A linear equation in $n > 1$ variables $x_1, x_2, x_3, \ldots, x_n$ has the form
$(*)$ $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$

here a_i is a constant for any $1 \leq i \leq n$ and b is a constant

a_i are the coefficients of $(*)$
x_i are the variables
a_1 is the leading coefficient
x_1 is the leading variable

Example: Linear Equations
a) $2x - 3y = 1$ (2 variables)
$\rightarrow$ represents a line of the xy plane
b) $x + y - z = 2$ (3 variables)
$\rightarrow$ represents a plane in the xyz coordinate system

Example: Non-Linear Equations
1) $\boxed{x^2} + y + z = 0 \leftarrow$ non-linear function x
2) $\boxed{xy} + z = -1 \leftarrow$ x and y are not independent of eachother
3) $\boxed{\cos x} + y + z = 3 \leftarrow$ non-linear function of x

<u>Definition (Solution and Solution Set of $(*)$):</u>

A solution of $(*)$ is a sequence of n real numbers $s_1, s_2, s_3, \ldots, s_n$ that satisfy $(*)$

The set of solutions of $(*)$ is called a solution set

Remark: The solution of $(*)$ is a linear equation in n variables does have a parametric representation.

Example: Find a parametric representation of each linear equation

a) $\boxed{1}$ $2x_1 - 3x_2 = 1$ (a linear equation in $n = 2$ represents a line in $\mathbb{R}^2$)
let $x_2 = t$, here $t \in \mathbb{R}$

$$\begin{aligned} \text{then } \boxed{1} &\Leftrightarrow 2x_1 - 3t = 1 \\ &\Leftrightarrow 2x_1 = 1 + 3t \\ &\Leftrightarrow x_1 = \frac{1}{2} + \frac{3}{2}t \end{aligned}$$

Then the solution set of $\boxed{1}$ is

$$\left\{(x_1, x_2)|x_1 = \frac{1}{2} + \frac{3}{2}t \text{ and } x_2 = t,\, t \in \mathbb{R}\right\}$$
$$\boxed{= \left\{\left(\frac{1}{2} + \frac{3}{2}t, t\right)|t \in \mathbb{R}\right\}} \text{ A paramaterization of the solution set}$$

b) $\boxed{2}$ $x_1 + 2x_2 - x_3 = 4$

Let $x_3 = t$, $t \in \mathbb{R}$ is a parameter
Let $x_2 = s$, $s \in \mathbb{R}$ is a parameter

Then

$$\begin{aligned} &\Leftrightarrow x_2 + 2s - t = 4 \\ &\Leftrightarrow x_2 = -2s + t + 4,\, t \in \mathbb{R} \text{ and } s \in \mathbb{R} \end{aligned}$$

Then the solution set of $\boxed{2}$ is

$$\{(x_1, x_2, x_3)|\, x_2 = -2s + t + 4,\, x_2 = s,\, x_3 = t,\, s \in \mathbb{R} \text{ and } t \in \mathbb{R}\}$$
$$\boxed{= \{(-2s + t + 4,\, s,\, t)|\, s \in \mathbb{R} \text{ and } t \in \mathbb{R}\}}$$

Definition (System of linear equations in $n \geq 1$ variables):

A system of $m \geq 1$ linear equation in $n \geq 1$ variables is a set of m equations, each of which is linear in the same n variables

(∗)

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

A solution of (∗) is a sequence of numbers $s_1, s_2, s_3, \ldots, s_n$ that is a solution of each equation of the system.

Remark (On the number of solutions of a system (∗) of linear equations):

For a given system of linear equations (∗) Precisely one of the following statements is true:

1) The system (∗) has exactly one solution (system is consistent)
2) The system (∗) has infinitely many solutions (system is consistent)
3) The system (∗) has no solutions (system is inconsistent)

Examples: Different Cases for Systems of Linear Equations

a)

$$(*) \quad \begin{cases} x + y = 0 & \text{①} \\ x - y = 1 & \text{②} \end{cases}$$

(∗) is a system of $m = 2$ linear equations in $n = 2$ variables
(∗) has one solution $(\frac{1}{2}, -\frac{1}{2})$ and the system is consistent

b)

$$(*) \quad \begin{cases} x + y = 1 & \text{①} \\ 2x + 2y = 2 & \text{②} \end{cases}$$

(∗) is a system of $m = 2$ linear equations and $n = 2$ variables
Note that ① and ② imply that: 2=2 which implies that (∗) has infinite solutions.

then $(*) \Leftrightarrow x+y=1$ hence the solution is

$$\begin{aligned}\{(x,y)\,|\,x+y=1\} &= \{(x,y)\,|\,y=1-x\}\\ &= \{(x,\,1-x)|\,x\in\mathbb{R}\}\end{aligned}$$

c)

$$(*)\quad \begin{cases} x+y=-1 & \text{①} \\ x+y=1 & \text{②} \end{cases}$$

$(*)$ is a system of $m=2$ linear equations and $n=2$ variables
Note that ① and ② imply that: $1=-1$ which is impossible, then $(*)$ has no solutions. The system is inconsistent.

Definition (system in a row-echelon form):

A system of linear equations is in row-echelon form when it has a stair-step pattern with leading coefficients of 1. To solve is we use a back substitution.

Examples: Row-Echelon Form

a)

$$(*)\quad \begin{cases} x-2y+3z=9 & \text{①} \\ \qquad\;\; y+3z=5 & \text{②} \\ \qquad\qquad\;\; z=2 & \text{③} \end{cases}$$

$(*)$ is in row-echelon form, it has a stair-step pattern and the leading coefficients are 1.

b)

$$(*)\quad \begin{cases} x+2y=1 & \text{①} \\ \qquad 3y=5 & \text{②} \end{cases}$$

$(*)$ has a stair-step pattern, but the leading coefficient of the second row isn't 1. Therefore, $(*)$ is not in row-echelon form.

c)

$$(*) \quad \begin{cases} x - y + z = 1 & \text{①} \\ \phantom{x -{}} y - z = 0 & \text{②} \\ \phantom{x -{}} y + 6z = 3 & \text{③} \end{cases}$$

The leading coefficients of $(*)$ are 1. $(*)$ does not have a stair-step pattern, therefore $(*)$ is not in row-echelon form.

Definition (equivalence of systems of linear equations:)

Two systems of linear equations are equivalent when they have the same solution set (the systems must have the same number of linear equations and the same variables).

Example: System Equivalency

$$(*) \quad \boxed{\begin{cases} 2x + 2y = 2 \\ x - y = 1 \end{cases}} \quad \text{①}$$

$$\Leftrightarrow \quad \boxed{\begin{cases} x + y = 1 \\ x - y = 1 \end{cases}} \quad \text{②}$$

$$\Leftrightarrow \quad \boxed{\begin{cases} x = 1 \\ y = 0 \end{cases}} \quad \text{③}$$

The solution set is $\{(1, 0)\}$ hence the systems ①, ②, and ③ are equivalent as they have the same solution set.

Remark: To solve a system that is not in a row-echelon form we write it as an equivalent system in row-echelon form.

Question: What are the possible operations that we can apply to the equations of a system $(*)$ to obtain an equivalent system?
Answer: The following are operations that we can apply to the equations in $(*)$ to obtain an equivalent system:
1) Interchanging the rows
2) Multiplying an equation by a non-zero constant
3) Adding a multiply of an equation to another equation

Remark: Rewriting a system of linear equations in a row-echelon form involves a chain of equivalent systems using operations 1), 2), and 3): the process is called a Gaussian Elimination.

Examples: Gaussian Elimination

a)

$$(*) \quad \begin{cases} x - 2y + 3z = 9 & (1) \\ -x + 3y \quad\quad = -4 & (2) \\ 2x - 5y + 5z = 17 & (3) \end{cases}$$

$(*)$ is a system of $m = 3$ linear equations in $n = 3$ variables.
To solve $(*)$ we apply a Gaussian Elimination:

$$(*) \Leftrightarrow \begin{cases} x - 2y + 3z = 9 & (1) \\ -x + 3y \quad\quad = -4 & (2) \\ \quad - \ y - \ z = -1 & (-2)(1) + (3) \to (3)' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & (1) \\ \quad\quad y + 3z = 5 & (1) + (2) \to (2)' \\ \quad - \ y - \ z = -1 & (3)' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & (1) \\ \quad\quad y + 3z = 5 & (2)' \\ \quad\quad\quad 2z = 4 & (2)' + (3)' \to (3)'' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & (1) \\ \quad\quad y + 3z = 5 & (2)' \\ \quad\quad\quad z = 2 & \frac{1}{2}(3)'' \to (3)''' \end{cases}$$

The system is in a row-echelon form, to solve it we use a back substitution, substite $\boxed{z = 2}$ in $(2)'$ yields:

$$\begin{aligned} y + 3(2) &= 5 \\ \Leftrightarrow \boxed{y = -1} \end{aligned}$$

substituting $y = -1$ and $z = 2$ in ① yields:

$$\begin{aligned} x - 2(-1) + 3(2) &= 9 \\ \Leftrightarrow x + 2 + 6 &= 9 \\ \implies \boxed{x = 1} \end{aligned}$$

The system is consistent, it has a unique solution and the solution set is $\{(1, -1, 2)\}$

b)

$$(*) \quad \begin{cases} x - 3y + z = 1 & ① \\ 2x - y - 2z = 2 & ② \\ x + 2y - 3z = -1 & ③ \end{cases}$$

$(*)$ is a system of $m = 3$ linear equations in $n = 3$ variables.
To solve $(*)$ we apply a Gaussian Elimination:

$$(*) \Leftrightarrow \begin{cases} x - 3y + z = 1 & ① \\ 2x - y - 2z = 2 & ② \\ 5y - 4z = -2 & (-1)① + ③ \to ③'' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 3y + z = 1 & ① \\ 5y - 4z = 0 & (-2)① + ② \to ②' \\ 5y - 4z = -2 & ③'' \end{cases}$$

The system is inconsistent and has no solutions because of the equations the bottom two equations $5y - 4z = 0$ and $5y - 4z = -2$ imply that $0 = -2$, which is impossible.

c)

$$(*) \quad \begin{cases} \quad\quad y - \; z = 0 & (1) \\ x \quad\quad - 3z = -1 & (2) \\ -x + 3y \quad\quad = 1 & (3) \end{cases}$$

$(*)$ is a system of $m = 3$ linear equations in $n = 3$ variables.
To solve $(*)$ we apply a Gaussian Elimination:

$$(*) \Leftrightarrow \begin{cases} x \quad\quad - 3z = -1 & (2) \\ \quad\quad y - \; z = 0 & (1) \\ -x + 3y \quad\quad = 1 & (3) \end{cases}$$

$$\Leftrightarrow \begin{cases} x \quad\quad - 3z = -1 & (2) \\ \quad\quad y - \; z = 0 & (1) \\ \quad\quad 3y - 3z = 0 & (2) + (3) \to (3)' \end{cases}$$

$$\Leftrightarrow \begin{cases} x \quad\quad - 3z = -1 & (2) \\ \quad\quad y - \; z = 0 & (1) \end{cases}$$

Let $z = t$
$(1) \implies y = t$
$(2) \implies x = -1 + 3t$
then

$$\begin{cases} x = -1 + 3t \\ y = t \\ z = t \end{cases} , \; t \in \mathbb{R}$$

Hence $(*)$ is a consistent system, it has infinitely many solutions. The solution set (on a paramaterized form) is:

$$\begin{aligned} &\{(x,\, y,\, z) |\, x = -1 + 3t,\, y = t,\, z = t,\, t \in \mathbb{R}\} \\ &= \{(-1 + 3t,\, t,\, t) |\, t \in \mathbb{R}\} \end{aligned}$$

This represents a line in space.