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Math 13: Exam 2 (solutions)

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Instructions:

(a) Solve the given *eight* problems.

(b) The test is closed notes and closed books. No phone or tablet is allowed.

(c) You may use well-known results from the course or the book unless you are specifically asked to reprove them.

(d) Please *show your work*. Always *justify your answers*, unless the omitted argument is trivial.

1 Find the value(s) of λ for which the given matrix is *singular*

$$A = \begin{bmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda - 2 \end{bmatrix}$$

<u>Answer:</u>

A is singular $\Leftrightarrow$ $\det(A) = 0$.

Then we must find the values of λ for which $\det(A) = 0$.

$$\Leftrightarrow \begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda-2 \end{vmatrix} = 0 \Leftrightarrow a_{11} C_{11} + a_{13} C_{13} = 0$$

$$\Leftrightarrow \lambda (-1)^{1+1} M_{11} + 1 \cdot (-1)^{1+3} M_{13} = 0$$

$$\Leftrightarrow \lambda \begin{vmatrix} \lambda & 3 \\ 2 & \lambda - 2 \end{vmatrix} + \begin{vmatrix} 0 & \lambda \\ 2 & 2 \end{vmatrix} = 0$$

$$\Leftrightarrow \lambda \left(\lambda(\lambda - 2) - 6 \right) - 2\lambda = 0$$

$$\Leftrightarrow \lambda \left[\lambda^2 - 2\lambda - 8 \right] = 0$$

$$\Leftrightarrow \lambda (\lambda - 4)(\lambda + 2) = 0$$

$$\Leftrightarrow \lambda = 0 \text{ or } \lambda = 4 \text{ or } \lambda = -2$$

Therefore A is a singular matrix $\Leftrightarrow$ $\lambda \in \{-2, 0, 4\}$

2 Evaluate the following determinant

$$\begin{array}{l} \begin{vmatrix} -2 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 \\ -4 & 4 & 6 & 1 \\ -1 & 1 & 0 & 5 \end{vmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$= \begin{vmatrix} -2 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 \\ -2 & 4 & 5 & 0 \\ -1 & 1 & 0 & 5 \end{vmatrix} \begin{array}{l} R_1 \\ R_2 \\ (-1)R_1 + R_3 \to R_3' \\ R_4 \end{array} = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{vmatrix} -2 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 \\ -2 & 4 & 5 & 0 \\ 9 & 1 & -5 & 0 \end{vmatrix} \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \\ (-5)R_1 + R_4 \to R_4' \end{array}$$

by expanding along column C_4

$$= a_{14} C_{14} = 1(-1)^{1+4} \cdot \begin{vmatrix} 1 & 2 & 2 \\ -2 & 4 & 5 \\ 9 & 1 & -5 \end{vmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} = - \begin{vmatrix} 1 & 2 & 2 \\ 0 & 8 & 9 \\ 9 & 1 & -5 \end{vmatrix} \begin{array}{l} R_1 \\ 2R_1 + R_2 \to R_2' \\ R_3 \end{array}$$

$$= - \begin{array}{c} \begin{array}{ccc} C_1' & C_2' & C_3' \end{array} \\ \begin{vmatrix} 1 & 2 & 2 \\ 0 & 8 & 9 \\ 0 & -17 & -23 \end{vmatrix} \end{array} \begin{array}{l} R_1 \\ R_2' \\ (-9)R_1 + R_3 \to R_3' \end{array}$$

by expanding along column C_1'

$$= - a_{11} C_{11} = -(1)(-1)^{1+1} \cdot \begin{vmatrix} 8 & 9 \\ -17 & -23 \end{vmatrix} = (-1)\left(8(-23) - (-17) \cdot 9\right)$$

$$= (-1)(-184 + 153)$$

$$= (-1)(-31)$$

$$= 31$$

3 Consider the subset of the vector space $M_{n,n}$, $n \geq 1$, defined as

$$W = \{A \in M_{n,n} \mid A^T = A\}$$

Determine whether W is a *subspace* of $M_{n,n}$.

Answer:

- Notice that $I_n \in M_{n,n}$ and that $I_n^T = I_n$ hence $I_n \in W$, then $W \neq \phi$ and we have $W \subset M_{n,n}$.

- Let A_1 and A_2 be in W, then

  $A_1 \in M_{n,n}$ and $A_1^T = A_1$

and

  $A_2 \in M_{n,n}$ and $A_2^T = A_2$.

we have $A_1 + A_2 \in M_{n,n}$ since $M_{n,n}$ is a vector space and hence, it is closed under vectors addition, we also have

$(A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2$, hence $A_1 + A_2 \in W$, and thus W is closed under vectors addition.

- Let $A_1 \in W$ and c a scalar, then

  $A_1 \in M_{n,n}$ and $A_1^T = A_1$, hence $cA_1 \in M_{n,n}$ since $M_{n,n}$ is a vector space, hence it is closed under scalar multiplication.

We also have $(cA_1)^T = cA_1^T = cA_1$, then $cA_1 \in W$, hence W is closed under scalar multiplication.

Conclusion: W is a subset of $M_{n,n}$ that is closed under vectors addition and scalar multiplication, hence W is a subspace of $M_{n,n}$ (by a theorem).

4 Let A and B be $n \times n$ matrices, $n \geq 1$.

(a) Prove that AB is nonsingular if and only if A and B are both non singular.

Answer: Note that AB is an $n\times n$ matrix, $n \geq 1$ since A and B are $n\times n$ matrices and we have

AB is non singular $\Leftrightarrow \det(AB) \neq 0 \Leftrightarrow \det(A)\det(B) \neq 0$

$\Leftrightarrow \det(A) \neq 0$ and $\det(B) \neq 0$

$\Leftrightarrow$ A is non singular and B is non singular.

(b) Prove that if $AB = I_n$, then $BA = I_n$.

Answer: Note that if $AB = I_n$ then $\det(AB) = \det(I_n) = 1$

$$\Leftrightarrow \det(A)\det(B) = 1 \Rightarrow \begin{cases} \det(A) \neq 0 \Rightarrow A \text{ is nonsingular, hence } A^{-1} \text{ exists} \\ \text{and} \\ \det(B) \neq 0 \Rightarrow B \text{ is nonsingular, hence } B^{-1} \text{ exists} \end{cases}$$

then $AB = I_n$

$\Rightarrow (AB)^{-1} = I_n^{-1} = I_n$

$\Leftrightarrow B^{-1}A^{-1} = I_n$

thus $BA = BI_nA = B\underbrace{(B^{-1}A^{-1})}_{I_n}A = \underbrace{(BB^{-1})}_{I_n}\underbrace{(A^{-1}A)}_{I_n}$

$= I_n \cdot I_n$

$= I_n$

5 Parts (*a*) and (*b*) of this problem are *independent.*

(a) Let A be a non singular $n \times n$ matrix, $n \geq 1$. Prove if matrix B is row equivalent to A, then B is also non singular.

Answer:

- Let A be an $n\times n$ nonsingular matrix, $n \geq 1$, then by a theorem A is row equivalent to I_n, hence there exist elementary matrices $E_1, E_2, \dots, E_k$, $k \geq 1$ (here E_i is of order n for $1 \leq i \leq k$) such that

$A = E_k E_{k-1} \cdots E_2 E_1 I_n$ (1)

- we have B is row equivalent to A then there exist elementary matrices $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{k'}$, $k' \geq 1$ (here $\tilde{E}_i$ is of order n for $1 \leq i \leq k'$) such that

$B = \tilde{E}_{k'} \tilde{E}_{k'-1} \cdots \tilde{E}_1 A$ (2)

substituting (1) in (2) yields:

$B = \underbrace{\tilde{E}_{k'} \cdots \tilde{E}_1 E_k \cdots E_1}_{\text{a finite product of } k+k' \text{ elementary}} I_n$

matrices of size $n\times n$, then B is row equivalent to I_n, thus B is non singular by a theorem.

(b) Prove that if matrix A is row equivalent to matrix B and matrix B is row equivalent to matrix C, then A is row equivalent to C.

Answer:

- Let A, B and C be $m\times n$ matrices $m \geq 1$ and $n \geq 1$, then
- A is row equivalent to B

$\Leftrightarrow$ there exist elementary matrices $E_1, E_2, \dots, E_k$, $k \geq 1$ of size $m\times m$ such that

$A = E_k E_{k-1} \cdots E_1 B$ (1)

- B is row equivalent to C

$\Leftrightarrow$ there exist elementary matrices $\tilde{E}_1, \tilde{E}_2, \dots, \tilde{E}_{k'}$, $k' \geq 1$ of size $m\times m$ such that

$B = \tilde{E}_{k'} \tilde{E}_{k'-1} \cdots \tilde{E}_1 C$ (2)

substituting (2) in (1) yields

$A = (E_k \cdots E_1)(\tilde{E}_{k'} \cdots \tilde{E}_1 C)$

$= \underbrace{(E_k \cdots E_1 \tilde{E}_{k'} \cdots \tilde{E}_1)}_{\text{a finite product of } k+k'} C$

elementary matrices of size $m\times m$, therefore, A is row equivalent to C.

6 Recall that a matrix A is skew-symmetric if and only if $A^T = -A$. Prove that if A is an $n \times n$, $n \geq 1$, skew-symmetric matrix and n is *odd* then A must be singular.

<u>Answer:</u>

Let A be an $n \times n$, $n \geq 1$, skew-symmetric matrix such that n is odd, then we have

$A^T = -A$

hence

$\det(A^T) = \det(-A)$

$\Leftrightarrow \det(A) = (-1)^n \det(A)$

since by a theorem we have $\det(A^T) = \det(A)$ and by a theorem we have $\det(cA) = c^n \det(A)$ for any $n \times n$ matrix, $n \geq 1$

$\Rightarrow \det(A) = (-1) \det(A)$

Note that since n is <u>odd</u> then $(-1)^n = -1$

hence $\det(A) = -\det(A)$

$\Leftrightarrow 2\det(A) = 0$

$\Leftrightarrow \det(A) = 0$

$\Leftrightarrow$ A is a singular matrix.

7 Let $\mathbb{R}^2$ be the set of all ordered pairs of real numbers equipped with the operations:

addition defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$$

and scalar multiplication defined by

$$c \odot (x_1, x_2) = (-cx_1, -cx_2),$$

here $c \in \mathbb{R}$ is a scalar. Note that both the operations of addition and scalar multiplication here are *non standard.* Is $\mathbb{R}^2$ in this case a vector space ? (*Justify your answer*)

Answer:

Notice that for $u = (x_1, x_2)$ and $v = (y_1, y_2)$, we have that $u \oplus v = (x_1, x_2) \oplus (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$ but $v \oplus u = (y_1, y_2) \oplus (x_1, x_2) = (y_1 + x_2, y_2 + x_1)$

$$\neq (x_1 + y_2, x_2 + y_1) = u \oplus v$$

hence the commutative property for the operation of vectors addition (axiom 2) fails, thus $(\mathbb{R}^2, \oplus, \odot)$ is not a vector space.

Note: one can also show that other axioms fail.

8 A subspace W of a vector space V is said to be a *proper* subspace if $W \neq V$ and $W \neq \{0_V\}$, here 0_V is the zero vector in V. What are the *proper* subspaces of $\mathbb{R}^2$ that contain the point $A(1,2)$? (*Justify your answer*)

Answer:

The proper subspaces of $(\mathbb{R}^2, +, \cdot)$ (here $\mathbb{R}^2$ is equipped with the standard operations of addition and scalar multiplications) are Lines passing through $(0,0)$. Since we are looking for the proper subspaces of $\mathbb{R}^2$ that contain point $A(1,2)$. Then it is the Line in $\mathbb{R}^2$ that passes through $(0,0)$ and point $A(1,2)$, it has the equation: $y = 2x$

Thus $W = \{(x,y) \mid y = 2x, x \in \mathbb{R}\}$

$= \{(x, 2x) \mid x \in \mathbb{R}\}$