一、填空题

1. 设 e^{-x^2} 是函数 $f(x)$ 的一个原函数，则 $\int f(2x)dx =$______________。（填 $\frac{1}{2}e^{-4x^2}+C$ ）

解析：由 e^{-x^2} 是函数 $f(x)$ 的一个原函数，则 $\int f(x)dx = e^{-x^2}+C_1$，

$\int f(2x)dx = \frac{1}{2}\int f(2x)d(2x) = \frac{1}{2}e^{-4x^2}+C, C=\frac{1}{2}C_1$。

二、计算（写出计算过程）

1. $\int \frac{x^4}{1+x^2}dx$

解 $\int \frac{x^4}{1+x^2}dx = \int \frac{x^4-1+1}{1+x^2}dx = \int (x^2-1)dx + \int \frac{dx}{1+x^2} = \frac{x^3}{3} - x + \arctan x + C$

2. $\int \frac{dx}{\sin x \cos x}$

解 $\int \frac{dx}{\sin x\cos x} = \int \frac{2dx}{\sin 2x} = \int \csc 2x d(2x) = \ln|\csc 2x - \cot 2x| + C$

也可进一步化简得 $\int \frac{dx}{\sin x \cos x} = \ln|\csc 2x - \cot 2x| + C = \ln\left|\frac{1-\cos 2x}{\sin 2x}\right| + C$

$= \ln\left|\frac{2\sin^2 x}{2\sin x\cos x}\right| + C = \ln|\tan x| + C$

3. $\int \frac{dx}{\sqrt[3]{2-3x}}$ 解 法一 利用凑微公式 $\int f(ax+b)dx = \frac{1}{a}\int f(ax+b)d(ax+b)$

$\int \frac{dx}{\sqrt[3]{2-3x}} = -\frac{1}{3}\int \frac{d(2-3x)}{\sqrt[3]{2-3x}} \xlongequal{t=2-3x} (-\frac{1}{3})\int t^{-\frac{1}{3}}dt = (-\frac{1}{3})\cdot\frac{3}{2}t^{\frac{2}{3}} + C = -\frac{1}{2}(2-3x)^{\frac{2}{3}} + C$

法二 第二换元法 令 $u=\sqrt[3]{2-3x}, x=\frac{1}{3}(2-u^3), dx=-u^2du$，则

$\int \frac{dx}{\sqrt[3]{2-3x}} = \int \frac{-u^2}{u}du = -\frac{u^2}{2} + C = -\frac{1}{2}(2-3x)^{\frac{2}{3}} + C$。

4. $\int \frac{x^2}{\sqrt{2-x}}dx$。 解 第二换元法

$\int \frac{x^2}{\sqrt{2-x}}dx \xlongequal{t=\sqrt{2-x}} \int \frac{(2-t^2)^2}{t}(-2t)dt = -2\int (4-4t^2+t^4)dt = -8t + \frac{8}{3}t^3 - \frac{2}{5}t^5 + C$

$= -8\sqrt{2-x} + \frac{8}{3}(2-x)^{\frac{3}{2}} - \frac{2}{5}(2-x)^{\frac{5}{2}} + C$

5. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

解 **第二换元法** $x = 2\sin t, t \in (-\frac{\pi}{2}, \frac{\pi}{2}), \sqrt{4-x^2} = 2\cos t, dx = 2\cos t dt$，则

$$\int \frac{dx}{x^2\sqrt{4-x^2}} = \int \frac{2\cos t dt}{4\sin^2 t 2\cos t} = \frac{1}{4}\int \frac{dt}{\sin^2 t} = \frac{1}{4}\int \csc^2 t dt = -\frac{1}{4}\cot t + C = -\frac{\sqrt{4-x^2}}{4x} + C 。$$

6. $\int \frac{dx}{x^2\sqrt{1+x^2}}$

解 **第二换元法** $x = \tan t, t \in (-\frac{\pi}{2}, \frac{\pi}{2}), \sqrt{1+x^2} = \sec t, dx = \sec^2 t dt$，则

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2 t dt}{\tan^2 t \cdot \sec t} = \int \frac{\cos t dt}{\sin^2 t} = \int \frac{d\sin t}{\sin^2 t} = -\frac{1}{\sin t} + C = -\frac{\sqrt{1+x^2}}{x} + C 。$$