

A linear equation in  $n > 1$  variables  $x_1, x_2, x_3, \dots, x_n$  has the form

$$(*) \quad a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b$$

here  $a_i$  is a constant for any  $1 \leq i \leq n$  and  $b$  is a constant

$a_i$  are the coefficients of  $(*)$

$x_i$  are the variables

$a_1$  is the leading coefficient

$x_1$  is the leading variable

### Example: Linear Equations

a)  $2x - 3y = 1$  (2 variables)

→ represents a line of the xy plane

b)  $x + y - z = 2$  (3 variables)

→ represents a plane in the xyz coordinate system

### Example: Non-Linear Equations

1)  $\boxed{x^2} + y + z = 0 \leftarrow$  non-linear function  $x$

2)  $\boxed{xy} + z = -1 \leftarrow$   $x$  and  $y$  are not independent of each other

3)  $\boxed{\cos x} + y + z = 3 \leftarrow$  non-linear function of  $x$

Definition (Solution and Solution Set of  $(*)$ ):

A solution of  $(*)$  is a sequence of  $n$  real numbers  $s_1, s_2, s_3, \dots, s_n$  that satisfy  $(*)$

The set of solutions of  $(*)$  is called a solution set

Remark: The solution of  $(*)$  is a linear equation in  $n$  variables does have a parametric representation.

### Example: Find a parametric representation of each linear equation

a)  $\boxed{1} \quad 2x_1 - 3x_2 = 1$  (a linear equation in  $n = 2$  represents a line in  $\mathbb{R}^2$ )

let  $x_2 = t$ , here  $t \in \mathbb{R}$

$$\begin{aligned}
\text{then } \boxed{1} &\Leftrightarrow 2x_1 - 3t = 1 \\
&\Leftrightarrow 2x_1 = 1 + 3t \\
&\Leftrightarrow x_1 = \frac{1}{2} + \frac{3}{2}t
\end{aligned}$$

Then the solution set of  $\boxed{1}$  is

$$\begin{aligned}
&\left\{ (x_1, x_2) \mid x_1 = \frac{1}{2} + \frac{3}{2}t \text{ and } x_2 = t, t \in \mathbb{R} \right\} \\
&\boxed{= \left\{ \left( \frac{1}{2} + \frac{3}{2}t, t \right) \mid t \in \mathbb{R} \right\}} \text{ A parameterization of the solution set}
\end{aligned}$$

b)  $\boxed{2} \quad x_1 + 2x_2 - x_3 = 4$

Let  $x_3 = t$ ,  $t \in \mathbb{R}$  is a parameter

Let  $x_2 = s$ ,  $s \in \mathbb{R}$  is a parameter

Then

$$\begin{aligned}
&\Leftrightarrow x_1 + 2s - t = 4 \\
&\Leftrightarrow x_1 = -2s + t + 4, t \in \mathbb{R} \text{ and } s \in \mathbb{R}
\end{aligned}$$

Then the solution set of  $\boxed{2}$  is

$$\begin{aligned}
&\{(x_1, x_2, x_3) \mid x_1 = -2s + t + 4, x_2 = s, x_3 = t, s \in \mathbb{R} \text{ and } t \in \mathbb{R}\} \\
&\boxed{= \{(-2s + t + 4, s, t) \mid s \in \mathbb{R} \text{ and } t \in \mathbb{R}\}}
\end{aligned}$$

Definition (System of linear equations in  $n \geq 1$  variables):

A system of  $m \geq 1$  linear equation in  $n \geq 1$  variables is a set of  $m$  equations, each of which is linear in the same  $n$  variables

(\*)

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n &= b_3 \\
 &\vdots \\
 a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m
 \end{aligned}$$

A solution of (\*) is a sequence of numbers  $s_1, s_2, s_3, \dots, s_n$  that is a solution of each equation of the system.

Remark (On the number of solutions of a system (\*) of linear equations):

For a given system of linear equations (\*) Precisely one of the following statements is true:

- 1) The system (\*) has exactly one solution (system is consistent)
- 2) The system (\*) has infinitely many solutions (system is consistent)
- 3) The system (\*) has no solutions (system is inconsistent)

### Examples: Different Cases for Systems of Linear Equations

a)

$$(*) \quad \begin{cases} x + y = 0 & \textcircled{1} \\ x - y = 1 & \textcircled{2} \end{cases}$$

(\*) is a system of  $m = 2$  linear equations in  $n = 2$  variables  
 (\*) has one solution  $(\frac{1}{2}, -\frac{1}{2})$ . The system is consistent.

b)

$$(*) \quad \begin{cases} x + y = 1 & \textcircled{1} \\ 2x + 2y = 2 & \textcircled{2} \end{cases}$$

(\*) is a system of  $m = 2$  linear equations and  $n = 2$  variables  
 observe that  $2 = 2$

then  $(*) \Leftrightarrow x + y = 1$  hence the solution is

$$\begin{aligned}\{(x, y) \mid x + y = 1\} &= \{(x, y) \mid y = 1 - x\} \\ &= \{(x, 1 - x) \mid x \in \mathbb{R}\}\end{aligned}$$

c)

$$(*) \quad \begin{cases} x + y = -1 & \textcircled{1} \\ x + y = 1 & \textcircled{2} \end{cases}$$

Note that  $\textcircled{1}$  and  $\textcircled{2}$  imply that:  $1 = -1$  which is impossible, then  $(*)$  has no solutions. The system is inconsistent.

Definition (system in a row-echelon form):

A system of linear equations is in row-echelon form when it has a stair-step pattern with leading coefficients of 1. To solve is we use a back substitution.

### Examples: Row-Echelon Form

a)

$$(*) \quad \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ y + 3z = 5 & \textcircled{2} \\ z = 2 & \textcircled{3} \end{cases}$$

$(*)$  is in row-echelon form, it has a stair-step pattern and the leading coefficients are 1.

b)

$$(*) \quad \begin{cases} x + 2y = 1 & \textcircled{1} \\ 3y = 5 & \textcircled{2} \end{cases}$$

$(*)$  has a stair-step pattern, but the leading coefficient of the second row isn't 1. Therefore,  $(*)$  is not in row-echelon form.

c)

$$(*) \quad \begin{cases} x - y + z = 1 & \textcircled{1} \\ y - z = 0 & \textcircled{2} \\ y + 6z = 3 & \textcircled{3} \end{cases}$$

The leading coefficients of  $(*)$  are 1.  $(*)$  does not have a stair-step pattern, therefore  $(*)$  is not in row-echelon form.

Definition (equivalence of systems of linear equations:)

Two systems of linear equations are equivalent when they have the same solution set (the systems must have the same number of linear equations and the same variables).

### Example: System Equivalency

$$\begin{aligned}
 (*) \quad & \boxed{\begin{cases} 2x + 2y = 2 \\ x - y = 1 \end{cases}} & \textcircled{1} \\
 \Leftrightarrow & \boxed{\begin{cases} x + y = 1 \\ x - y = 1 \end{cases}} & \textcircled{2} \\
 \Leftrightarrow & \boxed{\begin{cases} x = 1 \\ y = 0 \end{cases}} & \textcircled{3}
 \end{aligned}$$

The solution set is  $\{(1, 0)\}$  hence the systems  $\textcircled{1}$ ,  $\textcircled{2}$ , and  $\textcircled{3}$  are equivalent as they have the same solution set.

Remark: To solve a system that is not in a row-echelon form we write it as an equivalent system in row-echelon form.

**Question:** What are the possible operations that we can apply to the equations of a system  $(*)$  to obtain an equivalent system?

**Answer:** The following are operations that we can apply to the equations in  $(*)$  to obtain an equivalent system:

- 1) Interchanging the rows
- 2) Multiplying an equation by a non-zero constant
- 3) Adding a multiple of an equation to another equation

Remark: Rewriting a system of linear equations in a row-echelon form involves a chain of equivalent systems using operations 1), 2), and 3): the process is called a Gaussian Elimination.

### Examples: Gaussian Elimination

a)

$$(*) \quad \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ -x + 3y = -4 & \textcircled{2} \\ 2x - 5y + 5z = 17 & \textcircled{3} \end{cases}$$

(\*) is a system of  $m = 3$  linear equations in  $n = 3$  variables.

To solve (\*) we apply a Gaussian Elimination:

$$(*) \Leftrightarrow \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ -x + 3y = -4 & \textcircled{2} \\ -y - z = -1 & (-2)\textcircled{1} + \textcircled{3} \rightarrow \textcircled{3}' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ y + 3z = 5 & \textcircled{1} + \textcircled{2} \rightarrow \textcircled{2}' \\ -y - z = -1 & \textcircled{3}' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ y + 3z = 5 & \textcircled{2}' \\ 2z = 4 & \textcircled{2}' + \textcircled{3}' \rightarrow \textcircled{3}'' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 2y + 3z = 9 & \textcircled{1} \\ y + 3z = 5 & \textcircled{2}' \\ z = 2 & \frac{1}{2}\textcircled{3}'' \rightarrow \textcircled{3}''' \end{cases}$$

The system is in a row-echelon form, to solve it we use a back substitution, substitution  $\boxed{z = 2}$  in  $\textcircled{2}'$  yields:

$$y + 3(2) = 5$$

$$\Leftrightarrow \boxed{y = -1}$$

substituting  $y = -1$  and  $z = 2$  in ① yields:

$$x - 2(-1) + 3(2) = 9$$

$$\Leftrightarrow x + 2 + 6 = 9$$

$$\Rightarrow \boxed{x = 1}$$

The system is consistent, it has a unique solution and the solution set is  $\{(1, -1, 2)\}$

b)

$$(*) \quad \begin{cases} x - 3y + z = 1 & \text{①} \\ 2x - y - 2z = 2 & \text{②} \\ x + 2y - 3z = -1 & \text{③} \end{cases}$$

(\*) is a system of  $m = 3$  linear equations in  $n = 3$  variables.

To solve (\*) we apply a Gaussian Elimination:

$$(*) \Leftrightarrow \begin{cases} x - 3y + z = 1 & \text{①} \\ 2x - y - 2z = 2 & \text{②} \\ 5y - 4z = -2 & (-1)\text{①} + \text{③} \rightarrow \text{③}'' \end{cases}$$

$$\Leftrightarrow \begin{cases} x - 3y + z = 1 & \text{①} \\ 5y - 4z = 0 & (-2)\text{①} + \text{②} \rightarrow \text{②}' \\ 5y - 4z = -2 & \text{③}'' \end{cases}$$

The system is inconsistent and has no solutions because of the equations  $5y - 4z = 0$  and  $5y - 4z = -2$ . They are the same but equal different values.

c)

$$(*) \quad \begin{cases} y - z = 0 & \textcircled{1} \\ x - 3z = -1 & \textcircled{2} \\ -x + 3y = 1 & \textcircled{3} \end{cases}$$

(\*) is a system of  $m = 3$  linear equations in  $n = 3$  variables.

To solve (\*) we apply a Gaussian Elimination:

$$\begin{aligned} (*) &\Leftrightarrow \begin{cases} x - 3z = -1 & \textcircled{2} \\ y - z = 0 & \textcircled{1} \\ -x + 3y = 1 & \textcircled{3} \end{cases} \\ &\Leftrightarrow \begin{cases} x - 3z = -1 & \textcircled{2} \\ y - z = 0 & \textcircled{1} \\ 3y - 3z = 0 & \textcircled{2} + \textcircled{3} \rightarrow \textcircled{3}' \end{cases} \\ &\Leftrightarrow \begin{cases} x - 3z = -1 & \textcircled{2} \\ y - z = 0 & \textcircled{1} \end{cases} \end{aligned}$$

Let  $z = t$

$$\textcircled{1} \implies y = t$$

$$\textcircled{2} \implies x = -1 + 3t$$

then

$$\begin{cases} x = -1 + 3t \\ y = t \\ z = t \end{cases}, \quad t \in \mathbb{R}$$

Hence (\*) is a consistent system, it has infinitely many solutions. The solution set (on a parameterized form) is:

$$\begin{aligned} &\{(x, y, z) \mid x = -1 + 3t, y = t, z = t, t \in \mathbb{R}\} \\ &= \{(-1 + 3t, t, t) \mid t \in \mathbb{R}\} \end{aligned}$$

This represents a line in space.