To represent a matrix, we use one of these conventions:
1) an uppercase latter: A, B, C, D...
2) A representative element $[a_{ij}], [b_{ij}], [c_{ij}]$

3) A rectangular array: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{32} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$

Remark:
1) The matrices that we will consider have real valued entries: $a_{ij} \in \mathbb{R}$
for $1 \leq i \leq m$, $1 \leq j \leq n$
2) Matrices, depending on the applications, can have complex-valued entries $a_{ij} \in \mathbb{C}$

<u>Definition: Equality of Matrices</u>
Two matricies $A = [b_{ij}]$ and $B = [b_{ij}]$ are equal $\iff$
1) A and B have the same size
2) $a_{ij} = b_{ij}$ for any $1 \leq i \leq m$, $1 \leq j \leq n$

Example:

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 4 \\ -1 & 2 \\ 3 & 5 \end{bmatrix} = B$$

A and B have different sizes, hence are <u>not</u> equal

<u>Definitions: Row Matrix, Column Matrix</u>
A matrix that has one column is called a column matrix (or a column vector)

Example (Column Matrix):

$$A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

A matrix that has one row is called a row matrix (or a row vector)

Example (Row Matrix):

$B = \begin{bmatrix} 4 & 1 & 3 & 0 \end{bmatrix}$

Definition: Matrix Addition
If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matricies of sized $m * n$, then their sum is the $m * n$ matrix.

$A + B = [a_{ij} + b_{ij}]$ for $1 \leq i \leq m$, $1 \leq j \leq n$

Remark: The sim of two matrices of different sizes is undefined

Example:
a) Let $A = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & -6 \end{bmatrix}$
and $B = \begin{bmatrix} 0 & -2 & -4 \\ 1 & 0 & -1 \end{bmatrix}$
Since A and B have equal size, then their sum is the $2 * 3$ matrix

$$A + B = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 3 & -6 \end{bmatrix} + \begin{bmatrix} 0 & -2 & -4 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+0 & 2+(-2) & 4+4 \\ 0+1 & 3+0 & -6+(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 8 \\ 1 & 3 & -7 \end{bmatrix}$$

b) Let $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 4 \\ -1 & 2 & 5 \end{bmatrix}$
and $B = \begin{bmatrix} -1 & 0 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$

$\rightarrow$ the matrices have different sizes hence their sum is undefined

Example:
Consider the matrices A and B given in part a) on the previous examles, then it is easy to verify that: $A + B = B + A = \begin{bmatrix} -1 & 0 & 8 \\ 1 & 3 & -7 \end{bmatrix}$

$\rightarrow$ In general, if A and B are matricies of size $m * n$, then $A + B = B + A$, hence the oeration of matrix addition is commutative.

<u>Definition: Matrix Scalar Multilication</u>
If $A = [a_{ij}]$ is a matrix of size $m * n$ and $c \in \mathbb{R}$ is a scalar, then that scalar multiple of A is the $m * n$ matrix

$cA = c[a_{ij}] = [ca_{ij}]$ for $1 \leq i \leq m$, $1 \leq j \leq n$

Example:
Let $A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -4 & 1 \\ 4 & -2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$
then

$$-2A + 3B = -2 \begin{bmatrix} -1 & 3 & 0 \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & -4 & 1 \\ 4 & -2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -6 & 0 \\ 0 & -2 & -2 \\ -4 & -8 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -12 & 3 \\ 12 & -6 & 0 \\ 3 & 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -18 & 3 \\ 12 & -8 & -2 \\ -1 & -5 & 9 \end{bmatrix}$$

<u>Definition: Matrix Multiplication</u>
If $A = [a_{ij}]$ is an $m * n$ matrix
and $B = [b_{ij}]$ is an $n * p$ matrix

Then their product is the matrix AB of size $m * p$
here $AB = [C_{ij}]$, where

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} a_{ik}b_{kj} \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj} \end{aligned}$$

here $1 \leq i \leq m$, $1 \leq j \leq p$

Remark: On multilying a row by a column

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} xa & yb & zc \end{bmatrix}$$

Example:

Let $A = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & 5 \end{bmatrix}$
and $B = \begin{bmatrix} -2 & 3 \\ -4 & 1 \end{bmatrix}$
then

$$AB = \begin{bmatrix} 1 & 0 \\ -2 & 4 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ -4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-2) + 0(-4) & 1(3) + 0(1) \\ -2(-2) + 4(-4) & -2(3) + 4(1) \\ 3(-2) + 5(-4) & 3(3) + 5(1) \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 \\ -12 & -2 \\ -26 & 14 \end{bmatrix}$$

In general, matrix multiplication is <u>not</u> commutative, that is $AB \neq BA$

Remark: System of linear equation and matrices
A system $(*)$ of m linear equations in n variables can be reresented by matrix multiplication as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{32} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

$$\Leftrightarrow Ax = b$$

hence $(*) \Leftrightarrow Ax = b$

Example:
Consider the system

$$(*) \quad \begin{cases} x - 2y + 3z = 1 \\ -x + 4y + \ \ z = 4 \\ 2x - \ \ y \qquad = 3 \end{cases}$$

$(*)$ is a system of $m = 3$ linear equations in $n = 3$ variables

Notice that $(*) \Leftrightarrow \begin{bmatrix} 1 & -2 & 3 \\ -1 & 4 & 1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

Example:

Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix}$

and $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

and $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

Solve the matrix equation: $Ax = b$

We have $Ax = b$

$$\Leftrightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x_1 - 2x_2 + x_3 \\ 2x_1 + 3x_2 - x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} x_1 - 2x_2 + x_3 = 0 \\ 2x_1 + 3x_2 - x_3 = 0 \end{cases}$$

The corresonding augmented matrix is:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 3 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 7 & -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{3}{7} & 0 \end{bmatrix}$$

the the corresonding system of linear equations is:

$$\begin{cases} x_1 - 2x_2 + \phantom{\tfrac{3}{7}} x_3 = 0 & \quad 1 \\ x_2 - \frac{3}{7}x_3 = 0 & \quad 2 \end{cases}$$

Let $x_3 = t$, $t \in \mathbb{R}$ is a parameter
then
2 $\rightarrow x_2 = \frac{3}{7}x_3 = \frac{3}{7}t$
and 1 $\rightarrow x_1 = 2x_2 - x_3 = 2(\frac{3}{7}t) - t = \frac{6}{7}t - t = -\frac{t}{7}$

then $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7}t \\ \frac{3}{7}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{1}{7} \\ \frac{3}{7} \\ 1 \end{bmatrix}$

Then the solution set of $(*)$ is

$$\left\{ x = \begin{bmatrix} -\frac{1}{7} \\ \frac{3}{7} \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

hence $(*)$ has infinitely many solutions, and it is consistent.

Partitioned matrices:
We can represent the system $(*)$ or equivalently $Ax = b$ by partitioning matrix A and x as follows:

$$\Leftrightarrow \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 & \dots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & a_{32}x_3 & \dots & a_{2n}x_n \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_2 & \dots & a_{3n}x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1}x_1 & a_{m2}x_2 & a_{m3}x_3 & \dots & a_{mn}x_n \end{bmatrix} = b$$

$$\Leftrightarrow x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \\ \vdots \\ a_{m2} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{32} \\ \vdots \\ a_{m3} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ a_{3n} \\ \vdots \\ a_{mn} \end{bmatrix} = b$$

A_i is of size $m * 1$ for $1 \leq i \leq n$

$$\Leftrightarrow x_1A_1 + x_2A_2 + x_3A_3 + x_nA_n = b$$

here A_1, A_2, $A_3, \ldots,$ A_n for a partition of matrix A
(i.e. $A = \begin{bmatrix} A_1 & A_2 & A_3 & \ldots & A_n \end{bmatrix}$)

hence $Ax = x_1A_1 + x_2A_2 + x_3A_3 + \ldots + x_nA_n = b \qquad (2*)$
This is a linear combination of the $m * 1$ matricies Ai's $(1 \leq i \leq n)$

Remark:
System $(*)$: $Ax + b$ is consistent $\iff$ b can be exressed as such a linear combination as given in $(2*)$, where: x_1, x_2, $x_3, \ldots, x_n$ are solutions of the system.