Data Structures and Algorithms

P and NP

Introduction to Turing Machine

- Introduced by Alan Turing.
- A Turing machine is a formal computational model of finite-state computing machines.
- Has unlimited amount of time and memory available for computation.

- A Turing machine is a finite-state machine in which a transition reads and prints a symbol on the tape.
- A tape head may move in either direction.

- We will briefly discuss standard Turing machines.

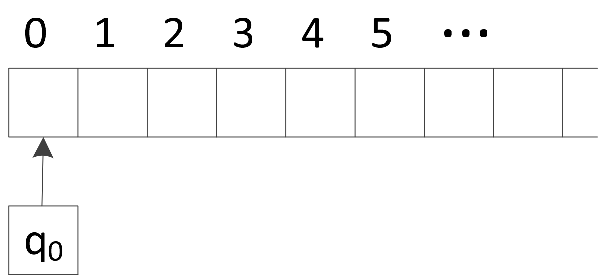
Introduction to Turing Machine

- Definition: A Turing machine is a quintuple M = (Q, Σ, Γ, δ, q_0), where

- Q is a finite set of states.
- Γ is a finite set called the *tape alphabet*. Γ contains a special symbol *B* that represents a blank.
- Σ is a subset of Γ – {*B*} called the *input alphabet*.
- δ is a partial function Q X Γ → Q X Γ X {*L*, *R*}.
- q_0 ϵ Q is a distinguished state called the *start state*.

Introduction to Turing Machine

- A tape extends indefinitely in one direction.
- Tape positions are numbered beginning with zero.
- A computation begins with the tape head in state q_0 scanning the leftmost position.



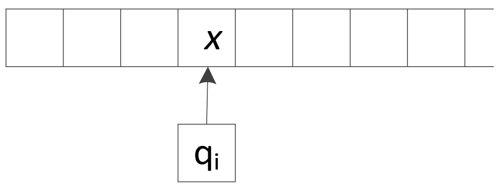
- Input string from Σ* is written on the tape beginning at position one. Position zero and all other positions are blanks.

Introduction to Turing Machine

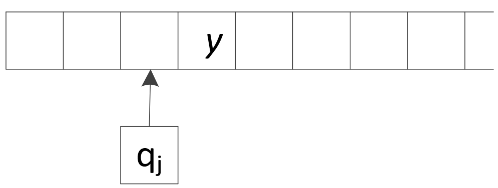
- A transition consists of three actions:
  - Change the state.
  - Write a symbol on the square scanned by the tape head.
  - Move the tap head. Direction of the move is indicated by *L* (left) or *R* (right).

Introduction to Turing Machine

- If the machine configuration is



and the transition is $\delta(q_i, x) = [q_j, y, L]$, then the new configuration is



- Here, the transition changed from q_i to q_j, tape symbol y was written replacing x, and the tape head was moved to the left by one position.

Introduction to Turing Machine

- A Turing machine *halts* when it encounters a state symbol pair for which no transition is defined.
- A transition from tape position zero may specify a move to the left (crossing the boundary of the tape). When this occurs, we say the computation *terminates abnormally*.
- When we say a computation *halts*, it means it terminates in a normal fashion.

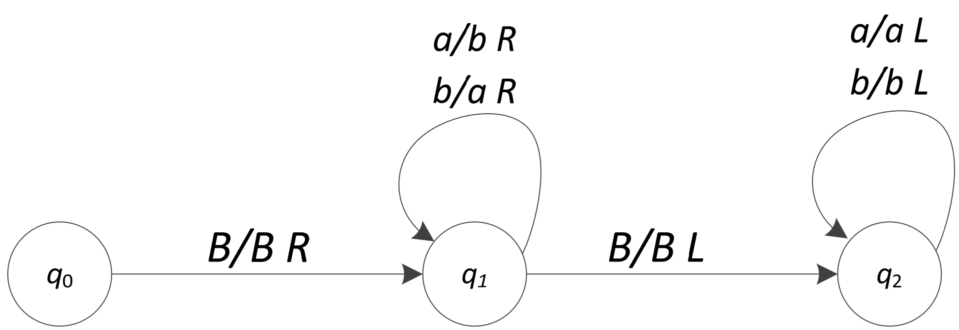
Introduction to Turing Machine

- Example: A transition function of a standard Turing machine with input alphabet {*a*, *b*}:

δ	**B**	**a**	**b**
$\mathbf{q_0}$	q_1, B, R		
$\mathbf{q_1}$	q_2, B, L	q_1, b, R	q_1, a, R
$\mathbf{q_2}$		q_2, a, L	q_2, b, L

Introduction to Turing Machine

- A Turing machine can be represented as a state diagram. In a state diagram, the transition $\delta(q_i, x) = [q_j, y, d]$, $d \in \{L, R\}$, is represented by an edge from q_i to q_j labeled *x/y d*. The state diagram corresponding to the above transition function is:



Introduction to Turing Machine

- A machine configuration consists of the state, the tape, and the position of the tape head.
- A configuration is denoted by uq_ivB, where uv is the string spelled on the tape from the left boundary to the rightmost nonblank symbol.
- The notation indicates the machine is in state q_i scanning the first symbol of v.

- The notation $uq_ivB \vdash xq_jyB$ indicates the configuration xq_jyB is obtained from uq_ivB by a single transition. Here, u, v, x, and y are strings.
- The notation $uq_ivB \vdash^* xq_jyB$ represents that xq_jyB can be obtained from uq_ivB by a finite number of, possibly zero, transitions.

Introduction to Turing Machine

- The following sequence of configurations, or transitions, show the computation generated by tracing the input *abab* by the above Turing machine:

$$q_0BababB \vdash Bq_1ababB \vdash Bbq_1babB \vdash Bbaq_1abB$$
$$\vdash Bbabq_1bB \vdash Bbabaq_1B \vdash Bbabq_2aB \vdash Bbaq_2baB$$
$$\vdash Bbq_2abaB \vdash Bq_2babaB \vdash q_2BbabaB$$

- This Turing machine exchanges *a*'s and *b*'s in the input string.

Introduction to Turing Machine

- Turing machines can be used as language acceptors.
- A computation accepts or rejects the input string.
- A Turing machine is augmented with final states.
- A Turing machine need not read the entire input to accept the string.

Introduction to Turing Machine

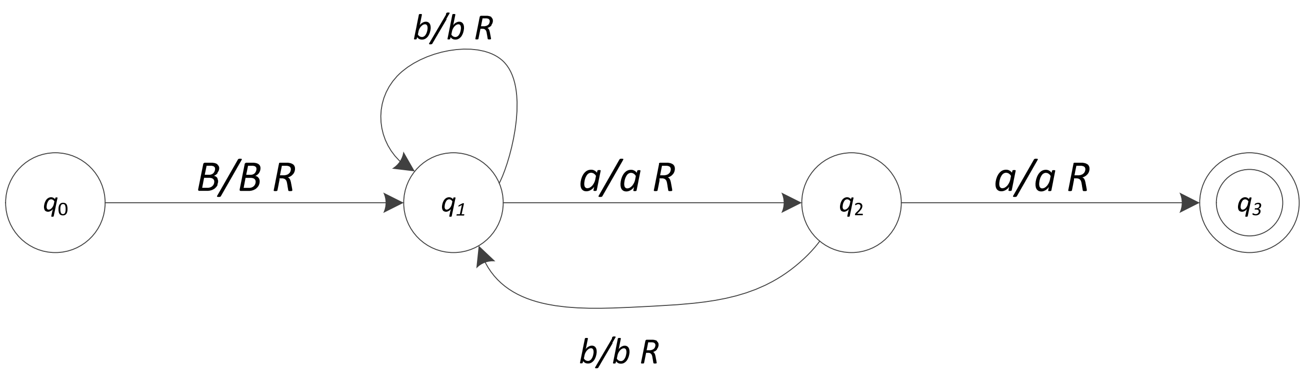
- Definition: Let M = (Q, Σ, Γ, δ, q_0, F) be a Turing machine. A string $u \in \Sigma^*$ is accepted by final state if the computation of M with input u halts in a final state. A computation that terminates abnormally rejects the input. The language of M, L(M), is the set of all languages accepted by M.

Introduction to Turing Machine

- A language accepted by a Turing machine is called a *recursively enumerable language*.
- If the Turing machine halts for all input string of a language, the language is said to be *recursive.*
- The computations of a Turing machine provide a decision procedure for membership in a recursive language.

Introduction to Turing Machine

- Example: The following Turing machine accepts the language (*a* U *b*)**aa*(*a* U *b*)*.



Introduction to Turing Machine

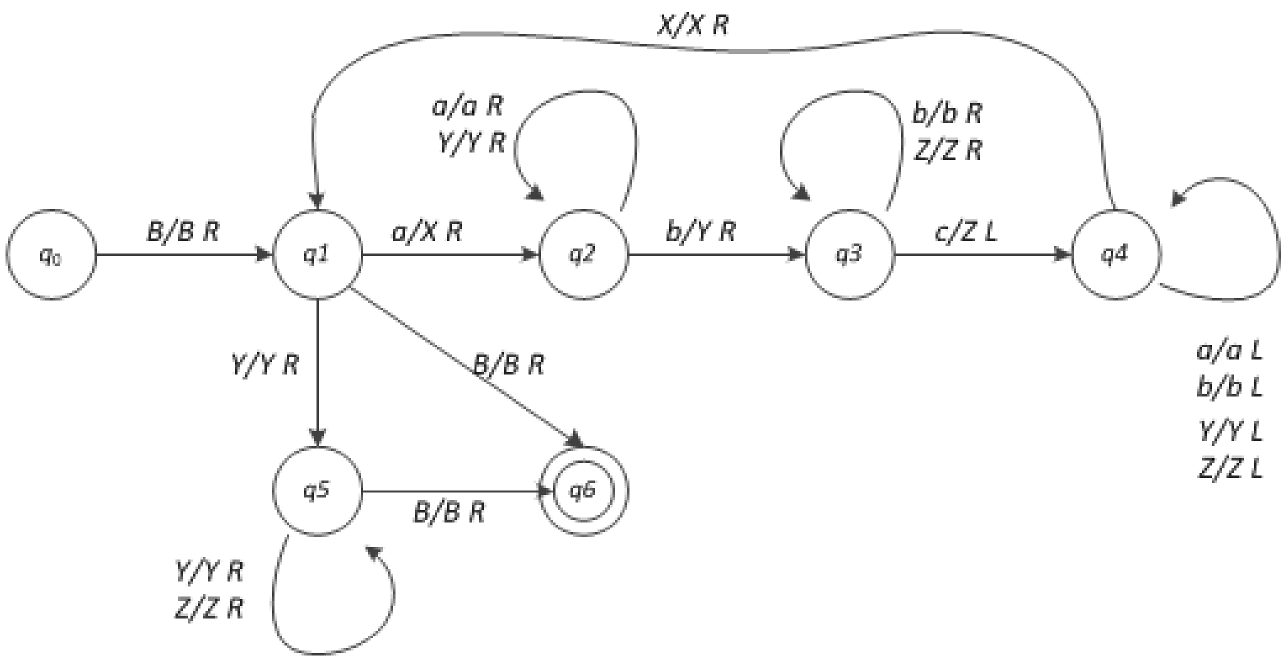
- The computation on the input *aabb* is:

$$
\begin{aligned}
& q_0BaabbB \\
\vdash\ & Bq_1aabbB \\
\vdash\ & Baq_2abbB \\
\vdash\ & Baaq_3bbB
\end{aligned}
$$

- Note that only the first half of the input string is examined before accepting it.

Introduction to Turing Machine

- Exercise: Construct a Turing machine that accepts the language $\{a^i b^i c^i \mid i \geq 0\}$.



Decision Problem

- Decision problem: A decision problem **P** is a set of questions each of which has a yes or no answer.

- Example: A decision problem $\mathbf{P_{SQ}}$: Determine whether an arbitrary number is a perfect square or not. This problem consists of the following questions:

  $\mathbf{p}_0$: Is 0 a perfect square?

  $\mathbf{p}_1$: Is 1 a perfect square?

  …

  Here, $\mathbf{p}_i$ is also called an instance of **P.**

Decision Problem

- A solution to a decision problem is an algorithm that determines the answer to every question $\mathbf{p}_i \in \mathbf{P}$.

- An algorithm that solves a decision problem should be
  - *complete* – it produces an answer, either positive or negative, to each question in the problem domain
  - *mechanistic* – it consists of a finite sequence of instructions each of which can be carried out without requiring insight, ingenuity, or guesswork
  - *deterministic* – when presented with identical input, it always produces the same result.

Decision Problem

- Decision problems:
  - *Unsolvable* (or *undecidable*)
  - *Solvable*:
    - *Tractable*: A decision problem is said to be tractable if there is at least one polynomially bounded algorithm that solves the problem. Such an algorithm is called an *efficient* algorithm.
    - *Intractable*: A decision problem is said to be intractable if there is no polynomially bounded algorithm (or no efficient algorithm) that solves the problem

Decision Problem

- Two examples of unsolvable problems: *the halting problem for Turing machines* and *the post correspondence problem*.
- The halting problem for Turing machines: Given an arbitrary Turing machine M with an input alphabet Σ and a string $w \in \Sigma^*$, will the computation of M with input w halt?

Decision Problem

- Note this problem is different from determining whether a particular Turing machine will halt for a given string.
- This problem requires a general algorithm that answers the halting question for every possible combination of Turing machine and input string.
- Theorem: The halting problem for Turing machines is undecidable.

Decision Problem

- Post correspondence problem: Instead of formally stating the problem, we will illustrate the problem as a simple game of manipulating dominoes.
- A domino consists of two strings from a fixed alphabet, one on the top half of the domino and the other on the bottom.

aba
bbaba

Decision Problem

- We are given a finite set of different types of dominoes.
- We assume that there are an unlimited number of each type of dominoes.
- The game begins when a domino is placed on a table. Another domino is placed to the immediate right of the domino. This process is repeated making a sequence of dominoes on the table.

Decision Problem

- The *top string* is obtained by concatenating the strings in the top halves of the sequence of dominoes.
- The *bottom string* is obtained by concatenating the strings in the bottom halves of the sequence of dominoes.

- The goal of the game (or the solution to a Post correspondence problem) is to come up with a sequence of dominoes where the top string is identical to the bottom string.

Decision Problem

- Example 1. Given the following two dominoes:

aaa	*baa*
aa	*abaaa*

The following sequence of dominoes is a solution:

aaa	*baa*	*aaa*
aa	*abaaa*	*aa*

Decision Problem

- Example 2. Given the following three dominoes:

ab	*bba*	*aba*
aba	*aa*	*bab*

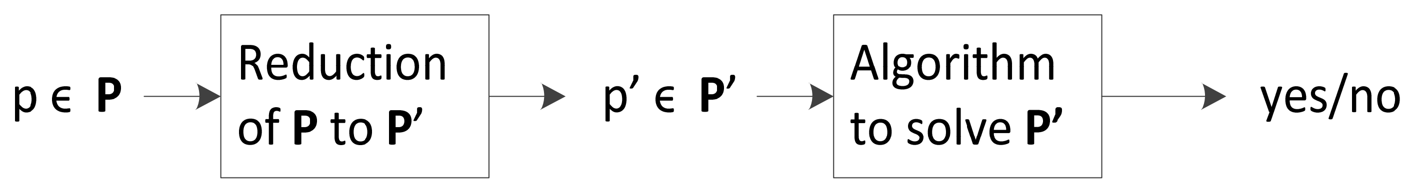
There is no solution.

- Theorem: There is no algorithm that determines whether an arbitrary finite set of dominoes has a solution.

- Since solvable problems are equivalent to recursive languages, *decision problems* and *languages* are used interchangeably.

Reducibility

- A decision problem **P** is Turing reducible to a problem **P'** if there is a Turing machine that takes any problem $\mathbf{p}_i \in$ **P** as input and produces an associated problem $\mathbf{p'}_i \in \mathbf{P'}$ where the answer to the original problem $\mathbf{p}_i$ can be obtained from the answer to $\mathbf{p'}_i$.



P and NP

- A language L is decidable in polynomial time if there is a standard (or deterministic) Turing machine M that accepts L in polynomial time, or $O(n^r)$, where r is a natural number independent of n.

- The family of languages decidable in polynomial time is denoted ***P*****.**

P and NP

- Nondeterministic computation:
  - A deterministic machine solves a decision problem by generating a solution.
  - A nondeterministic machine needs only determine if one of possibilities is a solution.

- A language *L* is said to be accepted in nondeterministic polynomial time if there is a nondeterministic Turing machine that accepts *L* in polynomial time, or $O(n^r)$, where *r* is a natural number independent of *n*.

P and NP

- The family of languages accepted in nondeterministic polynomial time is denoted ***NP*.**

- Another definition: A problem is in ***NP*** if it is “verifiable” in polynomial time.

- What “verifiable” means is that given a possible solution (which is also called ***certificate***) we can verify whether it is a solution or not in polynomial time.

P and NP

- ***P = NP* ?**

- Unsolved question.
- Since every deterministic machine is also nondeterministic, ***P*** $\subseteq$ ***NP*.**
- But it was never proved that ***NP*** $\subseteq$ ***P***. (If this is proved, then that proves ***P = NP***.)

P and NP

- If Q is reducible to L in polynomial time and $L \in$ ***P*****,** then $Q \in$ ***P***.

- A language L is called ***NP-hard*** if for every $Q \in$ ***NP*** Q is reducible to L in polynomial time.
- An ***NP-hard*** language that is also in ***NP*** is called ***NP-complete***.

- If there is an NP-complete language that is also in ***P***, then ***P = NP***.

P and NP

- Two examples of NP-complete problems: Hamiltonian cycle problem and traveling salesman problem.

Hamiltonian Cycle Problem

- A Hamiltonian cycle of an undirected graph $G = (V, E)$ is a simple cycle that contains each vertex in V.
- Note: A cycle is simple if a node, except the first node, is visited only once.
- A graph that contains a Hamiltonian cycle is called "Hamiltonian."
- ***Hamiltonian Cycle Problem***: Does a graph G have a Hamiltonian cycle?

Hamiltonian Cycle Problem

- It can be shown that the Hamiltonian cycle problem can be decidable by a Turing machine in *exponential* time, but not in *polynomial* time. This means Hamiltonian cycle problem is not in ***P***.

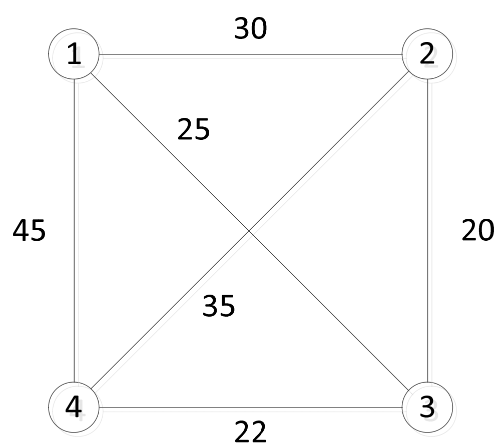
- But, it is decidable in nondeterministic polynomial time.
- Given a cycle in a graph, we can determine whether it is Hamiltonian cycle or not in polynomial time.
- So, Hamiltonian cycle problem is in ***NP***.
- In fact it is an ***NP-complete*** problem.

Traveling Salesman Problem

- Given a complete, non-negative weighted graph, find a Hamiltonian cycle of minimum weight.
- This problem is ***NP-complete***.
- Will briefly discuss three approximate algorithms.

Traveling Salesman Problem

- Consider the following graph:



minimum weight cycle = 1 → 2 → 4 → 3 → 1.

total weight = 30 + 35 + 22 + 25 = 112

Traveling Salesman Problem

- Nearest-neighbor strategy

NEAREST-TSP (G, f) /* f is a cost function, or a weight function */

select an arbitrary vertex s;

$v = s$; $Q = \{v\}$; $S = G.V - Q$; $C = \phi$;

while $S \mathrel{!=} \phi$

select an edge (v, w) of minimum weight, where $w \in S$;

$C = C \cup \{(v, w)\}$;

$Q = Q \cup \{w\}$;

$S = S - \{w\}$;

$v = w$;

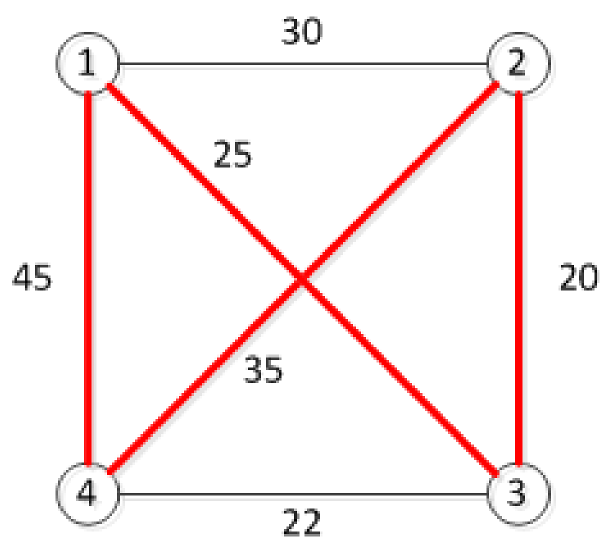
$C = C \cup \{(v, s)\}$;

return C;

Running time: $O(V^2)$

Traveling Salesman Problem

- Nearest-neighbor strategy



Starting at vertex 1: (1, 3), (3, 2), (2, 4), (4, 1)
Total weight = 25 + 20 + 35 + 45 = 125

Traveling Salesman Problem

- Shortest-link strategy

SHORTEST-LINK-TSP (G, f)

    $R = G.E$;

    $C = \phi$;

    while R != ϕ

        choose the shortest edge (v, w) from R;

        $R = R - \{(v, w)\}$;

        if (v, w) does not make a cycle with edges in C and (v, w) would not be the third edge in C incident on v or w

        then

            $C = C + \{(v, w)\}$;

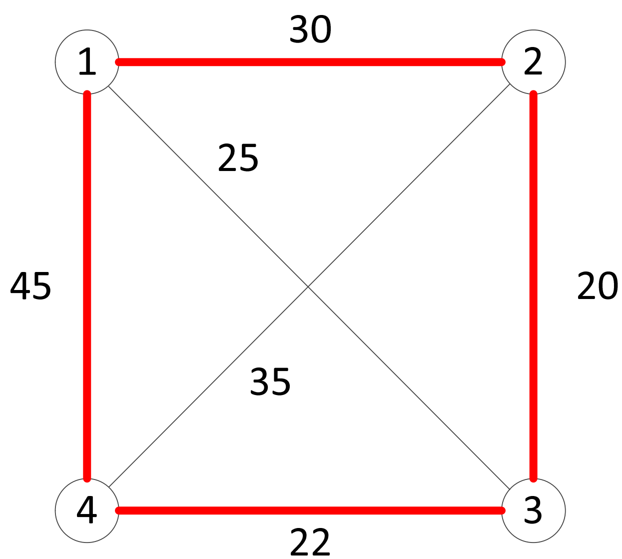
    add the edge connecting the end points of the path in C;

    return C;

Running time: $O(E \log V)$

Traveling Salesman Problem

- Shortest-link strategy



Edges added: (2, 3), (3, 4), (2, 1), (1, 4)

Total weight = 20 + 22 + 30 + 45 = 117

Traveling Salesman Problem

- In general, we cannot establish a bound on how much the weight of an approximate algorithm differ from the weight of a minimum tour.
- If we assume the triangle inequality holds on distances among vertices, we can develop an approximate algorithm that has an upper bound on the weight.
- Triangle inequality:

  $f(u, v) \le f(u, w) + f(w, v)$, for all $u, v, w \in G.V$.

- Euclidean distance has the triangle inequality property.

Traveling Salesman Problem

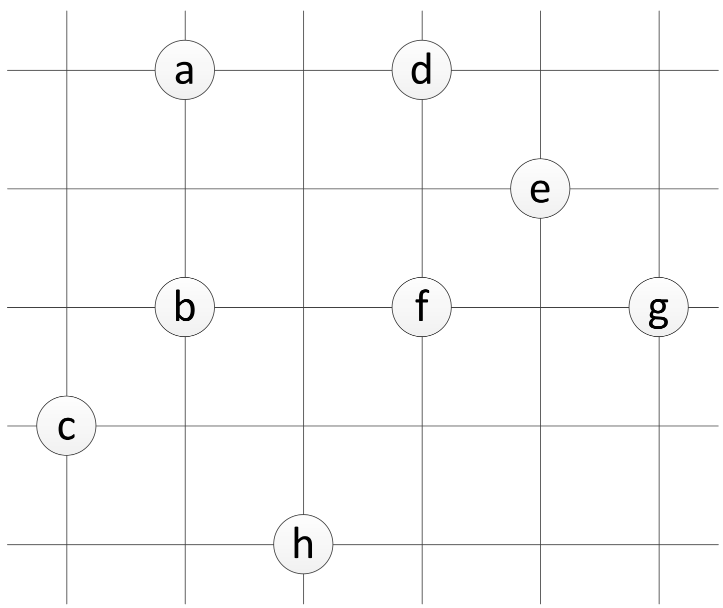
- The following approximate algorithm has an upper bound on the weight: total weight of a cycle is no more than the twice that of the minimum spanning tree's weight

APPROX-TSP-TOUR (G, f)

select a vertex $r \in G.V$ to be the root;
compute MST T from r using MST-PRIM(G, f, r);
let H be a list of vertices, ordered according to when they are
    first visited in a preorder tree walk of T;
return H

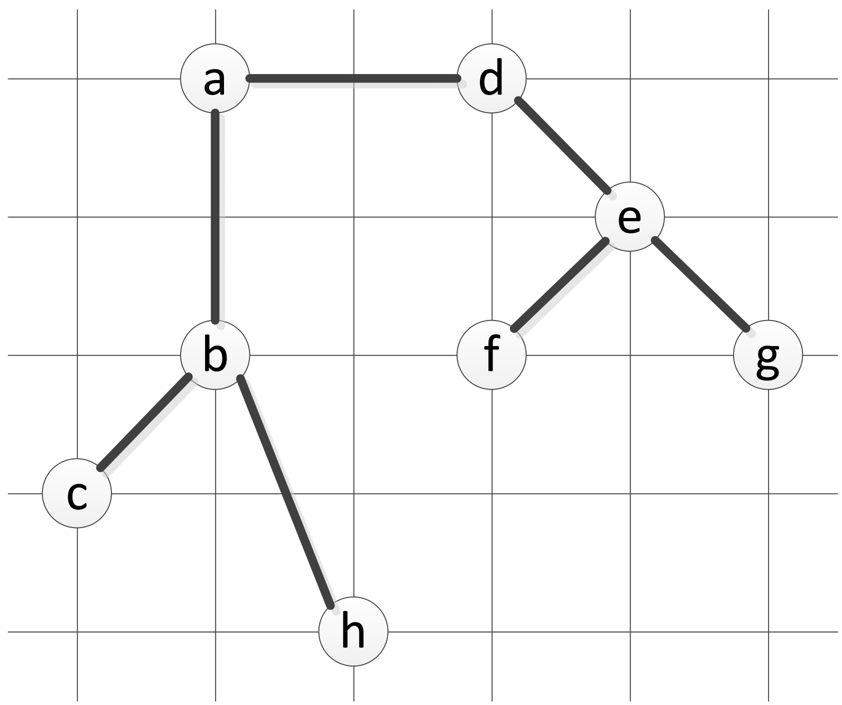
Traveling Salesman Problem

- Example (refer to Figure 35.2): Given the following complete graph (There are edges from each node to all other nodes though edges are not shown in the graph below).



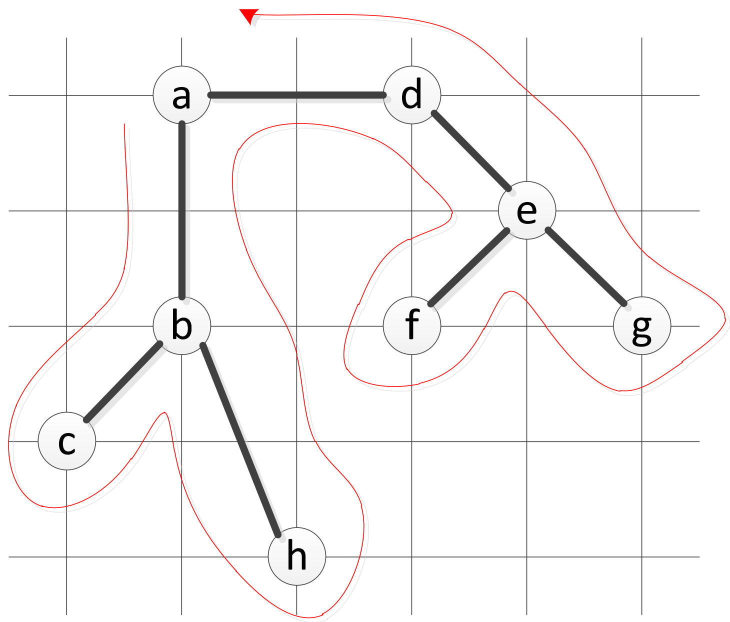
Traveling Salesman Problem

- A minimum spanning tree *T* (*a* is the root)



Traveling Salesman Problem

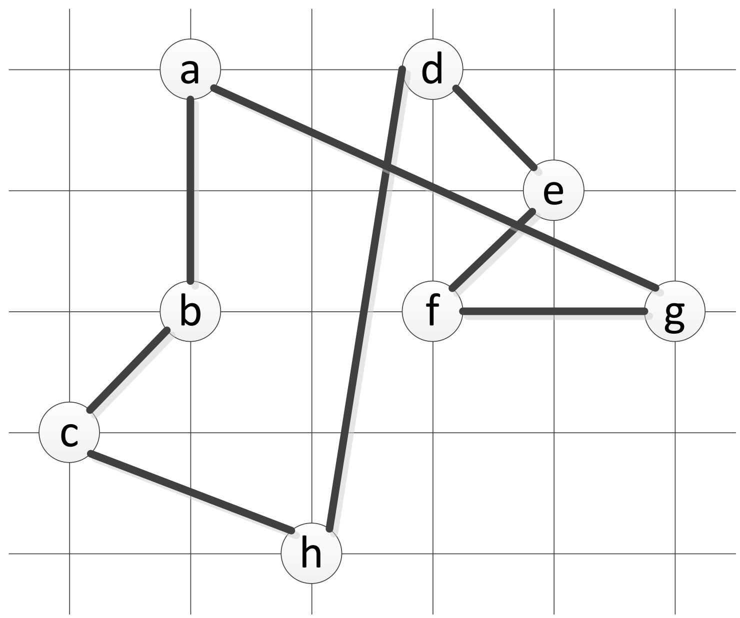
- A minimum spanning tree *T* (*a* is the root)



$$a \to b \to c \to h \to d \to e \to f \to g \to a$$

Traveling Salesman Problem

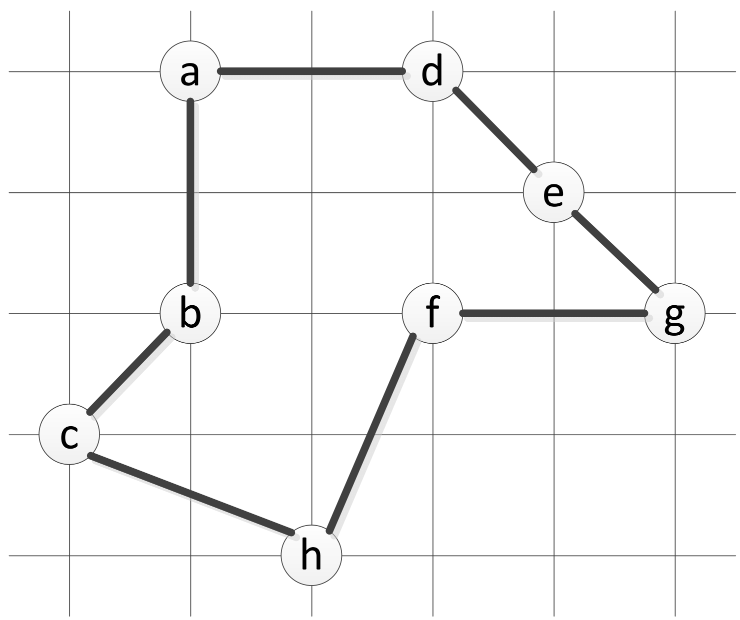
- *H* returned by APPROX-TSP-TOUR is



total weight = approx. 19.074

Traveling Salesman Problem

- An optimal tour (or Hamiltonian cycle with minimum weight)



total weight = approx. 14.715

Reference

- T.A. Sudkamp, “Languages and Machines,” 1988, Addison Wesley.
- T.H. Cormen, C.E. Leiserson, R.L. Rivest, C. Stein, “Introduction to Algorithms,” 3rd Ed., 2009, MIT Press.