高等数学模拟试卷（四）参考答案

1. 解：$\because x=1$是$f\left(x\right)$的可去间断点，$\therefore \lim\limits_{x\to 1}\dfrac{x^2+ax+2}{x-1}$存

又$\because \lim\limits_{x\to 1}\left(x-1\right)=0$，$\therefore$必有$\lim\limits_{x\to 1}\left(x^2+ax+2\right)=a+3=0$

$\therefore a=-3$

2. 解：当$x\to 0$时，$1-\cos\sqrt{x}\sim\dfrac{1}{2}x$，$\ln\left(1+\sqrt{x}\right)\sim\sqrt{x}$

$1-e^{\sqrt{x}}\sim-\sqrt{x}$，$\sqrt{1+\sqrt{x}}-1\sim\dfrac{1}{2}\sqrt{x}$，所以选 A

3. 解：$\because \lim\limits_{x\to\infty}\dfrac{x^2-3x+2}{x^2+x-6}=1$，$\therefore y=1$是其水平渐近线

$\because \lim\limits_{x\to 2}\dfrac{x^2-3x+2}{x^2+x-6}=\lim\limits_{x\to 2}\dfrac{2x-3}{2x+1}=\dfrac{1}{5}\neq\infty$，$\therefore x=2$不是垂直渐近线

$\because \lim\limits_{x\to -3}\dfrac{x^2-3x+2}{x^2+x-6}=\infty$，$\therefore x=-3$是垂直渐近线

所以共有 2 条渐近线

4. 解：$y'=\left(2ax-2\right)e^{ax^2-2x}$

$\because x=1$是函数的极值点

$\therefore y'\big|_{x=1}=\left(2a-2\right)e^{a-2}=0$，$\therefore a=1$

5. 解：对于 A：取$V_n=\dfrac{1}{n\sqrt{n}}$，$\because \sum\limits_{n=1}^{\infty}\dfrac{1}{n\sqrt{n}}$收敛，所以 A 绝对收敛

对于 B：取$V_n=\dfrac{1}{2}\cdot\left(\dfrac{1}{n}\right)^2=\dfrac{1}{2}\cdot\dfrac{1}{n^2}$，$\because \sum\limits_{n=1}^{\infty}\dfrac{1}{2n^2}$收敛，所以 B 绝对收敛

对于 C：取$V_n=\dfrac{n}{\sqrt{n^3}}=\dfrac{1}{\sqrt{n}}$，$\because \sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt{n}}$发散，$\sum\limits_{n=1}^{\infty}\dfrac{\left(-1\right)^n}{\sqrt{n}}$收敛

所以 C 条件收敛

6. 解：$\because f\left(x\right)$的一个原函数为$\dfrac{\sin x}{x}$，$\therefore \int f\left(x\right)dx=\dfrac{\sin x}{x}+c$

$$\int f(2x)dx=\frac{1}{2}\int f(2x)d2x=\frac{1}{2}\cdot\frac{\sin 2x}{2x}+c=\frac{\sin 2x}{4x}+c$$

7. 解：根据性质$(AB)^{-1}=B^{-1}A^{-1}$

$$(ABC)^{-1}=\left[(AB)C\right]^{-1}=C^{-1}(AB)^{-1}=C^{-1}B^{-1}A^{-1}$$

8. 解：根据性质可知 C 正确

9. 解：$\lim\limits_{x\to0}\left(\frac{2+x}{2-x}\right)^{\frac{k}{x}}=\lim\limits_{x\to0}e^{\left(\frac{2+x}{2-x}-1\right)\cdot\frac{k}{x}}=e^{k}=e^{2}$，$\therefore k=2$

10. 解：$\lim\limits_{x\to0}\frac{f(0-x)-f(0+x)}{x}=\left[(-1)-1\right]f'(0)=-2f'(0)=-2$

11. 解：方程两边对x求偏导数，得：

$$3z^{2}\frac{\partial z}{\partial x}-3yz-3xy\frac{\partial z}{\partial x}+3z+3x\frac{\partial z}{\partial x}+3x^{2}=0$$

$\therefore z^{2}\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}}+1=0$，即$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}}=-\frac{1}{z^{2}}$

由原方程可知，当$x=0,y=0$时$z=1$

$$\therefore\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=0}}=-\left.\frac{1}{z^{2}}\right|_{z=1}=-1$$

12. 解：令$\lim\limits_{n\to\infty}\left|\frac{(2x-1)^{n+1}}{n+2}\cdot\frac{n+1}{(2x-1)^{n}}\right|=|2x-1|<1$

$-1<2x-1<1$，$\therefore 0<x<1$

当$x=0$时，级数为$\sum\limits_{n=1}^{\infty}\frac{1}{n+1}$发散

当$x=1$时，级数为$\sum\limits_{n=1}^{\infty}\frac{(-1)^{n}}{n+1}$收敛

收敛域$(0,1]$

13. 解：$x\frac{dy}{dx}+y-x^{2}=0$，$\frac{dy}{dx}+\frac{1}{x}y=x$

通解$y=e^{-\int\frac{1}{x}dx}\left[\int xe^{\int\frac{1}{x}dx}dx+c\right]=\frac{1}{3}x^{2}+\frac{c}{x}$

14. 解：$A=\begin{pmatrix}3&2&1\\1&1&0\\2&0&a\end{pmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{pmatrix}1&1&0\\3&2&1\\2&0&a\end{pmatrix}\xrightarrow[r_3-2r_1]{r_2-3r_1}\begin{pmatrix}1&1&0\\0&-1&1\\0&-2&a\end{pmatrix}$

$\because$ 向量组$\alpha_1,\alpha_2,\alpha_3$线性无关，$\therefore$ 秩等于3

故应有$\dfrac{-1}{-2}\neq\dfrac{1}{a}$，$a\neq 2$

15. 解：$\lim\limits_{x\to 0}\dfrac{\int_0^x\left[t-\ln(1+t)\right]dt}{x^2\left(\sqrt{1+x}-1\right)}=\lim\limits_{x\to 0}\dfrac{\int_0^x\left[t-\ln(1+t)\right]dt}{\frac{1}{2}x^3}$

$$=\lim_{x\to 0}\frac{x-\ln(1+x)}{\frac{3}{2}x^2}=\lim_{x\to 0}\frac{1-\frac{1}{1+x}}{3x}$$

$$=\lim_{x\to 0}\frac{x}{3x(1+x)}=\frac{1}{3}$$

16. 解：方程两边对x求偏导数，得：$e^z\dfrac{\partial z}{\partial x}=1-2\dfrac{\partial z}{\partial x}$　　①

$\therefore\dfrac{\partial z}{\partial x}=\dfrac{1}{e^z+2}$

①式两边再对x求偏导数，得：$e^z\left(\dfrac{\partial z}{\partial x}\right)^2+e^z\dfrac{\partial^2 z}{\partial x^2}=-2\dfrac{\partial^2 z}{\partial x^2}$

$$\therefore\frac{\partial^2 z}{\partial x^2}=-\frac{e^z\left(\frac{\partial z}{\partial x}\right)^2}{e^z+2}=-\frac{e^z\left(\frac{1}{e^z+2}\right)^2}{e^z+2}=-\frac{e^z}{\left(e^z+2\right)^3}$$

17. 解：$\int(x\ln x)^2dx=\int x^2\ln^2 xdx=\dfrac{1}{3}\int\ln^2 xdx^3$

$$=\frac{1}{3}x^3\ln^2 x-\frac{1}{3}\int x^3d\ln^2 x=\frac{1}{3}x^3\ln^2 x-\frac{2}{3}\int x^2\ln xdx$$

$$=\frac{1}{3}x^3\ln^2 x-\frac{2}{9}\int\ln xdx^3$$

$$=\frac{1}{3}x^3\ln^2 x-\frac{2}{9}x^3\ln x+\frac{2}{9}\int x^3d\ln x$$

$$=\frac{1}{3}x^3\ln^2 x-\frac{2}{9}x^3\ln x+\frac{2}{9}\int x^2dx$$

$$=\frac{1}{3}x^3\ln^2 x-\frac{2}{9}x^3\ln x+\frac{2}{27}x^3+c$$

18. 解：令$\sqrt{2-x}=t$，则$x=2-t^2$，当$x=-1$时$t=\sqrt{3}$，当$x=1$时$t=1$

$$\int_{-1}^{1}\frac{dx}{(3-x)\sqrt{2-x}}=\int_{\sqrt{3}}^{1}\frac{d\left(2-t^{2}\right)}{\left(1+t^{2}\right)t}=\int_{\sqrt{3}}^{1}\frac{2t}{\left(1+t^{2}\right)t}dt$$

$$=-\int_{\sqrt{3}}^{1}\frac{1}{1+t^{2}}dt=-\arctan t\Big|_{\sqrt{3}}^{1}=\frac{\pi}{6}$$

19. 解： $\frac{\partial z}{\partial x}=f_1'\cdot y+f_2'\cdot 2x+\varphi'\frac{-y}{x^2}=yf_1'+2xf_2'-\frac{y}{x^2}\varphi'$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1'+y\left(f_{11}''\cdot x+f_{12}''\cdot 2y\right)+2x\left(f_{21}''\cdot x+f_{22}''\cdot 2y\right)-\frac{1}{x^2\varphi'}-\frac{y}{x^2}\varphi''\cdot\frac{1}{x}$$

$$=f_1'+xyf_{11}''+2\left(x^2+y^2\right)f_{12}''+4xyf_{22}''-\frac{1}{x^2\varphi'}-\frac{y}{x^3}\varphi''$$

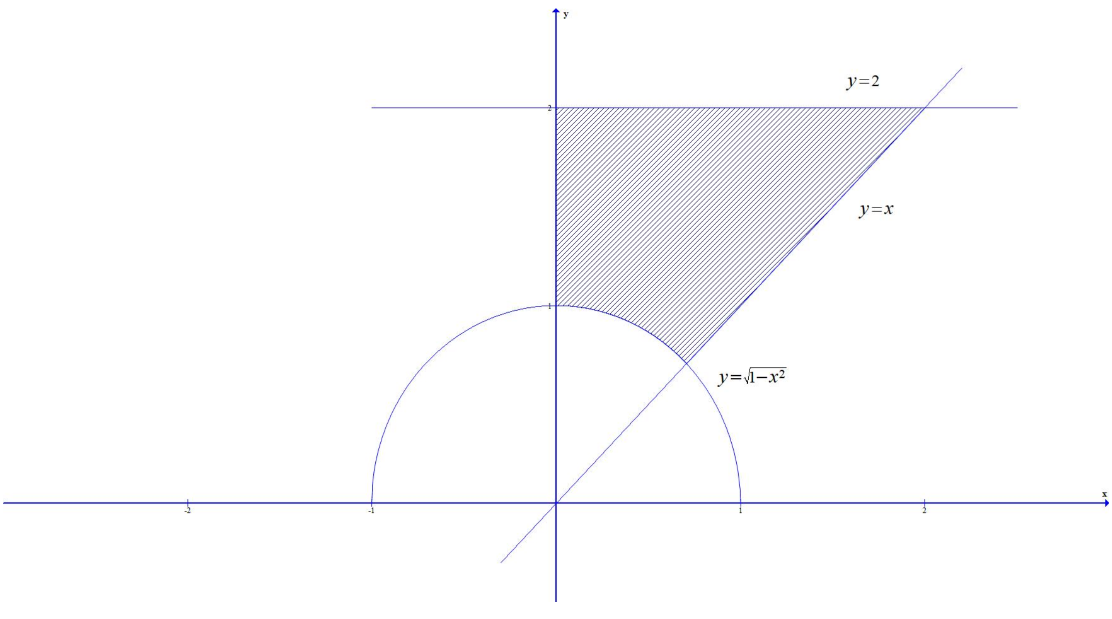
20. 解： $\iint\limits_{D}xydxdy=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_{1}^{\frac{2}{\sin\theta}}r\cos\theta r\sin\theta rdr$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\cdot\cos\theta\sin\theta\cdot\frac{1}{4}r^4\Big|_{1}^{\frac{2}{\sin\theta}}$$

$$=4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\theta\sin\theta\cdot\frac{1}{\sin^4\theta}d\theta-\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\theta\sin\theta d\theta$$

$$=4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos\theta}{\sin^3\theta}d\theta-\frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin\theta d\sin\theta$$

$$=-\frac{2}{\sin^2\theta}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}-\frac{1}{8}\sin^2\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}=\frac{31}{16}$$



21. 解：对应齐次方程的特征方程 $r^2+2r+3=0$，特征根 $r_{1,2}=-1\pm\sqrt{2}i$

通解 $\therefore \overline{y}=e^{-x}\left(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x\right)$

设原方程一个特解 $y^*=Ae^{2x}$， $y^{*}{'}=2Ae^{2x}$， $y^{*}{''}=4Ae^{2x}$

将 y^* 代入原方程，得：$11Ae^{2x}=e^{2x}$， $A=\dfrac{1}{11}$， $\therefore y^*=\dfrac{1}{11}e^{2x}$

原方程通解 $y=\overline{y}+y^*=e^{-x}\left(c_1\cos\sqrt{2}x+c_2\sin\sqrt{2}x\right)+\dfrac{1}{11}e^{2x}$

22. 解：$D=\begin{vmatrix}0&1&1&2\\1&0&3&4\\-1&2&4&5\\1&2&3&4\end{vmatrix}\overset{\substack{r_3+r_2\\r_4-r_2}}{=}\begin{vmatrix}0&1&1&2\\1&0&3&4\\0&2&7&9\\0&2&0&0\end{vmatrix}\overset{\text{按第一列展开}}{=}-\begin{vmatrix}1&1&2\\2&7&9\\2&0&0\end{vmatrix}$

$\overset{\text{按第三行展开}}{=}-2\begin{vmatrix}1&2\\7&9\end{vmatrix}=10$

23. 证明：令 $f(t)=\ln t$，则 $f(t)$ 在 $[2x,1+2x]$ 上连续，在 $(2x,1+2x)$ 内可导

由拉格朗日定理可知，至少存在一点 $\xi\in(2x,1+2x)$

使 $\dfrac{f(1+2x)-f(2x)}{1+2x-2x}=f'(\xi)$，即 $\ln(1+2x)-\ln 2x=\dfrac{1}{\xi}$

$\because 2x<\xi<1+2x$，且 $x>0$， $\therefore \dfrac{1}{1+2x}<\dfrac{1}{\xi}<\dfrac{1}{2x}$

$\therefore \dfrac{1}{1+2x}<\ln(1+2x)-\ln 2x<\dfrac{1}{2x}$，即 $\dfrac{1}{1+2x}<\ln\dfrac{1+2x}{2x}<\dfrac{1}{2x}$

24. 解： $f(0)=b$

$f(0-0)=\lim\limits_{x\to0^-}f(x)=\lim\limits_{x\to0^-}\dfrac{g(x)-\cos x}{x}=\lim\limits_{x\to0^-}\left[g'(x)+\sin x\right]=g'(0)$

$f(0+0)=\lim\limits_{x\to0^+}f(x)=\lim\limits_{x\to0^+}(ax+b)=b$

要使 $f(x)$ 在 $x=0$ 处连续，则应有 $f(0-0)=f(0+0)=f(0)$

$\therefore b=g'(0)$

$f_-'(0)=\lim\limits_{x\to0^-}\dfrac{f(x)-f(0)}{x}=\lim\limits_{x\to0^-}\dfrac{\dfrac{g(x)-\cos x}{x}-g'(0)}{x}$

$$=\lim_{x\to0^-}\frac{g(x)-\cos x-xg'(0)}{x^2}=\lim_{x\to0^-}\frac{g'(x)+\sin x-g'(0)}{2x}$$

$$=\lim_{x\to0-}\left[\frac{g'(x)-g'(0)}{x}\cdot\frac{1}{2}+\frac{\sin x}{2x}\right]=\frac{1}{2}g''(0)+\frac{1}{2}$$

$$f_+'(0)=\lim_{x\to0^+}\frac{f(x)-f(0)}{x}=\lim_{x\to0^+}\frac{ax}{x}=a$$

要使 $f(x)$ 在 $x=0$ 处可导，则应有 $f_-'(0)=f_+'(0)$，$\therefore a=\frac{1}{2}g''(0)+\frac{1}{2}$

25. 解：$\overline{A}=\begin{pmatrix}1&-1&1&-1&0\\1&-1&2&-3&1\\1&-1&3&-5&2\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-r_1}\begin{pmatrix}1&-1&1&-1&0\\0&0&1&-2&1\\0&0&2&-4&2\end{pmatrix}$

$\xrightarrow[r_3-2r_2]{r_1-r_2}\begin{pmatrix}1&-1&0&1&-1\\0&0&1&-2&1\\0&0&0&0&0\end{pmatrix}$

x_1,x_3 为约束变量，x_2,x_4 为自由变量

一般解 $\begin{cases}x_1=-1+x_2-x_4\\x_3=1+2x_4\end{cases}$，取 $\begin{cases}x_2=0\\x_4=0\end{cases}$ 得方程组得一个特解 $\eta^*=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}$

导出组一般解 $\begin{cases}x_1=x_2-x_4\\x_3=2x_4\end{cases}$，令 $\begin{cases}x_2=1\\x_4=0\end{cases}$，得 $\xi_1=\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$

令 $\begin{cases}x_2=0\\x_4=1\end{cases}$，得 $\xi_2=\begin{pmatrix}-1\\0\\2\\1\end{pmatrix}$

原方程组通解为 $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}+k_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix}+k_2\begin{pmatrix}-1\\0\\2\\1\end{pmatrix}$ (其中 k_1,k_2 为任意常数)