1. 解：① $12\left(\frac{dy}{dx}\right)^2-8\frac{dy}{dx}+1=0$、

② $a_{12}^2-a_{11}a_{22}=16-12>0$，双曲型。 则 $\begin{cases} C_1=y-\frac{1}{2}x \\ C_2=y-\frac{1}{6}x \end{cases}$ 令 $\begin{cases} \xi=y-\frac{1}{2}x \\ \eta=y-\frac{1}{6}x \end{cases}$ 则 $Q=\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{6} & 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}=\begin{pmatrix} -\frac{1}{2} & 1 \\ -\frac{1}{6} & 1 \end{pmatrix}\begin{pmatrix} 12 & 4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{6} \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & -\frac{2}{3} \\ -\frac{2}{3} & 0 \end{pmatrix}$ $\overline{b_1}=0$ $\overline{b_2}=0$ $\overline{c}=0$ $\overline{f}=0$

③ $\Rightarrow u_{\xi\eta}=0$. 则 $u=f(y-\frac{1}{2}x)+g(y-\frac{1}{6}x)$

2. $\begin{cases} u_{tt}=u_{xx}+2 & 0<x<L,\ t>0 \\ u|_{x=0}=0 & u|_{x=L}=L \\ u|_{t=0}=0 & u_t|_{t=0}=0 \end{cases}$ 解：令 $u(x,t)=v(x,t)+w(x)$

$\begin{cases} 0=w_{xx}+2 \\ w(0)=0,\ w(L)=L. \end{cases}$ $\Rightarrow w(x)=-x^2+(L+1)x$

$\begin{cases} v_{tt}=v_{xx} \\ v|_{x=0}=v|_{x=L}=0 \\ v|_{t=0}=-w(x).\ v_t|_{t=0}=0. \end{cases}$ 令 $v(x,t)=X(x)T(t)$ $\Rightarrow \begin{cases} X''+\lambda X=0 \\ T''+\lambda T=0 \end{cases}$ 则有 $\begin{cases} X''+\lambda X=0 \\ X(0)=X(L)=0 \end{cases}$ $\Rightarrow \lambda_n=\left(\frac{n\pi}{L}\right)^2$ $X_n(x)=\sin\frac{n\pi}{L}x$

$\Rightarrow T_n(t)=C\cdot\cos\sqrt{\lambda}t+D\sin\sqrt{\lambda}t=C\cdot\cos\frac{n\pi}{L}t+D\sin\frac{n\pi}{L}t$

$\Rightarrow v(x,t)=\sum_{n=1}^{\infty}\left(C\cdot\cos\frac{n\pi}{L}t+D\sin\frac{n\pi}{L}t\right)\cdot\sin\frac{n\pi}{L}x$

$\Rightarrow \begin{cases} -w(x)=\sum_{n=1}^{\infty}C\cdot\sin\frac{n\pi}{L}x \\ 0=D. \end{cases}$ 则 $C=\frac{2}{L}\int_0^L\left[x^2-(L+1)x\right]\sin\frac{n\pi}{L}x\,dx$

$=-\frac{2L^2\cos n\pi}{n\pi}+\frac{4L^2(\cos n\pi-1)}{n^3\pi^3}+\frac{2L(L+1)\cos n\pi}{n\pi}$

$=\frac{2L\cos n\pi}{n\pi}+\frac{4L^2(\cos n\pi-1)}{n^3\pi^3}$

$=\frac{(-1)^n\cdot 2n^2\pi^2L+4L^2((-1)^n-1)}{(n\pi)^3}$

则 $v(x,t)=\sum_{n=1}^{\infty}C\cdot\cos\frac{n\pi}{L}t\sin\frac{n\pi}{L}x.$ $u(x,t)=\sum_{n=1}^{\infty}C\cdot\cos\frac{n\pi}{L}t\sin\frac{n\pi}{L}x-x^2+(L+1)x$

3. 解：当 $\lambda\le0$ 时，$X(x)=0$

当 $\lambda>0$ 时，$X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x.$ $\Rightarrow \begin{cases} 0=B\sqrt{\lambda} \\ 0=-A\sqrt{\lambda}\sin\sqrt{\lambda}L+B\sqrt{\lambda}\cos\sqrt{\lambda}L \end{cases}$ $\Rightarrow \begin{cases} 0=B \\ 0=A\sin\sqrt{\lambda}L. \end{cases}$

$\Rightarrow \sqrt{\lambda}L=n\pi$ $\Rightarrow \lambda=\left(\frac{n\pi}{L}\right)^2.$ $\Rightarrow X(x)=\cos\frac{n\pi}{L}x.$

4. 解：$u(x,t)=f(x+t)+g(x-t)$

$\Rightarrow \begin{cases} 0=f(x)+g(x) \\ 0=f'(x)-g'(x) \end{cases}$ $\Rightarrow \begin{cases} f(x)+g(x)=0 \\ f(x)-g(x)=C \end{cases}$ $\Rightarrow \begin{cases} f(x)=\frac{1}{2}C \\ g(x)=-\frac{1}{2}C \end{cases}$

$h(t)=f'(t)+g'(-t)$ $\Rightarrow f(t)-g(-t)=\int_0^t h(z)dz+C$

则 $f(t)=g(t)+\int_0^t h(z)dz+C$

$= \frac{1}{2}c + \int_0^t h(z)dz$

则 $f(x-t) = \frac{1}{2}c + \int_0^{x-t} h(z)dz$. 又 $s(x-t) = -\frac{1}{2}c$

则 $u(x,t) = \int_0^{x-t} h(z)dz$

5, (1) $F^{-1}(e^{-y|\omega|})$, $y>0$. 解: $= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-y|\omega|} e^{j\omega x} d\omega$

$= \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-y|\omega| + j\omega x} d\omega$

$= \frac{1}{\pi}\int_{-\infty}^{0} e^{y\omega + j\omega x} d\omega + \frac{1}{\pi}\int_0^{+\infty} e^{-y\omega + j\omega x} d\omega$

$= \frac{1}{\pi}\frac{1}{y+jx} e^{y\omega + j\omega x}\Big|_{-\infty}^{0} + \frac{1}{\pi}\frac{1}{-y+jx} e^{-y\omega + j\omega x}\Big|_0^{+\infty}$

$= \frac{1}{\pi}\frac{1}{y+jx} - \frac{1}{\pi}\frac{1}{-y+jx}$

$= \frac{1}{\pi}\frac{y}{x^2+y^2}$

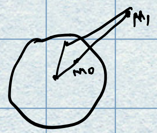
(2). $L^{-1}\left(\frac{1}{s(s+\omega)}(1-e^{-\frac{s}{a}x})\right)$ 解: $\frac{1}{s(s+\omega)} = \frac{\frac{1}{\omega}}{s} + \frac{-\frac{1}{\omega}}{s+\omega} = \frac{1}{\omega}\left(\frac{1}{s} - \frac{1}{s+\omega}\right)$.

$L^{-1}\left[\frac{1}{\omega}\left(\frac{1}{s} - \frac{1}{s+\omega}\right)\right] = (1-e^{-\omega t})\frac{1}{\omega}$ $t>0$

$L^{-1}\left[\frac{1}{\omega}\left(\frac{1}{s} - \frac{1}{s+\omega}\right)\cdot e^{-\frac{s}{a}x}\right] = (1-e^{-\omega(t-\frac{x}{a})})\frac{1}{\omega}$ $t>\frac{x}{a}$

$\Rightarrow$ 原式 $= (1-e^{-\omega t} - 1 + e^{-\omega(t-\frac{x}{a})})\frac{1}{\omega} = \frac{1}{\omega}(e^{-\omega(t-\frac{x}{a})} - e^{-\omega t})$ $t>\frac{x}{a}$.

6. 解:



则 $G(M,M_0) = \frac{1}{2\pi}\left(\ln\frac{1}{r_{MM_0}} - \ln\frac{R}{r_0}\frac{1}{r_{MM_1}}\right)$

7. 解: (1). $x^2y''(x) + xy'(x) + (4x^2 - n^2)y(x) = 0$.

令 $t=2x$. $\Rightarrow \frac{t^2}{4}y''(x) + \frac{t}{2}y'(x) + (t^2-n^2)y(x) = 0$

(2). $\Gamma(m+\frac{1}{2}+1) = \frac{1}{2}(1+\frac{1}{2})\cdots(m+\frac{1}{2})\Gamma(\frac{1}{2})$

$= \frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2m+1}{2}\sqrt{\pi}$

$= \frac{(2m+1)!!}{2^{m+1}}\sqrt{\pi}$

8. 解: $\int_{-1}^{1} x^n P_n(x)dx$.

解: $x^n = \sum_{k=0}^{n} C_k P_k(x)$. $P_n(x)$最高项系数为 $\frac{(2n)!}{2^n(n!)^2}$

则 $C_n = \frac{2^n(n!)^2}{(2n)!}$

$\Rightarrow = \int_{-1}^{1}\sum_{k=0}^{n} C_k P_k(x)P_n(x)dx$ 当 $k=n$ 时, $P_n(x)P_n(x) = \frac{2}{2n+1}$,

$= \int_{-1}^{1} C_n P_n(x) P_n(x)\,dx$ $\quad$ $k\neq n$ 时. $P_n(x)P_k(x)=0$

$= \dfrac{2^n(n!)^2}{(2n)!}\cdot\dfrac{2}{2m+1}$