The 1D Wave Equation

Aidan Higgins - 40366273

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CONTENTS

Abstract

In this report, the wave equation is solved numerically and analytically for homogeneous Dirichlet boundary conditions

1 INTRODUCTION

Waves play a vital role in a wide range of fields across maths and physics. The form a wave takes is defined by the wave equation which, in 1D, is a partial differential equation (PDE) given by:

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \frac{\partial^2 u(x,t)}{\partial t^2} \tag{1.0.1}$$

Where c is the speed of the wave and $u(x,t)$ is the amplitude at position x and time t. In this report, this equation is solved analytically, using separation of variables and D'Alembert's formula, and numerically, using finite difference, for $c = 2$, with homogeneous Dirichlet boundary conditions:

$$u(0,t) = u(L = 2\pi, t) = 0 \tag{1.0.2}$$

and the following initial conditions:

$$f(x) = u(x,0) = \begin{cases} 1 & |x-\pi| \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.0.3}$$

$$g(x) = u_t(x,0) = \sin\left(\frac{x}{2}\right) \tag{1.0.4}$$

2 Analytical

2.1 Separation of Variables

The goal of the separation of variables method is to split the PDE into multiple ordinary differential equations (ODEs), which are easier to solve. The first step is to assume the solution takes the form:

$$u(x,t) = \xi(x)\eta(t). \tag{2.1.1}$$

Subbing this into eq.(1.0.1) and dividing by $u(x,t)$ gives:

$$\frac{1}{\eta}\frac{\partial^2\eta}{\partial t^2} = \frac{4}{\xi}\frac{\partial^2\xi}{\partial x^2} = \lambda \tag{2.1.2}$$

Where λ is a constant that arises due to x and t being independent of each other. As the LHS only depends on t and the RHS only depends on x, both sides must be constant (if the LHS did change with t, then the RHS would have to too, contradicting the fact x is independent of t). This can be written in the form of two second order ODEs:

$$\frac{\partial^2\xi}{\partial x^2} = \lambda\xi \tag{2.1.3}$$

$$\frac{\partial^2\eta}{\partial t^2} = 4\lambda\eta \tag{2.1.4}$$

Starting with Eq.(2.1.3), the λ can either be positive, negative or zero. Starting with positive, the ODE has the general solution:

$$\xi(x) = A\cos(\omega x) + Bsin(\omega x) \tag{2.1.5}$$

Where the substitution $\lambda = +\omega^2$ was made for convenience. To find A and B, the boundary conditions are subbed in. Starting with $u(0,t) = 0$ (and therefore $\xi(0) = 0$), $\sin(0) = 0$ and $\cos(0) = 1$, and so $A = 0$. For $\xi(2\pi) = 0$:

$$B\sin(2\pi\omega) = 0 \tag{2.1.6}$$

$B = 0$ is trivial, so assuming $B \neq 0$:

$$\begin{aligned} \sin(2\pi\omega) &= 0 \\ 2\pi\omega &= n\pi \qquad\qquad \text{For n=1,2,3...} \\ \omega &= \frac{n}{2} \end{aligned}$$

Therefore:

$$\xi(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{nx}{2}) \tag{2.1.7}$$

For negative λ, the solution has the general form

$$\xi(x) = A\cosh(\omega x) + B\sinh(\omega x) \tag{2.1.8}$$

subbing in the boundary conditions

$$\begin{aligned} \xi(0) &= A = 0 \\ \Rightarrow \xi(x) &= B\sinh(\omega x), \qquad (2.1.9) \\ \xi(L) &= B\sinh(2\pi\omega) = 0 \end{aligned}$$

(2.1.10)

Since sinh is only zero at the origin, there is no non-zero ω that satisfies the above. Therefore, λ is not negative. Finally, for $\lambda = 0$ has general solution

$$\xi(x) = Ax + B \tag{2.1.11}$$

Subbing in eq.(1.0.2) gives

$$\begin{aligned} \xi(0) &= B = 0 \\ \xi(L) &= 2\pi A = 0 \Rightarrow A = 0 \end{aligned}$$

Therefore, there are only positive values of λ

Returning to Eq.(2.1.4), since this ODE is of the same form, $\eta(t)$ must also have the general solution Eq.(2.1.5):

$$\eta(t) = \sum_{n=1}^{\infty} A_n \cos(nt) + B_n \sin(nt) \tag{2.1.12}$$

and therefore:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos(nt) + B_n \sin(nt)) \sin(\frac{nx}{2}) \tag{2.1.13}$$

Where $\omega = \frac{n}{2}$ was used and the constant from Eq.(2.1.7) is taken into A_n and B_n above. To find A_n and B_n, the initial conditions are used. Starting with Eq.(1.0.3), subbing in $t = 0$ condition in gives:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{nx}{2}) \tag{2.1.14}$$

The initial condition is a sine Fourier transform of the constant A_n. To solve for A_n, both sides are multiplied by $\sin(\frac{mx}{2})$ and integrated w.r.t x:

$$\int_0^{2\pi} \sin(\frac{mx}{2})u(x,0)dx = \int_0^{2\pi} \sin(\frac{mx}{2})\sum_{n=1}^{\infty} A_n \sin(\frac{nx}{2}) \tag{2.1.15}$$

This is done to take advantage of the orthogonality of sine:

$$\int_0^{L} \sin(\frac{mx}{2})\sin(\frac{nx}{2}) = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n \end{cases} \tag{2.1.16}$$

Therefore, the only n that survives on the RHS of Eq.(2.1.15) is when $m = n$, leaving:

$$\pi A_m = \int_0^{2\pi} \sin(\frac{mx}{2})u(x,0)dx \tag{2.1.17}$$

This is simplified further by subbing in $u(x,0)$. Rearranging Eq.(1.0.3):

$$u(x,0) = \begin{cases} 1 & \pi - 2 \leq x \leq \pi + 2 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.18}$$

And so, A_m is given by the integral:

$$A_m = \frac{1}{\pi}\int_{\pi-2}^{\pi+2} \sin(\frac{mx}{2})dx \tag{2.1.19}$$

Which, when evaluated and simplified, gives:

$$A_m = \frac{4}{m\pi}\sin(n)\sin(\frac{n\pi}{2}) \tag{2.1.20}$$

For B_m, other initial condition is used. To use this, the time derivative of Eq.(2.1.13) taken

$$u(x,t) = \sum_{n=1}^{\infty}(nB_n\cos(nt) - nA_n\sin(nt))\sin(\frac{nx}{2}) \tag{2.1.21}$$

Using Eq.(1.0.4)

$$\sin(\frac{x}{2}) = \sum_{n=1}^{\infty} nB_n \sin(\frac{nx}{2}) \tag{2.1.22}$$

Using the same trick as before, both sides are multiplied by $\sin(\frac{mx}{2})$ and integrated over the length

$$\int_0^{2\pi} \sin(\frac{mx}{2})\sin(\frac{x}{2})dx = \int_0^{2\pi} \sin(\frac{mx}{2})\sum_{n=1}^{\infty} nB_n \sin(\frac{nx}{2}) \tag{2.1.23}$$

On the RHS, the only $m = n$ survives and on the LHS, the only $m = 1$ survives. Using Eq.(2.1.16) on the LHS gives the following:

$$B_m = \begin{cases} 1 & m = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.24}$$

Subbing this and Eq.(2.1.20) into Eq.(2.1.13) gives the final solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{nx}{2})(\frac{4}{n\pi}\sin(n)\sin(\frac{n\pi}{2})\cos(nt) + \delta_{n1}\sin(t)) \tag{2.1.25}$$

Where δ_{nm} is the Kronecker delta. The only terms above at risk of diverging are $\sin(\frac{n\pi}{2})$ and $\frac{1}{n}$. However, between 0 and 2π, it can be shown that as $N \to \infty$

$$\sum_{n=1}^{N} \frac{\sin(nx/2)}{n} = \frac{2\pi - x}{2} \tag{2.1.26}$$

and so the solution will remain stable for all x and t.

2.2 D'Alembert's Formula

It can be shown by the change of variable

$$\xi = x + ct \tag{2.2.1}$$

$$\eta = x - ct \tag{2.2.2}$$

Eq.(1.0.1) has the general formula

$$u(x,t) = F(x - ct) + G(x + ct) \tag{2.2.3}$$

This solution represents a wave F traveling in the positive x direction and a wave G traveling in the negative x direction. The derivative of this solution is

$$u_t(x,t) = -cF'(x - ct) + G'(x + ct) \tag{2.2.4}$$

Setting $t = 0$ equations for the general form of the initial conditions can be found

$$f(x) = F(x) + G(x) \tag{2.2.5}$$

$$g(x) = -cF'(x) + cG'(x) \tag{2.2.6}$$

Integrating both sides of Eq.(2.2.6) gives

$$-cF(x) + cG(x) = \int_0^x g(\zeta)d\zeta \tag{2.2.7}$$

Combining Eq.(2.2.5) with above produces formulas for $F(x)$ and $G(x)$.

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\zeta)d\zeta \tag{2.2.8}$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\zeta)d\zeta \tag{2.2.9}$$

Finally, subbing this into Eq.(1.0.1) gives D'Alembert's formula

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)d\zeta \tag{2.2.10}$$

This is much easier to use than the separation of variables above. Computing the integral above for Eq.(1.0.4) yields the term on the right

$$\sin(\frac{x}{2})\sin(t) \tag{2.2.11}$$

The $f(x-ct)$ and $f(x+ct)$ are found by simply subbing their respective arguments and rearranging for x

$$f(x-ct) = \begin{cases} 1 & \pi + 2(t-1) < x < \pi + 2(t+1), \\ 0 & \text{otherwise} \end{cases} \tag{2.2.12}$$

$$f(x+ct) = \begin{cases} 1 & \pi - 2(t+1) < x < \pi + 2(-t+1), \\ 0 & \text{otherwise} \end{cases} \tag{2.2.13}$$

However, at no point has the boundary condition been considered in this solution - D'Alembert's formula only solves the wave equation for boundaries at $\pm\infty$. $f(x-ct)$ and $f(x+ct)$ are rectangular pulses traveling to the right and left respectively and so, without boundaries, the pulses will leave the domain of the system. This seems like it would spell the end for this method in this example, but it can be worked around by making $f(x)$ odd and periodic, as seen in Fig.1 For $f(x)$ periodic over an infinite length,

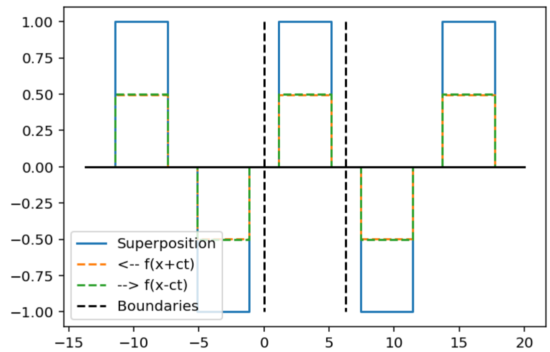


Figure 1: Caption

the solution will be exact. For finite length, the solution is exact until part of the pulse that starts furthest away from the region of interest has went past both boundaries. For example, for the system shown in Fig.1 the pulse given by

$$f(x-ct) = \begin{cases} 1 & 5\pi - 2(t+1) < x < 5\pi + 2(-t+1), \\ 0 & \text{otherwise} \end{cases} \tag{2.2.14}$$

will reach $x = 0$ in $\frac{5\pi-2}{2}$ units of time, since the leftmost part of the pulse is a distance of $5\pi - 2$ away from $x = 0$ and the pulse has a speed of 2 to the left(since the coefficient in front of t is -2)

3 Numerical

3.1 Deriving the Finite Difference Method

To solve the PDE numerically, x, t and u have to be discretised - this can be represented as a matrix with elements u_{ij}, where i is the jth step in space and j is the jth step in time. One way is to derive a formula from the finite difference forms of u_{tt} and u_{xx} the move a state of the system forward in time.
The first derivative of t is given by the 3 point central difference:

$$u_{tt}(i,j) = \frac{u(i,j+1) - 2u(i,j) + u(i,j-1)}{(\Delta t)^2} \tag{3.1.1}$$

Likewise for the second derivative of x:

$$u_{xx}(i,j) = \frac{u(i+1,j) - 2u(i,j) + u(i-1,j)}{(\Delta x)^2} \tag{3.1.2}$$

Subbing these into Eq.(1.0.1) and rearranging for $u(i,j+1)$ gives

$$u_{i,j+1} = \alpha(u_{i+1,j} + u_{i-1,j}) - 2(\alpha - 1)u_{i,j} - u_{i,j-1} \tag{3.1.3}$$

where $\alpha = 4(\frac{\Delta x}{\Delta t})^2$. Knowledge of the previous two time steps is needed for this formula, for $u_{i,j+1}$, $u_{i,j-1}$ is needed. This causes problems on the second time step as there as there is no step before the initial condition Eq.(1.0.3). To account for this, the step from $u_{i,1}$ to $u_{i,2}$ is found using Eq.(1.0.4) with the two point forward difference for $\frac{\partial u}{\partial t}$

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \sin(\frac{x}{2})$$

'

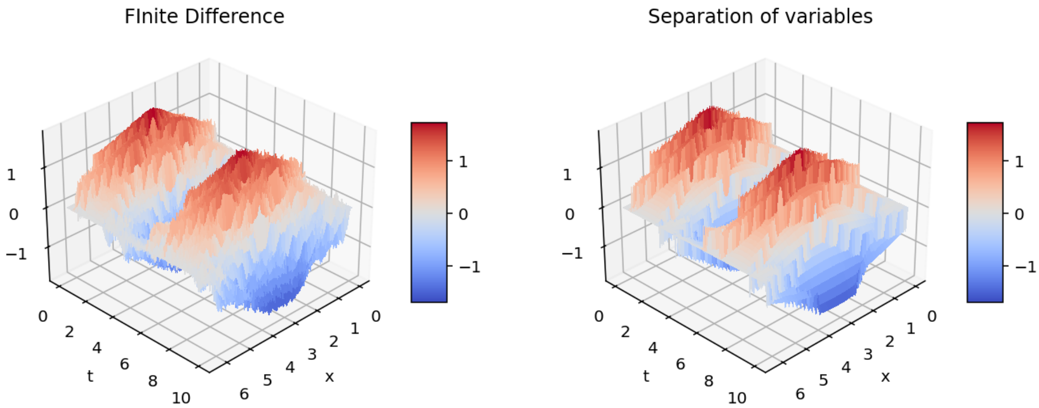
$$u_{i,2} = u_{i,1} + \Delta t \sin(\frac{x}{2}) \tag{3.1.4}$$

The Eqs.(3.1.3) and (3.1.4) are now enough to generate solutions to the wave equation. Practically, this can be done by updating the elements of a matrix according to these equations.

$$\begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & u_{i,j-1} & \cdots & \cdots \\ \cdots & u_{i-1,j} & u_{i,j} & u_{i+1,j} & \cdots \\ \cdots & \cdots & u_{i,j+1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

4 Results

Figs.2 and 3 show the 3D plots for all the methods discussed. The solutions take the form of the fundamental mode with two pulses of magnitude 0.5 oscillating between the



(a) Finite Difference

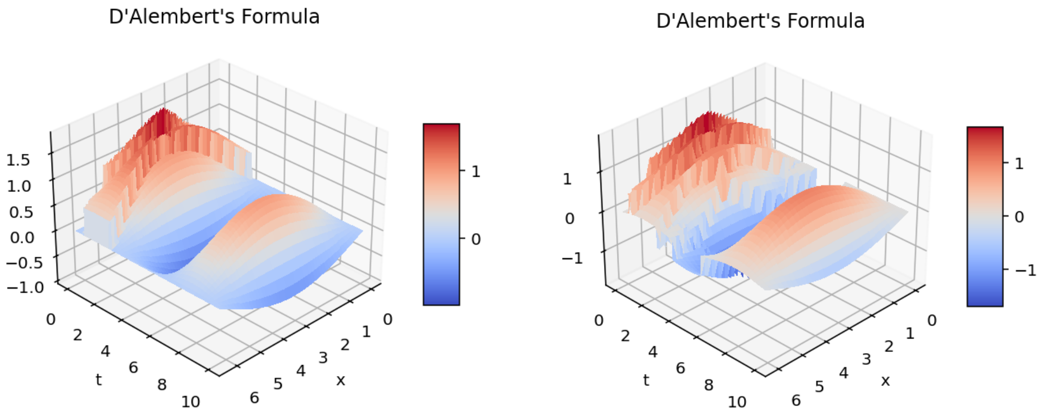
(b) Separation of variables

Figure 2: 3D plots of the finite difference and separation of variables solutions for the wave equation. Both are for the range $0 \leq x \leq 2\pi$ and the first 10 units of time. The sum in the separation of variables solution was done for the first 100 n

boundaries. When part of the pulse hits the boundary, it reflects and when the edges of a pulse cross over itself they change sign.
Figs. 3a and 3b show D'Alembert's solution for pulses centred at π and for pulses centred at $-\pi, \pi$ and 3π respectively. As predicted by the analytical solution, once the pulse hits the boundary, it will leave the system rather than reflect. Adding twos region on either side of the boundaries with negative pulses prolonged the solutions accuracy, lasting for $\frac{\pi-2}{2}$ and $\frac{3\pi-2}{2}$ units of time respectively. Once the pulses have left the boundaries, the solution becomes entirely defined by $\sin(\frac{x}{2})$.
Each of the solutions are plotted against each other in Fig.4 at 0.3 units of time, equating to 300 time steps. The numerical solution is much noisier than the analytical methods. The noise can be reduced by decreasing the size of Δx and therefore increasing the number of points along the length of the system. A GIF showing how each oth the solutions evolve with time cam be found on GitHub [1]

5 Conclusion

This report details two analytical and one numerical solution to a 1D wave equation, namely separation of variables, D'Alembert's solution and finite difference, producing 3D plots of each. The problem could alternatively be solved analytically with Sturm-Liouville theory, or by using canonical forms. Numerically, Crank-Nicholson could be applied to increase stability or a finite element method.



(a) Pulse at π

(b) Pulses at $-\pi, \pi, 3\pi$

Figure 3: 3D plots of the the D'Alembert solution to the wave equation. 2a is for a positive pulse centred at π. 2b is for a positive pulses centred at π and negative pulses at $-\pi$ and 3π. Both are for the range $0 \leq x \leq 2\pi$ and the first 10 units of time. The solutions are correct for the first $\frac{\pi-2}{2}$ and $\frac{3\pi-2}{2}$ units of time respectively.

References

[1] *Project Github*
URL: https://github.com/AidanHiggins461/PDE_Project/tree/main (accessed 30/11/2025)

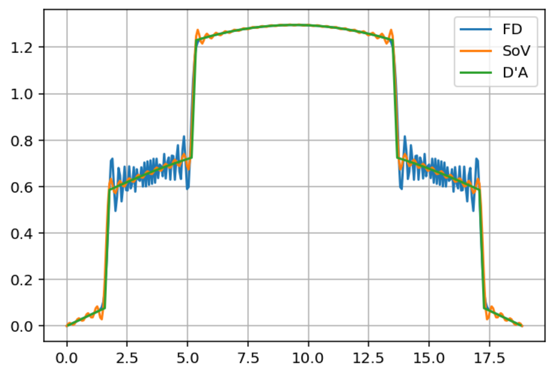


Figure 4: Comparison of 3 methods of solving the wave equation after 0.3 units of time. FD is finite difference, SoV is separation of variables and D'A is D'Alembert.