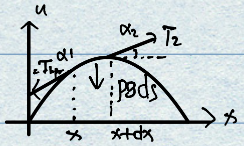
三类定解问题及定解条件的建立.

1. 波动方程的建立.

(细弦线水平横向振动)



水平合力为0. $\Rightarrow T_1\cos\alpha_1 = T_2\cos\alpha_2 \Rightarrow T_1 = T_2 = T$, $\cos\alpha_1 = \cos\alpha_2 \approx 1$

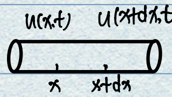
竖直合力 $F=ma$. $\Rightarrow T(\sin\alpha_2 - \sin\alpha_1) = \rho ds\, u_{tt}$

又α_1,α_2很小, $\Rightarrow \sin\alpha \approx \tan\alpha$. $\Rightarrow T(\tan\alpha_2 - \tan\alpha_1) = \rho ds\, u_{tt}$

$\Rightarrow T[u_x(x+dx,t) - u_x(x,t)] = \rho ds\, u_{tt}$.

$\Rightarrow T u_{xx} = \rho u_{tt} \quad \Rightarrow u_{tt} = \frac{T}{\rho} u_{xx} = a^2 u_{xx}$

(弹性细杆的纵向振动)

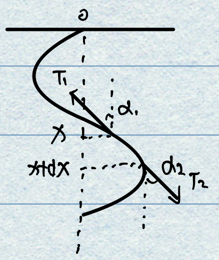


由牛二定律, $SY[u_x(x+dx,t) - u_x(x,t)] = \rho S dx\, u_{tt}$

$\Rightarrow SY u_{xx} = \rho S u_{tt}$

$\Rightarrow u_{tt} = \frac{Y}{\rho} u_{xx} = a^2 u_{xx}$

(细弦线垂直横向振动)



竖直方向 $F=ma \Rightarrow T_1\cos\alpha_1 - T_2\cos\alpha_2 = \rho g\, ds \Rightarrow T_2 - T_1 = -\rho g\, ds$

$\Rightarrow T_x = -\rho g$, 则 $T = -\rho g x + C$, 又当$x=0$时, $T=\rho g l$. $\Rightarrow T = \rho g(l-x)$

水平方向合力 $\Rightarrow T_2\sin\alpha_2 - T_1\sin\alpha_1 = \rho dx\, u_{tt}$

$\Rightarrow \rho g(l-x)(\tan\alpha_2 - \tan\alpha_1) = \rho dx\, u_{tt}$

$\Rightarrow \rho g(l-x)[u_x(x+dx,t) - u_x(x,t)] = \rho dx\, u_{tt}$

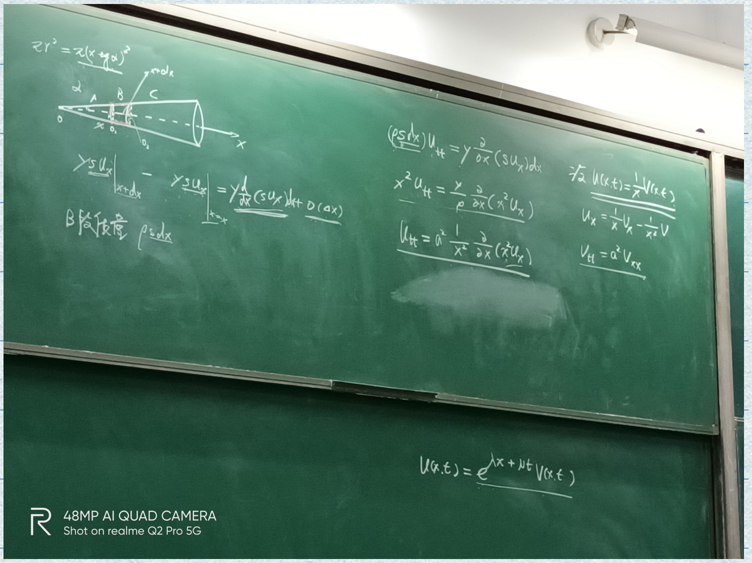
$\Rightarrow [g(l-x)u_x]_x = u_{tt}$

(定解条件) 边值条件 $\begin{cases} ①.\ u(0,t)=0 & u(l,t)=0 \\ ②.\ u_x(0,t)=0 & u_x(l,t)=0 \end{cases}$

③ $\begin{cases} u(0,t)=0 & u_x(l,t)=0 \\ u_x(0,t)=0 & u(l,t)=0. \end{cases}$

初值条件 $u(x,0)=\varphi(x)$ $u_t(x,0)=\psi(x)$

(圆锥杆纵向振动)



$$YS(x+dx)u_x(x+dx,t)-YS(x)u_x(x,t)=\rho S(x)dx\,u_{tt}$$

$$\Rightarrow \rho S\,u_{tt}=Y\frac{\partial}{\partial x}(S\,u_x)$$

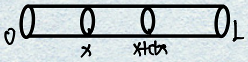
$$S=\pi\frac{x^2}{h^2}$$

$$\Rightarrow \rho\cdot\pi\cdot\frac{x^2}{h^2}u_{tt}=Y\frac{\pi}{h^2}\frac{\partial}{\partial x}(x^2u_x)$$

$$\Rightarrow u_{tt}=\frac{Y}{\rho}\frac{1}{x^2}\frac{\partial}{\partial x}(x^2u_x)$$

2. 热传导方程的建立

(侧面绝热)

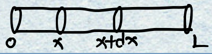


$$dQ=c\rho A dx\,[u(x,t+dt)-u(x,t)]=c\rho A u_t(x,t)\,dt\,dx$$

$$dQ_1=-ku_x(x,t)Adt \qquad dQ_2=-ku_x(x+dx,t)Adt$$

则 $dQ'=dQ_1-dQ_2$ $\Rightarrow u_t=a^2u_{xx}$ $a^2=\frac{k}{c\rho}$

(侧面进行热交换)



$$k\,dt\,[u_x(x+dx,t)S(x+dx,t)-u_x(x,t)S(x,t)]=c\rho S\,dx\,u_t\,dt+k_1(u-u_1)2\pi r\,dx\,dt$$

$$\Rightarrow u_t=\frac{k}{c\rho}u_{xx}-\frac{2k_1}{c\rho r}(u-u_1)$$