均方微积分复习　　均方极限

$$\underset{t\to t_0}{\text{l.i.m.}} X(t) = X. \Leftrightarrow \lim_{t\to t_0} d(X(t), X) = \lim_{t\to t_0} \|X(t) - X\| = 0.$$

Loève收敛准则(Loève′s criterion)

随机过程$\{X(t), t \in \mathbb{T}\}$在$t_0$处均方收敛的充要条件是

$$\lim_{\substack{s\to t_0\\ t\to t_0}} \mathsf{E}[X(s)X(t)].$$

存在.(将随机过程的收敛性问题转化为自相关函数的收敛性问题.)

均方连续

$$\underset{t\to t_0}{\text{l.i.m.}} X(t) = X(t_0) \Leftrightarrow \lim_{t\to t_0} \mathsf{E}[(X(t)-X(t_0))^2] = 0,$$

均方连续准则

随机过程$\{X(t), t \in \mathbb{T}\}$在$t_0 \in \mathbb{T}$处均方连续的充要条件是其自相关函数$R(s,t)$ 在(t_0, t_0)处连续.

均方微分

$$\underset{h\to 0}{\text{l.i.m.}} \frac{X(t_0+h) - X(t_0)}{h} := X'(t_0)$$

称$\{X'(t), t \in \mathbb{T}\}$为$\{X(t), t \in \mathbb{T}\}$的导数过程.

均方可导准则

随机过程$\{X(t), t \in \mathbb{T}\}$均方可导的充要条件是对于任意的$t \in \mathbb{T}$，下列极限(广义二阶导数)存在.

$$\underset{\substack{\Delta t\to 0\\ \Delta s\to 0}}{\text{l.i.m.}} \frac{R(t+\Delta t, t+\Delta s) - R(t+\Delta t, t) - R(t, t+\Delta s) + R(t,t)}{\Delta t \Delta s}.$$

$$\begin{aligned} R_{X'X}(s,t) &= \mathsf{E}[X'(s)X(t)] = \frac{\partial}{\partial s} R_X(s,t), \\ R_{XX'}(s,t) &= \mathsf{E}[X(s)X'(t)] = \frac{\partial}{\partial t} R_X(s,t), \\ R_{X'X'}(s,t) &= \mathsf{E}[X'(s)X'(t)] = \frac{\partial^2}{\partial s \partial t} R_X(s,t). \end{aligned}$$

$$m_{X'}(t) := \mathsf{E}[X'(t)] = m_X'(t).$$

均方积分

定义：Riemman均方积分

$$\int_a^b f(t)X(t)\mathrm{d}t := \underset{\Delta\to 0}{\text{l.i.m.}} \sum_{k=1}^n f(u_k)X(u_k)(t_k - t_{k-1}).$$

定义：Riemann-Stieltjes均方积分

$$\int_a^b f(t)\mathrm{d}X(t) := \underset{\substack{\Delta\to 0\\ n\to\infty}}{\text{l.i.m.}} \sum_{k=1}^n f(u_k)[X(t_k) - X(t_{k-1})].$$

定义：Ito积分

$$\int_a^b X(t)\mathrm{d}W(t) := \underset{\substack{\Delta\to 0\\ n\to\infty}}{\text{l.i.m.}} \sum_{k=1}^n X(t_{k-1})[W(t_k) - W(t_{k-1}]$$

均方可积准则

随机过程$f(t)X(t)$在区间$[a,b]$上均方可积的充分必要条件是以下二重积分存在，

$$\int_a^b \int_a^b f(s)f(t)R(s,t)\mathrm{d}s\mathrm{d}t,$$

且有

$$\mathsf{E}\left[\left(\int_a^b f(t)X(t)\mathrm{d}t\right)^2\right] = \int_a^b \int_a^b f(s)f(t)R(s,t)\mathrm{d}s\mathrm{d}t$$

其中$R(s,t)$是$X(t)$的自相关函数.

正态过程的导过程

设$\{X(t), t \in \mathbb{T}\}$是正态过程，且在$\mathbb{T}$上均方可导，则导过程$\{X'(t), t \in \mathbb{T}\}$也是正态过程

正态过程的积分过程

设$\{X(t), t \in \mathbb{T}\}$是正态过程，且在$\mathbb{T}$上均方可积，则

$$Y(t) = \int_a^t X(s)\mathrm{d}s, \quad (a, t \in \mathbb{T})$$

也是正态过程

均方可导 → 均方连续 → 均方可积

逆命题不成立

第五章均方微积分习题选解

习题1

有用的公式

1. $(X(t),Y(t))$是零均值的二维正态过程. 试证 $X^n(t)$也是二阶矩过程.

$$X \sim N(0,\sigma^2) \Rightarrow E[X^k] = \begin{cases} 0, & k\text{为奇数} \\ (k-1)!!, & k\text{为偶数} \end{cases}$$

$\{X(t)\}$为正态过程为二阶矩过 程

$$\frac{X(t)}{\sqrt{D(t)}} \sim N(0,1) \qquad \frac{X^2(t)}{D(t)} \sim \chi^2(1)$$

$$E\left[\frac{X^2(t)}{D(t)}\right] = 1 \qquad D\left[\frac{X^2(t)}{D(t)}\right] = 2 \qquad E\left[\frac{X^2(t)}{D(t)}\right]^2 = 3$$

$$E\left[X^2(t)\right]^2 = 3D^2(t) < +\infty$$

$X^2(t)$为二阶矩过程　　$X^n(t)$为二阶矩过程

习题2：先证均方可微

2. 设 $X(t)=At^2+Bt+C$, A,B,C 是相互独立的标准正态随机变量, 讨论随机过程 $\{X(t),-\infty<t<+\infty\}$ 的均方连续性, 均方可积性和均方可导性.

$$E[X(t)] = E[A]t^2 + E[B]t + E[C] = 0$$

$$R(s,t) = E[X(s)X(t)] = E[A^2t^2s^2 + B^2ts + C^2]$$

$$= t^2s^2 + ts + 1$$

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{R(t+h,t+k)-R(t+h,t)-R(t,t+k)+R(t,t)}{hk}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \frac{(t+h)^2(t+k)^2+(t+h)(t+k)-t^2(t+h)^2-t(t+h)-t^2(t+k)^2-t(t+k)+t^4+t^2}{hk}$$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \left(1+2kt+4t^2+h(k+2t)\right) = 4t^2+1 < \infty$$

X(t)可导 ⇒ X(t)连续 ⇒ X(t)可积

习题三

3. 设随机过程 $\{X(t),-\infty<t<+\infty\}$ 均值为零, 协方差函数

$$C(s,t)=e^{-a|t-s|}, \text{常数 } a>0.$$

讨论随机过程 $\{X(t),-\infty<t<+\infty\}$ 的均方连续性, 均方可积性和均方可导性.

$$C(s,t) = R(s,t) - m_X(t)m_X(s) = R(s,t)$$

$$\lim_{h\to 0,k\to 0} \frac{R(t+h,t+k)-R(t,t+h)-R(t,t+k)+R(t,t)}{ht}$$

$$= \lim_{h\to 0,k\to 0} \frac{e^{-a|h-k|} - e^{-ah} - e^{-ak} + 1}{hk}$$

$$\text{令 } h = k = \lim_{h\to 0,k\to 0} \frac{2-2e^{-ah}}{h^2} \to \infty$$

不可导　由于 $R(s,t) = e^{-a|t-s|}$ 在 (t,t) 连续 $\Rightarrow$

$X(t)$均方连续 $\Rightarrow$ $X(t)$均方可积

习题四

4. 设随机过程 $\{X(t),t\in T\}$ 均值为零, 协方差函数

$$C(s,t)=\frac{1}{a^2+(t-s)^2}, \text{常数 } a>0.$$

讨论随机过程 $\{X(t),t\in T\}$ 的均方连续性, 均方可积性和均方可导性.

$$C(s,t) = R(s,t) - m_X(t)m_X(s) = R(s,t)$$

由于 $R(s,t)$在 (t,t)连续 $\Rightarrow$ $X(t)$均方连续 $\Rightarrow$

$X(t)$均方可积

$$\lim_{h\to 0,k\to 0} \frac{R(t+h,t+k)-R(t,t+h)-R(t,t+k)+R(t,t)}{hk}$$

$$= \frac{\frac{1}{a^2+(h-k)^2} - \frac{1}{a^2+h^2} - \frac{1}{a^2+k^2} + \frac{1}{a^2}}{hk} \qquad \text{令 } h = k$$

$$= \lim_{h\to 0,k\to 0} \frac{2\left(\frac{1}{a^2} - \frac{1}{a^2+h^2}\right)}{h^2} = \lim_{h\to 0} \frac{2}{h^2}\frac{h^2}{a^2(a^2+h^2)}$$

$$= \frac{2}{a^4} < \infty \Rightarrow X(t) \text{ 均方可导}$$

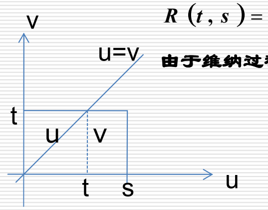
习题五

5. 设$\{W(t), t\geq 0\}$是参数为σ^2的维纳过程

$$X(t)=\frac{1}{t}\int_0^t W(u)du.$$

$$m_X(t)=E[X(t)]=\frac{1}{t}\int_0^t E[W(u)]du=\frac{1}{t}\int_0^t m_W(u)du=0$$

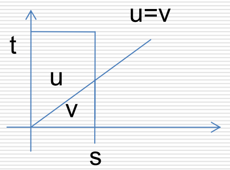
$$R(t,s)=E[X(t)X(s)]=\frac{1}{ts}\left(\int_0^t\int_0^s R(u,v)dudv\right)$$



由于维纳过程的 $R(t,s)=\sigma^2\min(t,s)$

$t\leq s$时 $\frac{\sigma^2}{ts}\int_0^t dv\int_0^v udu+\frac{\sigma^2}{ts}\int_0^t dv\int_v^s vdu$

$$=\frac{\sigma^2}{ts}\left(\int_0^t\frac{v^2}{2}dv+\int_0^t v(s-v)dv\right)=\frac{1}{ts}\cdot\frac{\sigma^2t^2}{6}(3s-t)$$



$t\geq s$时 根据 t,s的对称性 ·

有 $R(t,s)=\frac{1}{ts}\frac{\sigma^2s^2}{6}(3t-s)$

$$C(s,t)=\begin{cases}(3s-t)\dfrac{\sigma^2t}{6s} & t\leq s\\ (3t-s)\dfrac{\sigma^2s}{6t} & s\leq t\end{cases}$$

习题6：

6. 设$\{N(t), t\geq 0\}$是平均率为λ的泊松过程

$$X(t)=\frac{1}{t}\int_0^t N(u)du.$$

$$m_X(t)=E[X(t)]=t\int_0^t E[N(u)]du=\frac{\lambda t}{2}$$

$$E[N(s)N(t)]=\lambda\min\{s,t\}+\lambda^2st$$

$$C(s,t)=R(s,t)-m_X(t)m_X(s)$$

$$m_X(t)m_X(s)=\frac{\lambda^2}{4}ts$$

$$R(s,t)=E[X(t)X(S)]=\frac{1}{ts}\int_0^t\int_0^s E[N(u)N(v)]dudv$$

$$=\frac{1}{ts}\int_0^t\int_0^s\left[\lambda\min(u,v)+\lambda^2uv\right]dudv=\frac{\lambda^2ts}{4}+\frac{\lambda}{ts}\int_0^t\int_0^s\min(u,v)dudv$$

$$=\begin{cases}\dfrac{\lambda t}{6s}(3s-t) & t\leq s\\ \dfrac{\lambda s}{6t}(3t-s) & t>s\end{cases}$$

习题7：

7. 设$\{W(t), t\geq 0\}$是参数为σ^2的维纳过程

$$Y(t)=\frac{1}{L}\int_t^{t+L}W(u)du, \text{常数 } L>0.$$

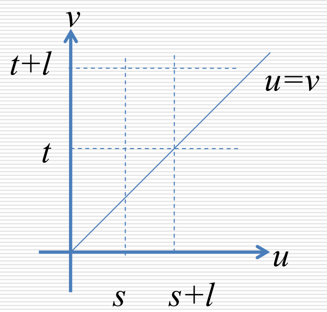
$$E[Y(t)]=\frac{1}{l}\int_0^l E[W(u)]du=0$$

$$C(s,t)=R(s,t)-m_Y(t)m_Y(s)=R(s,t)$$

$$R(s,t)=E[Y(t)Y(s)]=\frac{1}{l^2}\int_t^{t+l}\int_s^{s+l}E[w(u)w(v)]dudv$$

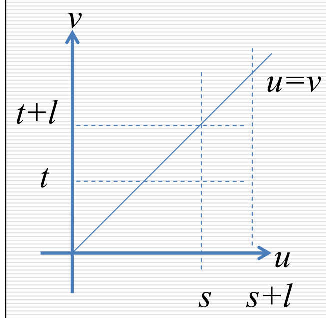
$$=\frac{\sigma^2}{l^2}\int_t^{t+l}\int_s^{s+l}\min(u,v)dudv=\frac{\sigma^2}{l^2}\int_0^t\int_0^s\min(u,v)dudv$$

当$t\geqslant s+l$时



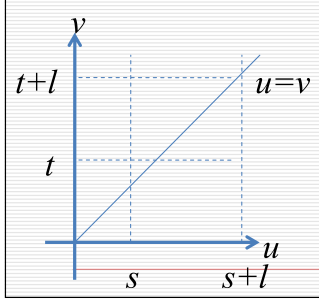
$$R(s,t)=\frac{\sigma^2}{l^2}\int_s^{s+l}\left[\int_t^{t+l}udv\right]du=\frac{\sigma^2}{2}(2s+l)$$

当$t+l\leqslant s$时



$$R(s,t)=\frac{\sigma^2}{l^2}\int_s^{s+l}\left[\int_t^{t+l}vdv\right]du=\frac{\sigma^2}{2}(2t+l)$$

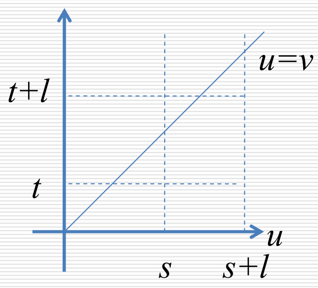
当$s\leqslant t\leqslant s+l$时



$$R(s,t)=\frac{\sigma^2}{l^2}\int_s^{s+l}\left[\int_t^{t+l}udv\right]du+\frac{\sigma^2}{l^2}\int_t^{s+l}\left[\int_t^u(v-u)dv\right]du$$

$$R(s,t)=\frac{\sigma^2}{2}(2s+l)-\frac{\sigma^2}{6l^2}(s+l-t)^3$$

当$0 \leq t \leq s \leq t+l$时



$$R(s,t)=\frac{\sigma^2}{l^2}\int_s^{s+l}\left[\int_t^{t+l} v dv\right]du+\frac{\sigma^2}{l^2}\int_s^{t+l}\left[\int_s^u (v-u)dv\right]du$$

$$=\frac{\sigma^2}{l^2}(2t+l)-\frac{\sigma^2}{6l^2}(t+l-s)^3$$

$$C(s,t)=\begin{cases}\frac{\sigma^2}{l^2}(2s+l)-\frac{\sigma^2}{6l^2}(t+l-s)^3 & 0\leq s\leq t\leq s+l\\ \frac{\sigma^2}{l^2}(2t+l) & s+l\leq t\\ \frac{\sigma^2}{l^2}(2t+l)-\frac{\sigma^2}{6l^2}(t+l-s)^3 & 0\leq s\leq t\leq s+l\\ \frac{\sigma^2}{l^2}(2t+l) & t+l\leq s\end{cases}$$

习题八

8. 设$\{N(t),t\geq 0\}$是平均率为λ的泊松过程

$$Y(t)=\frac{1}{L}\int_t^{t+L} N(u)du,\text{常数 } L>0,$$

$$E[Y(t)]=\frac{1}{L}\int_t^{t+l} E[N(u)]du=\frac{\lambda L}{2}+\lambda t$$

$$m_X(t)m_X(s)=\lambda^2\left(\frac{1}{2}+t\right)\left(\frac{L}{2}+s\right)$$

$$R(s,t)=E[Y(t)Y(s)]=\frac{1}{L^2}\int_t^{t+l}\int_s^{s+l}E[N(u)N(v)]dudv$$

$$=\frac{1}{L^2}\int_t^{t+l}\int_s^{s+l}[\lambda\min(u,v)+\lambda^2 uv]dudv$$

$$C(s,t)=\frac{1}{L^2}\int_t^{t+l}\int_s^{s+l}\lambda\min(u,v)dudv$$

同七题

$$C(s,t)=\begin{cases}\frac{\lambda}{2}(2s+L)+\frac{\lambda}{6L^2}(s+L-t)^3 & 0\leq s\leq t\leq s+L\\ \frac{\lambda}{2}(2s+L) & s+L\leq t\\ \frac{\lambda}{2}(2t+L)+\frac{\lambda}{6L^2}(t+L-t)^3 & 0\leq t\leq t\leq t+L\\ \frac{\lambda}{2}(2t+L) & t+L\leq s\end{cases}$$

习题9:

9. 设$\{W(t),t\geq 0\}$是参数为$\sigma^2=1$的维纳过程,令

$$X(t)=\int_0^t W(u)du,$$

求随机过程$\{X(t),t\geq 0\}$的

(1) 一维概率密度和特征函数;

(2) 二维概率密度和特征函数.

$X(t)=\int_0^t W(u)du$ **由于** $W(u)$**为正态过程故** $X(t)$**为正态过程**

$$E[X(t)]=\int_0^t E[W(u)]du=0$$

$$C(s,t)=R(s,t)=\int_0^s\int_0^t\sigma^2\min(u,v)dudv$$

$$=\begin{cases}\frac{\sigma^2 s^2}{6}(3t-s) & 0\leq s\leq t\\ \frac{\sigma^2 t^2}{6}(3s-t) & 0\leq t\leq s\end{cases}$$

$$D(t)=C(t,t)=\frac{\sigma^2}{3}t^3 \qquad X(t)\sim N\left(0,\frac{\sigma^2}{3}t^3\right)$$

$$C=\begin{bmatrix}\frac{\sigma^2 s^3}{3} & \frac{\sigma^2 s^2(3t-s)}{6}\\ \frac{\sigma^2 s^2(3t-s)}{6} & \frac{\sigma^2 t^3}{3}\end{bmatrix},\ s\leq t\text{时}$$

一维分布 $f(t,s)=\frac{1}{\sqrt{2\pi t\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2 t}\right\}$

二维特征函数 $\varphi(t,u)=e^{-\frac{\sigma^2 t}{2}u^2}$ **二维见第三章总结**

习题11

11. 随机电报信号$X(t)$的样本函数如图. 设均值和相关函数分别为

$m_X=0 \quad R_X(\tau)=e^{-2\lambda|\tau|}$

当$Y=\frac{1}{2T}\int_{-T}^{T}X(t)dt$时

证明:① $m_Y=0$

② $R_Y(t_1,t_2)=\frac{1}{2\lambda T}-\frac{1-e^{-4\lambda T}}{8\lambda^2 T^2}$

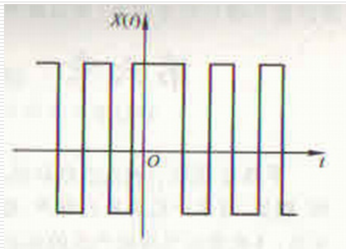
$$m_Y(t)=\frac{1}{2T}\int_{-T}^{T}m_X(t)dt=o$$

$$E[Y^2]=\frac{1}{4T^2}E\left[\left(\int_{-T}^{T}X(t)dt\right)^2\right]=\frac{1}{4T^2}E\left[\int_{-T}^{T}\int_{-T}^{T}X(t)X(s)dtds\right]$$

$E[X^2(t)]=R(t,t)=R_X(0)=1<+\infty\Rightarrow\{X(t)\}$**为二阶矩过程**

由$R_X(s,t)$**在**(t,t)**连续** $\Rightarrow\{X(t)\}$**均方连续** $\Rightarrow\{X(t)\}$**均方可积**

$$E[Y^2]=\frac{1}{4T^2}\int_{-t}^{t}\int_{-t}^{t}e^{-2\lambda|t-s|}dsdt$$



$$=\frac{1}{4T^2}\int_{-t}^{t}\int_{-t}^{t}e^{-2\lambda(s-t)}dsdt$$

$$=\frac{1}{2\lambda T}-\frac{1-e^{-4\lambda t}}{8\lambda^2 t^2}$$