THE LAW OF CONSERVATION OF INFORMATION

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ABSTRACT

Given that all physical laws are time-reversible and computable, information must be conserved as we run a simulation of the Universe forward in time. We would not be able to run the simulation backwards in time otherwise. But, how can we formulate the Law of conservation of Information mathematically?

1 Unitarity in Everettian Quantum Mechanics

In the Everettian formulation of Quantum Mechanics, the entire Universe may be identified with a single wave equation that obeys unitarity. That is, the time evolution of a quantum state must *conserve probability* in the sense that the sum of probabilities is always one.

It follows that a unitary operator describes the time evolution of the state of the Universe.

2 Conservation of Von Neumann entropy

If the quantum state of the Universe is given by a positive semi-definite matrix ρ then the Von Neumann entropy is given by:

$$S(\rho) = -\mathrm{Tr}(\rho \cdot \ln \rho) \tag{1}$$

which quantifies the total amount of statistical information in the Universe. Now, given that ρ can only undergo Unitary transformations:

$$\rho \mapsto U \cdot \rho \cdot U^* \tag{2}$$

we may deduce:

$$S(\rho) = S(U \cdot \rho \cdot U^*) \tag{3}$$

since $S(\cdot)$ only depends on the eigenvalues of ρ.

3 Proof

Given that ρ is positive semi-definite, it is diagonalisable:

$$\rho = V_\rho \lambda_\rho V_\rho^{-1} \tag{4}$$

where λ_ρ is the diagonal matrix of eigenvalues of ρ and V_ρ is the matrix of eigenvectors of ρ.

Moreover, given that $\rho^k = V_\rho \lambda_\rho^k V_\rho^{-1}$ we may deduce that the matrix exponential must satisfy:

$$e^\rho = V_\rho e^{\lambda_\rho} V_\rho^{-1} \tag{5}$$

so if we define $\rho := \log A$ for positive semi-definite A:

$$\text{eig}(A) = \exp(\text{eig}(\log A)) \tag{6}$$

Now, assuming that any quantum event has strictly non-zero probability of occurring i.e. $\min(\text{eig}(\rho)) > 0$, the previous equation generalises as follows:

$$\log \rho = V_\rho \log \lambda_\rho V_\rho^{-1} \tag{7}$$

and for unitary transformations U if we define $Q = V_\rho \cdot U$ we have:

$$\log U\rho U^* = Q \log \lambda_\rho \cdot Q^{-1} \tag{8}$$

so if we define the diagonal matrix $\Lambda = \lambda_\rho \cdot \log \lambda_\rho$, our analysis simplifies to:

$$\text{Tr}(Q\Lambda Q^{-1}) = \text{Tr}(\Lambda) \tag{9}$$

which concludes our proof.

References

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[3] Von Neumann, John (1932). Mathematische Grundlagen der Quantenmechanik. Berlin: Springer. ISBN 3-540-59207-5.; Von Neumann, John (1955). Mathematical Foundations of Quantum Mechanics. Princeton University Press.