The Boundary Value Problem: Using Shooting Methods to Solve the Damped Spring Mass System

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Abstract

Shooting methods are used to solve differential equations numerically. They involve starting with an initial guess for the solution and then iteratively updating the guess by ”shooting” from one point on the equation towards another point until a root (i.e., a solution) is found. In other words, they’re like aiming a gun at two points on an equation and calculating how far off course you need to go based on the intersection of those lines in order to hit your target. Different types of shooting methods can have different advantages depending on the properties of the differential equation being solved. For example, some may converge faster than others, while others might require fewer iterations to reach convergence. These methods are widely used in many fields such as engineering, physics, finance, and more generally in any field involving complex mathematical models that can be formulated as systems of ODEs.

1 Introduction

Numerical analysis has emerged as an indispensable tool for studying real-world phenomena whose behaviors are too intricate to analyze analytically. One particularly fascinating area where this technique proves highly beneficial is in simulating mechanical systems involving the movement of objects governed by nonlinear ordinary differential equations (ODE). Among the countless examples falling under this category lies the case of a physically dampened pendulum undergoing damping effects due to friction or air resistance. In particular, a pendulum’s complex motion is characterized by alterations in its amplitude and frequency when acted upon by external excitation forces or energy dissipation elements. These transformations stem from the interplay between the applied torques and inherent internal resistance components present in the system.

In order to understand the pendulum's behavior in response to certain forcing functions, we must rely on numerical approaches for solving ODEs. Two common methods for handling our ODEs involve shooting techniques and the application of root-finding methods. Specifically, we make use of the Fourth Order Runge-Kutta formula, along with either the Bisection Method or Newton's Method. This depth of methods allows for a comparison of accuracies, and a better understanding of the pendulum's motion.

Keywords Numerical Analysis, Boundary Value Problem, Shooting Methods, Pendulum Movement, Differential Equations, Newton's Method, 4th Order Runge-Kutta Method, Bisection Method . . . .

2 Addressing the Problem

The following Ordinary Differential Equation pertains to the motion of a pendulum:

$$\theta''(t) + c\theta'(t) + \frac{g}{L}\sin\theta = 0 \quad 0 \leq t \leq b \quad \theta(0) = \alpha \quad \theta(b) = \beta \tag{1}$$

This is an Ordinary Differential Equation that describes the angular motion of a pendulum of length L and damping constant c where g represents the constant of gravity. One approach to finding a numerical solution of this differential equation is as follows: first, make an educated guess as to the slope of the curve at $t = 0$, say p, then use this along with the given initial conditions to compute a candidate solution $\theta p(t)$ of the differential equation. Next, compare the final value of $\theta p(t)$ to the desired terminal condition β. If the difference between $\theta p(t)$ and β does not fall within a desired tolerance value, E, adjust the guess p until

$$|F(p)| := |\beta - \theta p(b)| < E \tag{2}$$

Using the Bisection Method with tolerance $E = 0.005$, repeat this process until convergence is achieved and a desired p is reached. To improve upon this, Newton's Method can be used instead, employing the derivative of F evaluated at the current estimate p_n to determine p_{n+1}. This approach allows us to achieve faster convergence rates than simple root-finding techniques such as bisection alone. Finally, based on our solved differential equations, we can comment upon the physics of this simple harmonic oscillator system. In addition to this, to further understand the dynamics of the system and our approximation methods we can compare the results we find with values of c besides the initial, $c = 0.1$.

3 Steps to Solution

3.1 Deriving an Iterative System

To find a solution to this Second Order Boundary Value Problem via Shooting Method, we must convert the given equation into a system of two First Order ODEs by setting $\theta' = \omega$.

$$\begin{cases} \theta'(t) = \omega(t) \\ \omega'(t) = -c\omega(t) - \frac{g}{L}\sin\theta \\ \theta(0) = \alpha \\ \omega(0) = p \end{cases} \tag{3}$$

3.2 Fourth Order Runge-Kutta Approximation

From here, we can apply side by side iterative numerical methods to solve for ω and θ. In this case we will implement two side by side Fourth Order Runge-Kutta Approximation Methods with step size h, initial conditions, and F_1, F_2 as follows:

$$\begin{aligned} h = \frac{b-a}{n}, \quad \theta_0 = \frac{9\pi}{10}, \quad \omega_0 = p \\ F_1 = \omega(t) \\ F_2 = -c\omega(t) - \frac{g}{L}\sin\theta \end{aligned} \tag{4}$$

Then we can write the rest of RK-4 as follows:

$$
\begin{aligned}
k_1 &= hF_1(\omega(t), \theta(t)) \\
l_1 &= hF_2(\omega(t), \theta(t)) \\
\\
k_2 &= hF_1(\omega(t) + \frac{k_1}{2}, \theta(t) + \frac{l_1}{2}) \\
l_2 &= hF_2(\omega(t) + \frac{k_1}{2}, \theta(t) + \frac{l_1}{2}) \\
\\
k_3 &= hF_1(\omega(t) + \frac{k_2}{2}, \theta(t) + \frac{l_2}{2}) \\
l_3 &= hF_2(\omega(t) + \frac{k_2}{2}, \theta(t) + \frac{l_2}{2}) \\
\\
k_4 &= hF_1(\omega(t) + k_3, \theta(t) + l_3) \\
l_4 &= hF_2(\omega(t) + k_3, \theta(t) + l_3) \\
\\
\theta_{t+1} &= \theta(t) + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \\
\omega_{t+1} &= \omega(t) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
\tag{5}
$$

3.3 Applying Bisection and Newton's Method

We must use a root finding method for p such that:

$$
|F(p)| := |\beta - \theta p(b)| < E \tag{6}
$$

We can use RK-4 in tandem with the bisection method, $F(p_0)F(p_1) < 0$, with $F(p_0)$ and $F(p_1)$ taken from calling RK-4 twice:

$$
\begin{aligned}
F(p_0) = \beta - \theta p_0(b), F(p_1) = \beta - \theta p_1(b), \\
b = 2, \quad \beta = -2
\end{aligned}
\tag{7}
$$

The points p_0 and p_1 have to be carefully selected to ensure that their product is negative and, consequentially, p_0 and p_1 are under-shots and over-shots. However, as long as these points are properly selected, the bisection method in tandem with RK-4 will converge to a proper p.

Employing Newton's Method would be more effective:

$$p_{n+1} = p_n - \frac{\theta(t,p)}{\frac{\partial \theta'}{\partial p}(t,p)} \tag{8}$$

However, approximating $\theta'(t)$ with respect to p requires another application of RK-4.

$$\theta''(t,p) = F(t, \theta(t,p), \theta'(t,p)) \tag{9}$$

First we must differentiate $F(t, \theta(t,p), \theta'(t,p))$ with respect to p using mixed partial sums.

$$\begin{aligned} \frac{\partial \theta''}{\partial p}(t,p) = \frac{\partial F}{\partial p} &= \frac{\partial f}{\partial t}(t, \theta(t,p), \theta'(t,p))\frac{\partial t}{\partial p} \\ &+ \frac{\partial F}{\partial \theta}(t, \theta(t,p), \theta'(t,p))\frac{\partial \theta}{\partial p}(t,p) \\ &+ \frac{\partial F}{\partial \theta}(t, \theta(t,p), \theta'(t,p))\frac{\partial \theta'}{\partial p}(t,p) \end{aligned} \tag{10}$$

Note that $\frac{\partial t}{\partial p} = 0$ because t does not depend on p. By making the substitution $z(t,p) = \frac{\partial \theta}{\partial p}$, we can simplify the notation and create an equation for a Second Order ODE:

$$\begin{aligned} z(t,p) &= \frac{\partial \theta}{\partial p} \\ z''(t,p) = \frac{\partial F}{\partial \theta}z(t,p) + \frac{\partial F}{\partial \theta'}z'(t,p), &\qquad z(0,p) = 0,\ z'(0,p) = 1. \end{aligned} \tag{11}$$

Note that for the pendulum equation we have:

$$z''(t,p) = \frac{-g}{L}z(t,p) - cz'(t,p), \qquad z(0,p) = 0,\ z'(0,p) = 1. \tag{12}$$

Thus, by employing a substitution of variable, $\delta(t) = z'(t,p)$, we have a system of First Order ODEs and can use RK-4 to approximate the solution to $z(t,p)$.

$$\begin{cases} z'(t) = \delta(t) \\ \delta'(t) = \frac{-g}{L}z(t) - c\delta(t) \\ z(0) = 0 \\ \delta(0) = 1 \end{cases} \tag{13}$$

Now we can use Newton's Method to find a p that satisfies a desired error tolerance:

$$P_k = p_{k-1} - \frac{\theta(t, p_{k-1}) - \beta}{z(t, p_{k-1})} \tag{14}$$

3.4 Implementing the Algorithm

We started by calling a Bisection Method function in MATLAB with inputs p_0 and p_1. In the case of our pendulum, we knew that the initial angular velocity, p, must be positive and is unlikely to exceed 100. As such, $p_0 = 0$ and $p_1 = 100$. This function begins by computing RK-4 with p_0 and p_1, then runs these points through bisection method code, eventually choosing p_i until reaching a p_n that gives an accurate initial angular velocity.

We then ran the p value given from Bisection Method through our Newton's Method code. After a few iterations, Newton's method displayed an accurate answer.

4 Proving the Methods

In our approximation using the Bisection Method, we would like it to measure as close to the result of the real function itself. Below is a direct comparison of the graph of the actual function, $y''(t) - \frac{y'(t)+y(t)}{2} = 0$. We know this equation can be solved as $y(t) = e^t$, so we can compare with that instead.

As can be observed above, when compared to the provided differential equation's exact solution, the result of the Bisection Method's approximation is indistinguishable. Therefore, the results of our Shooting Method code are correct for the implementation of the Bisection Method, as well as Newton's Method.

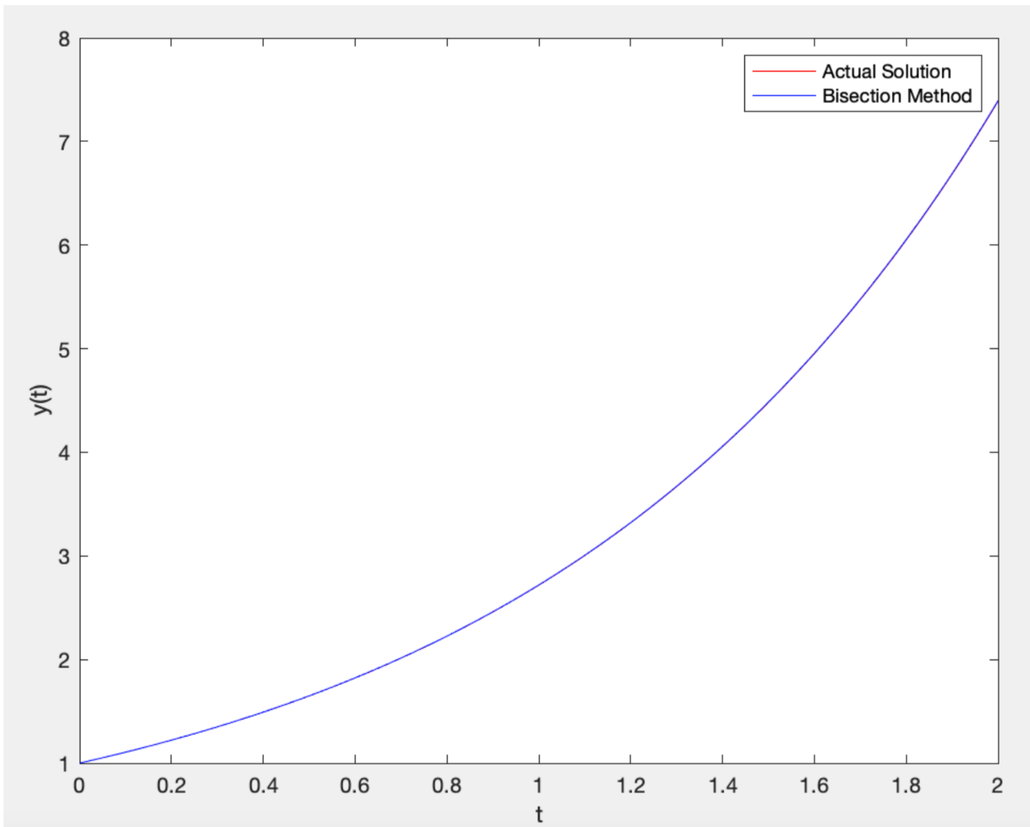


Figure 1: Bisection Approximation Comparison

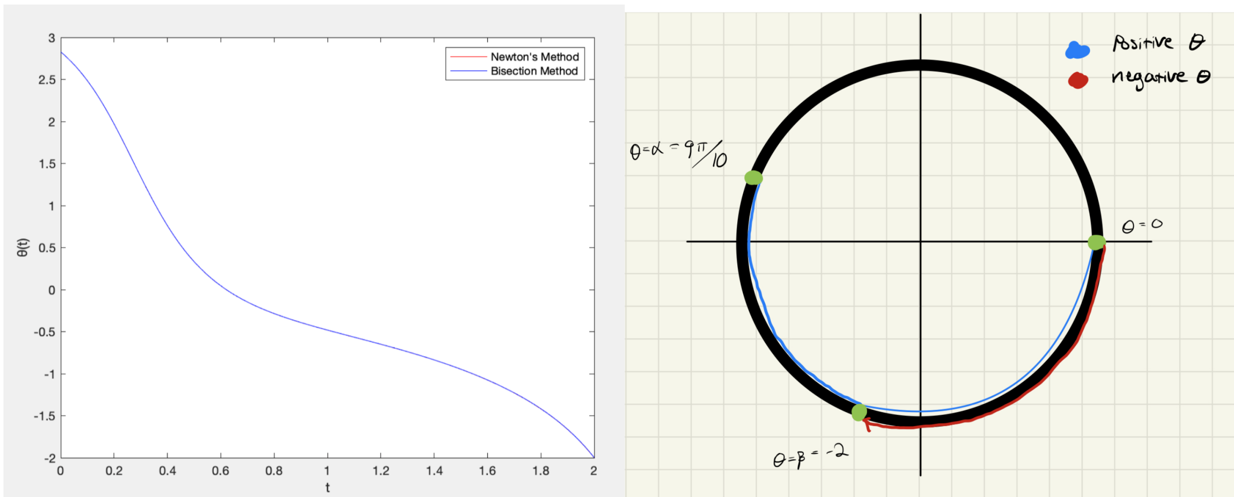


Figure 2: Pendulum Movement Visualization

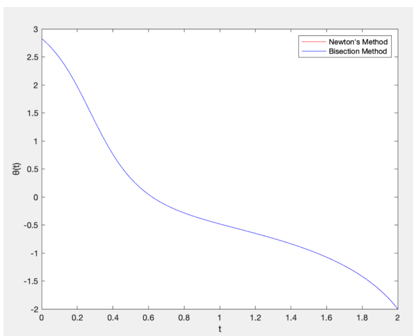
5 Results

5.1 Addressing the movement of the pendulum

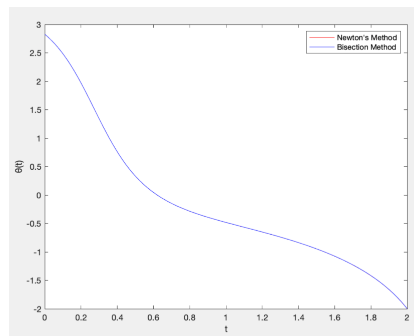
The pendulum will follow a pattern as visualized here:

When released from rest at $9\pi/10$, the pendulum begins to move towards the negative direction due to the gravitational force acting on it. As it swings, its angle increases until its maximum displacement is reached, and then decreases until it passes through a minimum point. Shortly after this minimum is reached and when 2 seconds have elapsed, the pendulum reaches a θ-value of $\beta = -2$.

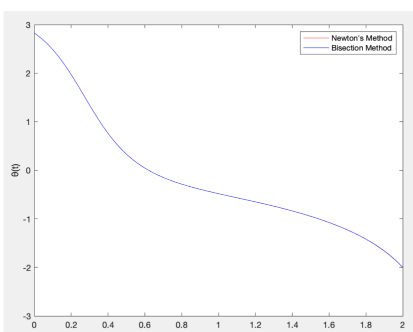
Next we can address how changing the value of c in the computation affects behavior.



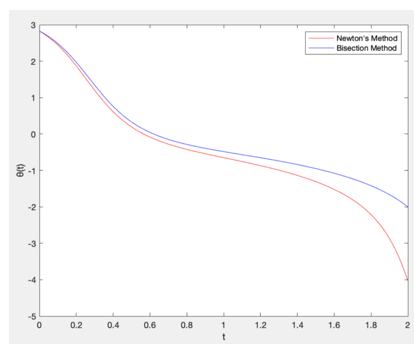
(a) $c = 0.1$



(b) $c = 0.15$



(c) $c = 0.2$



(d) $c = 0.25$

Figure 3: Comparing values of c

6 Conclusion

Altogether, the methods of approximation, MATLAB scripting, and reformatting of the Ordinary Differential Equation to a system of equations worked together to provide a deeper understanding of the problem at hand. We were able to execute on our knowledge of Numerical Analysis to pull together a result that allowed for a clear visualization of this dampened pendulum motion problem. The integration of mathematical methods and scientific analysis allows for the deeper understanding and solving of scientific problems in our world.

List of Figures