I) The Algebra of Matrices

Theorem 1 (Properties of matrix addition and scalar multiplication)**.** *Let A, B and C be $m * n$ matricies and c, d be scalars*

1) $A + B = B + A$ (Commutative property of matrix addition)
2) $A + (B + C) = (A + B) + C$ (Associative property of matrix addition)
3) $(cd)A = c(dA)$ (Associative property of scalar multilcation of matricies)
4) $c(A + B) = cA + cB$ (Distributive property of scalar multiplication of matricies)
5) $(c + d)A = cA + dA$ (Distributive property of scalar multiplication of matricies)

Proofs of 1), 2), and 5):

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \leq i \leq m$, $1 \leq j \leq n$
then

$$\begin{aligned} A + B &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= B + A \end{aligned}$$

□

Proof. Let $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ for $1 \leq i \leq m$, $1 \leq j \leq n$
then

$$\begin{aligned} A + (B + C) &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij} + b_{ij} + c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \text{ since the operation of addition is associative in } \mathbb{R} \\ &= (A + B) + C \end{aligned}$$

□

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ for $1 \leq i \leq m$, $1 \leq j \leq n$ and let $c \in \mathbb{R}$ be a scalar
then

$$\begin{aligned} c(A + B) &= c[a_{ij} + b_{ij}] \\ &= [c(a_{ij} + b_{ij})] \\ &= [ca_{ij} + cb_{ij}] \text{ by the distributive property of multilication with respect to addition in } \mathbb{R} \\ &= [ca_{ij}] + [cb_{ij}] \\ &= c[a_{ij}] + c[b_{ij}] \\ &= cA + cB \end{aligned}$$

□

Definition 1 (Zero matrix)**.** The $m*n$ matrix denoted by 0_{mn} whose entries are all 0's is called the zero matrix of size $m*n$

$$0_{mn} = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 \end{bmatrix}$$

Note: $0_{mn} = a_{ij}$ where $a_{ij} = 0$ for any $1 \le i \le m$, $1 \le j \le m$

Theorem 2 (Properties of the zero matrix)**.** *If A is an $m*n$ matrix and $c \in \mathbb{R}$ is a scalar, then*

1) $A + 0_{mn} = A$
2) $A + (-A) = 0_{mn}$
3) If $cA = 0_{mn}$ then $c = 0$ or $A = 0_{mn}$

Proof of 1):

Proof. Let $A = [a_{ij}]$ for $1 \le i \le m$, $1 \le j \le n$
then

$$\begin{aligned} &\Leftrightarrow \quad c[a_{ij}] = [0] \\ &\Leftrightarrow \quad [ca_{ij}] = [0] \\ &\Leftrightarrow \quad ca_{ij} = 0 \end{aligned}$$

$c = 0$ or $a_{ij} = 0_{mn}$ for all $1 \le i \le m$, $1 \le j \le n$
$\implies A = 0_{mn}$ □

Example. Let $A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix}$ and x be a $3*2$ matrix. Solve the matrix equation $3x + 2A = B$ $(*)$

let $x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$ where $x_{ij} \in \mathbb{R}$ for $1 \le i \le m$ and $1 \le j \le n$

then

$$\begin{aligned}
(*) \; &\Leftrightarrow 3x + 2A = B \\
&\Leftrightarrow 3x + 0_{mn} = B - 2A \\
&\Leftrightarrow 3x = B - 2A \\
&\Leftrightarrow \frac{1}{3}(3x) = (B - 2A)\frac{1}{3} \\
&\Leftrightarrow (\frac{1}{3} \cdot 3)x = \frac{1}{3}(B - 2A) \\
&\Leftrightarrow x = \frac{1}{3}(B - 2A) \\
&\Leftrightarrow x = \frac{1}{3}(\begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} - 2\begin{bmatrix} -2 & -1 \\ 1 & 0 \\ 3 & -4 \end{bmatrix}) \\
&\Leftrightarrow x = \frac{1}{3}(\begin{bmatrix} 0 & 3 \\ 2 & 0 \\ -4 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ -2 & 0 \\ -6 & 8 \end{bmatrix}) \\
&\Leftrightarrow x = \frac{1}{3}(\begin{bmatrix} 4 & 5 \\ 0 & 0 \\ -10 & 7 \end{bmatrix}) \\
&\Leftrightarrow x = \begin{bmatrix} 4/3 & 5/3 \\ 0 & 0 \\ -10/3 & 7/3 \end{bmatrix}
\end{aligned}$$

Theorem 3 (Properties of matrix multilication)**.** *Let A, B and C, be matricies (with sizes such that the matrix products given below are defined) and let $\alpha \in \mathbb{R}$ be a scalar*

1) $A(BC) = (AB)C$ (Associative property of multiplication)
2) $A(B + C) = AB + AC$ (Distributive property of matrix mult. w.r.t matrix addition)
3) $(A + B)C = AC + BC$ (Distributive property of matrix mult. w.r.t matrix addition)
4) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

Proof of 2):

Proof. Let $A = [a_{ij}]$ be an $m * n$ matrix
and let $B = [b_{ij}]$ and $C = [c_{ij}]$ be $n * p$ matricies,
then $B + C = [b_{ij} + c_{ij}]$ is an $n * p$ matrix
hence $A(B + C)$ is a well defined matrix of $m * p$
let $A(B + C) = [d_{ij}]$

Recall: If $A = [a_{ij}]$ is an $m * n$ matrix, and $B = [b_{ij}]$ be an $n * p$ matrix, then AB is an $m * p$ matrix and $AB = [c_{ij}]$ where $cij = \sum_{k=1}^{n} a_{ik}b_{kj}$

here,

$$\begin{aligned} d_{ij} &= \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + a_{ik}c_{kj} \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\ &= q_{ij} + r_{ij} \end{aligned}$$

where $AB = [q_{ij}] = [\sum_{k=1}^{n} a_{ik}b_{kj}]$
and $AC = [r_{ij}] = [\sum_{k=1}^{n} a_{ik}c_{kj}]$
hence $A(B + C) = AB + AC$ $\square$

Remark. Let A and B be matricies, with sizes such that products given below are well defined

1) $(A+B)^2 \neq A^2+2AB+B^2$ in general since $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2$. However, you must be careful since $AB \neq BA$ in general.

2) $(A-B)^2 \neq A^2-2AB+B^2$ in general since $(A+B)^2 = (A+B)(A+B) = A^2-AB-BA+B^2$

Definition 2 (Diagonal of a matrix)**.** Let $A = [a_{ij}]$ be an $n * n$ matrix, then its diagonal consists of the entries a_{ii} for any $1 \leq i \leq n$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$

and its trace is the sum of all diagonal entries, that is:

$$\begin{aligned} Trace(A) &= a_{11} + a_{22} + a_{33} + \ldots + a_{nn} \\ &= \sum_{i=1}^{n} a_{ii} \end{aligned}$$

Definition 3 (Identity Matrix)**.** An $n*n$ matrix, $n \geq 1$ that has 1 on all its diagonal entries and 0's elsewhere is called the indentity matrix of order n (size $n * n$). It is denoted by I_n

$$I_n = [a_{ij}] \text{ where } a_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \quad \text{where } 1 \leq i \leq n,\ 1 \leq j \leq n$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 1 \end{bmatrix}$$

Example. $I_1 = \begin{bmatrix} 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark. The matrix I_n ($n \geq 1$) serves as the identity element for matrix multiplication

Theorem 4 (Properties of the identity matrix)**.** *If A is a matrix of size $m * n$, then*

$$\begin{aligned} &1)\, AI = A \\ &2)\, IA = A \end{aligned}$$

Example. Consider the $4 * 4$ matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Show that there exists a positive integer n such that $A^n = 0_{44}$

$$A^2 = AA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = A^2A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = A^3A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 0_{44}$$

$\boxed{\therefore n = 4}$

Exercises:
Prove 1), 2), 3) for properties of zero matrix
Prove 1), 3), 4) for properties of matrix multiplication