高等数学模拟试卷（一）参考答案

1. 解：$\lim\limits_{x\to0}\dfrac{x-\sin ax}{x^2\ln(1+bx)}=\lim\limits_{x\to0}\dfrac{x-\sin ax}{-bx^3}=\lim\limits_{x\to0}\dfrac{1-a\cos ax}{-3bx^2}$

$\because\lim\limits_{x\to0}\left(-3bx^2\right)=0$，$\therefore$必有$\lim\limits_{x\to0}\left(1-a\cos ax\right)=0$，即$1-a=0$，$\therefore a=1$

于是有$\lim\limits_{x\to0}\dfrac{1-\cos x}{-3bx^2}=\lim\limits_{x\to0}\dfrac{\frac{1}{2}x^2}{-3bx^2}=-\dfrac{1}{6b}=1$，$\therefore b=-\dfrac{1}{6}$

2. 解：因为各函数在非分段点都是连续的，所以只要在分断点处连续即可保证在$(-\infty,+\infty)$上连续。

对于 A：$f(0)=1$

$$f(0-0)=\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{\sin x}{|x|}=\lim_{x\to0^-}\frac{\sin x}{-x}=-1$$

$$f(0+0)=\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{\sin x}{|x|}=\lim_{x\to0^-}\frac{\sin x}{x}=1$$

因为$f(0-0)\neq f(0+0)$，所以$f(x)$在$x=0$处不连续

从而$f(x)$在$(-\infty,+\infty)$上不连续

对于 B：$g(0)=0$

$$g(0-0)=\lim_{x\to0-}g(x)=\lim_{x\to0^-}\sin x=0$$

$$g(0+0)=\lim_{x\to0^+}g(x)=\lim_{x\to0^+}(\cos x-1)=0$$

因为$g(0-0)=g(0+0)$，所以$g(x)$在$x=0$处连续

从而$g(x)$在$(-\infty,+\infty)$上连续

3. 解：解题依据：当$f(x)$在$x=x_0$处可导时

$$\lim_{\Delta x\to0}\frac{f(x_0+\alpha\Delta x)-f(x_0+\beta\Delta x)}{\Delta x}=(\alpha-\beta)f'(x_0)$$

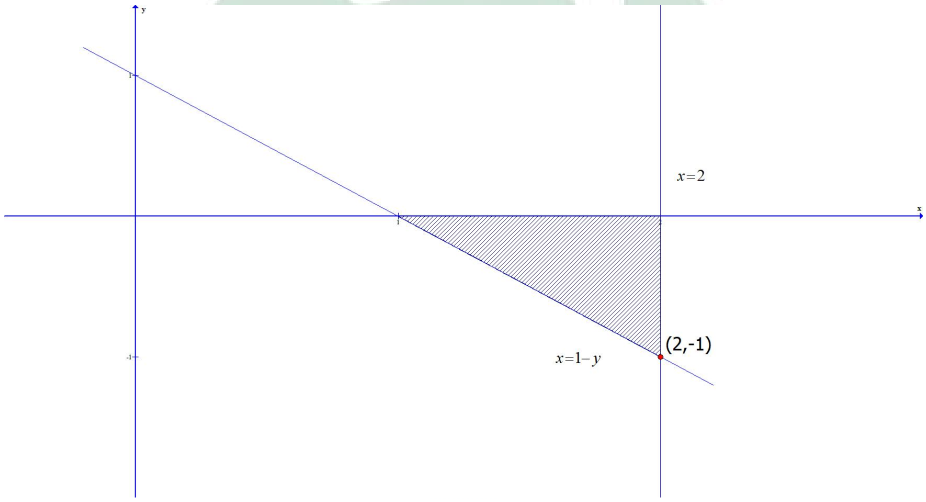
所以$\lim\limits_{x\to 0}\dfrac{f(a+x)-f(a-x)}{x}=[1-(-1)]f'(a)=2f'(a)$

4. 解：$\lim\limits_{x\to\infty}\dfrac{1+e^{-x^2}}{1-e^{-x^2}}=1$，所以$y=1$是其水平渐近线

当$x=0$时，分母等于 0，$\because \lim\limits_{x\to 0}\dfrac{1+e^{-x^2}}{1-e^{-x^2}}=\infty$，所以$x=0$是其垂直渐近线

5. 解：积分区域$\begin{cases}1-y\le x\le 2\\ -1\le y\le 0\end{cases}$

$\int_{-1}^{0}dy\int_{1-y}^{2}f(x,y)dx=\int_{1}^{2}dx\int_{1-x}^{0}f(x,y)dy$



6. 解：对于 A：取$V_n=\dfrac{n}{n\sqrt{n}}=\dfrac{1}{\sqrt{n}}$，$\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$发散，所以 A 发散

对于 B：$\because \lim\limits_{n\to\infty}\sqrt{\dfrac{2n(n+1)}{3n^2+2n+1}}=\sqrt{\dfrac{2}{3}}\equiv 0$，所以 B 发散

对于 C：$\sum\limits_{n=1}^{\infty}\dfrac{2n+(-1)^n}{n^2}=\sum\limits_{n=1}^{\infty}\dfrac{2n}{n^2}+\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}$发散

7. 解：$D_2=\begin{vmatrix}2a_1 & 6a_2\\ b_1 & 3b_2\end{vmatrix}=2\begin{vmatrix}a_1 & 3a_2\\ b_1 & 3b_2\end{vmatrix}=6\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}=6k$

8. 解：$2A=\begin{pmatrix}2 & 0 & 2x\\ 4 & 2 & 2\end{pmatrix}$

由$2A=B$可知，$\begin{pmatrix}2 & 0 & 2x\\ 4 & 2 & 2\end{pmatrix}=\begin{pmatrix}2 & 0 & 2\\ 4 & 2 & y\end{pmatrix}$

$\therefore \begin{cases} 2x = 2 \\ y = 2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 2 \end{cases}$

9. 解：$\lim\limits_{x\to\infty}\left(\dfrac{1+x}{2+x}\right)^{2x} = \lim\limits_{x\to\infty} e^{\left(\frac{1+x}{2+x}-1\right)\cdot 2x} = \lim\limits_{x\to\infty} e^{\frac{-2x}{2+x}} = e^{-2}$

10. 解：$y' = 2^x + x\cdot 2^x \ln 2$

令 $y'=0$，得 $2^x + x\cdot 2^x\ln 2 = 0$，即 $1+x\ln 2 = 0$，$\therefore x = -\dfrac{1}{\ln 2}$

当 $x < -\dfrac{1}{\ln 2}$ 时，$y' < 0$，所以 y 单调递减

当 $x > -\dfrac{1}{\ln 2}$ 时，$y' > 0$，所以 y 单调递增

故当 $x = -\dfrac{1}{\ln 2}$ 时，y 取得极小值

11. 解：原式 $= \int_{-2}^{2} x^3\sqrt{4-x^2}\,dx + \int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi$

12. 解：$z = \ln\sqrt{x^2+2y} = \dfrac{1}{2}\ln\left(x^2+2y\right)$

$\dfrac{\partial z}{\partial x} = \dfrac{1}{2}\cdot\dfrac{2x}{x^2+2y} = \dfrac{x}{x^2+2y}$，$\dfrac{\partial z}{\partial y} = \dfrac{1}{x^2+2y}$

$\therefore dz\Big|_{\substack{x=1\\y=2}} = \dfrac{x}{x^2+2y}\Bigg|_{\substack{x=1\\y=2}} dx + \dfrac{1}{x^2+2y}\Bigg|_{\substack{x=1\\y=2}} dy = \dfrac{1}{5}dx + \dfrac{1}{5}dy$

13. 解：令 $\lim\limits_{n\to\infty}\left|\dfrac{(x+1)^{n+1}}{\sqrt{n+2}}\cdot\dfrac{\sqrt{n+1}}{(x+1)^n}\right| = |x+1| < 1$，$\therefore -2 < x < 0$

当 $x=-2$ 时，级数为 $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n\cdot(-1)^n}{\sqrt{n+1}} = \sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n+1}}$ 发散

当 $x=0$ 时，级数为 $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{\sqrt{n+1}}$ 收敛

所以收敛域为 $(-2,0]$

14. 解：$B^T = \begin{pmatrix} 3 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$AB^T=\begin{pmatrix}1&0&2\\0&1&0\end{pmatrix}\begin{pmatrix}3&0\\0&1\\-1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

15. 解：$\lim\limits_{x\to0}\dfrac{\arctan x-\arcsin x}{x^2\arcsin x}=\lim\limits_{x\to0}\dfrac{\arctan x-\arcsin x}{x^3}=\lim\limits_{x\to0}\dfrac{\dfrac{1}{1+x^2}-\dfrac{1}{\sqrt{1-x^2}}}{3x^2}$

$$=\lim_{x\to0}\frac{\sqrt{1-x^2}-1-x^2}{3x^2}=\lim_{x\to0}\frac{\dfrac{-x}{\sqrt{1-x^2}}-2x}{6x}=-\frac{1}{2}$$

16. 解：$\dfrac{dx}{dt}=6t+2$

$e^y\sin t-y+1=0$ 两边对 t 求导，得

$e^y\dfrac{dy}{dt}\sin t+e^y\cos t-\dfrac{dy}{dt}=0$，$\therefore\dfrac{dy}{dt}=\dfrac{-e^y\cos t}{e^y\sin t-1}$

$$\left.\frac{dy}{dx}\right|_{t=0}=\left.\frac{dy}{dt}\middle/\frac{dx}{dt}\right|_{t=0}=\left.\frac{-e^y\cos t}{\left(e^y\sin t-1\right)\left(6t+2\right)}\right|_{t=0}=\frac{e^y}{2}$$

由原方程可知，当 $t=0$ 时 $y=1$，$\therefore\left.\dfrac{dy}{dx}\right|_{t=0}=\left.\dfrac{e^y}{2}\right|_{y=1}=\dfrac{e}{2}$

17. 解：$\int x^2\cos^2 xdx=\int x^2\cdot\dfrac{1+\cos 2x}{2}dx=\dfrac{1}{2}\int x^2dx+\dfrac{1}{2}\int x^2\cos 2xdx$

$$=\frac{1}{6}x^3+\frac{1}{4}\int x^2d\sin 2x=\frac{1}{6}x^3+\frac{1}{4}x^2\sin 2x-\frac{1}{4}\int\sin 2xdx^2$$

$$=\frac{1}{6}x^3+\frac{1}{4}x^2\sin 2x-\frac{1}{2}\int x\sin 2xdx$$

$$=\frac{1}{6}x^3+\frac{1}{4}x^2\sin 2x+\frac{1}{4}\int xd\cos 2x$$

$$=\frac{1}{6}x^3+\frac{1}{4}x^2\sin 2x+\frac{1}{4}x\cos 2x-\frac{1}{4}\int\cos 2xdx$$

$$=\frac{1}{6}x^3+\frac{1}{4}x^2\sin 2x+\frac{1}{4}x\cos 2x-\frac{1}{8}\sin 2x+c$$

18. 解：令 $x=\sin t$，当 $x=0$ 时 $t=0$，当 $x=\dfrac{\sqrt{2}}{2}$ 时 $t=\dfrac{\pi}{4}$

$$\int_0^{\frac{\sqrt{2}}{2}}\frac{dx}{\left(3-2x^2\right)\sqrt{1-x^2}}=\int_0^{\frac{\pi}{4}}\frac{d\sin t}{\left(3-2\sin^2 t\right)\cos t}=\int_0^{\frac{\pi}{4}}\frac{1}{3-2\sin^2 t}dt$$

$$=\int_0^{\frac{\pi}{4}}\frac{dt}{3\left(\sin^2 t+\cos^2 t\right)-2\sin^2 t}=\int_0^{\frac{\pi}{4}}\frac{dt}{3\cos^2 t+\sin^2 t}$$

$$=\int_0^{\frac{\pi}{4}}\frac{dt}{\cos^2 t\left(3+\tan^2 t\right)}=\int_0^{\frac{\pi}{4}}\frac{1}{\left(\sqrt{3}\right)^2+\tan^2 t}d\tan t$$

$$=\frac{1}{\sqrt{3}}\arctan t\frac{\tan t}{\sqrt{3}}\Bigg|_0^{\frac{\pi}{4}}=\frac{\pi}{6\sqrt{3}}$$

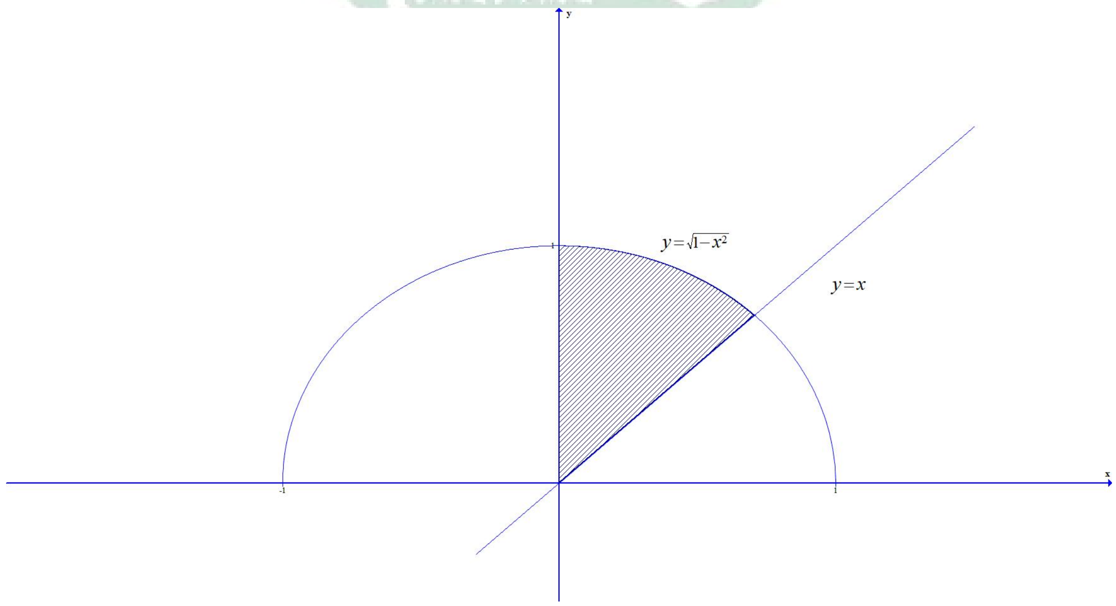
19. 解：$\frac{\partial z}{\partial x}=f_1'\cdot 1+f_2'\cdot 2x=f_1'+2xf_2'$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}''+f_{12}''\cdot(-2y)+2x\left[f_{21}''+f_{22}''\cdot(-2y)\right]$$

$$=f_{11}''+\left(2x-2y\right)f_{12}''-4xyf_{22}''$$

20. 解：$\iint\limits_D ydxdy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^1 r\sin\theta rdr=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\cdot\sin\theta\cdot\frac{1}{3}r^3\Big|_0^1$

$$=-\frac{1}{3}\cos\theta\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{\sqrt{2}}{6}$$



21. 解：对应齐次方程 $y''+3y'-4y=0$，特征方程 $r^2+3r-4=0$，特征根 $r_1=1,r_2=-4$

通解 $\overline{y}=c_1e^x+c_2e^{-4x}$

设原方程的一个特解 $y^*=x\left(ax+b\right)e^x=\left(ax^2+bx\right)e^x$

$y^*{}' = (2ax+b)e^x + (ax^2+bx)e^x$

$y^*{}'' = 2ax^x + (2ax+b)e^x + (2ax+b)e^x + (ax^2+bx)e^x$

将 y^* 代入原方程得 $(10ax+2a+5b)e^x = (x+1)e^x$

$\begin{cases} 10a = 1 \\ 2a+5b = 1 \end{cases} \Rightarrow \begin{cases} a = \dfrac{1}{10} \\ b = \dfrac{4}{25} \end{cases}$，$\therefore y^* = \left(\dfrac{1}{10}x^2 + \dfrac{4}{25}x\right)e^x$

$y = \overline{y} + y^* = c_1e^x + c_2e^{-4x} + \left(\dfrac{1}{10}x^2 + \dfrac{4}{25}x\right)e^x$

22. 解：$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 2 & 1 & 0 & 3 \\ 1 & 2 & 1 & -1 \end{pmatrix} \xrightarrow[r_4-r_1]{\substack{r_2+r_1 \\ r_3-2r_1}} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{\substack{r_3+r_2 \\ r_4-r_2}} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

$\xrightarrow{r_4+\frac{1}{2}r_3} \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

所以向量组得一个极大无关组为 $\alpha_1,\alpha_2,\alpha_3$，向量组得秩 $r(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = 3$

23. 证明：令 $F(x) = x - \ln(1+x)$，则 $F(0) = 0$

$F'(x) = 1 - \dfrac{1}{1+x} = \dfrac{x}{1+x}$

当 $x > 0$ 时，$F'(x) > 0$，所以 $F(x)$ 单调递增

$F(x) > F(0) = 0$，即 $x - \ln(1+x) > 0$，所以 $x > \ln(1+x)$

令 $G(x) = x^2 - (1+x)\ln^2(1+x)$，则 $G(0) = 0$

$G'(x) = 2x - \ln^2(1+x) - 2\ln(1+x)$，$G'(0) = 0$

$G''(x) = 2 - \dfrac{2\ln(1+x)}{1+x} - \dfrac{2}{1+x} = \dfrac{2[x - \ln(1+x)]}{1+x}$

因为当 $x > 0$ 时，$x - \ln(1+x) > 0$

所以当 $x > 0$ 时，$G''(x) > 0$，所以 $G'(x)$ 单调递增，$G'(x) > G'(0) = 0$

又由$G'(x)>0$可知，$G(x)$单调递增，故$G(x)>G(0)=0$

所以$x^2-(1+x)\ln^2(1+x)>0$，即$(1+x)\ln^2(1+x)<x^2$

24. 解：⑴设切点为$\left(a,\sqrt{a}\right)$

$$y'=\frac{1}{2\sqrt{x}}，\ k_{切}=y'|_{x=a}=\frac{1}{2\sqrt{a}}$$

切线方程为$y-\sqrt{a}=\dfrac{1}{2\sqrt{a}}(x-a)$

因为切线过点$(0,1)$，所以$1-\sqrt{a}=\dfrac{1}{2\sqrt{a}}(0-a)$，解得$a=4$

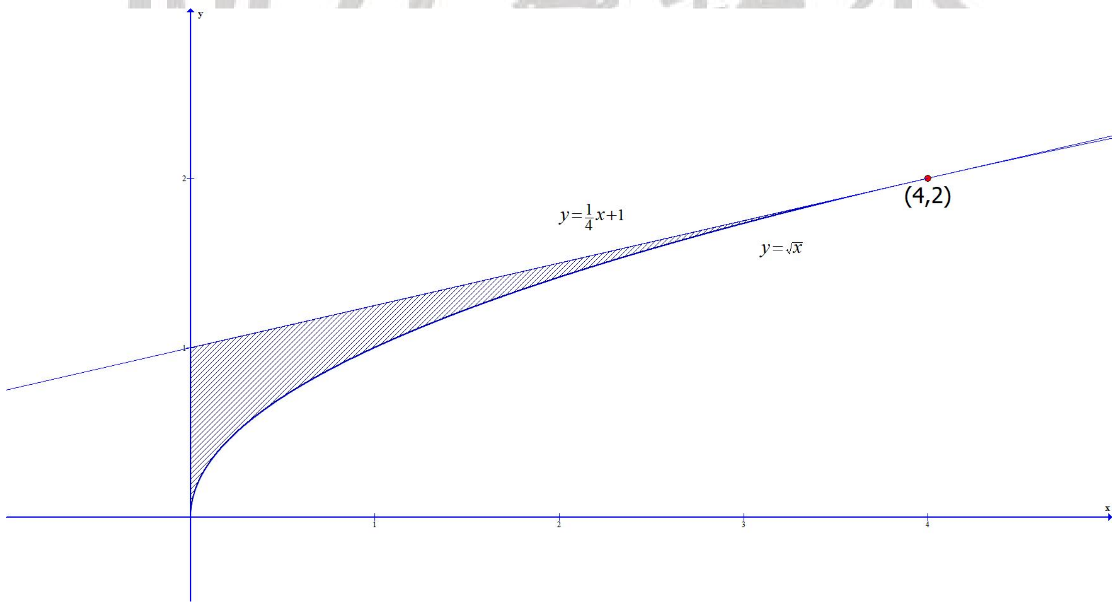
所以切线方程为$y=\dfrac{1}{4}x+1$

⑵ $S=\int_0^4\left(\frac{1}{4}x+1-\sqrt{x}\right)dx=\left(\frac{1}{8}x^2+x-\frac{2}{3}x^{\frac{3}{2}}\right)\Big|_0^4=\frac{2}{3}$

⑶ $V_x=\pi\int_0^4\left(\frac{1}{4}x+1\right)^2dx-\pi\int_0^4\left(\sqrt{x}\right)^2dx$

$$=\pi\int_0^4\left(\frac{x+4}{4}\right)^2dx-\pi\int_0^4 xdx$$

$$=\frac{\pi}{16}\int_0^4(x+4)^2d(x+4)-\frac{\pi}{2}x^2\Big|_0^4=\frac{\pi}{48}(x+4)^3\Big|_0^4-8\pi=\frac{4\pi}{3}$$



25. 解：$\overline{A}=\begin{pmatrix}1 & 0 & 1 & 2 & -1\\ 2 & 1 & 3 & 1 & -4\\ 1 & 2 & 3 & -4 & -5\\ -1 & -1 & -2 & 1 & 3\end{pmatrix}\xrightarrow{\substack{r_2-2r_1\\ r_3-r_1\\ r_4+r_1}}\begin{pmatrix}1 & 0 & 1 & 2 & -1\\ 0 & 1 & 1 & -3 & -2\\ 0 & 2 & 2 & -6 & -4\\ 0 & -1 & -1 & 3 & 2\end{pmatrix}$

$$\xrightarrow{\substack{r_3+2r_2\\ r_4+r_2}}\begin{pmatrix}1 & 0 & 1 & 2 & -1\\ 0 & 1 & 1 & -3 & -2\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

x_1,x_2 为约束变量，x_3,x_4 为自由变量

一般解$\begin{cases}x_1=-1-x_3-2x_4\\ x_2=-2-x_3+3x_4\end{cases}$

取 $x_3=x_4=0$，得原方程得一个特解 $\eta^*=\begin{pmatrix}-1\\ -2\\ 0\\ 0\end{pmatrix}$

导出组一般解$\begin{cases}x_1=-x_3-2x_4\\ x_2=-x_3+3x_4\end{cases}$

令$\begin{cases}x_3=1\\ x_4=0\end{cases}$，得 $\xi_1=\begin{pmatrix}-1\\ -1\\ 1\\ 0\end{pmatrix}$，令$\begin{cases}x_3=0\\ x_4=1\end{cases}$，得 $\xi_2=\begin{pmatrix}-2\\ 3\\ 0\\ 1\end{pmatrix}$

原方程组通解$\begin{pmatrix}x_1\\ x_2\\ x_3\\ x_4\end{pmatrix}=\begin{pmatrix}-1\\ -2\\ 0\\ 0\end{pmatrix}+k_1\begin{pmatrix}-1\\ -1\\ 1\\ 0\end{pmatrix}+k_2\begin{pmatrix}-2\\ 3\\ 0\\ 1\end{pmatrix}$，（$k_1,k_2$ 为任意常数）