1.解: ① $\left(\frac{dy}{dx}\right)^2-5\frac{dy}{dx}+4=0$

② $a_{12}^2-a_{11}a_{22}=\frac{25}{4}-4>0$ 双曲型.

$\begin{cases} C_1=y-x \\ C_2=y-4x \end{cases}$ 令 $\begin{cases} \xi=y-x \\ \eta=y-4x \end{cases}$ $\Rightarrow Q=\begin{pmatrix} -1 & 1 \\ -4 & 1 \end{pmatrix}$

③ $\begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{12}} & \overline{a_{22}} \end{pmatrix}=Q\begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{pmatrix}Q^T=\begin{pmatrix} -1 & 1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}=\begin{pmatrix} 0 & -\frac{9}{2} \\ -\frac{9}{2} & 0 \end{pmatrix}$

$\overline{b_1}=L\xi-C\xi=7$ $\overline{b_2}=L\eta-C\eta=7$. $\overline{c}=c=0$ $\overline{f}=f=\sin x=\sin\frac{\xi-\eta}{3}$

$\Rightarrow -9u_{\xi\eta}+7u_\xi+7u_\eta=\sin\frac{\xi-\eta}{3}$

2.解: 令 $u(x,t)=X(x)T(t)$

$\Rightarrow \begin{cases} X''+\lambda X=0 \\ T''+\lambda a^2T=0 \end{cases}$ 则有 $\begin{cases} X''+\lambda X=0 \\ X(0)=X'(\pi)=0 \end{cases}$ $\Rightarrow$ 当$\lambda>0$时. $\lambda_n=\left(n+\frac{1}{2}\right)^2$ $n=0,1,\cdots$

$X_n(x)=\sin\left(n+\frac{1}{2}\right)x$

$T_n(t)=C\cdot\cos\sqrt{\lambda}at+D\cdot\sin\sqrt{\lambda}at.=C\cdot\cos\left(n+\frac{1}{2}\right)at+D\cdot\sin\left(n+\frac{1}{2}\right)at$

$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}\left(C\cdot\cos\left(n+\frac{1}{2}\right)at+D\sin\left(n+\frac{1}{2}\right)at\right)\sin\left(n+\frac{1}{2}\right)x$

$\Rightarrow \begin{cases} x^3=\sum_{n=1}^{\infty}C\cdot\sin\left(n+\frac{1}{2}\right)x \\ 0=D. \end{cases}$ 则 $C=\frac{2}{\pi}\int_0^{\pi}x^3\sin\left(n+\frac{1}{2}\right)x\,dx$

$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}C\cdot\cos\left(n+\frac{1}{2}\right)at\sin\left(n+\frac{1}{2}\right)x$

3.解: $\begin{cases} u_t=a^2u_{xx} \\ u|_{x=0}=0,\quad u|_{x=1}=T_0 \\ u|_{t=0}=(T_0+1-x)x \end{cases}$ ① 令 $u(x,t)=v(x,t)+w(x,t)$

令 $w(x,t)=Ax+B.$ $\Rightarrow \begin{cases} 0=B \\ T_0=A+B \end{cases}$ $\Rightarrow \begin{cases} A=T_0 \\ B=0 \end{cases}$

$\Rightarrow w(x,t)=T_0x.$

② $\begin{cases} v_t=a^2v_{xx} \\ v|_{x=0}=v|_{x=1}=0 \\ v|_{t=0}=(T_0+1-x)x \end{cases}$

③ 令 $v(x,t)=X(x)T(t)$ $\Rightarrow \begin{cases} X''+\lambda X=0 \\ T'+\lambda a^2T=0 \end{cases}$

则有 $\begin{cases} X''+\lambda X=0 \\ X(0)=X(1)=0 \end{cases}$ $\Rightarrow \lambda_n=(n\pi)^2.$ $X_n(x)=\sin n\pi x$

$\Rightarrow T_n(t)=C\cdot e^{-\lambda a^2t}=C\cdot e^{-n\pi a^2t}$

$$\Rightarrow V(x,t)=\sum_{n=1}^{\infty} C\cdot e^{-n\pi a^2 t}\cdot \sin n\pi x$$

$$\Rightarrow (T_0+1-x)x=\sum_{n=1}^{\infty} C\cdot \sin n\pi x \qquad 则\ C=2\int_0^1 (T_0+1-x)x\sin n\pi x\,dx$$

$$\Rightarrow u(x,t)=V(x,t)+W(x,t)$$

4.解: $\tilde{f}(\omega)=\int_{-\infty}^{+\infty} f(x)e^{-j\omega x}dx=\int_{-1}^{1}(1-x^2)e^{-j\omega x}dx$

$$=-\frac{1}{j\omega}\int_{-1}^{1}(1-x^2)\,de^{-j\omega x}$$

$$=-\frac{1}{j\omega}(1-x^2)e^{-j\omega x}\Big|_{-1}^{1}+\frac{-2}{j\omega}\int_{-1}^{1}e^{-j\omega x}\cdot x\,dx$$

$$=-\frac{2}{\omega^2}\int_{-1}^{1}x\,de^{-j\omega x}$$

$$=-\frac{2}{\omega^2}xe^{-j\omega x}\Big|_{-1}^{1}+\frac{2}{\omega^2}\int_{-1}^{1}e^{-j\omega x}dx$$

$$=-\frac{4}{\omega^2}\cdot\frac{e^{j\omega}+e^{-j\omega}}{2}-\frac{2}{j\omega^3}e^{-j\omega x}\Big|_{-1}^{1}$$

$$=-\frac{4}{\omega^2}\cos\omega+\frac{4}{\omega^3}\sin\omega$$

$$=\frac{4}{\omega^3}(\sin\omega-\omega\cos\omega)$$

6.解: $J_{\frac{1}{2}}(x)=\sqrt{\frac{2}{\pi x}}\sin x$

7.解: (1). $G(M,M_0)=\frac{1}{4\pi}\left(\frac{1}{r_{MM_0}}-\frac{1}{r_{MM_1}}\right)$

(2). $\int[2x^3J_0(x)+4x^2J_1(x)]\,dx$

$[x^nJ_n(x)]'=x^nJ_{n-1}(x)$ $\quad =\int 2x^3J_0(x)dx+\int 4x^2J_1(x)dx$

$$\Rightarrow \int 2x^3J_0(x)dx=2\int x^2\cdot x^1J_0(x)dx=2\int x^2\,dx^1J_1(x)$$

$$=2x^3J_1(x)-4\int x^2J_1(x)dx$$

$$\Rightarrow \quad =2x^3J_1(x)$$

8.解: (1) $x^2y''+xy'+(x^2-n^2)y=0$

$$J_n(x)=\sum_{m=0}^{\infty}(-1)^m\frac{x^{n+2m}}{2^{n+2m}\,m!\,\Gamma(n+m+1)}$$

(2) $(1-x^2)y''-2xy'+n(n+1)y=0$

$$P_n(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m} \qquad M=\begin{cases}\frac{n}{2} & n=2k\\ \frac{n-1}{2} & n=2k+1\end{cases}$$

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

5.