- Solutions of set 2 practice problems -

Exercise 1:

a) Prove that if A is an $n\times n$ matrix that is idempotent and invertible then $A = I_n$

Answer:

Let A be an $n\times n$ idempotent matrix, then we have $A^2 = A$, hence $A.A = A$, thus $A^{-1}(AA) = A^{-1}A$ (since A is invertible then A^{-1} exists), then $(A^{-1}A)A = A^{-1}A$

$\Leftrightarrow \; I_n A = I_n \Rightarrow A = I_n$.

b) Prove that if A and B are idempotent matrices and $AB = BA$, then AB is idempotent.

Answer:

we have $(AB)^2 = (AB)(AB) = A(BA)B$

$= A(AB)B$ (since $AB = BA$)

$= (AA)(BB)$

$= A^2 . B^2$

$= A \; B$ (since A is idempotent then $A^2 = A$ and since B is idempotent then $B^2 = B$)

Therefore AB is an idempotent matrix.

Exercise 2:

Let A and B be $n \times n$ matrices, $n \geq 1$.

Prove that if $AB = I$, then $BA = I$.

- $AB = I \Rightarrow \det(AB) = \det I = 1 \Leftrightarrow \det(A)\det(B) = 1$
  $\Rightarrow \det(A) \neq 0$ and $\det(B) \neq 0$ ∴ A and B are invertible!

Note that $AB = I \Rightarrow A^{-1}(AB) = A^{-1}I \Rightarrow (A^{-1}A)B = A^{-1}$
$\Rightarrow B = A^{-1}$. Thus $BA = A^{-1}A = I$.

Therefore, if $AB = I$ then $BA = I$.

Exercise 3:

(a) Recall that a square matrix A is *orthogonal* if and only if $A^T = A^{-1}$. Prove that if A is *orthogonal* then $Det(A) = \pm 1$

- Let A be an orthogonal matrix, then

$A^T = A^{-1} \Rightarrow \det(A^T) = \det(A^{-1})$

$\Leftrightarrow \det(A) = \dfrac{1}{\det(A)}$ since $\det(A^T) = \det(A)$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$\Leftrightarrow [\det(A)]^2 = 1$

$\Leftrightarrow \det(A) = \pm 1$.

(b) Recall that a square matrix A is *idempotent* if and only if $A^2 = A$. Prove that if A is *idempotent* then $Det(A)$ is either 0 or 1

- Let A be an idempotent matrix, then

$A^2 = A \Rightarrow \det(A^2) = \det(A)$

$\Leftrightarrow (\det(A))^2 = \det(A)$ since for any $n \geq 1$, we have $\det(A^n) = (\det(A))^n$

$\Leftrightarrow (\det(A))^2 - \det(A) = 0$

$\Leftrightarrow \det(A)[\det(A) - 1] = 0$

$\Leftrightarrow \det(A) = 0$ or $\det(A) = 1$

Exercise 4 :

Prove that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

Answer:

- A is row equivalent to $B \Leftrightarrow$ there exist a finite sequence of elementary matrices: $E_1, E_2, E_3, \dots, E_k$ $(k \geq 1)$ such that $A = E_k E_{k-1} \dots E_3 E_2 E_1 B$ (1)
- B is row equivalent to $C \Leftrightarrow$ there exist a finite sequence of elementary matrices: $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \dots, \tilde{E}_{k'}$ $(k' \geq 1)$ such that $B = \tilde{E}_{k'} \tilde{E}_{k'-1} \dots \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 C$ (2).

Note that substituting (2) in (1) yields:

$$\begin{aligned} A &= E_k E_{k-1} \dots E_3 E_2 E_1 B \\ &= E_k E_{k-1} \dots E_3 E_2 E_1 (\tilde{E}_{k'} \tilde{E}_{k'-1} \dots \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 C) \\ &= \underbrace{E_k E_{k-1} \dots E_3 E_2 E_1 \tilde{E}_{k'} \tilde{E}_{k'-1} \dots \tilde{E}_3 \tilde{E}_2 \tilde{E}_1}_{\text{A finite product of elementary matrices}} C \end{aligned}$$

which implies that A is row equivalent to C.

Exercise 5:

Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers equipped with the operations:

addition defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 y_1, x_2 y_2)$$

and scalar multiplication defined by

$$c(x_1, x_2) = (cx_1, cx_2),$$

here $c \in \mathbb{R}$ is a scalar. Note that the operation addition here is *non standard*. Is $\mathbb{R}^2$ in this case a vector space? (*Justify your answer*)

- Let's find the identity element for this non standard operation of addition:

Let $(x_1, x_2) \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$, then

$(x_1, x_2) + (a, b) = (x_1, x_2)$

$\Leftrightarrow (ax_1, bx_2) = (x_1, x_2)$

$\Rightarrow ax_1 = x_1$ and $bx_2 = x_2$

$\Rightarrow a = 1$ and $b = 1$.

hence $(1, 1)$ is the identity element for addition here.

- Let's find the additive inverse of the element (x_1, x_2):

$(x_1, x_2) + (a, b) = (1, 1) \Leftrightarrow (ax_1, bx_2) = (1, 1)$

$\Rightarrow ax_1 = 1$ and $bx_2 = 1 \Rightarrow a = \frac{1}{x_1}$ and $b = \frac{1}{y_1}$, $x_1 \neq 0$, $y_1 \neq 0$

hence $\left(\frac{1}{x_1}, \frac{1}{y_1}\right)$ is the additive inverse of (x_1, x_2).

Note that this implies that $(0, 0) \in \mathbb{R}^2$ does <u>not</u> have an additive inverse under this <u>non</u> standard operation of addition, hence $\mathbb{R}^2$ equipped with the above operations is <u>not</u> a vector space.

6 Let A be an $n \times n$ invertible matrix, $n \geq 1$. Prove that $Det(A) = \pm 1$ when all the entries of A and A^{-1} are integers.

- Let A be an $n \times n$, $n \geq 1$ invertible matrix and assume that $A = [a_{ij}]$ and $A^{-1} = [\tilde{a}_{ij}]$ have entries that are integers, that is $a_{ij} \in \mathbb{Z}$ and $\tilde{a}_{ij} \in \mathbb{Z}$ for any $1 \leq i,j \leq n$.
- we have $A A^{-1} = A^{-1} A = I_n$, here I_n is the identity matrix of order n, $n \geq 1$, then

$$\det(A A^{-1}) = \det(I_n) = 1$$

$$\Leftrightarrow \det(A) \cdot \det(A^{-1}) = 1$$

Note that since the entries of A are <u>integers</u> then

$$\det(A) = \sum_{i=1}^{n} \underset{\in \mathbb{Z}}{a_{ij}} \; \underset{\in \mathbb{Z}}{C_{ij}}$$ ← an expansion of cofactors with respect to the j^{th} column

is an integer and

$$\det(A^{-1}) = \sum_{i=1}^{n} \underset{\in \mathbb{Z}}{\tilde{a}_{ij}} \; \underset{\in \mathbb{Z}}{\tilde{C}_{ij}}$$

<u>Remarks</u>:

$C_{ij} = (-1)^{i+j} \underbrace{M_{ij}}_{\in \mathbb{Z}} \in \mathbb{Z}$

and

$\tilde{C}_{ij} = (-1)^{i+j} \underbrace{\tilde{M}_{ij}}_{\in \mathbb{Z}} \in \mathbb{Z}$

is also an integer. Therefore $\det(A)$ and $\det(A^{-1})$ are integers. Then, either we have $\det(A) = 1$ and $\det(A^{-1}) = 1$

or $\det(A) = -1$ and $\det(A^{-1}) = -1$

Hence, $\det(A) = \pm 1$.

Exercise 7:

Let A be a non singular matrix. Prove that if B is row equivalent to A, then B is also non singular.

Answer:

- Let A be a non singular matrix, then by a theorem A is row equivalent to I_n, hence there exist a finite sequence of elementary matrices $E_1, E_2, E_3, \ldots, E_k$ $(k \geq 1)$ such that
$A = E_k E_{k-1} E_{k-2} \ldots E_2 E_1 I_n$ (1)

- B is row equivalent to A then there exist a finite sequence of elementary matrices $\tilde{E}_1, \tilde{E}_2, \tilde{E}_3, \ldots, \tilde{E}_{k'}$ $(k' \geq 1)$ such that
$B = \tilde{E}_{k'} \tilde{E}_{k'-1} \ldots \tilde{E}_3 \tilde{E}_2 \tilde{E}_1 A$ (2).

Substituting (1) in (2) yields:

$$B = \tilde{E}_{k'} \tilde{E}_{k'-1} \tilde{E}_{k'-2} \ldots \tilde{E}_2 \tilde{E}_1 A$$

$$= \tilde{E}_{k'} \tilde{E}_{k'-1} \tilde{E}_{k'-2} \ldots \tilde{E}_2 \tilde{E}_1 (E_k E_{k-1} E_{k-2} \ldots E_2 E_1 I_n)$$

$$= \underbrace{\tilde{E}_{k'} \tilde{E}_{k'-1} \tilde{E}_{k'-2} \ldots \tilde{E}_2 \tilde{E}_1 E_k E_{k-1} E_{k-2} \ldots E_2 E_1}_{\text{A finite product of elementary matrices}} I_n$$

which implies (by a theorem) that B is non singular.

Exercise 8:

a) Prove that in a given vector space V, the zero vector is unique.

Answer:

Let V be a vector space and assume that V has two zero vectors 0_V and $\tilde{0}_V$, we want to show that $0_V = \tilde{0}_V$.

Let $u \in V$, then $u + 0_V = u = u + \tilde{0}_V$

hence $u + 0_V = u + \tilde{0}_V$

thus $-u + (u + 0_V) = -u + (u + \tilde{0}_V)$ here $-u$ is the additive inverse of u in V

$\Leftrightarrow (-u+u) + 0_V = (-u+u) + \tilde{0}_V$

$\Leftrightarrow 0_V + 0_V = \tilde{0}_V + \tilde{0}_V$ Note that $-u + u = 0_V = \tilde{0}_V$

$\Rightarrow 0_V = \tilde{0}_V$

Therefore the zero vector in V is unique.

b) Prove that in a given vector space V, the additive inverse of a vector is unique.

Answer:

Let $u \in V$ and assume that u has two additive inverses $-u$ and $-\tilde{u}$, we want to show that $-u = -\tilde{u}$.

we have $u + (-u) = u + (-\tilde{u}) = 0_V$ here 0_V is the zero vector in V

then $u + (-u) = u + (-\tilde{u})$

hence $-u + (u + (-u)) = -u + (u + (-\tilde{u}))$

thus $(-u+u) + (-u) = (-u+u) + (-\tilde{u})$

hence $0_V + (-u) = 0_V + (-\tilde{u})$

therefore $-u = -\tilde{u}$, and as a result, the additive inverse of a vector is unique in vector space V.