CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

Handed out on Thursday, 2019-10-10
First submission and questions due on Monday, 2019-10-14
You will receive feedback from the TA.
Final submission due on Monday, 2019-10-21

4 Bottleneck Paths

Let $G = (V, E)$ be a directed graph with an edge capacity function $c : E \to \mathbb{R}^+$. For a path $p = u_0 u_1 \dots u_t$ define its *capacity* to be

$$c(p) := \min_{1 \le i \le t} c(\{u_{i-1}, u_i\}) \ . \tag{1}$$

> **Maximum Capacity Path Problem (MCP).** Given a directed graph $G = (V, E)$, an edge capacity function $c : E \to \mathbb{R}^+$, and two vertices $s, t \in V$, compute the path p^* maximizing $c(p)$. We denote by p^* the optimal path and by $c^* := c(p^*)$ its cost.

Exercise 1. Suppose the edges $e_1, \dots, e_m$ are sorted by their cost. Show how to solve MCP in time $O(n + m)$.

Exercise 2. Give an algorithm for MCP of running time $O(m \log \log m)$. **Hint:** Using the median-of-medians algorithm, you can determine an edge e such that at most $m/2$ edges are cheaper than e and at most $m/2$ edges are more expensive than e. Can you determine, in time $O(n + m)$, whether $c^* < c(e)$, $c^* = c(e)$, or $c^* > c(e)$? Iterate to shrink the set of possible values for c^* to $m/4$, $m/8$, and so on.

Exercise 3. Give an algorithm for MCP that runs in time $O(m \log\log\log m)$? How about $O(m \log\log\log\log m)$? How far can you get?

Exercise 4. Suppose we modify the Ford-Fulkerson method so that, in every round, it finds a path of *maximum capacity*, as opposed to be shortest-$s-t$-path method employed by Edmonds-Karp. Show that this algorithm terminates after a number of rounds that is polynomial in n and m.