HW4 STAT 312

1

Suppose that the probability density function of a random variable X is $f(x) = e^{-(x-4)}, x > 4$.

A

Find the following probabilities:

1

$P(X > 1)$

> $P(X > 1) = 1$ because f(x) is domain is $[4, \infty)$ so $X > 1$ is guaranteed

2

$P(2 < X < 5)$

> $P(2 < X < 5) = 1 - P(X \geq 5) = 1 - \int_5^\infty e^{-(x-4)}dx = 1 - e^{-(5-4)} = 1 - e^{-1} = 0.632$

3

$P(X > 5)$

> $P(X > 5) = P(X \geq 5) - P(5) = \int_5^\infty e^{-(x-4)}dx - 0 = e^{-(5-4)} = e^{-1} = 0.368$

4

Find x such that $P(X < x) = 0.9$

> $0.9 = \int_4^x e^{-(x-4)}dx \rightarrow 0.9 = -e^{-(x-4)} + e^{-(4-4)} \rightarrow ln(0.1) = -(x-4) \rightarrow x = 4 - ln(0.1) = 6.303$

B

Find $F(x)$

$$F(x) = \begin{cases} 0 & \text{for} \quad x \leq 4 \\ 1 - e^{-(x-4)} & \text{for} \quad 4 < x \end{cases}$$

C

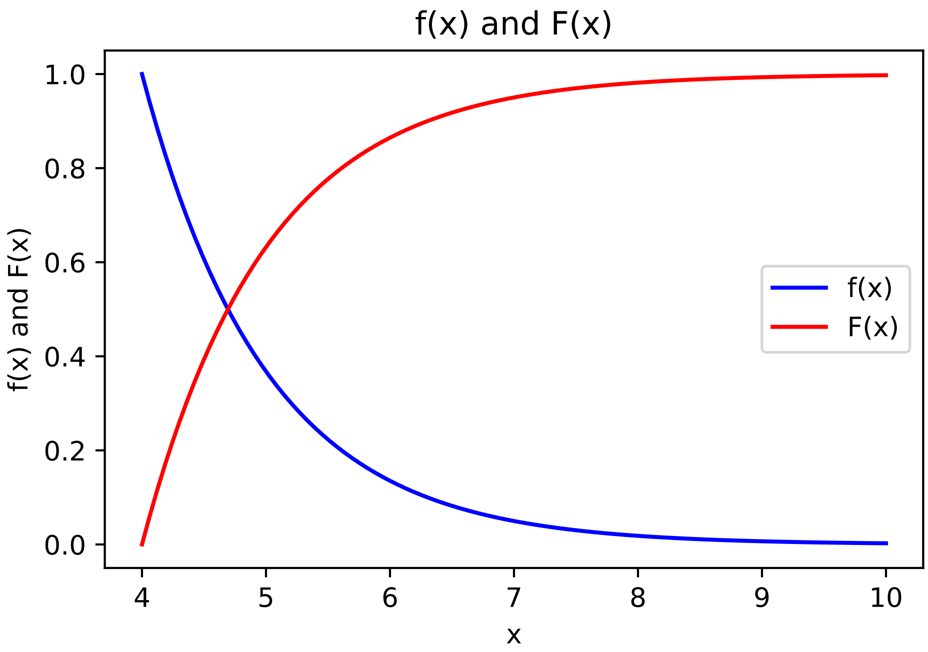
Graph $F(x)$ and $f(x)$

I used python to graph $F(x)$ and $f(x)$

```python
import matplotlib.pyplot as plt
import numpy as np

x = np.linspace(4, 10, 100)
f = np.exp(-(x-4))
F = 1 - np.exp(-(x-4))

plt.plot(x, f, color='blue')
plt.plot(x, F, color='red')
plt.xlabel('x')
plt.ylabel('f(x) and F(x)')
plt.title('f(x) and F(x)')
plt.legend(['f(x)', 'F(x)'])
plt.show()
```



d

D

Find the mean of X

$$E(X) = \int_4^\infty x \cdot e^{-(x-4)} dx = -(x+1)e^-(x-4)|_4^\infty = (4+1)e^0 = 5$$

2

Let X be a continuous random variable with cumulative distribution function (cdf)

$$F(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{6}x^2 + \frac{1}{6}x & \text{for} \quad 0 \leq x < 2 \\ 1 & \text{for} \quad 2 \leq x \end{cases}$$

A

1

$P(X > 1)$

$$P(X > 1) = 1 - P(X \leq 1) = 1 - F(1) = 1 - (\frac{1}{6} + \frac{1}{6}) = \frac{2}{3}$$

2

$P(-1 < x < 1.5)$

$$P(-1 < x < 1.5) = F(1.5) - F(-1) = (\frac{1}{6}\frac{9}{4} + \frac{1}{6}\frac{3}{2}) - 0 = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

3

$P(X > 3)$

$$P(X > 3) = 1 - P(X \leq 3) = 1 - F(3) = 1 - 1 = 0$$

B

Find the probability density function of X

$$f(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{3}x + \frac{1}{6} & \text{for} \quad 0 \leq x < 2 \\ 0 & \text{for} \quad 2 \geq x \end{cases}$$

C

Find the mean, variance and standard deviation of X

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_0^2 x(\frac{1}{3}x + \frac{1}{6})dx = \frac{1}{9}x^3 + \frac{1}{12}x^2|_0^2 = \frac{8}{9} + \frac{1}{3} = \frac{11}{9}$$

$$V(X) = \int_{-\infty}^{\infty}(x - E(X))^2 \cdot f(x)dx = \int_0^2 (x - \frac{11}{9})^2(\frac{1}{3}x + \frac{1}{6})dx = \int_0^2 \frac{1}{3}x^3 - \frac{35}{54}x^2 + \frac{22}{243}x + \frac{121}{486}dx = \frac{x^4}{12} - \frac{35}{162}x^3 + \frac{11}{243}x^2 - \frac{121}{486}x = \frac{23}{81}$$

$$\sigma = \sqrt{(V(X))} = \sqrt{\frac{23}{81}} = \frac{\sqrt{23}}{9}$$