① 贝塞尔方程

1) 方程形式：$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0$.

2) 解的形式：
$$\begin{cases} \text{当}n\text{为分数时,} & y=aJ_n(x)+bJ_{-n}(x) \\ \text{当}n\text{为整数时,} & y=aJ_n(x)+bY_n(x) \end{cases}$$

② 贝塞尔函数

1) n阶第一类Bessel函数：$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}\,m!\,\Gamma(n+m+1)}$

2) n阶第二类Bessel函数：$Y_n(x)$.

3) 半奇阶 Bessel函数 $J_{\frac12}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ $\qquad J_{-\frac12}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

证：
$$J_{\frac12}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{\frac12+2m}}{2^{\frac12+2m}\,m!\,\Gamma(\frac12+m+1)}$$

其中 $\Gamma(\frac32+m)=\frac12(1+\frac12)\cdots(m+\frac12)\Gamma(\frac12)$
$$=\frac12\cdot\frac32\cdots\frac{2m+1}{2}\cdot\sqrt{\pi}$$
$$=\frac{1\cdot3\cdots(2m+1)}{2^{m+1}}\sqrt{\pi}$$

$$\Rightarrow J_{\frac12}(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{\frac12+2m}\cdot2^{m+1}}{2^{\frac12+2m}\,m!\,1\cdot3\cdots(2m+1)\cdot\sqrt{\pi}}$$
$$=\sum_{m=0}^{\infty}(-1)^m\frac{x^{\frac12+2m}\,2^{2m+1}}{2^{\frac12+2m}(2m+1)!\sqrt{\pi}}$$
$$=\sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^{2m+1}\qquad=\sqrt{\frac{2}{\pi x}}\sin x$$

4) 母函数：$w(z)=e^{\frac{x}{2}(z-\frac1z)}=\sum_{n=-\infty}^{+\infty}J_n(x)z^n$，用于证 $J_n(x+y)=\sum_{k=-\infty}^{+\infty}J_k(x)J_{n-k}(y)$

5) 递推关系式：
$$\begin{cases}
(x^nJ_n(x))'=x^nJ_{n-1}(x) & ① \quad \text{证①②} \\
(x^{-n}J_n(x))'=-x^{-n}J_{n+1}(x) & ②. \\
xJ_n'(x)+nJ_n(x)=xJ_{n-1}(x) & ①\text{求导展开}\Rightarrow③ \\
xJ_n'(x)-nJ_n(x)=-xJ_{n+1}(x) & ②\text{求导展开}\Rightarrow④ \\
J_{n-1}(x)+J_{n+1}(x)=\frac{2}{x}nJ_n(x) & ③-④ \\
J_{n-1}(x)-J_{n+1}(x)=2J_n'(x) & ③+④
\end{cases}$$

例：求 $J_{3/2}(x)$, $J_{-3/2}(x)$

解：$J_{3/2}(x)=\frac{2}{x}\cdot\frac12J_{\frac12}(x)-J_{-\frac12}(x)$
$$=\frac1x\cdot\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left(\frac{\sin x}{x} - \cos x \right)$$

$$J_{-3/2}(x) = \frac{2}{x}\left(-\frac{1}{2}\right) J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\frac{1}{x}\sqrt{\frac{2}{\pi x}} \cos x - \sqrt{\frac{2}{\pi x}} \sin x$$

例: $\int x^3 J_0(x) dx. \quad = \int x^2 \cdot x J_0(x)\, dx. \quad = \int x^2\, d\, x J_1(x)$

$$= x^3 J_1(x) - 2\int x^2 J_1(x) dx$$

$$= x^3 J_1(x) - 2x^2 J_2(x) + C$$

$$= \int x^4 (x^{-1} J_{-2}(x))\, dx \quad = \int x^4\, d\, x^{-1} J_{-1}(x)$$

$$= x^3 J_{-1}(x) - 4\int x^2 J_{-1}(x)$$

$$= x^3 J_{-1}(x) - 4 \int x^2\, d J_0(x)$$

$$= x^3 J_{-1}(x) - 4x^2 J_0(x) + 8\int x J_0(x) dx$$

$$= x^3 J_{-1}(x) - 4x^2 J_0(x) + 8 x J_1(x)$$