第七讲：定积分

$f(x)$在$[a, b]$上连续，$g(x)$在$[a, b]$上具有连续导数且$g(a)=b$，$g(b)=a$

$$\int_a^b f(x)dx=-\int_a^b f(g(x))g'(x)dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b [f(x)-f(g(x))g'(x)]dx$$

令$x=g(t)$　　　　x：$a\to b$　　　t：$b\to a$

$$\int_a^b f(x)dx=\int_b^a f(g(t))g'(t)dt=-\int_a^b f(g(t))g'(t)dt=-\int_a^b f(g(x))g'(x)dx$$

$$\int_a^b f(x)dx=\frac{\int_a^b f(x)dx-\int_a^b f(g(x))g'(x)dx}{2}=\frac{1}{2}\int_a^b [f(x)-f(g(x))g'(x)]dx$$

第七讲：定积分

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大部分情况下我们是通过求定积分对应的不定积分来求定积分
如果定积分对应的不定积分难求、甚至不能用初等函数表达
我们可以利用这两个公式将难求的、甚至不能用初等函数表达的不定积分转化成能用初等函数表达的、易求的不定积分

取$g(x)=a+b-x$ 区间再现公式

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

$b=c$

$a=-c$

取$g(x)=-x$

$$\int_{-c}^c f(x)dx=\int_{-c}^c f(-x)dx$$

$$\int_{-c}^c f(x)dx=\frac{1}{2}\int_{-c}^c [f(x)+f(-x)]dx$$

取$g(x)=\frac{ab}{x}$

$$\int_a^b f(x)dx=\int_a^b \frac{ab}{x^2}f\left(\frac{ab}{x}\right)dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b \left[f(x)+\frac{ab}{x^2}f\left(\frac{ab}{x}\right)\right]dx$$

$b=c$

$a=\frac{1}{c}$

取$g(x)=\frac{1}{x}$

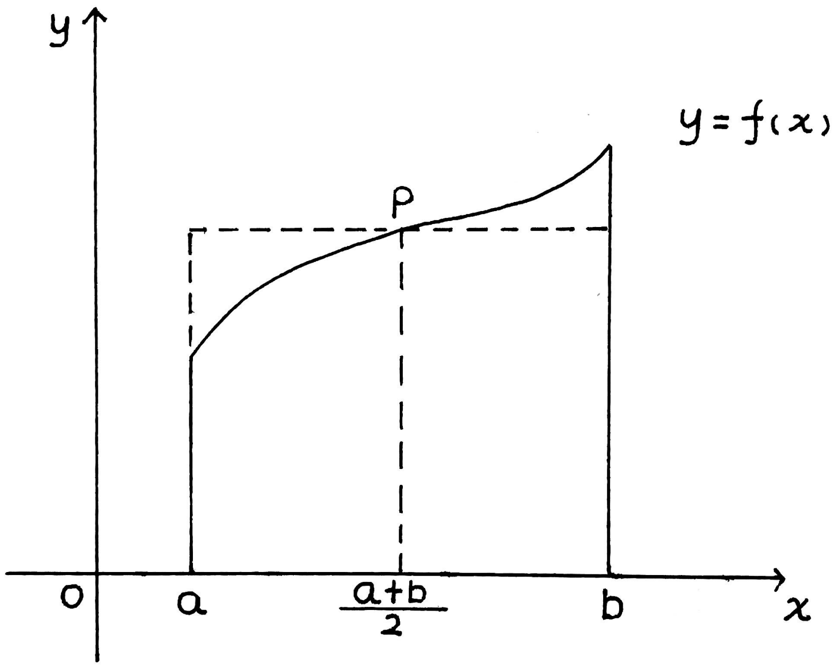
$$\int_{\frac{1}{c}}^c f(x)dx=\int_{\frac{1}{c}}^c \frac{1}{x^2}f\left(\frac{1}{x}\right)dx$$

$$\int_{\frac{1}{c}}^c f(x)dx=\frac{1}{2}\int_{\frac{1}{c}}^c \left[f(x)+\frac{1}{x^2}f\left(\frac{1}{x}\right)\right]dx$$

第七讲：定积分

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

若$f(x)+f(a+b-x)=2f(\frac{a+b}{2})$，则$\int_a^b f(x)dx=(b-a)f(\frac{a+b}{2})$



$y=f(x)$关于$(\frac{a+b}{2}, f(\frac{a+b}{2}))$中心对称

第七讲：定积分

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx \qquad \int_a^b f(x)\,dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]\,dx$$

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\ln(1+\tan x)+\ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)\right]dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\ln(1+\tan x)+\ln\left(1+\frac{1-\tan x}{1+\tan x}\right)\right]dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} \ln(1+\tan x+1-\tan x)\,dx$$

$$= \frac{1}{2}\cdot\frac{\pi}{4}\cdot\ln 2 \qquad y=\ln(1+\tan x)\text{关于点}\left(\frac{\pi}{8},\frac{\ln 2}{2}\right)\text{中心对称}$$

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta = \int_0^{\frac{\pi}{4}} \ln(1+\tan\theta)\,d\theta$$

第七讲：定积分

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}dx=\frac{1}{2}\int_2^4\left(\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}+\frac{\sqrt{\ln[9-(6-x)]}}{\sqrt{\ln[9-(6-x)]}+\sqrt{\ln[(6-x)+3]}}\right)dx$$

$$=\frac{1}{2}\int_2^4\left(\frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}+\frac{\sqrt{\ln(x+3)}}{\sqrt{\ln(x+3)}+\sqrt{\ln(9-x)}}\right)dx$$

$$=\frac{1}{2}\cdot 2$$

$y=\dfrac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(x+3)}}$ 关于点$(3,\dfrac{1}{2})$中心对称

第七讲：定积分

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\lambda}} dx$$

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+(\tan x)^{\lambda}} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{1+(\tan x)^{\lambda}} + \frac{1}{1+(\cot x)^{\lambda}}\right) dx$$

$$\tan\left(\frac{\pi}{2}-x\right) = \cot x$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{1}{1+(\tan x)^{\lambda}} + \frac{(\tan x)^{\lambda}}{(\tan x)^{\lambda}+1}\right) dx$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}$$

$y = \dfrac{1}{1+(\tan x)^{\lambda}}$ 关于点$(\dfrac{\pi}{4},\dfrac{1}{2})$中心对称

第七讲：定积分

$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}dx$$

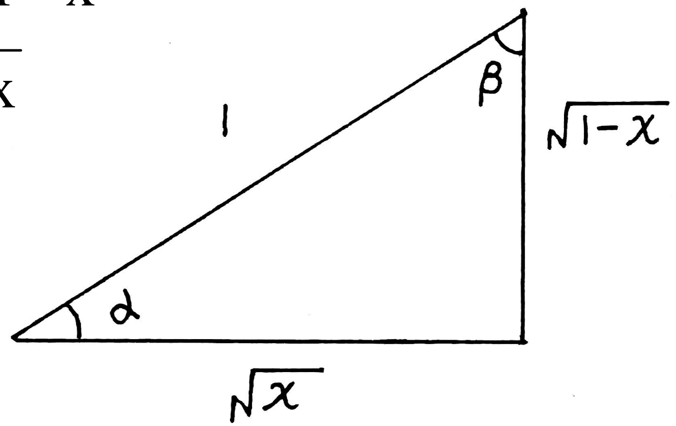
$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

$$\int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}dx = \frac{1}{2}\int_0^1\left(\frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}+\frac{\arcsin\sqrt{1-x}}{\sqrt{1-x+x^2}}\right)dx = \frac{1}{2}\int_0^1 \frac{\frac{\pi}{2}}{\sqrt{1-x+x^2}}dx$$

$$\sqrt{1-(1-x)+(1-x)^2} = \sqrt{1-x+x^2}$$

$$\alpha = \arcsin\sqrt{1-x}$$

$$\beta = \arcsin\sqrt{x}$$



$y=\sqrt{1-x+x^2}$关于$x=\frac{1}{2}$对称

$y=1-x+x^2$关于$x=\frac{1}{2}$对称

第七讲：定积分

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\left(x\ln x+\frac{1}{4x}\right)e^{\left(x-\frac{1}{x}\right)^2}dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]dx$$

$$=\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}}\left(te^{2t}+\frac{1}{4}\right)e^{\left(e^t-e^{-t}\right)^2}dt$$

令$t=\ln x\Rightarrow x=e^t$

$$=\frac{1}{2}\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}}\left[\left(te^{2t}+\frac{1}{4}\right)e^{\left(e^t-e^{-t}\right)^2}+\left(-te^{-2t}+\frac{1}{4}\right)e^{\left(e^{-t}-e^t\right)^2}\right]dt$$

$$=\frac{1}{2}\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}}\left(te^{2t}-te^{-2t}+\frac{1}{2}\right)e^{\left(e^t-e^{-t}\right)^2}dt$$

第七讲：定积分

$$\int_0^1\left(1-2x\sqrt{1-x^2}\right)^n dx$$

消去根号

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]dx$$

$$=\int_0^{\frac{\pi}{2}}(1-2\sin\theta\cos\theta)^n\cos\theta d\theta$$

令$x=\sin\theta \quad \theta\in[0,\frac{\pi}{2}]$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[(1-2\sin\theta\cos\theta)^n\cos\theta+\left(1-2\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)\right)^n\cos\left(\frac{\pi}{2}-\theta\right)\right]d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(1-2\sin\theta\cos\theta)^n(\cos\theta+\sin\theta)d\theta$$

$$=\int_0^{\frac{\pi}{2}}(1-2\sin\theta\cos\theta)^n d(\sin\theta-\cos\theta)$$

$$1-2\sin\theta\cos\theta=(\sin\theta-\cos\theta)^2$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin\theta-\cos\theta)^{2n}d(\sin\theta-\cos\theta)$$

第七讲：定积分

$$\int_{-\pi}^{\pi}\frac{x\sin x\arctan e^{x}}{1+\cos^{2}x}dx$$ （第五届初赛）

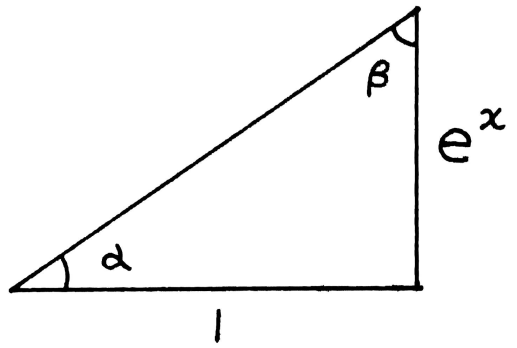
$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\left[\frac{x\sin x\arctan e^{x}}{1+\cos^{2}x}+\frac{-x\sin(-x)\arctan e^{-x}}{1+\cos^{2}(-x)}\right]dx$$

$$\alpha=\arctan e^{x}$$

$$\beta=\arctan e^{-x}$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\frac{x\sin x\left(\arctan e^{x}+\arctan e^{-x}\right)}{1+\cos^{2}x}dx$$



$$=\frac{1}{2}\cdot\frac{\pi}{2}\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^{2}x}dx$$

$y=\frac{x\sin x}{1+\cos^{2}x}$是偶函数

$$=\frac{1}{2}\cdot\frac{\pi}{2}\cdot 2\int_{0}^{\pi}\frac{x\sin x}{1+\cos^{2}x}dx$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}\cdot 2\cdot\frac{1}{2}\int_{0}^{\pi}\left[\frac{x\sin x}{1+\cos^{2}x}+\frac{(\pi-x)\sin(\pi-x)}{1+\cos^{2}(\pi-x)}\right]dx$$

$$\frac{1}{2}\int_{-\pi}^{\pi}\left(\frac{x\sin x}{1+\cos^{2}x}+\frac{(-x)\sin(-x)}{1+\cos^{2}(-x)}\right)dx$$

$$=\frac{1}{2}\cdot\frac{\pi}{2}\cdot 2\cdot\frac{1}{2}\int_{0}^{\pi}\frac{\pi\sin x}{1+\cos^{2}x}dx=\frac{\pi}{4}\int_{0}^{\pi}\frac{-\pi d\cos x}{1+\cos^{2}x}$$

$$\int_{-\pi}^{\pi}\frac{x\sin x}{1+\cos^{2}x}dx$$

第七讲：定积分

$$\int_0^{\frac{\pi}{2}} \ln\sin\theta d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} (\ln\sin\theta + \ln\cos\theta) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2\theta d\theta - \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln 2 d\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} \ln\sin 2\theta d2\theta - \frac{\pi\ln 2}{4}$$

$$= \frac{1}{4}\int_0^{\pi} \ln\sin\theta d\theta - \frac{\pi\ln 2}{4}$$

$$= \frac{1}{4}\cdot 2\int_0^{\frac{\pi}{2}} \ln\sin\theta d\theta - \frac{\pi\ln 2}{4}$$

$$\ln\sin\theta + \ln\cos\theta = \ln(\sin\theta\cos\theta) = \ln\sin 2\theta - \ln 2$$

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]dx$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$y = \ln\sin x$关于$x = \frac{\pi}{2}$对称

$$\int_0^{\frac{\pi}{2}} \ln\sin\theta d\theta = -\frac{\pi\ln 2}{2} \qquad \int_0^{\frac{\pi}{2}} \ln\cos\theta d\theta = -\frac{\pi\ln 2}{2}$$

第七讲：定积分

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x)+f(a+b-x)]dx$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\int_0^{\pi} \ln(1+\cos\theta)d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (\ln(1+\cos\theta)+\ln(1-\cos\theta))d\theta$$

$$= \frac{1}{2}\int_0^{\pi} 2\ln\sin\theta d\theta$$

$y = 2\ln\sin x$ 关于 $x = \frac{\pi}{2}$ 对称

$$= \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} 2\ln\sin\theta d\theta = -\pi\ln 2$$

$$\ln(1+\cos\theta)+\ln(1-\cos\theta) = \ln\left(1-\cos^2\theta\right) = \ln\sin^2\theta = 2\ln\sin\theta$$

$$\int_0^{\pi} \ln(1-\cos\theta)d\theta = \int_0^{\pi} \ln(1+\cos\theta)d\theta = -\pi\ln 2$$

第七讲：定积分

$$\int_0^{\pi} \ln(1+\cos\theta)\,d\theta = \int_0^{\pi} \ln 2\,d\theta + \int_0^{\pi} 2\ln\cos\frac{\theta}{2}\,d\theta$$

$$= \pi\ln 2 + \int_0^{\pi} 4\ln\cos\frac{\theta}{2}\,d\frac{\theta}{2}$$

$$= \pi\ln 2 + \int_0^{\frac{\pi}{2}} 4\ln\cos\theta\,d\theta = -\pi\ln 2$$

$$\ln(1+\cos\theta) = \ln\left(2\cos^2\frac{\theta}{2}\right) = \ln 2 + 2\ln\cos\frac{\theta}{2}$$

$$n \in N^+ \quad \int_0^{\pi} \ln(n+\cos\theta)\,d\theta$$

第七讲：定积分

$$\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\left(x\ln x+\frac{1}{4x}\right)e^{\left(x-\frac{1}{x}\right)^2}dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b\left[f(x)+\frac{ab}{x^2}f\left(\frac{ab}{x}\right)\right]dx$$

$$=\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\left[\left(x\ln x+\frac{1}{4x}\right)e^{\left(x-\frac{1}{x}\right)^2}+\frac{1}{x^2}\left(\frac{1}{x}\ln\frac{1}{x}+\frac{x}{4}\right)e^{\left(\frac{1}{x}-x\right)^2}\right]dx$$

$$=\frac{1}{2}\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}}\left[\left(x-\frac{1}{x^3}\right)\ln x+\frac{1}{2x}\right]e^{\left(x-\frac{1}{x}\right)^2}dx$$

第七讲：定积分

$$\int_{\frac{1}{3}}^{3}\left(x-\frac{1}{x}\right)^{2n}dx$$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b\left[f(x)+\frac{ab}{x^2}f\left(\frac{ab}{x}\right)\right]dx$$

$$=\frac{1}{2}\int_{\frac{1}{3}}^{3}\left[\left(x-\frac{1}{x}\right)^{2n}+\frac{1}{x^2}\left(\frac{1}{x}-x\right)^{2n}\right]dx$$

$$=\frac{1}{2}\int_{\frac{1}{3}}^{3}\left(x-\frac{1}{x}\right)^{2n}\left(1+\frac{1}{x^2}\right)dx$$

$$=\frac{1}{2}\int_{\frac{1}{3}}^{3}\left(x-\frac{1}{x}\right)^{2n}d\left(x-\frac{1}{x}\right)$$

第七讲：定积分

$$\int_a^b f(x)dx = \frac{1}{2}\int_a^b [f(x) + \frac{ab}{x^2} f(\frac{ab}{x})]dx$$

推广到无穷限的反常积分

$\forall k > 0$

令 $ab = k$ 且 $b \to +\infty \Rightarrow a \to 0^+$ $\qquad \Rightarrow \int_0^{+\infty} f(x)dx = \frac{1}{2}\int_0^{+\infty} [f(x) + \frac{k}{x^2} f(\frac{k}{x})]dx$

令 $ab = k$ 且 $b \to 0^- \Rightarrow a \to -\infty$ $\qquad \Rightarrow \int_{-\infty}^{0} f(x)dx = \frac{1}{2}\int_{-\infty}^{0} [f(x) + \frac{k}{x^2} f(\frac{k}{x})]dx$

第七讲：定积分

$$\int_0^{+\infty} \frac{x^2}{(1+x^2)^2}dx$$

$$\int_0^{+\infty} f(x)dx = \frac{1}{2}\int_0^{+\infty} [f(x)+\frac{k}{x^2}f(\frac{k}{x})]dx$$

$$k=1$$

$$\int_0^{+\infty} \frac{x^2}{(1+x^2)^2}dx = \frac{1}{2}\int_0^{+\infty}\left(\frac{x^2}{(1+x^2)^2}+\frac{1}{x^2}\cdot\frac{\frac{1}{x^2}}{\left(1+\frac{1}{x^2}\right)^2}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\left(\frac{x^2}{(1+x^2)^2}+\frac{1}{(1+x^2)^2}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{1}{1+x^2}dx=\frac{\pi}{4}$$

第七讲：定积分

$$\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^\alpha)}dx \qquad \int_0^{+\infty} f(x)dx=\frac{1}{2}\int_0^{+\infty}[f(x)+\frac{k}{x^2}f(\frac{k}{x})]dx \qquad k=1$$

$$\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^\alpha)}dx=\frac{1}{2}\int_0^{+\infty}\left(\frac{1}{(1+x^2)(1+x^\alpha)}+\frac{1}{x^2}\cdot\frac{1}{\left(1+\frac{1}{x^2}\right)\left(1+\frac{1}{x^\alpha}\right)}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\left(\frac{1}{(1+x^2)(1+x^\alpha)}+\frac{x^\alpha}{(1+x^2)(1+x^\alpha)}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{1}{1+x^2}dx=\frac{\pi}{4}$$

$$\int_0^{+\infty}\frac{1}{(1+x^2)(1+x^{-2})}dx=\int_0^{+\infty}\frac{x^2}{(1+x^2)^2}dx \qquad \int_0^{\frac{\pi}{2}}\frac{\sec^2\theta d\theta}{(1+\tan^2\theta)(1+\tan^\alpha\theta)}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{1+\tan^\alpha\theta}$$

第七讲：定积分

$$a>0\quad \int_0^{+\infty}\frac{\ln x}{x^2+a^2}dx \qquad \int_0^{+\infty}f(x)dx=\frac{1}{2}\int_0^{+\infty}[f(x)+\frac{k}{x^2}f(\frac{k}{x})]dx \quad k=a^2$$

$$\int_0^{+\infty}\frac{\ln x}{x^2+a^2}dx=\frac{1}{2}\int_0^{+\infty}\left(\frac{\ln x}{x^2+a^2}+\frac{a^2}{x^2}\cdot\frac{\ln\frac{a^2}{x}}{\left(\frac{a^2}{x}\right)^2+a^2}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\left(\frac{\ln x}{x^2+a^2}+\cdot\frac{\ln a^2-\ln x}{x^2+a^2}\right)dx$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{\ln a^2}{x^2+a^2}dx=\frac{\pi\ln a}{2a}$$

第七讲：定积分

$a>0\quad \int_0^{+\infty}\frac{\ln x}{x^2+a^2}dx$

$$\int_a^b f(x)dx=\frac{1}{2}\int_a^b[f(x)+f(a+b-x)]dx$$

$$\int_0^{\frac{\pi}{2}}\frac{\ln(a\tan\theta)\cdot a\sec^2\theta}{(a\tan\theta)^2+a^2}d\theta=\int_0^{\frac{\pi}{2}}\frac{\ln(a\tan\theta)}{a}d\theta$$

$$=\int_0^{\frac{\pi}{2}}\frac{\ln a+\ln\tan\theta}{a}d\theta$$

$$=\frac{\pi\ln a}{2a}+\frac{1}{a}\int_0^{\frac{\pi}{2}}\ln\tan\theta d\theta$$

$$\int_0^{\frac{\pi}{2}}\ln\tan\theta d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}(\ln\tan\theta+\ln\cot\theta)d\theta=0$$

第七讲：定积分 > 参数微分法

含参变量求导公式

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} f(x,\ y)dy=\int_{\alpha(x)}^{\beta(x)} f_x(x,\ y)dy+f[x,\ \beta(x)]\cdot\beta'(x)-f[x,\ \alpha(x)]\cdot\alpha'(x)$$

$$\beta(x)=b \qquad \alpha(x)=a$$

$$\frac{d}{dx}\int_{a}^{b} f(x,\ y)dy=\int_{a}^{b} f_x(x,\ y)dy$$

先积分后求导等于先求导后积分

第七讲：定积分 > 参数微分法

$\int_a^b f(x,\alpha)\big|_{\alpha=\alpha_1} dx$ 其中$f(x,\alpha)$是关于x，α的二元函数，α_1是常数

先积分后求导等于先求导后积分

$\varphi(\alpha)=\int_a^b f(x,\alpha)dx$

$\varphi(\alpha_1)=\int_a^b f(x,\alpha)\big|_{\alpha=\alpha_1} dx$

$\varphi'(\alpha)=\int_a^b f_\alpha(x,\alpha)dx$

$\varphi(\alpha_2)=\int_a^b f(x,\alpha)\big|_{\alpha=\alpha_2} dx$ 易求

$\varphi(\alpha_1)=\varphi(\alpha_2)+\int_{\alpha_2}^{\alpha_1}\varphi'(\alpha)d\delta$

$\int_a^b f_\alpha(x,\alpha)dx$

$$\int_a^b f(x,\alpha)\big|_{\alpha=\alpha_1} dx=\int_a^b f(x,\alpha)\big|_{\alpha=\alpha_2} dx+\int_{\alpha_2}^{\alpha_1}\left(\int_a^b f_\alpha(x,\alpha)dx\right)d\alpha$$

第七讲：定积分 > 参数微分法

$$\int_0^{\frac{\pi}{2}}\frac{\arctan\sin\theta}{\sin\theta}d\theta \qquad \varphi(1) \qquad \varphi(0)$$

$$\int\frac{\pi}{2\sqrt{\alpha^2+1}}d\alpha=\frac{\pi}{2}\ln\left(\alpha+\sqrt{\alpha^2+1}\right)+C$$

$$\varphi(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\arctan(\alpha\sin\theta)}{\sin\theta}d\theta$$

$$\varphi'(\alpha)=\int_0^{\frac{\pi}{2}}\frac{\partial}{\partial\alpha}\left(\frac{\arctan(\alpha\sin\theta)}{\sin\theta}\right)dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+(\alpha\sin\theta)^2}d\theta=\frac{\pi}{2\sqrt{\alpha^2+1}}$$

齐次化

$$\varphi(1)=\varphi(0)+\int_0^1\varphi'(\alpha)d\alpha=0+\int_0^1\frac{\pi}{2\sqrt{\alpha^2+1}}d\alpha=\frac{\pi}{2}\ln\left(1+\sqrt{2}\right)$$

$$\int\frac{1}{1+(\alpha\sin\theta)^2}d\theta=\int\frac{1}{(\alpha^2+1)\sin^2\theta+\cos^2\theta}d\theta=\int\frac{\sec^2\theta}{(\alpha^2+1)\tan^2\theta+1}d\theta=\int\frac{d\tan\theta}{(\alpha^2+1)\tan^2\theta+1}$$

$$=\frac{1}{\sqrt{\alpha^2+1}}\arctan\left(\sqrt{\alpha^2+1}\tan\theta\right)+C$$

第七讲：定积分 > 参数微分法

$$n\in N^{+}\quad \int_0^{\pi}\ln(n+\cos\theta)d\theta \qquad \varphi(n) \qquad \varphi(1)=-\pi\ln 2 \qquad \int\frac{\pi}{\sqrt{\alpha^2-1}}d\alpha=\pi\ln\left|\alpha+\sqrt{\alpha^2-1}\right|+C$$

$$\varphi(\alpha)=\int_0^{\pi}\ln(\alpha+\cos\theta)d\theta \qquad \alpha\geq 1$$

$$\varphi'(\alpha)=\int_0^{\pi}\frac{\partial}{\partial\alpha}\ln(\alpha+\cos\theta)\theta=\int_0^{\pi}\frac{1}{\alpha+\cos\theta}d\theta=\frac{\pi}{\sqrt{\alpha^2-1}}$$

$$\varphi(n)=\varphi(1)+\int_1^{n}\varphi'(\alpha)d\alpha=-\pi\ln 2+\int_1^{n}\frac{\pi}{\sqrt{\alpha^2-1}}d\alpha=-\pi\ln 2+\pi\ln\left(n+\sqrt{n^2-1}\right)$$

$$\int\frac{1}{\alpha+\cos\theta}d\theta=\int\frac{\frac{2}{1+u^2}}{\alpha+\frac{1-u^2}{1+u^2}}du=\int\frac{2}{(\alpha-1)u^2+\alpha+1}du=\frac{2}{\sqrt{\alpha^2-1}}\arctan\sqrt{\frac{\alpha-1}{\alpha+1}}u+C$$

令$u=\tan\frac{\theta}{2}$

$$=\frac{2}{\sqrt{\alpha^2-1}}\arctan\sqrt{\frac{\alpha-1}{\alpha+1}}\tan\frac{\theta}{2}+C$$

第七讲：定积分 > 参数微分法

$a，b>0\ \int_0^1 \frac{x^b - x^a}{\ln x}dx$ $\qquad \varphi(b)-\varphi(a) \qquad \varphi(\alpha_2)$

$$\varphi(\alpha)=\int_0^1 \frac{x^\alpha}{\ln x}dx$$

$$\varphi'(\alpha)=\int_0^1 \frac{\partial}{\partial\alpha}\left(\frac{x^\alpha}{\ln x}\right)dx=\int_0^1 x^\alpha dx=\frac{1}{\alpha+1}$$

$$\varphi(b)-\varphi(a)=\int_a^b \varphi'(\alpha)d\alpha=\int_a^b \frac{1}{\alpha+1}d\alpha=\ln\frac{b+1}{a+1}$$

第七讲：定积分 > 参数微分法

a，$b<0$ $\int_0^{+\infty}\frac{\sin x}{x}(e^{ax}-e^{bx})dx$ $\varphi(a)-\varphi(b)$ $\int \sin xe^{\alpha x}dx=\frac{\alpha\sin x-\cos x}{\alpha^2+1}e^{\alpha x}+C$

$$\varphi(\alpha)=\int_0^{+\infty}\frac{\sin x}{x}e^{\alpha x}dx \quad \alpha<0$$

$$\varphi'(\alpha)=\int_0^{+\infty}\frac{\partial}{\partial\alpha}\left(\frac{\sin x}{x}e^{\alpha x}\right)dx=\int_0^{+\infty}\sin xe^{\alpha x}dx=\frac{1}{\alpha^2+1}$$

$$\varphi(a)-\varphi(b)=\int_b^a\varphi(\alpha)d\alpha=\int_b^a\frac{1}{\alpha^2+1}d\alpha=\arctan a-\arctan b$$

$$\int \sin xe^{\alpha x}dx=(A\sin x+B\cos x)e^{\alpha x}+C$$

待定A B

$$\sin xe^{\alpha x}=(\alpha A\sin x+\alpha B\cos x+A\cos x-B\sin x)e^{\alpha x}$$

$$\begin{cases}\alpha A-B=1\\ \alpha B+A=0\end{cases}$$

第七讲：定积分 > 利用二重积分

$$\int_0^1 \frac{x^b - x^a}{\ln x}dx$$

把 x 看作常数 y 是变量

$$\frac{x^b - x^a}{\ln x} = \left[\frac{x^y}{\ln x}\right]_a^b = \int_a^b \frac{\partial}{\partial y}\left(\frac{x^y}{\ln x}\right)dy = \int_a^b x^y dy$$

$$\int_0^1 \frac{x^b - x^a}{\ln x}dx = \int_0^1\left(\int_a^b x^y dy\right)dx = \int_a^b\left(\int_0^1 x^y dx\right)dy = \int_a^b \frac{1}{y+1}dy = \ln\frac{b+1}{a+1}$$

利用牛顿-莱布尼茨公式将被积函数转换成一个定积分
这样原定积分就转换成了一个二重积分
然后交换积分次序求出这个二重积分

$$\int_0^{\frac{\pi}{2}} \frac{\arctan \sin\theta}{\sin\theta}d\theta \qquad a，b<0 \int_0^{+\infty} \frac{\sin x}{x}\left(e^{ax} - e^{bx}\right)dx \qquad n \in N^+ \int_0^{\pi} \ln(n+\cos\theta)d\theta$$

第七讲：定积分 > 利用二重积分

$$\int_0^{+\infty} e^{-x^2}\,dx$$

$$x\in(0,+\infty)\quad \int_0^{+\infty} e^{-xt^2}\,dt=\frac{1}{2}\sqrt{\frac{\pi}{x}}$$

$$\left(\int_0^{+\infty} e^{-x^2}\,dx\right)\left(\int_0^{+\infty} e^{-x^2}\,dx\right)=\left(\int_0^{+\infty} e^{-x^2}\,dx\right)\left(\int_0^{+\infty} e^{-y^2}\,dy\right)$$

第七届初赛

$$=\iint_D e^{-\left(x^2+y^2\right)}dxdy \qquad D=\{(x,\ y)\,|\,0\le x,\ y\}$$

$$=\iint_D e^{-r^2}\,rdrd\theta \qquad D=\{(r,\ \theta)\,|\,0\le\theta\le\frac{\pi}{2}\}$$

$$=\int_0^{+\infty} e^{-r^2}\,rdr\cdot\int_0^{\frac{\pi}{2}} d\theta$$

$$=\left[\frac{-e^{-r^2}}{2}\right]_0^{+\infty}\cdot\frac{\pi}{2}=\frac{\pi}{4}$$

$$\int_0^{+\infty} e^{-x^2}\,dx=\frac{\sqrt{\pi}}{2}$$

第七讲：定积分 > 利用级数

$$\int_a^b f(x)dx = \int_a^b \sum_{n=1}^{\infty} u_n(x)dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)dx$$

$$f(x) = \sum_{n=1}^{\infty} u_n(x)$$

$$\int_a^b u_n(x)dx$$

转换

第七讲：定积分 > 利用级数

$$\int_0^{+\infty}\frac{x}{1+e^x}dx$$

当$x>0$时

$$\frac{x}{1+e^x}=\frac{e^{-x}x}{e^{-x}+1}=e^{-x}x\cdot\frac{1}{1-(-e^{-x})}=e^{-x}x\sum_{k=0}^{\infty}(-e^{-x})^k=\sum_{k=0}^{\infty}(-1)^k e^{-(k+1)x}x$$

$$\int_0^{+\infty}\frac{x}{1+e^x}dx=\int_0^{+\infty}\left(\sum_{k=0}^{\infty}(-1)^k e^{-(k+1)x}x\right)dx=\sum_{k=0}^{\infty}(-1)^k\int_0^{+\infty}e^{-(k+1)x}xdx$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{1}{(k+1)^2}=\frac{\pi^2}{12}$$

$$\int e^{-(k+1)x}xdx=\frac{-(k+1)x-1}{(k+1)^2}e^{-(k+1)x}+C$$

第七讲：定积分 > 利用级数

$$\int_1^{+\infty} \frac{\ln x}{x(1+x)} dx \qquad \int_0^{+\infty} \frac{x}{1+e^x} dx$$

令 $\ln x = t \Rightarrow x = e^t \qquad x:1 \to +\infty \quad t:0 \to +\infty$

$$\int_1^{+\infty} \frac{\ln x}{x(1+x)} dx = \int_0^{+\infty} \frac{t}{e^t\left(1+e^t\right)} e^t dt = \int_0^{+\infty} \frac{t}{1+e^t} dt$$

$$\frac{\ln x}{x(1+x)} = \frac{\ln x}{x^2} \cdot \frac{1}{1-\left(-\frac{1}{x}\right)} = \frac{\ln x}{x^2} \sum_{k=0}^{\infty} \left(-\frac{1}{x}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k \ln x}{x^{k+2}}$$

第七讲：定积分 > 利用级数

$$\int_0^1 \ln(1-x)\ln x dx$$

$$\ln(1-x)\ln x=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(-x)^k}{k}\ln x=\sum_{k=1}^{\infty}\frac{-x^k\ln x}{k}$$

$$\int_0^1 \ln(1-x)\ln x dx=\int_0^1\left(\sum_{k=1}^{\infty}\frac{-x^k\ln x}{k}\right)dx=\sum_{k=1}^{\infty}\int_0^1\frac{-x^k\ln x}{k}dx=\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}$$

$$\int_0^1\frac{-x^k\ln x}{k}dx=\frac{-1}{k}\int_0^1 x^k\ln x dx=\frac{-1}{k(k+1)}\int_0^1\ln x dx^{k+1}=\frac{-1}{k(k+1)}\left(x^{k+1}\ln x\Big|_0^1-\int_0^1 x^{k+1}\cdot\frac{1}{x}dx\right)$$

$$=\frac{-1}{k(k+1)}\left(0-\frac{1}{k+1}\right)$$

$$=\frac{1}{k(k+1)^2}$$

第七讲：定积分 > 利用级数

$$\int_0^1 \ln(1-x)\ln x dx$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}$$ 裂项

$$\frac{1}{k(k+1)^2}=\frac{1}{k(k+1)}\cdot\frac{1}{k+1}=\left(\frac{1}{k}-\frac{1}{k+1}\right)\frac{1}{k+1}=\frac{1}{k(k+1)}-\frac{1}{(k+1)^2}=\left(\frac{1}{k}-\frac{1}{k+1}\right)-\frac{1}{(k+1)^2}$$

$$\sum_{k=1}^{\infty}\frac{1}{k(k+1)^2}=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}\right)-\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}=1-\left(\frac{\pi^2}{6}-1\right)=2-\frac{\pi^2}{6}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$$

第七讲：定积分 > 利用递推公式

$\int_a^b f_n(x)dx \quad I_n$

$I_k = \int_a^b f_k(x)dx \quad k=1,2,\cdots\cdots$

找出数列$\{I_k\}$的递推公式

进而得到$\{I_k\}$的通项公式

产生递推公式　分部积分法

第七讲：定积分 > 利用递推公式

$$n\in N^{+}\quad \int_0^{\frac{\pi}{2}} \sin^n x dx \qquad 令 I_k=\int_0^{\frac{\pi}{2}} \sin^k x dx$$

$$\int_0^{\frac{\pi}{2}} \sin^k x dx=\int_0^{\frac{\pi}{2}} \sin^{k-1} x\cdot \sin x dx \qquad k\ge 2$$

$$=-\int_0^{\frac{\pi}{2}} \sin^{k-1} x d\cos x$$

$$=-\left(\sin^{k-1} x\cdot\cos x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\cos x\cdot(k-1)\sin^{k-2} x\cos x dx\right)$$

$$=(k-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{k-2} x dx$$

$$=(k-1)\left(\int_0^{\frac{\pi}{2}}\sin^{k-2} x dx-\int_0^{\frac{\pi}{2}}\sin^k x dx\right)$$

$$I_k=(k-1)(I_{k-2}-I_k)$$

$$I_k=\frac{k-1}{k}I_{k-2}$$

当n是偶数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}I_0=\frac{(n-1)!!}{n!!}\frac{\pi}{2}$$

当n是奇数时

$$I_n=\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}I_1=\frac{(n-1)!!}{n!!}$$

第七讲：定积分 > 利用递推公式

$n\in N^+$ 且 $a>0$ $\quad \int_0^{+\infty}\frac{1}{(x^2+a^2)^n}dx$

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

令 $x=a\tan\theta \quad \theta\in(0,\frac{\pi}{2}) \qquad x:0\to+\infty \quad \theta:0\to\frac{\pi}{2}$

$$\int_0^{+\infty}\frac{1}{(x^2+a^2)^n}dx=\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta}{(a^2\tan^2\theta+a^2)^n}d\theta=\frac{1}{a^{2n}}\int_0^{\frac{\pi}{2}}\cos^{2n-2}\theta d\theta=\frac{1}{a^{2n}}\int_0^{\frac{\pi}{2}}\sin^{2n-2}\theta d\theta$$

第七讲：定积分 > 利用递推公式

$n \in N^+ \quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2n} x} dx$ 令 $I_k = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2k} x} dx$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2k} x} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin x}{\sin^{2k+1} x} dx$$

$$I_k = 2^k - (2k+1)(I_{k+1} - I_k)$$

$$= -\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2k+1} x} d\cos x$$

$$-2^k \qquad = -\left(\frac{1}{\sin^{2k+1} x} \cdot \cos x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \cdot \frac{-(2k+1)\cos x}{\sin^{2k+2} x} dx \right)$$

$$= 2^k - (2k+1) \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin^{2k+2} x} dx$$

$$= 2^k - (2k+1) \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2k+2} x} dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^{2k} x} dx \right)$$

第七讲：定积分 > 利用递推公式

凑差分

$$I_k = 2^k - (2k+1)(I_{k+1} - I_k)$$

$$(2k+1)I_{k+1} - 2kI_k = 2^k$$

$$\frac{2k+1}{2k}I_{k+1} - I_k = \frac{2^k}{2k}$$

$$\frac{(2k-1)!!}{(2k-2)!!}$$

$$I_1 = 1$$

$$\frac{(2k-1)!!}{(2k-2)!!}\frac{2k+1}{2k}I_{k+1} - \frac{(2k-1)!!}{(2k-2)!!}I_k = \frac{(2k-1)!!}{(2k-2)!!}\frac{2^k}{2k}$$

$$\frac{(2k+1)!!}{(2k)!!}I_{k+1} - \frac{(2k-1)!!}{(2k-2)!!}I_k = \frac{(2k-1)!!}{(2k)!!}2^k$$

对k从1到$n-1$求和

$$\frac{(2n-1)!!}{(2n-2)!!}I_n - \frac{1!!}{0!!}I_1 = \sum_{k=1}^{n-1}\frac{(2k-1)!!}{(2k)!!}2^k$$

$$I_n = \frac{(2n-2)!!}{(2n-1)!!} + \frac{(2n-2)!!}{(2n-1)!!}\sum_{k=1}^{n-1}\frac{(2k-1)!!}{(2k)!!}2^k$$

第七讲：定积分 > 利用递推公式

$$n\in N^{+}\quad \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sin^{2n}x}dx=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\csc^{2n}xdx$$

$$-\csc^{2}xdx=d\cot x$$

凑微分消去三角

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\csc^{2n-2}x\cdot\csc^{2}xdx$$

$$\csc^{2n-2}x=\left(\csc^{2}x\right)^{n-1}=\left(\cot^{2}x+1\right)^{n-1}$$

$$=-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\cot^{2}x+1\right)^{n-1}d\cot x$$

令 $\cot x=t$

$$=-\int_{1}^{0}\left(t^{2}+1\right)^{n-1}dt$$

$$x:\frac{\pi}{4}\to\frac{\pi}{2}\quad t:1\to 0$$

$$=-\int_{1}^{0}\sum_{k=0}^{n-1}C_{n-1}^{k}t^{2k}dt=-\sum_{k=0}^{n-1}C_{n-1}^{k}\int_{1}^{0}t^{2k}dt$$

$$=\sum_{k=0}^{n-1}C_{n-1}^{k}\frac{1}{2k+1}$$

第七讲：定积分 > 利用递推公式

$n \in N^{+}\ \int_0^{\pi} \cos nx \cdot \cos^n x dx$ 令 $I_k = \int_0^{\pi} \cos kx \cdot \cos^k x dx$

$$\int_0^{\pi} \cos kx \cdot \cos^k x dx = \frac{1}{k}\int_0^{\pi} \cos^k x d\sin kx$$

$$= \frac{1}{k}\left(\cos^k x \cdot \sin kx \Big|_0^{\pi} - \int_0^{\pi} \sin kx \cdot \left(-k\cos^{k-1} x \cdot \sin x\right)dx \right)$$

$$= \int_0^{\pi} \sin kx \cdot \sin x \cdot \cos^{k-1} x dx \qquad \cos(k-1)x = \cos kx \cdot \cos x + \sin kx \cdot \sin x$$

$$= \int_0^{\pi} \left(\cos(k-1)x - \cos kx \cdot \cos x\right)\cos^{k-1} x dx$$

$$= \int_0^{\pi} \cos(k-1)x \cdot \cos^{k-1} x dx - \int_0^{\pi} \cos kx \cdot \cos^k x dx$$

$$I_k = I_{k-1} - I_k \qquad I_k = \frac{1}{2} I_{k-1} \qquad I_n = \left(\frac{1}{2}\right)^n I_0 = \left(\frac{1}{2}\right)^n \pi$$