Data Structures and Algorithms

Chapter 9

Priority Queues

- Each element in a queue is associated with a *key*.
- When an element is removed, an element with a *minimal* (or *maximal*) *key* is removed.
- Usually keys are numbers.
- Objects can be used as keys as far as there is a total ordering among those objects.

Priority Queues
ADT

- insert(*k*, *v*): Create an entry with key *k* and value *v* in the priority queue.
- min( ): Returns (but does not remove) an entry (*k*, *v*) with the minimum key. Returns null if the priority queue is empty.
- removeMin( ): Removes and returns an entry (*k*, *v*) with the minimum key. Returns null if the priority queue is empty.
- size( ): Returns the number of entries in the priority queue.
- isEmpty( ): Returns true if the priority queue is empty. Returns false, otherwise.

Priority Queues
ADT

- Illustration

Method	Return Value	Priority Queue Contents
insert(17, A)		{(17, A)}
insert(4, P)		{(4, P) , (17, A)}
insert(15, X)		{(4, P) , (15, X), (17, A)}
size( )	3	{(4, P) , (15, X), (17, A)}
isEmpty( )	false	{(4, P) , (15, X), (17, A)}
min( )	(4, P)	{(4, P) , (15, X), (17, A)}
removeMin( )	(4, P)	{(15, X), (17, A)}
removeMin( )	(15, X)	{(17, A)}
removeMin( )	(17, A)	{ }
removeMin( )	null	{ }
size( )	0	{ }
isEmpty( )	true	{ }

Priority Queues
Implementation

- An element in a priority queue has *key* and *value*.
- *Entry* interface is used to store a key-value pair.

```
public interface Entry<K,V> {
  K getKey();
  V getValue();
}
```

Priority Queues
Implementation

- *PriorityQueue* interface

```
public interface PriorityQueue<K,V> {
    int size();
    boolean isEmpty();
    Entry<K,V> insert(K key, V value) throws
                                    IllegalArgumentException;
    Entry<K,V> min();
    Entry<K,V> removeMin();
}
```

Priority Queues
Implementation

- Keys must have *total ordering*.
- Total ordering: there is a linear ordering among all keys.
- Total ordering of a comparison rule, ρ, satisfies the following properties:
  - Comparability property: k_1 ρ k_2 or k_2 ρ k_1.
  - Antisymmetric property: If k_1 ρ k_2 and k_2 ρ k_1, then $k_1 = k_2$.
  - Transitive property: If k_1 ρ k_2 and k_2 ρ k_3, then k_1 ρ k_3.
- If keys have total ordering, *minimal key* is well defined
- key_{min} is a key such that: key_{min} ρ k, for all k
- Note: it will be easy to understand if you replace ρ with ≤ (or any other familiar relation)

Priority Queues
Implementation

- Two ways to compare objects in Java
  - *compareTo* and *compare*
- *compareTo* is defined in *java.util.Comparable* interface.
- A class must override and implement the *compareTo* method.
- *Ordering* defined in the *compareTo* method is called *natural ordering*.
- Usage: *a.compareTo*(*b*) returns
  - a negative number, if a < b
  - zero, if a = b
  - a positive number, if a > b
- Many Java classes implemented *Comparable* interface.

Priority Queues
Implementation

- *compare* is defined in *java.util.Comparator* interface.
- Use this to compare not by natural ordering
- Need to write a separate customized comparator
- Example: To compare strings by length (natural ordering is lexicograhic ordering).
- First, write a customized comparator method

```
public class StringLengthComparator implements Comparator<String> {
   public int compare(String a, String b){
               if (a.length() < b.length()) return -1;
               else if (a.length() == b.length()) return 0;
               else return 1;
   }
}
```

Priority Queues
Implementation

- Then, use it as follows:

```
public class ComparatorTest {
    public static void main(String[] args) {
            StringLengthComparator c = new StringLengthComparator();
            String s1 = "tiger";
            String s2 = "sugar";
            String s3 = "coffee";
            String s4 = "cat";
            System.out.println("Compare s1 and s2: " + c.compare(s1, s2)); // 0
            System.out.println("Compare s1 and s3: " + c.compare(s1, s3)); // -1
            System.out.println("Compare s1 and s4: " + c.compare(s1, s4)); // 1
    }
}
```

Priority Queues
AbstractPriorityQueue Base Class

- Provides common features for different concrete implementations.
- An entry in a queue is implemented as *PQEntry*:

```
protected static class PQEntry<K,V> implements Entry<K,V> {
    private K k;  // key
    private V v;  // value
    public PQEntry(K key, V value) {
        k = key;
        v = value;
    }
    public K getKey() { return k; }
    public V getValue() { return v; }
    protected void setKey(K key) { k = key; }
    protected void setValue(V value) { v = value; }
}
```

Priority Queues
Implementing Using a Heap

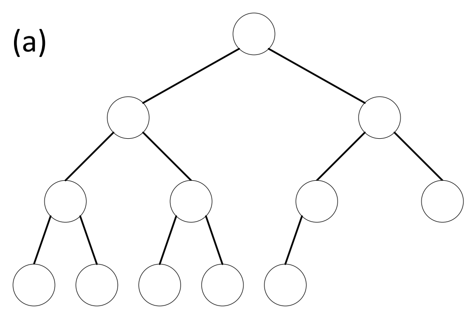
- Implementation with an unsorted list
- Implementation with a sorted list

- We will focus on implementation with *heap*.

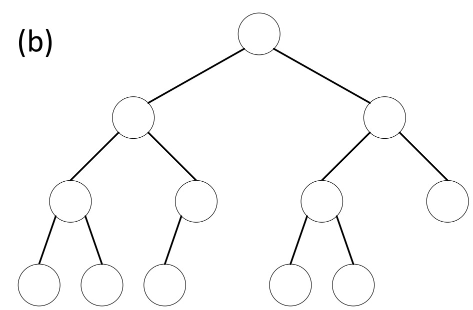
- *Heap* is a binary tree with the following properties:
  - *Heap-order property*: In a heap T, for every position p, except the root, the key stored at p is greater than or equal to the key stored at p's parent. (*minimum-oriented* heap)
  - *Complete binary tree property*: A heap is a complete binary tree.

Priority Queues
Implementing Using a Heap

- Complete binary tree
  - Levels 0, 1, . . ., h – 1 of T have the maximal number of nodes (in other words, level i has 2^i nodes, where $0 \leq i \leq h - 1$), and
  - Nodes at level h are in the leftmost possible positions at that level.



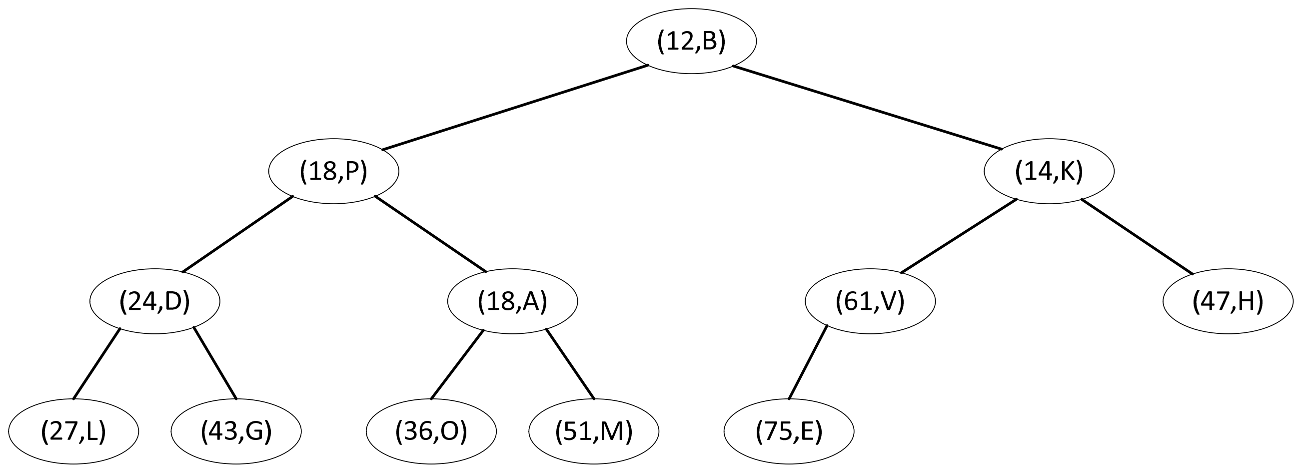
yes



no

Priority Queues
Implementing Using a Heap

- Priority queue implemented using a heap example:



- Height of a heap with *n* entries is $h = \lfloor \log n \rfloor$

Priority Queues

Implementing Using a Heap

- Adding an entry to a heap
  - Step 1: Add new entry at the “end” of the heap
  - Step 2: Reorganize the heap (because adding new entry may violate the heap-order property)

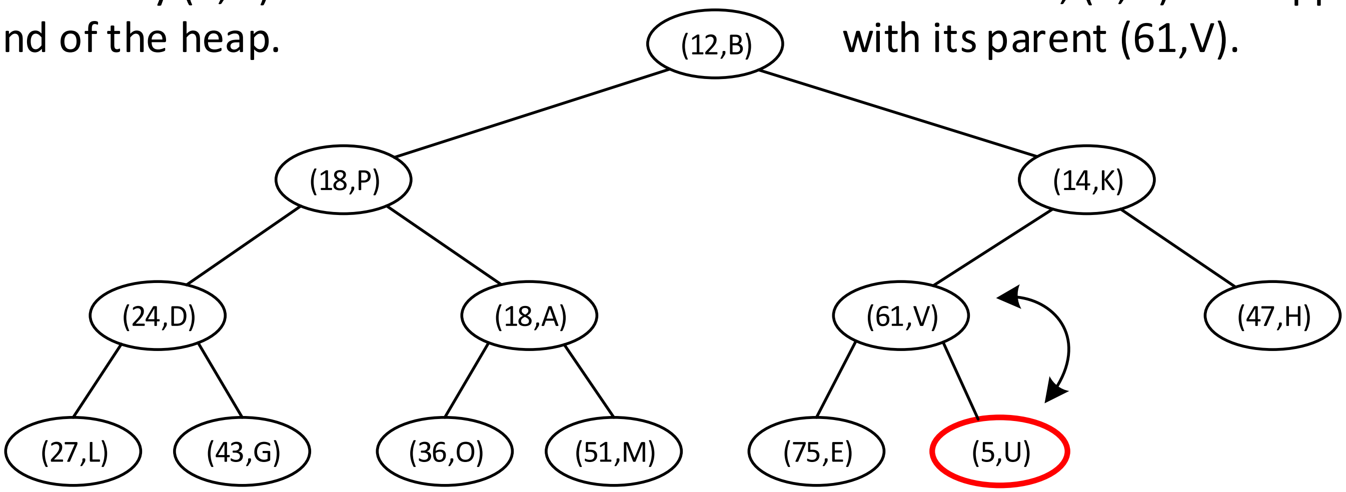
- Reorganization is done by *up-heap bubbling*.

Priority Queues
Implementing Using a Heap

- Illustration

New entry (5,U) is added to the end of the heap.

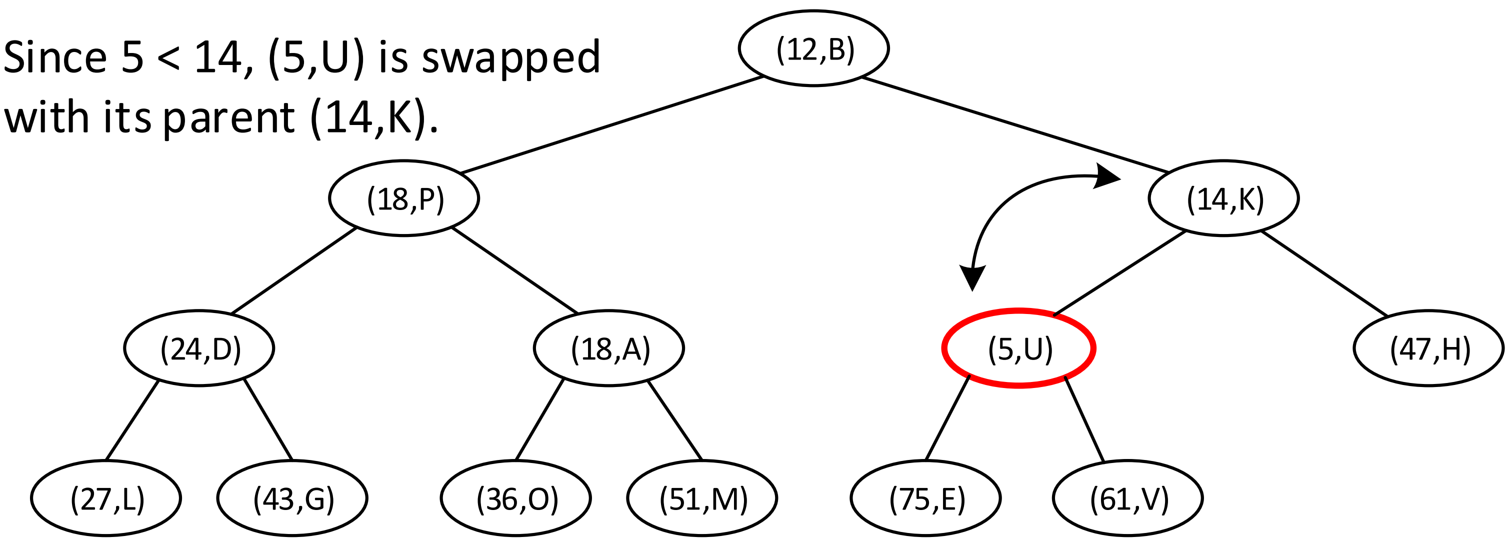
Since 5 < 61, (5,U) is swapped with its parent (61,V).



Priority Queues
Implementing Using a Heap

- Illustration

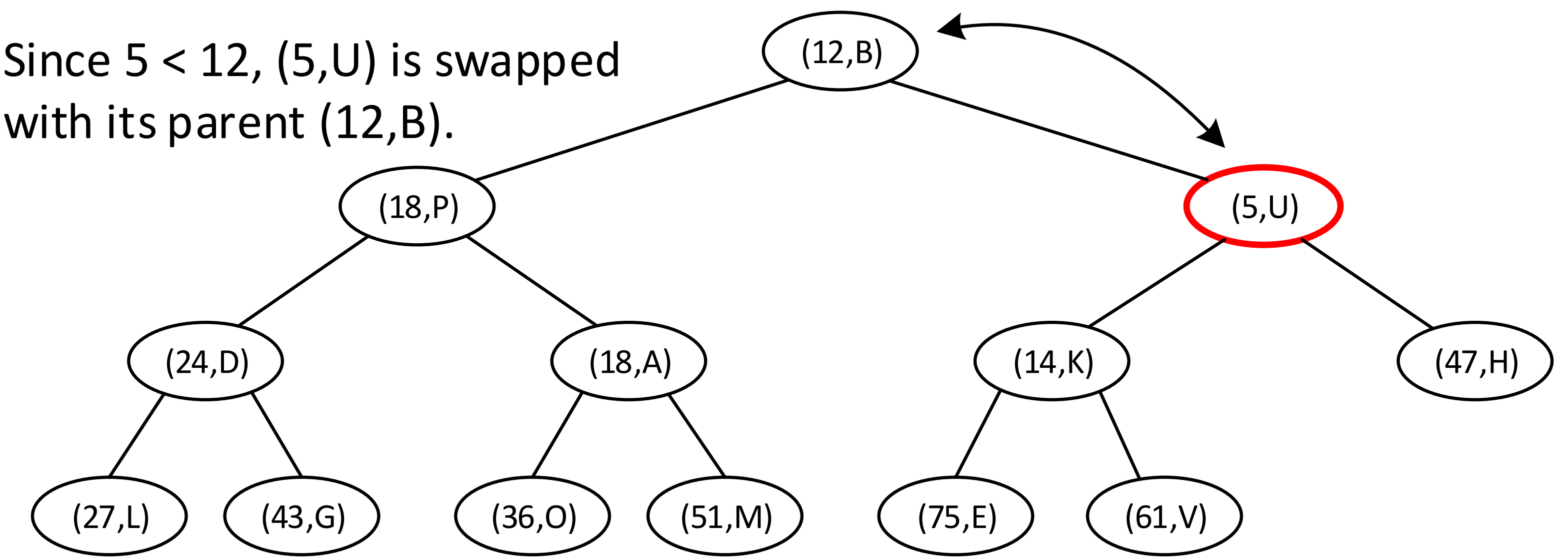
Since 5 < 14, (5,U) is swapped with its parent (14,K).



Priority Queues
Implementing Using a Heap

- Illustration

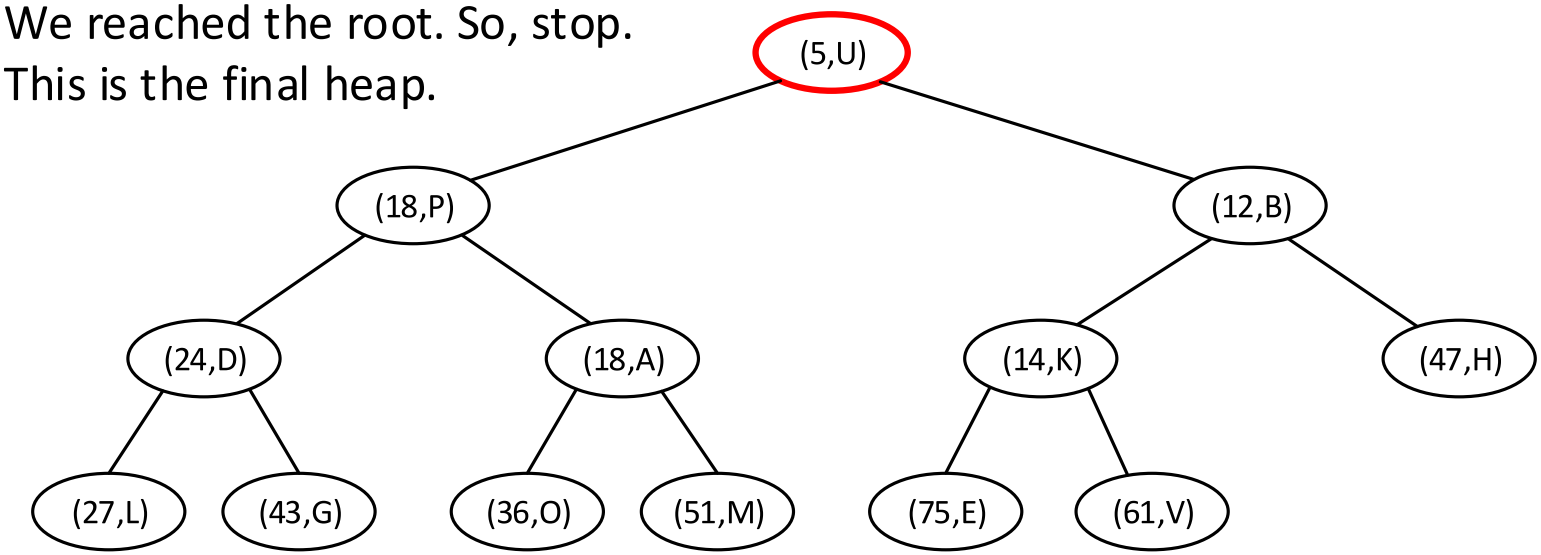
Since 5 < 12, (5,U) is swapped with its parent (12,B).



Priority Queues
Implementing Using a Heap

- Illustration

We reached the root. So, stop.
This is the final heap.



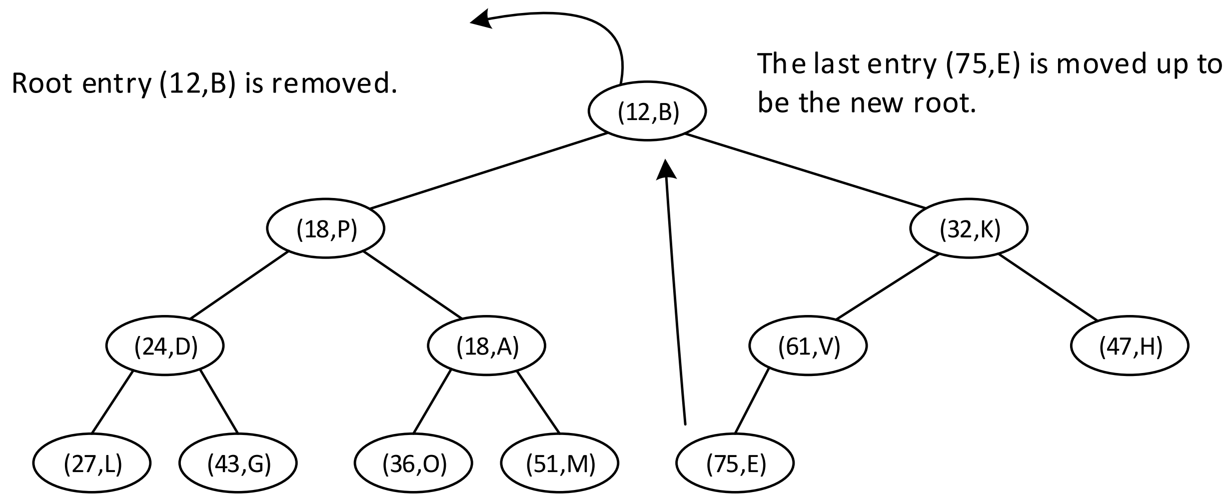
Priority Queues
Implementing Using a Heap

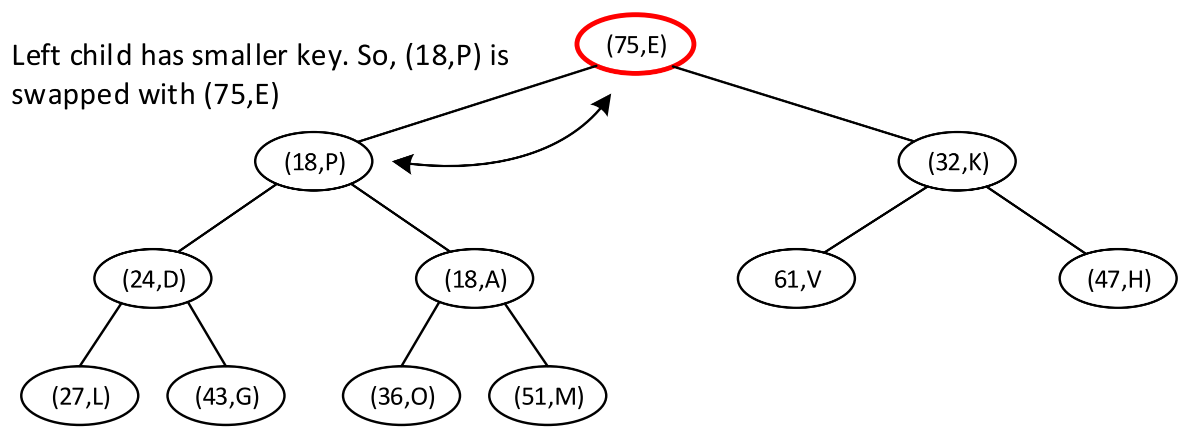
- Removing the entry with minimal key
  - Step1: Remove the root
  - Step 2: Last node is move up to the root and perform *down-heap bubbling*.

- Down-heap bubbling is opposite of up-heap bubbling.

Priority Queues
Implementing Using a Heap

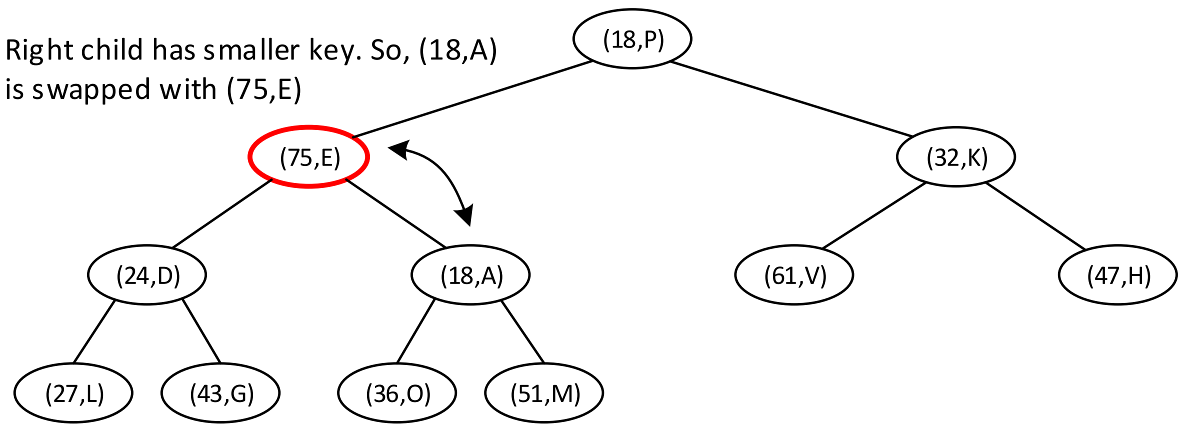
- Illustration

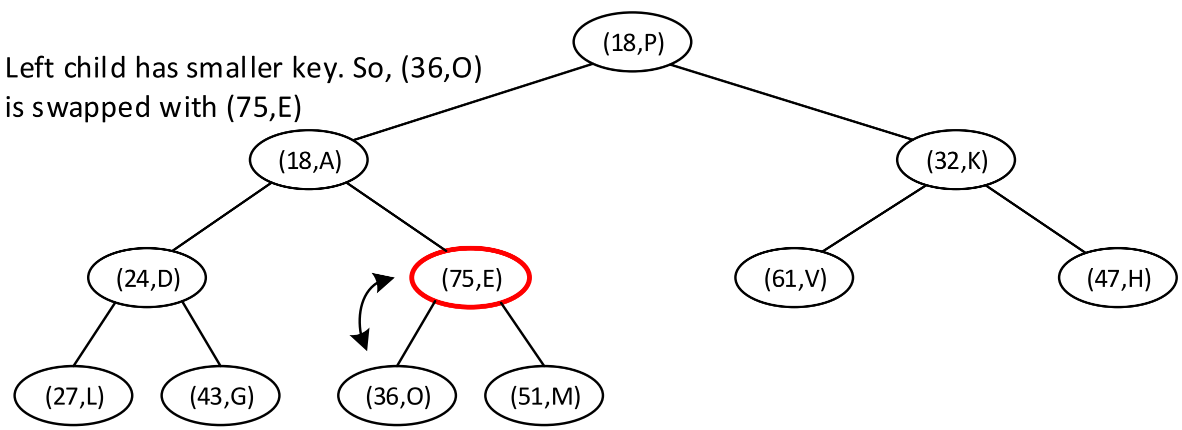




Priority Queues
Implementing Using a Heap

- Illustration

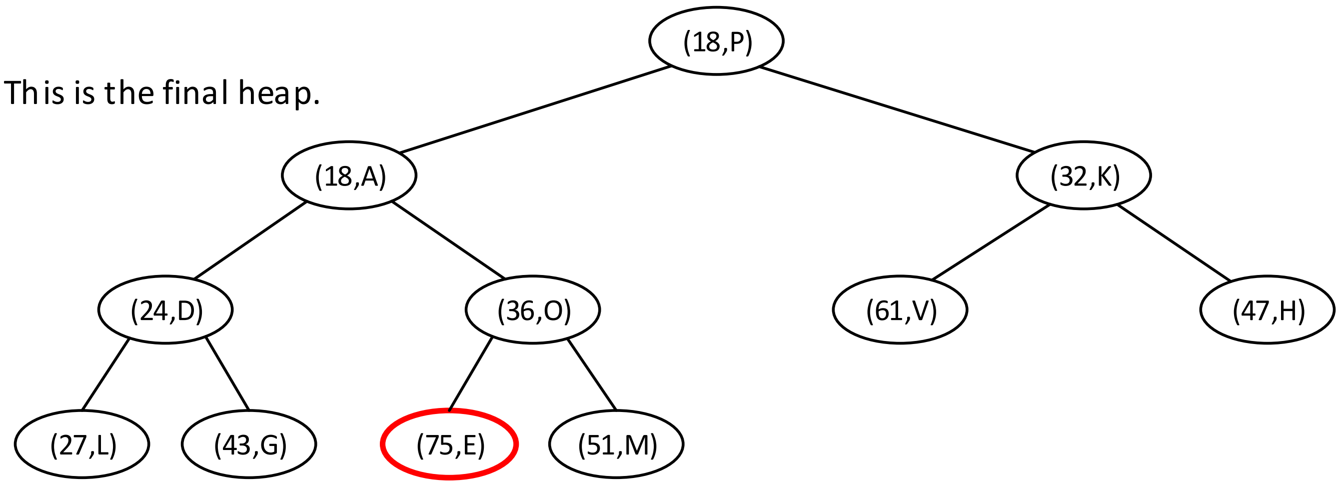




Priority Queues
Implementing Using a Heap

- Illustration

This is the final heap.



Priority Queues
Array-Based Heap

- The level number of a position *p*, *f*(*p*), is defined as follow:

  - If *p* is the root, *f*(*p*) = 0
  - If *p* is the left child of position *q*, *f*(*p*) = 2**f*(*q*) + 1
  - If *p* is the right child of position *q*, *f*(*p*) = 2**f*(*q*) + 2

- The level number is used as the index in an array where the entry with position *p* is stored.

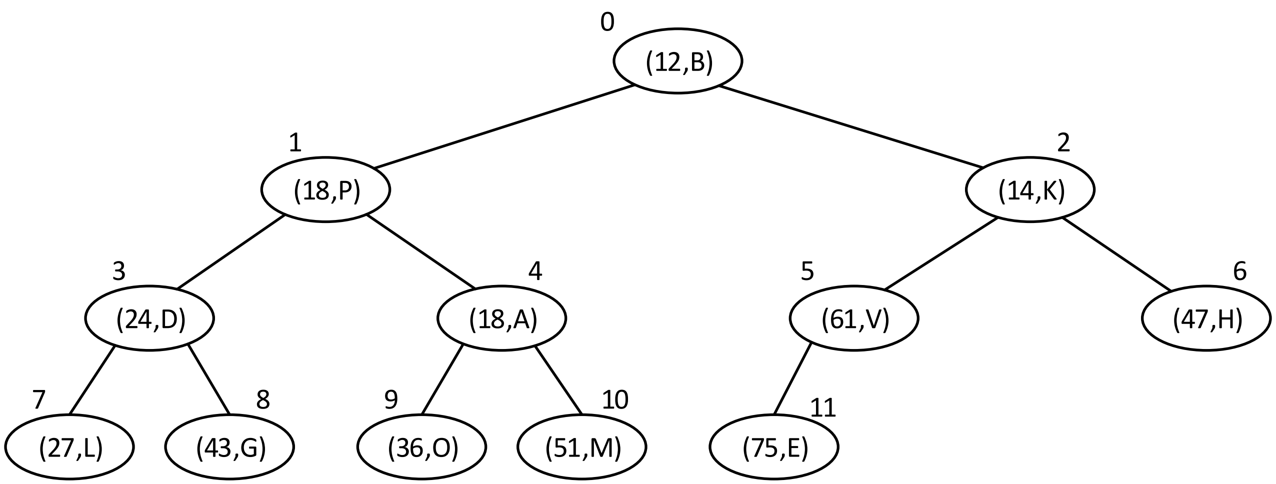
Priority Queues
Array-Based Heap

- Then, the entry at position *p* is stored in *A*[*f*(*p*)].

- Index of the root node is 0.
- Index of left child of *p* = 2**f*(*p*) + 1
- Index of right child of *p* = 2**f*(*p*) + 2
- Index of parent of *p* = $\lfloor (f(p)-1)/2 \rfloor$

Priority Queues
Array-Based Heap

- Example



(12,B)	(18,P)	(14,K)	(24,D)	(18,A)	(61,V)	(47,H)	(27,L)	(43,G)	(36,O)	(51,M)	(75,E)
0	1	2	3	4	5	6	7	8	9	10	11

Priority Queues
Array-Based Heap

- *HeapPriorityQueue* class implements a priority queue using a heap.
- A heap is implemented using *ArrayList*.
- Will briefly discuss *upheap*, *downheap*, *insert*, and *removeMin* methods.

- *HeapPriorityQueue.java* code

Priority Queues
Analysis of Heap-Based Priority Queue

- insertion:
  - *upheap* method takes $O(\log n)$
  - So, insertion takes $O(\log n)$
- removeMin:
  - *downheap* method takes $O(\log n)$
  - So, removeMin takes $O(\log n)$

Method	Running Time
size, isEmpty	$O(1)$
min	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$

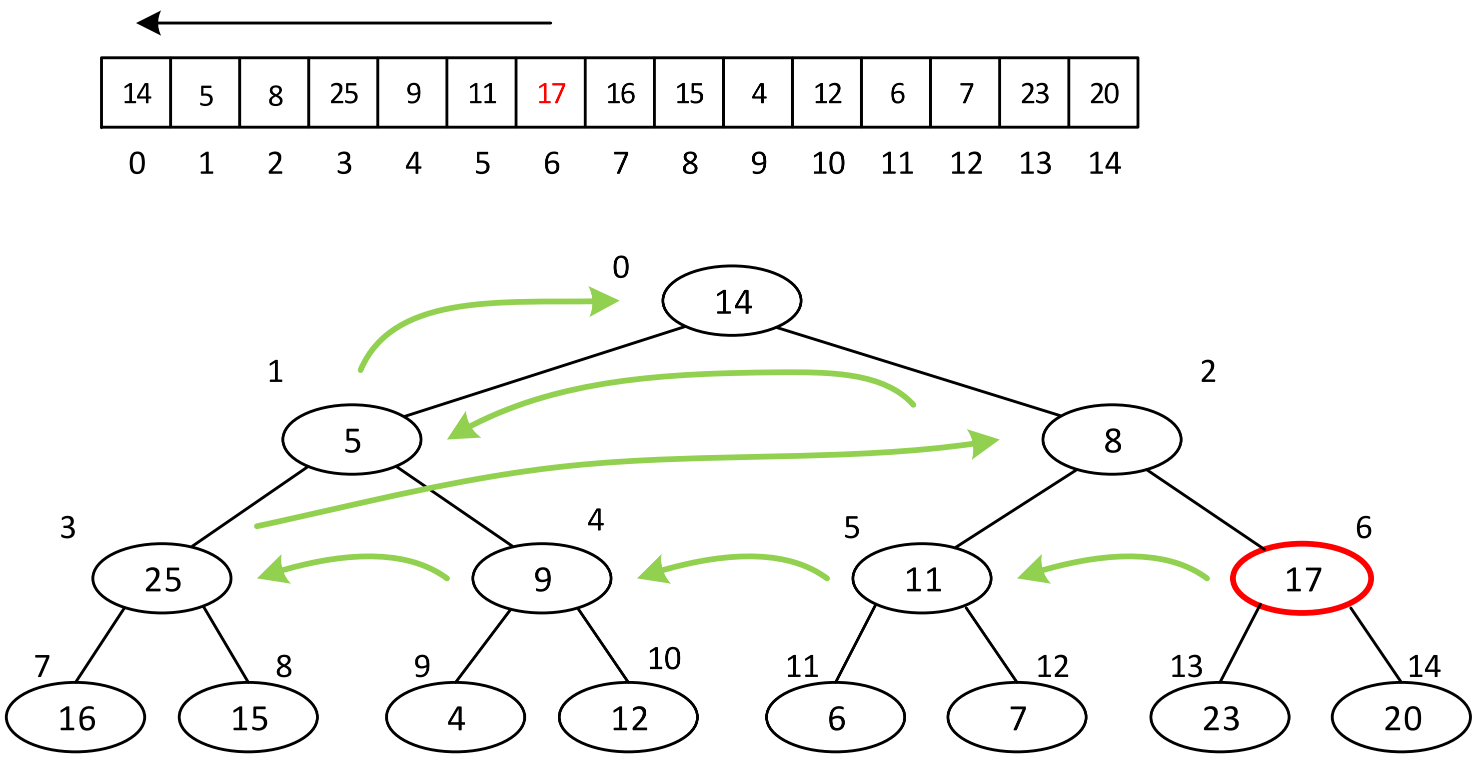
Priority Queues
Bottom-up Heap Construction

- Given n elements, we can build a heap with n successive insertions => takes $O(n \log n)$ time.

- O(n) time algorithm
  – Begin at the parent of the last node, move backward to the root.
  – At each node, perform down-heap bubbling.

Priority Queues
Bottom-up Heap Construction

- Illustration



Priority Queues

Bottom-up Heap Construction

- Java implementation

```
public HeapPriorityQueue(K[ ] keys, V[ ] values) {
  super();
  for (int j=0; j < Math.min(keys.length, values.length); j++)
      heap.add(new PQEntry<>(keys[j], values[j]));
  heapify();
}

protected void heapify() {
  int startIndex = parent(size()-1);    // start at PARENT of last entry
  for (int j=startIndex; j >= 0; j--)   // loop until processing the root
      downheap(j);
}
```

Priority Queues
Java's Priority Queue

- *java.util.PriorityQueue*
- An entry is a single element.
- Some operations in Java's *PriorityQueue*
  - add(E e): Inserts the specified element *e* to the priority queue.
  - isEmpty( ): Returns true if the priority queue contains no element.
  - peek( ): Retrieves, but does not remove, a minimal element from the priority queue.
  - remove( ): Removes a minimal element from the priority queue.
  - size( ): Returns the number of elements in the priority queue.

Priority Queues
Heap-Sort

- Uses array-based heap data structure.
- In-place sorting: no additional storage is used.
- Uses a *maximum-oriented* heap.
- *maximum-oriented* heap: In a heap *T*, for every position *p*, except the root, the key stored at *p* is *smaller* than or equal to the key stored at *p*'s parent.
- Sorting steps:
  1. Given *n* elements are inserted into a maximum-oriented heap.
  2. Repeat the following until only one node is left in the heap:

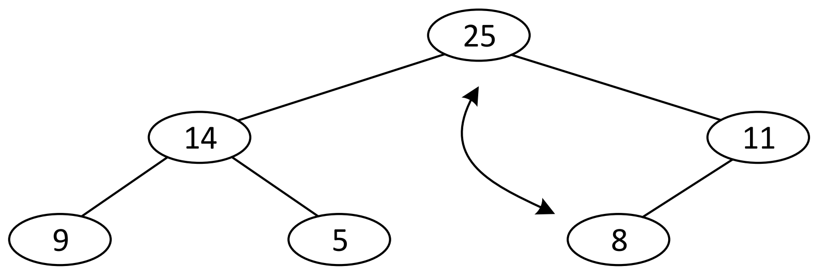
     Root is swapped with the last node, heap size is decremented, perform down-heap bubbling.

Priority Queues
Sorting with Priority Queue

- Illustration

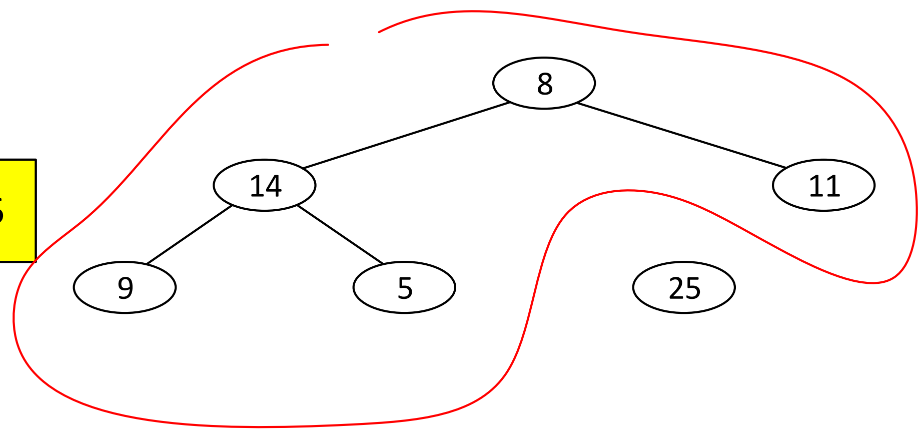
The maximum-oriented heap after the first step.
The root node and the last node is swapped.
Heap size is decremented.

25	14	11	9	5	8



Down-heap bubbling is applied on the root.

8	14	11	9	5	25

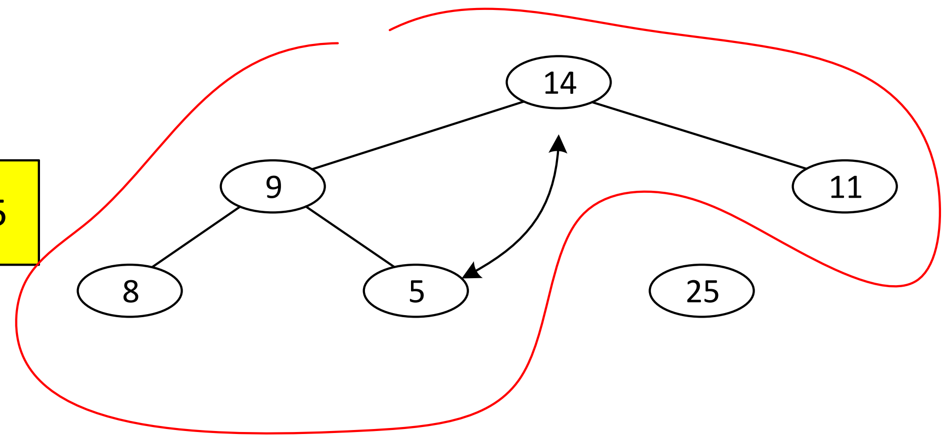


Priority Queues
Sorting with Priority Queue

- Illustration

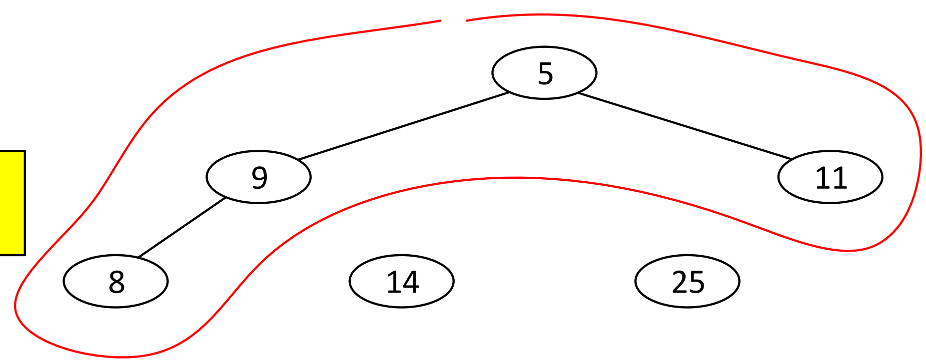
The root node is swapped with the last node. Heap size is decremented.

14	9	11	8	5	25



Down-heap bubbling is applied on the root.

5	9	11	8	14	25

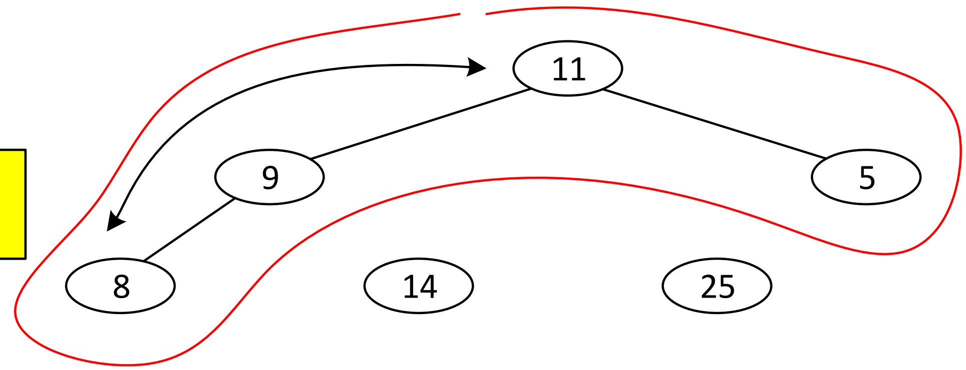


Priority Queues
Sorting with Priority Queue

- Illustration

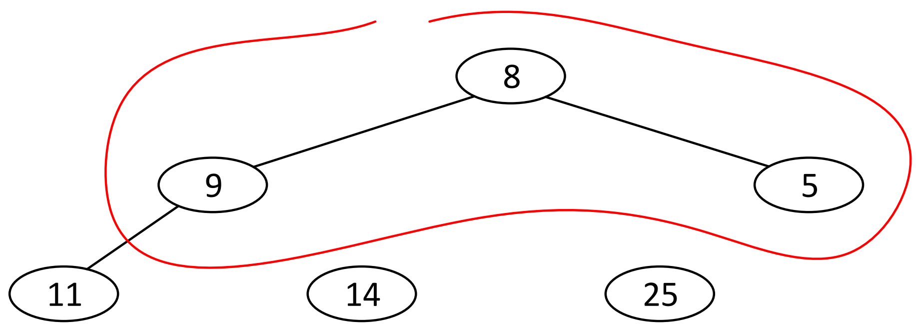
The root node is swapped with the last node. Heap size is decremented.

11	9	5	8	14	25



Down-heap bubbling is applied on the root.

8	9	5	11	14	25



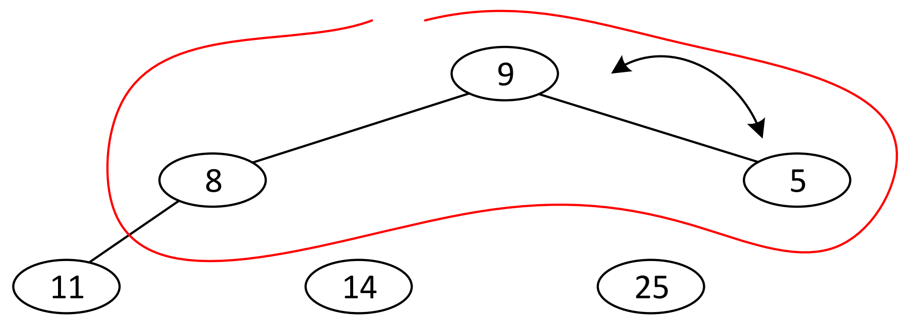
Priority Queues

Sorting with Priority Queue

- Illustration

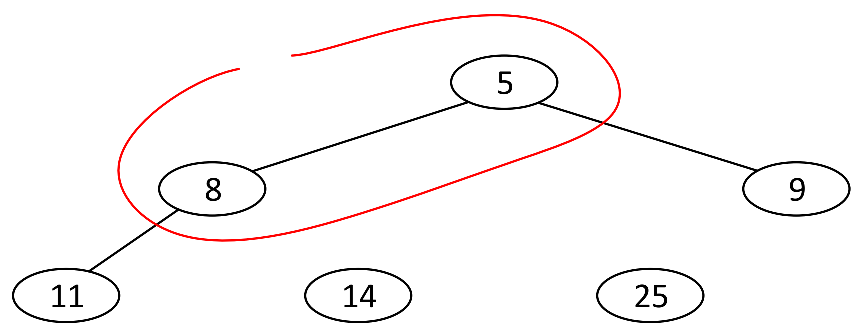
The root node is swapped with the last node. Heap size is decremented.

9	8	5	11	14	25



Down-heap bubbling is applied on the root.

5	8	9	11	14	25

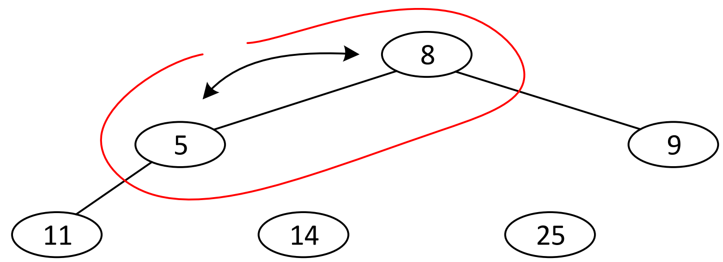


Priority Queues
Sorting with Priority Queue

- Illustration

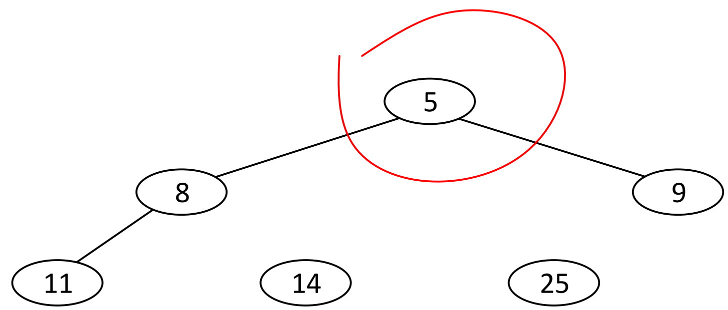
The root node is swapped with the last node. Heap size is decremented.

8	5	9	11	14	25



At this time the array is sorted.

5	8	9	11	14	25



Priority Queues
Adaptable Priority Queue

- Can remove arbitrary entry (not just the root).
- Can replace the key of an entry.
- Can replace the value of an entry.

References

- M.T. Goodrich, R. Tamassia, and M.H. Goldwasser, "Data Structures and Algorithms in Java," Sixth Edition, Wiley, 2014.