Homework #6 — **Coffee Automaton**

CS 217 @ SJTU — ACM Class, Zhiyuan College, SJTU

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Exercise 1

Let $\nu(G)$ denote the size of a maximum matching of G.

For every cycle $x_1, \cdots, x_k$ in the bipartite graph (every cycle has an even length), we can modify the cycle by shaking the weights to $x_1+\epsilon, x_2-\epsilon, x_3+\epsilon, \cdots, x_k-\epsilon$ until some weight becomes 0 or 1.

Repeat the process above until we get a integral graph.

Therefore, $\nu(G) = \text{val}(MLP(G))$ for all bipartite graphs G.

Exercise 2

For every edge $e = (u, v)$ in the match we choose both u and v, then we can obtain a vertex cover with size $2\nu(G)$.

Therefore, $\tau(G) \le 2\nu(G)$.

Exercise 3

Let $C = \{v \in V | x_v \ge \frac{1}{2}\}$, then C is a vertex cover, and $tau(G) \le |C| \le 2\text{opt}(\text{VCLP}(G))$.

Exercise 5

> **Definition 4.** For $x \in \mathbb{R}^n$ let $I(x) := \{i \in [m+n] | a_i x = b_i\}$ be the set of indices of the constraints that are *tight*", i.e., satisfied with equality (we include non-negativity constraints here).

Suppose C is a vertex cover, and F be the set of tight edges.

From **Definition 4**, since we have we translated the constraint $x \ge 0$ into n constraints $-x_i \le 0$ and integrated them into A, which implies

- $i \in I(x)$ iff e_i is a tight edge for $1 \le i \le m$
- $i \in I(x)$ iff $v_{i-m} \notin C$ for $1 \le i \le n$

1. If $y_u \in \{0,1\}$ for all $u \in V$ and C is a minimal vertex cover, then $I(x)$ contains vertices that are not covered and all tight edges. We have $\text{rank}(A_{I(x)}) = n$ so y is a basic solution of VCLP(G).

2. If y is a basic solution of VCLP(G), y is the solution of $A_{I(y)}y = b_{I(y)}$. Then $I(y)$ contains the vertices that are not covered and all tight edges. To satisfy all the constraints, y must be an integral vector, that is, $y_u \in \{0,1\}$ for all $u \in V$, and C is a vertex cover of G. To show C is a minimal vertex cover, consider removing some vertex u from C, the constraints can't be satisfied, then $C\backslash\{u\}$ can't be a vertex cover. Therefore, C is a minimal vertex cover.

In conclusion, y is a basic solution of VCLP(G) iff $y_u \in \{0,1\}$ for all $u \in V$ and C is a minimal vertex cover.