Asymptotic incompressibility and its application to rare-event modelling

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Abstract

The objective of this article is to present a theory of algorithmic information that allows us to characterise rare events that are recurrent, of variable frequency, and unpredictable. Within this theory, it may be shown that such data-generating processes are asymptotically incompressible and therefore identifiable via an invariance theorem for algorithmically random data. While such phenomena define the worst case in rare-event modelling, a robust approach for distinguishing such cases which are intractable from cases which are algorithmically tractable has so far been lacking.

1 Unpredictable phenomena are asymptotically intractable

Given a rare-event process X, a scientist may only collect a finite number of observations $X_N = \{x_i\}_{i=1}^N$ from that process. Moreover, let's suppose $x_i = 1$ when the event of interest is observed and $x_i = 0$ otherwise. Then the location of the rare events, i.e. i where $x_i = 1$, are necessary and sufficient to define the binary encoding X_N.

This process is deterministic if there exists a computable function f such that:

$$x_{n+1} = f \circ x_n \tag{1}$$

and if there exists an asymptotic formula $\pi(N)$ such that for large N,

$$\pi(N) \sim \sum_{i=1}^{N} x_i \tag{2}$$

then we say that X is an unpredictable process if the average amount of information gained from the occurrence of each rare-event is given by the combinatorial entropy:

$$S_c = \frac{\log_2(2^N)}{\pi(N)} = \frac{N}{\pi(N)} \sim \ln(N) \tag{3}$$

where 2^N is exactly the number of possible binary encodings of length N and $\sim \ln(N)$ implies that the rate of entropy production is maximal. (3) also implies that X_N may not be compressed into fewer than $\pi(N) \cdot \ln(N)$ bits without being certain that information will be lost.

In fact, X_N is asymptotically incompressible in the sense that:

$$\mathbb{E}[K(X_N)] \sim \pi(N) \cdot \ln(N) \sim N \tag{4}$$

as (3) states that on average each observation x_i is a surprising event.

Now, a direct implication of (4) is that any approximation $\hat{f} \in F_\theta$ of f discovered using machine learning or another scientific method such that the Kronecker delta satisfies:

$$\forall n \in [1, N], \delta_{\hat{f}(x_n), x_{n+1}} = 1 \tag{5}$$

has an algorithmic complexity that scales as follows:

$$K(\hat{f}) \sim N \tag{6}$$

as such high levels of accuracy are due to memorisation and not discovering regularities in X_N. Given (4), (5), and (6) we may state that the process X is algorithmically random with respect to F_θ.

2 A derivation of the maximum entropy distribution

Let's define the sequence $\mathbb{P} = \{p_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $p_k \in \mathbb{P}$ if and only if $x_{p_k} = 1$. Then the $\ln(N)$ term in (4) implies that the elements of $\mathbb{P}$ are in some sense uniformly distributed in X_N.

Given that there are k distinct ways to sample uniformly from $[1, k]$ and a frequency of $\frac{1}{k}$ associated with the event $(k-1, k] \cap \mathbb{P} \neq \emptyset$ the average entropy rate has a natural interpretation.

By breaking $\sum_{k=1}^{N} \frac{1}{k}$ into $\pi(N)$ disjoint blocks of size $[p_k, p_{k+1}]$ where $p_k, p_{k+1} \in \mathbb{P}$:

$$\sum_{k=1}^{N} \frac{1}{k} \approx \sum_{k=1}^{\pi(N)} \sum_{n=p_k}^{p_{k+1}} \frac{1}{n} = \sum_{k=1}^{\pi(N)} (p_{k+1} - p_k) \cdot P(p_k) \approx \ln(N) \tag{7}$$

where $P(p_k) = \frac{1}{p_{k+1} - p_k} \sum_{n=p_k}^{p_{k+1}} \frac{1}{n}$.

So we see that (4) approximates the expected number of observations per rare event where $P(p_k)$ may be interpreted as the probability of a successful observation in a frequentist sense. This is consistent with John Wheeler's *it from bit* interpretation of entropy where entropy measures the average number of bits(i.e. yes/no questions) per rare event.

Interestingly, (7) may also be interpreted as the expected distance or waiting time between consecutive rare events as we have:

$$\mathbb{E}[|p_{n+1} - p_n|] = \sum_{k=1}^{\pi(N)} (p_{k+1} - p_k) \cdot P(p_k) \approx \ln(N) \tag{8}$$

Having clarified the rate of entropy production of X we may consider a practical method for identifying data-generating processes that are asymptotically incompressible.

3 An invariance theorem for algorithmically random data

Let's suppose we have a natural signal described by the process X:

$$x_n \in \{0,1\}, x_{n+1} = \varphi \circ x_n \tag{9}$$

If we should use machine learning to approximate φ given the datasets $X_N^{train} = \{x_i\}_{i=1}^{N}, X_N^{test} = \{x_i\}_{i=N+1}^{2N}$ such that for any $\hat{f} \in F_\theta$:

$$x_{n+1} = \hat{f} \circ x_n \Rightarrow \delta_{\hat{f}(x_n),x_{n+1}} = 1 \tag{10}$$

then X_N is asymptotically incompressible if for large N any solution to the empirical risk minimisation problem:

$$\hat{f} = \max_{f \in F_\theta} \frac{1}{N} \sum_{i=1}^{N} \delta_{f(x_n),x_{n+1}} \tag{11}$$

has an expected performance:

$$\frac{1}{N} \sum_{n=N+1}^{2N-1} \delta_{f(x_n),x_{n+1}} \leq \frac{1}{2} \tag{12}$$

Furthermore, if the dataset is imbalanced i.e. $\frac{1}{N}\sum_{i=1}^{N} x_i \neq \frac{1}{2}$ then we may generalise this result by introducing the auxiliary definitions:

$$y_n = x_{n+1} \tag{13}$$

$$\hat{y_n} = \hat{f} \circ x_n \tag{14}$$

$$\beta_n = \delta_{y_n,\hat{y_n}} \tag{15}$$

$$N_1 = \sum_{n=N+1}^{2N} \delta_{y_n,1} \tag{16}$$

$$N_0 = \sum_{n=N+1}^{2N} \delta_{y_n,0} \tag{17}$$

and so for large N, we have:

$$\frac{1}{2} \sum_{n=N+1}^{2N-1} \Big(\frac{\delta_{y_n,1} \cdot \beta_n}{N_1} + \frac{\delta_{y_n,0} \cdot \beta_n}{N_0}\Big) \leq \frac{1}{2} \tag{18}$$

which implies that the true-positive rate is an unbiased estimator of (18):

$$\frac{1}{N_1} \sum_{n=N+1}^{2N-1} \delta_{y_n,1} \cdot \beta_n \leq \frac{1}{2} \tag{19}$$

Finally, as these results are invariant to transformations that preserve the phase-space dimension of X, this theorem may be used as an overfitting test for algorithmically-random data.

4 Discussion

Given that the prime numbers are the only known rare-event process with a distribution satisfying (4):

$$\pi(N) \sim \frac{N}{\ln(N)} \tag{20}$$

the overfitting test (19) may be applied to the scientific problem of identifying cosmic signals communicated by civilisations within the Turing limit that are sufficiently advanced to do number theory.

References

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