Green's Theorem with the 2D Laplacian

Aidan Olson

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Objective

We want to understand how Green's Theorem is used to acquire the weak form of the 2D homogeneous Laplacian. This weak form is the dot product of the solution gradient and that of the basis function, v, all integrated over our domain, D. We then use the weak form in the Finite Element Method.

Application of Green's Theorem in Solving the Laplace Equation

To understand how Green's theorem is applied in the context of solving the Laplace equation ($\Delta u = 0$) in two dimensions, let's start with the standard form of the Laplace equation and see how it leads to an integral formulation using Green's theorem.

Laplace's Equation

The Laplace equation in two dimensions is given by:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

This equation states that the sum of the second partial derivatives of the function $u(x, y)$ with respect to x and y is zero everywhere in the domain. In our case, we aim to solve the Laplacian on a square domain with Dirichlet boundary conditions.

Weak Formulation

To solve this equation using the Finite Element Method, we first need to convert it into its weak form. The weak form is obtained by multiplying the equation by a test function $v(x, y)$ and integrating over the domain D. The test function v is arbitrary but should be sufficiently smooth (usually at least as smooth as the solution u) and satisfy the boundary conditions of the problem.

So, we start with:

$$\int_D \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) v \, dA = 0 \tag{2}$$

Applying Green's Theorem

Green's theorem can be used to convert the above integral into a boundary integral. However, for the Laplace equation, we can simplify the process by integrating by parts. The goal is to transfer one derivative from u to v. Doing this for each term, we get:

$$\int_D \frac{\partial^2 u}{\partial x^2} v \, dA + \int_D \frac{\partial^2 u}{\partial y^2} v \, dA = \int_D \nabla u \cdot \nabla v \, dA - \oint_{\partial D} \frac{\partial u}{\partial n} v \, ds \tag{3}$$

Here, ∂D is the boundary of the domain D, and $\frac{\partial u}{\partial n}$ represents the normal derivative of u at the boundary (i.e., the derivative of u in the direction normal to the boundary). The term $\nabla u \cdot \nabla v$ is the dot product of the gradients of u and v.

Homogeneous Boundary Conditions

For the homogeneous Laplacian ($\Delta u = 0$) with homogeneous boundary conditions (for example, $u = 0$ on ∂D), the boundary integral term vanishes, and we are left with:

$$\int_D \nabla u \cdot \nabla v \, dA = 0 \tag{4}$$

This is the weak form of the Laplace equation. It states that the integral of the dot product of the gradients of u and v over the domain D is zero for all test functions v. This form is amenable to numerical methods like FEM, where the domain D is discretized into finite elements, and the solution u is approximated within each element.

Conclusion

In summary, the application of Green's theorem (or integration by parts, which is a related concept) in deriving the weak form of the Laplace equation is a crucial step in preparing the equation for numerical solution via the Finite Element Method. This process involves transforming the original PDE into an integral equation that respects the boundary conditions and is suitable for approximation over a discretized domain.