Homework 7

Coffee Automaton

November 2019

Exercise 1

Prove three theorems separately(for simplicity and convenience, put the negative on the right), and we set that $x = (x_1, x_2, ..., x_n)^T$.

For (1), we set a function of x: $f(x) = (\frac{1}{2}x_1 + \frac{1}{2}|x_1|, ..., \frac{1}{2}x_n + \frac{1}{2}|x_n|)^T$. Then

$$
\begin{aligned}
&\exists x : Ax \leq b \\
&\qquad \Leftrightarrow \exists f(x), f(-x) \geq 0 : [A \ \ -A][\begin{matrix} f(x) \\ f(-x) \end{matrix}] \leq b \\
&\qquad \Leftrightarrow \neg(\exists y \geq 0 : [A - A]^T y \geq 0, b^T y < 0) \\
&\qquad \Leftrightarrow \forall y \geq 0 : y^T [A - A] \geq 0 \rightarrow b^T y \geq 0 \\
&\qquad \Leftrightarrow \forall y \geq 0 : y^T A \geq 0 \wedge y^T A \leq 0 \rightarrow y^T b \leq 0 \\
&\qquad \forall y \geq 0 : y^T A = 0 \rightarrow y^T b \leq 0 \\
&\qquad \Leftrightarrow \neg(\exists y \geq 0 : A^T y = 0, b^T y < 0)
\end{aligned}
$$

Then prove (2), adding a variable z.

$$
\begin{aligned}
&\exists x \geq 0 : Ax \leq b \\
&\qquad \Leftrightarrow \exists x, z : [AI_m][\begin{matrix} x \\ z \end{matrix}] = b \\
&\qquad \Leftrightarrow \neg(\exists y : [AI_m]^T y \geq 0, b^T y < 0) \\
&\qquad \Leftrightarrow \forall y : y^T [AI_m]^T y \geq 0 \rightarrow y^T b \geq 0 \\
&\qquad \Leftrightarrow \forall y : (y^T A \geq 0 \wedge y^T \geq 0) \rightarrow y^T b \geq 0 \\
&\qquad \Leftrightarrow \forall y \geq 0 : y^T A \geq 0 \rightarrow y^T b \geq 0 \\
&\qquad \Leftrightarrow \neg(\exists y \geq 0 : A^T y \geq 0, b^T y < 0)
\end{aligned}
$$

Then prove (3):

$$
\begin{aligned}
&\exists x \geq 0 : Ax = b \\
&\qquad \Leftrightarrow \exists x : I_n x \geq 0, Ax \leq b, Ax \geq b \\
&\qquad \Leftrightarrow \exists x : \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix} x = \begin{bmatrix} \text{b} \\ \text{-b} \\ 0 \end{bmatrix} \\
&\qquad \Leftrightarrow \neg(\overline{y} \geq 0 : \begin{bmatrix} \text{A} \\ \text{-A} \\ \text{-I}_n \end{bmatrix}^T \overline{y} = 0, \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}^T \overline{y} < 0) \\
&\qquad \Leftrightarrow \forall \overline{y} \geq 0 : \overline{y}^T \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix} = 0 \rightarrow \overline{y}^T \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \geq 0 \\
&\qquad \Leftrightarrow \forall \overline{y} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} \geq 0 : y^T A - y'^T A - y''^T I_n = 0 \rightarrow y^T b - y'^T b \geq 0 \\
&\qquad \Leftrightarrow \forall y, y', y'' \geq 0 : (y^T - y'^T)A = y''^T \rightarrow (y^T - y'^T)b \geq 0 \\
&\qquad \Leftrightarrow \forall y : y^T A \geq 0 \rightarrow y^T b \geq 0
\end{aligned}
$$

where $\overline{y} \in R^{2m+n}, y, y' \in R^m, y'' \in R^n$.

So we prove the 3 theorems can imply each other, which means they are equivalent.

Exercise 2 Denote the variable y, z

$$
Dual(G, s, t, c) : \quad
\begin{array}{lcl}
\textit{minimize} & \sum_{(u,v) \in E} c_{u,v} y_{u,v} & \\
\textit{subject to} & y_{u,v} - z_u + z_v \geq 0 & \forall (u,v) \in E, u \neq s, v \neq t \\
 & y_{s,v} + z_v \geq 1 & \forall (s,v) \in E \\
 & y_{u,t} - z_u \geq 0 & \forall (u,t) \in E \\
 & z_s - z_t \geq 1 & \\
 & y_{u,v} \geq 0 & \forall (u,v) \in E \\
 & z_v \in R & \forall v \in V \backslash \{s,t\}
\end{array}
$$

Exercise 3

It's obvious, because we can transform the above constrains into $y_{u,v} \geq z_u - z_v, y_{s,v} \geq 1 - z_v, y_{u,t} \geq z_u$, and to ger opt(D), we need minimize $c_{u,v} d_{u,v}$, which means we need minimize $y_{u,v}$.

So let

$$
\begin{aligned}
y_{u,v} &= max\{0, z_u - z_v\} \forall (u,v) \in E, u \neq s, v \neq t \\
y_{s,v} &= max\{0, 1 - z_v\} \forall (s,v) \in E \\
y_{u,t} &= max\{0, z_u\} \forall (u,t) \in E
\end{aligned}
$$

is the optimal solution, and the formulas for $y_{u,v}$ are just these.

Exercise 4 / 5
As showed in Exercise 3, we have $z_s - z_t \geq 1$, so the exercise 4 and 5 has optimal solution, and the solution has been shown in 3.

Exercise 6 For each solution in Exercise 5, we can collect all the y that $y_{u,v} = 1$, and $z_s = 1, z_t = 0$, then we can get a cut that $y_{u,v} = 1$ if and only if $edge(u, v)$ is in the cut, and we can set $z_u = 1$ if $u \in S$, and $z_u = 0$ if $u \in T$.

For clarity, if we have$e = (u, v)$, then we can write $f(e)$ as $f(u, v)$.

Exercise 7 Consider that the value of the flow is 1, and there is no capacity constrain. So assume there is P different path from s to t, each has cost c_p, which is the sum of cost of all the edges it visits. Obviously, the minimum c_p is d. and we can transform the problem MCF as $\sum c_p f_p$, where f_p if the value of flow that pass this path p(Compared with the origin sum, we split the flow of edges that have multiple paths, and add the entries that belong to same path). Now that any other path will bring a larger cost on the path, the minimum cost is all the flow go through the path of cost d. Considering that it is the same problem, we get that $opt(MCF) = d$.

Exercise 8

$$Dual(G, s, t, c): \begin{array}{lll} \textit{maximize} & z_t - z_s & \\ \textit{subject to} & z_v - z_u \leq c(u, v) & \forall (u, v) \in E \\ & z_v \geq 0 & \forall v \in V \end{array}$$

Exercise 9 It can be seen as a shortest path problem, and it's a problem to find the max flow, where $z_t - z_s$ shows the difference value from s to t.

Exercise 10 Let z_v be the shortest distance from s to v, $v \in G$, and we set $z_s = 0$.

So $\forall u, v \in E$, we have $z_v - z_u \leq c(u, v)$ (or we can find a shorter path from s to v). In exercise 7 we prove $opt(MCF) = z_t$, and by strong duality we have $opt(MCF) = opt(D)$. So $z_t = opt(D)$, we get a optimal solution of this dual program.