① 勒让德方程

1). 方程形式：$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$

2) 解的形式：$y=AP_n(x)+BQ_n(x)$

②.勒让德多项式

1). $P_n(x)=\sum_{m=0}^{\frac{n}{2}}(-1)^m\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$ $\quad m=\begin{cases}\frac{n}{2}, & n=2k\\ \frac{n-1}{2}, & n=2k+1\end{cases}$

2) $P_n(x)=\frac{1}{2^n n!}\frac{d^n}{x^n}(x^2-1)^n$

$P_0(x)=1 \quad P_1(x)=x \quad P_2(x)=\frac{1}{2}(3x^2-1)$ $\quad x=\cos\theta$

$P_3(x)=\frac{1}{2}(5x^3-3x)$

$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)$

3) 母函数：$G(x.z)=\frac{1}{\sqrt{1-2xz+z^2}}$

4). 推递：$xP_n(x)=\frac{n}{2n+1}P_{n-1}(x)+\frac{n+1}{2n+1}P_{n+1}(x)$

$P_{n-1}'(x)=xP_n'(x)-nP_n(x)$

$P_n'(x)=xP_{n-1}'(x)+nP_{n-1}(x)$

5) 正交性. $\int_{-1}^{1}P_m(x)P_n(x)dx=\begin{cases}0 & m\neq n\\ \frac{2}{2n+1} & m=n\end{cases}$

6) 展开 $f(x)=\sum_{k=0}^{n}C_kP_k(x)$

① $\int_{-1}^{1}(1-x^2)[P_n'(x)]^2dx$

解：$=\int_{-1}^{1}(1-x^2)P_n'(x)dP_n(x)=(1-x^2)P_n'(x)P_n(x)\Big|_{-1}^{1}-\int_{-1}^{1}P_n(x)[(1-x^2)P_n''(x)-2xP_n'(x)]dx$

又 $(1-x^2)P_n''(x)-2xP_n'(x)+n(n+1)P_n(x)=0.$ $\Rightarrow$. $=-\int_{-1}^{1}P_n(x)[-n(n+1)P_n(x)]dx$

$=\int_{-1}^{1}n(n+1)P_n(x)P_n(x)dx$

$=n(n+1)\frac{2}{2n+1}$

②. $\int_{-1}^{1}x^nP_n(x)dx=\int_{-1}^{1}\sum_{k=0}^{n}C_kP_kP_n(x)dx=C_n\int_{-1}^{1}P_n(x)P_n(x)dx$

$P_n(x)=\sum_{m=0}^{M}(-1)^M\frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}x^{n-2m}$ $\quad m=0$时. $P_n(x)=\frac{(2n)!}{2^n n!n!}x^n$

$=\frac{2^n(n!)^2}{(2n)!}\cdot\frac{2}{2n+1}$