ICIS workshop 'Finding Structure in Time' (June 30, 2018)

Exercise 1: Simple Phase-Space Reconstruction

The aim of this exercise is to perform a simple delay embedding procedure. After this exercise you should (1) be convinced that delay-embedding phase-space reconstruction makes sense, (2) have witnessed how it is done, and (3) understand its basic limitation. The idea is to see how the limit-cycle dynamics of a two-dimensional system can be reconstructed from just one of its variables by using the delay embedding technique. We will use the well-known predator-prey model as a case study.

Specific assignments

1. Open the Excel file 'Exercise 1.xlsx', which contains three worksheets. Have a look at the first worksheet, called 'Predator-Prey'. This is what a simple version of the predator-prey model looks like. You can see two columns of data, representing two time series, one for the number of predators (**F** for foxes) and one for the number of prey (**R** for rabbits). The units are arbitrary. The *time series* of **F** and **R** are drawn in the figure panel on the right. The figure panel on the left shows the *phase space*.

   The phase space is created by plotting the equal-time values of **F** and **R** as points in a two-dimensional space. Doing this for all values of the two time series creates a trajectory in space, forming a spatial shape called an *attractor*. The *limit-cycle* attractor (i.e. closed-loop; 'circular' pattern) you see here is typical for this kind of system. It reflects that the variables **F** and **R** are actually coupled. This means that their individual behavior is mutually constraining, leading to this self-sustaining limit-cycle dynamics.

   But enough about this. There is tons of material available on the internet, showing how this works and what it means, and also about applications of this model.

Imagine you don't know the system you are studying very well and that you certainly don't have a model that describes its behavior. Next, imagine that you only measured a single variable of that system. What to do? Well, Taken's theorem says that it is possible to reconstruct the behavior of the entire multi-dimensional system from this single variable (up to a limit, see below). What you are going to do now is show that it is possible to reconstruct the limit-cycle pattern from just one of the variables **F** and **R**.

2. Go to worksheet 'F based'. You will see two empty figure panels and three empty data columns labeled '**F**', '**Rsur (100)**', and '**Rsur (500)**'. First copy the entire set of values in the column labeled '**F**' on worksheet 'Predator-Prey' (A2:A4001), and paste it at the same position in the column '**F**' on worksheet 'F based'. This is the initial variable.

3. Next, you are going to create a surrogate dimension '**Rsur (100)**' based on the initial variable '**F**' with a delay of 100 time points. Copy the entire set of values of '**F**' in the

column with label ‘**Rsur (100)**’, but instead of starting in cell A2 now start in cell A101. (Just leave the first 99 cells empty.) Go to the top of the worksheet and look to the left figure panel to see the attractor you’ve reconstructed.

4. Now create another surrogate dimension ‘**Rsur (500)**’ based on the initial variable ‘**F**’ with a delay of 500 time points. Copy the entire set of values of ‘**F**’ in the column with label ‘**Rsur (500)**’, now starting in cell A501. (Just leave the first 499 cells empty this time.) Go to the top of the worksheet again and look to the right figure panel to see this reconstructed attractor.

<u>Note:</u> As you will have noticed, the reconstructed attractors are not exactly the same as the one of the actual model on worksheet ‘Predator-Prey’. Nevertheless, you can clearly see that they all are limit-cycles, just like the original one, only somewhat warped. This is the one major limitation of the delay embedding technique. The reconstructed attractor is what mathematicians call *topologically equivalent* to the ‘actual’ one. For us that means that it has the same dynamical properties. That already is quite a lot if you think about it! With a bit of moving and stretching you could make it exactly equal by the way.

<u>Note:</u> This was just a two-dimensional system, and we knew from the start the number of dimensions was two. The technique can be performed in higher dimensional embedding spaces though, and for empirical data it generally is much higher. In these cases visualization becomes a problem of course, but it is something that you would rarely do anyway. This is just a step in between that single (or pair of) measured variable(s) and (cross-)recurrence quantification analysis. If you don’t know the number of dimensions --and you usually don’t, there are methods and heuristics to help you decide how many are necessary to create a big enough space. The same is true for the size of the delay that is necessary. You’ll see how this works in the next exercise.

5. If there is enough time and you can’t get enough, repeat steps 2 - 4, now using the variable **‘R**’. You can use worksheet ‘R based’ for this.