0.1 Elementary Matrices

44 in 2.4
1) Prove that A is idempotent if and only if A^T is idempotent
2) Prove that if A is an $n \times n$ matrix that is idempotent and invertible, then $A = I_n$
3) Prove that if A and B are idempotent and $AB = BA$, then AB is idempotent
4) Prove that if A is row-equivalent to B and B is row-equivalent to C, then A is row-equivalent to C
5) Prove that if matrix A is row equivalent to matrix B then matrix B is row equivalent to matrix A
6) Let A be a nonsingular matrix. Prove that if B is row-equivalent to A, then B is also non-singular.

0.2 The Determinant of a Matrix

Theorem 1 (Determinant of an Upper or Lower Triangular Matrix). *If A is triangular (upper or lower matrix of order $n \geq 1$, and if $A = [a_{ij}]$ for $1 \leq n$, $1 \leq j \leq n$) then its determinant is the product of the entries on the main diagonal that is:*

$$\det(A) = |A| = a_{11}a_{22}a_{33}\dots a_{nn}$$
$$= \prod_{i=1}^{n} a_{ii}$$

0.3 Determinants and Elementary Row Oerations

Theorem 2 (Elementary Row Operations and Determinants). *Let A and B be square matrices of the same size, then*
1) When B is obtained from A by interchanging two rows of A, then $\det(A) = -\det(B)$
2) When B is obtained from A by adding a multiple of a row of A to another row, then $\det(B) = \det(A)$
3) When B is obtained from A by multiplying a row of A by a non-zero contstant c, then $\det(B) = c\det(A)$

Theorem 3 (Conditions That Yield a Zero Determinant). *If A is a square matrix and any of the conditions given below is true, then* $\det(A) = 0$
1) An entire row (or an entire column) consists of zeros
2) Two rows (or two columns) are equal
3) One row (or one column) is a multiple of another row (is a multiple of another column)

0.4 Properties of Determinants

Theorem 4 (Determinant of a Finite Matrix Product). *If A and B are square matrices of order $n \geq 1$ then:* $\det(AB) = \det(A)\det(B)$

Remark. If $A_1A_2A_3\dots A_k (k \geq 1)$
If these are matrices of the same order $n \geq 1$, then

$$\det(A_1A_2A_3\dots A_k) = \det(A_1)\det(A_2)\det(A_3)\dots\det(A_k)$$
$$\Leftrightarrow \det(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} \det(A_i)$$

Theorem 5 (Determinant of a Scalar Multiple of a Matrix)**.** *If A is a square matrix of order $n \geq 1$ and c is a scalar, then*

$$\det(cA) = c^n \det(A)$$

1) Let A and B be $n \times n$ matrices such that $AB = I$. Prove that $|A| \neq 0$ and $B \neq 0$
2) Let A and B be $n \times n$ matrices such that AB is singular. Prove that either A or B is singular
3) Let A be an $n \times n$ matrix in which the entires of each row sum to zero. Find $|A|$
4) Prove that the determinant of an invertible matrix A is equal to ± 1 when all of the entries of A and A^{-1} are integers
5) If A is a square matrix, then $\det(A) = \det(A^T)$
6) A square matrix is skew-symmetric when $A^T = -A$. Prove that if A is an $n \times n$ skew-symmetric matrix, then $|A| = (-1)^n|A|$
7) Let A be a skew-symmetric matrix of odd order. Use the result of exercise 69 (6) to prove that $|A| = 0$
8) Prove that the $n \times n$ identity matrix is orthogonal
9) Prove that if A is an orthogonal matrix, then $|A| = \pm 1$
10) If A is an idempotent matrix ($A^2 = A$), then prove that the determinant of A is either 0 or 1
11) Let S be an $n \times n$ singular matrix. Prove that for any $n \times n$ matrix B, the matrix SB is also singular