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Math 13: Exam 1 (Solutions)

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Instructions:

(a) Solve the given *ten* problems.

(b) The test is closed notes and closed books. No phone or tablet is allowed.

(c) You may use well-known results from the course or the book unless you are specifically asked to reprove them.

(d) Please *show your work*. Always *justify your answers*, unless the omitted argument is trivial.

1 Solve the system using a *Gaussian elimination*:

$$(*)\begin{cases} x+y-5z=3 \\ x \quad\; -2z=1 \\ 2x-y-z=0 \end{cases}$$

(*) is a system of $m=3$ Linear equations in $n=3$ variables

and determine whether the system is consistent or inconsistent.

The augmented matrix associated to system (*) is

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 1 & 0 & -2 & 1 \\ 2 & -1 & -1 & 0 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

we use a Gaussian elimination to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 2 & -1 & -1 & 0 \end{array}\right]\begin{array}{l} R_1 \\ -R_1+R_2 \longrightarrow R_2' \\ R_3 \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & -1 & 3 & -2 \\ 0 & -3 & 9 & -6 \end{array}\right]\begin{array}{l} R_1 \\ R_2' \\ -2R_1+R_3 \longrightarrow R_3' \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & -3 & 9 & -6 \end{array}\right]\begin{array}{l} R_1 \\ -R_2' \longrightarrow R_2'' \\ R_3' \end{array} \qquad \left[\begin{array}{ccc|c} 1 & 1 & -5 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} R_1 \\ R_2'' \\ 3R_2'+R_3' \longrightarrow R_3'' \end{array}$$

hence we have $\begin{cases} x+y-5z=3 & (1) \\ \quad\;\, y-3z=2 & (2) \end{cases}$

Let $z=t$, here $t\in\mathbb{R}$ is a parameter, then $(2) \Rightarrow y = 2+3z = 2+3t$

and $(1) \Rightarrow x = -y+5z+3 = -(2+3t)+5t+3 = -2-3t+5t+3 = 1+2t$

then the solution set is $\{(1+2t,\ 2+3t,\ t) \mid t\in\mathbb{R}\}$ and hence, we have infinitely many solutions, the system is consistent.

2 Find the values of k such that the given system of linear equations

$$(*)\quad \begin{cases} x+y+kz=3 \\ x+ky+z=2 \\ kx+y+z=1 \end{cases}$$

$(*)$ is a system of $m=3$ Linear equations in $n=3$ variables

has

(a) Exactly one solution.

The augmented matrix associated to system $(*)$ is

$$\begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ \boxed{1} & k & 1 & 2 \\ k & 1 & 1 & 1 \end{array}\right] & \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \end{array} \qquad \begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & k-1 & 1-k & -1 \\ \boxed{k} & 1 & 1 & 1 \end{array}\right] & \begin{array}{l} R_1 \\ (-1)R_1+R_2 \to R_2' \\ R_3 \end{array} \end{array}$$

we use a Gaussian elimination

$$\begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & k-1 & 1-k & -1 \\ 0 & \boxed{1-k} & 1-k^2 & 1-3k \end{array}\right] & \begin{array}{l} R_1 \\ R_2' \\ (-k)R_1+R_3 \to R_3' \end{array} \end{array} \qquad \begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & k-1 & 1-k & -1 \\ 0 & 0 & -k^2-k+2 & -3k \end{array}\right] & \begin{array}{l} R_1 \\ R_2' \\ R_2'+R_3' \to R_3'' \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & k-1 & 1-k & -1 \\ 0 & 0 & k^2+k-2 & 3k \end{array}\right] & \begin{array}{l} R_1 \\ R_2' \\ (-1)R_3'' \to R_3''' \end{array} \end{array} \qquad \begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & -1 & 1-2k & -1-3k \\ 0 & 0 & k^2+k-2 & 3k \end{array}\right] & \begin{array}{l} R_1 \\ -kR_1+R_2' \to R_2'' \\ R_3''' \end{array} \end{array}$$

$$\begin{array}{cl} \left[\begin{array}{cccc} 1 & 1 & k & 3 \\ 0 & 1 & 2k-1 & 1+3k \\ 0 & 0 & k^2+k-2 & 3k \end{array}\right] & \begin{array}{l} R_1 \\ (-1)R_2'' \to R_2''' \\ R_3''' \end{array} \end{array}$$

Then the system $(*)$ has exactly one solution if and only if $k^2+k-2 \neq 0 \Leftrightarrow (k+2)(k-1) \neq 0$ $\Leftrightarrow k \neq -2$ and $k \neq 1$

(b) Infinitely many solutions.

In order for system (*) to have infinitely many solution we must have in R_3''': $k^2+k-2=0$ and $3k=0$

$\Leftrightarrow$ $k=-2$ or $k=1$ and $k=0$

which is impossible.

There is no value of k for which system (*) has infinitely many solutions.

(c) No solution.

- Notice that if $k=-2$, then

$$(*) \Leftrightarrow \begin{aligned} x+y-2z&=3 \\ y-5z&=-5 \\ 0&=-6 \text{ impossible} \end{aligned}$$

hence (*) has no solutions

- Notice that if $k=1$, then

$$(*) \Leftrightarrow \begin{aligned} x+y+z&=3 \\ y+z&=4 \\ 0&=3 \text{ impossible} \end{aligned}$$

hence (*) has no solutions

Conclusion:

System (*) has no solutions iff $k=-2$ or $k=1$.

3 Let

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(a) Show that $A^2 - 2A + 5I_2 = 0_{22}$

- $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix}$

then $A^2 - 2A + 5I_2 = \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - 2\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{22}$$

(b) Show that for every matrix satisfying the matrix equation given in part a), the inverse of A is $A^{-1} = \frac{1}{5}(2I_2 - A)$.

- we have $A \cdot \frac{1}{5}(2I_2 - A) = \frac{2}{5} A I_2 - \frac{1}{5} A^2$

$\boxed{\begin{array}{l} \underline{\text{Note}}\text{: since } A^2 - 2A + 5I_2 = 0_{22} \\ \text{then } A^2 = 2A - 5I_2 \end{array}}$

$$= \frac{2}{5} A I_2 - \frac{1}{5}(2A - 5I_2)$$

$$= \frac{2}{5} A - \frac{2}{5} A + I_2$$

$$= I_2$$

then $\frac{1}{5}(2I_2 - A)$ is a right inverse for matrix A.

- we have $\frac{1}{5}(2I_2 - A) A = \frac{2}{5} I_2 A - \frac{1}{5} A^2$

$$= \frac{2}{5} A - \frac{1}{5}(2A - 5I_2)$$

$$= \frac{2}{5} A - \frac{2}{5} A + I_2$$

$$= I_2$$

then $\frac{1}{5}(2I_2 - A)$ is a left inverse for matrix A.

Since $\frac{1}{5}(2I_2 - A)$ is both a Left and a right inverse for matrix A, then A is invertible and $A^{-1} = \frac{1}{5}(2I_2 - A)$

4 Let A be an $n \times n$ matrix, $n \geq 1$ and let

$$B = A + A^T \quad \text{and } C = A - A^T$$

(a) Show that B is *symmetric.*

- Let A be an $n\times n$ matrix, $n\geq 1$.
  then A^T is an $n\times n$ matrix and
  $B = A + A^T$ is also an $n\times n$ matrix, $n\geq 1$
  thus, we have
  $B^T = (A + A^T)^T = A^T + (A^T)^T$
  $= A^T + A$
  $= A + A^T$
  $= B$
  hence B is a symmetric matrix.

(b) Show that C is *skew-symmetric.*

- Let A be an $n\times n$ matrix, $n\geq 1$,
  then A^T is an $n\times n$ matrix and
  $C = A - A^T$ is an $n\times n$ matrix, $n\geq 1$
  thus, we have
  $C^T = (A - A^T)^T = A^T - (A^T)^T$
  $= A^T - A$
  $= -(A - A^T)$
  $= -C$
  hence C is skew-symmetric.

(c) Show that every $n \times n$ matrix can be represented as a sum of a symmetric matrix and a skew-symmetric matrix.

- Let A be an $n\times n$ matrix, $n\geq 1$, then

$$A = \frac{1}{2}\left(A + A^T + A - A^T\right)$$

$$= \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \frac{1}{2}B + \frac{1}{2}C$$

since B is symmetric then $\frac{1}{2}B$ is also a symmetric matrix and since C is skew-symmetric then $\frac{1}{2}C$ is also skew symmetric. As a result A is the sum of a symmetric and a skew symmetric matrix.

5 A matrix is said to be skew-symmetric if $A^T = -A$. Prove that if a matrix is skew-symmetric then its diagonal entries must be all 0.

Let $A = [a_{ij}]$ be an $n \times n$, $n \geq 1$ skew symmetric matrix

then $A^T = -A \iff [a_{ji}] = -[a_{ij}]$ for any $1 \leq i \leq n$ and any $1 \leq j \leq n$

hence on the diagonal entries we have

$$a_{ii} = -a_{ii} \quad \text{for any } 1 \leq i \leq n$$

$\iff 2a_{ii} = 0$

$\iff a_{ii} = 0$ for any $1 \leq i \leq n$

then the diagonal entries of matrix A must be all 0.

6 Let A and B be two $n \times n$ symmetric matrices, $n \geq 1$. Prove that the product AB is symmetric if and only if $AB = BA$.

$(\Longrightarrow)$ Let A and B be $n\times n$ symmetric matrices, $n \geq 1$

then $A^T = A$ and $B^T = B$.

we have AB is a symmetric $n\times n$, $n\geq 1$ matrix

then $(AB)^T = AB$

$\Longleftrightarrow$ $B^T A^T = AB$

$\Longleftrightarrow$ $BA = AB$ since $B^T = B$ and $A^T = A$.

$(\Longleftarrow)$ Let A and B be $n\times n$ symmetric matrices, $n \geq 1$

then $A^T = A$ and $B^T = B$.

Assume that $AB = BA$, we want to show that AB is a symmetric $n\times n$ matrix, $n \geq 1$.

we have $(AB)^T = B^T A^T$

$= B\,A$ since $B^T = B$ and $A^T = A$

$= AB$ since $BA = AB$

hence AB is symmetric.

7 Prove that if A and B are *diagonal* matrices of the same size, then

$$AB = BA$$

• Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be diagonal matrices of size $n \times n$, $n \geq 1$

then $a_{ij} = 0$ for $i \neq j$ and

$b_{ij} = 0$ for $i \neq j$ for any $1 \leq i \leq n$

and any $1 \leq j \leq n$.

we have AB is an $n \times n$ matrix and

$AB = [c_{ij}]$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = \begin{cases} \sum_{k=1}^{n} a_{kk} b_{kk} & \text{if } i = k \text{ and } j = k \\ 0 & \text{if } i \neq k \text{ or } j \neq k \end{cases}$$

$$= \begin{cases} \sum_{k=1}^{n} b_{kk} a_{kk} & \text{if } i = k \text{ and } j = k \\ 0 & \text{if } i \neq k \text{ and } j \neq k \end{cases}$$

$$= \tilde{c}_{ij}$$

$$= \sum_{k=1}^{n} b_{ik} a_{kj}$$

here $BA = [\tilde{c}_{ij}]$.

then $AB = BA$

8 Let A be a nonsingular matrix of order n, $n \geq 1$. Prove that for any integer $m \geq 1$, we have that A^m is nonsingular and that $(A^m)^{-1} = (A^{-1})^m$.

Let A be a nonsingular matrix of order $n \geq 1$, we use mathematical induction on $m \geq 1$.

• For $m = 1$, we have $(A)^{-1} = A^{-1}$ True

• Assume that $(A^m)^{-1} = (A^{-1})^m$ for some $m \geq 1$,

we must show that $(A^{m+1})^{-1} = (A^{-1})^{m+1}$.

we have $(A^{m+1})^{-1} = (A^m A)^{-1} = A^{-1} (A^m)^{-1}$

$= A^{-1} (A^{-1})^m$ by the hypothesis of induction

$= (A^{-1})^{m+1}$

Therefore, A^m is non singular and $(A^m)^{-1} = (A^{-1})^m$.

9 An $n \times n$ matrix, $n \geq 1$ is said to be *idempotent* if and only if $A^2 = A$. Let A be an idempotent matrix of order $n \geq 1$.

(a) Show that $I_n - A$ is also idempotent.

- $I_n - A$ is an $n \times n$ matrix, $n \geq 1$ and we have

$$(I_n - A)^2 = (I_n - A)(I_n - A) = I_n (I_n - A) - A(I_n - A)$$

$$= I_n^2 - I_n A - A I_n + A^2 = I_n - A - A + A^2$$

$$= I_n - A - A + A \quad \text{since } A \text{ is idempotent then } A^2 = A$$

$$= I_n - A$$

therefore $I_n - A$ is idempotent.

(b) Show that $I_n + A$ is nonsingular and that $(I_n + A)^{-1} = I_n - \frac{1}{2} A$.

- we have $(I_n + A)(I_n - \frac{1}{2} A)$

$$= I_n (I_n - \tfrac{1}{2} A) + A(I_n - \tfrac{1}{2} A)$$

$$= I_n^2 - \tfrac{1}{2} I_n A + A I_n - \tfrac{1}{2} A^2$$

$$= I_n^2 - \tfrac{1}{2} A + A - \tfrac{1}{2} A \quad \text{since } A \text{ is idempotent then } A^2 = A$$

$$= I_n - A + A = I_n$$

then $I_n - \frac{1}{2} A$ is a right inverse for $I_n + A$.

we also have $(I_n - \frac{1}{2} A)(I_n + A)$

$$= I_n (I_n + A) - \tfrac{1}{2} A (I_n + A)$$

$$= I_n^2 + I_n A - \tfrac{1}{2} A I_n - \tfrac{1}{2} A^2$$

$$= I_n + A - \tfrac{1}{2} A - \tfrac{1}{2} A \quad \text{since } A \text{ is idempotent then } A^2 = A$$

$$= I_n + A - A = I_n$$

then $I_n - \frac{1}{2} A$ is a left inverse for $I_n + A$

Conclusion: Since $I_n - \frac{1}{2} A$ is a left and a right inverse of $I_n + A$ then $(I_n + A)^{-1} = I_n - \frac{1}{2} A$

10 Let A and B be nonsingular matrices of size $n \times n$, $n \geq 1$.

(a) Prove that $(AB)^T$ is a nonsingular matrix

Since A is a non singular $n \times n$, $n \geq 1$ matrix, then by a theorem A^T is non singular and $(A^T)^{-1} = (A^{-1})^T$

Since B is a non singular $n \times n$, $n \geq 1$ matrix, then by a theorem B^T is non singular and $(B^T)^{-1} = (B^{-1})^T$

we have $(AB)^T = B^T A^T$ it is the product of two non singular matrices hence $(AB)^T$ is non singular. Note that if $(AB)^T$ was singular then either B^T or A^T is singular and we obtain a contradiction.

(b) Prove that $((AB)^T)^{-1} = (A^{-1})^T(B^{-1})^T$

Notice that AB is a $n \times n$ matrix, $n \geq 1$ and we have $(AB)^T = B^T A^T$, then

$$((AB)^T)^{-1} = (B^T A^T)^{-1}$$
$$= (A^T)^{-1} (B^T)^{-1}$$
$$= (A^{-1})^T (B^{-1})^T.$$

Remark:

On part a) we can also proceed as follows:

since A^T is non singular and $(A^T)^{-1} = (A^{-1})^T$

and B^T is non singular and $(B^T)^{-1} = (B^{-1})^T$

Then $(AB)^T(A^{-1})^T(B^{-1})^T = (AB)^T(B^{-1}A^{-1})^T$

$$= (B^{-1}A^{-1}AB)^T = I_n^T = I_n$$

hence $(A^{-1})^T(B^{-1})^T$ is a right inverse for $(AB)^T$

we also have: $(A^{-1})^T(B^{-1})^T(AB)^T = (B^{-1}A^{-1})^T(AB)^T$

$$= (ABB^{-1}A^{-1})^T = I_n^T = I_n$$

then $(A^{-1})^T(B^{-1})^T$ is a Left inverse for $(AB)^T$,

therefore $(AB)^T$ is a non singular matrix and

$((AB)^T)^{-1} = (A^{-1})^T(B^{-1})^T$