HW4

1

Proof by contradiction. Assume L_{343} is decidable. Then $\exists M_{343}$ which decides L_{343}.

Create a machine M which takes a string y as input

M on input y

- Check if y is "CSDS 343 is fun" or "MATH 343 is fun"
- If y is neither string we reject
- Run M_{343} on M
- If M_{343} accepts, M rejects
- If M_{343} rejects, M accepts

Proof

Now suppose we run M on input "CSDS 343 is fun" or "MATH 343 is fun"

If M_{343} accepts M then M must accept "CSDS 343 is fun". However, M does the opposite of M_{343} thus M would reject "CSDS 343 is fun". This creates a contradiction.

If M_{343} rejects M then M must reject "CSDS 343 is fun". However, M does the opposite of M_{343} thus M would accept "CSDS 343 is fun". This creates a contradiction.

Thus we have a contradiction and we know M is a makeable machine as long as M_{343} is decidable. Therefore M_{343} is undecidable.

2

Proof by contradiction. So assume $DIFF\_BY\_1$ is decidable. Then $\exists M_{DIFF}$ which decides $DIFF\_BY\_1$

Create a TM M which takes a TM $< m >$ as input

M on input $< m >$

- Run M_{DIFF} on $< M, m >$
- If M_{DIFF} accepts, M rejects
- If M_{DIFF} rejects, M accepts

Create a TM M' which takes y as input

M' on input string y

- If $y = M'$ reject

- Else run M on y
  - If M accepts, M' accepts
  - If M rejects, M' rejects

Proof

Now suppose we run M on M', which will then run M_{DIFF} on $< M, M' >$

If M_{DIFF} accepts $< M, M' >$, then M will reject on M'. However this creates a contradiction because M_{DIFF} says $< M, M' >$ are different by 1, however M and M' both accept / reject on the same input (they are not different by 1) because M' runs M and rejects on input M'. Thus they are not different by 1 and M_{DIFF} should reject.

If M_{DIFF} rejects $< M, M' >$, then M will accept on M'. However this creates a contradiction because M_{DIFF} says $< M, M' >$ are not different by 1, however M and M' both accept / reject on the same input except for input M' where M' rejects and M accepts. Thus they are different by one and M_{DIFF} should accept.

Thus we have a contradiction and we know both M and M' are makeable machines as long as M_{DIFF} is decidable. Therefore M_{DIFF} is undecidable.

3

A Prove $A_{TM} \leq_M L_{add}$

In order to prove this, we must prove there exists a function f such that $x \in A_{TM} \leftrightarrow f(x) \in L_{add}$

Remember $x =< M, w >$ and $f(x) =< M' >$

M' on input y:

- Run M on w
- If M accepts:
  - Run M_{add} on y // $< M_{add} >\in L_{add}$ we made in class
  - If M_{add} accepts: M' accepts
  - If M_{add} rejects: M' rejects
- If M rejects:
  - M' rejects

Proof

If $x \in A_{TM}$ show $f(x) \in L_{add}$

- If $x \in A_{TM}$ then M accepts w
  - Since M accepts w, M' will run M_{add} on y and accept / reject accordingly. So M' is the same as M_{add} when M accepts w. Since $< M_{add} >\in L_{add}$ and M' mimics M_{add} when M accepts w, $< M' >\in L_{add}$.

If $x \notin A_{TM}$ show $f(x) \notin L_{add}$

- If $x \notin A_{TM}$, M will either reject on input w or run forever on input w.
- Suppose M rejects w
  - Since M rejects w, M' will always reject. Thus M' will reject valid strings of $L(M_{add})$ so $< M' >\notin L_{add}$.
- Suppose M runs forever on w
  - Since M runs forever on w, M' will always run forever. Thus M' will run forever on valid string of $L(M_{add})$ so $< M' >\notin L_{add}$.

If $f(x) \in L_{add}$ show $x \in A_{TM}$

- If $f(x) \in L_{add}$ then $L(M') = a^n b^m c^{n+m}, n, m \geq 0$. In order for $L(M')$ to equal $a^n b^m c^{n+m}, n, m \geq 0$, then M' must always run M_{add}. M' only runs M_{add} when M accepts w. If M accepts w then $< M, w >= x \in A_{TM}$

If $f(x) \notin L_{add}$ show $x \notin A_{TM}$

- If $f(x) \notin L_{add}$ then M' either runs forever on y or rejects y when $y \in L(M_{add})$
- Suppose M' rejects a valid y
  - If M' rejected a valid y then it must be because M rejected w because if M accepted w, M' would run M_{add} which would accept y so M' would accept on y. Thus if M' rejects a valid y it is because M rejected w. If M rejects w then $< M, w >= x \notin A_{TM}$
- Suppose M' ran forever on a valid y
  - If M' ran forever on a valid y then it must be because M ran forever on w because if M had accepted w then M' would run M_{add} which would recognize y and accept. Thus if M' ran forever on a valid y it is because M ran forever on w. If M ran forever on w then $< M, w >= x \notin A_{TM}$

Thus we have proved $x \in A_{TM} \leftrightarrow f(x) \in L_{add}$ therefore $A_{TM} \leq_m L_{add}$

B Prove $\bar{A_{TM}} \leq_M L_{add}$

In order to prove this, we must prove there exists a function f such that $x \in \bar{A_{TM}} \leftrightarrow f(x) \in L_{add}$

Remember $x =< M, w >$ and $f(x) =< M' >$

f on input x:

If x is not of the form $< M, w >$ then $f(x) = M_{add}$

Else create M' which takes a string y

M' on input y:

- Run M on w for $|y|$ steps
- If M does not accept with $|y|$ steps:
  - Run M_{add} on y for count steps // $< M_{add} >\in L_{add}$ we made in class
  - If M_{add} accepts: M' accepts
  - If M_{add} rejects: M' rejects
- Else:
  - Reject

Proof

If $x \in \bar{A_{TM}}$ show $f(x) \in L_{add}$

- If $x \in \bar{A_{TM}}$ then M rejects w
  - Since $x \in \bar{A_{TM}}$ then M will not accept w in finite steps (will either reject or run forever, either will result in M not accepting in finite steps). Thus we run a machine $< M_{add} >\in L_{add}$ and M' will accept / reject. Thus when $x \in \bar{A_{TM}}, L(M') = L(M_{add})$ so $< M' >= f(x) \in L_{add}$.
  - Alternatively, $x \in \bar{A_{TM}}$ if x is not of the form $< M, w >$ in which case

If $x \notin \bar{A_{TM}}$ show $f(x) \notin L_{add}$

- If $x \notin A_{TM}$, M will accept w in finite steps s. $\forall y$ where $|y| > s$, M' will detect these strings and reject. This means M' will reject infinitely many valid strings y where $|y| > s$ so $M' \notin L_{add}$.

If $f(x) \in L_{add}$ show $x \in A_{TM}$

- If $f(x) \in L_{add}$ then $L(M') = a^n b^m c^{n+m}, n, m \geq 0$. In order for $L(M')$ to equal $a^n b^m c^{n+m}, n, m \geq 0$, then M' must always run M_{add}. M' only runs M_{add} when M accepts w. If M accepts w then $< M, w >= x \in A_{TM}$

If $f(x) \notin L_{add}$ show $x \notin A_{TM}$

- If $f(x) \notin L_{add}$ then M' either runs forever on y or rejects y when $y \in L(M_{add})$
- Suppose M' rejects a valid y
  - If M' rejected a valid y then it must be because M rejected w because if M accepted w, M' would run M_{add} which would accept y so M' would accept on y. Thus if M' rejects a valid y it is because M rejected w. If M rejects w then $< M, w >= x \notin A_{TM}$
- Suppose M' ran forever on a valid y
  - If M' ran forever on a valid y then it must be because M ran forever on w because if M had accepted w then M' would run M_{add} which would recognize y and accept. Thus if M' ran forever on a valid y it is because M ran forever on w. If M ran forever on w then $< M, w >= x \notin A_{TM}$

Thus we have proved $x \in A_{TM} \leftrightarrow f(x) \in L_{add}$ therefore $A_{TM} \leq_m L_{add}$