CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Thursday, 2019-10-30
First submission and questions due on Thursday, 2019-11-07
You will receive feedback from the TA.
Final submission due on Thursday, 2019-11-14

6 Matching LP and Vertex Cover LP

Let $G = (V, E)$ be a graph and consider the Vertex Cover Linear Program VCLP(G):

$$\begin{array}{lll} \text{VCLP}(G): & \text{minimize} & \sum_{u \in V} y_u \\ & \text{subject to} & y_u + y_v \geq 1 \quad \forall \text{ edges } \{u, v\} \in E \\ & & \mathbf{y} \geq \mathbf{0} \end{array}$$

Every vertex cover of G corresponds to a feasible solution $\mathbf{y} \in \text{sol}(\text{VCLP}(G))$, but not vice versa. However, every $\mathbf{y} \in \text{sol}(\text{VCLP}(G)) \cap \{0, 1\}^V$ does. Let $\tau(G)$ denote the size of a minimum vertex cover of G. In class, we showed that $\tau(G) = \text{val}(\text{VCLP}(G))$ for all *bipartite* graphs G. We achieved this by taking an arbitrary feasible solution $\mathbf{y}$ and "shaking" it until it becomes integral, while making sure its value does not go up.

Next, recall the Matching Linear Program MLP(G):

$$\begin{array}{lll} \text{MLP}(G): & \text{maximize} & \sum_{e \in E} x_e \\ & \text{subject to} & \sum_{e \in E: u \in e} x_e \leq 1 \quad \forall\, u \in V \\ & & \mathbf{x} \geq \mathbf{0} \end{array}$$

Every matching of G corresponds to a feasible solution $\mathbf{x} \in \text{sol}(\text{MLP}(G))$, but not vice versa. However, every $\mathbf{x} \in \text{sol}(\text{MLP}(G)) \cap \{0, 1\}^E$ does.

Exercise 1. Let $\nu(G)$ denote the size of a maximum matching of G. Obviously, $\text{val}(\text{MLP}(G)) \geq \nu(G)$ for all graphs. Show that $\nu(G) = \text{val}(\text{MLP}(G))$ for all *bipartite* graphs G.

Exercise 2. We know that $\nu(G) = \tau(G)$ for all bipartite graphs (Kőnig's Theorem) and $\nu(G) \leq \tau(G)$ for all graphs (since every matched edge must be covered by a distinct vertex). Show that $\tau(G) \leq 2\,\nu(G)$ for all graphs G.

Exercise 3. Show that $\tau(G) \leq 2\,\text{opt}(\text{VCLP}(G))$ for all graphs G (including non-bipartite graphs).

Basic Solutions. Recall our definition of basic solutions. Let P be the following linear program.

$$P: \qquad \begin{array}{ll} \text{maximize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & A\mathbf{x} \ \leq \mathbf{b} \end{array}$$

where we translated the constraint $\mathbf{x} \geq 0$ into n constraints $-x_i \leq 0$ and integrated them into A, so the n last rows of A form the negative identity matrix $-I_n$. We introduce some notation: $\mathbf{a}_i$ is the i^{th} row of i; for $I \subseteq [m+n]$ let A_I be the matrix consisting of the rows $\mathbf{a}_i$ for $i \in I$.

Definition 4. *For $\mathbf{x} \in \mathbb{R}^n$ let $I(\mathbf{x}) := \{i \in [m+n] \mid \mathbf{a}_i\mathbf{x} = b_i\}$ be the set of indices of the constraints that are "tight", i.e., satisfied with equality (we include non-negativity constraints here). We call $\mathbf{x} \in \mathbb{R}^n$ a* basic point *if $\text{rank}(A_{I(x)}) = n$. If $\mathbf{x}$ is a basic point and feasible, we call it a* basic feasible solution *or simply a* basic solution.

We can define the same concept for minimization programs.

We say a set $C \subseteq V$ is a minimal vertex cover of $G = (V, E)$ if (1) it is a vertex cover and (2) it is minimal, i.e., for every $u \in C$ the set $C \setminus \{u\}$ is not a vertex cover anymore.

Exercise 5. Let G be a bipartite graph. Show that $\mathbf{y} \in \mathbb{R}^V$ is a basic solution of $\text{VCLP}(G)$ if and only if (1) $y_u \in \{0, 1\}$ for all $u \in V$ and (2) the set $C := \{u \in V \mid y_u = 1\}$ is a minimal vertex cover.

Hint. Suppose $e_1, \ldots, e_k$ form a cycle in G. Note that every edge corresponds to a constraint of the VCLP, and thus this cycle corresponds to a submatrix A_I with $|I| = k$. Show that the k rows of A_I are linearly dependent.

Hint. Suppose C is a minimal vertex cover. Let F be the set of "tight" edges, i.e., the edges $e \in E$ incident to exactly one $u \in C$. What does minimality of C say about the relation between C and F? Does this help you to show that the set of tight constraints of VCLP has rank n?

Hint. Conversely, suppose $\mathbf{y}$ is a basic solution. Look at the vertices u with $y_0 = 0$ and the "tight edges", those $e = \{u, v\}$ for which $y_u + y_v = 1$.