第六讲：不定积分 > 换元法 > 倒代换

当被积函数是分式时，并且当分母次数明显高于分子次数时，我们可以考虑倒代换
倒代换可以起到对分母降次的作用，但是倒代换后分子次数一般会增加
那么我们怎么对分子进行降次呢？

1. 多项式的除法
2. 因式分解
3. 二项式展开

这样我们就对分子和分母都进行了降次，从而达到简化分式的目的

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$$\int \frac{dx}{x^8(1+x^2)} \qquad 令 x=\frac{1}{t}$$

$$\int \frac{dx}{x^8(1+x^2)} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^8}\left(1+\frac{1}{t^2}\right)} = \int \frac{-t^8 dt}{1+t^2} = \int\left(1-t^2+t^4-t^6-\frac{1}{1+t^2}\right)dt$$

$$= t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}-\arctan t + C$$

$$= \frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}-\frac{1}{7x^7}-\arctan\frac{1}{x}+C$$

$$-t^8 = 1-t^8-1 = (1-t^4)(1+t^4)-1 = (1+t^2)(1-t^2)(1+t^4)-1$$

$$= (1+t^2)(1-t^2+t^4-t^6)-1$$

$$\begin{array}{r|l}
 & -t^6+t^4-t^2+1 \\
t^2+1 & -t^8 \\
 & -t^8-t^6 \\ \hline
 & t^6 \\
 & t^6+t^4 \\ \hline
 & -t^4 \\
 & -t^4-t^2 \\ \hline
 & t^2 \\
 & t^2+1 \\ \hline
 & -1
\end{array}$$

$$-t^8 = (1+t^2)(1-t^2+t^4-t^6)-1$$

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$$\int \frac{dx}{(x^2-1)^n}$$

令$x^2-1=\frac{1}{t} \Rightarrow x=\pm\sqrt{1+\frac{1}{t}}$

$$\int \frac{dx}{(x^2-1)^n}=\int t^n \cdot \frac{\pm 1}{2\sqrt{1+\frac{1}{t}}} \cdot \left(-\frac{1}{t^2}\right) dt=\mp\int \frac{t^{n-2}}{2\sqrt{1+\frac{1}{t}}} dt$$

令$x=\frac{1}{t}$

$$\int \frac{dx}{(x^2-1)^n}=\int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t^2}-1\right)^n}=\int \frac{-t^{2n-2}dt}{(1-t^2)^n} \qquad \int \frac{dx}{(x-1)^n(x+1)^n}=\int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t}-1\right)^n\left(\frac{1}{t}+1\right)^n}=\int \frac{-t^{2n-2}dt}{(1-t)^n(1+t)^n}$$

令$x+1=\frac{1}{t}$

$$\int \frac{dx}{(x^2-1)^n}=\int \frac{dx}{(x-1)^n(x+1)^n}=\int \frac{-\frac{1}{t^2}dt}{\left(\frac{1}{t}-2\right)^n\left(\frac{1}{t}\right)^n}=\int \frac{-t^{2n-2}dt}{(1-2t)^n}$$

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$$\int\frac{-t^{2n-2}dt}{(1-2t)^n}=\int\frac{-\left(\frac{1-s}{2}\right)^{2n-2}\left(-\frac{1}{2}\right)ds}{s^n}=\left(\frac{1}{2}\right)^{2n-1}\int\frac{(1-s)^{2n-2}ds}{s^n}$$

令$1-2t=s\Rightarrow t=\frac{1-s}{2}$

$$=\left(\frac{1}{2}\right)^{2n-1}\int\frac{\sum_{k=0}^{2n-2}(-1)^kC_{2n-2}^{k}s^kds}{s^n}=\left(\frac{1}{2}\right)^{2n-1}\int\sum_{k=0}^{2n-2}(-1)^kC_{2n-2}^{k}s^{k-n}ds$$

变量代换简化分母

$$=\left(\frac{1}{2}\right)^{2n-1}\left[\int\sum_{\substack{k=0\\k\neq n-1}}^{2n-2}(-1)^kC_{2n-2}^{k}s^{k-n}ds+\int(-1)^{n-1}C_{2n-2}^{n-1}s^{-1}ds\right]$$

$$=\left(\frac{1}{2}\right)^{2n-1}\left[\sum_{\substack{k=0\\k\neq n-1}}^{2n-2}(-1)^kC_{2n-2}^{k}\frac{s^{k-n+1}}{k-n+1}+(-1)^{n-1}C_{2n-2}^{n-1}\ln|s|\right]+C$$

$1-2t=s$且$x+1=\frac{1}{t}\Rightarrow s=\frac{x-1}{x+1}$

$$=\left(\frac{1}{2}\right)^{2n-1}\left[\sum_{\substack{k=0\\k\neq n-1}}^{2n-2}(-1)^k\frac{C_{2n-2}^{k}}{k-n+1}\left(\frac{x-1}{x+1}\right)^{k-n+1}+(-1)^{n-1}C_{2n-2}^{n-1}\ln\left|\frac{x-1}{x+1}\right|\right]+C$$

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有时候我们作倒代换后可以凑微分，不用对分子进行降次

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$$a \neq 0, \int \frac{dx}{x(x^n + a)} \qquad 令x = \frac{1}{t}$$

$$\int \frac{dx}{x(x^n + a)} = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\left(\frac{1}{t^n} + a\right)} = \int \frac{-t^{n-1}dt}{1 + at^n} = \int \frac{-\frac{1}{an}d(1 + at^n)}{1 + at^n}$$

$$= -\frac{1}{an}\ln\left|1 + at^n\right| + C$$

$$= -\frac{1}{an}\ln\left|\frac{x^n + a}{x^n}\right| + C$$

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$$\int \frac{1}{(1+x^4)\sqrt[4]{1+x^4}}dx \qquad 令x=\frac{1}{t}$$

当$x>0$时 $$\int \frac{1}{(1+x^4)\sqrt[4]{1+x^4}}dx=\int \frac{-\frac{1}{t^2}}{\left(1+\frac{1}{t^4}\right)\sqrt[4]{1+\frac{1}{t^4}}}dt=\int \frac{-t^3}{t(1+t^4)\sqrt[4]{1+\frac{1}{t^4}}}dt=\int \frac{-t^3}{(1+t^4)\sqrt[4]{t^4+1}}dt$$

$$=\int \frac{-t^3}{(1+t^4)^{\frac{5}{4}}}dt=\int \frac{-\frac{1}{4}d(1+t^4)}{(1+t^4)^{\frac{5}{4}}}=(1+t^4)^{-\frac{1}{4}}+C=\frac{x}{\sqrt[4]{1+x^4}}+C$$

当$x<0$时 $$\int \frac{1}{(1+x^4)\sqrt[4]{1+x^4}}dx=\frac{x}{\sqrt[4]{1+x^4}}+C$$

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$$\int \frac{1}{(1+x^4)\sqrt[4]{1+x^4}}dx \qquad 令x=\frac{1}{t} \qquad t=\frac{t}{|t|}\cdot|t|$$

$$\int \frac{1}{(1+x^4)\sqrt[4]{1+x^4}}dx=\int \frac{-t^3}{t(1+t^4)\sqrt[4]{1+\frac{1}{t^4}}}dt=\int \frac{-t^3}{\frac{t}{|t|}\cdot|t|(1+t^4)\sqrt[4]{1+\frac{1}{t^4}}}dt=\int \frac{|t|}{t}\cdot\frac{-t^3}{(1+t^4)\sqrt[4]{1+\frac{1}{t^4}}}dt$$

$$=\int \frac{|t|}{t}\cdot\frac{-\frac{1}{4}d(1+t^4)}{(1+t^4)^{\frac{5}{4}}}=\frac{|t|}{t}\cdot(1+t^4)^{-\frac{1}{4}}+C=\frac{x}{|x|}\cdot\frac{1}{\sqrt[4]{1+\frac{1}{x^4}}}+C=\frac{x}{\sqrt[4]{1+x^4}}+C$$

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$$\int \frac{dx}{x\sqrt{x^2-1}} \qquad 令x=\frac{1}{t}$$

当$x>0$时 $$\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-1}}=\int \frac{-dt}{\sqrt{1-t^2}}=-\arcsin t+C=-\arcsin\frac{1}{x}+C$$

当$x<0$时 $$\int \frac{dx}{x\sqrt{x^2-1}}=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-1}}=\int \frac{dt}{\sqrt{1-t^2}}=\arcsin t+C=\arcsin\frac{1}{x}+C$$

$$-\frac{|x|}{x}\arcsin\frac{1}{x}+C$$

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$$\int \frac{dx}{(x+1)^3\sqrt{x^2+2x}}$$

令$x+1=t$

平移

$$\int \frac{dx}{(x+1)^3\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)^3\sqrt{(x+1)^2-1}} = \int \frac{dt}{t^3\sqrt{t^2-1}}$$

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当被积函数只含有三角函数时

我们一般通过凑微分 $-\sin x dx = d\cos x \quad \cos x dx = d\sin x \quad \sec^2 x dx = d\tan x$

把积分变换成 $\int f(\sin x)d\sin x \quad \int f(\cos x)d\cos x \quad \int f(\tan x)d\tan x$ 这样的形式

把被积函数化成关于sinx或cosx或tanx的函数

积分变量对应地化成sinx或cosx或tanx

这样我们就可以通过换元消去三角

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$\int R(\sin x, \cos x)dx$ 被积函数是关于$\sin x, \cos x$的三角有理式

$R(\sin x, -\cos x) = -R(\sin x, \cos x)$，即$R(\sin x, \cos x)$是关于$\cos x$的奇函数，则令$t = \sin x$或$\csc x$

$\int R(\sin x, \cos x)dx = \int f(\sin x)d\sin x = \int f(t)dt$

$R(-\sin x, \cos x) = -R(\sin x, \cos x)$，即$R(\sin x, \cos x)$是关于$\sin x$的奇函数，则令$t = \cos x$或$\sec x$

$\int R(\sin x, \cos x)dx = \int f(\cos x)d\cos x = \int f(t)dt$

$R(-\sin x, -\cos x) = R(\sin x, \cos x)$，则令$t = \tan x$或$\cot x$

$\int R(\sin x, \cos x)dx = \int f(\tan x)d\tan x = \int f(t)dt$

$\int R(\sin x, \cos x)dx$ 转化成了有理函数$f(t)$的不定积分

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$$\int \frac{dx}{\sin x\cos^4 x}$$

$\dfrac{1}{\sin x\cos^4 x}$ 关于sinx的奇函数

故可化 $\int f(\cos x)\,d\cos x$

$$\int \frac{dx}{\sin x\cos^4 x}=\int \frac{\sin x dx}{\sin^2 x\cos^4 x}=\int \frac{-d\cos x}{\sin^2 x\cos^4 x}=\int \frac{-d\cos x}{(1-\cos^2 x)\cos^4 x}=\int \frac{-dt}{(1-t^2)t^4}$$

令 $\cos x = t$

$$\int \frac{dx}{\sin x\cos^4 x}=\int \frac{\cos x dx}{\sin x\cos^5 x}=\int \frac{d\sin x}{\sin x\cos^5 x}=\int \frac{d\sin x}{\pm\sin x(1-\sin^2 x)^{\frac{5}{2}}}=\int \frac{dt}{\pm t(1-t^2)^{\frac{5}{2}}}$$

令 $\sin x = t$

$$\cos x=\pm(1-\sin^2 x)^{\frac{1}{2}}$$

$$\cos^5 x=\pm(1-\sin^2 x)^{\frac{5}{2}}$$

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倒代换对分母进行降次

$$\int\frac{-dt}{(1-t^2)t^4}=\int\frac{\frac{1}{s^2}ds}{\left(1-\frac{1}{s^2}\right)\frac{1}{s^4}}=\int\frac{s^4ds}{s^2-1}\qquad 令t=\frac{1}{s}$$

$$\begin{array}{r} s^2+1 \\ s^2-1\overline{)s^4\qquad\qquad} \\ \underline{s^4-s^2} \\ s^2\qquad \\ \underline{s^2-1} \\ 1\qquad \end{array}$$

$$=\int\left[s^2+1+\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)\right]ds$$

$$=\frac{1}{3}s^3+s+\frac{1}{2}(\ln|s-1|-\ln|s+1|)+C\quad=\frac{1}{3}s^3+s+\frac{1}{2}\ln\left|\frac{s-1}{s+1}\right|+C$$

$$=\frac{1}{3\cos^3x}+\frac{1}{\cos x}+\frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+C$$

$$\frac{s^4}{s^2-1}=\frac{(s^4-1)+1}{s^2-1}=s^2+1+\frac{1}{s^2-1}=s^2+1+\frac{1}{2}\left(\frac{1}{s-1}-\frac{1}{s+1}\right)$$

直接对分式进行分解达到对分母进行降次的目的

$$\frac{-1}{(1-t^2)t^4}=\frac{(t^2-1)-t^2}{(1-t^2)t^4}=-\frac{1}{t^4}+\frac{1}{(t^2-1)t^2}=-\frac{1}{t^4}+\frac{1}{t^2-1}-\frac{1}{t^2}=-\frac{1}{t^4}+\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)-\frac{1}{t^2}$$

$$\frac{-1}{(1-t^2)t^4}=\frac{at^3+bt^2+ct+d}{t^4}+\frac{e}{t-1}+\frac{f}{t+1}$$

待定系数法

分式分解定理

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$$\frac{1}{\sin^3 x+3\sin x}\text{关于}\sin x\text{的奇函数}$$

故可化 $\int f(\cos x)\,d\cos x$

$$\int\frac{dx}{\sin^3 x+3\sin x}=\int\frac{\sin x\,dx}{\sin^4 x+3\sin^2 x}=\int\frac{-d\cos x}{(1-\cos^2 x)(4-\cos^2 x)}$$

$$=\int\frac{-dt}{(1-t^2)(4-t^2)}$$

令 $\cos x=t$

$$=\int\frac{1}{3}\left(\frac{1}{t^2-1}-\frac{1}{t^2-4}\right)dt$$

$$=\frac{1}{6}\ln\left|\frac{1-t}{1+t}\right|-\frac{1}{12}\ln\left|\frac{2-t}{2+t}\right|+C=\frac{1}{6}\ln\left|\frac{1-\cos x}{1+\cos x}\right|-\frac{1}{12}\ln\left|\frac{2-\cos x}{2+\cos x}\right|+C$$

$$\int\frac{dx}{x^2-a^2}=\int\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

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$\sec^2 x = \tan^2 x + 1 \qquad \sec^2 x dx = d\tan x$

$$\int \sec^6 x dx = \int \sec^4 x \cdot \sec^2 x dx = \int \sec^4 x d\tan x = \int (1+\tan^2 x)^2 d\tan x$$

$$= \int (1+t^2)^2 dt \qquad 令 \tan x = t$$

$$= \int (1+2t^2+t^4) dt$$

$$= t + \frac{2t^3}{3} + \frac{t^5}{5} + C$$

$$= \tan x + \frac{2\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$$

$\sec^6 x = \dfrac{1}{\cos^6 x} = R(\sin x, \cos x)$

$R(\sin x, \cos x) = R(-\sin x, -\cos x)$

故可化$\int f(\tan x) d\tan x$

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$$\frac{1}{5+8\sin\theta\cos\theta}=R(\sin\theta,\cos\theta)$$

$$R(\sin\theta,\cos\theta)=R(-\sin\theta,-\cos\theta)$$

故可化 $\int f(\tan\theta)\,d\tan\theta$

$$\int\frac{1}{5+8\sin\theta\cos\theta}d\theta=\int\frac{\sec^2\theta d\theta}{5\sec^2\theta+8\tan\theta}$$

$$=\int\frac{\sec^2\theta d\theta}{5\tan^2\theta+5+8\tan\theta}$$

$$=\int\frac{d\tan\theta}{5\tan^2\theta+5+8\tan\theta}$$

令 $\tan\theta=t$

$$=\int\frac{dt}{5t^2+5+8t}$$

$$=\int\frac{dt}{5\left(t+\frac{4}{5}\right)^2+\frac{9}{5}}=\int\frac{\frac{5}{9}dt}{\frac{25}{9}\left(t+\frac{4}{5}\right)^2+1}=\int\frac{\frac{1}{3}d\left(\frac{5}{3}t+\frac{4}{3}\right)}{\left(\frac{5}{3}t+\frac{4}{3}\right)^2+1}=\frac{1}{3}\arctan\left(\frac{5}{3}t+\frac{4}{3}\right)+C$$

$$=\frac{1}{3}\arctan\left(\frac{5}{3}\tan\theta+\frac{4}{3}\right)+C$$

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$ab \neq 0 \quad \int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x} = \int \frac{\sec^2 x dx}{a^2\tan^2 x + b^2} = \int \frac{d\tan x}{a^2\tan^2 x + b^2}$

$\frac{1}{a^2\sin^2 x + b^2\cos^2 x} = R(\sin x, \cos x)$

$R(\sin x, \cos x) = R(-\sin x, -\cos x)$

故可化 $\int f(\tan x) d\tan x$

$= \int \frac{dt}{a^2t^2 + b^2}$ 令 $\tan x = t$

$$= \int \frac{\frac{1}{b^2}dt}{\left(\frac{a}{b}\right)^2 t^2 + 1} = \int \frac{\frac{1}{ab} d\left(\frac{a}{b}t\right)}{\left(\frac{a}{b}t\right)^2 + 1} = \frac{1}{ab}\arctan\left(\frac{a}{b}t\right) + C$$

$$= \frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right) + C$$

$ab \neq 0 \quad \int \frac{dx}{a^2\sin^2 x - b^2\cos^2 x} = \frac{1}{2ab}\ln\left|\frac{a\tan x - b}{a\tan x + b}\right| + C$

$ab \neq 0 \quad \int \frac{dx}{a\sin^2 x + b} \qquad \int \frac{dx}{a\cos^2 x + b}$

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前面我们是通过凑微分的方法来消去三角，我们还可以利用万能代换来消去三角

$$\text{令}u=\tan\frac{x}{2}\Rightarrow\cos x=\frac{1-u^2}{1+u^2}\quad\sin x=\frac{2u}{1+u^2}\quad\tan x=\frac{2u}{1-u^2}\quad\cot x=\frac{1-u^2}{2u}\quad dx=\frac{2}{1+u^2}du$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\Big/\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)=\left(1-\tan^2\frac{x}{2}\right)\Big/\left(1+\tan^2\frac{x}{2}\right)=\frac{1-u^2}{1+u^2}$$

$$\sin x=2\sin\frac{x}{2}\cos\frac{x}{2}=2\sin\frac{x}{2}\cos\frac{x}{2}\Big/\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)=2\tan\frac{x}{2}\Big/\left(1+\tan^2\frac{x}{2}\right)=\frac{2u}{1+u^2}$$

$$\tan x=\frac{\sin x}{\cos x}=\frac{2u}{1-u^2}\qquad \cot x=\frac{\sin x}{\cos x}=\frac{1-u^2}{2u}$$

$$du=d\tan\frac{x}{2}=\frac{1}{2}\sec^2\frac{x}{2}dx=\frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx=\frac{1}{2}(1+u^2)dx$$

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万能代换实际上是第二类换元　　　凑微分法是第一类换元

$$\frac{x}{2}=\arctan u \Rightarrow \tan\frac{x}{2}=\tan(\arctan u)=u\text{但}\tan\frac{x}{2}=u \nRightarrow \frac{x}{2}=\arctan u$$

$$\text{故}u=\tan\frac{x}{2} \not\Leftrightarrow \frac{x}{2}=\arctan u$$

$$u=\tan\frac{x}{2} \Leftrightarrow \tan(\arctan u)=\tan\frac{x}{2}$$

$$\arctan u\text{与}\frac{x}{2}\text{不一定相等}$$

$$\arctan u\text{与}\frac{x}{2}\text{相差}\pi\text{的整数倍}$$

$$\text{当}x\in((2k-1)\pi,(2k+1)\pi)\text{时}\Rightarrow \frac{x}{2}\in(k\pi-\frac{\pi}{2},\ k\pi+\frac{\pi}{2})\Rightarrow \frac{x}{2}-k\pi\in(-\frac{\pi}{2},\frac{\pi}{2})$$

$$\text{此时}\frac{x}{2}-k\pi=\arctan u \Leftrightarrow \tan\left(\frac{x}{2}-k\pi\right)=\tan(\arctan u)\Leftrightarrow \tan\frac{x}{2}=u$$

$$\text{故当}x\in((2k-1)\pi,(2k+1)\pi)\text{时，令}\tan\frac{x}{2}=u\text{，实际上是令}x=2\arctan u+2k\pi$$

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$$\int \frac{1}{\cos x+\sin x+2}dx=\int \frac{\frac{2}{1+u^2}}{\frac{1-u^2}{1+u^2}+\frac{2u}{1+u^2}+2}du=\int \frac{2}{u^2+2u+3}du \qquad 令u=\tan\frac{x}{2}$$

$$=\int \frac{2}{(u+1)^2+2}du$$

$$=\int \frac{du}{\left(\frac{u+1}{\sqrt{2}}\right)^2+1}$$

$$=\int \frac{\sqrt{2}d\left(\frac{u+1}{\sqrt{2}}\right)}{\left(\frac{u+1}{\sqrt{2}}\right)^2+1}=\sqrt{2}\arctan\frac{u+1}{\sqrt{2}}+C=\sqrt{2}\arctan\frac{\tan\frac{x}{2}+1}{\sqrt{2}}+C$$

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$$\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\cos x+\sin x+2}dx = \int_{1}^{0} \frac{2}{u^2+2u+3}du \quad ?? \qquad 令 u=\tan\frac{x}{2}$$

$$\begin{array}{lll} x & \frac{\pi}{2}\to\pi & \pi\to 2\pi \\ u & 1\to+\infty & -\infty\to 0 \end{array}$$

无穷间断点

$$\int_{\frac{\pi}{2}}^{2\pi} \frac{1}{\cos x+\sin x+2}dx = \int_{1}^{+\infty} \frac{2}{u^2+2u+3}du + \int_{-\infty}^{0} \frac{2}{u^2+2u+3}du$$

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$$\int_{\frac{\pi}{2}}^{2\pi}\frac{1}{\cos x+\sin x+2}dx=\int_{\frac{\pi}{2}}^{\pi}\frac{1}{\cos x+\sin x+2}dx+\int_{\pi}^{2\pi}\frac{1}{\cos x+\sin x+2}dx$$

$x=2\arctan u$

$$\tan\frac{x}{2}=u$$

$$\int_{\frac{\pi}{2}}^{\pi}\frac{1}{\cos x+\sin x+2}dx=\int_{1}^{+\infty}\frac{2}{u^2+2u+3}du$$

$x=2\arctan u+2\pi$

$$\int_{\pi}^{2\pi}\frac{1}{\cos x+\sin x+2}dx=\int_{-\infty}^{0}\frac{2}{u^2+2u+3}du$$

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$$\int \frac{1}{\sin x+2\cos x}dx=\int \frac{\frac{2}{1+u^2}}{\frac{2u}{1+u^2}+2\cdot\frac{1-u^2}{1+u^2}}du \qquad 令u=\tan\frac{x}{2}$$

$$=\int \frac{1}{-u^2+u+1}du$$

$$=\int \frac{1}{\left(\frac{1+\sqrt{5}}{2}-u\right)\left(u-\frac{1-\sqrt{5}}{2}\right)}du=\int \frac{1}{\sqrt{5}}\left(\frac{1}{\frac{1+\sqrt{5}}{2}-u}+\frac{1}{u-\frac{1-\sqrt{5}}{2}}\right)du$$

$$=\frac{1}{\sqrt{5}}\left(-\ln\left|\frac{1+\sqrt{5}}{2}-u\right|+\ln\left|u-\frac{1-\sqrt{5}}{2}\right|\right)+C=\frac{1}{\sqrt{5}}\ln\left|\frac{\tan\frac{x}{2}-\frac{1-\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}-\tan\frac{x}{2}}\right|+C$$

第六讲：不定积分 > 换元法 > 三角代换 > 消去三角

$$\int \frac{1}{\sin x+2\cos x}dx$$

$$\sin x+2\cos x=\sqrt{1^2+2^2}\left(\frac{1}{\sqrt{1^2+2^2}}\sin x+\frac{2}{\sqrt{1^2+2^2}}\cos x\right)$$

设 $\frac{2}{\sqrt{1^2+2^2}}=\sin\varphi \quad \varphi\in(0,\frac{\pi}{2}) \Rightarrow \frac{1}{\sqrt{1^2+2^2}}=\cos\varphi$

$$\sin x+2\cos x=\sqrt{5}(\cos\varphi\sin x+\sin\varphi\cos x)=\sqrt{5}\sin(x+\varphi)$$

$$=\int\frac{1}{\sqrt{5}\sin(x+\varphi)}dx=\int\frac{\sin(x+\varphi)dx}{\sqrt{5}\sin^2(x+\varphi)}=\frac{1}{\sqrt{5}}\int\frac{-d\cos(x+\varphi)}{1-\cos^2(x+\varphi)}$$

令 $\cos(x+\varphi)=t$

$$=\frac{1}{\sqrt{5}}\int\frac{-dt}{1-t^2}=\frac{1}{2\sqrt{5}}\ln\left|\frac{t-1}{t+1}\right|+C=\frac{1}{2\sqrt{5}}\ln\left|\frac{\cos(x+\varphi)-1}{\cos(x+\varphi)+1}\right|+C=\frac{1}{2\sqrt{5}}\ln\left|\frac{\cos\left(x+\arcsin 2/\sqrt{5}\right)-1}{\cos\left(x+\arcsin 2/\sqrt{5}\right)+1}\right|+C$$

第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

通过三角代换来产生三角目的是为了消去根号

$\sqrt{a^2-x^2} \Rightarrow -a \le x \le a \Rightarrow$ 令$x=a\sin t$或$a\cos t$

令$x=a\sin t$，$t\in[-\frac{\pi}{2},\frac{\pi}{2}] \Rightarrow \sqrt{a^2-x^2}=a\cos t$

令$x=a\cos t$，$t\in[0,\ \pi] \Rightarrow \sqrt{a^2-x^2}=a\sin t$

$x=a\sin t$，$t\in[-\frac{\pi}{2},0]\cup[\frac{\pi}{2},\ \pi]$

$\Rightarrow \sqrt{a^2-x^2}=\pm a\cos t$

$\sqrt{a^2+x^2} \Rightarrow$ 对x无限制 $\Rightarrow$ 令$x=a\tan t$或$a\cot t$

令$x=a\tan t$，$t\in(-\frac{\pi}{2},\frac{\pi}{2}) \Rightarrow \sqrt{a^2+x^2}=a\sec t$

令$x=a\cot t$，$t\in(0,\ \pi) \Rightarrow \sqrt{a^2+x^2}=a\csc t$

令$x=a\tan t$，$t\in[0,\frac{\pi}{2})\cup(\frac{\pi}{2},\ \pi]$

$\Rightarrow \sqrt{a^2-x^2}=\pm a\sec t$

$\sqrt{x^2-a^2} \Rightarrow a\le x$或$x\le -a \Rightarrow$ 令$x=a\sec t$或$a\csc t$

令$x=a\sec t$，$t\in(0,\frac{\pi}{2})\cup(\pi,\frac{3\pi}{2}) \Rightarrow \sqrt{x^2-a^2}=a\tan t$

令$x=a\csc t$，$t\in(0,\frac{\pi}{2})\cup(\pi,\frac{3\pi}{2}) \Rightarrow \sqrt{x^2-a^2}=a\cot t$

第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int (3-x^2)^{\frac{3}{2}}\,dx$$

降次

$$|x| \le \sqrt{3} \Rightarrow \text{令} x=\sqrt{3}\sin t \quad t\in[-\frac{\pi}{2},\frac{\pi}{2}]$$

$$=\int (3-3\sin^2 t)^{\frac{3}{2}}\cdot\sqrt{3}\cos t\,dt=\int 9\cos^4 t\,dt$$

$$\cos^4 t=(\cos^2 t)^2=\left(\frac{1+\cos 2t}{2}\right)^2=\frac{1+2\cos 2t+\cos^2 2t}{4}$$

$$=\frac{1+2\cos 2t+\frac{1+\cos 4t}{2}}{4}=\frac{4\cos 2t+3+\cos 4t}{8}$$

$$=\int 9\cdot\frac{4\cos 2t+3+\cos 4t}{8}dt=\frac{9}{4}\sin 2t+\frac{27}{8}t+\frac{9}{32}\sin 4t+C$$

$$t=\arcsin\frac{x}{\sqrt{3}}$$

$$\sin 2t=2\sin t\cos t=2\sin t\sqrt{1-\sin^2 t}=2\cdot\frac{x}{\sqrt{3}}\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}=\frac{2x}{3}\sqrt{3-x^2}$$

$$\sin 4t=2\sin 2t\cos 2t=2\sin 2t(1-2\sin^2 t)=2\cdot\frac{2x}{3}\sqrt{3-x^2}\cdot\left(1-2\left(\frac{x}{\sqrt{3}}\right)^2\right)=\frac{4}{9}(3x-2x^3)\sqrt{3-x^2}$$

第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int \frac{1}{x^2\sqrt{x^2+3}}dx \qquad 令x=\sqrt{3}\tan t \quad t\in(-\frac{\pi}{2},\frac{\pi}{2})$$

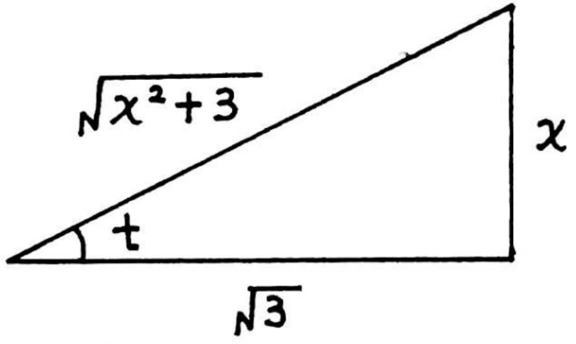
若令$x=\sqrt{3}\tan t \quad t\in(0,\frac{\pi}{2})\cup(\frac{\pi}{2},\ \pi)$

$$\sqrt{x^2+3}=\sqrt{3\tan^2 t+3}=\sqrt{3\sec^2 t}=\pm\sqrt{3}\sec t$$

需讨论麻烦

$$=\int \frac{\sqrt{3}\sec^2 t}{3\tan^2 t\sqrt{3\tan^2 t+3}}dt=\int \frac{\sec t}{3\tan^2 t}dt$$

$$=\int \frac{\cos tdt}{3\sin^2 t}=\int \frac{d\sin t}{3\sin^2 t}=-\frac{1}{3\sin t}+C=-\frac{\sqrt{x^2+3}}{3x}+C$$



$$|\sin t|=\left|\frac{x}{\sqrt{x^2+3}}\right|$$

$$x>0\Rightarrow t\in(0,\frac{\pi}{2}) \quad \Rightarrow \sin t>0\Rightarrow \sin t=\frac{x}{\sqrt{x^2+3}}$$

$$x<0\Rightarrow t\in(-\frac{\pi}{2},0)\Rightarrow \sin t<0\Rightarrow \sin t=\frac{x}{\sqrt{x^2+3}}$$

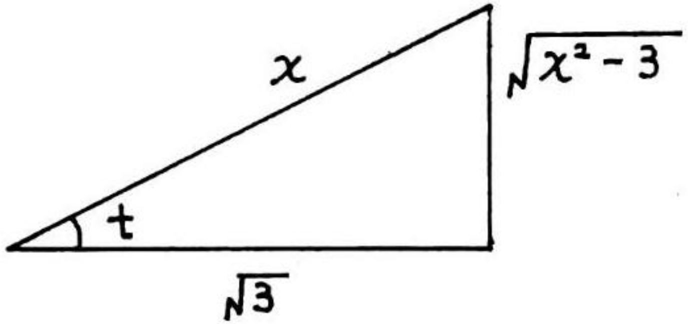
第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int \frac{x^2}{(x^2-3)^{\frac{3}{2}}}dx \qquad |x|>\sqrt{3} \qquad \text{令} x=\sqrt{3}\sec t \qquad t\in(0,\frac{\pi}{2})\cup(\pi,\frac{3\pi}{2})$$

$$=\int \frac{3\sec^2 t\cdot\sqrt{3}\sec t\tan t dt}{(3\tan^2 t)^{\frac{3}{2}}}=\int \frac{\sec^3 t dt}{\tan^2 t} \qquad (x^2-3)^{\frac{3}{2}}=(3\tan^2 t)^{\frac{3}{2}}=3^{\frac{3}{2}}\tan^3 t$$

$$=\int \frac{dt}{\sin^2 t\cos t}=\int \frac{\cos t dt}{\sin^2 t\cos^2 t}=\int \frac{d\sin t}{\sin^2 t(1-\sin^2 t)} \qquad \text{令}\sin t=s$$

$$=\int \frac{ds}{s^2(1-s^2)}=\int\left(\frac{1}{s^2}+\frac{1}{1-s^2}\right)ds=-\frac{1}{s}+\frac{1}{2}\ln\left|\frac{s+1}{s-1}\right|+C=-\frac{1}{\sin t}+\frac{1}{2}\ln\left|\frac{\sin t+1}{\sin t-1}\right|+C$$



$$|\sin t|=\left|\frac{\sqrt{x^2-3}}{x}\right|$$

$$x>0\Rightarrow t\in(0,\frac{\pi}{2}) \quad \Rightarrow \sin t>0\Rightarrow \sin t=\frac{\sqrt{x^2-3}}{x}$$

$$x<0\Rightarrow t\in(\pi,\frac{3\pi}{2})\Rightarrow \sin t<0\Rightarrow \sin t=\frac{\sqrt{x^2-3}}{x}$$

$$\sqrt{Ax^2+Bx+C} \xrightarrow{\text{配方}} \sqrt{a^2-x^2} \text{或} \sqrt{a^2+x^2} \text{或} \sqrt{x^2-a^2}$$

再作三角代换即可消去根号

第六讲：不定积分 > 换元法 > 三角代换 > 产生三角

$$\int\frac{dx}{\sqrt{1-x+x^2}}=\int\frac{dx}{\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}}=\int\frac{\frac{4}{3}dx}{\sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}}=\int\frac{\frac{2}{\sqrt{3}}d\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}}\qquad 令t=\frac{2x-1}{\sqrt{3}}$$

$$=\frac{2}{\sqrt{3}}\int\frac{dt}{\sqrt{t^2+1}}\qquad 令t=\tan\theta\quad \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$$

$$=\frac{2}{\sqrt{3}}\int\frac{\sec^2\theta d\theta}{\sqrt{\tan^2\theta+1}}=\frac{2}{\sqrt{3}}\int\sec\theta d\theta$$

欧拉替换法

$$=\frac{2}{\sqrt{3}}\int\frac{d\theta}{\cos\theta}=\frac{2}{\sqrt{3}}\int\frac{\cos\theta d\theta}{\cos^2\theta}$$

$$=\frac{2}{\sqrt{3}}\int\frac{d\sin\theta}{1-\sin^2\theta}=\frac{1}{\sqrt{3}}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|+C$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int R(x, \sqrt{ax^2+bx+c})dx$$

其中$a \neq 0$且$\Delta = b^2 - 4ac \neq 0$　$R(x, y)$表示以x，y的为变量的有理函数

对于这样一种形式的不定积分我们用欧拉代换都可以将被积函数有理化

欧拉第一代换

$a > 0$　令$\sqrt{ax^2+bx+c} = t + \sqrt{a}x$或$t - \sqrt{a}x$

欧拉第二代换

$c > 0$　令$\sqrt{ax^2+bx+c} = xt + \sqrt{c}$或$xt - \sqrt{c}$

欧拉第三代换

$\Delta > 0$　令$\sqrt{ax^2+bx+c} = t(x-\alpha)$或$t(x-\beta)$

其中α，β是$ax^2+bx+c=0$的两互异实根

第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第一代换

$a>0$ 令 $\sqrt{ax^2+bx+c}=t+\sqrt{a}x$ 或 $t-\sqrt{a}x$

$$\sqrt{ax^2+bx+c}=t+\sqrt{a}x \Rightarrow ax^2+bx+c=t^2+2\sqrt{a}xt+ax^2$$

$$\Rightarrow x=\frac{c-t^2}{2\sqrt{a}t-b} \qquad \sqrt{ax^2+bx+c}=\frac{\sqrt{a}t^2-bt+c\sqrt{a}}{2\sqrt{a}t-b} \qquad dx=\frac{-2\sqrt{a}t^2+2bt-2c\sqrt{a}}{\left(2\sqrt{a}t-b\right)^2}dt$$

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{c-t^2}{2\sqrt{a}t-b},\frac{\sqrt{a}t^2-bt+c\sqrt{a}}{2\sqrt{a}t-b}\right)\frac{-2\sqrt{a}t^2+2bt-2c\sqrt{a}}{\left(2\sqrt{a}t-b\right)^2}dt$$

第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第二代换

$c>0$ 令 $\sqrt{ax^2+bx+c}=xt+\sqrt{c}$ 或 $xt-\sqrt{c}$

$$\sqrt{ax^2+bx+c}=xt+\sqrt{c}\Rightarrow ax^2+bx+c=x^2t^2+2\sqrt{c}xt+c$$

$$\Rightarrow x=\frac{2\sqrt{c}t-b}{a-t^2} \qquad \sqrt{ax^2+bx+c}=\frac{\sqrt{c}t^2-bt+a\sqrt{c}}{a-t^2} \qquad dx=\frac{2\sqrt{c}t^2-2bt+2a\sqrt{c}}{\left(a-t^2\right)^2}dt$$

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{2\sqrt{c}t-b}{a-t^2},\frac{\sqrt{c}t^2-bt+a\sqrt{c}}{a-t^2}\right)\frac{2\sqrt{c}t^2-2bt+2a\sqrt{c}}{\left(a-t^2\right)^2}dt$$

第六讲：不定积分 > 换元法 > 欧拉替换法

欧拉第三代换

$\Delta>0$ 令 $\sqrt{ax^2+bx+c}=t(x-\alpha)$ 或 $t(x-\beta)$

其中 α，β 是 $ax^2+bx+c=0$ 的两互异实

令 $t=\sqrt{\dfrac{a(x-\beta)}{x-\alpha}}$ 或 $\sqrt{\dfrac{a(x-\alpha)}{x-\beta}}$

$$\sqrt{ax^2+bx+c}=t(x-\alpha)\Rightarrow ax^2+bx+c=t^2(x-\alpha)^2$$

$$\Rightarrow a(x-\beta)(x-\alpha)=t^2(x-\alpha)^2\Rightarrow a(x-\beta)=t^2(x-\alpha)$$

$$\Rightarrow x=\frac{\alpha t^2-a\beta}{t^2-a}\qquad \sqrt{ax^2+bx+c}=\frac{a(\alpha-\beta)t}{t^2-a}\qquad dx=\frac{2a(\beta-\alpha)t}{(t^2-a)^2}dt$$

$$\int R(x,\sqrt{ax^2+bx+c})dx=\int R\left(\frac{\alpha t^2-a\beta}{t^2-a},\frac{a(\alpha-\beta)t}{t^2-a}\right)\frac{2a(\beta-\alpha)t}{(t^2-a)^2}dt$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int \frac{dx}{\sqrt{1-x+x^2}}$$

令$\sqrt{1-x+x^2}=t+x \Rightarrow 1-x+x^2=t^2+2xt+x^2$

$$\Rightarrow x=\frac{1-t^2}{2t+1} \qquad \sqrt{1-x+x^2}=\frac{t^2+t+1}{2t+1} \qquad dx=\frac{-2t^2-2t-2}{(2t+1)^2}dt$$

$$\int \frac{dx}{\sqrt{1-x+x^2}}=\int \frac{\frac{-2(t^2+t+1)}{(2t+1)^2}dt}{\frac{t^2+t+1}{2t+1}}=\int \frac{-2dt}{2t+1}=-\ln|2t+1|+C=-\ln\left|2\sqrt{1-x+x^2}-2x+1\right|+C$$

$a>0 \quad b^2-4ac\neq 0$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$-\ln\left|2\sqrt{1-x+x^2}-2x+1\right|=-\ln\left|\frac{\left(2\sqrt{1-x+x^2}-2x+1\right)\left(2\sqrt{1-x+x^2}+2x-1\right)}{2\sqrt{1-x+x^2}+2x-1}\right|$$

$$=-\ln\left|\frac{\left(2\sqrt{1-x+x^2}\right)^2-(2x-1)^2}{2\sqrt{1-x+x^2}+2x-1}\right|=-\ln\left|\frac{3}{2\sqrt{1-x+x^2}+2x-1}\right|=\ln\left|2\sqrt{1-x+x^2}+2x-1\right|-\ln 3$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$\int \frac{dx}{\sqrt{1-x+x^2}}$$

令 $\sqrt{1-x+x^2} = xt+1 \Rightarrow x^2 - x + 1 = x^2t^2 + 2xt + 1$

$$\Rightarrow x = \frac{2t+1}{1-t^2} \qquad \sqrt{1-x+x^2} = \frac{t^2+t+1}{1-t^2} \qquad dx = \frac{2\left(t^2+t+1\right)}{\left(1-t^2\right)^2}dt$$

$$\int \frac{dx}{\sqrt{1-x+x^2}} = \int \frac{\frac{2\left(t^2+t+1\right)}{\left(1-t^2\right)^2}dt}{\frac{t^2+t+1}{1-t^2}} = \int \frac{2\,dt}{1-t^2} = \ln\left|\frac{t+1}{t-1}\right| + C = \ln\left|\frac{tx+x}{tx-x}\right| + C = \ln\left|\frac{\sqrt{1-x+x^2}-1+x}{\sqrt{1-x+x^2}-1-x}\right| + C$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$\ln\left|\frac{\sqrt{1-x+x^2}-1+x}{\sqrt{1-x+x^2}-1-x}\right|=\ln\left|\frac{\left(\sqrt{1-x+x^2}-1+x\right)\left(\sqrt{1-x+x^2}+1+x\right)}{\left(\sqrt{1-x+x^2}-1-x\right)\left(\sqrt{1-x+x^2}+1+x\right)}\right|$$

$$=\ln\left|\frac{\left(\sqrt{1-x+x^2}+x\right)^2-1}{\left(\sqrt{1-x+x^2}\right)^2-(1+x)^2}\right|=\ln\left|\frac{2x\sqrt{1-x+x^2}+2x^2-x}{-3x}\right|=\ln\left|2\sqrt{1-x+x^2}+2x-1\right|-\ln 3$$

第六讲：不定积分 > 换元法 > 欧拉替换法

$$n>2\quad \int\frac{dx}{x\sqrt{-x^2+nx-1}}\qquad -x^2+nx-1=0\quad \alpha=\frac{n-\sqrt{n^2-4}}{2}\quad \beta=\frac{n+\sqrt{n^2-4}}{2}$$

令 $\sqrt{-x^2+nx-1}=t(x-\alpha)\Rightarrow -x^2+nx-1=t^2(x-\alpha)^2$　　$\alpha<x<\beta$

$$t=\frac{\sqrt{-x^2+nx-1}}{x-\alpha}=\sqrt{\frac{-(x-\beta)(x-\alpha)}{(x-\alpha)^2}}=\sqrt{\frac{\beta-x}{x-\alpha}}$$

$$\Rightarrow -(x-\beta)(x-\alpha)=t^2(x-\alpha)^2\Rightarrow -(x-\beta)=t^2(x-\alpha)$$

$$\Rightarrow x=\frac{\alpha t^2+\beta}{t^2+1}\qquad \sqrt{-x^2+nx-1}=\frac{-(\alpha-\beta)t}{t^2+1}\qquad dx=\frac{-2(\beta-\alpha)t}{(t^2+1)^2}dt$$

$$=\int\frac{\frac{-2(\beta-\alpha)t}{(t^2+1)^2}dt}{\frac{\alpha t^2+\beta}{t^2+1}\cdot\frac{-(\alpha-\beta)t}{t^2+1}}=\int\frac{-2dt}{\alpha t^2+\beta}=\int\frac{-\frac{2}{\beta}dt}{\frac{\alpha}{\beta}t^2+1}=\int\frac{-2\sqrt{\frac{1}{\alpha\beta}}d\left(\sqrt{\frac{\alpha}{\beta}}t\right)}{\left(\sqrt{\frac{\alpha}{\beta}}t\right)^2+1}$$

$$\sqrt{\frac{\alpha}{\beta}}t=\sqrt{\frac{\alpha\beta-\alpha x}{\beta x-\alpha\beta}}=\sqrt{\frac{1-\alpha x}{\beta x-1}}=\sqrt{\frac{2-\left(n-\sqrt{n^2-4}\right)x}{\left(n+\sqrt{n^2-4}\right)x-2}}$$

$$=-2\arctan\sqrt{\frac{\alpha}{\beta}}t+C=-2\arctan\sqrt{\frac{2-\left(n-\sqrt{n^2-4}\right)x}{\left(n+\sqrt{n^2-4}\right)x-2}}+C$$

第六讲：不定积分 > 待定系数法 > 有理分式

理论依据

分式分解定理

设 $\dfrac{P(x)}{Q(x)}$ 为有理真分式，其中 $Q(x)=Q_1(x)Q_2(x)$ 且 $Q_1(x)$，$Q_2(x)$ 互素

则存在唯一一组多项式 $P_1(x)$，$P_2(x)$ 使得 $\dfrac{P(x)}{Q(x)}=\dfrac{P_1(x)}{Q_1(x)}+\dfrac{P_2(x)}{Q_2(x)}$

其中 $\dfrac{P_1(x)}{Q_1(x)},\dfrac{P_2(x)}{Q_2(x)}$ 为真分式

分式分解定理的推论

设 $\dfrac{P(x)}{Q(x)}$ 为有理真分式，其中 $Q(x)=Q_1(x)Q_2(x)\cdots Q_m(x)$ 且 $Q_1(x)$，$Q_2(x),\cdots$，$Q_m(x)$ 互素

则存在唯一一组多项式 $P_1(x)$，$P_2(x),\cdots$，$P_m(x)$ 使得 $\dfrac{P(x)}{Q(x)}=\dfrac{P_1(x)}{Q_1(x)}+\dfrac{P_2(x)}{Q_2(x)}+\cdots+\dfrac{P_m(x)}{Q_m(x)}$

其中 $\dfrac{P_1(x)}{Q_1(x)},\dfrac{P_2(x)}{Q_2(x)},\cdots,\dfrac{P_m(x)}{Q_m(x)}$ 为真分式

第六讲：不定积分 > 待定系数法 > 有理分式

$a>0 \quad b^2-4ac\neq 0$

$$\int \frac{1}{ax^2+bx+c}dx=\begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\arctan\dfrac{2ax+b}{\sqrt{4ac-b^2}}+C & b^2<4ac \qquad \displaystyle\int\frac{1}{t^2+1}dt \\ \dfrac{1}{\sqrt{b^2-4ac}}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|+C & b^2>4ac \qquad \displaystyle\int\frac{1}{t^2-1}dt \end{cases}$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int\frac{1}{(x^2+1)(x^2+x+1)}dx$$

分式分解定理 省去一些中间过程

$$\begin{cases}M=A\\N=2B\\P=C+D\\Q=2D\end{cases}$$

$$\frac{1}{(x^2+1)(x^2+x+1)}=\frac{M+Nx}{x^2+1}+\frac{P+Qx}{x^2+x+1}=A\frac{1}{x^2+1}+B\frac{2x}{x^2+1}+C\frac{1}{x^2+x+1}+D\frac{2x+1}{x^2+x+1}$$

$$=A(\arctan x)'+B\left(\ln\left(x^2+1\right)\right)'+C\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right)'+D\left(\ln\left(x^2+x+1\right)\right)'$$

$$\frac{1}{(x^2+1)(x^2+x+1)}=A\frac{1}{x^2+1}+B\frac{2x}{x^2+1}+C\frac{1}{x^2+x+1}+D\frac{2x+1}{x^2+x+1}=\frac{(A+2Bx)(x^2+x+1)+(C+2Dx+D)(x^2+1)}{(x^2+1)(x^2+x+1)}$$

$x=0,-1,\ i$

赋值法

$$\begin{cases}A+C+D=1\\(A-2B)+2(C-D)=1\\(A+2Bi)i=Ai-2B=1\end{cases}\Rightarrow\begin{cases}A=0\\B=-1/2\\C=D=1/2\end{cases}$$

$$\int\frac{1}{(x^2+1)(x^2+x+1)}dx=-\frac{1}{2}\ln\left(x^2+1\right)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+\frac{1}{2}\ln\left(x^2+x+1\right)+C$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{x^3-1}dx$$

$$x^3-1=(x-1)\left(1+x+x^2\right)$$

$$\frac{1}{x^3-1}=\frac{A}{x-1}+\frac{P+Qx}{1+x+x^2}=A\frac{1}{x-1}+B\frac{1}{1+x+x^2}+C\frac{2x+1}{1+x+x^2}$$

$$\begin{cases} P=B+C \\ Q=2C \end{cases}$$

$$=A(\ln|x-1|)'+B\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right)'+C\left(\ln\left(1+x^2\right)\right)'$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1}{x^5+1}dx$$

实系数多项式的虚根成对存在

$$(x-\alpha)(x-\overline{\alpha})=x^2-(\alpha+\overline{\alpha})x+\alpha\overline{\alpha}$$

$$(\alpha+\overline{\alpha})^2-4\alpha\overline{\alpha}=(\alpha-\overline{\alpha})^2<0$$

复数域上多项式因式分解定理

每个次数≥ 1的复系数多项式在复数域上都可以唯一地分解成一次因式的乘积

其标准分解式为：$f(x)=a_n(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_s)^{k_s}$

其中a_n为$f(x)$的首项系数，α_1，$\alpha_2,\cdots$，α_s是互异的复数

k_1，$k_2,\cdots$，k_s是正整数，$k_1+k_2+\cdots+k_s=n=\partial(f(x))$

实数域上多项式因式分解定理

每个次数≥ 1的实系数多项式在实数域上都可以唯一地分解成一次因式和二次不可约因式的乘积

其标准分解式为：$f(x)=a_n(x-c_1)^{k_1}\cdots(x-c_s)^{k_s}(x^2+p_1x+q_1)^{j_1}\cdots(x^2+p_rx+q_r)^{j_r}$

其中a_n为$f(x)$的首项系数，$c_1,\cdots$，c_s，$p_1,\cdots$，p_r，$q_1,\cdots$，q_r是实数且$p_i^2-4q_i<0$　$i=1,\cdots$，r

$k_1,\cdots$，k_s，$j_1,\cdots$，j_r是正整数，$k_1+\cdots+k_s+2(j_1+\cdots+j_r)=n=\partial(f(x))$

第六讲：不定积分 > 待定系数法 > 有理分式

$x^n - a \qquad a \neq 0$

n次复系数多项式在复数域恰有n个根

$$x = \begin{cases} \sqrt[n]{a} e^{\frac{2k}{n}\pi i} & k = 0, \cdots, n-1 \quad a > 0 \\ \sqrt[n]{-a} e^{\frac{2k+1}{n}\pi i} & k = 0, \cdots, n-1 \quad a < 0 \end{cases}$$

$$\left(\sqrt[n]{a} e^{\frac{2k}{n}\pi i} \right)^n = a e^{2k\pi i} = a(\cos 2k\pi + i \sin 2k\pi) = a$$

$$\left(\sqrt[n]{-a} e^{\frac{2k+1}{n}\pi i} \right)^n = -a e^{(2k+1)\pi i} = -a[\cos(2k+1)\pi + i \sin(2k+1)\pi] = a$$

$e^{\frac{2k}{n}\pi i}\ (k = 0, \cdots, n-1)$ 是 $x^n - 1$ 的所有复根　　$e^{\frac{2k+1}{n}\pi i}\ (k = 0, \cdots, n-1)$ 是 $x^n + 1$ 的所有复根

$$x^n - 1 = \prod_{k=0}^{n-1} \left(x - e^{\frac{2k}{n}\pi i} \right) \qquad x^n + 1 = \prod_{k=0}^{n-1} \left(x - e^{\frac{2k+1}{n}\pi i} \right)$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$x^5+1=\left(x-e^{\frac{1}{5}\pi i}\right)\left(x-e^{\frac{3}{5}\pi i}\right)\left(x-e^{\frac{5}{5}\pi i}\right)\left(x-e^{\frac{7}{5}\pi i}\right)\left(x-e^{\frac{9}{5}\pi i}\right)$$

$$=\left(x-e^{\frac{1}{5}\pi i}\right)\left(x-e^{\frac{3}{5}\pi i}\right)\left(x-e^{\frac{5}{5}\pi i}\right)\left(x-e^{-\frac{3}{5}\pi i}\right)\left(x-e^{-\frac{1}{5}\pi i}\right)$$

$$=\left[x^2-\left(e^{\frac{1}{5}\pi i}+e^{-\frac{1}{5}\pi i}\right)x+e^{\frac{1}{5}\pi i}e^{-\frac{1}{5}\pi i}\right]\left[x^2-\left(e^{\frac{3}{5}\pi i}+e^{-\frac{3}{5}\pi i}\right)x+e^{\frac{3}{5}\pi i}e^{-\frac{3}{5}\pi i}\right]\left(x-e^{\frac{5}{5}\pi i}\right)$$

$$=\left(x^2-2\cos\frac{1}{5}\pi\cdot x+1\right)\left(x^2-2\cos\frac{3}{5}\pi\cdot x+1\right)(x+1)$$

赋值法

$$x=i\Rightarrow i+1=-4\cos\frac{1}{5}\pi\cos\frac{3}{5}\pi(i+1)\Rightarrow\cos\frac{1}{5}\pi\cos\frac{3}{5}\pi=-\frac{1}{4}$$

$$x=1\Rightarrow 2=\left(2-2\cos\frac{1}{5}\pi\right)\left(2-2\cos\frac{3}{5}\pi\right)2\Rightarrow 1-\cos\frac{1}{5}\pi-\cos\frac{3}{5}\pi+\cos\frac{1}{5}\pi\cos\frac{3}{5}\pi=\frac{1}{4}\Rightarrow\cos\frac{1}{5}\pi+\cos\frac{3}{5}\pi=\frac{1}{2}$$

$\cos\frac{1}{5}\pi,\cos\frac{3}{5}\pi$是方程$t^2-\frac{1}{2}t-\frac{1}{4}=0$的两根 $\quad\cos\frac{1}{5}\pi=\frac{1+\sqrt5}{4}\quad\cos\frac{3}{5}\pi=\frac{1-\sqrt5}{4}$

$$x^5+1=\left(x^2-\frac{1+\sqrt5}{2}x+1\right)\left(x^2-\frac{1-\sqrt5}{2}x+1\right)(x+1)$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$x^5+1=\left(x^2-\frac{1+\sqrt{5}}{2}x+1\right)\left(x^2-\frac{1-\sqrt{5}}{2}x+1\right)(x+1)$$

$$\frac{1}{x^5+1}=\frac{Px+Q}{x^2-\frac{1+\sqrt{5}}{2}x+1}+\frac{Mx+N}{x^2-\frac{1-\sqrt{5}}{2}x+1}+\frac{E}{x+1}$$

$$\begin{cases} P=2A \\ Q=B-\frac{1+\sqrt{5}}{2}A \\ M=2C \\ N=D-\frac{1-\sqrt{5}}{2}C \end{cases}$$

$$=A\frac{2x-\frac{1+\sqrt{5}}{2}}{x^2-\frac{1+\sqrt{5}}{2}x+1}+B\frac{1}{x^2-\frac{1+\sqrt{5}}{2}x+1}+C\frac{2x-\frac{1-\sqrt{5}}{2}}{x^2-\frac{1-\sqrt{5}}{2}x+1}+D\frac{1}{x^2-\frac{1-\sqrt{5}}{2}x+1}+E\frac{1}{x+1}$$

$$=A\left(\ln\left|x^2-\frac{1+\sqrt{5}}{2}x+1\right|\right)'+B\frac{1}{x^2-\frac{1+\sqrt{5}}{2}x+1}+C\left(\ln\left|x^2-\frac{1-\sqrt{5}}{2}x+1\right|\right)'+D\frac{1}{x^2-\frac{1-\sqrt{5}}{2}x+1}+E(\ln|x+1|)'$$

$a>0$ 且 $b^2-4ac<0$

$$\int\frac{1}{ax^2+bx+c}dx=\frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}+C$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int\frac{1}{x^5-1}dx \qquad \int\frac{1}{x^n+1}dx \qquad \int\frac{1}{x^n-1}dx$$

$$x^5-1=\left(x-e^{\frac{0}{5}\pi i}\right)\left(x-e^{\frac{2}{5}\pi i}\right)\left(x-e^{\frac{4}{5}\pi i}\right)\left(x-e^{\frac{6}{5}\pi i}\right)\left(x-e^{\frac{8}{5}\pi i}\right)$$

$$=\left(x-e^{\frac{0}{5}\pi i}\right)\left(x-e^{\frac{2}{5}\pi i}\right)\left(x-e^{\frac{4}{5}\pi i}\right)\left(x-e^{-\frac{4}{5}\pi i}\right)\left(x-e^{-\frac{2}{5}\pi i}\right)$$

$$=\left(x-e^{\frac{0}{5}\pi i}\right)\left[x^2-\left(e^{\frac{2}{5}\pi i}+e^{-\frac{2}{5}\pi i}\right)x+e^{\frac{2}{5}\pi i}e^{-\frac{2}{5}\pi i}\right]\left[x^2-\left(e^{\frac{4}{5}\pi i}+e^{-\frac{4}{5}\pi i}\right)x+e^{\frac{4}{5}\pi i}e^{-\frac{4}{5}\pi i}\right]$$

$$=(x-1)\left(x^2-2\cos\frac{2}{5}\pi\cdot x+1\right)\left(x^2-2\cos\frac{4}{5}\pi\cdot x+1\right)$$

$$x=i \quad x=-1\Rightarrow\cos\frac{2}{5}\pi=\frac{\sqrt{5}-1}{4} \qquad \cos\frac{4}{5}\pi=\frac{-\sqrt{5}-1}{4}$$

$$x^5-1=(x-1)\left(x^2-\frac{\sqrt{5}-1}{2}x+1\right)\left(x^2+\frac{\sqrt{5}+1}{2}x+1\right)$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{-x^2-2}{(x^2+x+1)^2}dx \qquad \frac{1}{x^2+x+1} \qquad \frac{2x+1}{x^2+x+1} \qquad \left(\frac{Ax+B}{x^2+x+1}\right)'$$

猜测

$$\frac{-x^2-2}{(x^2+x+1)^2} = \left(\frac{Ax+B}{x^2+x+1}\right)' + C\frac{2x+1}{x^2+x+1} + D\frac{1}{x^2+x+1}$$

$$= \left(\frac{Ax+B}{x^2+x+1}\right)' + C\left(\ln(x^2+x+1)\right)' + D\left(\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}\right)'$$

$$\frac{-x^2-2}{(x^2+x+1)^2} = \frac{A(x^2+x+1)-(Ax+B)(2x+1)+(2Cx+C+D)(x^2+x+1)}{(x^2+x+1)^2}$$

$$\begin{cases} 0=2C \\ -1=A-2A+C+D+2C \\ 0=A-(2B+A)+C+D+2C \\ -2=A-B+C+D \end{cases} \qquad \begin{cases} A=-1 \\ B=-1 \\ C=0 \\ D=-2 \end{cases}$$

$$\int \frac{-x^2-2}{(x^2+x+1)^2}dx = \frac{-x-1}{x^2+x+1} - \frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$$

第六讲：不定积分 > 待定系数法 > 有理分式

$$\int \frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2}dx$$

$$\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2}=\frac{P+Qx+Rx^2+Sx^3}{(1+x^2)^2}+\frac{M+Nx}{x^2}$$

分式分解定理

$$\frac{P+Qx+Rx^2+Sx^3}{(1+x^2)^2}=\left(\frac{A+Bx}{1+x^2}\right)'+C\frac{2x}{1+x^2}+D\frac{1}{1+x^2}$$

猜测

$$\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2}=\left(\frac{A+Bx}{1+x^2}\right)'+C\frac{2x}{1+x^2}+D\frac{1}{1+x^2}+M\frac{1}{x^2}+N\frac{1}{x}$$

$$\frac{1+x+x^2+x^3+x^4+x^5}{x^2(1+x^2)^2}=\left(\frac{A+Bx}{1+x^2}\right)'+C\frac{2x}{1+x^2}+D\frac{1}{1+x^2}+M\frac{1}{x^2}+N\frac{1}{x}$$

$$=\left(\frac{A+Bx}{1+x^2}\right)'+C\left(\ln\left(1+x^2\right)\right)'+D(\arctan x)'+M\left(-\frac{1}{x}\right)'+N(\ln x)'$$

$$=\left(\frac{1-x}{2\left(1+x^2\right)}\right)'+0\left(\ln\left(1+x^2\right)\right)'-\frac{1}{2}(\arctan x)'+\left(-\frac{1}{x}\right)'+(\ln x)'$$

$$1+x+x^2+x^3+x^4+x^5$$

$$=x^2\left[B\left(1+x^2\right)-2x(A+Bx)\right]+2Cx^3\left(1+x^2\right)+Dx^2\left(1+x^2\right)+M\left(1+x^2\right)^2+Nx\left(1+x^2\right)^2$$

$$\begin{cases}1=M\\1=N\\1=B+D+2M\\1=-2A+2C+2N\\1=-B+D+M\\1=2C+N\end{cases}\qquad\begin{cases}A=1/2\\B=-1/2\\C=0\\D=-1/2\\M=1\\N=1\end{cases}$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$a>0 \quad b^2-4ac\neq 0$

$$\int \frac{1}{\sqrt{ax^2+bx+c}}dx=\frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|+C$$

$b^2+4ac>0$

$a>0 \quad b^2+4ac\neq 0$

$$\int \frac{1}{\sqrt{-ax^2+bx+c}}dx=\frac{1}{\sqrt{a}}\arcsin\frac{2ax-b}{\sqrt{b^2+4ac}}+C \qquad \int \frac{1}{\sqrt{1-t^2}}dt$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int \frac{x}{\sqrt{1-x+x^2}}dx \qquad \frac{1}{\sqrt{1-x+x^2}} \qquad \left(\sqrt{1-x+x^2}\right)' = \frac{x-\frac{1}{2}}{\sqrt{1-x+x^2}}$$

$$\frac{x}{\sqrt{1-x+x^2}} = A\frac{x-\frac{1}{2}}{\sqrt{1-x+x^2}} + B\frac{1}{\sqrt{1-x+x^2}}$$

猜测

$$\begin{cases} A=1 \\ -\frac{1}{2}A+B=0 \end{cases} \qquad \begin{cases} A=1 \\ B=\frac{1}{2} \end{cases}$$

$$\frac{x}{\sqrt{1-x+x^2}} = \frac{x-\frac{1}{2}}{\sqrt{1-x+x^2}} + \frac{1}{2}\frac{1}{\sqrt{1-x+x^2}}$$

$$= \left(\sqrt{1-x+x^2}\right)' + \frac{1}{2}\left(\ln\left|2x-1+2\sqrt{1-x+x^2}\right|\right)'$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int \frac{x}{\sqrt{1-x-x^2}}dx \qquad \frac{1}{\sqrt{1-x-x^2}} \qquad \left(\sqrt{1-x-x^2}\right)' = \frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}}$$

$$\frac{x}{\sqrt{1-x-x^2}} = A\frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}} + B\frac{1}{\sqrt{1-x-x^2}}$$

猜测

$$\begin{cases} -A=1 \\ -\frac{1}{2}A+B=0 \end{cases} \qquad \begin{cases} A=-1 \\ B=-\frac{1}{2} \end{cases}$$

$$\frac{x}{\sqrt{1-x-x^2}} = -\frac{-x-\frac{1}{2}}{\sqrt{1-x-x^2}} - \frac{1}{2}\frac{1}{\sqrt{1-x-x^2}}$$

$$= -\left(\sqrt{1-x-x^2}\right)' - \frac{1}{2}\left(\arcsin\frac{2x+1}{\sqrt{5}}\right)'$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$a^2+b^2\neq 0$ $\displaystyle\int\frac{c\sin x+d\cos x}{a\sin x+b\cos x}dx$ 待定 m n

$$c\sin x+d\cos x=m(a\sin x+b\cos x)+n(a\sin x+b\cos x)'$$

$$c\sin x+d\cos x=m(a\sin x+b\cos x)+n(a\cos x-b\sin x)$$

$$\begin{cases}c=am-bn\\ d=bm+an\end{cases}\qquad \begin{cases}m=\dfrac{ac+bd}{a^2+b^2}\\ n=\dfrac{ad-bc}{a^2+b^2}\end{cases}$$

$$c\sin x+d\cos x=\frac{ac+bd}{a^2+b^2}(a\sin x+b\cos x)+\frac{ad-bc}{a^2+b^2}(a\sin x+b\cos x)'$$

$$\frac{c\sin x+d\cos x}{a\sin x+b\cos x}=\frac{ac+bd}{a^2+b^2}+\frac{ad-bc}{a^2+b^2}\frac{(a\sin x+b\cos x)'}{a\sin x+b\cos x}$$

$$=\frac{ac+bd}{a^2+b^2}(x)'+\frac{ad-bc}{a^2+b^2}(\ln|a\sin x+b\cos x|)'$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int e^{\lambda x}(p\sin kx+q\cos kx)dx=e^{\lambda x}(m\sin kx+n\cos kx)+C$$

$$\int e^{x}(2\sin 4x+\cos 4x)dx \qquad \int 2e^{x}\sin 4x dx+\int e^{x}\cos 4x dx$$

待定 m n

$$\int e^{x}(2\sin 4x+\cos 4x)dx=e^{x}(m\sin 4x+n\cos 4x)+C$$

$$e^{x}(2\sin 4x+\cos 4x)=e^{x}(m\sin 4x+n\cos 4x+4m\cos 4x-4n\sin 4x)$$

$$2e^{x}\sin 4x+e^{x}\cos 4x=(m-4n)e^{x}\sin 4x+(n+4m)e^{x}\cos 4x$$

$$\begin{cases}2=m-4n\\1=n+4m\end{cases} \qquad \begin{cases}m=\dfrac{6}{17}\\n=-\dfrac{7}{17}\end{cases}$$

$$\int e^{ax}\sin bx dx \qquad \int e^{ax}\cos bx dx$$

$$\int e^{x}(2\sin 4x+\cos 4x)dx=e^{x}\left(\frac{6}{17}\sin 4x-\frac{7}{17}\cos 4x\right)+C$$

第六讲：不定积分 > 待定系数法 > 非有理分式

$$\int e^{\lambda x}[P_1(x)\cos kx+P_2(x)\sin kx]dx=e^{\lambda x}[Q_1(x)\cos kx+Q_2(x)\sin kx]+C$$

其中$P_1(x)$，$P_2(x)$，$Q_1(x)$，$Q_2(x)$是多项式且$\partial(Q_1(x))$，$\partial(Q_2(x))\le\max\{\partial(P_1(x))$，$\partial(P_2(x))\}$

$$\int xe^x(\cos 4x+2\sin 4x)dx$$

待定 a b c d

$$\int xe^x(\cos 4x+2\sin 4x)dx=e^x[(ax+b)\cos 4x+(cx+d)\sin 4x]+C$$

$$xe^x(\cos 4x+2\sin 4x)=e^x[(ax+b)\cos 4x+(cx+d)\sin 4x+a\cos 4x-4(ax+b)\sin 4x+c\sin 4x+4(cx+d)\cos 4x]$$

$$xe^x\cos 4x+2xe^x\sin 4x=(ax+4cx+b+a+4d)e^x\cos 4x+(cx-4ax+d+c-4b)e^x\sin 4x$$

$$\begin{cases}1=a+4c\\0=b+a+4d\\2=c-4a\\0=d+c-4b\end{cases}\qquad\begin{cases}a=-7/17\\b=31/17^2\\c=6/17\\d=22/17^2\end{cases}$$

$$\int xe^x(\cos 4x+2\sin 4x)dx=e^x\left[\left(-\frac{7}{17}x+\frac{31}{17^2}\right)\cos 4x+\left(\frac{6}{17}x+\frac{22}{17^2}\right)\sin 4x\right]+C$$

第六讲：不定积分 > 待定系数法

$$\int g(x)e^{f(x)}dx$$

$$\int g(x)e^{f(x)}dx = h(x)e^{f(x)} + C$$

待定 h(x)

$$g(x)e^{f(x)} = (h'(x) + f'(x)h(x))e^{f(x)}$$

$$g(x) = h'(x) + f'(x)h(x)$$

猜测 h(x)

$$h(x) = e^{-f(x)}\left(\int g(x)e^{f(x)}dx + C\right)$$

第六讲：不定积分 > 待定系数法

$$\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx \quad \text{(第三届决赛)}$$

$$\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx=h(x)e^{x+\frac{1}{x}}+C$$

待定 $h(x)$

$$\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}=\left[\left(1-\frac{1}{x^2}\right)h(x)+h'(x)\right]e^{x+\frac{1}{x}}$$

$$1+x-\frac{1}{x}=\left(1-\frac{1}{x^2}\right)h(x)+h'(x) \qquad h(x)=x$$

$$\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx=\int\left[x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}+e^{x+\frac{1}{x}}\right]dx=\int\left[x\left(e^{x+\frac{1}{x}}\right)'+(x)'e^{x+\frac{1}{x}}\right]dx=xe^{x+\frac{1}{x}}+C$$

第六讲：不定积分 > 待定系数法

$$\int \frac{e^{-\sin x}\sin 2x}{(1-\sin x)^2}dx$$ （第九届初赛） 消去三角 令 $\sin x = t$

$$=\int \frac{e^{-\sin x}2\sin x\cos x}{(1-\sin x)^2}dx=\int \frac{e^{-\sin x}2\sin x d\sin x}{(1-\sin x)^2}=\int \frac{e^{-t}2tdt}{(1-t)^2}=h(t)e^{-t}+C$$

待定 h(t)

$$\frac{e^{-t}2t}{(1-t)^2}=(-h(t)+h'(t))e^{-t} \qquad \frac{2t}{(1-t)^2}=-h(t)+h'(t)$$ 猜测 $h(t)=\frac{a+bt}{1-t}$

$$\Rightarrow -h(t)+h'(t)=-\frac{a+bt}{1-t}+\frac{a+b}{(1-t)^2}=\frac{-(a+bt)(1-t)+a+b}{(1-t)^2}\Rightarrow a=2 \quad b=0 \qquad h(t)=\frac{2}{1-t}$$

$$\int \frac{e^{-t}2tdt}{(1-t)^2}=\int\left[\frac{2}{1-t}(-e^{-t})+\frac{2}{(1-t)^2}e^{-t}\right]dt=\int\left[\frac{2}{1-t}(e^{-t})'+\left(\frac{2}{1-t}\right)'e^{-t}\right]dt=\frac{2}{1-t}e^{-t}+C$$

$$=\frac{2}{1-\sin x}e^{-\sin x}+C$$

第六讲：不定积分 > 待定系数法

$$\int \frac{e^{\sin 2x}\sin^2 x}{e^{2x}}dx$$

$$\int \frac{e^{\sin 2x}\sin^2 x}{e^{2x}}dx = \int e^{\sin 2x - 2x}\sin^2 x dx = h(x)e^{\sin 2x-2x} + C$$

待定 h(x)

$$e^{\sin 2x-2x}\sin^2 x = e^{\sin 2x-2x}[(2\cos 2x-2)h(x)+h'(x)]$$

$$\sin^2 x = (2\cos 2x - 2)h(x) + h'(x)$$

$$\frac{1-\cos 2x}{2} = (2\cos 2x-2)h(x)+h'(x)$$

$$h(x) = -\frac{1}{4}$$

第六讲：不定积分 > 待定系数法

$$\int\left(x^{3}+x^{2}+\frac{1}{3}\right)e^{x^{3}}dx$$

$$\int\left(x^{3}+x^{2}+\frac{1}{3}\right)e^{x^{3}}dx=h(x)e^{x^{3}}+C$$

待定 h(x)

$$\left(x^{3}+x^{2}+\frac{1}{3}\right)e^{x^{3}}=\left(3x^{2}h(x)+h'(x)\right)e^{x^{3}} \qquad x^{3}+x^{2}+\frac{1}{3}=3x^{2}h(x)+h'(x)$$

猜测$h(x)=ax+b\Rightarrow 3x^{2}h(x)+h'(x)=3ax^{3}+3bx^{2}+a\Rightarrow a=b=\frac{1}{3}$ $\qquad h(x)=\frac{x+1}{3}$

第六讲：不定积分 > 待定系数法

$$\int \frac{1+\sin x}{1+\cos x} e^{x} dx$$

待定 h(x)

$$\int \frac{1+\sin x}{1+\cos x} e^{x} dx = \int \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^{2}\frac{x}{2}} e^{x} dx = \int \left(\frac{1}{2}\sec^{2}\frac{x}{2} + \tan\frac{x}{2}\right) e^{x} dx = e^{x} h(x) + C$$

$$\left(\frac{1}{2}\sec^{2}\frac{x}{2} + \tan\frac{x}{2}\right) e^{x} = (h(x) + h'(x)) e^{x}$$

$$\frac{1}{2}\sec^{2}\frac{x}{2} + \tan\frac{x}{2} = h(x) + h'(x) \qquad h(x) = \tan\frac{x}{2}$$

第六讲：不定积分 > 待定系数法

$$\int\left(te^{2t}-te^{-2t}+\frac{1}{2}\right)e^{\left(e^{t}-e^{-t}\right)^{2}}dt=e^{\left(e^{t}-e^{-t}\right)^{2}}h(t)+C$$

待定 h(t)

$$\left(te^{2t}-te^{-2t}+\frac{1}{2}\right)e^{\left(e^{t}-e^{-t}\right)^{2}}=\left[2\left(e^{t}-e^{-t}\right)\left(e^{t}+e^{-t}\right)h(t)+h'(t)\right]e^{\left(e^{t}-e^{-t}\right)^{2}}$$

$$te^{2t}-te^{-2t}+\frac{1}{2}=2\left(e^{2t}-e^{-2t}\right)h(t)+h'(t)$$

$$h(t)=\frac{t}{2}$$

第六讲：不定积分 > 待定系数法

$$\int\left[\left(x-\frac{1}{x^3}\right)\ln x+\frac{1}{2x}\right]e^{\left(x-\frac{1}{x}\right)^2}dx=h(x)e^{\left(x-\frac{1}{x}\right)^2}+C$$

待定 h(x)

$$\left[\left(x-\frac{1}{x^3}\right)\ln x+\frac{1}{2x}\right]e^{\left(x-\frac{1}{x}\right)^2}=\left[2\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)h(x)+h'(x)\right]e^{\left(x-\frac{1}{x}\right)^2}$$

$$\left(x-\frac{1}{x^3}\right)\ln x+\frac{1}{2x}=2\left(x-\frac{1}{x^3}\right)h(x)+h'(x)$$

$$h(x)=\frac{1}{2}\ln x$$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u'v\,dx = -\int uv'\,dx + uv$$

$$u'v = -uv' + (uv)'$$

$$\int u^{(n)}v\,dx = (-1)^n \int uv^{(n)}\,dx + \sum_{k=1}^{n}(-1)^{n-k} v^{(n-k)}u^{(k-1)}$$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u^{(n)}vdx=(-1)^n\int uv^{(n)}dx+\sum_{k=1}^{n}(-1)^{k-1}v^{(k-1)}u^{(n-k)}$$

当v是一个$n-1$次多项式函数时，$v^{(n)}=0\Rightarrow(-1)^n\int uv^{(n)}dx=C$

$$\int u^{(n)}vdx=C+\sum_{k=1}^{n}(-1)^{k-1}v^{(k-1)}u^{(n-k)}$$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int u^{(n)} v dx = (-1)^n \int u v^{(n)} dx + \sum_{k=1}^{n} (-1)^{n-k} v^{(n-k)} u^{(k-1)}$$

$$u^{(n)} v = (-1)^n u v^{(n)} + \left(\sum_{k=1}^{n} (-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)'$$

设 $(-1)^{n-k} v^{(n-k)} u^{(k)} = I_k$

$$\left(\sum_{k=1}^{n} (-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)' = \sum_{k=1}^{n} \left((-1)^{n-k} v^{(n-k)} u^{(k-1)} \right)'$$

$$= \sum_{k=1}^{n} \left((-1)^{n-k} v^{(n-k)} u^{(k)} + (-1)^{n-k} v^{(n-k+1)} u^{(k-1)} \right)$$

$$= \sum_{k=1}^{n} \left((-1)^{n-k} v^{(n-k)} u^{(k)} - (-1)^{n-k+1} v^{(n-k+1)} u^{(k-1)} \right)$$

$$= \sum_{k=1}^{n} (I_k - I_{k-1}) = I_n - I_0 = (-1)^0 v^{(0)} u^{(n)} - (-1)^n v^{(n)} u^{(0)}$$

凑差分

消项

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$m \in N^{+}$ $\int x^{m} e^{x} dx$ $\qquad \int u^{(n)} v dx = (-1)^{n} \int u v^{(n)} dx + \sum_{k=1}^{n} (-1)^{n-k} v^{(n-k)} u^{(k-1)}$

$$= \int (e^{x})^{(m+1)} x^{m} dx$$

$$= (-1)^{m+1} \int e^{x} (x^{m})^{(m+1)} dx + \sum_{k=1}^{m+1} (-1)^{m+1-k} (x^{m})^{(m+1-k)} (e^{x})^{(k-1)}$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(m-(m+1-k))!} x^{k-1} e^{x} + C$$

$$= \sum_{k=1}^{m+1} (-1)^{m+1-k} \frac{m!}{(k-1)!} x^{k-1} e^{x} + C$$

$$= \sum_{k=0}^{m} (-1)^{m-k} \frac{m!}{k!} x^{k} e^{x} + C$$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$m\in N^{+}\quad \int(\ln x)^{m}dx$

令 $\ln x=t\Rightarrow x=e^{t}$

$$\int(\ln x)^{m}dx=\int t^{m}e^{t}dt$$

m，$n\in N^{+}\quad \int x^{n}(\ln x)^{m}dx$

$$\int x^{n}(\ln x)^{m}dx=\int(e^{t})^{n}t^{m}de^{t}=\int e^{(n+1)t}t^{m}dt$$

$$=\int e^{s}\left(\frac{s}{n+1}\right)^{m}d\left(\frac{s}{n+1}\right)=\left(\frac{1}{n+1}\right)^{m+1}\int e^{s}s^{m}ds$$

令 $\ln x=t\Rightarrow x=e^{t}$　　令 $(n+1)t=s\Rightarrow t=\dfrac{s}{n+1}$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$m\in N^{+}\quad \int x^{m}\sin x dx$

$$\int u^{(n)}vdx=(-1)^{n}\int uv^{(n)}dx+\sum_{k=1}^{n}(-1)^{n-k}v^{(n-k)}u^{(k-1)}$$

$$=\int\left(\sin\left(x-\frac{(m+1)\pi}{2}\right)\right)^{(m+1)}x^{m}dx$$

$$=(-1)^{m+1}\int\sin\left(x-\frac{(m+1)\pi}{2}\right)(x^{m})^{(m+1)}dx+\sum_{k=1}^{m+1}(-1)^{m+1-k}(x^{m})^{(m+1-k)}\left(\sin\left(x-\frac{(m+1)\pi}{2}\right)\right)^{(k-1)}$$

$$=\sum_{k=1}^{m+1}(-1)^{m+1-k}\frac{m!}{(m-(m+1-k))!}x^{k-1}\sin\left(x-\frac{(m+1)\pi}{2}+\frac{(k-1)\pi}{2}\right)+C$$

$$=\sum_{k=1}^{m+1}(-1)^{m+1-k}\frac{m!}{(k-1)!}x^{k-1}\sin\left(x+\frac{(k-m-2)\pi}{2}\right)+C$$

$$=\sum_{k=0}^{m}(-1)^{m-k}\frac{m!}{k!}x^{k}\sin\left(x+\frac{(k-m-1)\pi}{2}\right)+C$$

第六讲：不定积分 > 分部积分法 > 分部积分法的推广

$$\int e^{x}\sin x dx \qquad \int u^{(n)}v dx=(-1)^{n}\int uv^{(n)}dx+\sum_{k=1}^{n}(-1)^{n-k}v^{(n-k)}u^{(k-1)}$$

$$\int e^{x}\sin x dx=\int (e^{x})''\sin x dx=(-1)^{2}\int e^{x}(\sin x)''dx+\sum_{k=1}^{2}(-1)^{2-k}(\sin x)^{(2-k)}(e^{x})^{(k-1)}$$

$$=-\int e^{x}\sin x dx-e^{x}\cos x+e^{x}\sin x$$

$$\int e^{x}\sin x dx=\frac{-e^{x}\cos x+e^{x}\sin x}{2}+C$$

$$\int \sin\ln x dx \qquad 令\ln x=t\Rightarrow x=e^{t}\Rightarrow\int \sin\ln x dx=\int \sin t de^{t}=\int e^{t}\sin t dt$$

$$\int e^{\arccos x}dx \qquad 令\arccos x=t\Rightarrow x=\cos t\Rightarrow\int e^{\arccos x}dx=\int e^{t}d\cos t=-\int e^{t}\sin t dt$$

当原不定积分不易直接求出时

我们设原不定积分为 I ，我们找出原不定积分的一个对偶式 J

通过求出 I，J 两组不同的线性组合，从而求出 I，J

$$\begin{cases} AI+BJ=P \\ CI+DJ=Q \end{cases} \Rightarrow \begin{cases} I=\dfrac{PD-QB}{AD-BC} \\ J=\dfrac{QA-PC}{AD-BC} \end{cases} \qquad \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD-BC \neq 0$$

所以我们可以将计算 I 的问题转化成计算 AI+BJ 和 CI+DJ 的问题

这样我们避开了对原不定积分的直接计算

第六讲：不定积分 > 组合法

$ab \neq 0 \quad \int \frac{\sin x}{(a\sin x + b\cos x)^2}dx \qquad$ 令$u = \tan\frac{x}{2}$

$$= \int \frac{\frac{2u}{1+u^2}\frac{2}{1+u^2}}{\left(a\frac{2u}{1+u^2} + b\frac{1-u^2}{1+u^2}\right)^2}du$$

$$= \int \frac{4u}{(2au + b - bu^2)^2}du$$

间接计算

第六讲：不定积分 > 组合法

$ab \neq 0$ $\int \frac{\sin x}{(a\sin x + b\cos x)^2} dx$ 对偶式 $\int \frac{\cos x}{(a\sin x + b\cos x)^2} dx$

设 $I = \int \frac{\sin x}{(a\sin x + b\cos x)^2} dx$ $J = \int \frac{\cos x}{(a\sin x + b\cos x)^2} dx$ 线性组合

$$aI + bJ = \int \frac{a\sin x + b\cos x}{(a\sin x + b\cos x)^2} dx = \int \frac{1}{a\sin x + b\cos x} dx = \frac{1}{2\sqrt{a^2 + b^2}} \ln\left|\frac{a\cos x - b\sin x - \sqrt{a^2 + b^2}}{a\cos x - b\sin x + \sqrt{a^2 + b^2}}\right| + C$$

$$-bI + aJ = \int \frac{-b\sin x + a\cos x}{(a\sin x + b\cos x)^2} dx = \int \frac{d(a\sin x + b\cos x)}{(a\sin x + b\cos x)^2} = -\frac{1}{a\sin x + b\cos x} + C$$

$$d(a\sin x + b\cos x) = (-b\sin x + a\cos x)dx$$

第六讲：不定积分 > 组合法

$ab \neq 0$ $\int \frac{\sin x}{(a\sin x + b\cos x)^2}dx$ $\exists \varphi \in (0,2\pi]$使得$\frac{a}{\sqrt{a^2+b^2}} = \cos\varphi$ $\frac{b}{\sqrt{a^2+b^2}} = \sin\varphi$

$$= \int \frac{\sin x}{(a^2+b^2)\sin^2(x+\varphi)}dx$$

令$x+\varphi = t$

简化分母

平移

$$= \int \frac{\sin(t-\varphi)}{(a^2+b^2)\sin^2 t}dt$$

$$= \int \frac{\sin t\cos\varphi - \cos t\sin\varphi}{(a^2+b^2)\sin^2 t}dt$$

$$= \int \left(\frac{\cos\varphi}{a^2+b^2}\frac{1}{\sin t} - \frac{\sin\varphi}{a^2+b^2}\frac{\cos t}{\sin^2 t}\right)dt$$

第六讲：不定积分 > 组合法

$ab \neq 0 \quad \int \frac{\sin^2 x}{a\sin x + b\cos x}dx$ 令$u = \tan\frac{x}{2}$

$$= \int \frac{\left(\frac{2u}{1+u^2}\right)^2 \frac{2}{1+u^2}}{a\frac{2u}{1+u^2} + b\frac{1-u^2}{1+u^2}}du$$

$$= \int \frac{8u^2}{(2au + b - bu^2)(1+u^2)^2}du$$

第六讲：不定积分 > 组合法

$$ab \neq 0 \quad \int \frac{\sin^2 x}{a\sin x + b\cos x}dx \qquad \text{对偶式} \int \frac{\cos^2 x}{a\sin x + b\cos x}dx$$

$$\text{设} I = \int \frac{\sin^2 x}{a\sin x + b\cos x}dx \qquad J = \int \frac{\cos^2 x}{a\sin x + b\cos x}dx$$

$$I + J = \int \frac{\sin^2 x + \cos^2 x}{a\sin x + b\cos x}dx = \int \frac{1}{a\sin x + b\cos x}dx = \frac{1}{2\sqrt{a^2+b^2}}\ln\left|\frac{a\cos x - b\sin x - \sqrt{a^2+b^2}}{a\cos x - b\sin x + \sqrt{a^2+b^2}}\right| + C$$

$$a^2 I - b^2 J = \int \frac{a^2\sin^2 x - b^2\cos^2 x}{a\sin x + b\cos x}dx = \int (a\sin x - b\cos x)dx = -a\cos x - b\sin x + C$$

第六讲：不定积分 > 组合法

$ab \neq 0 \quad \int \frac{\sin^2 x}{a\sin x + b\cos x} dx$　　$\exists \varphi \in (0,2\pi]$使得$\frac{a}{\sqrt{a^2+b^2}} = \cos\varphi \quad \frac{b}{\sqrt{a^2+b^2}} = \sin\varphi$

$$= \int \frac{\sin^2 x}{\sqrt{a^2+b^2}\sin(x+\varphi)} dx$$

令$x+\varphi = t$

$$= \int \frac{\sin^2(t-\varphi)}{\sqrt{a^2+b^2}\sin t} dt$$

$$= \int \frac{(\sin t\cos\varphi - \cos t\sin\varphi)^2}{\sqrt{a^2+b^2}\sin t} dt$$

$$= \int \frac{\sin^2 t\cos^2\varphi - 2\sin t\cos t\sin\varphi\cos\varphi + \cos^2 t\sin^2\varphi}{\sqrt{a^2+b^2}\sin t} dt$$

$$= \int \left(\frac{\cos^2\varphi}{\sqrt{a^2+b^2}}\sin t - \frac{2\sin\varphi\cos\varphi}{\sqrt{a^2+b^2}}\cos t + \frac{\sin^2\varphi}{\sqrt{a^2+b^2}}\frac{\cos^2 t}{\sin t} \right) dt$$

第六讲：不定积分 > 组合法

$$\int\frac{\sin x}{1+\sin x\cos x}dx \qquad 对偶式\int\frac{\cos x}{1+\sin x\cos x}dx \qquad \sin x\cos x=\frac{(\sin x+\cos x)^2-1}{2}=\frac{1-(\sin x-\cos x)^2}{2}$$

$$设I=\int\frac{\sin x}{1+\sin x\cos x}dx \qquad J=\int\frac{\cos x}{1+\sin x\cos x}dx$$

$$I+J=\int\frac{\sin x+\cos x}{1+\sin x\cos x}dx=\int\frac{d(\sin x-\cos x)}{1+\sin x\cos x}=\int\frac{d(\sin x-\cos x)}{1+\frac{1-(\sin x-\cos x)^2}{2}}=\int\frac{2d(\sin x-\cos x)}{3-(\sin x-\cos x)^2}$$

$$=\frac{1}{\sqrt{3}}\ln\left|\frac{\sin x-\cos x+\sqrt{3}}{\sin x-\cos x-\sqrt{3}}\right|+C$$

$$I-J=\int\frac{\sin x-\cos x}{1+\sin x\cos x}dx=\int\frac{-d(\sin x+\cos x)}{1+\sin x\cos x}=\int\frac{-d(\sin x+\cos x)}{1+\frac{(\sin x+\cos x)^2-1}{2}}=\int\frac{-2d(\sin x+\cos x)}{1+(\sin x+\cos x)^2}$$

$$=-2\arctan(\sin x+\cos x)+C$$

第六讲：不定积分 > 组合法

$\int \frac{\sin^3 x}{1+\sin x\cos x}dx$ 对偶式 $\int \frac{\cos^3 x}{1+\sin x\cos x}dx$ $\sin x\cos x = \frac{(\sin x+\cos x)^2-1}{2} = \frac{1-(\sin x-\cos x)^2}{2}$

设 $I = \int \frac{\sin^3 x}{1+\sin x\cos x}dx \quad J = \int \frac{\cos^3 x}{1+\sin x\cos x}dx$

$$I+J = \int \frac{\sin^3 x+\cos^3 x}{1+\sin x\cos x}dx = \int \frac{(\sin x+\cos x)(\sin^2 x-\sin x\cos x+\cos^2 x)}{1+\sin x\cos x}dx = \int \frac{1-\sin x\cos x}{1+\sin x\cos x}d(\sin x-\cos x)$$

$$= \int \frac{1-\frac{1-(\sin x-\cos x)^2}{2}}{1+\frac{1-(\sin x-\cos x)^2}{2}}d(\sin x-\cos x) = \int \left(\frac{4}{3-(\sin x-\cos x)^2}-1\right)d(\sin x-\cos x)$$

$$= \frac{4}{2\sqrt{3}}\ln\left|\frac{\sin x-\cos x+\sqrt{3}}{\sin x-\cos x-\sqrt{3}}\right| -(\sin x-\cos x)+C$$

$$I-J = \int \frac{\sin^3 x-\cos^3 x}{1+\sin x\cos x}dx = \int \frac{(\sin x-\cos x)(\sin^2 x+\sin x\cos x+\cos^2 x)}{1+\sin x\cos x}dx = \int (\sin x-\cos x)dx$$

$$= -\cos x-\sin x+C$$

$$\int \frac{\sin^2 x}{1+\sin x\cos x}dx \qquad \int \frac{\sin x}{1+\sin x\cos x}dx \qquad \int \frac{\sin^3 x}{1+\sin x\cos x}dx$$

$$=\int \frac{\tan^2 x\sec^2 x}{\left(\sec^2 x+\tan x\right)\sec^2 x}dx$$

$$=\int \frac{\tan^2 x}{\left(\tan^2 x+1+\tan x\right)\left(\tan^2 x+1\right)}d\tan x \qquad 令\tan x=u$$

$$=\int \frac{u^2}{\left(u^2+1+u\right)\left(u^2+1\right)}du$$

第六讲：不定积分 > 组合法

$n>1 \quad \int \frac{\sin^2 x}{1+n\sin x}dx$ 　　对偶式$\int \frac{1}{1+n\sin x}dx$

设$I=\int \frac{\sin^2 x}{1+n\sin x}dx \qquad J=\int \frac{1}{1+n\sin x}dx$

$$J-n^2 I=\int \frac{1-n^2\sin^2 x}{1+n\sin x}dx=\int (1-n\sin x)dx=x+n\cos x+C$$

$$J=\int \frac{1}{1+n\sin x}dx=\int \frac{\frac{2}{1+u^2}}{1+n\frac{2u}{1+u^2}}du=\int \frac{2}{1+u^2+2nu}du=\frac{1}{\sqrt{n^2-1}}\ln\left|\frac{u+n-\sqrt{n^2-1}}{u+n+\sqrt{n^2-1}}\right|+C$$

令$u=\tan\frac{x}{2}$

$$=\frac{1}{\sqrt{n^2-1}}\ln\left|\frac{\tan\frac{x}{2}+n-\sqrt{n^2-1}}{\tan\frac{x}{2}+n+\sqrt{n^2-1}}\right|+C$$

第六讲：不定积分 > 组合法

$a \neq 0$ $\int \frac{dx}{x(x^n+a)}$ $\frac{1}{x} = \frac{x^n+a}{x(x^n+a)}$ 对偶式 $\int \frac{dx}{x}$

设 $I = \int \frac{dx}{x(x^n+a)}$ $J = \int \frac{dx}{x}$

$$J - aI = \int \frac{x^n}{x(x^n+a)} dx = \int \frac{x^{n-1} dx}{x^n+a} = \frac{1}{n} \int \frac{d(x^n+a)}{x^n+a} = \frac{1}{n} \ln|x^n+a| + C$$

$$J = \ln|x| + C$$

第六讲：不定积分 > 组合法

$$\int \frac{x^2 \arctan x}{1+x^2}dx \qquad \arctan x = \frac{(1+x^2)\arctan x}{1+x^2} \qquad \text{对偶式} \int \arctan x dx$$

$$\text{设} I = \int \frac{x^2 \arctan x}{1+x^2}dx \qquad J = \int \arctan x dx$$

$$J - I = \int \frac{\arctan x}{1+x^2}dx = \int \arctan x d(\arctan x) = \frac{1}{2}\arctan^2 x + C$$

$$J = \int \arctan x dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C$$

$$\text{对偶式} \int \frac{\arctan x}{1+x^2}dx$$

$$\int e^{x}(2\sin 4x+\cos 4x)dx=I+2J$$

设$I=\int e^{x}\cos 4xdx \qquad J=\int e^{x}\sin 4xdx$

$$\left(e^{x}\cos 4x\right)'=e^{x}\cos 4x-4e^{x}\sin 4x \Rightarrow e^{x}\cos 4x+C=I-4J$$

$$\left(e^{x}\sin 4x\right)'=e^{x}\sin 4x+4e^{x}\cos 4x \Rightarrow e^{x}\sin 4x+C=J+4I$$

第六讲：不定积分 > 组合法

$$\int xe^{x}(\cos 4x + 2\sin 4x)dx = I + 2J$$

设$I = \int xe^{x}\cos 4xdx \quad J = \int xe^{x}\sin 4xdx$

$$(xe^{x}\cos 4x)' = e^{x}\cos 4x + xe^{x}\cos 4x - 4xe^{x}\sin 4x \qquad \Rightarrow xe^{x}\cos 4x = \int e^{x}\cos 4xdx + I - 4J$$

$$(xe^{x}\sin 4x)' = e^{x}\sin 4x + xe^{x}\sin 4x + 4xe^{x}\cos 4x \qquad \Rightarrow xe^{x}\sin 4x = \int e^{x}\sin 4xdx + J + 4I$$