Linear Algebra Homework 9

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5.1:

- 15. For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues). Visit goo.gl/7Qss2u for a solution.

  The characteristic polynomial of a matrix A is defined by $p(\lambda) = \det(\lambda I - A)$. For any matrix B, $\det(B) = \det(B^T)$. Applying this property to the characteristic polynomial, we have:

  $$p(\lambda) = \det(\lambda I - A) = \det((\lambda I - A)^T) = \det(\lambda I - A^T).$$

  So, the characteristic polynomial of A is $\det(\lambda I - A)$ and that of A^T is $\det(\lambda I - A^T)$. Since these are equal, A and A^T have the same characteristic polynomial. Thus, A and A^T have the same eigenvalues, as these are determined by the roots of the characteristic polynomial. $\square$

- 17. Let T be a linear operator on a finite-dimensional vector space V, and let c be any scalar.

  (a) Determine the relationship between the eigenvalues and eigenvectors of T (if any) and the eigenvalues and eigenvectors of $U = T - cI$. Justify your answers.

  Given v as an eigenvector of T, then $T(v) = \lambda v$. Applying U to v:

  $$U(v) = (T - cI)(v) = T(v) - cI(v) = \lambda v - cv = (\lambda - c)v.$$

  Therefore, v is also an eigenvector of U with the eigenvalue $\lambda - c$. The eigenvectors of T and U remain the same. Each eigenvalue λ of T corresponds to the eigenvalue $\lambda - c$ of U. This means every eigenvalue of T is scaled by $-c$ in U, while the eigenvectors are unchanged. $\square$

  (b) Prove that T is diagonalizable if and only if U is diagonalizable.

  1. **($\Rightarrow$) T is diagonalizable.** Since T is diagonalizable, there exists a basis of V consisting of eigenvectors of T. These same vectors are eigenvectors of U with eigenvalues $\lambda - c$, where λ are eigenvalues of T. Hence, U is also diagonalizable.
  2. **($\Leftarrow$) U is diagonalizable.** Similarly, if U is diagonalizable with a basis of eigenvectors corresponding to eigenvalues $\lambda - c$, these eigenvectors are also eigenvectors of T with eigenvalues λ. Thus, T is diagonalizable. $\square$

- 20. Let A be an $n \times n$ matrix with characteristic polynomial

  $$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0.$$

  Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

We can rewrite out polynomial as

$$f(t) = \det(tI - A)$$

And, at $t = 0$,

$$f(0) = \det(-A) = (-1)^n \det(A)$$

Since $f(0)$ is the constant term a_0 of $f(t)$, it follows that

$$f(0) = (-1)^n \det(A) = a_0$$

Thus,

$$\det(A) = a_0.$$

A matrix A is invertible if and only if:

$$\det(A) \neq 0$$

So, given $\det(A) = a_0$, A is invertible if and only if:

$$a_0 \neq 0 \square$$

5.2:

- 3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

  (b) $V = P_2(\mathbb{R})$ and T is defined by $T(a_2x^2 + bx + c) = cx^2 + bx + a$.

  Consider the standard basis $\beta = \{1, x, x^2\}$. So,

  $$T(1) = 1, \quad T(x) = x, \quad T(x^2) = x^2$$

  Each element of β is an eigenvector with eigenvalue $\lambda = 1$ since T acts as the identity operator on these elements. Since all basis elements are eigenvectors with the same eigenvalue, T is diagonalizable. So,

  $$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

  The basis $\beta = \{1, x, x^2\}$ confirms T's diagonalization, without any change of basis yet needed. $\square$

- 8. Suppose that $A \in M_{n\times n}(\mathbb{F})$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(E_{\lambda_1}) = n-1$. Prove that A is diagonalizable.

  Vectors in E_{λ_2} are linearly independent of those in E_{λ_1} due to a different eigenvalue ($\lambda_2 \neq \lambda_1$). The sum of the dimensions of E_{λ_1} and E_{λ_2} is n (since $\dim(E_{\lambda_1}) = n - 1$ and $\dim(E_{\lambda_2}) = 1$).

  Because A has n linearly independent eigenvectors that span $\mathbb{F}^n$, it forms a complete basis, making A diagonalizable:

  $$\dim(E_{\lambda_1}) + \dim(E_{\lambda_2}) = n$$

  Thus, A is diagonalizable as there exists a basis for $\mathbb{F}^n$ consisting entirely of eigenvectors of A, making it possible to express A diagonalizedly. $\square$

- 14. Let $A \in M_{n\times n}(\mathbb{F})$. Recall from Exercise 15 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

  (a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

  Consider the matrix

  $$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

  The characteristic polynomial for both A and A^t is:

  $$\chi(\lambda) = \lambda^2,$$

  So eigenvalue $\lambda = 0$ with algebraic multiplicity 2.

  For A, solving $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives:

  $$y = 0,$$

  so E_0 is spanned by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

  For A^t, solving $A^t\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ gives:

  $$x = 0,$$

  so E'_0 is spanned by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

  Even though A and A^t share the same eigenvalues, their eigenspaces corresponding to these eigenvalues are different:

  - E_0 for A: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$
  - E'_0 for A^t: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ □

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

A matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix D such that:

$$A = PDP^{-1}$$

Recall our transformation properties of matrices:

- $(AB)^t = B^t A^t$
- For a diagonal matrix D, $D^t = D$

Starting with the diagonalizability of A:

$$A = PDP^{-1}$$

Taking the transposition of t on both sides:

$$A^t = (PDP^{-1})^t = (P^{-1})^t DP^t$$

Since $D^t = D$ and $(P^{-1})^t = (P^t)^{-1}$, we simplify to:

$$A^t = (P^t)^{-1} DP^t$$

This shows $A^t = (P^t)^{-1} DP^t$ represents A^t in a diagonalized form where P^t is the invertible matrix of eigenvectors and D remains the diagonal matrix of eigenvalues. Thus, A^t is diagonalizable. $\square$