

习题一

一、判断题 (1) \checkmark ; (2) \times

二、单项选择题 C; A

三、填空题

1 导数,常; 2 阶; 3 初始; 4、 xy 或 $\ln(xy)$

四、计算题:

1、

$$\frac{2x}{1-x^2} dx = \frac{1}{y+y^2} dy$$

$$\int \frac{2x}{1-x^2} dx = \int \frac{1}{y+y^2} dy$$

$$-\ln|1-x^2| + c' = \ln \left| \frac{y}{1+y} \right|$$

$$\frac{y(1-x^2)}{1+y} = c$$

故通解为: $y(1-x^2) = c(1+y)$ (c 为任意常数)

2、

$$-\frac{x}{\sqrt{1-x^2}} dx = \frac{1}{y} dy; y \neq 0$$

$$\int -\frac{x}{\sqrt{1-x^2}} dx = \int \frac{1}{y} dy$$

$$(1-x^2)^{\frac{1}{2}} + c_1 = \ln|y|, y \neq 0$$

$$y = ce^{(1-x^2)^{\frac{1}{2}}}$$

$$x = -1, y = 2, c = 2$$

$$\text{故特解为: } y = 2e^{(1-x^2)^{\frac{1}{2}}}$$

3、

$$\frac{1}{x} dx = \frac{1}{y \ln y} dy, y \neq 1$$

$$\int \frac{1}{x} dx = \int \frac{1}{y \ln y} dy$$

$$\ln|x| + c_1 = \ln|\ln y|, y \neq 1$$

$$\ln y = cx,$$

故通解为: $y = e^{cx}$ (c 为任意常数)

习题二

一、判断题 (1) \checkmark ; (2) \checkmark

二、 C

三、

1、

$$u = \frac{y}{x}, y = ux, \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = u + \tan u$$

$$\cot u du = \frac{1}{x} dx, \int \cot u du = \int \frac{1}{x} dx$$

$$\ln |\sin u| = \ln |x| + c_1$$

$$\sin u = cx,$$

$$\text{通解为: } \sin\left(\frac{y}{x}\right) = cx$$

2、

$$y = ux, \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = u \ln u$$

$$\frac{1}{u(\ln u - 1)} du = \frac{1}{x} dx, \int \frac{1}{u(\ln u - 1)} du = \int \frac{1}{x} dx$$

$$\ln |\ln u - 1| = \ln |x| + c_1$$

$$\ln u - 1 = cx,$$

$$\text{通解为: } \ln \frac{y}{x} - 1 = cx$$

3、

$$y = ux, \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = u + \frac{2}{u}, u du = \frac{2}{x} dx$$

$$\frac{1}{2} u^2 = \ln x^2 + c_1$$

$$y^2 = 2x^2 \ln x^2 + cx^2, x=1, y=6, c=36$$

$$\text{特解为: } y^2 = 2x^2 \ln x^2 + 36x^2$$

4、

$$u = \frac{x}{y}, x = uy, \frac{dx}{dy} = u + y \frac{du}{dy}$$

$$u + y \frac{du}{dy} = u - \frac{1}{u}, -u du = \frac{1}{y} dy$$

$$\int -u du = \int \frac{1}{y} dy, \text{ 则 } -\frac{1}{2} u^2 = \ln |y| + c_1$$

$$\text{于是通解为: } x^2 + y^2 \ln y^2 + c = 0$$

习题三

一 C; C; B

二

1

$$\begin{aligned}P(x) &= 2x, Q(x) = e^{-x^2} \\y &= e^{-\int 2x dx} \left(\int e^{-x^2} e^{\int 2x dx} dx + c \right) \\&= e^{-x^2} (x + c)\end{aligned}$$

2

$$\begin{aligned}P(x) &= \tan x, Q(x) = \sin 2x \\y &= e^{-\int \tan x dx} \left(\int \sin 2x e^{\int \tan x dx} dx + c \right) \\&= e^{\ln |\cos x|} \left(\int \frac{\sin 2x}{\cos x} |\cos x| dx + c \right) \\&= -2(\cos x)^2 + c \cos x\end{aligned}$$

用凑微分法

3

$$\begin{aligned}y &= e^{-\int 2x dx} \left(\int 8xe^{\int 2x dx} dx + c \right) \\&= e^{-x^2} \left(\int 8xe^{x^2} dx + c \right) \\&= e^{-x^2} (4e^{x^2} + c), x=0, y=2, c=-2 \\ \text{特解为: } y &= e^{-x^2} (4e^{x^2} - 2) = 4 - 2e^{-x^2}\end{aligned}$$

4、

$$\begin{aligned}z &= y^{-1}, \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}, \frac{dy}{dx} = -y^2 \frac{dz}{dx} \\-y^2 \frac{dz}{dx} + \frac{y}{x} &= 2y^2 \ln x \\ \frac{dz}{dx} - \frac{1}{x} z &= -2 \ln x \\z &= e^{\int \frac{1}{x} dx} \left(\int -2 \ln x e^{-\int \frac{1}{x} dx} dx + c \right) \\&= x[-(\ln x)^2 + c] \\&= -x(\ln x)^2 + cx \\ \text{故通解为: } &(-x(\ln x)^2 + cx) y = 1\end{aligned}$$

习题四

1、

$$\begin{aligned}y' &= \int (x + \sin x) dx = \frac{1}{2} x^2 - \cos x + c_1 \\ \text{通解为: } y &= \int \left(\frac{1}{2} x^2 - \cos x + c_1 \right) dx = \frac{1}{6} x^3 - \sin x + c_1 x + c_2\end{aligned}$$

2、

$$y' = p, y'' = \frac{dp}{dx}$$

$$\frac{dp}{dx} + \frac{1}{x} p = -1$$

$$p = e^{-\int \frac{1}{x} dx} \left(\int -e^{\int \frac{1}{x} dx} dx + c_1 \right)$$

$$= \frac{1}{x} \left(-\frac{1}{2} x^2 + c_1 \right) = \frac{c_1}{x} - \frac{1}{2} x$$

$$\text{通解为: } y = -\frac{1}{4} x^2 + c_1 \ln|x| + c_2$$

3、

$$y' = p, y'' = \frac{dp}{dx}$$

$$\frac{dp}{dx} = \frac{x}{p}, p dp = x dx$$

$$p^2 = x^2 + c_1$$

$$y' = \pm \sqrt{x^2 + c_1}, y'(1) = 1, c_1 = 0$$

$$y' = x$$

$$y = \frac{1}{2} x^2 + c_2, y(1) = -1, c_2 = -\frac{3}{2}$$

$$\text{特解为: } y = \frac{1}{2} x^2 - \frac{3}{2}$$

4、

$$y' = p, y'' = \frac{dp}{dy} \frac{dy}{dx} = p \frac{dp}{dy}$$

$$yp \frac{dp}{dy} - p^2 = 0, p \neq 0, y \neq 0$$

$$\frac{1}{p} dp = \frac{1}{y} dy, \ln|p| = \ln|y| + \ln c_1, p = 0$$

$$p = c_1 y, \text{ 则 } y' = \frac{dy}{dx} = c_1 y,$$

$$\text{这样 } \ln|y| = c_1 x + c_2$$

$$\text{故通解为: } y = c_2 e^{c_1 x}$$

习题五

一

D; D;

二

1 $y = c_1 e^{x^2} + c_2 x e^{x^2};$

2 $y = c_1 e^x + c_2 e^{x^2} + x + 2 + x e^x \ln|x|$

三

1

$$r^2 - r - 6 = 0, r_1 = -2, r_2 = 3$$

通解为: $y = c_1 e^{-2x} + c_2 e^{3x}$

2

$$r^2 + 12r + 36 = 0$$

$$r_1 = r_2 = -6$$

通解为: $y = (c_1 + c_2 x) e^{-6x}$

3

$$r^2 + r + 5 = 0$$

$$r_1 = -2 - i, r_2 = -2 + i$$

通解为: $y = e^{-2x} (c_1 \cos x + c_2 \sin x)$

四、

$$r^2 + 4r + 4 = 0, r_1 = -2, r_2 = -2$$

通解为: $y = (c_1 + c_2 x) e^{-2x},$

$x = 0$ 时, $y = -1, y' = 4,$

于是 $c_1 = -1, c_2 = 2$

故特解为: $y = (2x - 1) e^{-2x}$

习题六

一、 1 $x(ax^3 + bx^2 + cx + d);$

2 $e^{3x} (c_1 \cos x + c_2 \sin x);$

3 $x(ax + b) + cxe^{-x}$

二

1

$$r^2 - 2r - 3 = 0, r_1 = -1, r_2 = 3$$

令 $y^* = x(ax + b)e^{3x},$

可解得 $a = \frac{1}{8}, b = \frac{3}{16}$

$$y = c_1 e^{-x} + c_2 e^{3x} + \left(\frac{1}{8}x^2 + \frac{3}{16}x\right)e^{3x}$$

2

$$r^2 - 6r + 9 = 0, r_1 = 3, r_2 = 3$$

$$\text{令 } y^* = ax^2 e^{3x}, \text{ 得 } a = 3$$

$$y = (c_1 + c_2 x) e^{3x} + 3x^2 e^{3x}$$

3 、

$$r^2 + 4 = 0,$$

$$r_1 = 2i, r_2 = -2i$$

$$\text{令 } y^* = x(a \cos 2x + b \sin 2x),$$

$$\text{可解得 } a = -\frac{1}{8}, b = 0$$

$$y = (c_1 \cos 2x + c_2 \sin 2x) - \frac{1}{8} x \cos 2x$$

4

$$\varphi'(x) = e^x + x\varphi(x) - \int_0^x \varphi(t) dt - x\varphi(x)$$

$$\varphi''(x) = e^x - \varphi(x), \varphi(0) = 1, \varphi'(0) = 1$$

$$r^2 + 1 = 0, r_1 = i, r_2 = -i$$

$$\text{令 } y^* = c_1 e^x, \text{ 可解得 } c_1 = \frac{1}{2}$$

$$\text{故 } \varphi(x) = c_2 \cos x + c_3 \sin x + \frac{1}{2} e^x$$

$$\text{又由于 } \varphi(0) = 1, \varphi'(0) = 1, \text{ 可得}$$

$$c_2 = \frac{1}{2}, c_3 = \frac{1}{2},$$

$$\text{故 } \varphi(x) = \frac{1}{2} \cos x + \frac{1}{2} \sin x + \frac{1}{2} e^x$$

第七章复习题

一、判断题 (1) \times ; (2) \checkmark

二

C; A; C

三

$$1 \quad y = \frac{1}{2} c_1 x^2 + c_2 x - e^{-x} + c_3;$$

$$2 \quad y^* = x(ax + b) + cxe^{-4x}$$

四

1

$$\frac{1}{1+y^2} dy = \frac{2x}{1+x^2} dx, \arctan y = \ln(1+x^2) + c$$

$$y=0, x=1, c=-\ln 2$$

$$\text{特解: } \arctan y = \ln \frac{1+x^2}{2}$$

2

$$y=ux, \frac{dy}{dx} = u + x \frac{du}{dx}$$

$$u + x \frac{du}{dx} = u + 3 \tan u, \cot u du = \frac{3}{x} dx$$

$$\ln |\sin u| = \ln |x^3| + \ln c$$

$$\sin u = cx^3,$$

$$\text{通解为: } \sin \frac{y}{x} = cx^3$$

3

$$z = y^{-1}, \frac{dz}{dx} = -y^{-2} \frac{dy}{dx}$$

$$\frac{dy}{dx} = -y^2 \frac{dz}{dx}$$

$$\frac{dz}{dx} - 2xz = -2x$$

$$z = e^{\int 2x dx} \left(\int -2xe^{-\int 2x dx} dx + c \right) = e^{x^2} \left(\int -2xe^{-x^2} dx + c \right) = 1 + ce^{x^2}$$

$$\text{通解为: } y = \frac{1}{1 + ce^{x^2}}$$

4

$$r^2 - 3r + 2 = 0, r_1 = 1, r_2 = 2$$

$$y^* = e^x (a_1 \sin x + a_2 \cos x), a_1 = -1, a_2 = -1$$

$$\text{通解为: } y = c_1 e^x + c_2 e^{2x} - e^x (\sin x + \cos x)$$

习题七

一. $\times \times \times \quad \checkmark \checkmark \checkmark \checkmark$

二. A D C

三. xoy 面 $(-2, 3, 0)$ $-2\vec{a}$ $\vec{a} + \vec{b}$ $\vec{a} - \vec{b}$ $2\sqrt{3}$ yoz 坐标面

四. $\cos \alpha = \frac{-1}{2}, \cos \beta = \frac{-\sqrt{2}}{2}, \cos \gamma = \frac{1}{2}$ $(\frac{1}{2}, \frac{\sqrt{2}}{2}, -\frac{1}{2})$

五. (1) $(-1, 3, 3)$ (2) $2\sqrt{3}$ (3) $\cos \alpha = \frac{-\sqrt{3}}{3}, \cos \beta = \frac{\sqrt{3}}{3}, \cos \gamma = \frac{\sqrt{3}}{3}$

习题八

一. $\times \times \times \times \checkmark$

二. C D

三. 1. $(-4, 2, -4)$ 2. $-10, 2$

3. 7 4. $\frac{\pi}{4}$ 5. $2\sqrt{2}$

四. $S = \frac{15}{2}$

五. $\pm \frac{1}{\sqrt{93}} (5, -8, 2)$

习题九

一. $\times \checkmark \times$

二. CDDCC

三. 1. ± 2 2. $x^2 + y^2 + z^2 = 3$ 3. $y^2 + z^2 = 5x$ 4. $\frac{\pi}{3}$

四. 1. 由 xoz 面上的曲线 $z = 2x^2$ 绕 z 轴旋转得到的

2. 由 xoy 面上的曲线 $\frac{x^2}{9} + \frac{y^2}{4} = 1$ 绕 x 轴旋转得到的

习题十

一. $\times \checkmark \times$

二. BD

三. 1. 点 $(-\frac{4}{3}, -\frac{17}{3})$, 过点 $(-\frac{4}{3}, -\frac{17}{3}, 0)$ 平行于 z 轴的直线

2. $\begin{cases} x^2 + y^2 = 1 \\ z = 3 \end{cases}, (0, 0, 3), 1$

3. $\begin{cases} y = (x-1)^2 \\ z = 2x-1 \end{cases}$

$$\text{四. } \begin{cases} x = \frac{3}{\sqrt{2}} \cos \alpha \\ y = \frac{3}{\sqrt{2}} \cos \alpha \\ z = 3 \sin \alpha \end{cases}$$

$$\text{五. 在 } xoy \text{ 平面的投影曲线 } \begin{cases} x^2 + y^2 + x + y = 1 \\ z = 0 \end{cases}$$

$$\text{在 } yoz \text{ 平面的投影曲线 } \begin{cases} x^2 + (1 - y - z)^2 = z \\ x = 0 \end{cases}$$

$$\text{在 } xoz \text{ 平面的投影曲线 } \begin{cases} x^2 + (1 - x - z)^2 = z \\ y = 0 \end{cases}$$

习题十一

一. DCC

二. 1. $3x - 7y + 5z - 14 = 0$

2. $(1, -1, 3)$

3. $\frac{10}{3}$

4. $-4, 3$

三. $x + 7y + 8z + 12 = 0$

四. $9x - y + 3z - 16 = 0$

五. 面方程: $y = 3x$ 或 $x + 3y = 0$

习题十二

一. D B A C

二. 1. $\frac{x-1}{0} = \frac{y-2}{1} = \frac{z-3}{0}$

2. $\frac{x-1}{-2} = \frac{y-1}{1} = \frac{z-1}{3}$, 参数方程: $x = 1 - 2t, y = 1 + t, z = 1 + 3t$

3. -1

三. 直线方程: $\frac{x-1}{9} = \frac{y-1}{2} = \frac{z-1}{-5}$

四. $x + 5y + z - 1 = 0$

第八章复习题

一. $\times \sqrt{\quad} \sqrt{\quad} \times \times$

二. BBB

三. 1. 0 2. $(x-3)^2 + (y+1)^2 + (z-1)^2 = 21$ 3. $(x+y)^2 + (z+1)^2 = 3/2$

4. 2 5. $x = z^2 + y^2, z^4 = x^2 + y^2$

6. $\frac{x-2}{12} = \frac{y-3}{20} = \frac{z-1}{23}$

四. $(-1, 6, 3)$ $\alpha = \arcsin \frac{5}{\sqrt{19 \times 35}} = \arcsin \frac{\sqrt{665}}{133}$ $\frac{x+1}{1} = \frac{y-6}{3} = \frac{z-3}{2}$

五. $(2, 9, 6)$

六. $(x+1)^2 + (y-2)^2 + (z-1)^2 = 49$

习题十三

一. $\times \quad \checkmark \quad \times$

二. D C

三、

1、 $f(x, y) = xy$

2、 0

3、 $\{(x, y) \mid \sin(x^2 + y^2) - 1 = 0\}$

四、

1、 13

2、 6

3、 $\frac{1}{2}$

4 五、 由于 $\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x}} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x}{1+x^2} = 0,$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ y=x^2}} \frac{x^2 y}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2}$$

所以极限不存在

习题十四

一. $\times \quad \times$

二、 D B

三、 1、 -8

2、 0

四、

$$1、 \frac{\partial z}{\partial x} = \frac{2x}{y^3} \cot \frac{x^2}{y^3}; \quad \frac{\partial z}{\partial y} = -\frac{3x^2}{y^4} \cot \frac{x^2}{y^3}$$

$$2、 \frac{\partial z}{\partial x} = \frac{15x^2 \sqrt{\ln^3(x^3 + y^2)}}{2(x^3 + y^2)}; \quad \frac{\partial z}{\partial y} = \frac{5y \sqrt{\ln^3(x^3 + y^2)}}{x^3 + y^2}$$

$$4、 \frac{\partial u}{\partial x} = \frac{y^2}{z^2} x^{\frac{y^2}{z^2}-1}; \quad \frac{\partial u}{\partial y} = \frac{2y \ln x}{z^2} x^{\frac{y^2}{z^2}}; \quad \frac{\partial u}{\partial z} = -\frac{2y^2 \ln x}{z^3} x^{\frac{y^2}{z^2}}$$

五、

$$1、 \frac{\partial z}{\partial x} = y^x \ln y; \quad \frac{\partial^2 z}{\partial x \partial y} = y^{x-1}(1 + x \ln y)$$

$$2、 \frac{\partial z}{\partial x} = 2x \ln(x^2 + y) + \frac{2x^3}{x^2 + y}; \quad \frac{\partial^2 z}{\partial x \partial y} = \frac{2xy}{(x^2 + y)^2}$$

习题十五

一、 × ×

二、 D C

$$三、 1、 \frac{dz}{dt} = \frac{6t - 12t^2}{\sqrt{1 - (3t^2 - 4t^3 + 2)^2}}$$

$$2、 du = yzx^{yz-1}dx + zx^{yz} \ln x dy + yx^{yz} \ln x dz$$

$$3、 \frac{dz}{dx} = \frac{3x^2 + 2e^{2x}}{1 + (x^3 + e^{2x})^2}$$

四、

$$1、 \Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = -\frac{5}{42};$$

$$dz = -0.125$$

$$2、 \frac{\partial z}{\partial x} = \frac{1}{y} f_1' + y^2 f_2'; \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2} f_1' + 2xy f_2'$$

$$3、 \frac{\partial z}{\partial x} = 2xf' + yg'; \quad \frac{\partial^2 z}{\partial x \partial y} = 6xy^2 f'' + g' + yg''$$

4、 令 $u = 2x + y, \quad v = 3x - 2y$ 则

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = 2vu^{v-1} + 3u^v \ln u \\ &= 2(3x-2y)(2x+y)^{3x-2y-1} + 3(2x+y)^{3x-2y} \ln(2x+y) \end{aligned}$$

五、证明：

$$\begin{aligned} x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= x[y + F(u) - \frac{y}{x} F'(u)] + y[x + F'(u)] \\ &= xy + xF(u) - yF'(u) + xy + yF'(u) \\ &= z + xy \end{aligned}$$

习题十六

一、 1. \times 2. \times

二、 D B C

三、 1. 3 2. $\frac{y}{1+e^u}$

四、 1. $\frac{6x^2 y^2 - 3e^{3x}}{\cos y - 4x^3 y}$

$$a) \quad z_x = -\frac{y}{x} \quad z_y = \frac{xz}{e^z - xy}$$

$$b) \quad z_x = -\frac{2z + 2y^2 e^{-2xy^2}}{2x + ye^{-z}} \quad z_y = \frac{e^{-z} - 4xye^{-2xy^2}}{2x + ye^{-z}}$$

习题十七

一、 $\times \quad \times$

二、 C C

三、 1. $-\frac{2\sqrt{5}}{5}$ 2. $(3, -12, -6)$ 3. $-\frac{1}{18}(1, 2, 3)$

四、 1. $\frac{7\sqrt{5}}{50}$ 2. $\frac{\sqrt{2}}{2}(-6e^{-4} + 1)$

3. $\frac{3-2\sqrt{2}}{2}$ 4. 0

习题十八

一、 × ×

二、 B A

三、 1. $x+2y-3z+14=0$

2. $x+6y+10z-17=0$

四、 1. $x-\frac{3}{2}=\frac{y-4}{4}=\frac{z-1}{-12}$ $x+4y-12z-\frac{11}{2}=0$

2. $x-\frac{\pi}{2}+1=y-1=\frac{z-2\sqrt{2}}{\sqrt{2}}$ $x+y+\sqrt{2}z-\frac{\pi}{2}-4=0$

3. $12x+18y+z-30=0$ $\frac{x-1}{12}=\frac{y-1}{18}=\frac{z}{1}$

习题十九

一、 × √

二、 B A D

三、 1. 36 2. 18

四、 1. (1,3)为极大值点, 极大值为 10

2. $-e^{-1}-4$ 3. 极大值6, 极大小值-2

五、 $x=6, y=6, z=3$

复习题

一、 √ ×

二、 D C

三、 1. xy 2. $\sin(x^2+y^2)-1=0$

3. $\{(x,y)|1\leq x^2+y^2<6\text{ 且 }x^2+y^2\neq 5\}$ 4. 0

四、 1. $\frac{\partial u}{\partial x}=2xy^3zf_1'+yf_2'+2xf_3'$

2. $dz=\frac{1}{3x^2z^2+4y^2z}[(2x-2xz^3)dz-(4yz^2+3y^2)dy]$

3. $\sqrt{2}(5e^{-5}-16)$

习题二十

一、 1. $\frac{2}{3}\pi R^3$ 2. 0 3. 6π

二、 A B

三、 1. $0 \leq I \leq \pi^2$

2. $36\pi \leq I \leq 100\pi$

习题二十一

一、 1. $\frac{23}{40}$ 2. $\frac{9}{16}$ 3. $-\frac{243}{20}$ 4. $\frac{8}{3}(1-\cos 1)$

二、 1. $\int_0^2 dy \int_y^2 f(x, y) dx$

习题二十二

一、 1. $\pi(1-e^{-1})$ 2. $\frac{2}{3}R^3(\frac{\pi}{2}-\frac{2}{3})$

二、 1. $14a^4$ 2. $\frac{2\pi}{3}(|b|^3-|a|^3)$

三、 $\pi(1-\cos 1)$

四、 $\frac{3}{32}\pi a^4$

习题二十三

一、 1. $2\pi a^2$ 2. 0

二、 1. $\int_0^R dx \int_0^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} f(x, y, z) dz$

2. $\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{x^2+2y^2}^{2-x^2} f(x, y, z) dz$

三、 1. $\frac{1}{2}(\ln 2 - \frac{5}{8})$ 2. $\frac{14}{45}$

四、 64π

习题二十四

一、 1. $\int_0^{2\pi} d\theta \int_0^1 d\rho \int_\rho^1 (\rho \cos \theta + \rho \sin \theta) \rho dz$ 2. $\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r^3 \sin \varphi dr$

$$\text{二、1、原式} = \iiint_{\Omega} z \rho^2 d\rho d\theta dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} d\rho \int_0^1 z \rho^2 dz = \frac{16}{9}$$

$$\text{2、原式} = \iiint_{\Omega} \rho^3 d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^2 d\rho \int_{\frac{1}{2}\rho^2}^2 \rho^3 dz = \frac{16\pi}{3}$$

$$\text{三、原式} = \iiint_{\Omega} r^3 \sin\varphi dr d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{2a\cos\varphi} r^3 \sin\varphi dr = \frac{8\pi a^4}{5} \left(1 - \frac{\sqrt{2}}{8}\right)$$

$$\text{四、1、原式} = \iiint_{\Omega} r^3 \sin\varphi dr d\varphi d\theta = \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} r^3 \sin\varphi dr = \frac{\pi}{10}$$

$$\text{2、原式} = \iiint_{\Omega} z \rho d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^2 d\rho \int_{\frac{1}{2}\rho^2}^{\sqrt{8-\rho^2}} z \rho dz = \frac{28\pi}{3}$$

习题二十五

$$\begin{aligned} \text{一、} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy = \iint_D \sqrt{1 + x^2 + y^2} dxdy = \iint_D \sqrt{1 + \rho^2} \rho d\rho d\theta \\ &= \int_0^{\frac{\pi}{2}} d\theta \int_0^1 \sqrt{1 + \rho^2} \rho d\rho = \frac{\pi}{6} (2\sqrt{2} - 1) \end{aligned}$$

二、

$$M = \iint_D x|y| dxdy = 2 \iint_{D_1} xy dxdy = 2 \iint_{D_1} \rho^3 \cos\theta \sin\theta d\rho d\theta = 2 \int_0^{\frac{\pi}{3}} d\theta \int_1^{2\cos\theta} \rho^3 \cos\theta \sin\theta d\rho = \frac{9}{8} \quad \text{三、将扇形}$$

形顶点放在坐标原点，取 y 轴为中心轴，则质心为 $(0, \bar{y})$

$$\bar{y} = \frac{1}{A} \iint_D y dxdy, A = \frac{1}{2} a^2 \times 2\alpha = \alpha a^2$$

$$\iint_D y dxdy = \iint_D \rho^2 \sin\theta d\rho d\theta = \iiint_{\Omega} z \rho d\rho d\theta dz = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} d\theta \int_0^a \rho^2 \sin\theta d\rho = \frac{2a^3}{3} \sin\alpha$$

$$\bar{y} = \frac{2a \sin\alpha}{3\alpha}, \quad \text{质心为} \left(0, \frac{2a \sin\alpha}{3\alpha}\right)$$

$$\text{四、} I_y = \iint_D x^2 dxdy = \iint_D \rho^3 \cos^2\theta d\rho d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2R\cos\theta} \rho^3 \cos^2\theta d\rho = \frac{5\pi R^4}{4}$$

$$\text{五、(1)} \quad V = \iint_D (x^2 + y^2) dxdy = \int_{-a}^a dx \int_{-a}^a (x^2 + y^2) dy = \frac{8a^4}{3}$$

$$(2) \quad \bar{x} = 0, \bar{y} = 0, \bar{z} = \frac{1}{V} \iiint_{\Omega} z dv = \frac{1}{V} \int_{-a}^a dx \int_{-a}^a dy \int_0^{x^2+y^2} z dz = \frac{7a^2}{15}$$

$$\text{质心为} \left(0, 0, \frac{7a^2}{15}\right)$$

$$(3) I_z = \iiint_{\Omega} (x^2 + y^2) \rho dv = \int_{-a}^a dx \int_{-a}^a dy \int_0^{x^2+y^2} (x^2 + y^2) \rho dz = \frac{112}{45} a^6 \rho$$

第十章 复习题

一、

$$1、 \int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y) dx \quad 2、 e + e^{-1} - 2 \quad 3、 \frac{4\pi R^5}{15} \quad 4、 4\pi R^3$$

二、 $B \quad C \quad A$

$$三、 \text{原式} = \iint_D \theta d\rho d\theta = \int_0^{\frac{\pi}{4}} d\theta \int_1^3 \theta d\rho = \frac{\pi^2}{16}$$

$$四、 \text{原式} = \iiint_{\Omega} \rho^3 d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^4 d\rho \int_{\frac{1}{2}\rho^2}^8 \rho^3 dz = \frac{4^5}{3} \pi$$

$$五、 A = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy = \iint_D \sqrt{1 + \frac{c^2}{a^2} + \frac{c^2}{b^2}} dxdy = \frac{1}{2} \sqrt{a^2 b^2 + b^2 c^2 + a^2 c^2}$$

$$六、 \text{原式} = \iiint_{\Omega} r^4 \sin^2 \varphi \cos \varphi \sin \theta dr d\varphi d\theta = \int_0^{\pi} d\theta \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sqrt{2}} r^4 \sin^2 \varphi \cos \varphi \sin \theta dr = \frac{4}{15}$$

习题二十六

$$一、 1、 \int_0^{2\pi} a^3 t(1+t^2) dt \quad 2、 \sqrt{2}$$

二、 $B \quad A$

$$三、 1、 \text{原式} = \sqrt{2} + 1 + 1 = \sqrt{2} + 2$$

2、

$$\text{原式} = \int_0^2 \frac{1}{(e^t \cos t)^2 + (e^t \sin t)^2 + (e^t)^2} \sqrt{(x')^2 + (y')^2 + (z')^2} dt = \int_0^2 \frac{\sqrt{3}e^t}{2e^{2t}} dt = \frac{\sqrt{3}}{2} (1 - \frac{1}{e^2})$$

$$3、 \text{原式} = \int_{\overline{OA}} (x+y) ds + \int_{\overline{AB}} (x+y) ds + \int_{\overline{OB}} (x+y) ds = \int_0^1 x dx + \sqrt{2} + \int_0^1 y dy = 1 + \sqrt{2}$$

$$4、 \text{原式} = \oint_L R^{2n} ds = R^{2n} \square_S = 2\pi R^{2n+1}$$

$$5、 AB \text{ 的方程为 } \frac{x}{0} = \frac{y}{0} = \frac{z}{1}, \text{ 即参数方程为 } x=0, y=0, z=t$$

同理可得 BC, CD 的参数方程分别为

$$x=t, y=0, z=2 \quad x=1, y=t, z=2$$

$$I = \int_{AB} x^2 y z ds + \int_{BC} x^2 y z ds + \int_{CD} x^2 y z ds = 0 + 0 + \int_0^3 2t dt = 9$$

习题二十七

一、1、 $-\frac{39}{4}$

2、 $\int_0^1 (10t^3 + 5t^2 + 9t + 2)dt, \quad \frac{32}{3}$

二、 $B \quad C$

三、1、(1) 原式 $=\int_0^1 2xdx=1$

(2) 原式 $=\int_0^1 [(x+x^2)+(x-x^2)\cdot 2x]dx=1$

(3) 原式 $=\int_0^1 (0-y)dy+\int_0^1 (x+1)dx=1$

2、圆弧的参数方程为: $x=\cos t, y=\sin t$

原式 $=\int_0^\pi [\cos t \sin^2 t \cos t - \cos^2 t \sin t (-\sin t)]dt = \frac{\pi}{4}$

3、圆的参数方程为: $x=a+a\cos t, y=a\sin t$

原式 $=-\int_0^{2\pi} a(1+\cos t)a\sin t(-a\sin t)dt = \pi a^3$

习题二十八

一、1、 $\oint_L (3x+y)dx + (2y-x)dy \quad \iint_D -2dxdy \quad -4\pi$

2、 $x\frac{\partial F}{\partial y} = y\frac{\partial F}{\partial x}$

二、 $D \quad D$

三、1、 $P=x^2y, Q=y^2x$

$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\sigma = \iint_D (y^2 - x^2) d\sigma = \iint_D \rho^3 (\sin^2 \theta - \cos^2 \theta) d\rho d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \int_0^{2\cos\theta} \rho^3 (\sin^2 \theta - \cos^2 \theta) d\rho = -\pi$

2、 $I = \frac{1}{R^2} \oint_L xdy - ydx = \frac{1}{R^2} \iint_D 2d\sigma = \frac{2}{R^2} \times \pi R^2 = 2\pi$

3、 $P=2x-y+4, Q=3x+5y-6$

$I = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\sigma = \iint_D 4d\sigma = 4\sigma = 12$

四、 $P = 2x \cos y - y^2 \sin x, Q = 2y \cos x - x^2 \sin y$

$$\frac{\partial P}{\partial y} = -2x \sin y - 2y \sin x, \frac{\partial Q}{\partial x} = -2y \sin x - 2x \sin y$$

$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \therefore$ 积分与路径无关

$$\text{原式} = \int_0^2 2x dx + \int_0^3 (2y \cos 2 - 4 \sin y) dy = 9 \cos 2 + 4 \cos 3$$

习题二十九

三、1、

2、 Σ 的方程为: $z = 4 - x^2 - y^2$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy = \sqrt{1 + 4x^2 + 4y^2} dxdy$$

$$\text{原式} = \iint_{D_{xy}} (2 - x^2 - y^2) \sqrt{1 + 4x^2 + 4y^2} dxdy = \frac{37\pi}{10}$$

3、 Σ 的方程为: $z = \sqrt{3(x^2 + y^2)}$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy = \frac{2}{\sqrt{3}} dxdy$$

$$\text{原式} = \iint_{D_{xy}} (x^2 + y^2) \frac{2}{\sqrt{3}} dxdy = \frac{2}{\sqrt{3}} \iint_{D_{xy}} \rho^3 d\rho d\theta = \frac{2}{\sqrt{3}} \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \rho^3 d\rho = 3\sqrt{3}\pi$$

3、 $\Sigma: z = -\sqrt{a^2 - x^2 - y^2}$

$$ds = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy = \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy$$

原式=

$$\begin{aligned} & \iint_{D_{xy}} (x + y - \sqrt{a^2 - x^2 - y^2}) \cdot \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy \\ &= \iint_{D_{xy}} \frac{ax}{\sqrt{a^2 - x^2 - y^2}} dxdy - \iint_{D_{xy}} \frac{ay}{\sqrt{a^2 - x^2 - y^2}} dxdy - \iint_{D_{xy}} adxdy \\ &= -a\sigma = -\pi a^3 \end{aligned}$$

习题三十

一、1、 $\iint_{\Sigma} (P \cos \alpha + Q \cos \beta + R \cos \gamma) ds$ 2、0

二、1、C 2、C

三、1、原式 = $2 \iint_{D_{xy}} (2-x-y) dx dy = 2 \int_0^2 dx \int_0^{2-x} (2-x-y) dy = \frac{8}{3}$

2、 $\Sigma: z = -\sqrt{a^2 - x^2 - y^2}$

原式 = $-\iint_{D_{xy}} x^2 y^2 (-\sqrt{a^2 - x^2 - y^2}) dx dy = \iint_{D_{xy}} \rho^5 \cos^2 \theta \sin^2 \theta \sqrt{a^2 - \rho^2} d\rho d\theta$

= $\int_0^{2\pi} d\theta \int_0^a \rho^5 \cos^2 \theta \sin^2 \theta \sqrt{a^2 - \rho^2} d\rho = \frac{2\pi a^7}{105}$

3、

原式 = $\frac{1}{8}$

四、(1) $\vec{n} = (3, 2, 2\sqrt{3}) \quad \vec{e}_{\vec{n}} = \frac{\vec{n}}{|\vec{n}|} = (\frac{3}{5}, \frac{2}{5}, \frac{2\sqrt{3}}{5}) = (\cos \alpha, \cos \beta, \cos \gamma)$

原式 = $\iint_{\Sigma} \left[\frac{2\sqrt{3}}{5} R + \frac{3}{5} P + \frac{2}{5} Q \right] ds$

(2) $\vec{n}' = (-2x, -2y, -1) \quad$ 外侧法向量 $\vec{n} = (2x, 2y, 1)$

$\vec{e}_{\vec{n}} = \frac{\vec{n}}{|\vec{n}|} = (\frac{2x}{\sqrt{1+4x^2+4y^2}}, \frac{2y}{\sqrt{1+4x^2+4y^2}}, \frac{1}{\sqrt{1+4x^2+4y^2}}) = (\cos \alpha, \cos \beta, \cos \gamma)$

原式 = $\iint_{\Sigma} \left[\frac{R}{\sqrt{1+4x^2+4y^2}} + \frac{2xP}{\sqrt{1+4x^2+4y^2}} + \frac{2yQ}{\sqrt{1+4x^2+4y^2}} \right] ds$

习题三十一

一、1、 108π 2、 $ye^{xy} - x \sin xy - 2z \cos(xz^2)$

3、(2, 4, 6)

二、1、原式 = $\iiint_{\Omega} (z^2 + x^2 + y^2) dv = \iiint_{\Omega} r^4 \sin \varphi dr d\theta d\varphi$

= $\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^a r^4 \sin \varphi dr = \frac{2\pi a^5}{5}$

2、原式 = $\iiint_{\Omega} (1+1+1) dv = 3V = 81\pi$

3、原式 = $\iiint_{\Omega} (4z - 2y + y) dv = \int_0^1 dx \int_0^1 dy \int_0^1 (4z - y) dz = \frac{3}{2}$

三、1、 -20π

2、0

3、 9π

第十一章 复习题

一、 1、 $\frac{3}{2}$ 2、 $-\pi$ 3、 $\iiint_V \frac{\partial P}{\partial x} dv$ 4、 $\frac{4\pi a^3}{3}$

二、 **B**

三、 1、 π 2、 $-3\pi ab$ 3、 288π

四、 $I = \sqrt{3}\pi R^2$

五、 $I = \frac{15}{2}$

习题 三十二 常数项级数的概念与性质

一、 \times \times \sqrt \times

二 **D B A**

三 1、 1

2、 $u_1 - u_{n+1} = u_1$;

3、 $\frac{1}{(2n+1)(2n-1)}$

4、 2

四 发散; 发散; 发散; 发散; 发散

五 \because 级数 $\sum_{n=1}^{\infty} (n+1)(u_{n+1} - u_n)$ 收敛

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} s_n &= 2(u_2 - u_1) + 3(u_3 - u_2) + \cdots + (n+1)(u_{n+1} - u_n) \text{ 存在} \\ &= -(u_1 + u_2 + u_3 + \cdots + u_n) + (n+1)u_{n+1} - u_1 \end{aligned}$$

而 $\lim_{n \rightarrow \infty} nu_n = 0$, 得到级数 $\sum_{n=1}^{\infty} u_n$ 的部分和收敛, 得到此级数收敛.

习题三十三 正项级数及审敛法

一 \times \sqrt \sqrt

二 1、 $p < -2$;

2、 $+\infty$

3、 $\alpha > \frac{1}{2}$

三 1、 $\lim_{n \rightarrow \infty} \frac{1+n^2}{\frac{1}{n}} = 1$, 此级数发散;

2、 $\lim_{n \rightarrow \infty} \frac{\sin \frac{\pi}{2^n}}{\frac{\pi}{2^n}} = 1$, 此级数收敛;

3、 $\lim_{n \rightarrow \infty} \frac{\tan \frac{\pi}{\sqrt{n^3 + n + 1}}}{\frac{\pi}{\sqrt{n^3 + n + 1}}} = 1$, 此级数收敛;

4、 $\alpha > 1$ 时收敛, $\alpha \leq 1$ 时发散

四、 1 发散; 2 收敛; 3 收敛

五、 收敛

六、级数 $\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$, $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{2(n+1)n^n}{(n+1)^{n+1}} = \lim_{n \rightarrow \infty} 2 \left[\left(1 - \frac{1}{n+1}\right)^{-(n+1)} \right]^{-1} \left(1 - \frac{1}{n+1}\right)^{-1} = \frac{2}{e}$

此级数收敛, 得 $\lim_{n \rightarrow \infty} \frac{2^n n!}{n^n} = 0$

习题 三十四 交错级数, 绝对收敛与条件收敛

一 C D C C

二 1 绝对收敛; 2 发散;

3 $|a| \leq 1$ 时绝对收敛, $|a| > 1$ 发散;

4 绝对收敛; 5 条件收敛;

6 条件收敛

三 $|u_n v_n| \leq \frac{u_n^2 + v_n^2}{2}$, $(u_n + v_n)^2 \leq 2(u_n^2 + v_n^2)$, 即可得到级数收敛.

习题三十五 幂级数

一 B D D A B

二 1、 $[-3, 3)$;

2、 $(-\sqrt{2}, \sqrt{2})$;

3、 $[4, 6)$

三 1 $(-1, 1)$, $s(x) = \frac{x^2}{(1-x)^2}$;

2、

$$(-1,1), \quad s(x) = \frac{1}{2} \ln \left| \frac{x+1}{x-1} \right|,$$

$$x = \frac{\sqrt{2}}{2}, \quad \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

习题三十六 函数展开成泰勒级数

一 1、 $\sum_{n=1}^{\infty} (n+1)! x^{n-1};$

2、 $a_n = \frac{(-1)^n}{2^{2n+2}};$

3、 $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(4n+1)(2n)!};$

4、 $\sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n$

5、 $\sum_{n=1}^{\infty} \frac{nx^{n-1}}{(n+1)!}, \quad 1$

二 1、 $\frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} 4^n \frac{x^{2n}}{(2n)!}, x \in R;$

2、 $x + \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)(n+2)} x^{n+2}, x \in (-1,1]$

三、 $\frac{1}{x} = \frac{1}{3(1+\frac{x-3}{3})} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{n+1}} (x-3)^n, x \in (-10,6)$

四、

$$\frac{1}{x^2+3x+2} = \frac{1}{1+x} - \frac{1}{2+x} = -\frac{1}{3(1-\frac{x+4}{3})} + \frac{1}{2(1-\frac{x+4}{2})}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}} \right) (x+4)^n, x \in (-6, -2)$$

五 $\left. \frac{(\ln x)^{(n)}}{n!} \right|_{x=2} = (-1)^{n-1} \frac{1}{n2^n}, \ln x = \ln 2 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n2^n} (x-2)^n, x=1, \ln 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n}$

习题三十七傅里叶级数

一、 B B A

二、 $s(-\frac{3}{2}\pi) = 1 + \frac{\pi}{2}, s(\pi) = 1, s(2\pi) = 1, s(\frac{197}{2}\pi) = 1 + \frac{\pi}{2}$

三、

$$x = 2k\pi, f(x) = \frac{1}{2}, x = (2k+1)\pi, f(x) = \frac{1}{2}$$

$$b_n = \frac{1}{\pi} \left[\int_{-\pi}^0 x \sin nx dx + \int_0^{\pi} (x+1) \sin nx dx \right] = \frac{2}{\pi} \int_0^{\pi} x \sin nx dx + \frac{1}{\pi} \int_0^{\pi} \sin nx dx$$

$$= \frac{-2}{\pi n} x \cos nx \Big|_0^{\pi} + \frac{2}{\pi n^2} \sin nx \Big|_0^{\pi} - \frac{1}{\pi} \frac{\cos nx}{n} \Big|_0^{\pi} = \begin{cases} \frac{2\pi+2}{n\pi}, n=1,3,\dots \\ -\frac{2}{n}, n=2,4,\dots \end{cases}$$

$$a_0 = \frac{1}{\pi} \left[\int_{-\pi}^0 x dx + \int_0^{\pi} (x+1) dx \right] = \frac{1}{\pi} \int_0^{\pi} dx = 1$$

$$a_n = \frac{1}{\pi} \left[\int_{-\pi}^0 x \cos nx dx + \int_0^{\pi} (x+1) \cos nx dx \right] = \frac{2}{\pi} \int_0^{\pi} x \cos nx dx + \frac{1}{\pi} \int_0^{\pi} \cos nx dx$$

$$= \frac{2}{\pi n} x \sin nx \Big|_0^{\pi} + \frac{2}{\pi n^2} \cos nx \Big|_0^{\pi} + \frac{1}{\pi} \frac{\sin nx}{n} \Big|_0^{\pi} = (-1)^n \frac{4}{n^2 \pi},$$

$$f(x) = \frac{1}{2} + \left(\frac{2\pi+2}{\pi} \cos x - \frac{4}{n^2 \pi} \sin x \right) + \dots (x \in R, x \neq k\pi)$$

四、

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx = \frac{2}{\pi} \left[\frac{\sin nx}{n} \right]_0^{\pi} - \frac{4}{n\pi} \int_0^{\pi} x \sin nx dx$$

$$= (-1)^n \frac{4}{n^2}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{2\pi^2}{3}$$

$$b_n = 0$$

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx, x \in [-\pi, \pi] \quad f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^2} \cos nx, x \in [-\pi, \pi]$$

$$x=0, \frac{\pi^2}{3} + 4(-1 + \frac{1}{2^2} + \dots) = 0$$

$$x=\pi, \frac{\pi^2}{3} + 4(1 + \frac{1}{2^2} + \dots) = \pi^2$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

复习题

一 $\sqrt{\quad} \sqrt{\quad} \times \times \times$

二 C C D

三 1、 $[-1,1)$;

2、 $a>1$;

3、 2 ;

4、 $1 \quad 2$;

5、 $x = \frac{\pi}{2} - \frac{4}{\pi} (\cos x + \frac{1}{3^2} \cos 3x + \dots), x \in [0, \pi]$;

6、 $\sum_{n=0}^{\infty} \frac{1}{2^{2n+2}} x^{2n+1}$

四、 1 发散; 2 收敛; 3 收敛; 4 发散

五、 1 条件收敛; 2 条件收敛

六、 $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \frac{3^{n+1} + 5^{n+1}}{3^n + 5^n} \frac{n}{n+1} = 5, R = \frac{1}{5}, [-\frac{1}{5}, \frac{1}{5})$

七、 $s(x) = \arctan x, x \in (-1, 1]$

八、 $2e$

自测题一

一、单项选择题

(1)	(2)	(3)	(4)	(5)
D	C	B	A	A

二、填空题

(6)	(7)	(8)	(9)	(10)
$y = \frac{1}{3}x^3 + \sin x + C_1x + C_2$	$2dx + 2dy$	10	$\frac{x-1}{0} = \frac{y-1}{1} = \frac{z-2}{2}$	$\frac{2\pi}{\quad}$

三、计算题 (每小题 6 分, 共 48 分)

11、解: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2}{\sqrt{4-x^2-y^2}-2} = \lim_{(x,y) \rightarrow (0,0)} \frac{x^2+y^2(\sqrt{4-x^2-y^2}+2)}{4-x^2-y^2-4}$

$$= \lim_{(x,y) \rightarrow (0,0)} -(\sqrt{4-x^2-y^2}+2)$$

$$= -4$$

12、设 $z = x^3y^2 - y^{\sin y} - x^2 \sin x$, 求 $\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}}$

解: $\frac{\partial z}{\partial x} = 3x^2y^2 - 2x \sin x - x^2 \cos x$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y \quad \left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}} = 6$$

13、求通过点 $P(-1, -2, 1)$ 、 $Q(1, -2, -3)$ 且垂直于平面 $x - 2y + 3z - 4 = 0$ 平面方程

解: $\overrightarrow{PQ} = (2, 0, -4),$

于是所求平面的法向量为: $\vec{n} = (2, 0, -4) \times (1, -2, 3) = (-8, -10, -4) = -2(4, 5, 2)$

故所求平面方程为: $4(x+1)+5(y+2)+2(z-1)=0$,

$$\text{即 } 4x+5y+2z+12=0$$

14、计算 $I = \int_L (x+y+z)ds$, 其中 L 为折线 ABC , 这里 $A(0,0,0), B(0,0,2), C(1,0,2)$

$$\text{解: } AB: \frac{x}{0} = \frac{y}{0} = \frac{z}{1}, \quad \text{即 } AB: \begin{cases} x=0 \\ y=0, t: 0 \rightarrow 2 \\ z=t \end{cases}$$

$$BC: \frac{x}{1} = \frac{y}{0} = \frac{z-2}{0}, \quad \text{即 } BC: \frac{x}{1} = \frac{y}{0} = \frac{z-2}{0}: \begin{cases} x=t \\ y=0, t: 0 \rightarrow 1 \\ z=2 \end{cases}$$

$$\begin{aligned} I &= \int_{AB} (x+y+z)ds + \int_{BC} (x+y+z)ds \\ &= \int_0^2 t\sqrt{0+0+1}dt + \int_0^1 (t+2)\sqrt{1+0+0}dt \\ &= \frac{9}{2} \end{aligned}$$

15、计算 $\int_0^1 dx \int_x^1 e^{\frac{x}{y}} dy$

$$\begin{aligned} \text{解: } \int_0^1 dx \int_x^1 e^{\frac{x}{y}} dy &= \int_0^1 dy \int_0^y e^{\frac{x}{y}} dx \\ &= \int_0^1 y(e-1)dy, \\ &= \frac{e-1}{2} \end{aligned}$$

16、计算 $I = \iiint_{\Omega} (x^2 + y^2)dv$, Ω 为平面曲线 $\begin{cases} x^2 = 2z \\ y = 0 \end{cases}$ 绕 Z 轴旋转一周形成的曲面 Σ 与平面 $z=2$ 围成的区域

解: (1) 旋转曲面 Σ 为 $2z = x^2 + y^2$

$$(2) \iiint_{\Omega} (x^2 + y^2)dv = \iiint_{\Omega} \rho^3 d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^2 d\rho \int_{\frac{1}{2}\rho^2}^2 \rho^3 dz = \frac{16\pi}{3}$$

17、计算 $I = \oiint_{\Sigma} (x-y+1)dydz + (y-z+2)dzdx + (z-x+3)dxdy$, 其中 Σ 是球面

$x^2 + y^2 + z^2 = 2x$ 的外侧

解: 令 $P = x - y + 1$, $Q = y - z + 2$, $R = z - x + 3$, Ω 是球面 $x^2 + y^2 + z^2 = 2x$ 围成的闭区域,

由高斯公式, (2 分)

$$I = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \iiint_{\Omega} (1+1+1) dv = \iiint_{\Omega} 3 dv = 4\pi$$

18、判断级数 $\sum_{n=1}^{\infty} \frac{n^3-1}{2^n}$ 的敛散性

解: 令 $u_n = \frac{n^3-1}{2^n}$ (1 分)

$$\text{由于 } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^3-1}{2^{n+1}}}{\frac{n^3-1}{2^n}} = \frac{1}{2} < 1$$

故 级数 $\sum_{n=1}^{\infty} \frac{n^3-1}{2^n}$ 收敛。

四、综合应用题

19、设曲线积分 $\int_L 2xe^y dx + e^y f(x) dy$ 与路径无关, 其中 $f(x)$ 具有连续的导数, 且 $f(0) = 0$.

(1) 求函数 $f(x)$ 的表达式; (2) 求 $\int_{(0,0)}^{(1,1)} 2xe^y dx + e^y f(x) dy$.

解: (1) 令 $P = 2xe^y$, $Q = e^y f(x)$, 由于曲线积分 $\int_L 2xe^y dx + e^y f(x) dy$ 与路径无关,

则 $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, 于是, $2xe^y = e^y f'(x)$, 即 $f'(x) = 2x$

则 $f(x) = x^2 + C$, 由于 $f(0) = 0$, 于是 $C = 0$, 故 $f(x) = x^2$

(2)

$$\begin{aligned} \int_{(0,0)}^{(1,1)} 2xe^y dx + e^y f(x) dy &= \int_{(0,0)}^{(1,1)} 2xe^y dx + e^y x^2 dy = \int_0^1 2xdx + \int_0^1 e^y dy \\ &= e \end{aligned} \quad (10 \text{ 分})$$

20、解 (1) 由于 $\lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^2}{(2n+1)(2n+2)} \right| = 0$, 故收敛域为 $(-\infty, +\infty)$

$$(2) \quad y'(x) = \left(\sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} \right)' = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!}$$

$$y''(x) = \left(\sum_{n=1}^{\infty} \frac{x^{2n-1}}{(2n-1)!} \right)' = 1 + \sum_{n=2}^{\infty} \frac{x^{2n-2}}{(2n-2)!} = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!}$$

$$\text{于是 } y'' - y = 1 + \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} - \sum_{n=1}^{\infty} \frac{x^{2n}}{(2n)!} = 1$$

(3) 由 $r^2 - 1 = 0$ 得 $r = \pm 1$ ，于是微分方程的对应齐次方程的通解为

$$Y(x) = C_1 e^x + C_2 e^{-x}$$

又显然 $y^* = -1$ 是微分方程的一个特解，于是微分方程的通解为

$$y(x) = C_1 e^x + C_2 e^{-x} - 1$$

由于 $y(0) = 0, y'(0) = 0$ ，于是 $C_1 = C_2 = \frac{1}{2}$ ，

$$\text{所以 } y(x) = \frac{e^x + e^{-x}}{2} - 1$$

自测题二

一、单项选择题

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
B	B	A	C	A	B	D	C	C	B

二、填空题（本大题共 5 小题，每小题 3 分，共 15 分）

(11)	(12)	(13)	(14)	(15)
-2	$\int_0^2 dy \int_0^{\frac{y}{2}} f(x, y) dx$	0	$(-3, 3)$	0

三、求解下列各题

(16) 解：平面 $3x - y + z - 2 = 0$ 的法向量 $\vec{n}_1 = (3, -1, 1)$ ， $\vec{PQ} = (-2, 2, 4)$ ，

由题意得所求平面的法向量

$$\vec{n} = \vec{n}_1 \times \vec{PQ} = (3, -1, 1) \times (-2, 2, 4) = (-6, -14, 4) = -2(3, 7, -2),$$

故所求平面方程为 $3(x-1) + 7(y+2) - 2(z+1) = 0$ ，

$$\text{即 } 3x + 7y - 2z + 9 = 0$$

(17) 解：设 $F(x, y, z) = x + 2y + z - ye^{xyz}$ ，则 $F_x = 1 - y^2 ze^{xyz}$ ，

$$F_y = 2 - e^{xyz} - xye^{xyz}, \quad F_z = 1 - xy^2 e^{xyz}$$

$$\text{于是 } \left. \frac{\partial z}{\partial x} \right|_{\substack{x=1 \\ y=0}} = - \frac{F_x}{F_z} \bigg|_{\substack{x=1 \\ y=0}} = -1, \quad \left. \frac{\partial z}{\partial y} \right|_{\substack{x=1 \\ y=0}} = - \frac{F_y}{F_z} \bigg|_{\substack{x=1 \\ y=0}} = -1$$

故
$$dz \Big|_{\substack{x=1 \\ y=0}} = -dx - dy$$

(18) 解: 令 $u(x, y) = 2x - y$, $v(x, y) = 3x - 2y$ 则 $u(1, 1) = 1$, $v(1, 1) = 1$,

于是
$$\frac{\partial z}{\partial x} \Big|_{\substack{x=1 \\ y=1}} = \left(\frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} \right) \Big|_{\substack{x=1 \\ y=1}} = (2vu^{v-1} + 3u^v \ln u) \Big|_{\substack{x=1 \\ y=1}} = 2$$

$$\frac{\partial z}{\partial y} \Big|_{\substack{x=1 \\ y=1}} = \left(\frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} \right) \Big|_{\substack{x=1 \\ y=1}} = (-vu^{v-1} - 2u^v \ln u) \Big|_{\substack{x=1 \\ y=1}} = -1$$

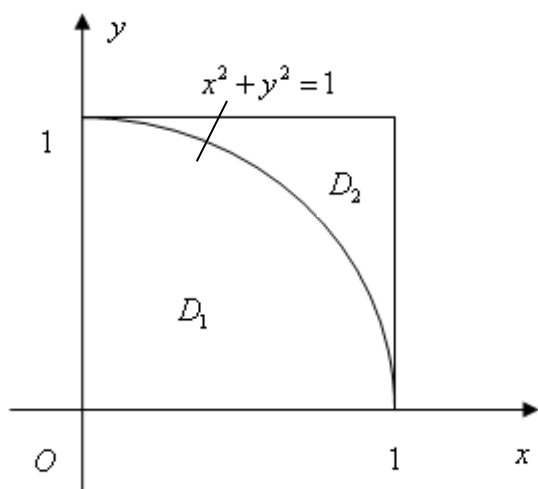
(19) 解: 解方程组 $\begin{cases} f_x(x, y) = 6 - 6x = 0 \\ f_y(x, y) = -2 - 2y = 0 \end{cases}$ 得驻点 $(1, -1)$

又 $A = f_{xx}(1, -1) = -6 < 0$, $B = f_{xy}(1, -1) = 0$, $C = f_{yy}(1, -1) = -2$,

则 $AC - B^2 > 0$, 于是函数在 $(1, -1)$ 处有极大值 $f(1, -1) = 4$

(20) 计算二重积分 $\iint_D |x^2 + y^2 - 1| d\sigma$, 其中 $D = \{(x, y) | 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

解 如图所示, 把 D 分成 D_1 与 D_2 两部分,



$$\iint_D |x^2 + y^2 - 1| d\sigma$$

$$= \iint_{D_1} |x^2 + y^2 - 1| d\sigma + \iint_{D_2} |x^2 + y^2 - 1| d\sigma,$$

$$\text{由于 } \iint_{D_2} |x^2 + y^2 - 1| d\sigma = \int_0^{\frac{\pi}{2}} d\theta \int_0^1 (1 - \rho^2) \rho d\rho = \frac{\pi}{8}$$

$$\iint_{D_1} |x^2 + y^2 - 1| d\sigma$$

$$= \int_0^1 dx \int_{\sqrt{1-x^2}}^1 (x^2 + y^2 - 1) dy$$

$$= \int_0^1 \left(x^2 - \frac{2}{3} + \frac{2}{3} (1 - x^2)^{\frac{3}{2}} \right) dx$$

$$= \frac{\pi}{8} - \frac{1}{3}$$

$$\text{因此, } \iint_D |x^2 + y^2 - 1| d\sigma = \frac{\pi}{8} + \frac{\pi}{8} - \frac{1}{3} = \frac{\pi}{4} - \frac{1}{3}$$

(21) 解:
$$\iiint_{\Omega} z dv = \iiint_{\Omega} z \rho d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} d\rho \int_{\frac{1}{2}\rho^2}^1 z \rho dz$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} \frac{1}{2} \rho (1 - \frac{1}{4} \rho^4) d\rho = \frac{2}{3} \pi$$

或:
$$\iiint_{\Omega} z dv = \int_0^1 z dz \iint_{x^2 + y^2 \leq 2z} dx dy = \int_0^1 2\pi z^2 dz = \frac{2}{3} \pi$$

(22) 解: L 的方程为 $\frac{x-1}{3} = \frac{y-2}{0} = \frac{z+2}{-4}$,

$$\text{即 } L \text{ 的参数方程为 } \begin{cases} x = 3t + 1 \\ y = 2 \\ z = -4t - 2 \end{cases} \quad (0 \leq t \leq 1)$$

$$\begin{aligned} \int_L (x + y + z) ds &= \int_0^1 (3t + 1 + 2 - 4t - 2) \sqrt{9 + 0 + 16} dt \\ &= \frac{5}{2} \end{aligned}$$

(23) 解: 令 $P = x^3 - y$, $Q = x - y^3$, 由格林公式得

$$\begin{aligned} &\oint_L (x^3 - y) dx + (x - y^3) dy \\ &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) d\sigma = \iint_D 2 d\sigma \\ &= 2 \times \frac{1}{2} = 1 \end{aligned}$$

或:

AB 的方程为 $y = 0$, x 从 0 变到 1, BC 的方程为 $x = 1$, y 从 0 变到 1,

CA 的方程为 $y = x$, x 从 1 变到 0,

$$\begin{aligned} &\oint_L (x^3 - y) dx + (x - y^3) dy \\ &= \int_{AB} (x^3 - y) dx + (x - y^3) dy + \int_{BC} (x^3 - y) dx + (x - y^3) dy + \int_{CA} (x^3 - y) dx + (x - y^3) dy \\ &= \int_0^1 x^3 dx + \int_0^1 (1 - y^3) dx + \int_1^0 (x^3 - x + x - x^3) dx = 1 \end{aligned}$$

(24) 解: 设 $\Sigma_1: \begin{cases} z = 1 \\ x^2 + y^2 \leq 1 \end{cases}$ 取下侧, 记由 Σ, Σ_1 所围立体为 Ω , 则高斯公式可得

$$\begin{aligned} \iint_{\Sigma + \Sigma_1} (x-1)^3 dydz + (y-1)^3 dzdx + (z-1) dxdy &= - \iiint_{\Omega} (3(x-1)^2 + 3(y-1)^2 + 1) dxdydz \\ &= - \iiint_{\Omega} (3x^2 + 3y^2 + 7 - 6x - 6y) dxdydz \\ &= - \iiint_{\Omega} (3x^2 + 3y^2 + 7) dxdydz \\ &= - \int_0^{2\pi} d\theta \int_0^1 r dr \int_{r^2}^1 (3r^2 + 7) dz = -4\pi \end{aligned}$$

$$\text{在 } \Sigma_1: \begin{cases} z = 1 \\ x^2 + y^2 \leq 1 \end{cases} \text{ 取下侧上, } \iint_{\Sigma_1} (x-1)^3 dydz + (y-1)^3 dzdx + (z-1) dxdy = \iint_{\Sigma_1} (1-1) dxdy = 0,$$

$$\text{所以 } \iint_{\Sigma} (x-1)^3 dydz + (y-1)^3 dzdx + (z-1) dxdy = \iint_{\Sigma + \Sigma_1} (x-1)^3 dydz + (y-1)^3 dzdx + (z-1) dxdy = -4\pi$$

(25) 解: 由于 $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$, $-1 < x < 1$

$$\text{故 } \frac{1}{x} = \frac{1}{1+(x-1)} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n, \quad 0 < x < 2$$

四、证明题

(26) 已知平面区域 $D = \{(x, y) | 0 \leq x \leq \pi, 0 \leq y \leq \pi\}$, L 为 D 的正向边界. 证明:

$$(1) \oint_L x e^{\sin y} dy - y e^{-\sin x} dx = \oint_L x e^{-\sin y} dy - y e^{\sin x} dx; \quad (2) \oint_L x e^{\sin y} dy - y e^{-\sin x} dx \geq 2\pi^2.$$

证明 (1) 左边 $= \int_0^\pi \pi e^{\sin y} dy - \int_\pi^0 \pi e^{-\sin x} dx = \pi \int_0^\pi (e^{\sin x} + e^{-\sin x}) dx,$

$$\text{右边} = \int_0^\pi \pi e^{-\sin y} dy - \int_\pi^0 \pi e^{\sin x} dx = \pi \int_0^\pi (e^{\sin x} + e^{-\sin x}) dx$$

$$\text{故 } \oint_L x e^{\sin y} dy - y e^{-\sin x} dx = \oint_L x e^{-\sin y} dy - y e^{\sin x} dx$$

$$(2) \text{ 由于 } e^{\sin x} + e^{-\sin x} \geq 2,$$

$$\text{所以 } \oint_L x e^{\sin y} dy - y e^{-\sin x} dx = \pi \int_0^\pi (e^{\sin x} + e^{-\sin x}) dx \geq 2\pi^2$$

自测题三

一、单项选择题

(1)	(2)	(3)	(4)	(5)
D	D	C	A	c

二、填空题

(6)	(7)	(8)	(9)	(10)
<u>2</u>	$-\sqrt{2}$	<u>$x - y - 3z + 16 = 0$</u>	1	0

三、计算题

11、解: $\lim_{(x,y) \rightarrow (1,0)} \frac{3-(xy)^2 - e^{xy}}{x^3 + y^3} = \frac{3-0-e^0}{1^3+0^3} = 2$

12、设 $z = x^2 y^2 - 3y^3 - x^2 \cos x$ ，求 $\left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}}$ ；

解： $\frac{\partial z}{\partial x} = 2xy^2 - 2x \cos x + x^2 \sin x$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xy \quad \left. \frac{\partial^2 z}{\partial x \partial y} \right|_{\substack{x=1 \\ y=1}} = 4$$

13、解： $\overrightarrow{PQ} = (-1, 3, 4)$ ，

于是所求平面的法向量为： $\vec{n} = (2, 3, -5) \times (-1, 3, 4) = (27, -3, 9) = 3(9, -1, 3)$

故所求平面方程为： $9(x+1) - (y+1) + 3(z+7) = 0$ ，

$$\text{即 } 9x - y + 3z + 29 = 0$$

14、解：令 $O(0,0)$ 、 $A(1,1)$ 、 $B(1,0)$

$OA: y = x(0 \leq x \leq 1)$ ， $AB: x = 1(0 \leq y \leq 1)$ ， $BO: y = 0(0 \leq x \leq 1)$

$$\begin{aligned} \int_L (x-y) ds &= \int_{OA} (x-y) ds + \int_{AB} (x-y) ds + \int_{BO} (x-y) ds \\ &= \int_0^1 (x-x) \sqrt{1+1} dx + \int_0^1 (1-y) \sqrt{1+0} dy + \int_0^1 (x-0) \sqrt{1+0} dx \\ &= 1 \end{aligned}$$

$$\begin{aligned} 15、\text{解：} \quad \int_{-1}^1 dx \int_0^1 y e^{xy} dy &= \int_0^1 dy \int_{-1}^1 y e^{xy} dx, \\ &= \int_0^1 (e^y - e^{-y}) dy = e + \frac{1}{e} - 2 \end{aligned}$$

$$16、\text{解：} \quad \iiint_{\Omega} z^2 dx dy dz = \iiint_{\Omega} z^2 \rho d\rho d\theta dz = \int_0^{2\pi} d\theta \int_0^2 d\rho \int_{\rho^2}^4 z^2 \rho dz = 64\pi$$

17、解：令 $P = x + 2y + 1$ ， $Q = y + 3z + 2$ ， $R = z + 4x + 3$ ， Ω 是平面 $|x|=1$ ， $|y|=1$ ，

$|z|=1$ 围成的闭区域，

由高斯公式，

$$I = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dv = \iiint_{\Omega} (1+1+1) dv = \iiint_{\Omega} 3 dv = 24$$

18、级数 $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n}$ 是否收敛？若收敛，是条件收敛，还是绝对收敛？

解: 令 $u_n = (-1)^n \frac{n^2}{3^n}$ (1 分)

$$\text{由于 } \lim_{n \rightarrow \infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n \rightarrow \infty} \frac{\frac{(n+1)^2}{3^{n+1}}}{\frac{n^2}{3^n}} = \frac{1}{3} < 1$$

故 级数 $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{3^n}$ 收敛且绝对收敛。

四、综合应用题

19、解: (1) 方程两边求导得

$$f(x) + \frac{1}{2} f'(x) = 2x$$

(2) 令 $y = f(x)$

$y' + 2y = 4x$ 为一阶线性微分方程, 其中 $P(x) = 2, Q(x) = 4x$

代入公式

$$y = e^{-\int P(x) dx} \left(\int Q(x) e^{\int P(x) dx} dx + C \right) = e^{-\int 2 dx} \left(\int 4x e^{\int 2 dx} dx + C \right) = 2x - 1 + C e^{-2x}$$

由 $f(0) = 0$ 得 $C = 1$. 原方程的解为

$$y = 2x - 1 + e^{-2x}$$

20、设函数 $f(u)$ 具有二阶连续导数, 函数 $z = f(e^x \sin y)$ 满足方程 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (z+1)e^{2x}$,

若 $f(0) = 0, f'(0) = 0$, 求函数 $f(u)$ 的表达式.

解 令 $u = e^x \sin y$, 则 $\frac{\partial z}{\partial x} = f'(u)e^x \sin y, \frac{\partial z}{\partial y} = f'(u)e^x \cos y$,

$$\frac{\partial^2 z}{\partial x^2} = f'(u)e^x \sin y + f''(u)e^{2x} \sin^2 y, \frac{\partial^2 z}{\partial y^2} = -f'(u)e^x \sin y + f''(u)e^{2x} \cos^2 y$$

代入 $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (z+1)e^{2x}$ 得 $f''(u) - f(u) = 1$

齐次方程 $f''(u) - f(u) = 0$ 的通解为 $f(u) = C_1 e^u + C_2 e^{-u}$, 方程 $f''(u) - f(u) = 1$ 的一个特解为 $f^*(u) = -1$, 故方程 $f''(u) - f(u) = 1$ 的通解为

$$f(u) = C_1 e^u + C_2 e^{-u} - 1.$$

由 $f(0) = 0, f'(0) = 0$ 得 $C_1 = C_2 = \frac{1}{2}$, 从而函数 $f(u)$ 的表达式为 $f(u) = \frac{e^u + e^{-u}}{2} - 1$

21、设 $a_n = \frac{1}{\pi} \int_0^{n\pi} x |\sin x| dx$, ($n=1, 2, \dots$), 分别求级数 $\sum_{n=1}^{\infty} \frac{1}{4a_n - 1}$ 与 $\sum_{n=1}^{\infty} \frac{(-1)^n}{4a_n - 1}$ 的和.

解 令 $x = n\pi - t$, 则

$$a_n = \frac{1}{\pi} \int_0^{n\pi} x |\sin x| dx = \frac{1}{\pi} \int_0^{n\pi} (n\pi - t) |\sin t| dt = n \int_0^{n\pi} |\sin t| dt - \frac{1}{\pi} \int_0^{n\pi} t |\sin t| dt$$

$$\text{所以 } a_n = \frac{n}{2} \int_0^{n\pi} |\sin t| dt = n^2 \quad (n=1, 2, \dots)$$

(1) 级数 $\sum_{n=1}^{\infty} \frac{1}{4a_n - 1} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$ 的部分和数列为

$$\begin{aligned} S_n &= \sum_{k=1}^n \frac{1}{4k^2 - 1} = \frac{1}{2} \sum_{k=1}^n \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right) \\ &= \frac{1}{2} \left[\left(1 - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right] \\ &= \frac{1}{2} \left(1 - \frac{1}{2n+1} \right) \end{aligned}$$

$$\text{所以 } \lim_{n \rightarrow \infty} S_n = \frac{1}{2}, \text{ 即 } \sum_{n=1}^{\infty} \frac{1}{4a_n - 1} = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$$

$$(2) \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4a_n - 1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{2n-1} - \frac{(-1)^n}{2n+1} \right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} + \frac{1}{2}$$

考虑幂级数 $\varphi(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} x^{2n-1}$, $-1 \leq x \leq 1$, 则逐项求导, 得

$$\varphi'(x) = \sum_{n=1}^{\infty} (-1)^n x^{2n-2} = \frac{-1}{1+x^2}, \quad -1 < x < 1$$

$$\text{于是 } \varphi(x) = \varphi(0) + \int_0^x \varphi'(x) dx = \int_0^x \frac{-1}{1+x^2} dx = -\arctan x \quad -1 \leq x \leq 1$$

$$\text{所以 } \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} = -\frac{\pi}{4}, \text{ 故 } \sum_{n=1}^{\infty} \frac{(-1)^n}{4a_n - 1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1} = -\frac{\pi}{4} + \frac{1}{2}$$