Homework 3

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Question

Use Verlet Algorithm to solve the pendulum problem:

1. Initial velocity $v_0 = 0$, initial angle $\theta_0 = \pi/4$, find out the angle and angular velocity in window $[0, 2000]$ with an interval of 0.2.
2. Plot the trajectory of angular velocity ~ angle in the phase space.
3. Display the total energy as a function of time.

$$\begin{cases} \theta_i = \theta_{i-1} + h v_{i-\frac{1}{2}}, & i = 1, 2, \cdots, \dfrac{T}{h} \\ v_{i+\frac{1}{2}} = v_{i-\frac{1}{2}} - h \sin\theta_i, & \theta_0 = \dfrac{\pi}{4},\ T = 2000,\ h = 0.2 \\ \theta_0 = \theta_0 \\ v_{\frac{1}{2}} = -\dfrac{h}{2} \sin\theta_0 \end{cases}$$

Tips

Tip 1: Use `v=zeros(N, 1)` to initialize θ, v, t.

Tip 2: Use `for i = 1:N` to do the loop.

Tip 3: Use the following program structure.

1. Initialize variables
2. Initialize θ_0, $v_{\frac{1}{2}}$
3. Define θ_i, $v_{i+\frac{1}{2}}$ recursively

Tip 4: Use the angle and angular velocity of the same moment to calculate the total energy and to plot the phase space, $v_i = \frac{1}{2}\left(v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}}\right)$.

Solution

根据题设信息设定程序变量

```
% initialize variables
% params init
v_0 = 0;
theta_0 = pi / 4;
T = 2000;
h = 0.2;
```

根据给定的采样时间范围以及采样时间间隔生成θ，v，t的数组

```
% init var arrays
```

```
t = 0:0.2:2000;
v_midi = zeros(size(t));
theta = zeros(size(t));
```

由 $v_{\frac{1}{2}} = -\frac{h}{2}\sin\theta_0$ 及题设条件，设定数组的第一个元素，用于后续迭代循环

```
% initialize theta_0 and v_mid0
v_midi(1) = -h / 2 * sin(theta_0);
theta(1) = theta_0;
```

进行迭代，求解各采样点处的单摆运动状态

```
% define theta_i and v_midi recursively
[~, max_i] = size(t);
for i = 2:max_i
    theta(i) = theta(i-1) + h * v_midi(i-1);
    v_midi(i) = v_midi(i-1) + h * (-sin(theta(i)));
end
```

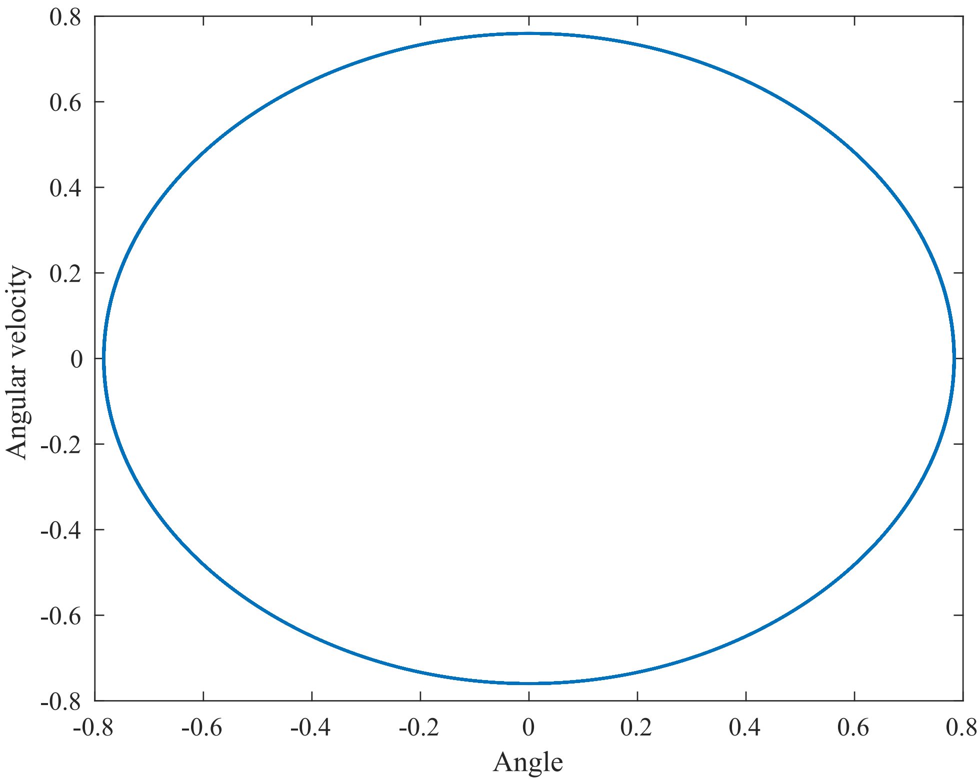
在对单摆进行分析时，需要 θ， v 的采样在同一时间点 t 处进行，根据 $v_i = \frac{1}{2}\left(v_{i+\frac{1}{2}} + v_{i-\frac{1}{2}}\right)$，将程序中的

v_midi 数组转换至数组 t 对应的采样点

```
% align v with theta to the same sampling points in t
v_next_midi = circshift(v_midi, -1);
v_next_midi = v_next_midi(1:end-1);
v_form_midi = v_midi(1:end-1);
v = (v_form_midi + v_next_midi) ./ 2;
v = [v_0, v]; % concatenate v_i
```

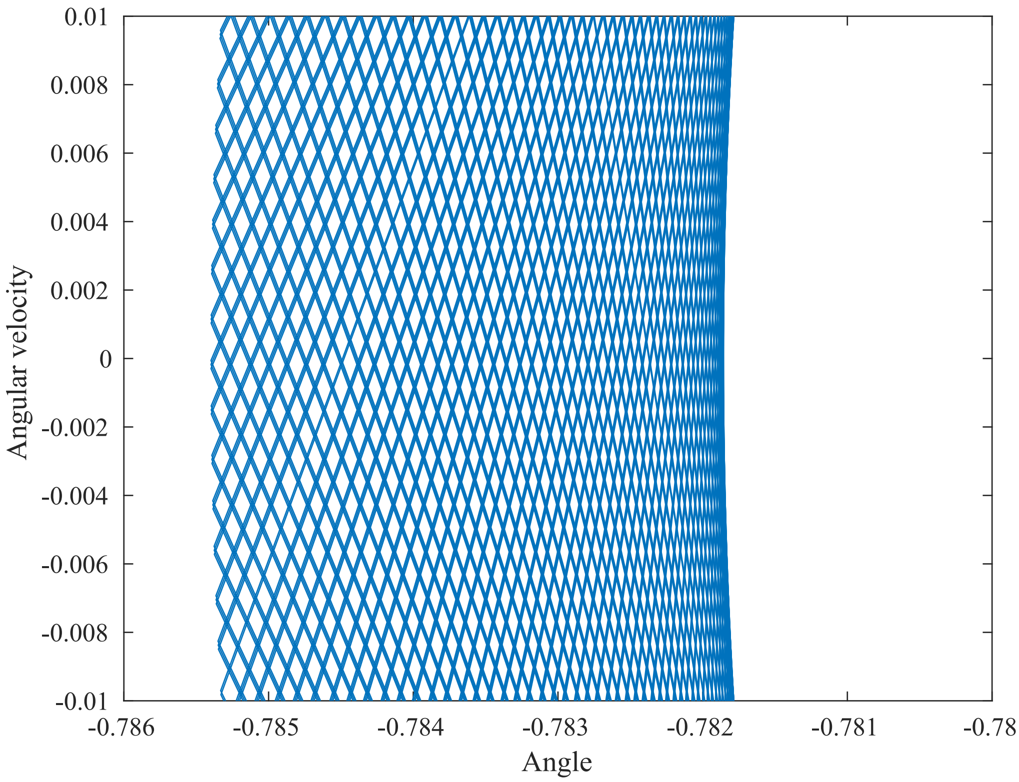
至此，已经通过 Verlet 算法得到了单摆在所有采样时间点处的角度 θ 及角速度 v，绘制两者在相空间中的关系图

```
% plot the trajectory of angular velocity ~ angle in the phase space
figure;
plot(theta, v);
xlabel("Angle");
ylabel("Angular velocity");
```



局部放大图像

```
% plot the trajectory of angular velocity ~ angle in the phase space
figure;
plot(theta, v);
xlabel("Angle");
ylabel("Angular velocity");
xlim([-0.786, -0.78]);
ylim([-0.01, 0.01]);
```



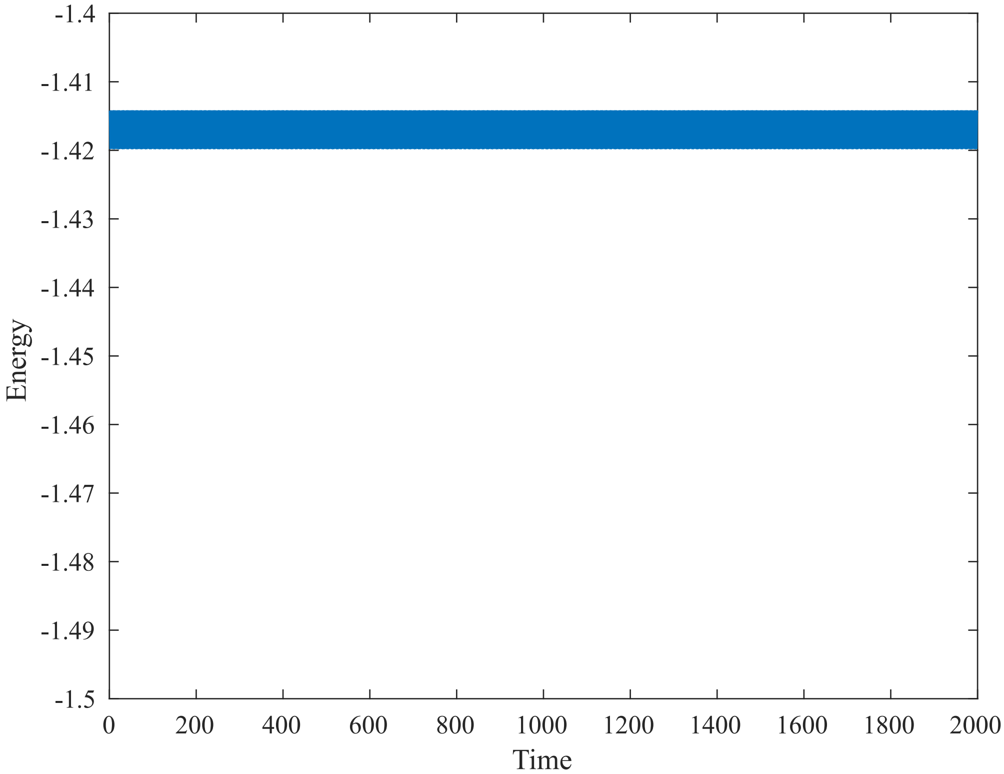
由上图可见计算结果在一个较小范围内波动

不妨令 $\frac{1}{\omega}$ 为单位时间，$\frac{1}{2}mL^2\omega^2$ 为单位能量，单摆的总能量如下

$$
\begin{aligned}
\mathrm{E} &= \frac{1}{2}mv^2 + V(r) \\
&= \frac{1}{2}m(L\theta')^2 - mgL\cos\theta \\
&= \frac{1}{2}mL^2\left(\theta'^2 - \frac{2g}{L}\cos\theta\right) \\
&= \frac{1}{2}mL^2(\theta'^2 - 2\omega^2\cos\theta) \\
&= \theta'^2 - 2\cos\theta
\end{aligned}
$$

由上述推导结果，可以由角度 θ 及 角速度 v 计算单摆总能量，并绘制能量在题设时间范围内的变化情况

```
% Calculate energy and plot
E = v .^ 2 - 2 * cos(theta);
figure;
plot(t, E);
xlabel("Time");
ylabel("Energy");
ylim([-1.5, -1.4]);
```



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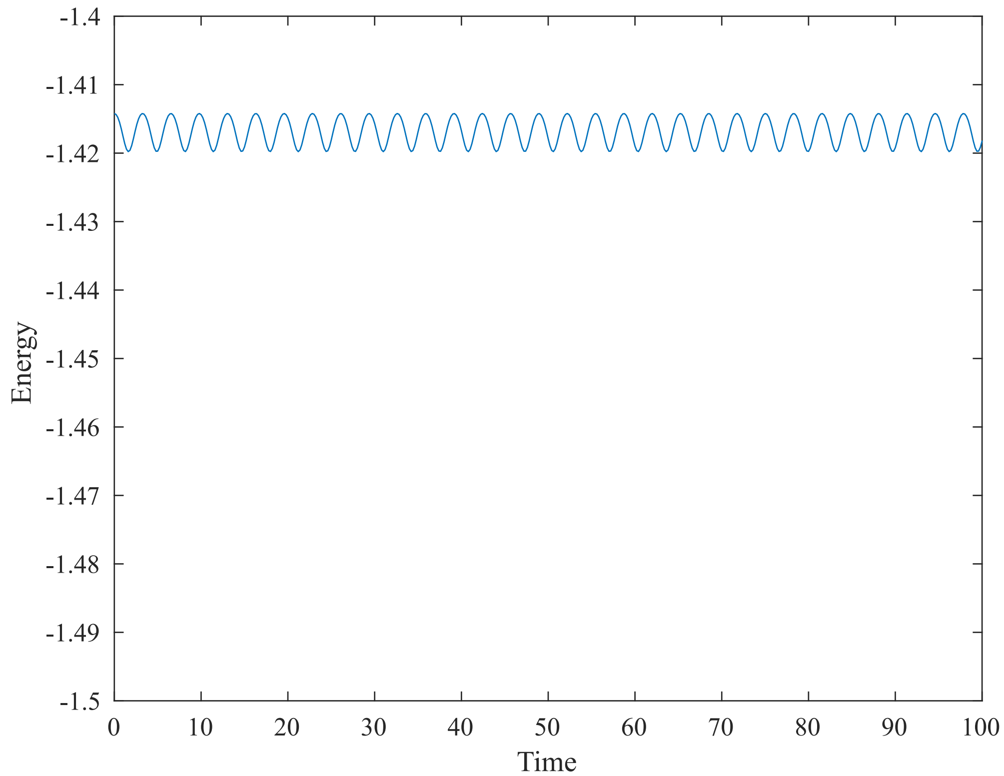
```
figure;
plot(t, E);
xlabel("Time");
ylabel("Energy");
xlim([0, 100]);
ylim([-1.5, -1.4]);
```



由上述能量变化图像可知，Verlet 算法求解结果中的系统能量虽有波动，但能够保持不变的趋势，符合物理推导结果。