Some Probability

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Let X be a random variable that is -1 with probability $1-p$ and 1 with probability p. However, Hoeffding inequalities are in terms of some random variable H where H is 0 with probability $1-p$ and 1 with probability p. If $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} H_1 =$ the sample proportion, we have for $\epsilon > 0$

$$P(\hat{p} \leq p - \epsilon) \leq e^{-2\epsilon^2 n}$$

$$P(\hat{p} \geq p + \epsilon) \leq e^{-2\epsilon^2 n}$$

Then, for each of these, we can multiply both inequalities by 2 and and subtract 1 to get

$$P(2\hat{p} - 1 \leq 2p - 1 - 2\epsilon) \leq e^{-2\epsilon^2 n}$$

$$P(2\hat{p} - 1 \geq 2p - 1 + 2\epsilon) \leq e^{-2\epsilon^2 n}$$

But, $2\hat{p} - 1$ is just the random variable $\bar{X}$, the mean of a sample of size n, and $2p-1$ is just $\mathbb{E}[X]$. Then, we have that

$$P(\bar{X} \leq \mathbb{E}[X] - \epsilon) \leq e^{-1/2\epsilon^2 n}$$

$$P(\bar{X} \geq \mathbb{E}[X] + \epsilon) \leq e^{-1/2\epsilon^2 n}$$

Then, we can combine these. Consider the probability of at least one of these events happening. Then, by the union bound this is

$$P(\bar{X} \leq \mathbb{E}[X] - \epsilon) \cup P(\bar{X} \geq \mathbb{E}[X] + \epsilon) \leq P(\bar{X} \leq \mathbb{E}[X] - \epsilon) + P(\bar{X} \geq \mathbb{E}[X] + \epsilon) \leq 2e^{-1/2\epsilon^2 n}.$$

We can write this as one statement as $\bar{X} \leq \mathbb{E}[X] - \epsilon$ or $\bar{X} \geq \mathbb{E}[X] + \epsilon$, which is equivalent to $\bar{X} - \mathbb{E}[X] \leq -\epsilon$ or $\bar{X} - \mathbb{E}[X] \geq \epsilon$, which is the same as $|\bar{X} - \mathbb{E}[X]| \geq \epsilon$. Then, we have that

$$P(|\bar{X} - \mathbb{E}[X]| \geq \epsilon) \leq 2e^{-1/2\epsilon^2 n}.$$

Calculating weights stochastically in the l_0 case is the same as trying find the sign of $\mathbb{E}[X]$.