1. 解: ① $12\left(\frac{dy}{dx}\right)^2-8\frac{dy}{dx}+1=0.$

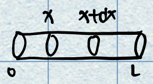
② $a_{12}^2-a_{11}a_{22}=16-12>0.$ 双曲型

③ $\begin{cases} C_1=y-\frac{1}{2}x \\ C_2=y-\frac{1}{6}x \end{cases}$ 则令 $\begin{cases} \xi=y-\frac{1}{2}x \\ \eta=y-\frac{1}{6}x. \end{cases}$ $Q=\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{6} & 1 \end{pmatrix}$

④. $\begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}=Q\begin{pmatrix} 12 & 4 \\ 4 & 1 \end{pmatrix}Q^T=\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{6} & 1 \end{pmatrix}\begin{pmatrix} 12 & 4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & -\frac{2}{3} \\ -\frac{2}{3} & 0 \end{pmatrix}$

$\overline{b_1}=L\xi-C\xi=0$ $\overline{b_2}=L\eta-C\eta=0.$ $\overline{c}=0$ $\overline{f}=0$

$\Rightarrow u_{\xi\eta}=0.$ 则 $u=f(y-\frac{1}{2}x)+g(y-\frac{1}{6}x)$

2. 解. 

(1) $dQ=CA\rho dx[u(x,t+dt)-u(x,t)]=CA\rho dx\,u_t dt$

$dQ_1=-kAu_x(x,t)dt$ $dQ_2=-kAu_x(x+dx,t)dt$

又 $dQ'=dQ_1-dQ_2$ $\Rightarrow u_t=\frac{k}{C\rho}u_{xx}=a^2u_{xx}$

$\Rightarrow \begin{cases} u_t=a^2u_{xx} \\ u|_{x=0}=u|_{x=L}=0 \\ u|_{t=0}=\varphi(x) \end{cases}$ 令 $u(x,t)=X(x)T(t)$ $\Rightarrow \begin{cases} X''+\lambda X=0 \\ T'+\lambda a^2T=0 \end{cases}$ 则有 $\begin{cases} X''+\lambda X=0 \\ X(0)=X(L)=0 \end{cases}$ 则 $\lambda_n=\left(\frac{n\pi}{L}\right)^2$

$X_n(x)=\sin\frac{n\pi}{L}x$

$T_n(t)=C\cdot e^{-\lambda a^2t}=C\cdot e^{-\left(\frac{n\pi a}{L}\right)^2t}$

$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}C\cdot e^{-\left(\frac{n\pi a}{L}\right)^2t}\sin\frac{n\pi}{L}x$

$\varphi(x)=\sum_{n=1}^{\infty}C\cdot\sin\frac{n\pi}{L}x.$ $\Rightarrow C=\frac{2}{L}\int_0^L\varphi(x)\sin\frac{n\pi}{L}x.dx$

(2) 当 $u|_{t=0}=D$ 时, $C=\frac{2}{L}\int_0^L D\cdot\sin\frac{n\pi}{L}x.dx=(1-(-1)^n)\frac{2D}{n\pi}$

则 $u(x,t)=\sum_{n=1}^{\infty}[1-(-1)^n]\frac{2D}{n\pi}\cdot e^{-\left(\frac{n\pi a}{2}\right)^2t}\sin\frac{n\pi}{L}x.$

3. 解: 令 $u(x,t)=V(x,t)+W(x,t)$ 令 $W(x,t)=Ax+B.$

$\Rightarrow \begin{cases} u_0=B \\ \frac{q_0}{k}=AL+B, \end{cases}$ $\Rightarrow \begin{cases} A=\frac{q_0-ku_0}{Lk} \\ B=u_0 \end{cases}$ 则 $W(x,t)=\frac{q_0-ku_0}{Lk}x+u_0$

$\Rightarrow \begin{cases} V_t=a^2V_{xx} \\ V|_{x=0}=V|_{x=L}=0 \\ V|_{t=0}=u_0 \end{cases}$ 令 $V(x,t)=X(x)T(t)$ $\Rightarrow \begin{cases} X''+\lambda X=0 \\ T'+\lambda a^2T=0 \end{cases}$ 有 $\begin{cases} X''+\lambda X=0 \\ X(0)=X(L)=0 \end{cases}$ $\lambda_n=\left(\frac{n\pi}{L}\right)^2$

$X_n(x)=\sin\frac{n\pi}{L}x$

$T_n(t)=C\cdot e^{-\lambda a^2}=C\cdot e^{-\left(\frac{n\pi a}{L}\right)^2t}$

则 $V(x,t)=\sum_{n=1}^{\infty}C\cdot e^{-\left(\frac{n\pi a}{L}\right)^2t}\cdot\sin\frac{n\pi}{L}x$

$\Rightarrow u_0=\sum_{n=1}^{\infty} C\cdot\sin\frac{n\pi}{L}x.$ 则 $C=\frac{2}{L}\int_0^L u_0\sin\frac{n\pi}{L}x\,dx=\frac{2u_0}{n\pi}[1-(-1)^n]$

$\Rightarrow v(x,t)=\sum_{k=1}^{\infty}[1-(-1)^n]\frac{2u_0}{n\pi}\cdot e^{-(\frac{n\pi a}{L})^2 t}\sin\frac{n\pi}{L}x$

则 $u(x,t)=v(x,t)+w(x,t)$

4.解: $\frac{1}{(s+1)(s-3)^2}=\frac{\frac{1}{16}}{s+1}+\frac{-\frac{1}{16}}{s-3}+\frac{\frac{1}{4}}{(s-3)^2}$

$\Rightarrow f(x)=\frac{1}{16}e^{-x}-\frac{1}{16}e^{3x}+\frac{1}{4}xe^{3x}$

5.解: 令 $u(x,t)=v(x,t)+w(x,t)$

$$\begin{cases} v_{tt}=a^2v_{xx} & x\in R,\ t>0 \\ v|_{t=0}=x \\ v_t|_{t=0}=\sin x \end{cases}$$

$\Rightarrow v(x,t)=\frac{1}{2}[x+at+x-at]+\frac{1}{2a}\int_{x-at}^{x+at}\sin\xi\,d\xi$

$=x+\frac{1}{a}\sin x\sin at$

$$\begin{cases} w_{tt}=a^2w_{xx}+x+at & x\in R,\ t>0 \\ w|_{t=0}=0 \\ w_t|_{t=0}=0. \end{cases}$$

$\Rightarrow w(x,t)=\int_0^t\frac{1}{2a}\int_{x-a(t-\tau)}^{x+a(t-\tau)}(\xi+a\tau)\,d\xi\,d\tau$

$=\frac{1}{4}xt^2+\frac{1}{6}at^3$

$\Rightarrow u(x,t)=x+\frac{1}{a}\sin x\sin at+\frac{1}{4}xt^2+\frac{1}{6}at^3$

6.解: $\int J_3(x)dx$ $\quad [x^{-n}J_n(x)]'=-x^{-n}J_{n+1}(x)$

$=\int(-x^2)(-x^{-2})J_3(x)dx$

$=\int -x^2\,d\,x^{-2}J_2(x)$

$=-J_2(x)+2\int x^{-1}J_2(x)dx$

$=-J_2(x)-2x^{-1}J_1(x)+C$

7.解: $$\begin{cases} \Delta G=-\delta(M-M_0) & x>0,\ y>0 \\ G|_{x=y=0}=0. \end{cases}$$

$G(M,M_0)=\frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_0}}-\ln\frac{1}{r_{MM_1}}-\ln\frac{1}{r_{MM_2}}+\ln\frac{1}{r_{MM_3}}\right)$