, \varepsilon_2 = 10$)-separated DP where $\varepsilon_1 \in \{50, 100, 250, 500\}$. The corresponding clip S and standard deviation of noise σ in the central DP algorithm ensures that if N = 10,000,000 and we have an expected batch size of 20,000 then after 100 rounds, the resulting model would be ($\varepsilon_C = 1.90, \delta = 10^{-9}$)-central DP.

$\texttt{PrivUnit}_2(\cdot,\gamma,p)$			$oxed{ t ScalarDP(\cdot,arepsilon_2,k,r_{ ext{max}})}$			Central DP	
ε	$\gamma(.99\varepsilon_1)$	$p = \frac{e^{.01\varepsilon_1}}{1 + e^{.01\varepsilon_1}}$	ε_2	$k = \lceil e^{\varepsilon_2/3} \rceil$	$r_{\rm max}$	S	σ
5000	0.07492	1.0	10	29	2	30	0.002
1000	0.03347	1.0	10	29	2	30	0.002
500	0.02361	0.993	10	29	2	30	0.002
100	0.01038	0.731	10	29	2	30	0.002
						1	

Table 3: Parameters in CIFAR10 Experiments over 200 rounds with plots in Figure 3b.

CIFAR10 We now present results on the CIFAR10 dataset [1]. We use the same CNN model architecture as in the Tensorflow tutorial [2] with an Adam optimizer and d = 1,756,426. We preprocess the data as in the Tensorflow tutorial so that the inputs are 24×24 with 3 channels.

Similar to the set up for MNIST, we shuffle the training images over an expected batch size of 75 devices, each with 500 images. Each device will do 5 local epochs over their training data where the local batch size is set to be the full set of 500 images. As in the mechanism (23), use ScalarDP($\cdot, \varepsilon_2 = 10, k = \lceil e^{\varepsilon_2/3} \rceil, r_{\text{max}} = 2$) to sample the magnitude of the updates and PrivUnit₂($\cdot, \gamma(.99 \cdot \varepsilon_1), p(0.01 \cdot \varepsilon_1)$) with various ε_1 for the direction. See Table 3 for the privacy parameters that we set in each experiment. We present the results in Figure 3b for mechanisms that satisfy ($\varepsilon_1, \varepsilon_2 = 10$)-separated DP where $\varepsilon_1 \in \{100, 500, 1000, 5000\}$. The corresponding clip S and standard deviation of noise σ in the central DP algorithm ensures that if N = 10,000,000 and an expected batch size of 15,000 then after 200 rounds, the resulting model would be ($\epsilon_C = 1.76, \delta = 10^{-9}$)-central DP.

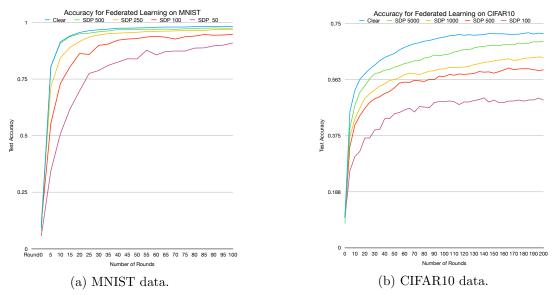


Figure 3. Accuracy plots for image classification comparing our private federated learning approach (labeled SDP with the corresponding ε_1 parameter) and federated learning with clear model updates (labeled Clear). For the SDP experiments, we fix all the parameters except the privacy parameter for PrivUnit₂, which varies.

5.2 Pretrained models

We now present results for more complex models that are pretrained on some publicly available dataset, rather than initialized with a random state. We see this as being the more realistic setting, where federated learning is used to fine tune the model.

$\texttt{PrivUnit}_2(\cdot,\gamma,p)$			$oxed{ t ScalarDP(\cdot, arepsilon_2, k, r_{ ext{max}})}$			Central DP		
ε		$\gamma(.99\varepsilon_1)$	$p = \frac{e^{.01\varepsilon_1}}{1 + e^{.01\varepsilon_1}}$	$arepsilon_2$	$k = \lceil e^{\varepsilon_2/3} \rceil$	$r_{\rm max}$	S	σ
500	00	0.08857	1.0	10	29	10	100	0.005
50	0	0.02793	0.993	10	29	10	100	0.005
10	0	0.01227	0.731	10	29	10	100	0.005
50)	0.00851	0.622	10	29	10	100	0.005

Table 4. Parameters in pretrained ResNet50v2 experiments over 100 rounds with corresponding plots in Figure 4a.

Image classification on Flickr over 100 classes We ran further experiments on a pretrained Resnet50v2 that used ImageNet data, from [50]. We then started with this pretrained model to then run our private federated learning system to train the last (softmax) layer and the last convolution layer of Resnet50v2 [41] with Flickr data [3]. We include not just the last layer to ensure that our algorithms can handle more complex non-linear functions like convolutions. The total number of trainable parameters is d = 1, 255, 524.

We ran our experiments on a subset of the Flickr dataset. The dataset was obtained by choosing 100 arbitrary classes and choosing 2000 random images from each class (i.e. 200,000 images in total). These were then shuffled, followed by a 9:1 split into training and testing set. Each device on a given epoch/day has access to 128 images from the training set and it trains on that for 15 epochs with a local batch size being the full 128 examples. The expected number of updates each round is 100. The accuracy at the end of each round is computed on a random subset of 512 images from the testing set.

Again following the mechanism (23), for the magnitude of each update we use ScalarDP($\cdot, \varepsilon_2 = 10, k = \lceil e^{\varepsilon_2/3} \rceil, r_{\text{max}} = 10$), while for the unit direction we use PrivUnit₂($\cdot, \gamma(.99 \cdot \varepsilon_1), p(0.01 \cdot \varepsilon_1)$) while varying ε_1 . See Table 4 for the privacy parameters that we set in each experiment. We present the results in Figure 4a for mechanisms that satisfy ($\varepsilon_1, \varepsilon_2 = 10$)-separated DP where $\varepsilon_1 \in \{50, 100, 500, 5000\}$. The corresponding clip S and standard deviation of noise σ in the central DP algorithm ensures that if N = 10,000,000 and an expected batch size of 20,000 then after 100 rounds, the resulting model would be ($\varepsilon_C = 1.90, \delta = 10^{-9}$)-central DP.

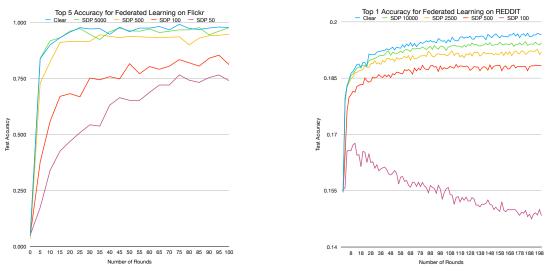
$\mathtt{PrivUnit}_2(\cdot,\gamma,p)$			$oxed{ t ScalarDP(\cdot, arepsilon_2, k, r_{ ext{max}})}$			Central DP	
ε	$\gamma(.99\varepsilon_1)$	$p = \frac{e^{.01\varepsilon_1}}{1 + e^{.01\varepsilon_1}}$	$arepsilon_2$	$k = \lceil e^{\varepsilon_2/3} \rceil$	$r_{\rm max}$	S	σ
1000	0.03848	1.0	10	29	5	100	0.001
2500	0.01923	1.0	10	29	5	100	0.001
500	0.00856	0.993	10	29	5	100	0.001
100	0.00376	0.731	10	29	5	100	0.001

Table 5. Parameters in next word prediction experiments on Reddit comments from November 2017 over 200 rounds with corresponding plots in Figure 4b.

Next Word Prediction We further demonstrate our private federated learning system for next word prediction. We pretrain an LSTM on Wikipedia data [67] taken from October 1, 2016. Our model architecture consists of one LSTM cell with a word encoding that takes 25,003 tokens (including an unknown, end of sentence, and beginning of sentence tokens) to 256 values. We use the NLTK [51] tokenization procedure to tokenize each sentence and then each word. The LSTM cell has 256 units, which leads to 526,336 trainable parameters within the RNN. Then we decode back into 25,003 tokens. In total, there are 13,352,875 trainable parameters in the LSTM.

We then use the pretrained LSTM to train on the Reddit dataset of comments from the month of November 2017 [4], again using the NLTK tokenization procedure. We allocate 1000 sentences on each device where each sentence contains no out of vocabulary words. For local training, we use a local minibatch of 100 sentences and train for 10 local epochs. The expected batch size for each round is 200. In this experiment, we use update parameters ScalarDP(\cdot , $\varepsilon_2 = 10$, $k = \lceil e^{\varepsilon_2/3} \rceil$, $r_{\text{max}} = 5$) and PrivUnit₂(\cdot , $\gamma(.99 \cdot \varepsilon_1)$, $p(0.01 \cdot \varepsilon_1)$) with various ε_1 . See Table 5 with the privacy parameters we set in each experiment. We present results in Figure 4b for mechanisms that satisfy (ε_1 , $\varepsilon_2 = 10$)-separated DP where $\varepsilon_1 \in \{100, 500, 2500, 10000\}$. The corresponding clip S and standard deviation of noise σ in the central DP algorithm ensures that if N = 10,000,000 and an expected batch size

of 100,000 then after 200 rounds, the resulting model would be ($\epsilon_C = 2.95, \delta = 10^{-9}$)-central DP.



(a) Top 5 accuracy image classification: top 100 (b) Next Word Prediction: Reddit data with preclasses from Flickr data with Resnet50v2 model pretrained LSTM on Wikipedia data with initial accutrained on ImageNet. racy roughly 15.5%.

Figure 4. Accuracy plots for pretrained models comparing our private federated learning approach (labeled SDP with the corresponding ε_1 parameter) with various privacy parameter ε and federated learning with clear model updates (labeled Clear).

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A Technical proofs

A.1 Proof of Theorem 2

Let $u \in \{-1, +1\}^d$ and $U \sim \mathsf{Uni}(\{-1, +1\}^d)$. The vector $V \in \{-1, 1\}^d$ sampled as in (16), has p.m.f.

$$p(v \mid u) \propto \begin{cases} 1/\mathbb{P}(\langle U, u \rangle > \kappa) & \text{if } \langle v, u \rangle > \kappa \\ 1/\mathbb{P}(\langle U, u \rangle < \kappa) & \text{if } \langle v, u \rangle \leq \kappa. \end{cases}$$

The event that $\langle U,u\rangle=\kappa$ when $\frac{d+\kappa+1}{2}\in\mathbb{Z}$ implies that U and u match in exactly $\frac{d+\kappa+1}{2}$ coordinates; the number of such matches is $\binom{d}{(d+\kappa+1)/2}$. Computing the binomial sum, we have

$$\mathbb{P}(\langle U, u \rangle > \kappa) = \frac{1}{2^d} \sum_{\ell = \lceil \frac{d + \kappa + 1}{2} \rceil}^d \binom{d}{\ell} \quad \text{and} \quad \mathbb{P}(\langle U, u \rangle \le \kappa) = \frac{1}{2^d} \sum_{\ell = 0}^{\lceil \frac{d + \kappa + 1}{2} \rceil - 1} \binom{d}{\ell}.$$

As $\mathbb{P}(\langle U, u \rangle > \kappa)$ is decreasing in κ for any $u, u' \in \{-1, +1\}^d$ and $v \in \{-1, 1\}^d$ we have

$$\frac{p(v \mid u)}{p(v \mid u')} \le \frac{p_0}{1 - p_0} \cdot \frac{\mathbb{P}(\langle U, u' \rangle \le \kappa)}{\mathbb{P}(\langle U, u \rangle > \kappa)} = e^{\varepsilon_0} \cdot \frac{\sum_{\ell=0}^{d\tau-1} \binom{d}{\ell}}{\sum_{\ell=d\tau}^{d} \binom{d}{\ell}},$$

where $\tau = \lceil (d+\kappa)/2 \rceil$. Bounding this by $e^{\varepsilon + \varepsilon_0}$ gives the result.

A.2 Proof of Corollary 4.1

Using Theorem 2, we seek to bound the quantity (17) for various κ values. We first analyze the case when $\kappa \leq \sqrt{3/2d+1}$. We use the following claim to bound each term in the summation in this case.

Claim A.1 (See Problem 1 in [45]). For even $d \geq 2$, we have

$$\binom{d}{d/2} \le \frac{2^{d+1/2}}{\sqrt{3d+2}}$$

Thus, when $\kappa < \sqrt{\frac{3d+2}{2}}$,

$$\log\left(\sum_{\ell=0}^{d\tau-1} \binom{d}{\ell}\right) - \log\left(\sum_{\ell=d\tau}^{d} \binom{d}{\ell}\right) = \log\left(1/2 + \frac{1}{2^d} \cdot \sum_{\ell=d/2}^{d\tau-1} \binom{d}{\ell}\right) - \log\left(1/2 - \frac{1}{2^d} \sum_{\ell=d/2}^{d\tau-1} \binom{d}{\ell}\right)$$

$$\leq \log\left(1 + \kappa \cdot \left(\sqrt{\frac{2}{3d+2}}\right)\right) - \log\left(1 - \kappa \cdot \left(\sqrt{\frac{2}{3d+2}}\right)\right).$$

Hence, to ensure ε -differential privacy, it suffices to have

$$\varepsilon \ge \log\left(1 + \kappa \cdot \sqrt{\frac{2}{3d+2}}\right) - \log\left(1 - \kappa \cdot \sqrt{\frac{2}{3d+2}}\right),$$

so Eq. (18) follows.

We now consider the case where $\epsilon = \Omega(\log(d))$. We use the following claim.

Claim A.2 (Lemma 4.7.2 in [9]). Let $Z \sim \text{Bin}(d, 1/2)$. Then for $0 < \lambda < 1$, we have

$$\mathbb{P}(Z \ge d\lambda) \ge \frac{1}{\sqrt{8d\lambda(1-\lambda)}} \exp\left(-dD_{\mathrm{kl}}\left(\lambda \| 1/2\right)\right)$$

We then use this to obtain the following bound:

$$\log \left(\sum_{\ell=0}^{d\tau-1} {d \choose \ell} \right) - \log \left(\sum_{\ell=d\tau}^{d} {d \choose \ell} \right) \le \frac{1}{2} \log \left(8d\tau (1-\tau) \right) + dD_{kl} \left(\tau \| 1/2 \right)$$

To ensure ε -differential privacy, it is thus sufficient, for $\tau := \frac{\lceil \frac{d+\kappa+1}{2} \rceil}{d}$, to have

$$\varepsilon \geq \frac{1}{2}\log\left(8 \cdot d\tau (1-\tau)\right) + d \cdot D_{\mathrm{kl}}\left(\tau \| 1/2\right),\,$$

which implies the final claim of the corollary.

A.3 Proof of Lemma 4.3

The proof for privacy follows from randomized response being ε -DP and the fact that DP is closed under post-processing. To prove that Z is unbiased, we note that

$$\mathbb{E}[Z] = \mathbb{E}[a(\widehat{J} - b)] = a\mathbb{E}[\mathbb{E}[\widehat{J} \mid J]] - ab = r_{\text{max}}/k \cdot \mathbb{E}[J] = r$$

To prove the second equation, we use the following conditional expectations

$$\mathbb{E}[(Z-r)^2] = \mathbb{E}\left[\operatorname{Var}\left[Z\mid J\right]\right] + \operatorname{Var}\left[\mathbb{E}\left[Z\mid J\right]\right]$$

We have $\operatorname{Var}[Z\mid J]=a^2\cdot\operatorname{Var}[\widehat{J}\mid J].$ Further, we have

$$\mathbb{E}[\widehat{J}^2 \mid J] = \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \cdot J^2 + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \sum_{j=0}^{k} j^2$$

Furthermore, we have

$$\mathbb{E}[\widehat{J} \mid J] = \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \cdot J + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \sum_{i=0}^{k} j$$

Putting this together, we have

$$\begin{aligned} \operatorname{Var}[\widehat{J} \mid J] &= \mathbb{E}[\widehat{J}^2 \mid J] - \mathbb{E}[\widehat{J} \mid J]^2 \\ &= \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \left((e^{\varepsilon} - 1)J^2 + \sum_{j=1}^k j^2\right) - \left(\frac{1}{e^{\varepsilon} + k}\right)^2 \left((e^{\varepsilon} - 1)J + \frac{k(k+1)}{2}\right)^2 \\ &= \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right)^2 \cdot \left(\left(\frac{e^{\varepsilon} + k}{e^{\varepsilon} - 1} - 1\right) \cdot J^2 - \frac{k(k+1)}{e^{\varepsilon} - 1} \cdot J + \frac{k(k+1)(2k+1)(e^{\varepsilon} + k)}{6(e^{\varepsilon} - 1)^2} - \frac{k^2(k+1)^2}{4(e^{\varepsilon} - 1)^2}\right) \end{aligned}$$

We then have the following conditional variance of Z

$$\begin{aligned} \operatorname{Var}[Z \mid J] &= a^2 \cdot \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right)^2 \cdot \left(\left(\frac{k + 1}{e^{\varepsilon} - 1}\right) \cdot J^2 - \frac{k(k + 1)}{e^{\varepsilon} - 1} \cdot J + \frac{k(k + 1)(2k + 1)(e^{\varepsilon} + k)}{6(e^{\varepsilon} - 1)^2} - \frac{k^2(k + 1)^2}{4(e^{\varepsilon} - 1)^2}\right) \\ &= \frac{r_{\max}^2}{(e^{\varepsilon} - 1)^2} \cdot \left(\frac{(k + 1)(e^{\varepsilon} - 1)}{k^2} \cdot J^2 - \frac{(k + 1)(e^{\varepsilon} - 1)}{k}J + \frac{(k + 1)(2k + 1)(e^{\varepsilon} + k)}{6k} - \frac{(k + 1)^2}{4}\right) \end{aligned}$$

Next we compute the expectation of J. Note that if kr/r_{max} is an integer, then $J = kr/r_{\text{max}}$

$$\mathbb{E}[J] = \frac{kr}{r_{\text{max}}}$$

We have $J \leq k$ and $\mathbb{E}[J^2] \leq (\lceil kr/r_{\text{max}} \rceil)^2$. Thus, we have

$$\begin{split} \frac{(e^{\varepsilon}-1)^2}{r_{\max}^2} \cdot \mathbb{E}[\operatorname{Var}[Z\mid J]] \\ &= \frac{(k+1)(e^{\varepsilon}-1)}{k^2} \cdot \mathbb{E}[J^2] - \frac{r(k+1)(e^{\varepsilon}-1)}{r_{\max}} + \frac{(k+1)(2k+1)(e^{\varepsilon}+k))}{6k} - \frac{(k+1)^2}{4} \end{split}$$

We then compute the conditional expectation of Z

$$\mathbb{E}[Z\mid J] = \frac{r_{\max}}{k} \cdot J \implies \mathrm{Var}[\mathbb{E}[Z\mid J]] = \frac{r_{\max}^2}{k^2} \cdot \mathrm{Var}[J] = \frac{r_{\max}^2}{k^2} \left(\mathbb{E}[J^2] - \left(\frac{kr}{r_{\max}}\right)^2 \right)$$

Putting this all together, we have

$$\begin{split} &\mathbb{E}[(Z-r)^2 \\ &= \left(\frac{r_{\max}}{e^{\varepsilon}-1}\right)^2 \cdot \left(\frac{(k+1)(e^{\varepsilon}-1)}{k^2} \cdot \mathbb{E}[J^2] - \frac{r(k+1)(e^{\varepsilon}-1)}{r_{\max}} + \frac{(k+1)(2k+1)(e^{\varepsilon}+k))}{6k} - \frac{(k+1)^2}{4}\right) \\ &\quad + \frac{r_{\max}^2}{k^2} \left(\mathbb{E}[J^2] - \left(\frac{kr}{r_{\max}}\right)^2\right) \\ &= \frac{r_{\max}^2 \cdot (k+1)}{(e^{\varepsilon}-1)^2} \cdot \left(\frac{(2k+1)(e^{\varepsilon}+k))}{6k} - \frac{k+1}{4}\right) - r_{\max} \cdot \left(\frac{r(k+1)}{e^{\varepsilon}-1}\right) + \frac{r_{\max}^2}{k^2} \mathbb{E}[J^2] \cdot \left(\frac{e^{\varepsilon}+k}{e^{\varepsilon}-1}\right) - r^2 \end{split}$$

A.4 Proof of Lemma 4.5

To see that ScalarRelDP($\cdot, k, \alpha, \nu; \varepsilon, r_{\text{max}}$) is differentially private, we point out that randomized response is ε -DP and then DP is closed under post-processing. To see that Z is unbiased, note that

$$\mathbb{E}[\tilde{J} \mid J = i] = \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \nu^{i} \mathbb{1} \left\{ i \ge 1 \right\} + \frac{1}{e^{\varepsilon} + k} \sum_{i=1}^{k} \nu^{j},$$

and thus for $r \in E_{i^*}$,

$$\begin{split} \mathbb{E}[Z] &= a \cdot (\mathbb{E}[\hat{J}] - b) = \alpha \cdot \left(\frac{e^{\varepsilon} + k}{e^{\varepsilon} - 1}\right) \left(\left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \mathbb{E}[\nu^{J} \mathbb{1} \left\{J \ge 1\right\}] + \frac{1}{e^{\varepsilon} + k} \cdot \sum_{j=1}^{k} \nu^{j} - \frac{1}{e^{\varepsilon} + k} \cdot \sum_{j=1}^{k} \nu^{j}\right) \\ &= \alpha \cdot \mathbb{E}\left[\nu^{J} \mathbb{1} \left\{J \ge 1\right\}\right] \\ &= \alpha \cdot \left[\mathbb{1} \left\{i^{*} = 0\right\} \cdot \left(\frac{r}{\alpha \nu}\right) \cdot \nu + \mathbb{1} \left\{i^{*} \ge 1\right\} \cdot \left(\left(\frac{\nu^{i^{*} + 1} \alpha - r}{\nu^{i^{*}} (\nu - 1) \alpha}\right) \cdot \nu^{i^{*}} + \left(\frac{r - \nu^{i^{*}} \alpha}{\nu^{i^{*}} (\nu - 1) \alpha}\right) \cdot \nu^{i^{*} + 1}\right)\right] \\ &= \mathbb{1} \left\{i^{*} = 0\right\} \cdot r + \mathbb{1} \left\{i^{*} \ge 1\right\} \cdot r = r \end{split}$$

Consider the following mean squared error terms

$$\mathbb{E}\left[\left(r-\alpha\nu^{J}\cdot\mathbb{1}\left\{J\geq1\right\}\right)^{2}\mid r\in E_{0}\right]=r^{2}\left(1-\frac{r}{\alpha\nu}\right)+\left(r-\alpha\nu\right)^{2}\left(\frac{r}{\alpha\nu}\right)=r\left(\alpha\nu-r\right)\leq(\nu\alpha)^{2}$$

And for $i^* \geq 1$

$$\mathbb{E}\left[(r - \alpha \nu^{J} \cdot \mathbb{1}\left\{J \ge 1\right\})^{2} \mid r \in E_{i^{*}}\right] \le (\nu^{i^{*}+1}\alpha - \nu^{i^{*}}\alpha)^{2} = \nu^{2i^{*}}(\nu^{i^{*}} - 1)^{2}\alpha^{2}.$$

We then bound the conditional expectation

$$\operatorname{Var}(Z \mid J = i^*) = a^2 \operatorname{Var}(\tilde{J} \mid J = i^*) = \alpha^2 \cdot \left(\frac{e^{\varepsilon} + k}{e^{\varepsilon} - 1}\right) \cdot \operatorname{Var}(\tilde{J} \mid J = i^*),$$

Now we note that

$$\operatorname{Var}(\tilde{J} \mid J = i^*) = \left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \cdot \nu^{2i^*} \mathbb{1} \left\{i^* \ge 1\right\} + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \sum_{j=1}^k \nu^{2j}$$

$$- \left(\left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + k}\right) \cdot \nu^{i^*} \mathbb{1} \left\{i^* \ge 1\right\} + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \sum_{j=1}^k \nu^j\right)^2$$

$$\le \left(\frac{(k+1)(e^{\varepsilon} - 1)}{(e^{\varepsilon} + k)^2}\right) \cdot \nu^{2i^*} \mathbb{1} \left\{i^* \ge 1\right\} + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \nu^{2k} \cdot \sum_{j=0}^{k-1} \nu^{-2j}$$

$$= \left(\frac{(k+1)(e^{\varepsilon} - 1)}{(e^{\varepsilon} + k)^2}\right) \cdot \nu^{2i^*} \mathbb{1} \left\{i^* \ge 1\right\} + \left(\frac{1}{e^{\varepsilon} + k}\right) \cdot \nu^{2k} \cdot \left(\frac{1 - \nu^{-2k}}{1 - \nu^{-2}}\right).$$

Now, if $r \in E_{i^*}$, we have that $\nu^{i^*} \alpha \leq (r \vee \alpha) \leq \nu^{i^*+1} \alpha$, and thus

$$\begin{split} \frac{\mathbb{E}[(Z-r)^2]}{(r\vee\alpha)^2} &\leq \frac{\alpha^2}{(r\vee\alpha)^2} \cdot \left(\frac{e^{\varepsilon}+k}{e^{\varepsilon}-1}\right)^2 \cdot \left[\frac{(k+1)(e^{\varepsilon}-1)}{(e^{\varepsilon}+k)^2} \nu^{2(i^*+1)} + \frac{\nu^{2k}}{e^{\varepsilon}+k} \left(\cdot \frac{1-\nu^{-2k}}{1-\nu^{-2}}\right)\right] \\ &\quad + \frac{\alpha^2}{(r\vee\alpha)^2} \cdot \nu^{2i^*} (\nu-1)^2 \\ &\leq \frac{(k+1)}{(e^{\varepsilon}-1)} \nu^2 + \left(\frac{\nu^{2k} \cdot (e^{\varepsilon}+k)}{(e^{\varepsilon}-1)^2}\right) \left(\frac{1-\nu^{-2k}}{1-\nu^{-2}}\right) + (\nu-1)^2, \end{split}$$

as desired.

A.5 Proof of Lemma 4.6

For shorthand, define the radius $R = \|\nabla \ell(\theta^*; X)\|_2$, $\gamma = \gamma(\varepsilon_1)$, and write

$$Z_1 = \mathtt{PrivUnit}_2\left(U; \gamma, p = 1/2\right) \qquad Z_2 = \mathtt{ScalarDP}(R, \varepsilon_2; k = \lceil e^{\varepsilon_2/3} \rceil, r_{\max})$$

so that $Z=Z_1Z_2$. Using that $\mathbb{E}[(Z_2-R)^2\mid R]\leq O(r_{\max}^2e^{-2\varepsilon_2/3})$ by Lemma 4.4, we have

$$\mathbb{E}[Z(\theta^{\star}; X)Z(\theta^{\star}; X)^{\top}] = \mathbb{E}\left[\mathbb{E}\left[Z_{1}Z_{1}^{\top} \mid U\right] \cdot \mathbb{E}\left[Z_{2}^{2} \mid R\right]\right]
= \mathbb{E}\left[\mathbb{E}\left[Z_{1}Z_{1}^{\top} \mid U\right] \cdot \mathbb{E}\left[(Z_{2} - R)^{2} \mid R\right]\right] + \mathbb{E}\left[R^{2} \cdot \mathbb{E}[Z_{1}Z_{1}^{\top} \mid U]\right]
\leq O\left(r_{\max}^{2} e^{-2\varepsilon_{2}/3}\right) \cdot \mathbb{E}\left[\mathbb{E}\left[Z_{1}Z_{1}^{\top} \mid U\right]\right] + \mathbb{E}\left[R^{2} \cdot \mathbb{E}[Z_{1}Z_{1}^{\top} \mid U]\right].$$
(24)

We now focus on the term $\mathbb{E}[Z_1Z_1^\top \mid U]$. Recall that V is uniform on $\{v \in \mathbb{S}^{d-1} : \langle v, u \rangle \geq \gamma\}$ with probability $\frac{1}{2}$ and uniform on the complement $\{v \in \mathbb{S}^{d-1} : \langle v, u \rangle < \gamma\}$ otherwise (Eq. (12)). Then for $W \sim \mathsf{Uni}(\mathbb{S}^{d-1})$, we obtain $\mathbb{E}[VV^\top \mid u] = \frac{1}{2}\mathbb{E}[WW^\top \mid \langle W, u \rangle \geq \gamma] + \frac{1}{2}\mathbb{E}[WW^\top \mid \langle W, u \rangle < \gamma]$, where

$$\mathbb{E}[WW^\top \mid \langle W, u \rangle \geq \gamma] \preceq uu^\top + \frac{1 - \gamma^2}{d}(I_d - uu^\top) \quad \text{and} \quad \mathbb{E}[WW^\top \mid \langle W, u \rangle \leq \gamma] \preceq uu^\top + \frac{1}{d}(I_d - uu^\top).$$

Both of these are in turn bounded by $uu^{\top} + (1/d)I_d$. Using that the normalization m defined in Eq. (13) satisfies $m \gtrsim \min\{\varepsilon_1, \sqrt{\varepsilon_1}\}/\sqrt{d}$ by Proposition 3, we obtain

$$\mathbb{E}[Z_1 Z_1^\top \mid U = u] \preceq \frac{1}{m^2} u u^\top + \frac{1}{dm^2} I_d \preceq \frac{d}{\varepsilon_1 \wedge \varepsilon_1^2} u u^\top + \frac{1}{\varepsilon_1 \wedge \varepsilon_1^2} I_d.$$

Substituting this bound into our earlier inequality (24) and using that $RU = \nabla \ell(\theta^*; X)$, we obtain

$$\begin{split} \mathbb{E}[Z(\theta^{\star};X)Z(\theta^{\star};X)^{\top}] & \leq O\left(\frac{dr_{\max}^{2}e^{-2\varepsilon_{2}/3}}{\varepsilon_{1}\wedge\varepsilon_{1}^{2}}\right)\mathbb{E}[UU^{\top} + (1/d)I_{d}] \\ & + O\left(\frac{d}{\varepsilon_{1}\wedge\varepsilon_{1}^{2}}\right) \cdot \mathbb{E}\left[\nabla\ell(\theta^{\star};X)\nabla\ell(\theta^{\star};X)^{\top} + (1/d)\left\|\nabla\ell(\theta^{\star};X)\right\|_{2}^{2}I_{d}\right]. \end{split}$$

Noting that $\operatorname{tr}(\operatorname{Cov}(W)) = \mathbb{E}[\|W\|_2^2]$ for any random vector W gives the lemma.

B Proofs of utility in private sampling mechanisms

B.1 Proof of Proposition 2

We divide the proof into two cases: whether $\varepsilon \geq 1$ or $\varepsilon \leq 1$.

Case 1, where $\varepsilon \geq 1$. Let us first consider the case that $\varepsilon > \frac{d}{128}$. For any set A and $x, x' \in \mathbb{S}^{d-1}$, we have $\mathbb{P}(M(x) \in A) \geq e^{-\varepsilon} \mathbb{P}(M(x') \in A)$. Let V be a maximal $\frac{1}{2}$ -packing of \mathbb{S}^{d-1} , with cardinality $|V| \geq 2^d$, and let B_x denote the ℓ_2 -ball of radius $\frac{1}{2}$ centered at x. Then for any x_0 we have

$$1 \ge \sum_{x \in V} \mathbb{P}(M(x_0) \in B_x) \ge e^{-\varepsilon} \sum_{x \in V} \mathbb{P}(M(x) \in B_x).$$

Now if $\mathbb{E}[\|M(x) - x\|_2^2 \mid x] \le c^2 \frac{d}{\varepsilon} \le 128c^2$, for $c^2 \le \frac{1}{2048}$ we have $\mathbb{P}(M(x) \in B_x) \ge \frac{3}{4}$ by Markov's inequality. Thus $1 \ge (3/4)e^{-\varepsilon}2^d$, a contradiction if $d \log 2 - \log \frac{4}{3} > \varepsilon$. In particular, if $\varepsilon < d \log 2 - \log \frac{4}{3}$, there exists $x \in \mathbb{S}^{d-1}$ such that $\mathbb{E}[\|M(x) - x\|_2^2 \mid x] \ge \frac{d}{2048\varepsilon}$.

Now, we consider the alternative that $\varepsilon \le \frac{d}{128}$. Let us give a heuristic sketch of the result before

Now, we consider the alternative that $\varepsilon \leq \frac{d}{128}$. Let us give a heuristic sketch of the result before proving it. Roughly, Barber and Duchi [13, Proposition 4] implies that for any ε -differentially private estimator $\widehat{\theta}$ of the mean of a distribution supported on \mathbb{S}^{d-1} , there exists a distribution P on \mathbb{S}^{d-1} with mean $\theta(P)$ such that

$$\mathbb{E}[\|\widehat{\theta}(X_1,\dots,X_n) - \theta(P)\|_2^2] = \Omega\left(\frac{d^2}{n^2\varepsilon^2}\right)$$

for n large enough that $n \geq \frac{d}{\varepsilon}$. Now, if Z_i are ε -locally differentially private versions of X_i , then $\overline{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i$ is certainly differentially private; if $\|Z_i\|_2 \leq r$ with probability 1 for all i then

 $\mathbb{E}[\|\overline{Z}_n - \theta(P)\|_2^2] = O\left(\frac{r^2}{n}\right). \text{ Thus, we obtain that } r^2/n = \Omega\left(\frac{d^2}{n^2\varepsilon^2}\right) \text{ for } n \geq \frac{d}{\varepsilon}, \text{ or } r^2 = \Omega\left(\frac{d^2}{n\varepsilon^2}\right); \text{ the choice } n = \frac{d}{\varepsilon} \text{ then implies that } r^2 = \Omega\left(\frac{d}{\varepsilon}\right).$

We now make this sketch rigorous using the results of Barber and Duchi [13, Proof of Proposition 4]. Indeed, their result shows that for all $p \in [0,1]$ there exists a collection of 2^d distributions $\{P_v = (1-p) \cdot \mathsf{Uni}\left(\mathbb{S}^{d-1}\right) + p \cdot \{X = v\} : v \in \mathcal{V}, \|v\|_2 = 1\}$, each supported on \mathbb{S}^{d-1} with mean $\theta_v = p \cdot v$, such that if $\widehat{\theta}$ is any ε -differentially private estimator (including centralized private estimators) based on a sample of size n then

$$\frac{1}{2^d} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v} \left[\|\widehat{\theta} - \theta_v\|_2^2 \right] \ge \left(\frac{p}{2} \right)^2 \left(\frac{2}{(2^d - 1) \exp(-\varepsilon \lceil np \rceil)} + 2 \right)^{-1}.$$

We consider a few cases. In the first, let us assume that $d \geq 3$; we will choose n so that $\frac{d}{2n\varepsilon} \leq 1$. Then taking $p = \frac{d}{2n\varepsilon}$, we have $\varepsilon \lceil np \rceil \leq \frac{d}{2} + \varepsilon \leq \frac{65d}{128}$, and so

$$\frac{1}{2^d} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v} \left[\|\widehat{\theta} - \theta_v\|_2^2 \right] \ge \frac{1}{4} \cdot \left(\frac{d}{2n\varepsilon} \right)^2 \left(\frac{2}{(2^d - 1) \exp(-65d/128)} + 2 \right)^{-1}.$$

For $d \geq 3$, the final quantity has lower bound

$$\frac{1}{2^d} \sum_{v \in \mathcal{V}} \mathbb{E}_{P_v} \left[\|\widehat{\theta} - \theta_v\|_2^2 \right] \ge \frac{d^2}{64n^2 \varepsilon^2}.$$

Now, assume that M satisfies $\mathbb{E}[\|M(x) - x\|_2^2] \leq c^2 \frac{d}{\varepsilon}$ for all $x \in \mathbb{S}^{d-1}$. Let $X_i \stackrel{\text{iid}}{\sim} P$ for a distribution P supported on \mathbb{S}^{d-1} , letting $Z_i = M(X_i)$, the sample mean $\overline{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i$ is ε -differentially private and

$$\frac{d^2}{64n^2\varepsilon^2} \le \sup_{P} \mathbb{E}_P \left[\left\| \overline{Z}_n - \mathbb{E}_P[X] \right\|_2^2 \right] \le \frac{1}{n} + \frac{c^2 d}{n\varepsilon},$$

the supremum taken over distributions supported on \mathbb{S}^{d-1} . Rearranging, we have

$$c^2 \ge \frac{d}{64n\varepsilon} - \frac{\varepsilon}{d} \ge \frac{d}{64n\varepsilon} - \frac{1}{128}$$

Set $n = \left\lceil \frac{d}{2\varepsilon} \right\rceil \le \frac{d}{2\varepsilon} + 1$, so that using $\varepsilon \le \frac{d}{128}$, we obtain

$$c^2 \ge \frac{d}{64(d/2+\varepsilon)} - \frac{1}{128} = \frac{1}{32+64\varepsilon/d} - \frac{1}{128} \ge \frac{1}{32.5} - \frac{1}{128} = \frac{1}{64}.$$

Now let us consider the alternative case that $d \leq 3$. Here, using that $\varepsilon \leq \frac{d}{128} \leq \frac{3}{128}$, the results of Duchi et al. [30] give the result.

Case 2: where $\varepsilon \leq 1$. In this case, the mean-estimation lower bounds of Duchi et al. [31, Sec. 4] show that if Z_i are ε -locally differentially private versions of X_i , with $\mathbb{E}[Z_i \mid X_i] = X_i$, where $X_i \in \mathbb{S}^{d-1}$, then $\mathbb{E}[\|\overline{Z}_n - \mathbb{E}[X]\|_2^2] = \Omega\left(\frac{d}{n\varepsilon^2} \wedge \frac{1}{\sqrt{n\varepsilon^2}} \wedge 1\right)$. Now, if $\|Z_i\|_2 \leq r$ for all i, then we must have $r^2/n+1/n \geq \mathbb{E}[\|\overline{Z}_n - \mathbb{E}[X]\|_2^2] = \Omega\left(\frac{d}{n\varepsilon^2}\right)$ for all large enough n, so that $r^2 = \Omega\left(\frac{d}{\varepsilon^2}\right)$ as desired.

B.2 Proof of Lemma 4.1

Let $u \in \mathbb{S}^{d-1}$ and $U \in \mathbb{S}^{d-1}$ be a uniform random variable on the unit sphere. Then rotational symmetry implies that for some $\gamma_+ > 0 > \gamma_-$,

$$\mathbb{E}[U \mid \langle U, u \rangle \geq \gamma] = \gamma_+ \cdot u \text{ and } \mathbb{E}[U \mid \langle U, u \rangle < \gamma] = \gamma_- \cdot u$$

and similarly, the random vector V in Algorithm 1 satisfies

$$\mathbb{E}[V \mid u] = p\mathbb{E}[U \mid \langle U, u \rangle \ge \gamma] + (1 - p) \cdot \mathbb{E}[U \mid \langle U, u \rangle < \gamma] = p(\gamma_+ + \gamma_-) \cdot u.$$

By rotational symetry, we may assume $u=e_1$, the first standard basis vector, without loss of generality. We now compute the normalization constant. Letting $U \sim \mathsf{Uni}(\mathbb{S}^{d-1})$ have coordinates U_1, \ldots, U_d , we marginally $U_i \stackrel{\text{dist}}{=} 2B-1$ where $B \sim \mathsf{Beta}(\frac{d-1}{2}, \frac{d-1}{2})$. Now, we note that

$$\gamma_{+} = \mathbb{E}[U_1 \mid U_1 \geq \gamma] = \mathbb{E}[2B - 1 \mid B \geq \frac{1+\gamma}{2}]$$
 and $\gamma_{-} = \mathbb{E}[U_1 \mid U_1 < \gamma] = \mathbb{E}[2B - 1 \mid B < \frac{1+\gamma}{2}]$

and that if $B \sim \text{Beta}(\alpha, \beta)$, then for $0 \le \tau \le 1$, we have

$$\mathbb{E}[B \mid B \ge \tau] = \frac{1}{B(\alpha, \beta) \cdot \mathbb{P}(B \ge \tau)} \int_{\tau}^{1} x^{\alpha} (1 - x)^{\beta - 1} dx = \frac{B(\alpha + 1, \beta) - B(\tau; \alpha + 1, \beta)}{B(\alpha, \beta) - B(\tau; \alpha, \beta)}$$

and similarly $\mathbb{E}[B \mid B < \tau] = \frac{B(\tau; \alpha + 1, \beta)}{B(\tau; \alpha, \beta)}$. Using that $B(\tau; \alpha + 1, \alpha) = B(\alpha + 1, \alpha) \left[\frac{B(\tau; \alpha, \alpha)}{B(\alpha, \alpha)} - \frac{\tau^{\alpha}(1 - \tau)^{\alpha}}{\alpha B(\alpha, \alpha)}\right]$ and $B(\alpha + 1, \alpha) = \frac{1}{2}B(\alpha, \alpha)$, then substituting in our calculation for γ_+ and γ_- , we have for $\tau = \frac{1 + \gamma}{2}$ and $\alpha = \frac{d-1}{2}$ that

$$\gamma_{+} = \frac{1}{\alpha \cdot 2^{d-1}} \cdot \frac{(1 - \gamma^{2})^{\alpha}}{B(\alpha, \alpha) - B(\tau; \alpha, \alpha)} \quad \text{and} \quad \gamma_{-} = \frac{-1}{\alpha 2^{d-1}} \cdot \frac{(1 - \gamma^{2})^{\alpha}}{B(\tau; \alpha, \alpha)}$$

Consequently, $\mathbb{E}[V \mid u] = (p\gamma_+ + (1-p)\gamma_-)u$, and if $Z = \text{PrivUnit}_2(u, \gamma, p)$ we have $\mathbb{E}[Z] = u$ as desired.

B.3 Proof of Proposition 3

Recall from the proof of Lemma 4.1 that $Z = \frac{1}{p\gamma_+ + (1-p)\gamma_-}V$, where $\mathbb{E}[U \mid \langle U, u \rangle \geq \gamma] = \gamma_+ \cdot u$ and $\mathbb{E}[U \mid \langle U, u \rangle < \gamma] = \gamma_- \cdot u$ for $U = (U_1, \dots, U_d) \sim \mathsf{Uni}(\mathbb{S}^{d-1})$, so that $\gamma_+ = \mathbb{E}[U_1 \mid U_1 \geq \gamma] \geq \gamma$ and, using $\mathbb{E}[|U_1|] \leq \mathbb{E}[U_1^2]^{1/2} \leq 1/\sqrt{d}$, we have $\gamma_- = \mathbb{E}[U_1 \mid U_1 \leq \gamma] \in [-\mathbb{E}[|U_1|], 0] \in [-1/\sqrt{d}, 0]$. Summarizing, we always have

$$\gamma \le \gamma_+ \le 1$$
, $-\frac{1}{\sqrt{d}} \le \gamma_- \le 0$, and $|\gamma_+| > |\gamma_-|$.

As a consequence, as the norm of V is 1 and $\|Z\|_2 = 1/(p\gamma_+ + (1-p)\gamma_-)$, which is decreasing to $1/\gamma_+$ as $p \uparrow 1$, we always have $\|Z\|_2 \le \frac{2}{\gamma_+ + \gamma_-}$ and we may assume w.l.o.g. that $p = \frac{1}{2}$ in the remainder of the derivation.

Now, we consider three cases in the inequalities (14), deriving lower bounds on $\gamma_+ + \gamma_-$ for each.

1. First, assume $5 \le \varepsilon \le 2 \log d$. Let $\gamma = \gamma_0 \sqrt{2/d}$ for some $\gamma_0 \ge 1$. Then the choice $\gamma_0 = 1$ guarantees that second inequality of (14b) holds, while we note that we must have $\gamma_0^2 \le \frac{d}{d-1}\varepsilon$, as otherwise $\frac{d-1}{2}\log(1-\gamma_0^2\frac{2}{d}) \le -\frac{d-1}{d}\gamma_0^2 < -\varepsilon$, contradicting the inequality (14b). For $\gamma_0 \in$

 $[1, \sqrt{\frac{d}{d-1}\varepsilon}]$, we have $\log(1-\gamma_0^2\frac{2}{d}) \ge -\frac{8\gamma_0^2}{3d}$ for sufficiently large d, and solving the first inequality in Eq. (14b) we see it is sufficient that

$$\frac{4(d-1)}{3d}\gamma_0^2 \le \varepsilon - \log \frac{2d\varepsilon}{d-1} - \log 6 \text{ or } \gamma^2 \le \frac{6\varepsilon - 6\log 6 - 3\log \frac{2d\varepsilon}{d-1}}{4(d-1)}$$

With this choice of γ , we obtain $\gamma_+ + \gamma_- \ge c\sqrt{\frac{d}{\varepsilon}}$ for a numerical constant c.

- 2. For $d \ge \varepsilon \ge 5$, it is evident that (for some numerical constant c) the choice $\gamma = c\sqrt{\varepsilon/d}$ satisfies inequality (14b). Thus $\gamma_+ + \gamma_- \ge c\sqrt{\frac{d}{\varepsilon}}$.
- 3. Finally, we consider the last case that $\varepsilon \leq 5$. In this case, the choice $\gamma^2 = \pi(e^{\varepsilon}-1)^2/(2d(e^{\varepsilon}+1)^2)$ satisfies inequality (14a). We need to control the difference $\gamma_+ + \gamma_- = \mathbb{E}[U_1 \mid U_1 \geq \gamma] + \mathbb{E}[U_1 \mid U_1 \leq \gamma]$. In this case, let $p_+ = \mathbb{P}(\langle U, u \rangle \geq \gamma)$ and $p_- = \mathbb{P}(\langle U, u \rangle < \gamma)$, so that Lemma C.3 implies that $p_+ \leq \frac{1}{2} e^{-2}\gamma\sqrt{\frac{d-1}{2\pi}}$ and $p_- \geq \frac{1}{2} + \gamma\sqrt{\frac{d-1}{2\pi}}$. Then

$$\mathbb{E}[U_1 \mid U_1 \ge \gamma] + \mathbb{E}[U_1 \mid U_1 < \gamma] = \frac{1}{p_+} \mathbb{E}[U_1 \cdot \mathbb{1} \{U_1 \ge \gamma\}] + \frac{1}{p_-} \mathbb{E}[U_1 \cdot \mathbb{1} \{U_1 < \gamma\}]$$

$$= \left(\frac{1}{p_+} - \frac{1}{p_-}\right) \mathbb{E}[U_1 \cdot \mathbb{1} \{U_1 \ge \gamma\}] \ge \left(1 - \frac{p_+}{p_-}\right) \mathbb{E}[U_1 \mid U_1 \ge 0],$$

where the second equality follows from the fact that $\mathbb{E}[U_1] = 0$. Using that $\mathbb{E}[U_1 \mid U_1 \geq 0] \geq cd^{-1/2}$ for a numerical constant c and

$$1 - \frac{p_+}{p_-} \ge \frac{4e^{-2}\sqrt{\frac{2(d-1)}{\pi}}}{1 + 2e^{-2}\gamma\sqrt{\frac{2(d-1)}{\pi}}} = \Omega\left(\frac{e^{\varepsilon} - 1}{e^{\varepsilon} + 1}\right)$$

by our choice of γ , we obtain $\gamma_+ + \gamma_- \gtrsim (e^{\varepsilon} - 1)\sqrt{d}$.

Combining the three cases above, we use that V in Alg. 1 has norm $\|V\|_2 = 1$ and

$$\left\|Z\right\|_{2} \leq \frac{2}{\gamma_{+} + \gamma_{-}} \left\|V\right\|_{2} \leq c\sqrt{d \cdot \max\left\{\varepsilon^{-1}, \varepsilon^{-2}\right\}}$$

to obtain the first result of the proposition.

The final result of the proposition is immediate by the bound on $||Z||_2$.

B.4 Proof of Lemma 4.2

Without loss of generality, assume that $u \in \{-1,1\}^d$, as it is clear that $\mathbb{E}[\widehat{U} \mid u] = u$ in Algorithm 2. We now show that given u, $\mathbb{E}[V \mid u = u] = m \cdot u$. Consider $U \sim \mathsf{Uni}(\{-1,+1\}^d)$, in which case

$$\mathbb{E}[V\mid u=u] = p\mathbb{E}[U\mid \langle U,u\rangle > \kappa] + (1-p)\mathbb{E}[U\mid \langle U,u\rangle \leq \kappa].$$

For constants κ_+, κ_- , uniformity of U implies that

$$\mathbb{E}[U \mid \langle U, u \rangle > \kappa] = \kappa_+ \cdot u$$
 and $\mathbb{E}[U \mid \langle U, u \rangle \leq \kappa] = \kappa_- \cdot u$.

By symmetry it is no loss of generality to assume that $u=(1,1,\cdots,1)$, so $\langle U,u\rangle=\sum_{\ell=1}^d U_\ell$. We then have for $\tau=\lceil\frac{d+\kappa+1}{2}\rceil/d$ that

$$\mathbb{E}[U_1 \mid \langle U, u \rangle > \kappa] = \frac{1}{2^d \cdot \mathbb{P}(\langle U, u \rangle > \kappa)} \cdot \sum_{\ell = d\tau}^d \left(\begin{pmatrix} d-1 \\ \ell-1 \end{pmatrix} - \begin{pmatrix} d-1 \\ \ell \end{pmatrix} \right) = \frac{\binom{d-1}{d\tau - 1}}{\sum_{\ell = d\tau}^d \binom{d}{\ell}} =: \kappa_+$$

and

$$\mathbb{E}[U_1 \mid \langle U, u \rangle \leq \kappa] = \frac{1}{2^d \cdot \mathbb{P}(\langle U, u \rangle \leq \kappa)} \cdot \sum_{\ell=0}^{d\tau-1} \left(\binom{d-1}{\ell-1} - \binom{d-1}{\ell} \right) = \frac{-\binom{d-1}{d\tau-1}}{\sum_{\ell=0}^{d\tau-1} \binom{d}{\ell}} =: \kappa_-.$$

Putting these together and setting $m = p\kappa_+ + (1-p)\kappa_-$, we have $\mathbb{E}[(1/m)V \mid u] = u$.

B.5 Proof of Proposition 4

Using the notation of the proof of Lemma 4.2, the debiasing multiplier m satisfies $1/m = \frac{1}{p\kappa_+ + (1-p)\kappa_-} \le \frac{2}{\kappa_+ + \kappa_-}$, as $p \ge \frac{1}{2}$. Thus, we seek to lower bound $\kappa_+ + \kappa_-$ by lower bounding κ_+ and κ_- individually. We consider two cases: the case that $\varepsilon \ge \log d$ and the case that $\varepsilon < \log d$.

Case 1: when $\varepsilon \ge \log d$. We first focus on lower bounding

$$\kappa_{+} = \frac{\binom{d-1}{d\tau - 1}}{\sum_{\ell = d\tau}^{d} \binom{d}{\ell}} = \tau \cdot \frac{\binom{d}{d\tau}}{\sum_{\ell = d\tau}^{d} \binom{d}{\ell}} = \tau \cdot \left(\sum_{\ell = d\tau}^{d} \binom{d}{d\tau}^{-1} \binom{d}{\ell}\right)^{-1}.$$

We argue that eventually the terms in the summation become small. For $\ell \geq d\tau$ defining

$$r := \frac{d\tau + 1}{d - d\tau} \le \frac{\ell + 1}{d - \ell} = \frac{\binom{d}{\ell}}{\binom{d}{\ell + 1}} \quad \text{implies} \quad \binom{d}{\ell} \ge r \binom{d}{\ell + 1},$$

We then have

$$\sum_{\ell=d\tau}^d \binom{d}{d\tau}^{-1} \binom{d}{\ell} = \sum_{i=0}^{d-d\tau} \binom{d}{d\tau}^{-1} \binom{d}{d\tau+i} \le \sum_{i=0}^{d-d\tau} \binom{d}{d\tau}^{-1} \binom{d}{d\tau} \left(\frac{1}{r}\right)^i = \sum_{i=0}^{d-d\tau} \frac{1}{r^i} \le \frac{1}{1-1/r},$$

so that

$$\kappa_{+} \ge 1 - 1/r = 1 - \frac{d - d\tau}{d\tau + 1} = \frac{2d\tau - d + 1}{d\tau + 1} = \begin{cases} 2 \cdot \frac{\kappa + 2}{d + \kappa + 3} & \text{for odd } d + \kappa \\ 2 \cdot \frac{\kappa + 3}{d + \kappa + 4} & \text{for even } d + \kappa \end{cases} \ge \frac{\kappa}{d}. \tag{25}$$

We now lower bound κ_- , where we use the fact that $d\tau - 1 \ge \frac{d-1}{2}$ for $\kappa \in \{0, 1, \dots, d\}$ to obtain

$$\kappa_{-} = \frac{-\binom{d-1}{d\tau - 1}}{\sum_{\ell=0}^{\tau \cdot d - 1} \binom{d}{\ell}} = -\tau \frac{\binom{d}{d\tau}}{\sum_{\ell=0}^{d\tau - 1} \binom{d}{\ell}} \ge -\tau \frac{\binom{d}{d\tau}}{2^{d-1}}.$$

Using Stirling's approximation and that $\tau = \lceil \frac{d+\kappa+1}{2} \rceil/d$, we have

$$\begin{pmatrix} d \\ d\tau \end{pmatrix} = C(\kappa, d) \sqrt{\frac{1}{d}} \cdot 2^d \cdot \exp\left(-\frac{\kappa^2}{2d}\right),$$

where $C(\kappa, d)$ is upper and lower bounded by positive universal constants. Hence, we have for a constant $c < \infty$,

$$\kappa_{-} \ge -c \cdot \left(\frac{1}{\sqrt{d}} \cdot \exp\left(\frac{-\kappa^2}{2d}\right)\right).$$
(26)

An inspection of the bound (19) shows that the choice $\kappa = c\sqrt{\varepsilon d}$ for some (sufficiently small) constant c immediately satisfies the sufficient condition for Algorithm 2 to be private. Substituting this choice of κ into the lower bounds (25) and (26) gives that the normalizer $m^{-1} \leq \frac{2}{\kappa_+ + \kappa_-} \lesssim \sqrt{d/\varepsilon}$, which is first result of Proposition 4.

Case 2: when $\varepsilon < \log d$. In this case, we use the bound (18) to obtain the result. Let us first choose κ to saturate the bound (18), for which it suffices to choose $\kappa = c \min\{\sqrt{d}, \varepsilon \sqrt{d}\}$ for a numerical constant c > 0. We assume for simplicity that d is even, as extending the argument is simply notational. Defining the shorthand $s_{\tau} := \frac{1}{2^{d-1}} \sum_{\ell=d/2}^{d\tau-1} \binom{d}{\ell}$, we recall the definitions of κ_+ and κ_- to find

$$\kappa_{+} = \tau \cdot \begin{pmatrix} d \\ d\tau \end{pmatrix} \cdot \frac{1}{1 - s_{\tau}} \quad \text{and} \quad \kappa_{-} = \tau \cdot \begin{pmatrix} d \\ d\tau \end{pmatrix} \cdot \frac{1}{1 + s_{\tau}}.$$

Using the definition of the debiasing normalizer $m = p\kappa_+ + (1-p)\kappa_- \ge \frac{1}{2}(\kappa_+ + \kappa_-)$, we obtain

$$m \ge \frac{\tau}{2^d} \binom{d}{d\tau} \left(\frac{1}{1 - s_\tau} - \frac{1}{1 + s_\tau} \right) = \frac{\tau}{2^d} \binom{d}{d\tau} \frac{2s_\tau}{1 - s_\tau^2}.$$

By definition of s_{τ} , we have $s_{\tau} \geq \frac{d\tau - d/2}{2^{d-1}} {d \choose d\tau} \geq \frac{\kappa}{2^d} {d \choose d\tau}$, so that

$$m \gtrsim \tau \kappa \left(2^{-d} \binom{d}{d\tau}\right)^2 \gtrsim \frac{1}{d} \kappa \exp\left(-\frac{\kappa^2}{d}\right),$$

where the second inequality uses Stirling's approximation. Our choice of $\kappa = c \min\{\sqrt{d}, \varepsilon\sqrt{d}\}$ thus yields $m \gtrsim \frac{1}{\sqrt{d}} \min\{1, \varepsilon\}$, which gives the second result of Proposition 4.

C Uniform random variables and concentration

In this section, we collect a number of results on the concentration properties of variables uniform on the unit sphere \mathbb{S}^{d-1} , which allow our analysis of the mechanism $PrivUnit_2$ for privatizing vectors in the ℓ_2 ball in Algorithm 1. For a vector $u \in \mathbb{S}^{d-1}$, that is, satisfying $||u||_2 = 1$, and $a \in [0,1]$ we define the spherical cap

$$C(a,u) := \left\{ v \in \mathbb{S}^{d-1} \mid \langle v, u \rangle > a \right\}.$$

There are a number of bounds on the probability that $U \in C(a, u)$ for a fixed $u \in \mathbb{S}^{d-1}$ where $U \sim \mathsf{Uni}(\mathbb{S}^{d-1})$, which the following lemma summarizes.

Lemma C.1. Let U be uniform on the unit sphere \mathbb{S}^{d-1} in \mathbb{R}^d . Then for $\sqrt{2/d} \leq a \leq 1$ and $u \in \mathbb{S}^{d-1}$,

$$\frac{1}{6a\sqrt{d}}(1-a^2)^{\frac{d-1}{2}} \le \mathbb{P}(U \in C(a,u)) \le \frac{1}{2a\sqrt{d}}(1-a^2)^{\frac{d-1}{2}}.$$
 (27)

For all $0 \le a \le 1$, we have

$$\frac{(1-a)}{2d}(1-a^2)^{\frac{d-1}{2}} \le \mathbb{P}(U \in C(a,u))$$
 (28)

and for $a \in [0, 1/\sqrt{2}]$,

$$\mathbb{P}(U \in C(a, u)) \le (1 - a^2)^{\frac{d}{2}}.$$

Proof The first result is [19, Exercise 7.9]. The lower bound of the second is due to [58, Lemma 4.1], while the third inequality follows for all $a \in [0, 1/\sqrt{2}]$ by [12, Proof of Lemma 2.2].

We will use the lower bound in our privacy analysis and use the upper bound in bounding the probability of a breach in Proposition 1. Summarizing the upper bounds, we get

$$\mathbb{P}(\langle U, v_0 \rangle > a) \le \begin{cases} \sqrt{1 - a^2}^d & \text{if } a \in [0, 1/\sqrt{2}] \\ \frac{\sqrt{1 - a^2}^{d-1}}{2a\sqrt{d}} & \text{if } a \in [\sqrt{2/d}, 1] \end{cases}$$
 (29)

We also require a slightly different lemma for small values of the threshold a in Lemma C.1.

Lemma C.2. Let $\gamma \geq 0$ and U be uniform on \mathbb{S}^{d-1} . Then

$$\gamma \sqrt{\frac{d-1}{2\pi}} \exp\left(-\frac{1}{4d-4}-1\right) \mathbbm{1}\left\{\gamma \leq \sqrt{2/(d-3)}\right\} \leq \mathbb{P}(\langle U,u \rangle \in [0,\gamma]) \leq \gamma \sqrt{\frac{d-1}{2\pi}}$$

Proof For any fixed unit vector $u \in \mathbb{S}^{d-1}$ and $U \sim \mathsf{Uni}(\mathbb{S}^{d-1})$, we have that marginally $\langle U, u \rangle \sim 2B-1$ for $B \sim \mathsf{Beta}(\frac{d-1}{2}, \frac{d-1}{2})$. Thus we have

$$\mathbb{P}(\langle U, u \rangle \in [0, \gamma]) = \mathbb{P}\left(\frac{1}{2} \le B \le \frac{1+\gamma}{2}\right)$$

We will upper and lower bound the last probability above. Letting $\alpha = \frac{d-1}{2}$ for shorthand, we have

$$\mathbb{P}\left(\frac{1}{2} \le B \le \frac{1+\gamma}{2}\right) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2} \int_{\frac{1}{2}}^{\frac{1+\gamma}{2}} t^{\alpha-1} (1-t)^{\alpha-1} dt$$

$$\stackrel{(i)}{=} \frac{\Gamma(2\alpha)}{2\Gamma(\alpha)^2} \int_0^{\gamma} \left(\frac{1+u}{2}\right)^{\alpha-1} \left(\frac{1-u}{2}\right)^{\alpha-1} du = \frac{\Gamma(2\alpha)}{2^{2\alpha-1}\Gamma(\alpha)^2} \int_0^{\gamma} (1-u^2)^{\alpha-1} du,$$

where equality (i) is the change of variables u = 2t - 1. Using Stirling's approximation, we have that

$$\log \Gamma(2\alpha) - 2\log \Gamma(\alpha) = (2\alpha - 1)\log 2 + \frac{1}{2}\log \frac{\alpha}{\pi} + \operatorname{err}(\alpha),$$

where $\operatorname{err}(\alpha) \in [-\frac{1}{8\alpha}, -\frac{1}{8\alpha+1}]$. When $\gamma \leq \sqrt{1/(\alpha-1)}$, we have

$$\int_0^{\gamma} (1 - u^2)^{\alpha - 1} du \ge \int_0^{\gamma} \left(1 - \left(\sqrt{\frac{1}{\alpha - 1}} \right)^2 \right)^{\alpha - 1} du \ge \gamma e^{-1}$$

Otherwise, if $\gamma > \sqrt{1/(\alpha - 1)}$ we have the trivial bound $\int_0^{\gamma} (1 - u^2)^{\alpha - 1} du \ge 0$. Furthermore, for $\gamma \ge 0$ we have $\int_0^{\gamma} (1 - u^2)^{\alpha - 1} du \le \gamma$. Putting this all together, we have

$$\exp\left(-\frac{1}{8\alpha}\right) \cdot \left(\sqrt{\frac{\alpha}{\pi}}\right) \cdot \left(\gamma e^{-1}\right) \cdot \mathbb{1}\left\{\gamma \le (\alpha - 1)^{-\frac{1}{2}}\right\}$$

$$\le \exp(\operatorname{err}(\alpha))\sqrt{\frac{\alpha}{\pi}} \int_0^{\gamma} (1 - u^2)^{\alpha - 1} du = \mathbb{P}\left(\frac{1}{2} \le B \le \frac{1 + \gamma}{2}\right) \le \gamma \sqrt{\frac{\alpha}{\pi}}.$$

Substituting $\alpha \mapsto \frac{d-1}{2}$ yields the desired upper and lower bounds on $\mathbb{P}(\langle U, u \rangle \in [0, \gamma])$.

As a consequence of Lemma C.2, we have the following result.

Lemma C.3. Let $\gamma \in [0, \sqrt{2/(d-3)}]$ and U be uniform on \mathbb{S}^{d-1} . Then

$$\frac{1}{2}-\gamma\sqrt{\frac{d-1}{2\pi}}\leq \mathbb{P}(U\in C(\gamma,u))\leq \frac{1}{2}-\gamma\sqrt{\frac{d-1}{2\pi}}e^{-\frac{4d-3}{4d-4}}.$$

Proof We have

Ensure: V

$$\mathbb{P}\left(U \in C(\gamma, u)\right) = 1 - \mathbb{P}\left(\langle U, u \rangle < \gamma\right) = 1/2 - \mathbb{P}\left(\langle U, u \rangle \in [0, \gamma)\right).$$

We then use Lemma C.2

D Efficient sampling of unit vectors in PrivUnit₂

We now describe tools for performing precise sampling from the distribution (12). This will require sampling from a conditional beta distribution which turns out to be unstable as the dimension d grows large. For our setting, we want our algorithms to scale with the number of parameters in state of the art neural nets, which requires d to be in the thousands or even millions.

We first present a way to sample (12) in Algorithm 5 with procedure Sample and then show how we can sample a conditional beta distribution.

```
Algorithm 5 Sampling unit vectors according to (12): Sample
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 \begin{aligned} & \mathbf{Require:} \ u \in \mathbb{S}^{d-1} \ \text{and} \ \gamma \in [0,1]. \\ & \mathbf{Sample} \ Y = \mathsf{Bern}(1/2) \\ & \mathbf{if} \ Y = 0 \ \mathbf{then} \\ & \mathbf{Sample} \ B' = 2B - 1 \ \text{where} \ B \sim \mathsf{Beta} \left( \frac{d-1}{2}, \frac{d-1}{2} \right) \ \text{conditioned on} \ B \geq \frac{\gamma+1}{2} \\ & \mathbf{Set} \ \widehat{U} \leftarrow B' \cdot u \\ & \mathbf{Draw} \ U \sim \mathsf{Uni}(\mathbb{S}^{d-1}). \ \# \ \mathsf{Can} \ \mathsf{do} \ \mathsf{this} \ \mathsf{by} \ \mathsf{sampling} \ \mathsf{a} \ d \ \mathsf{standard} \ \mathsf{gaussians} \ \mathsf{and} \ \mathsf{then} \ \mathsf{normalizing}. \\ & \mathbf{Set} \ V \leftarrow \sqrt{1 - B'^2} \cdot \frac{(I - u u^\intercal) U}{\|(I_d - u u^\intercal) \cdot U\|_2} \\ & \mathbf{Set} \ V \leftarrow \widehat{U} + V \end{aligned}   \mathbf{else}   \mathbf{Do} \ \mathsf{rejection} \ \mathsf{sampling}, \ \mathsf{i.e.} \\ & \mathbf{Bool} \leftarrow \mathbf{True} \\ & \mathbf{while} \ \mathsf{Bool} \ \mathsf{do} \\ & \mathbf{Draw} \ U \sim \mathsf{Uni}(\mathbb{S}^{d-1}). \\ & \mathbf{if} \ \langle U, u \rangle < \gamma \ \mathbf{then} \\ & V \leftarrow U \\ & \mathbf{Bool} = \mathbf{False} \end{aligned}
```

D.1 Sampling conditional beta distributions with large shape parameters

The main difficulty with directly using Sample is that for large d, sampling 2B-1 where $B \sim \text{Beta}\left(\frac{d-1}{2},\frac{d-1}{2}\right)$ conditional on $B \geq \frac{\gamma+1}{2}$ is not stable. We next propose an efficient way to sample a conditional beta using the continued fraction representation of a incomplete beta function and then applying the inverse sampling technique.

Let $\tau = \frac{\gamma+1}{2}$ for simplicity, and define $p_{\tau}(x)$ to be the density of $B \sim \mathsf{Beta}(\alpha, \beta)$ conditional on $B \geq \tau$. Then we have the density

$$p_{\tau}(x) = \frac{x^{\alpha - 1}(1 - x)^{\beta - 1}}{B(\alpha, \beta)} \cdot \frac{1}{\mathbb{P}(B \ge \tau)} \cdot \mathbb{1} \left\{ x \in [\tau, 1] \right\},\,$$

and its CDF is

$$F_{\tau}(x) = \frac{\int_{\tau}^{x} t^{\alpha-1} (1-t)^{\beta-1} dt}{B(\alpha,\beta)} \cdot \frac{1}{\mathbb{P}(B \ge \tau)}$$

$$= \frac{\int_{0}^{x} t^{\alpha-1} (1-t)^{\beta-1} dt - \int_{0}^{\tau} t^{\alpha-1} (1-t)^{\beta-1} dt}{B(\alpha,\beta)} \cdot \frac{1}{\mathbb{P}(B \ge \tau)}$$

$$= \frac{B(x;\alpha,\beta) - B(\tau;\alpha,\beta)}{B(\alpha,\beta) - B(\tau;\alpha,\beta)},$$
(30)

where $B(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$ denotes the incomplete beta function, and we have used that $\mathbb{P}(B\geq\tau)=\frac{B(\alpha,\beta)-B(\tau;\alpha,\beta)}{B(\alpha,\beta)}$. To sample from this, we use the inverse CDF transform, that is, we draw $U\sim \mathsf{Uni}[0,1]$ and then set $B=F_\tau^{-1}(U)$.

However, computing an incomplete beta function with large shape parameters in standard packages has precision issues. Instead, we will use a continue fraction approach to computing an incomplete beta function.

D.1.1 Continued Fractions Preliminaries

Consider the general form of a continued fraction

$$r_0 + \frac{a_1}{r_{1+}} + \frac{a_2}{r_{2+}} + \cdots + \frac{a_{n-1}}{r_{n-1+}} + \frac{a_n}{r_n} := r_0 + \frac{a_1}{r_1 + \frac{a_2}{r_2 + \frac{a_3}{r_2}}} \cdot \cdot \cdot + \frac{a_n}{r_n}$$

We then have the following recurrence relation for continued fractions, which is from [42] but we prove here for completeness.

Claim D.1. Let $y_n = r_0 + \frac{a_1}{r_1 +} + \cdots + \frac{a_n}{r_n}$. We define $A_{-1} = 1$, $A_0 = r_0$, $R_{-1} = 0$ and $R_0 = 1$. For

$$A_n = r_n A_{n-1} + a_n A_{n-2} \qquad \& \qquad R_n = r_n R_{n-1} + a_n r_{n-2}$$
(31)

we have

$$y_n = \frac{A_n}{R_n} \tag{32}$$

Proof We prove this with induction. Let n = 1 so that

$$y_1 = r_0 + \frac{a_1}{r_1} = \frac{r_0 r_1 + a_1}{r_1} = \frac{A_1}{r_1}$$

We now prove the induction step. Consider the function

$$y_{n-1}(x) = r_0 + \frac{a_1}{r_{1+}} + \dots + \frac{a_{n-1}}{r_{n-1} + x}.$$

With this notation, we have $y_n = y_{n-1}(a_n/r_n)$. By induction, we have

$$y_{n-1}(a_n/r_n) = \frac{\left(r_{n-1} + \frac{a_n}{r_n}\right) A_{n-2} + a_{n-1}A_{n-3}}{\left(r_{n-1} + \frac{a_n}{r_n}\right) R_{n-2} + a_{n-1}R_{n-3}}$$

$$= \frac{r_{n-1}r_n A_{n-2} + a_n A_{n-2} + a_{n-1}r_n A_{n-3}}{r_{n-1}r_n R_{n-2} + a_n R_{n-2} + a_{n-1}r_n R_{n-3}}$$

$$= \frac{r_n \left(r_{n-1}A_{n-2} + a_{n-1}A_{n-3}\right) + a_n A_{n-2}}{r_n \left(r_{n-1}R_{n-2} + a_{n-1}R_{n-3}\right) + a_n R_{n-2}}$$

$$= \frac{r_n \left(A_{n-1}\right) + a_n A_{n-2}}{r_n \left(R_{n-1}\right) + a_n R_{n-2}} = \frac{A_n}{R_n}.$$

This proves the recurrence.

We also present a more stable recurrence relation than the one presented above

$$y_n = C_n D_n y_{n-1}$$
 where $C_n = \frac{A_n}{A_{n-1}}$ & $D_n = \frac{R_{n-1}}{R_n}$ (33)

To obtain a good estimate for the limit y_{∞} we continue with finite n until the product C_nD_n is close to 1.

D.1.2 Using continued fractions to sample a conditional beta random variable

Recall that the incomplete beta function is defined as the following

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha - 1} (1 - t)^{\beta - 1} dt$$

which can then be written as a continued fraction

$$B(x; \alpha, \beta) = \frac{x^{\alpha} (1 - x)^{\beta}}{\alpha} \left(\underbrace{\frac{1}{1 + \frac{d_1}{1 + \frac{d_2}{1 + \dots}}}_{CF(x, \alpha, \beta)}} \right) \quad \text{where } \begin{cases} d_{2k} = \frac{k(\beta - k)x}{(\alpha + 2k - 1)(\alpha + 2k)} \\ d_{2k+1} = \frac{-(\alpha + k)(\alpha + \beta + k)x}{(\alpha + 2k)(\alpha + 2k + 1)} \end{cases}$$
(34)

With this formulation of the regularized incomplete beta function, we use the recurrence relation for continued fractions in (33) to find a good estimate for $I(x, \alpha, \beta)$. We first simplify the CDF for $B \sim \text{Beta}\left(\frac{d-1}{2}, \frac{d-1}{2}\right)$ conditional on $B \geq \tau$.

$$F_{\tau}(x) = \mathbb{P}\left[B \le x \mid B \ge \tau\right]$$

$$= \mathbb{P}\left[B \ge 1 - x \mid B \le 1 - \tau\right]$$

$$= 1 - \frac{B\left(1 - x; \frac{d-1}{2}, \frac{d-1}{2}\right)}{B\left(1 - \tau; \frac{d-1}{2}, \frac{d-1}{2}\right)}$$

$$= 1 - \left(\frac{x(1 - x)}{\tau(1 - \tau)}\right)^{\frac{d-1}{2}} \frac{CF\left(1 - x; \frac{d-1}{2}, \frac{d-1}{2}\right)}{CF\left(1 - \tau; \frac{d-1}{2}, \frac{d-1}{2}\right)}$$
(35)

To sample from a conditional beta, we then use the inverse sampling technique, i.e. we draw $U \sim \mathsf{Uni}[0,1]$ and solve the following equation for x

$$U = F_{\tau}(x)$$
.

Then the resulting x is drawn according to the conditional beta distribution. Note that in Algorithm 5, we use B'=2B-1 where $B\sim \text{Beta}\left(\frac{d-1}{2},\frac{d-1}{2}\right)$ conditioned on $B\geq \frac{\gamma+1}{2}$. The CDF for B' is then $F_{\tau}\left(\frac{x+1}{2}\right)$. Hence to sample B' directly, we sample $U\sim \text{Uni}[0,1]$ and solve for $x\geq \gamma$ such that

$$\log(1 - U) = \log\left(1 - F_{\tau}\left(\frac{x+1}{2}\right)\right)$$

$$\implies \log(1 - U) = \left(\frac{d-1}{2}\right) \cdot \left(\log(1 - x^{2}) - \log(1 - \gamma^{2})\right) + \log\left(\frac{CF\left(\frac{1-x}{2}; \frac{d-1}{2}, \frac{d-1}{2}\right)}{CF\left(\frac{1-\gamma}{2}; \frac{d-1}{2}, \frac{d-1}{2}\right)}\right)$$

A natural algorithm is a simple binary search (though given that $\frac{\partial}{\partial x}F_{\tau}(x)$ is efficiently computable, Newton's method may be more effective).