0.1 Elementary Matrices

Definition 1 (Elementary Matrix). An $n \times n$ matrix E, $n \geq 1$ is an elementary matrix when it can be obtained from the identity matrix I_n by a single elementary row operation.

Theorem 1 (Representing Elementary Row Operations). *Let E be the elementary matrix obtained by performing an elementary row operation on I_n for $n \geq 1$. If that same elementary row operation is performed on an $m \times n$ matrix A, then the resulting matrix is the matrix EA.*

Definition 2 (Row Equivalence of Matrices). Let A and B be $m \times n$ matrices. Then matrix B is said to be row-equivalent to matrix A if and only if there exists a finite number of elementary matrices: $E_1, E_2, E_3, \ldots, E_k$, for $k \geq 1$ such that:

$$B = E_k E_{k-1} E_{k-2} \ldots E_3 E_2 E_1 A$$

Theorem 2 (Elementary Matrices are Invertible). *If E is an elementary matrix, then E^{-1} exists and it is also an elementary matrix*

Theorem 3 (A Property of Invertible Matrices). *A square matrix A is invertible if an only if it can be written as a finite product of elementary matrices*

Remark. Let A be an $n \times n$ matrix, $n \geq 1$, then by Theorem 3:
A is invertible $\Leftrightarrow$ There exists $E_1, E_2, E_3, \ldots, E_k$, $k \geq 1$ elementary matrices of size $n \times n$ such that:
$A = E_k E_{k-1} E_{k-2} \ldots E_1 I_n \Leftrightarrow A$ is row-equivalent to I_n

Theorem 4 (On Some Equivalent Conditions). *If A is an $n \times n$ matrix, $n \geq 1$, then the statements given below are equivalent:*
1) A is invertible
2) $Ax = b$ has a unique solution for every $n \geq 1$ column matrix b
3) $Ax = 0$ has only the trivial solution
4) A is row equivalent to I_n
5) A can be written as a finite roduct of elementary matrices

Definition 3 (An LU-Factorization). If the $n \times n$ matrix A $(n \geq 1)$ can be written as the roduct of a lower triangular matrix L and an upper triangular matrix U then,

$$A = LU$$

Is an LU factorization of matrix A as a pproduct of two matrices.

Remark. If matrix A reduces to an upper triangular matrix using only the row operation of adding a multiple of one row to another row below it, then A has an LU-Factorization. We have:

$$E_k E_{k-1} E_{k-2} \ldots E_2 E_1 A = U(*)$$

Note: E_i is obtained by only aplying the row operation of adding a muiltiple of one row on I_n
hence,

$$\begin{aligned}(*) \Leftrightarrow A &= (E_1^{-1}E_2^{-1}E_3^{-1}\dots E_{k-1}^{-1}E_k^{-1})U \\ &= LU\end{aligned}$$

0.2 The Determinant of a Matrix

Definition 4 (Determinant of a 2×2 matrix)**.** The determinant of the matrix

$$\begin{aligned}A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ \det(A) &= a_{11}a_{22} - a_{21}a_{12}\end{aligned}$$

Definition 5 (Minor and Cofactor)**.** If A is a square matrix, then the minor M_{ij} of the entry a_{ij} is the determinant of the matrix obtained by the deleting the i^{th} row and the i^{th} column of A

Definition 6 (Determinant of a Square $n \times n$ matrix $n \geq 2$)**.** If A is a square matrix of order $n \geq 2$, then the determinant of A is the sum of entries in the first row multilied by their respective cofactor.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

then

$$\begin{aligned}\det(A) = |A| &= \sum_{j=1}^{n} a_{1j}c_{1j} \\ &= a_{11}c_{11} + a_{12} + c_{12} + \dots + a_{1n}c_{1n}\end{aligned}$$

Remark. Definition 6 is inductive because it uses the determinant of a suqare matrix of order $n - 1$ to define the determinant of a matrix of order n

Theorem 5 (Finding the Determinant by Expansion of Cofactors)**.** *Let A be a square matrix of order $n \geq 2$, then the dterminant of A is:*

$$\begin{aligned}\det(A) = |A| &= \sum_{j=1}^{n} a_{ij}c_{ij} \\ &= a_{i1}c_{i1} + a_{i2}c_{i2} + \dots + a_{in}c_{in}\end{aligned}$$

or

$$\det(A) = |A| = \sum_{i=1}^{n} a_{ij}c_{ij}$$
$$= a_{1j}c_{1j} + a_{2j}c_{2j} + \ldots + a_{nj}c_{nj}$$

Remark. Keep in mind tat while expanding by cofactors, you do not need to find cofactors of zero entries since if $a_{ij} = 0$ then $a_{ij}c_{ij} = 0c_{ij} = 0$ hence the row (or column) containing most zeros is the best choice for expansion by cofactors

Definition 7 (Diagonal Matrix)**.** An $n \times n$ matrix, $n \geq 1$, in which all entries above and below the main diagonal are zero is called a diagonal matrix

Remark. $A = [a_{ij}]$ for $1 \leq i \leq n$, $1 \leq j \leq n$ is diagonal $\Leftrightarrow$ $a_{ij} = 0$ for $i \neq j$

Theorem 6 (Determinant of a Triangular Matrix)**.** *If A is a triangular matrix of order n, $n \geq 1$ then its determinant is the roduct of the entries in the diagonal, that is*

$$\det(A) = a_{11}a_{22}a_{33}\ldots a_{nn}$$
$$= \prod_{i=1}^{n} a_{ii}$$

Theorem 7 (Deterninant of an Upper or Lower Triangular Matrix)**.** *If A is triangular (upper or lower matrix of order $n \geq 1$, and if $A = [a_{ij}]$ for $1 \leq n$, $1 \leq j \leq n$) then its determinant is the product of the entries on the main diagonal that is:*

$$\det(A) = |A| = a_{11}a_{22}a_{33}\ldots a_{nn}$$
$$= \prod_{i=1}^{n} a_{ii}$$

0.3 Determinants and Elementary Row Operations

Theorem 8 (Elementary Row Operations and Determinants)**.** *Let A and B be square matrices of the same size, then*
1) When B is obtained from A by interchanging two rows of A, then $\det(A) = -\det(B)$
2) When B is obtained from A by adding a multiple of a row of A to another row, then $\det(B) = \det(A)$
3) When B is obtained from A by multiplying a row of A by a non-zero contstant c, then $\det(B) = c\det(A)$

Remark. 1) In Theorem 8, rules 1), 2), 3) remain valid if we replace "row" with "column"
2) Operations performed in the columns of a matrix are called elementary column operations. Two matrices are column equivalent when one can be obtained from the other by elementary column operations.

Theorem 9 (Conditions That Yield a Zero Deteriminant). *If A is a square matrix and any of the conditions given below is true, then* $\det(A) = 0$
1) An entire row (or an entire column) consists of zeros
2) Two rows (or two columns) are equal
3) One row (or one column) is a multiple of another row (or column)

0.4 Properties of Determinants

Theorem 10 (Determinant of a Finite Matrix Product). *If A and B are square matrices of order $n \geq 1$ then:* $\det(AB) = \det(A)\det(B)$

Remark. If $A_1A_2A_3 \dots A_k (k \geq 1)$
If these are matrices of the same order $n \geq 1$, then

$$\det(A_1A_2A_3 \dots A_k) = \det(A_1)\det(A_2)\det(A_3)\dots\det(A_k)$$
$$\Leftrightarrow \det(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \det(A_i)$$

Theorem 11 (Determinant of a Scalar Multiple of a Matrix). *If A is a square matrix of order $n \geq 1$ and c is a scalar, then*

$$\det(cA) = c^n \det(A)$$

Theorem 12 (Determinant of an Invertible Matrix). *A square matrix A is invertible (non-singular) iff* $\det(A) \neq 0$

Theorem 13 (Determinant of an Inverse Matrix). *If A is an $n \times n$, $n \geq 1$ matrix, then* $\det(A^{-1}) = \frac{1}{\det(A)}$

Theorem 14 (Determinant of the Transpose of a Matrix). *If A is a square matrix, then* $\det(A^T) = \det(A)$

Theorem 15 (Equivalent Conditions for a Non-Singular Matrix). *If A is an $n \times n$, $n \geq 1$ matrix then the following statements are equivalent:*
1) A is invertible
2) $Ax = b$ has a unique solution for ever $n \times 1$ column matrix b
3)$Ax = 0$ has only the trivial solution
4) A is row-equivalent to I_n
5) A can be written as a finite product of elementary matrices
6) $\det(A) \neq 0$

0.5 Vector Spaces

Definition 8 (Vector Spaces). Let V be a set on which we have the operations: vector addition and vector scalar multiplication are defined. If the axioms listed below are satisfied

for any vectors u, v, and w and any scalars c and d in $\mathbb{R}\,(v,\, t,\, \cdot)$ is a vector space.
I) Axioms of Additon:
1) $u + v \in V$ (closure under addition)
2) $u + v = v + u$ (commutative property of vector addition)
3) $u + (v + w) = (u + v) + w$ (associative property of vector addition)
4) V has a zero vector $\vec{0}$ such that for every v in V, $u + \vec{0} = \vec{0} + u = u$ 5) For any u in V, there exists a vector in V denoted by: $-u$ such that: $u + (-u) = \vec{0}$

II) Axioms of Scalar Multiplication
6) $cu \in V$ (closure under scalar multiplication)
7) $c(u + v) = cu + cv$
8) $(c + d)u = cu + du$
9) $c(du) = (cd)u$
10) $1u = u$

Remark. If one of the ten axioms fails then set V equipped with the operation of addition and scalar multiplication is not a vector

Remark. $\mathbb{R}^n = \{(x_1, x_2, x_3, \ldots, x_n) | x_i \in \mathbb{R}$ for any $1 \leq i \leq n\}$
Then $(\mathbb{R}, +, \cdot)$ is a vector space (Verify it!)

Theorem 16 (Properties of Scalar Multilication)**.** *Let V be a vector space. Let $v \in V$ and c any scalar. The the following properties are true:*
1) $0v = \vec{0}$
2) $c\vec{0} = \vec{0}$
3) If $cv = 0$ *then either* $c = 0$ *or* $v = 0$
4) $(-1)v = -v$

0.6 Sub Spaces of a Vector Space

Definition 9 (Subspace of a Vector Space)**.** A non-empty subspace w of a vector space v is a subspace of v when w is a vector sace under the operations of addition and scalar multiplication defined in v

Theorem 17 (Test for a Subspace)**.** *If w is a non-emty subset of a vector space v, then w is a subspace of v if and only if the two closure conditions listed below hold:*
1) If u and v are in w, then $u + v \in w$
2) If u is in w and c is a scalar, then $cu \in w$

Remark. w is a subsace of vector space v
$\Leftrightarrow \emptyset \neq w \subseteq v$ and w is closed under vector addition and scalar multiplication

Remark. Sometimes we may encounter in applications sequences of nestled subspaces (one is contained with the others). For example, if we consider the vector space P_k of polynomials of degree less or equal to k, then

$$P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_k \subset \ldots \subset P_n \text{ for any } 0 \leq k \leq n$$

Remark. If w is a subspace of vectorspace v, then $\{0_v\} \subseteq w \subseteq v$
If $w = \{0_v\}$ or $w = v$ then w is a trivial subspace of v
If $\{0_v\} \subset w \subset v$, then w is a proper space of v

Theorem 18 (The Intersection of Two Subspaces is a Subspace)**.** *If u and w are subspaces of vector space v, then their intersection $u \cap w$ is also a subspace of v*

Remark. Let $w \subset \mathbb{R}^2$ (equipped with standard addition and scalar multiplication) then w is a subspace of $\mathbb{R}^2$
$\Leftrightarrow$ 1) $w = \{(0,0)\}$
or
2) $w = \mathbb{R}^2$
or
3) w is the set of all ordered pairs lying on a line passing through the origin