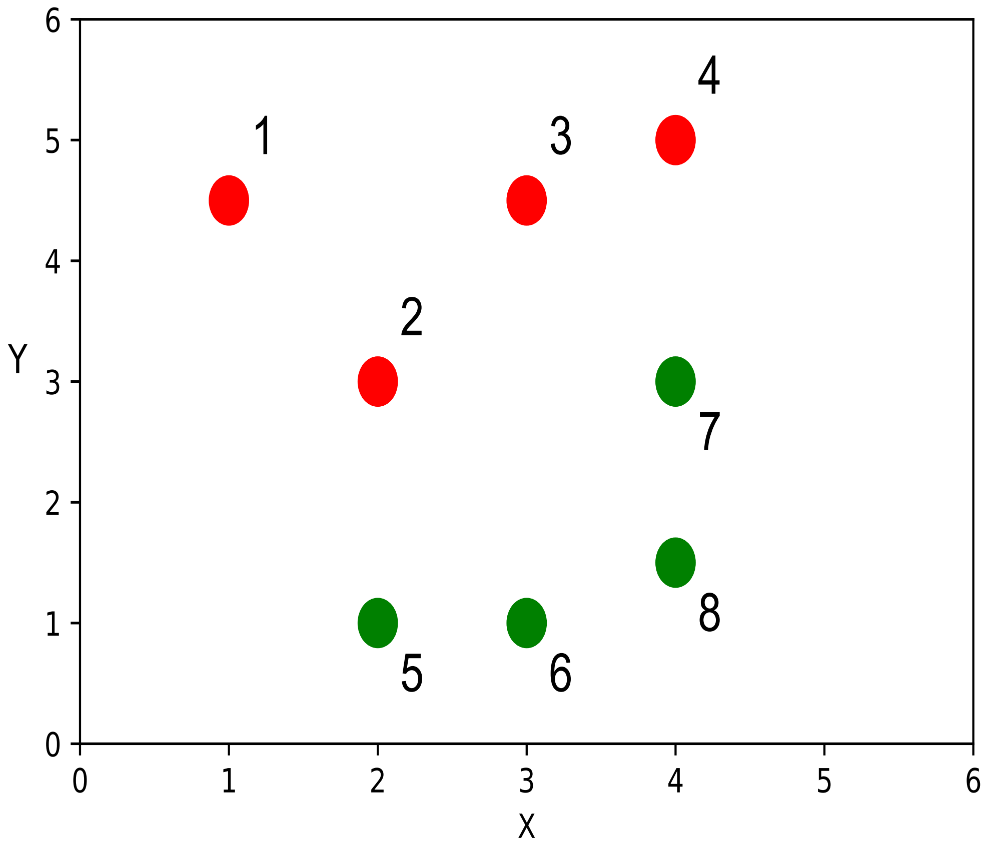
SUPPORT

VECTOR

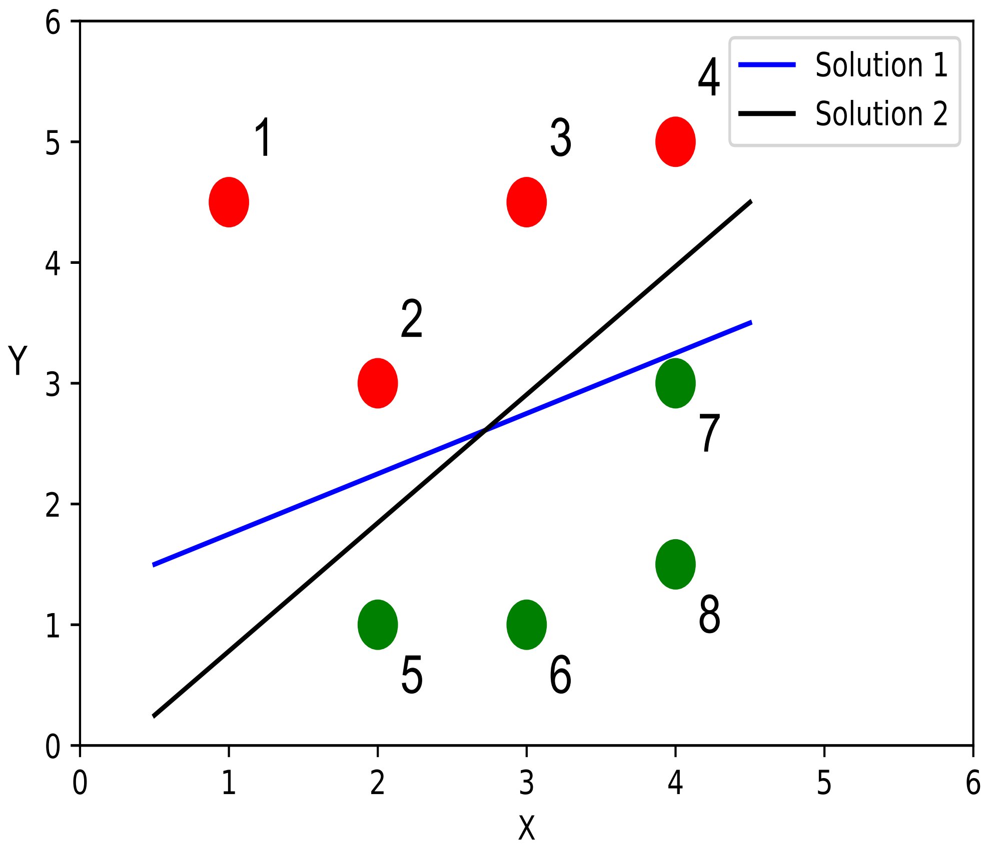
MACHINES

Overview



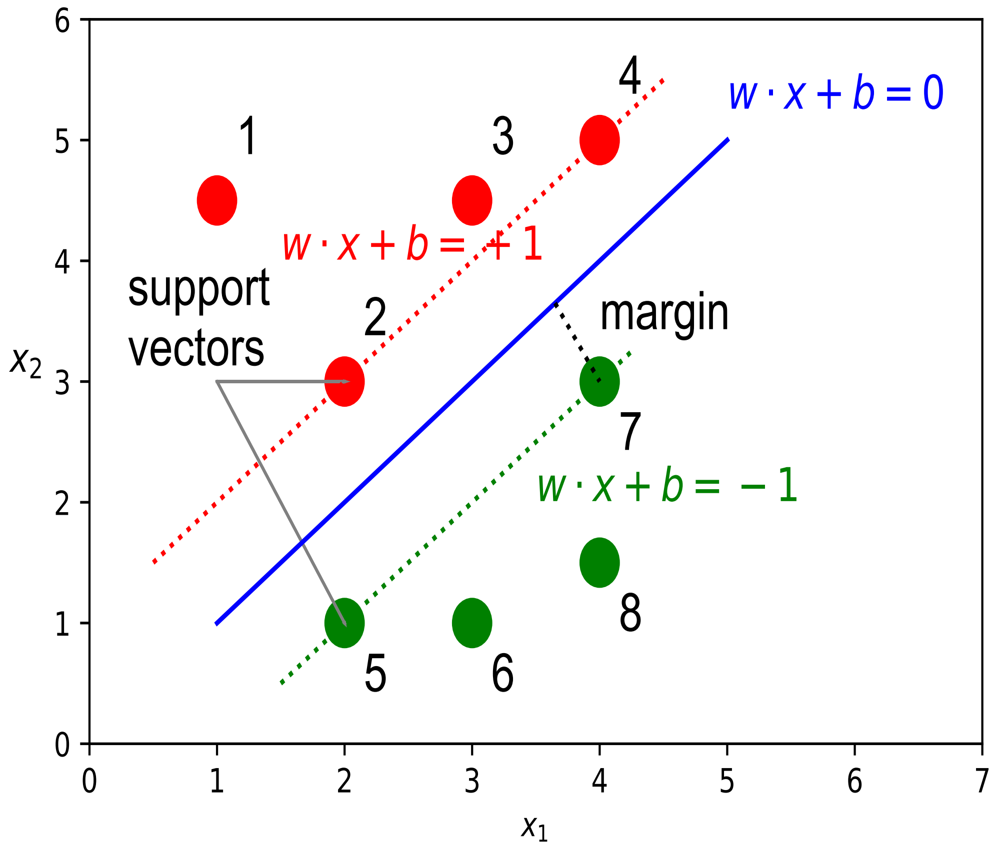
- supervised learning
- want to separate classes

How to Separate?



- many possibilities

SVM Intuition



- use "thickest" line
- maximize margins

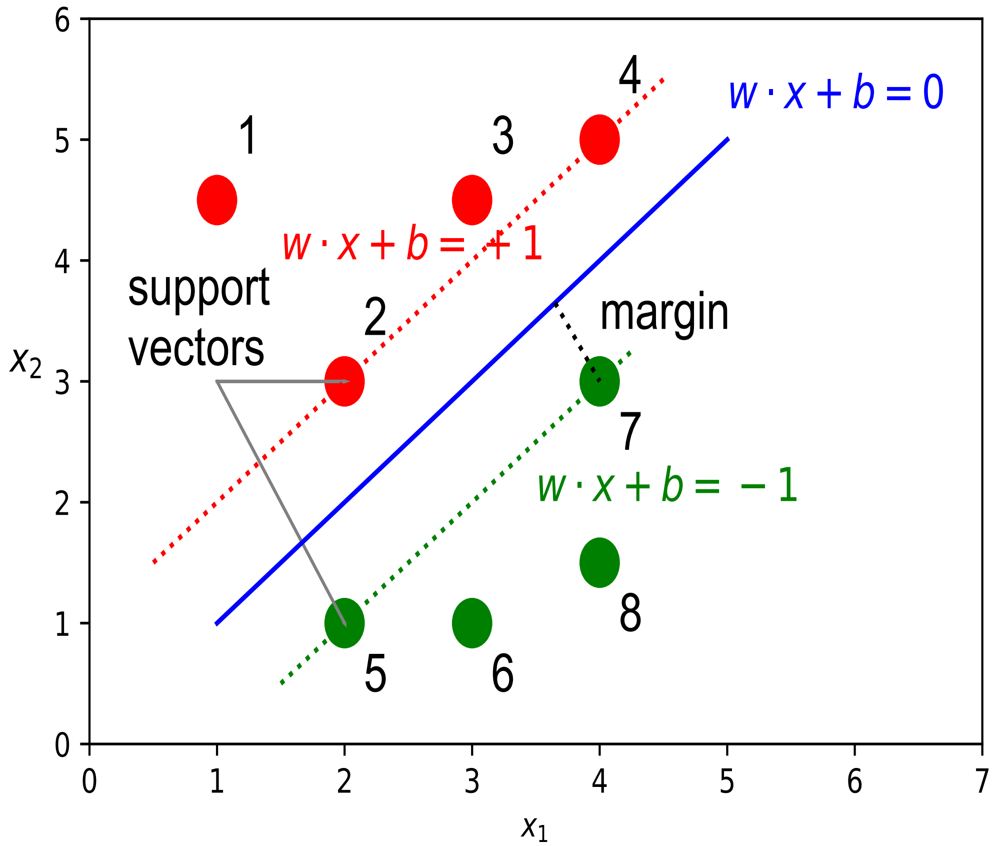
Binary Classification

- training set S with labels $\{-1, +1\}$
- find a classifier H

$$H : X \mapsto \{-1, +1\}$$

- low generalization error
- linear classification (based on hyperplanes)

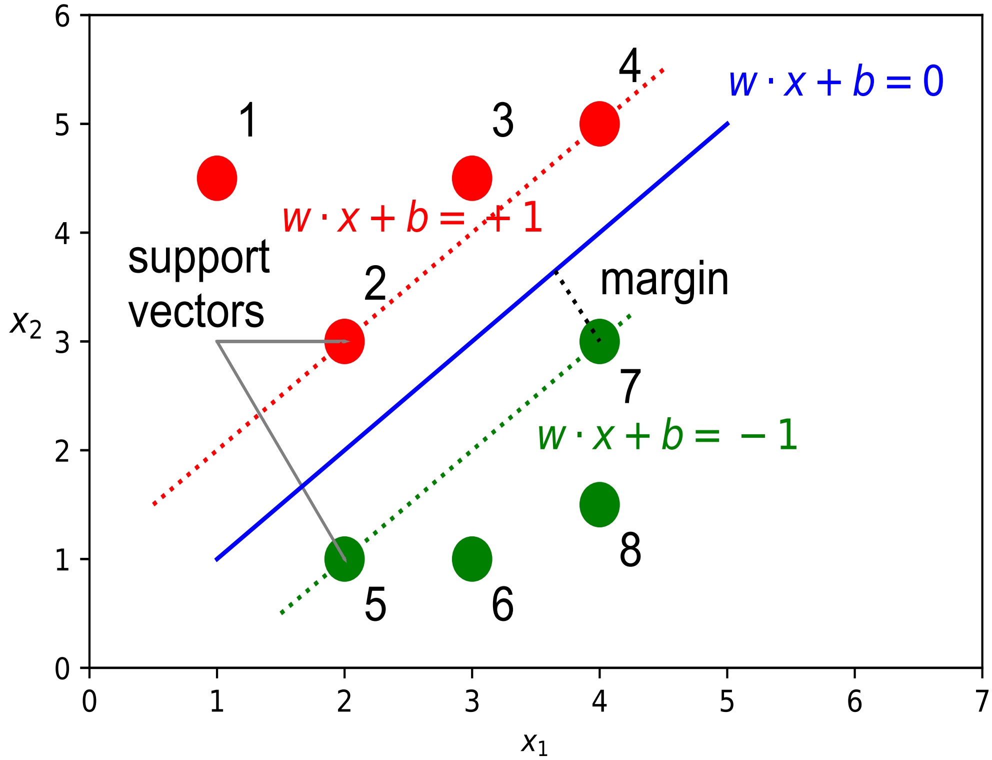
Linear Separation



$$H : X \mapsto \operatorname{sgn}\big(W \cdot X + b\big)$$

$$W \in R^N, b \in R$$

Optimal Hyperplane



- $|w \cdot x + b| = 1$ at support vectors
- max margin:$\min_x \frac{|w \cdot x + b|}{||w||} = \frac{1}{||w||}$

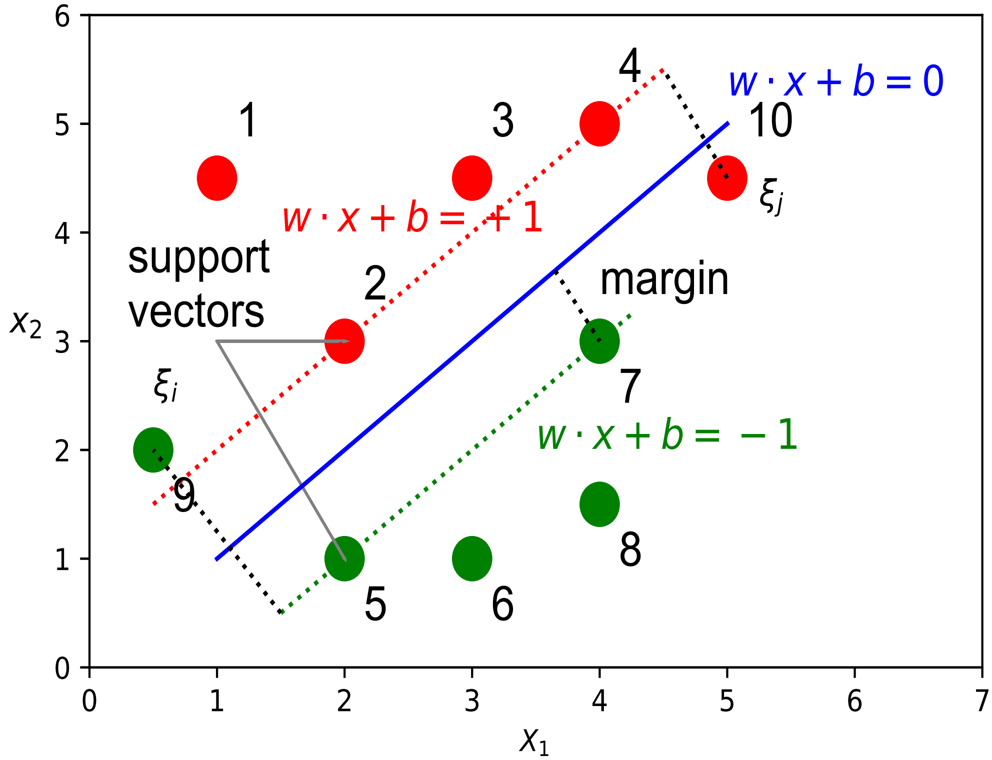
Optimization Problem

- (strictly) convex optimization:

$$\min_{W,b} \frac{1}{2}||W||^2 \text{ where } y_i(W \cdot X_i + b) \geq 1$$

- unique solution for linearly separable points
- only support vectors for solution

Soft Margin



- slack variables

$$y_i(W \cdot X_i + b) \geq 1 - \xi_i$$

Optimization Problem with Slack Variables

- still (strictly) convex optimization:

$$\min_{W,b} \frac{1}{2}||W||^2 + C\sum_{i=1}^{m}\xi_i$$

where $y_i\big(W \cdot X_i + b\big) \geq 1 - \xi_i$

- unique solution
- C is regularization parameter

Optimization Problem Alternative Formulation

- $y_i(W \cdot X_i + b) \geq 1 - \xi_i$ equivalent to $\xi_i = \max(0, 1 - y_i f(X_i))$
- can re-write optimization as

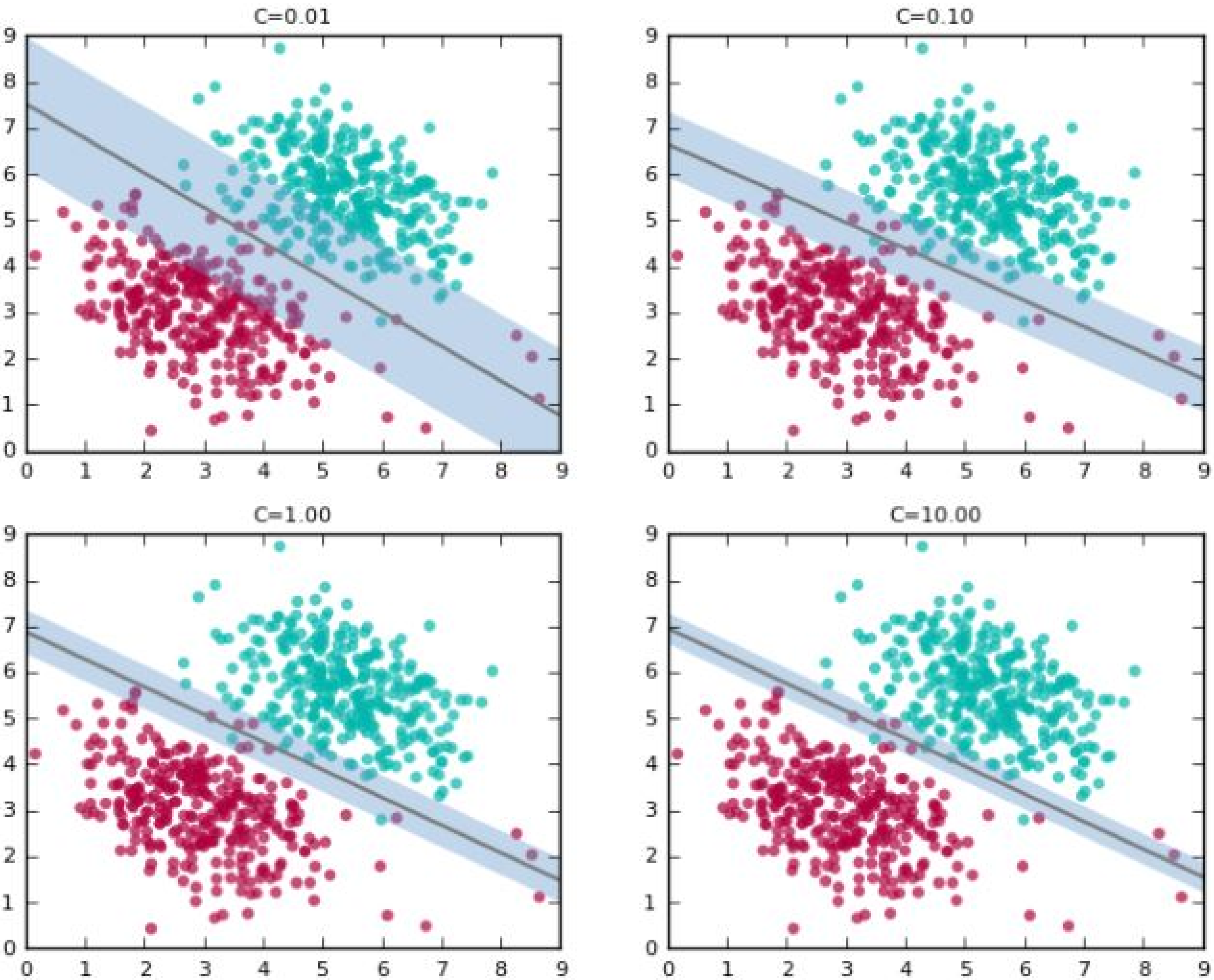
$$\min_W \frac{1}{2} \underbrace{||W||}_{\text{regularization}}{}^2 + C \sum_{i=1}^{m} \underbrace{\max(0, 1 - y_i f(X_i))}_{\text{loss function}}$$

- unique solution
- unconstrained optimization

The Meaning of C

- small C - ”soft” margin (ignore constraints)
- C - narrow margin (hard to ignore constraints)
- $C \mapsto \infty$ - hard margin (enforce all constraints
- for any C still a quadratic optimization
- unique minimum

Margin Width vs. C



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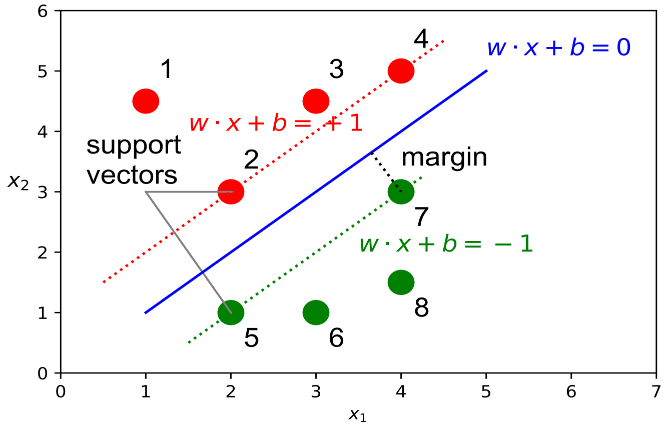
Loss Function



$$\min_{W} \frac{1}{2}||W||^2 + C\sum_{i=1}^{m} \max(0, 1 - y_i f(X_i))$$

1. outside margin: $y_i f(X_i) > 1$ - no contribution to loss
2. on the margin: $y_i f(X_i) = 1$ - no contribution to loss
3. violates margin: $y_i f(X_i) < 1$ - contributes to loss

Dual Optimization

- constrained optimization

$$\max_{\alpha} \sum_{i=1}^{m} \alpha_i = \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j (X_i \cdot X_j)$$

subject to: $\alpha_i \geq 0$ and $\sum_{i=1}^{m} \alpha_i y_i = 0$

- solution:

$$h(x) = \text{sgn} \left(\sum_{i=1}^{m} \alpha_i y_i (X_i \cdot X) + b \right)$$

with $b = y_i - \sum_{j=1}^{m} \alpha_j y_j (X_j \cdot X_i)$

for any support vectors X_i

Kernel Methods

- define $K : X \times X \mapsto R$ so that

$$K(X, X') = \phi(X) \cdot \phi(X')$$

- K is similarity measure
- easier to compute that dot product and $\Phi()$
- example: $K(X, Y) = (X \cdot Y)^2$

$$\Phi(x_1, x_2) = (x_1^2, x_1 x_2 \sqrt{2}, x_2^2)$$

$$\Phi(x_1, x_2) \cdot \Phi(y_1, y_2) = x_1^2 x_1^2 + 2 x_1 x_2 y_1 y_2 + x_2^2 y_2^2$$

$$= (x_1 y_1)^2 + 2 x_1 y_1 x_2 y_2 + (x_2 y_2)^2$$

$$= (x_1 y_1 + x_2 y_2)^2 = (X \cdot Y)^2 = K(X, Y)$$

Kernels for Non-Linear SVM

- allow separability in higher dimensions
- function like dot product

1. polynomial

$$K(X, X') \mapsto (X \cdot X' + C)^p$$

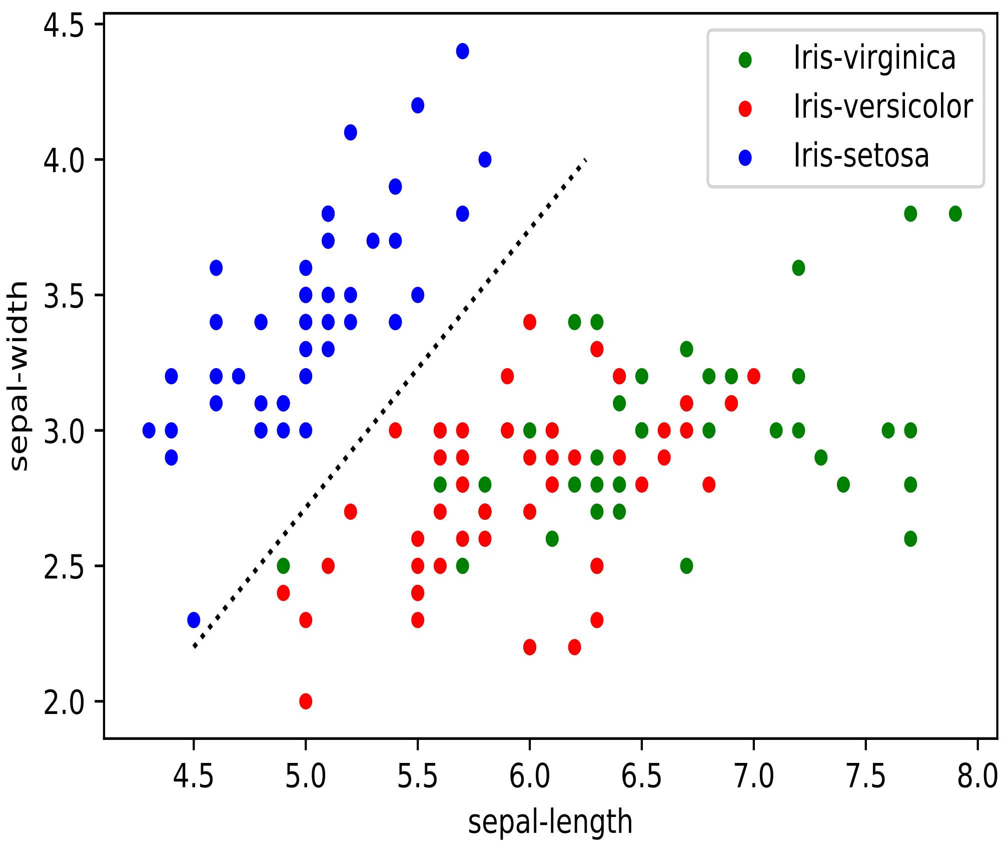
2. sigmoid

$$K(X, X') \mapsto \tanh(k * X \cdot X' - \delta)$$

3. Gauss (radial basis functions)

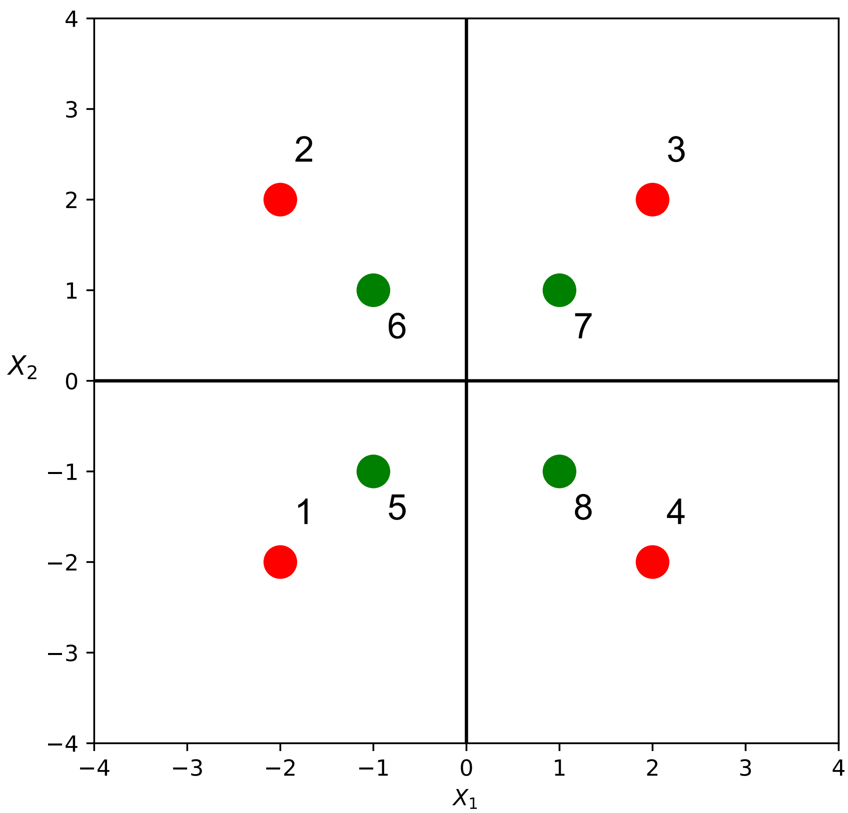
$$K(X, X') \mapsto \exp\left(-\nu(X - X')^2/2\sigma^2\right)$$

Linear Separability



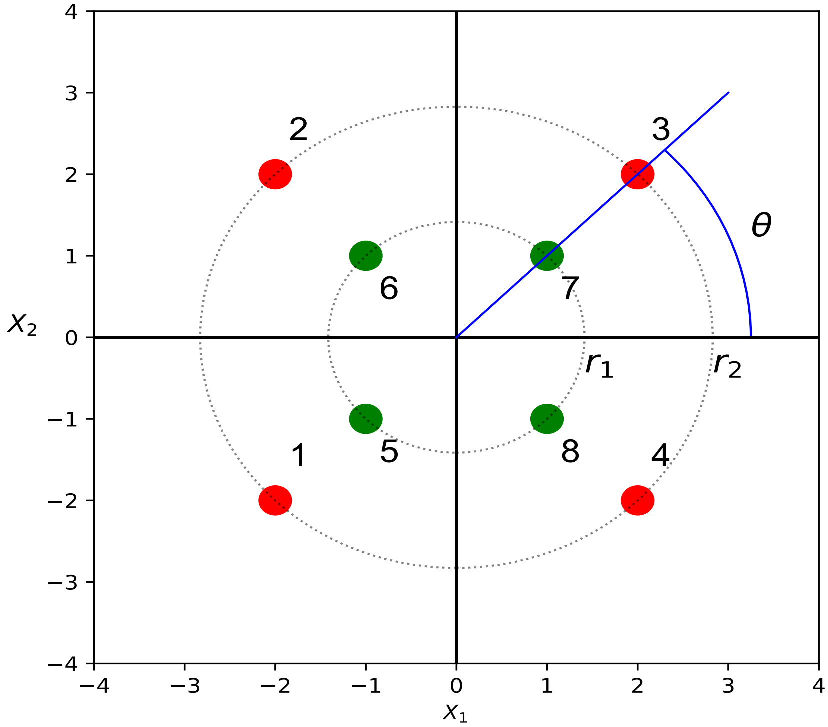
- draw a hyperplane
- difficult in many cases

Example of Difficulty



- non-separable in 2 dimensions

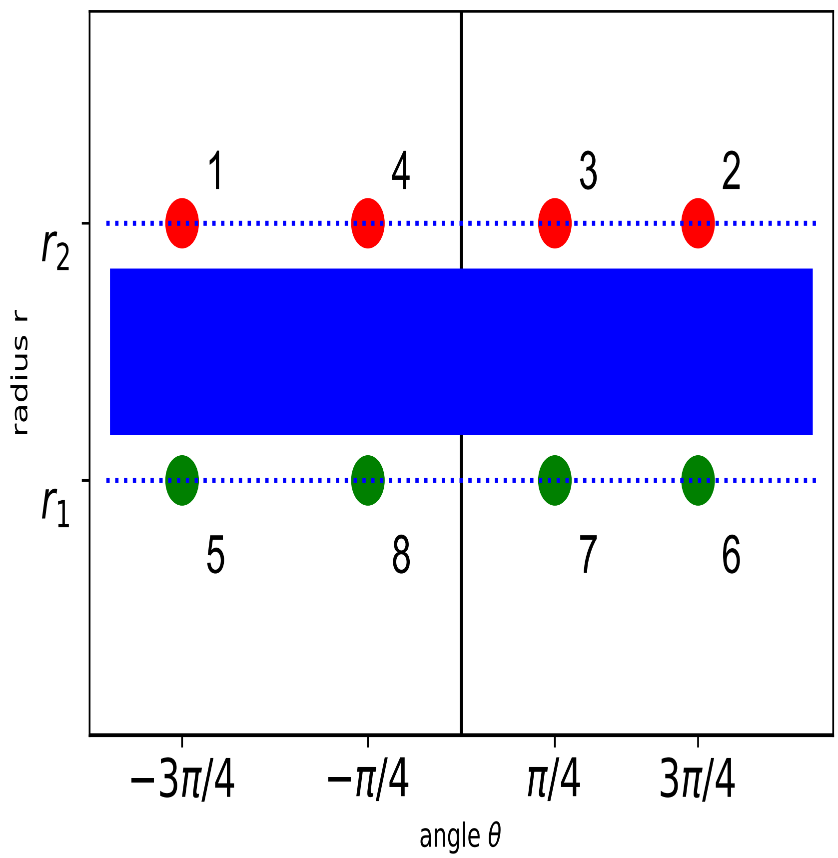
Mapping to Polar



$$\Phi : (x_1, x_2) \mapsto (\sqrt{x_1^2 + x_2^2}, \arccos \frac{x_1}{\sqrt{x_1^2 + x_2^2}})$$

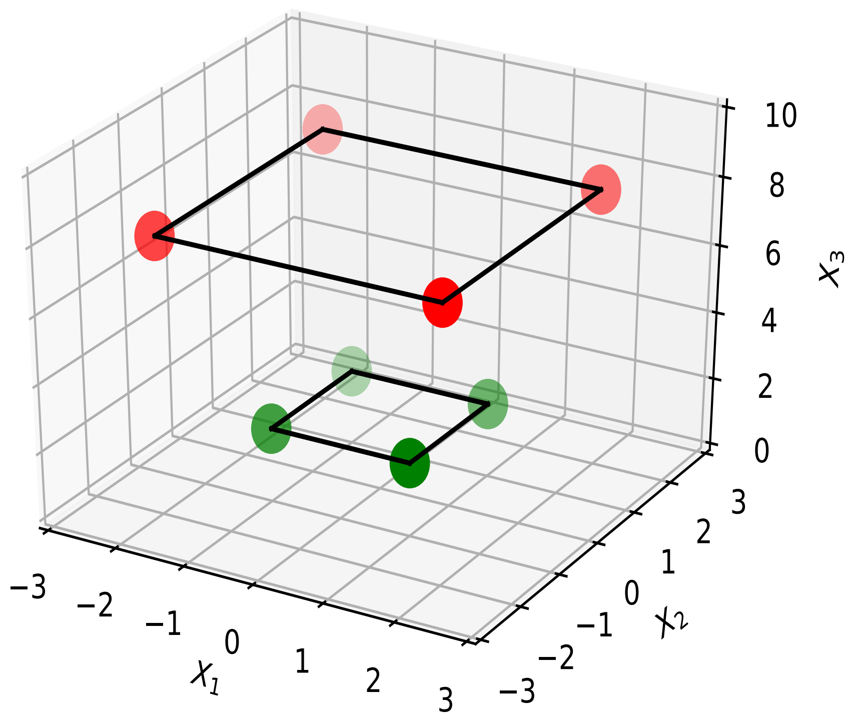
$$\Phi(1, 1) \mapsto (\sqrt{2}, \pi/4)$$

Linear Separation in Polar Coordinates



- separation by radius

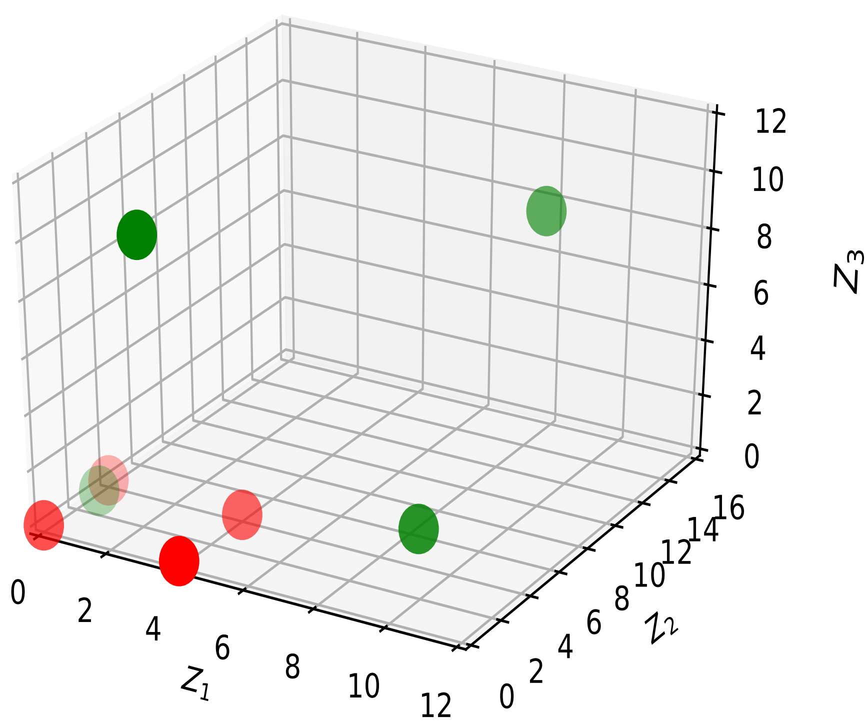
Alternative Solution



$$\Phi(x_1, x_2) \mapsto (x_1, x_2, \sqrt{x_1^2 + x_2^2})$$

- separable by z

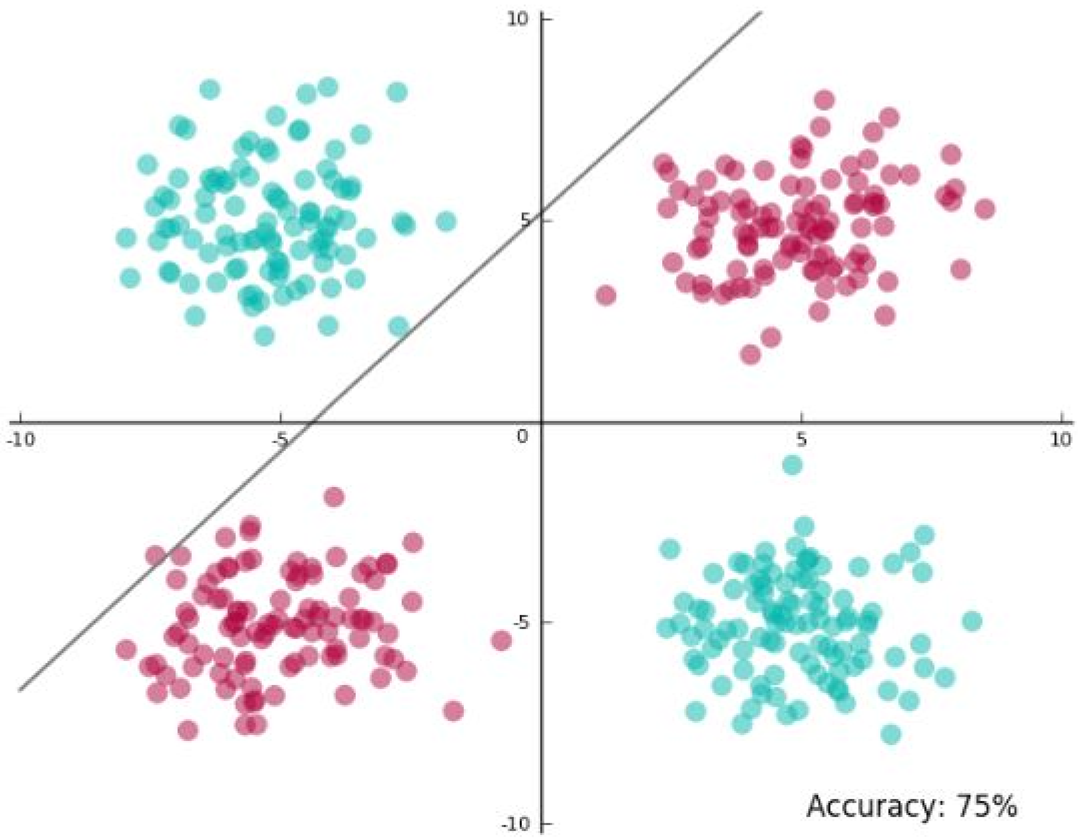
Another Solution



$$\Phi(x_1, x_2) \mapsto ((x_1+2)^2, \sqrt{2}(x_1+2)(x_2+2), (x_2+2)^2)$$

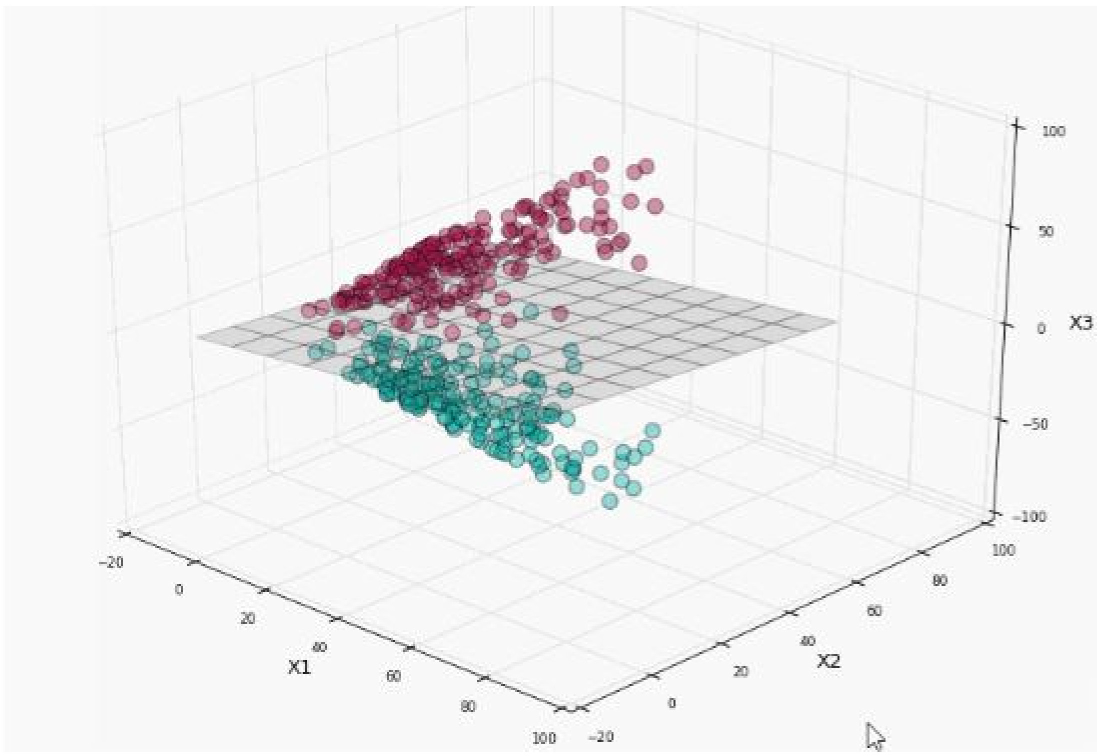
- separable in 3 dimensions

Kernel Example



- non-linearly separable
- similar to XOR

Using a Kernel Transformation



$$\Phi(x_1, x_2) \mapsto (x_1^2, x_2^2, \sqrt{2}x_1x_2)$$

- linearly separable in R^3

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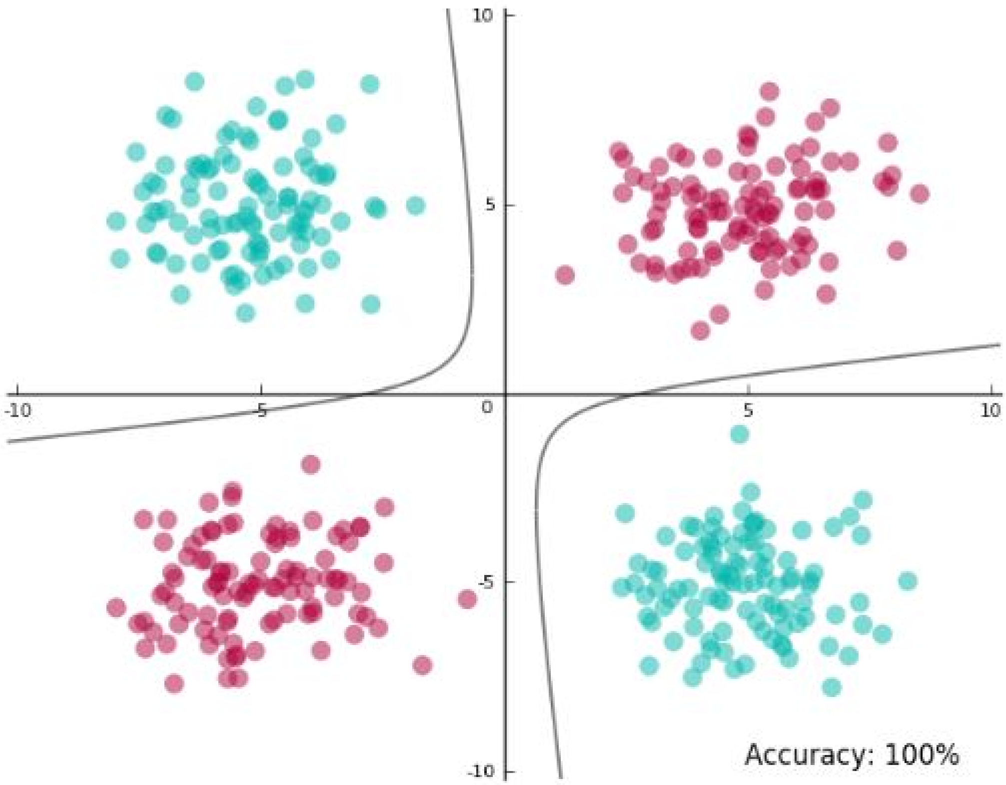
Projecting Onto Original Space



- separating boundary is not linear

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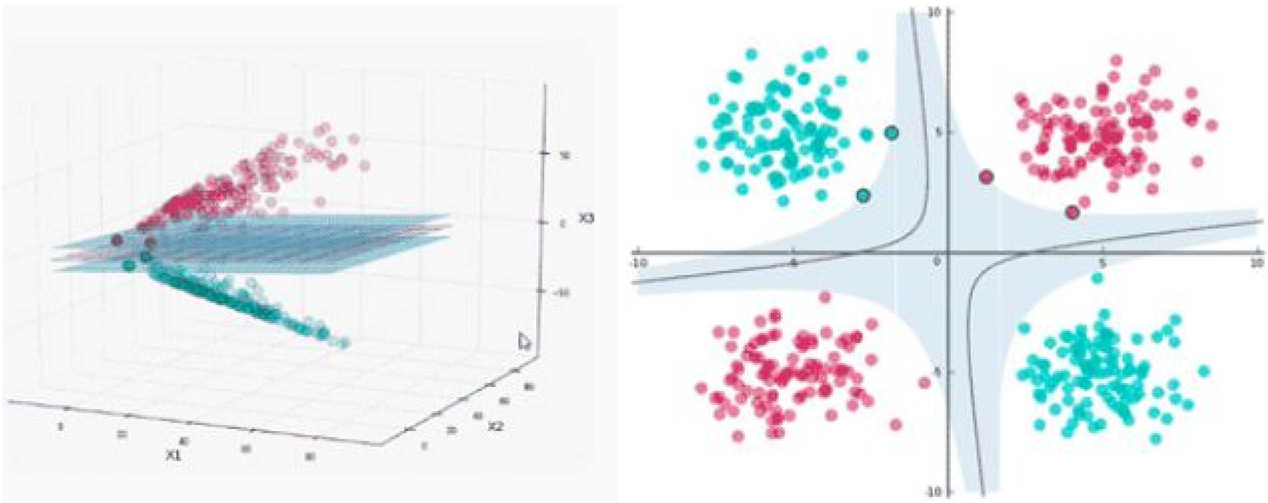
Kernel "Trick"



- efficient way to transform data
- distance in original space: $X \cdot Y$
- distance in new space: $K(X, Y)$
- $K(X, Y) = (X \cdot Y)^2$

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A Numerical Dataset

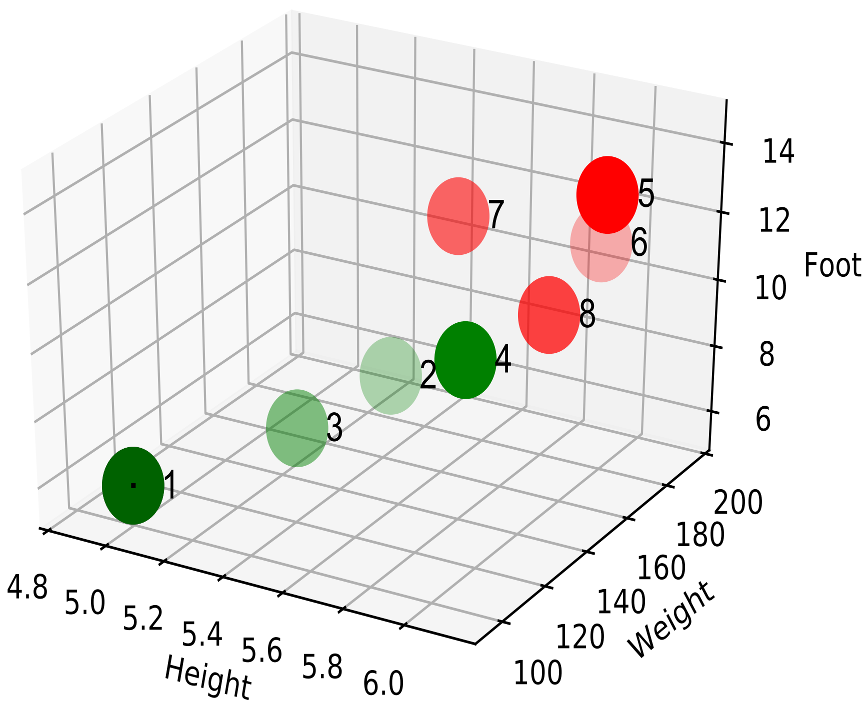
object x_i	Height (H)	Weight (W)	Foot (F)	Label (L)
x_1	5.00	100	6	green
x_2	5.50	150	8	green
x_3	5.33	130	7	green
x_4	5.75	150	9	green
x_5	6.00	180	13	red
x_6	5.92	190	11	red
x_7	5.58	170	12	red
x_8	5.92	165	10	red

Code for the Dataset

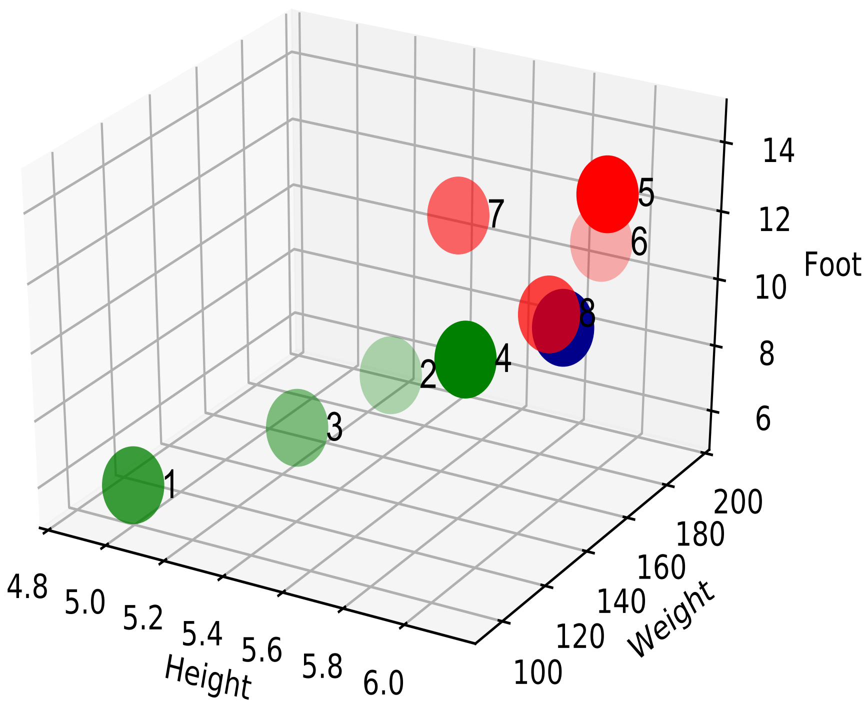
```
import pandas as pd
data = pd.DataFrame(
        {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green','green','green','green',
                   'red','red','red','red'],
        'Height': [5, 5.5, 5.33, 5.75,
                   6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150,
                   180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight',
                      'Foot', 'Label'] )
```

```
ipdb> data
  id Height Weight Foot Label
0  1  5.00    100    6  green
1  2  5.50    150    8  green
2  3  5.33    130    7  green
3  4  5.75    150    9  green
4  5  6.00    180   13    red
5  6  5.92    190   11    red
6  7  5.58    170   12    red
7  8  5.92    165   10    red
```

A Dataset Illustration

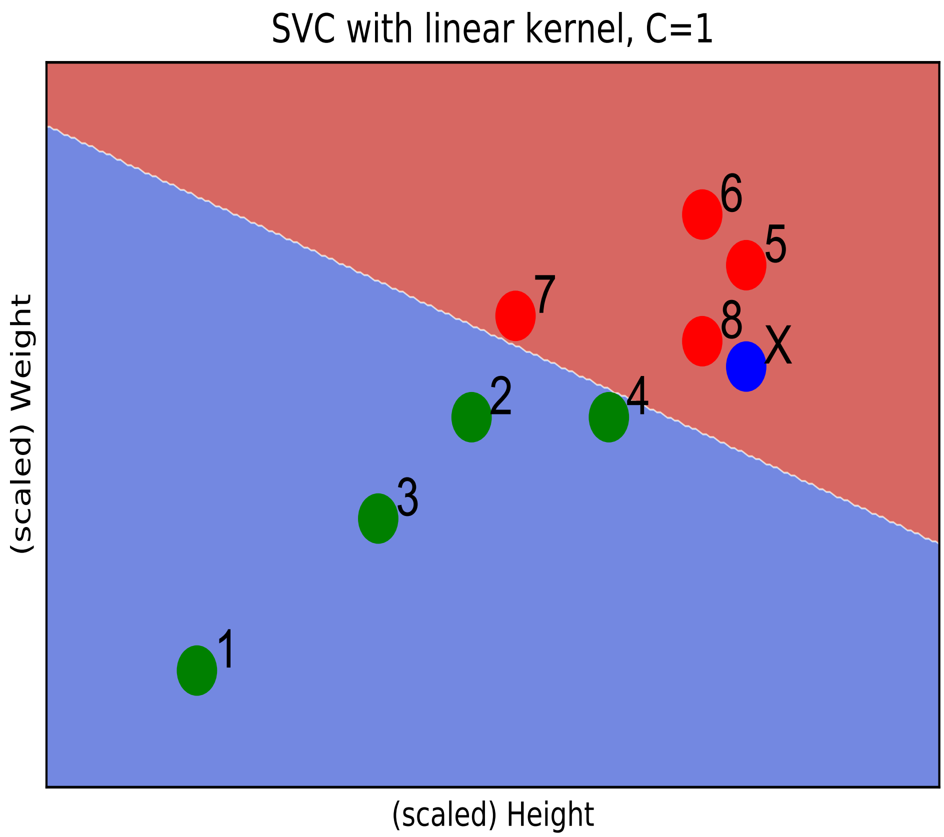


A New Instance



$$(H=6,\ W=160,\ F=10) \mapsto ?$$

A Linear SVM



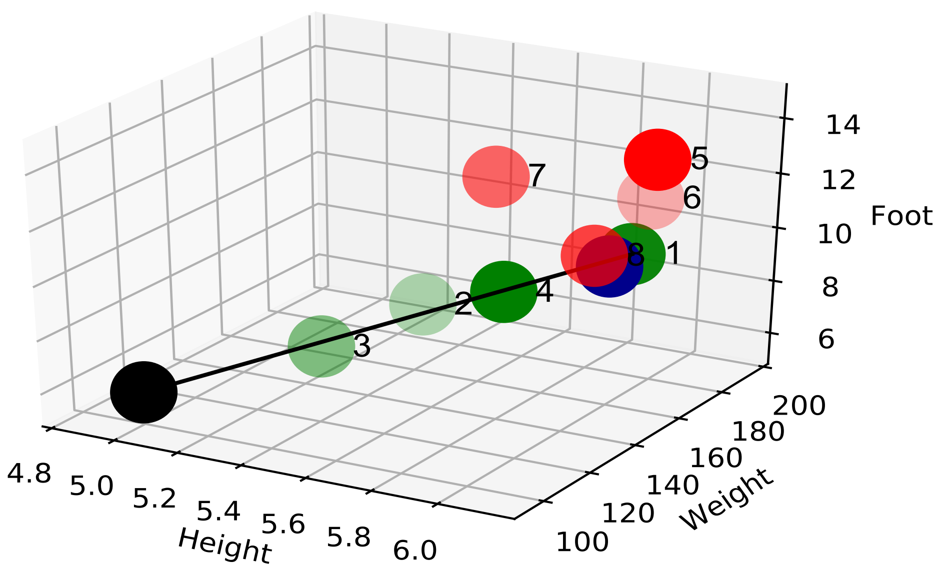
- predict(x^*)=red
- accuracy $= 100\%$

Python Code: Linear

```
import pandas as pd
import numpy as np
from sklearn import svm
from sklearn.preprocessing import StandardScaler

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                        'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )

X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

svm_classifier = svm.SVC(kernel='linear')
svm_classifier.fit(X,Y)

new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = svm_classifier.predict(new_x)
accuracy = svm_classifier.score(X, Y)
```

```
ipdb> predicted[0]
red
ipdb> accuracy
1.0
```

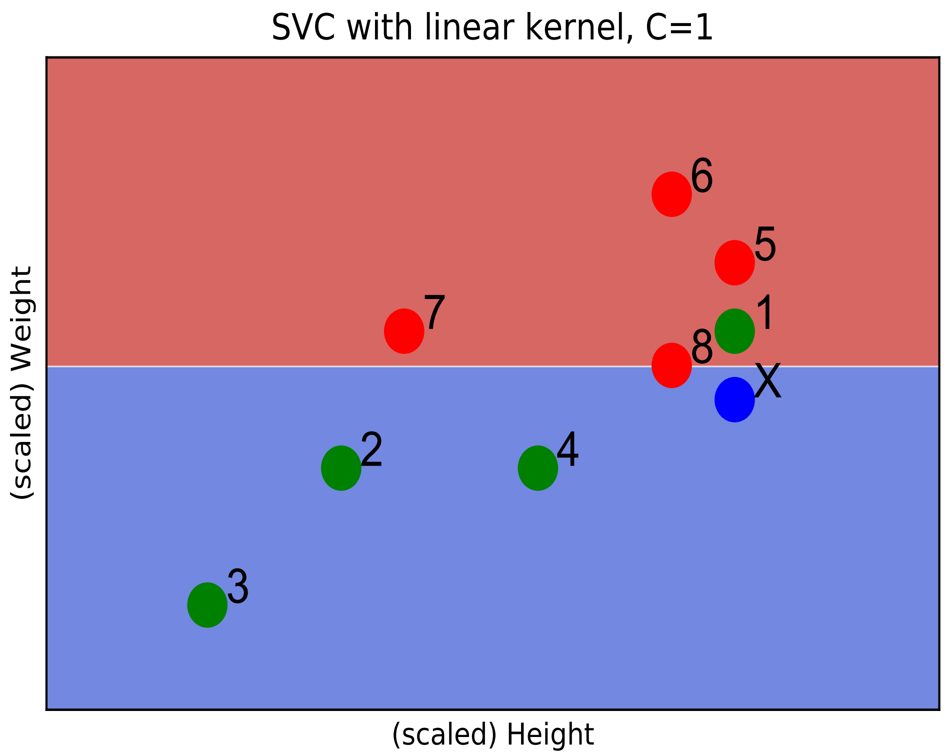
F/W/H Change



id	Height	Weight	Foot	Label
1	$5 \mapsto 6$	$100 \mapsto 170$	$6 \mapsto 10$	green

$$(\text{H}=6,\ \text{W}=160,\ \text{F}=10) \mapsto ?$$

A Linear SVM (modified dataset)



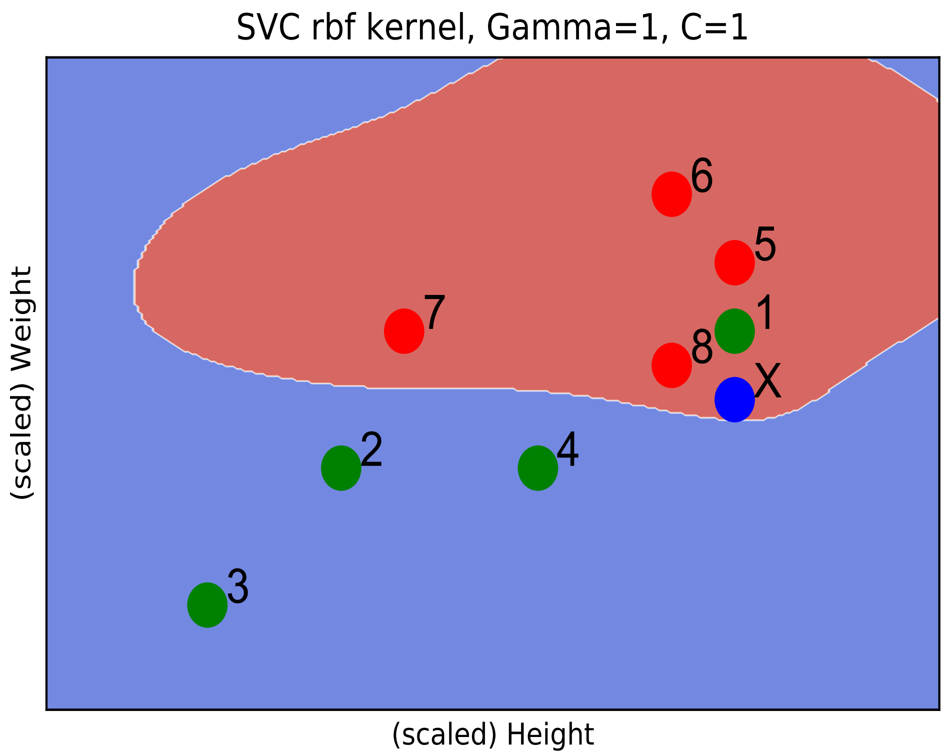
- predict(x^*)=green
- accuracy = 75%

Python Code: Linear (modified dataset)

```
import pandas as pd
import numpy as np
from sklearn import svm
from sklearn.preprocessing import StandardScaler

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                        'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )
data['Height'].iloc[0] = 6;
data['Weight'].iloc[0] = 170;
data['Foot'].iloc[0] = 10
X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

svm_classifier = svm.SVC(kernel='linear')
svm_classifier.fit(X,Y)
new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = svm_classifier.predict(new_x)
accuracy = svm_classifier.score(X, Y)
```

```
ipdb> predicted[0]
green
ipdb> accuracy
0.75
```

A Gaussian SVM (modified dataset)



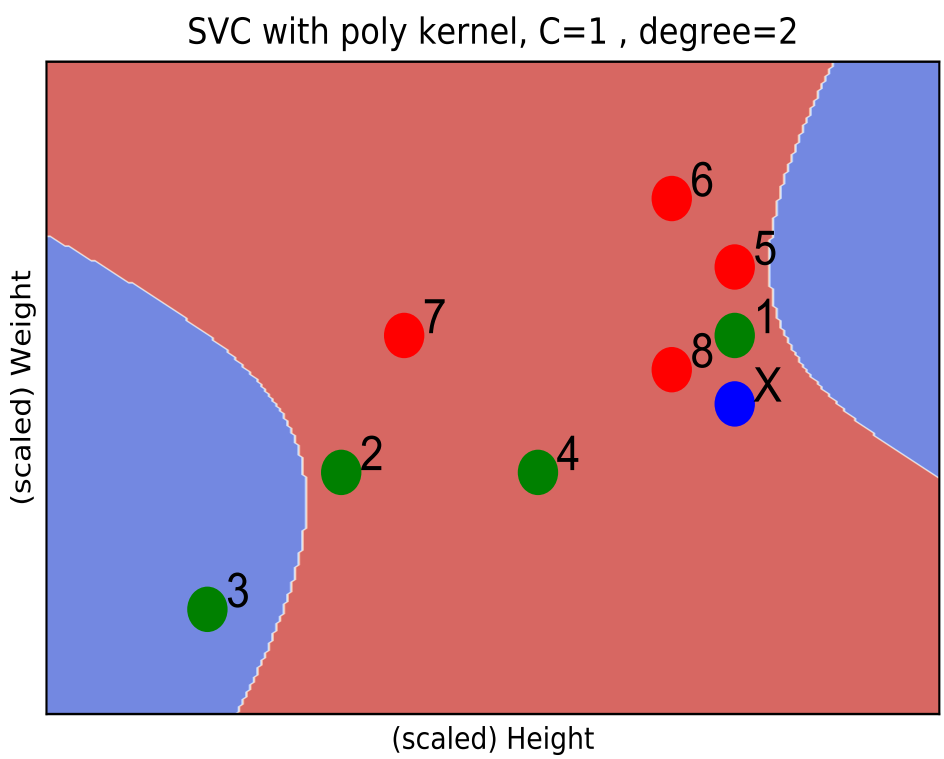
- predict(x^*)='red'
- accuracy = 87.5%

Python Code: Gaussian (modified dataset)

```
import pandas as pd
import numpy as np
from sklearn import svm
from sklearn.preprocessing import StandardScaler

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                           'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )
data['Height'].iloc[0] = 6;
data['Weight'].iloc[0] = 170;
data['Foot'].iloc[0] = 10
X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

svm_classifier = svm.SVC(kernel='rbf')
svm_classifier.fit(X,Y)
new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = svm_classifier.predict(new_x)
accuracy = svm_classifier.score(X, Y)
```

```
ipdb> predicted[0]
red
ipdb> accuracy
0.875
```

Polynomial (d=2) SVM (modified dataset)



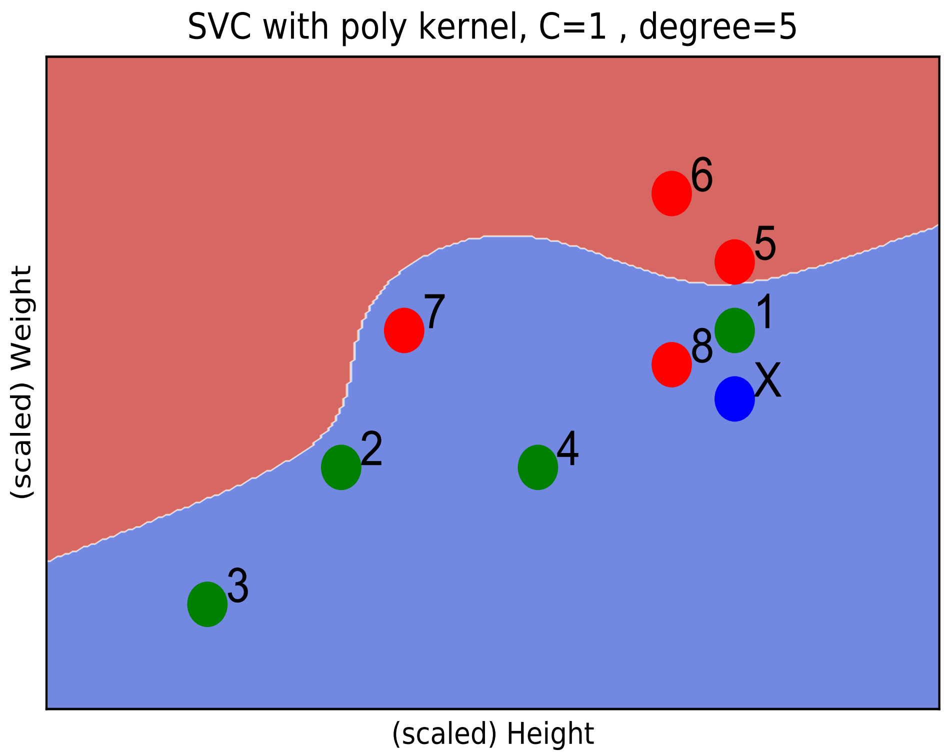
- predict(x^*)='red'
- accuracy = 62.5%

Python Code: Poly (d=2, modified dataset)

```
import pandas as pd
import numpy as np
from sklearn import svm
from sklearn.preprocessing import StandardScaler

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                              'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )
data['Height'].iloc[0] = 6;
data['Weight'].iloc[0] = 170;
data['Foot'].iloc[0] = 10
X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

svm_classifier = svm.SVC(kernel='poly', degree=2)
svm_classifier.fit(X,Y)
new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = svm_classifier.predict(new_x)
accuracy = svm_classifier.score(X, Y)
```

```
ipdb> predicted[0]
red
ipdb> accuracy
0.625
```

Polynomial (d=5) SVM (modified dataset)



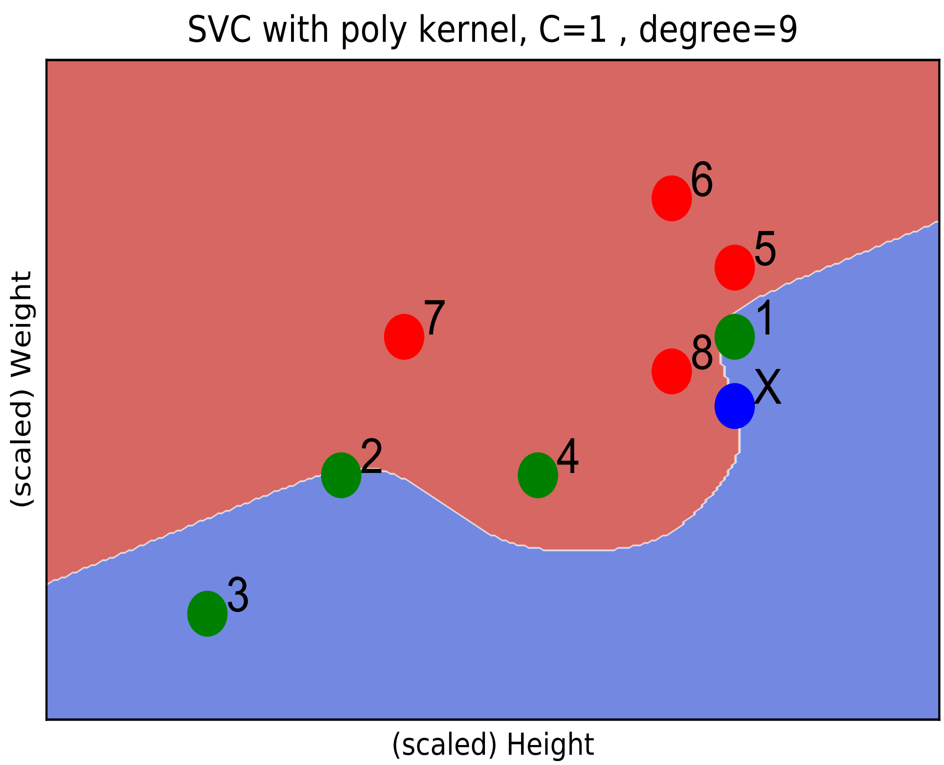
- predict(x^*)='green'
- accuracy = 75%

Python Code: Poly (d=5, modified dataset)

```
import pandas as pd
import numpy as np
from sklearn import svm
from sklearn.preprocessing import StandardScaler

data = pd.DataFrame( {'id': [ 1,2,3,4,5,6,7,8],
        'Label': ['green', 'green', 'green', 'green',
                            'red', 'red', 'red', 'red'],
        'Height': [5, 5.5, 5.33, 5.75, 6.00, 5.92,  5.58, 5.92],
        'Weight': [100, 150, 130, 150, 180, 190, 170, 165],
        'Foot': [6, 8, 7, 9, 13, 11, 12, 10]},
         columns = ['id', 'Height', 'Weight', 'Foot', 'Label'] )
data['Height'].iloc[0] = 6;
data['Weight'].iloc[0] = 170;
data['Foot'].iloc[0] = 10
X = data[['Height', 'Weight']].values
scaler = StandardScaler()
scaler.fit(X)
X = scaler.transform(X)
Y = data['Label'].values

svm_classifier = svm.SVC(kernel='poly', degree=5)
svm_classifier.fit(X,Y)
new_x = scaler.transform(np.asmatrix([6, 160]))
predicted = svm_classifier.predict(new_x)
accuracy = svm_classifier.score(X, Y)
```

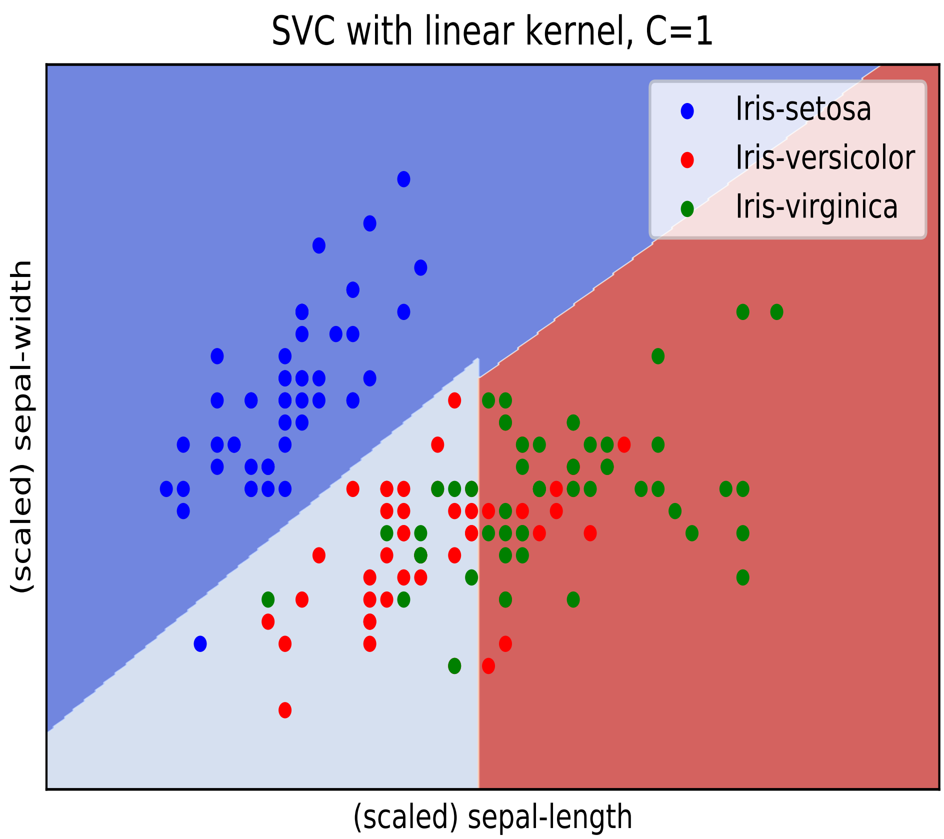
```
ipdb> predicted[0]
green
ipdb> accuracy
0.75
```

Polynomial (d=9) SVM (modified dataset)



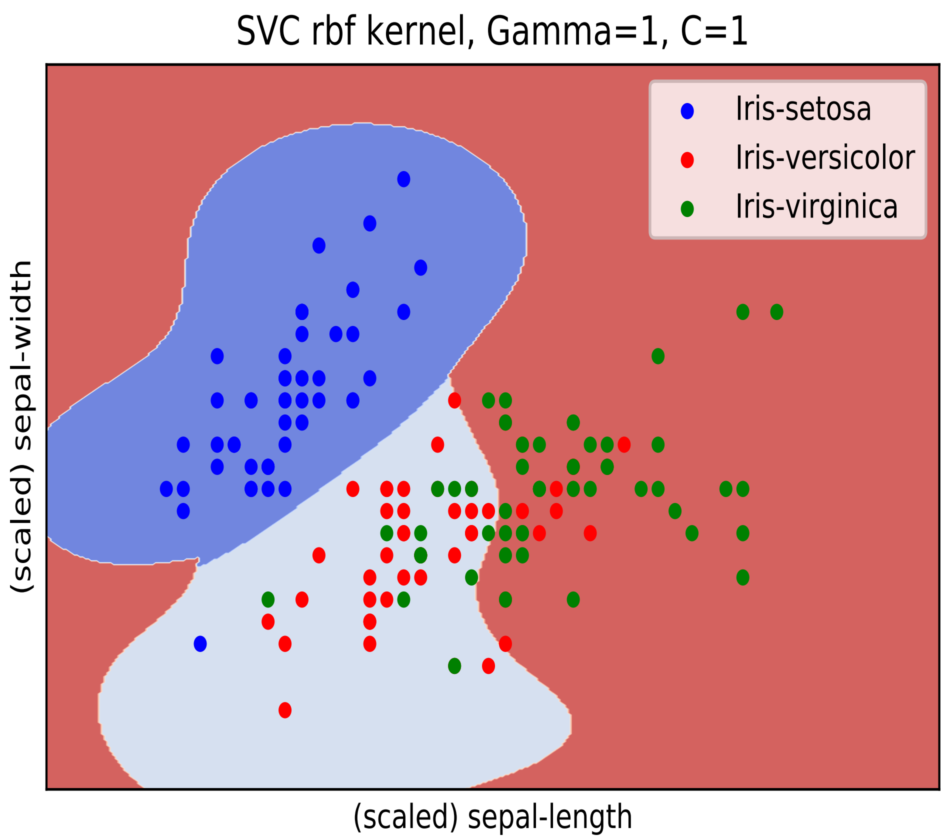
- predict(x^*)='green'
- accuracy = 87.5% (high d)

Iris: Linear SVM



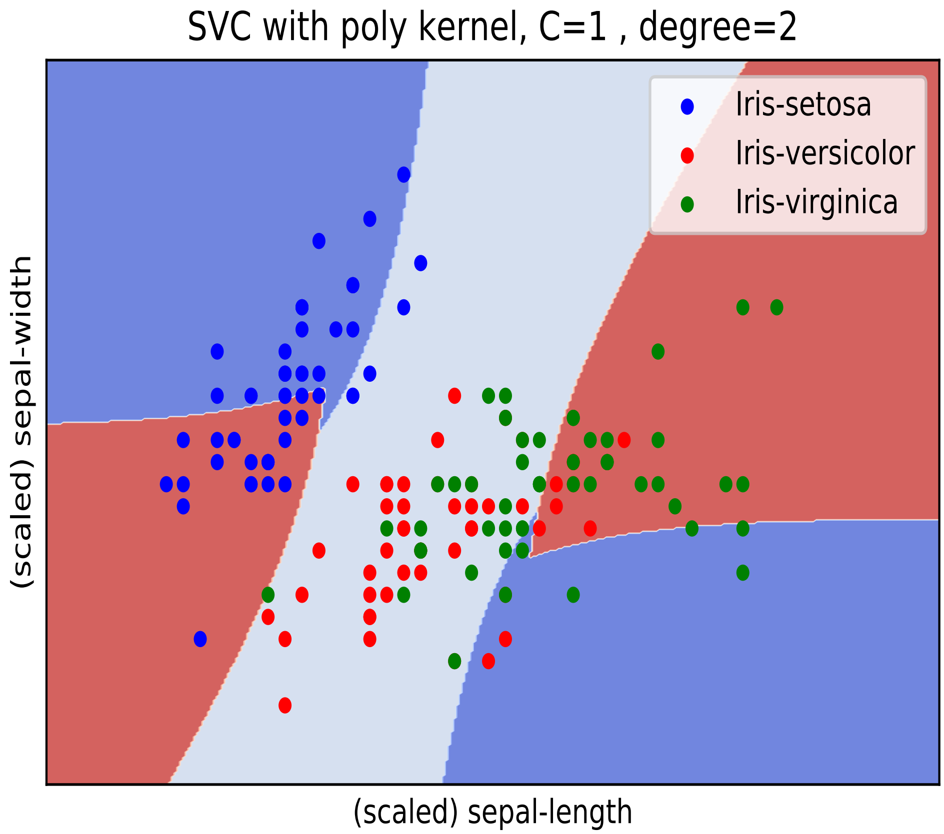
- accuracy = 80%

Iris: Gaussian SVM



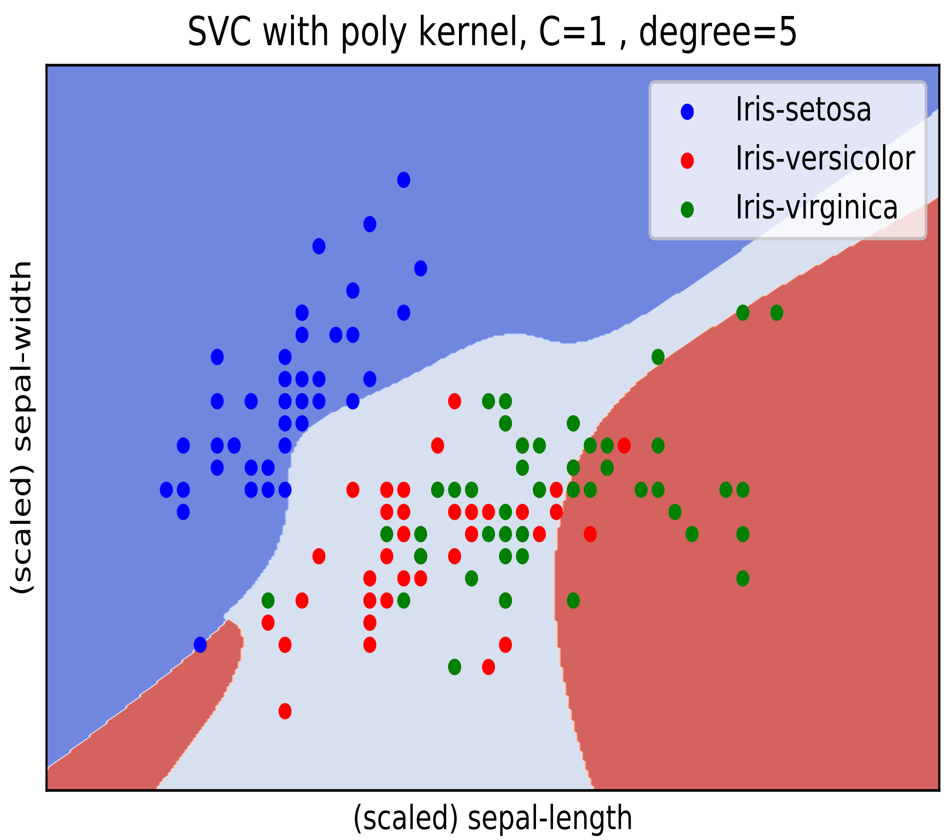
- accuracy = 80%

Iris: Poly SVM (d=2)



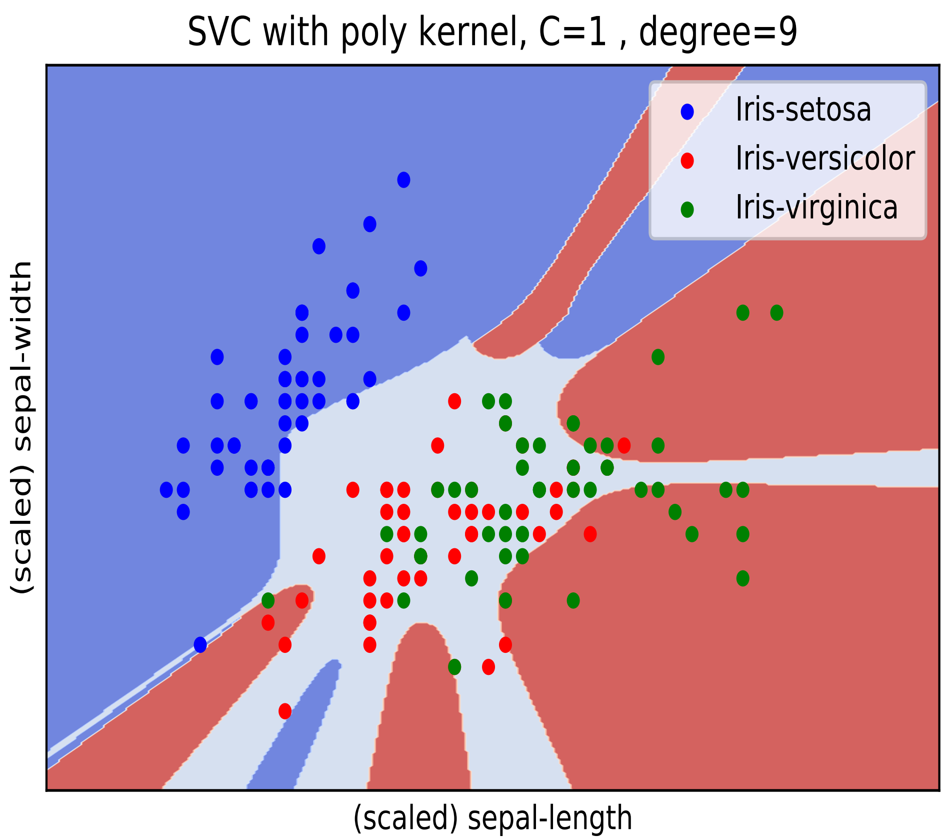
- accuracy = 47%

Iris: Poly SVM (d=5)



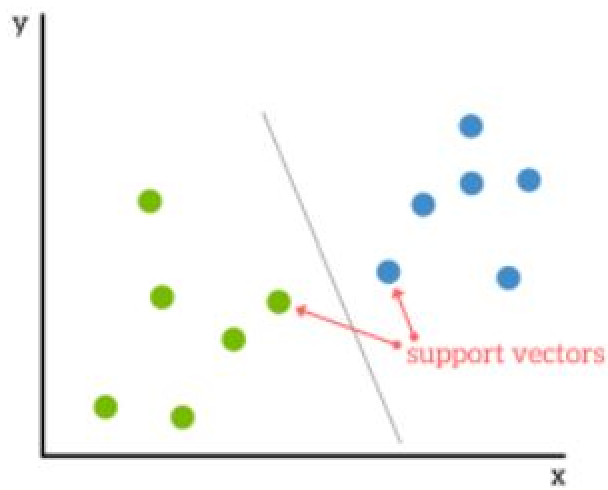
- accuracy = 75%

Iris: Poly SVM (d=9)



- accuracy = 64%

SVM Summary



- want to separate classes
- problem: how to find hyperplane

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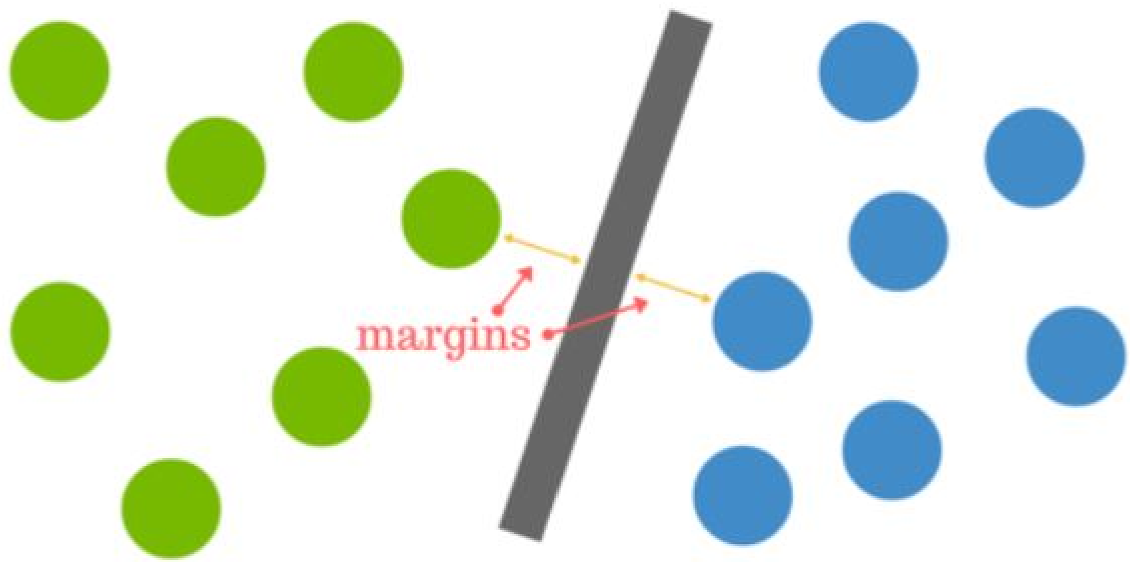
SVM Summary (cont'd)



- if points are separable, can compute margin
- mathematical optimization

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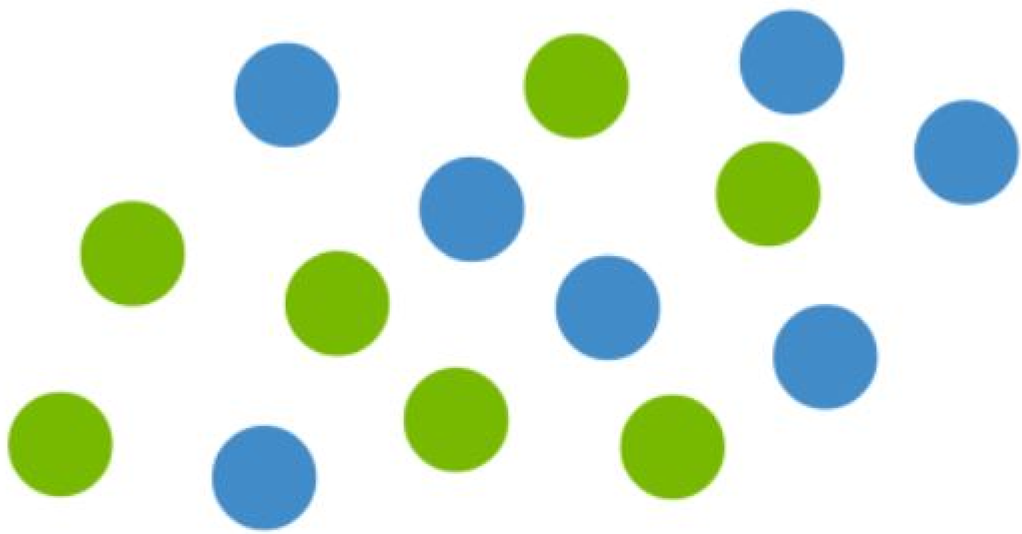
SVM Summary (cont'd)



- problem: what if points are not separable

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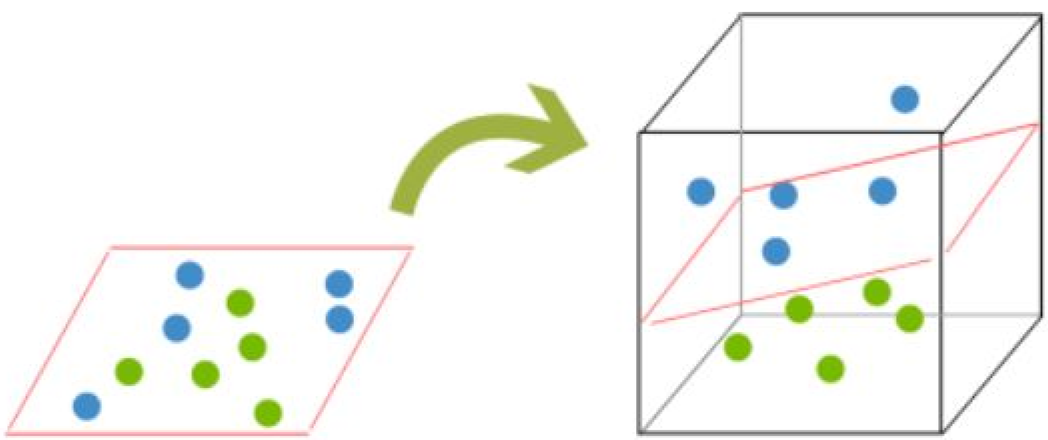
SVM Summary (cont'd)



- solution: "separate" in higher dimensional space
- classify points in new space
- computationally efficient for many kernel functions

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Advantages/Disadvantages

- advantages

(a) only support vectors are used in classification (efficient)

(b) can classify many non-linearly separable datasets

- disadvantages

(a) computationally intensive to compute support vectors

(b) inaccurate in large noisy datasets

Concepts Check:

(a) linear separability
(b) margin and support vectors
(c) soft vs. hard margins
(d) kernel transformations
(e) kernel "trick"
(f) linear, polynomial and Gaussian SVM