Definition (Matrix)
If m and n are positive integrers than an $m * n$ matrix is a rectangular array of the form:

$$\begin{matrix} a_{11}\, a_{12}\, a_{13} \;\dots\; a_{1n} \\ a_{21}\, a_{21}\, a_{23} \;\dots\; a_{2n} \\ a_{31}\, a_{32}\, a_{33} \;\dots\; a_{3n} \\ \vdots \\ a_{m1}\, a_{m2}\, a_{m3} \;\dots\; a_{mn} \end{matrix}$$

There are n columns
There are m rows

Notice that each entry number inside the array of the following form: a_{ij} for $1 \leq i \leq m$, $1 \leq j \leq n$

A matrix with $mat \geq 1$ cluns is said to be a matrix of size: $m * n$

When $m = n$ a matrix has the same number of rows and columns. We call it a square matrix. It is said to have size $n * n$ $(n \geq 1)$ or order n, $n \geq 1$

If a given matrix is square of size $n*n$, $n \geq 1$, then the entries a_{11}, a_{22}, $a_{33}, \dots$ a_{nn} are called the main diagonal entries.

Note: Notice that a typical diagonal entry is of the form a_{ii}, $1 \leq i \leq n$

Example:

Conider matrix $A = \begin{pmatrix} -2 & 1 & 0 \\ 4 & -1 & 3 \\ 7 & 2 & 6 \end{pmatrix}$

Note that A is a square matrix of size $3 * 3$. The diagonal entries are: $a_{11} = -2$, $a_{22} = -1$, $a_{33} = 6$

Definition Augmented matrix and coefficient matrix:

Given a system of m linear equations in n variables, then the matrix derived from the coefficients and the constant terms of the system is called: <u>the augmented matrix of the system</u>

Example:

$$(*) \begin{cases} 2x + y = 3 \\ -x + y = 1 \end{cases}$$

$(*)$ is a system of $n = 2$ linear equations in $n = 2$ variables.
The corresponding augmented matrix is:

$$\begin{pmatrix} 2 & 1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$$

The corresponding coefficient matrix is:

$$\begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

Recall: (Elementary row operations)
1) Interchanging two rows
2) Multiply one row by a non-zero constant
3) Adding a multiple of one row to another row

<u>Row-echelon form and reduced row-echolon form</u>

<u>form:</u>
A matrix in a row-echelon form has the following properties:

1) Any row consisting of zeros occur at the bottom of the matrix

2) For each row that does <u>not</u> consist entierely of zeros, the first non-zero entry is 1

3) For two successive (non-zero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

example:

$$\begin{pmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A matrix in a row-echelon form, but not in a reduced row-echelon form

4) A matrix is in a reduced row-echelon form when every column that has a leading 1, it has zeros in every position above and below its leading 1.

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix is in a reduced and row-echelon form, this implies that it is in a row-echelon form

Remark: To solve a system of linear equations, we can use a <u>Gaussian Elimination</u> with back substitution:

1) write the augmented matrix of the system
2) use elementary row operations to write the matrix in a row-echelon form:
3) write the system of linear equations corresponding to the matrix obtained in step 2)
4) Use a back substituion to solve the system

Example: Solve the given system using a Gaussian Elimination:

$$(*) \quad \begin{cases} x - 2y + 3z = 9 \\ -x + 3y \qquad = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Answer: $(*)$ is a system if $m = 3$ linear equations in $n - 3$ variables

The corresponding augmented matrix to $(*)$ is:

$$
* \begin{pmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{pmatrix} R_1
$$

$$
R_2
$$

$$
R_3
$$

$$
\Leftrightarrow \quad \begin{pmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 0 & -1 & -1 & -1 \end{pmatrix}
$$

$$
\Leftrightarrow \quad \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{pmatrix}
$$

$$
\Leftrightarrow \quad \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{pmatrix}
$$

$$
\Leftrightarrow \quad \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}
$$

The matrix is on a row-echelon form
The corresponding system is:

$$\begin{cases} x - 2y + 3z = 9 \\ \quad\quad y + 3z = 5 \\ \quad\quad\quad\quad z = 2 \end{cases}$$

We use a back substitution

2 $\rightarrow y = 5 - 3z = 5 - 3(2) = 5 - 6 = -1$
3 $\rightarrow x = 2y - 3z - 9 = 2(-1) - 3(2) - 9 = 1$

The system is consistent, it has a unique solution and the solution set is $\{(1, -1, 2)\}$

Remark: A Gauss-Jordan Elimination involves writing the matrix associated to the system in a reduced row-echelon form:

After applying elementary row-operation we showed that the augmented matrix to system $(*)$ can be written in a row-echelon form

$$2* \begin{pmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 9 & 19 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Then $x = 1$, $y = -2$, $z = 2$
The system is consisten and has a unique solution set $\{(1, -1, 2)\}$

Example: Solve the system using gauss-elimination