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**MATH 112: Introduction to Analysis**
**Fall 2023 Semester**
**Homework 8: Due Monday December 04, 11:00pm PST.**

**Instructions**

- Write your full name, "Homework 8", and the date at the top of the first page.
- Show all work and explain your reasoning. Write in complete sentences.
- Typeset your solutions in LaTeX.
- Submit a single .pdf file to Gradescope under the assignment "Homework 8".
- You must use Gradescope to electronically match problems to pages in your .pdf
- Questions? Email me or come to office hours.
- You are strongly encouraged to work together! Just write up your own solutions.

**Assignment** (4 Problems: 25 + 25 + 25 + 25 = 100 points total.)

☐ **Problem 1** [Series] Determine if the given series converges or not. Give a proof.

*PS: I know we did some of these a specific way in class, which I am not.*

- 1.1 [5 points] $\sum\limits_{k=0}^{\infty} \frac{4k}{k+1}$
  We know that for two series $a_k$ and $b_k$ with positive terms, if $\lim_{k\to\infty} \frac{a_k}{b_k} = c$ where $c$ is a finite rational and $c > 0$ (i.e., $b_k \in \Theta(a_k)$), then either both series converge or both diverge.

  Let's compare the given series with $\sum_{k=1}^{\infty} \frac{4k}{k} = \sum_{k=1}^{\infty} 4$. Let:

$$a_k := \frac{4k}{k+1} \quad \text{and} \quad b_k := 4$$

  Calculating the limit of $\frac{a_k}{b_k}$:

$$\lim_{k\to\infty} \frac{\frac{4k}{k+1}}{4} = \lim_{k\to\infty} \frac{k}{k+1}$$

$$= \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}}$$

$$= \lim_{k\to\infty} \left(\frac{1}{1} + \frac{1}{k}\right)$$

$$= \lim_{k\to\infty} \frac{1}{1} + \lim_{k\to\infty} \frac{1}{k} = 1$$

Therefore, since this limit is a finite and greater than 0, and the series $\sum_{k=1}^{\infty} 4$ diverges, $\boxed{\text{the given series } \sum_{k=0}^{\infty} \frac{4k}{k+1} \text{ also diverges.}}$

- 1.2 [5 points] $\sum\limits_{k=0}^{\infty} \frac{4^k}{k!}$

  Consider the exponential series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. This series converges for all real values of $x$. Specifically, for $x = 4$, the series becomes $\sum_{k=0}^{\infty} \frac{4^k}{k!}$, which is the series in question.

  Our test from before states that for two series $a_k$ and $b_k$ where $0 \leq a_k \leq b_k$ for all $k \geq n_0$ for some $n_0 \in \mathbb{Z}_+$, if $b_k$ converges, then $a_k$ also converges.

  Here, the series $\sum_{k=0}^{\infty} \frac{4^k}{k!}$ is non-negative and is the series for $e^4$, which we know to converge.[1] Therefore, by our test, $\boxed{\text{our series } \sum_{k=0}^{\infty} \frac{4^k}{k!} \text{ converges.}}$

- 1.3 [5 points] $\sum\limits_{k=0}^{\infty} \frac{(\mathbf{i}\pi)^k}{k!}$ where $\mathbf{i} \in \mathbb{C}$ is the imaginary unit and $\pi \in \mathbb{R}$.

  Like before, this series represents the exponential function for the complex number $i\pi$: The exponential function is defined as $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$. For our series, $z = i\pi$.

  We know that a series $a_k$ converges absolutely if the series of the absolute values of its terms, $\sum |a_k|$, converges.

  For $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$, let $z = a + bi$ be a complex number, where $a, b \in \mathbb{R}$. The absolute value of the term is $|\frac{z^k}{k!}|$. The magnitude of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$. Thus, the absolute value of the $k$-th term is:

$$\left|\frac{z^k}{k!}\right| = \frac{|z|^k}{k!}$$

  We compare this with $\sum_{k=0}^{\infty} \frac{|z|^k}{k!}$, which is $e^{|z|}$. Since $e^{|z|}$ converges for real $|z|$, this series converges. Since $e^{|z|}$ converges and bounds the terms $\frac{z^k}{k!}$ in magnitude, by comparison, the series $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges absolutely for all complex numbers $z$.

  We know that the exponential function $e^z$ converges absolutely for all complex numbers $z$. $\boxed{\text{Therefore, the series } \sum_{k=0}^{\infty} \frac{(i\pi)^k}{k!} \text{ converges.}}$

- 1.4 [10 points] $\sum\limits_{\ell=0}^{\infty} (-1)^\ell \frac{x^{2\ell}}{(2\ell)!}$ where $x \in \mathbb{R}$ satisfies $|x| \leq 112$.

  We again consider absolute convergence. The absolute value of the $\ell$-th term of our series is:

[1] A more general proof of this is given later.

$$\left|(-1)^{\ell}\frac{x^{2\ell}}{(2\ell)!}\right| = \frac{|x|^{2\ell}}{(2\ell)!}.$$

The series $\sum_{\ell=0}^{\infty}\frac{|x|^{2\ell}}{(2\ell)!}$ corresponds to the series for $e^{|x|^2}$, which converges for all real numbers, including $|x| \leq 112$.

Since $\sum_{\ell=0}^{\infty}\frac{|x|^{2\ell}}{(2\ell)!}$ converges and bounds the terms of our original series in magnitude, by comparison, $\boxed{\text{the series } \sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{x^{2\ell}}{(2\ell)!} \text{ converges}}$ absolutely for all $x \in \mathbb{R}$ with $|x| \leq 112$.

☐ **Problem 2** [Calculus] Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{1-x}.$$

In each problem below, show your work and cite results from MATH 111 *Calculus* as needed.

- 2.1 [5 points] Calculate $f'(x)$ and $f'(5)$.

  To find the derivative $f'(x)$ of the function $f(x) = \frac{1}{1-x}$, we rewrite $f(x)$ as $f(x) = (1-x)^{-1}$ and use the power rule for differentiation. Applying the power rule, which states that if $g(x) = x^n \Rightarrow g'(x) = nx^{n-1}$, we get:

  $$f'(x) = \frac{d}{dx}(1-x)^{-1} = -1 \cdot (1-x)^{-2} \cdot (-1)$$
  $$f'(x) = (1-x)^{-2}$$

  Substitute $x = 5$ into $f'(x)$:

  $$f'(5) = (1-5)^{-2} = (-4)^{-2}$$
  $$f'(5) = \frac{1}{16}$$

  So, $\boxed{f'(x) \text{ of } f(x) = \frac{1}{1-x} \text{ is } (1-x)^{-2}, \text{ and } f'(5) = \frac{1}{16}.}$

- 2.2 [5 points] Calculate $f''(x)$ and $f''(-1)$.

  To find the second derivative, we differentiate the first derivative $f'(x) = (1-x)^{-2}$ again. Applying the power rule and the chain rule, we get:

  $$f''(x) = \frac{d}{dx}[(1-x)^{-2}] = -2(1-x)^{-3} \cdot (-1)$$
  $$f''(x) = 2(1-x)^{-3}$$

  Substitute $x = -1$ into $f''(x)$:

$$f''(-1) = 2(1-(-1))^{-3} = 2(2)^{-3}$$
$$f''(-1) = \frac{2}{8} = \frac{1}{4}$$

So, $\boxed{f''(x) \text{ of } f(x) = \frac{1}{1-x} \text{ is } 2(1-x)^{-3}, \text{ and } f''(-1) = \frac{1}{4}.}$

- 2.3 [5 points] Calculate $f^{(k)}(x)$ and $f^{(k)}(0)$ for all $k \in \{0, 1, 2, 3, 4, 5\}$.

  $k = 0$

  - $f^{(0)}(x) = f(x) = \frac{1}{1-x}$
  - $f^{(0)}(0) = \frac{1}{1-0} = 1$

  $k = 1$

  - $f^{(1)}(x) = f'(x) = (1-x)^{-2}$
  - $f^{(1)}(0) = (1-0)^{-2} = 1$

  $k = 2$

  - $f^{(2)}(x) = f''(x) = 2(1-x)^{-3}$
  - $f^{(2)}(0) = 2(1-0)^{-3} = 2$

  $k = 3$

  - $f^{(3)}(x) = 6(1-x)^{-4}$
  - $f^{(3)}(0) = 6(1-0)^{-4} = 6$

  $k = 4$

  - $f^{(4)}(x) = 24(1-x)^{-5}$
  - $f^{(4)}(0) = 24(1-0)^{-5} = 24$

  $k = 5$

  - $f^{(5)}(x) = 120(1-x)^{-6}$
  - $f^{(5)}(0) = 120(1-0)^{-6} = 120$

- 2.4 [5 points] Propose an explicit formula for $f^{(k)}(x)$ in terms of $x$ and $k$, then prove that your formula is correct for all $k \in \mathbb{N}$ by induction.

  In general, $f^{(k)}(x)$ seems to be:

  $$f^{(k)}(x) = k!(1-x)^{-(k+1)},$$

  which is consistent with the specific values we calculated for $k \in \{0, 1, 2, 3, 4, 5\}$.

  **Base Case ($k = 1$):**

  We know that $f'(x) = (1-x)^{-2}$. This can be rewritten as $1! \cdot (1-x)^{-2}$, since $1! = 1$. Thus, the formula holds for $k = 1$.

  **Inductive Step**:

  Assume the formula holds for some $k = n$: that $f^{(n)}(x) = n! \cdot (1-x)^{-(n+1)}$. We need to show that $f^{(n+1)}(x) = (n+1)! \cdot (1-x)^{-((n+1)+1)}$.

Differentiate $f^{(n)}(x)$ to get $f^{(n+1)}(x)$:

$$\begin{aligned} f^{(n+1)}(x) &= \frac{d}{dx}[n! \cdot (1-x)^{-(n+1)}] \\ &= n! \cdot \frac{d}{dx}[(1-x)^{-(n+1)}] \\ &= n! \cdot (-(n+1)) \cdot (1-x)^{-(n+2)} \cdot (-1) \\ &= n! \cdot (n+1) \cdot (1-x)^{-(n+2)} \\ &= (n+1)! \cdot (1-x)^{-(n+2)} \end{aligned}$$

Therefore, by induction, $\boxed{\text{the formula } f^{(k)}(x) = k! \cdot (1-x)^{-(k+1)} \text{ is true for all } k \in \mathbb{N}_{\geq 1}.}$

- 2.5 [5 points] Find $\varepsilon > 0$ so $f(0.8) \approx_\varepsilon \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k$. Recall $\approx_\varepsilon$ from Midterm 2.

We'll nominate $\varepsilon := 1.7 \in \mathbb{R}$.
We must then show that $|f(0.8) - \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k < \varepsilon|$. Starting on the LHS:

$$\begin{aligned} |f(0.8) - \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k| &= |\frac{1}{1-0.8} - \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k| \\ &= |\frac{1}{0.2} - \sum_{k=0}^{4} \frac{k! \cdot (1-0)^{-(k+1)}}{k!}(0.8)^k| \\ &= |5 - \sum_{k=0}^{4} \frac{k! \cdot 1^{-(k+1)}}{k!}(0.8)^k| \end{aligned}$$

Which we can expand to

$$= \left|5 - \left(\tfrac{0! \cdot 1^{-(0+1)}}{0!}(0.8)^0 + \tfrac{1! \cdot 1^{-(1+1)}}{1!}(0.8)^1 + \tfrac{2! \cdot 1^{-(2+1)}}{2!}(0.8)^2 + \tfrac{3! \cdot 1^{-(3+1)}}{3!}(0.8)^3 + \tfrac{4! \cdot 1^{-(4+1)}}{4!}(0.8)^4\right)\right|$$

$$= \left|5 - \left(1 + 1^{-2}(0.8) + \frac{2 \cdot 1^{-(3)}}{2}(0.8)^2 + \frac{6 \cdot 1^{-(4)}}{6}(0.8)^3 + \frac{24 \cdot 1^{-(5)}}{24}(0.8)^4\right)\right|$$

$$= \left|5 - \left(1 + 0.8 + \frac{2}{2}(0.8)^2 + \frac{6}{6}(0.8)^3 + \frac{24}{24}(0.8)^4\right)\right|$$

$$= \left|4 - \left((0.8)^1 + (0.8)^2 + (0.8)^3 + (0.8)^4\right)\right|$$

Computing this, we find the above value to be less than $1.7 \in \mathbb{R}$, which allows us to conclude that

$$= |f(0.8) - \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k| < 1.7 = \varepsilon,$$

which completes our proof that $\boxed{\text{for } \varepsilon = 1.7 \in \mathbb{R}, f(0.8) \approx_\varepsilon \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(0.8)^k.}$

☐ **Problem 3** [Limits, Continuity, and Calculus] In L23, we defined the *limit* of a function. In this last week of MATH 112, we engage with this definition for three reasons:

(A) To gain further practice with nested quantifiers, implications, inequalities, and absolute values – four central topics in MATH 112 – as review for the final.
(B) To define *continuity* and *differentiability* of functions (supplementary reading L23) seen previously in MATH 111 without invoking the existence of infinitesimals "$dx$". In L10 we saw that infinitesimals do not exist in any Archimedean ordered field!
(C) To better appreciate the analogous logical formulation of a limit of a sequence L15.

Recall from L23: given any function $f : \Omega \to \mathbb{C}$, we say that $\lim\limits_{x \to x_0} f(x) = L$ if

$$\boxed{\forall \varepsilon \in \mathbb{R}_+ \ \exists \delta \in \mathbb{R}_+ \ \forall x \in \Omega \left( (\ 0 < |x - x_0| < \delta\ ) \ \Rightarrow \ (\ |f(x) - L| < \varepsilon\ ) \right)}$$

- 3.1 [15 points] Prove that if $f(x) = 5x - 2$ then $\lim\limits_{x \to 1} f(x) = 3$ is true.

  Consider $|f(x) - 3|$:

  $$|f(x) - 3| = |(5x - 2) - 3| = |5x - 5| = 5|x - 1|$$

  We want $5|x - 1| < \varepsilon$. Rearrange this:

  $$5|x - 1| < \varepsilon \Rightarrow |x - 1| < \frac{\varepsilon}{5}$$

  Define $\delta$ in terms of $\varepsilon$:

  $$\delta = \frac{\varepsilon}{5}$$

  With this $\delta$, if $0 < |x - 1| < \delta$, then:

  $$5|x - 1| < \varepsilon \Rightarrow |(5x - 2) - 3| < \varepsilon$$

  Therefore, $\boxed{\lim_{x \to 1} f(x) = 3 \text{ for } f(x) = 5x - 2.}$

- 3.2 [10 points] Prove that if $f(x) = 5x - 2$ then $\lim\limits_{x \to 1} f(x) = 11$ is false.

  We need to find an $\varepsilon > 0$ such that for any $\delta > 0$, there exists some $x$ with $0 < |x - 1| < \delta$ and $|f(x) - 11| \geq \varepsilon$. First, evaluate $f(1)$:

  $$f(1) = 5 \cdot 1 - 2 = 3$$

  Choose $\varepsilon$ smaller than the difference between 11 and $f(1)$. Take $\varepsilon = \frac{1}{2}|11 - 3| = 4$. Consider any $\delta > 0$ and choose $x = 1$. For this $x$, $0 < |x - 1| < \delta$ but $|f(x) - 11| = |3 - 11| = 8$, which is greater than $\varepsilon = 4$.

Since an $\varepsilon > 0$ exists (here, 4) for which no $\delta > 0$ can satisfy the limit condition, $\boxed{\text{the statement } \lim_{x\to 1} f(x) = 11 \text{ is false.}}$

☐ **Problem 4** [Taylor Approximation] Suppose that $f : \Omega \to \mathbb{R}$ is a function with domain $\Omega \subseteq \mathbb{R}$. Fix $x_0 \in \Omega$. The $n^{th}$ *Taylor polynomial approximation of $f$ near $x_0$* is the polynomial

$$\mathrm{Tay}_{n,f,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k.$$

For example, in Problem 2.5 above, we consider the function $f : \Omega \to \mathbb{R}$ defined on the domain $\Omega = \mathbb{R} \setminus \{1\}$ by the formula $f(x) = \frac{1}{1-x}$, then found that the $4^{th}$ Taylor polynomial approximation of $f$ near $x_0 = 0$ is the quartic polynomial $\mathrm{Tay}_{4,f,x_0}(x) = 1+x+x^2+x^3+x^4$. In L24, we will briefly encounter the statement of *Taylor's theorem with remainder* which quantifies how good of a job these Taylor polynomials do in approximating $f$ near $x_0$.

For each given $n \in \mathbb{N}$, each formula defining a function $f : \mathbb{R} \to \mathbb{R}$, and each $x_0 \in \mathbb{R}$ below, calculate $\mathrm{Tay}_{n,f,x_0}(x)$ using your background from MATH 111.

- 4.1 [5 points] $n = 2$, $f(x) = 1 + x^2 - x^3$, $x_0 = 0$

  $f(x)$ and its derivatives:
  1. $f(x) = 1 + x^2 - x^3$
  2. $f'(x) = 2x - 3x^2$
  3. $f''(x) = 2 - 6x$

  The derivatives at $x_0 = 0$:
  1. $f(0) = 1$
  2. $f'(0) = 2 \cdot 0 - 3 \cdot 0^2 = 0$
  3. $f''(0) = 2 - 6 \cdot 0 = 2$

  Plugging these values into the Taylor polynomial formula for $n = 2$:

$$\mathrm{Tay}_{2,f,0}(x) = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2$$

$$\mathrm{Tay}_{2,f,0}(x) = 1 + 0 \cdot x + \frac{2}{2}x^2$$

$$\mathrm{Tay}_{2,f,0}(x) = 1 + x^2$$

  Thus, the $2^{nd}$ Taylor polynomial approximation of $f(x) = 1 + x^2 - x^3$ near $x_0 = 0$ is $\mathrm{Tay}_{2,f,0}(x) = 1 + x^2$.

- 4.2 [5 points] $n = 2$, $f(x) = 1 + x^2 - x^3$, $x_0 = 1$

  $f(x)$ and its derivatives:

  1. $f(x) = 1 + x^2 - x^3$
  2. $f'(x) = 2x - 3x^2$
  3. $f''(x) = 2 - 6x$

  The derivatives at $x_0 = 1$:

  1. $f(1) = 1$
  2. $f'(1) = -1$
  3. $f''(1) = -4$

  Plug these values into the formula for $n = 2$:

$$\mathrm{Tay}_{2,f,1}(x) = \frac{f(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2$$

$$\mathrm{Tay}_{2,f,1}(x) = 1 - (x-1) - 2(x^2 - 2x + 1)$$

$$\mathrm{Tay}_{2,f,1}(x) = -2x^2 + 3x$$

  Thus, the $2^{nd}$ Taylor polynomial approximation of $f(x) = 1 + x^2 - x^3$ near $x_0 = 1$ is $\mathrm{Tay}_{2,f,1}(x) = -2x^2 + 3x$.

- 4.3 [5 points] $n = 3$, $f(x) = e^x$, $x_0 = 0$

  Since $f(x) = e^x$, all derivatives are also $e^x$. Thus:
  1. $f(x) = e^x$
  2. $f'(x) = e^x$
  3. $f''(x) = e^x$
  4. $f'''(x) = e^x$

  All derivatives at $x = 0$ are $e^0 = 1$:
  1. $f(0) = 1$
  2. $f'(0) = 1$
  3. $f''(0) = 1$
  4. $f'''(0) = 1$

  Plug these values into the formula for $n = 3$:

$$\mathrm{Tay}_{3,f,0}(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!} x^k$$

$$\text{Tay}_{3,f,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Thus, the $3^{rd}$ Taylor polynomial approximation of $f(x) = e^x$ near $x_0 = 0$ is $\text{Tay}_{3,f,0}(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$.

- 4.4 [5 points] $n = 4,\ f(x) = \cos x, \qquad x_0 = 0$

  $f(x)$ and its derivatives:

  1. $f(x) = \cos x$
  2. $f'(x) = -\sin x$
  3. $f''(x) = -\cos x$
  4. $f'''(x) = \sin x$
  5. $f^{(4)}(x) = \cos x$

  The derivatives at $x_0 = 0$:

  1. $f(0) = 1$
  2. $f'(0) = 0$
  3. $f''(0) = -1$
  4. $f'''(0) = 0$
  5. $f^{(4)}(0) = 1$

  Plug these values into the formula for $n = 4$:

$$\text{Tay}_{4,f,0}(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$$

$$\text{Tay}_{4,f,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

  Thus, the $4^{th}$ Taylor polynomial approximation of $f(x) = \cos x$ near $x_0 = 0$ is $\text{Tay}_{4,f,0}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

- 4.5 [5 points] $n = 4,\ f(x) = \frac{1}{112-x}, \qquad x_0 = 0$

  $f(x)$ and its derivatives:

  1. $f(x) = \frac{1}{112-x}$
  2. $f'(x) = \frac{1}{(112-x)^2}$
  3. $f''(x) = \frac{2}{(112-x)^3}$
  4. $f'''(x) = \frac{6}{(112-x)^4}$
  5. $f^{(4)}(x) = \frac{24}{(112-x)^5}$

The derivatives at $x_0 = 0$:

1. $f(0) = \frac{1}{112}$
2. $f'(0) = \frac{1}{112^2}$
3. $f''(0) = \frac{2}{112^3}$
4. $f'''(0) = \frac{6}{112^4}$
5. $f^{(4)}(0) = \frac{24}{112^5}$

Plug these values into the formula for $n = 4$:

$$\mathrm{Tay}_{4,f,0}(x) = \frac{1}{112} + \frac{1}{112^2}x + \frac{1}{112^3}x^2 + \frac{1}{112^4}x^3 + \frac{1}{112^5}x^4$$

Thus, the $4^{th}$ Taylor polynomial approximation of $f(x) = \frac{1}{112-x}$ near $x_0 = 0$ is $\mathrm{Tay}_{4,f,0}(x) = \frac{1}{112} + \frac{1}{112^2}x + \frac{1}{112^3}x^2 + \frac{1}{112^4}x^3 + \frac{1}{112^5}x^4$.