

PHYS 486

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Chapter 1

PHYS486

Quantum Mechanics lol. This is apparently a math language more than actual physics.

1.1 What is QM

- Classical Mechanics - $F = ma$, solve for x and p , position and momentum.
- Classical E & M - $\ddot{\vec{E}}(\vec{r}, t) - \nabla^2 \vec{E}(\vec{r}, t) = 0$

You have observable entities and you work with just the observable entities to figure out the exact numbers.

1.1.1 The Quantum State $|\psi\rangle$

This is a state vector.

The quantum state by definition is **not** observable.

An observable operator is something like position that describes the system that we can actually see.

$$\hat{A} |\psi\rangle = a |\psi\rangle$$

where \hat{A} is an observable and a is the measurement out.

This is just an eigenvalue equation.

1.1.2 Schrodinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H} |\psi(t)\rangle$$

\hat{H} is a matrix and $|\psi(t)\rangle$ is a vector.

1.2 Summary of "Central Phenomena"

Imagine an electron orbiting around a proton.

The electron **cannot** take arbitrary orbits.

The electron can only have **certain discrete** orbits.

1.2.1 Superpositions

Consider two discrete states in which an electron can orbit around a proton.

Because 0 and 1 are solutions for the electron, a superposition of 0 and 1 is also a solution for the electron.

$$\alpha |0\rangle + \beta |1\rangle$$

This is the electron being in both discrete states at the same time.

$$\sim \left[|0\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, SP \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]$$

1.3 Probabilistic Interpretation

Consider a system with the solution

$$\alpha |0\rangle + \beta |1\rangle$$

If we have a thing that measures the energy, we can get **either 0 or 1** with probabilities depending on the values α and β .

1.4 Entanglement

Consider atom A with state $\{|0\rangle_A, |1\rangle_A\}$ and atom B with state $\{|0\rangle_B, |1\rangle_B\}$.

It is completely legal to have an entangled state with the form

$$\alpha |0\rangle_A |0\rangle_B + \beta |1\rangle_A |1\rangle_B$$

1.5 Course Outline

1. Basic Rules
2. Wave Mechanics ("toy models")
3. Formalism (Midterm 1)
4. Simplest Real System (Hydrogen Atom) (Midterm 2)
5. Intro to Multiparticle Descriptions

Chapter 2

Actual Physics Now

2.1 Black Body Radiation

Consider an object that perfectly absorbs radiation and turns it into heat.

Place it in thermal equilibrium (finite T , constant E , absorption \rightarrow emission).

2.1.1 Classical Description

Consider the system as a harmonic oscillator.

$$\langle E \rangle = k_B T \Rightarrow I(\omega) \sim \frac{\omega^2 k_B T}{\pi^2 c^3} \quad \text{Rayleigh-Jeans Law}$$

This **does not** match experimental data.

While the classical description says that temperature will go up forever, the experiments show that the temperature stops increasing for large frequencies.

2.1.2 Planck, 1900

Complete guess that maybe energy is discretized. Maybe the harmonic oscillator comes in chunks of size $\hbar\omega$.

Now, energy can be written in the form

$$P(E) = \alpha e^{-\frac{E}{k_B T}} \longrightarrow \langle E \rangle = \frac{\hbar\omega}{\exp((\hbar\omega/k_B T) - 1)}$$
$$I(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 \exp((\hbar\omega/k_B T) - 1)}$$

This perfectly matches experimental data.

2.2 Photoelectric Effect

Consider a circuit that absorbs light with frequency ω and intensity J to create a potential.

As light intensity J increases, current I increases, but V_0 stays the same.

V_0 is dependent on ω and the properties of the material.

Einstein figured out that there is an energy threshold such that light will not be absorbed if it does not have a high enough energy.

The kinetic energy of the electron can be given in the form:

$$kE = \hbar\omega - W > -eV_0$$

2.3 Wave Particle Duality

Consider a photon with energy and momentum

$$E = \hbar\omega \quad p = \hbar k \left(= \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} \right)$$

The photon has no mass ($m = 0$), so consider the relativistic mass of the photon

$$E^2 = m^2c^4 + p^2c^2 \rightarrow p^2c^2 \rightarrow$$

$$E = pc \quad \omega = kc \quad f\lambda = c$$

2.3.1 De Broglie, 1924

Everything has a wavelength and frequency

$$\lambda = \frac{h}{p}$$

The wavelength for even electrons is extremely small, so this feature is very hard to observe even if its true for everything.

2.3.2 Davisson and Germer, 1927

Electrons act in the same wave-like nature as light when shot through a small slit (double slit experiment but for electrons).

This proved the De Broglie wavelength theory for particles.

2.4 Beginning of QM

The issue with quantum mechanics is that the math is derived only from experimental data. There is no classical physics basis or derivation behind quantum mechanics.

Multiple textbooks will bring up quantum mechanics in different ways. The professor recommends the Townsend QM textbook

Chapter 3

Ruleset

Motivated from experimental observations. These are base postulates with no derivations.

3.1 System State

The state of the physical system is a vector $|\Psi\rangle$ in Hilbert Space.

A Hilbert space ($\equiv H^2$) is a complex vector space with a well-behaved inner product.

(bra + ket = braket). You have to take the complex conjugate

$$\langle \Psi | \Psi \rangle \quad \text{"bra"} = (|\Psi\rangle^*)^T \quad \langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$$

3.2 Observables

Observables are operators in Hilbert Space, and they have to have real eigenvalues.

The measurement outcomes of the observable are the eigenvalues.

$$\hat{A} |\alpha_n\rangle = a_n |\alpha\rangle_n$$

I think \hat{A} is the observable of the state $|\alpha_n\rangle$ and a_n is the n -th eigenvalue of the observable.

3.2.1 Born's Rule

The probability for an outcome a_n is given by

$$P(a_n) = |\langle \alpha_n | \Psi \rangle|^2$$

Where α_n is an eigenvector of the operator.

3.2.2 Expected Value

Given an observable operator \hat{A} , the expected value is given by

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle$$

3.2.3 Waveform "Collapse"

Given a quantum state $|\Psi\rangle$, when you measure an observable \hat{A} , you return only a single result a_n

$$|\Psi\rangle \longrightarrow |\alpha_n\rangle$$

The system state collapses to just the individual eigenvector state.

3.3 Time Evolution (Schrodinger Equation)

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \quad \hat{H} = \text{Hamiltonian (Energy Operator)}$$

The Eigenvectors of \hat{H} are stationary states (do not change with time). This is the time-independent Schrodinger Equation

$$\hat{H} |\Psi\rangle = E_h |\Psi\rangle$$

Now you know the entirety of quantum mechanics. Everything else can be derived.

Chapter 4

Experiments

4.1 Double Slit

If you shine light through a small slit, you will see a diffraction pattern.

Imagine shining light at 2 slits and an attenuator at the other side that **measures** the location of each photon at the screen.

If you just shoot a single photon, the location will be random, but if you shoot multiple photons, there is a very distinct diffraction pattern that demonstrate the probability distribution of the list.

4.2 Stern-Gerlach

Consider a beam of atoms with varying spins. This beam of atoms is put into a magnetic field gradient.

If the magnetic moment is up, the atom will go up. If the magnetic moment is down, the atom will go down.

If we put the beam through this field, then half the atoms will go up, and half the atoms will go down (probabilistically).

This makes sense.

The quantum state of each atom is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle) \hat{z}$$

Consider a basis

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's call spin up +1 and spin down -1.

Because of those numbers, our observable operator is

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \hat{A} |\uparrow\rangle = +1 |\uparrow\rangle \quad \hat{A} |\downarrow\rangle = -1 |\downarrow\rangle$$

Now we have our quantum state and our observable operator matrix.

Because we have two possible eigenvectors, the superposition of our quantum state can be written as

$$\begin{aligned} |\Psi\rangle &= \left(\sum_k |k\rangle \langle k| \right) |\Psi\rangle \equiv \lambda_i, |k\rangle && \text{kth basis vector} \\ &= \sum_k c_k |k\rangle & c_k = \langle k|\Psi\rangle \end{aligned}$$

4.2.1 Born's Rule

$$P(+1) = |\langle \uparrow|\Psi\rangle|^2 = \frac{1}{2} |\langle \uparrow|\uparrow\rangle + \langle \uparrow|\downarrow\rangle|^2$$

Because we're using Dirac notation, we know exactly what inner products are parallel and what inner products are orthonormal.

$$P(+1) = |\langle \uparrow|\Psi\rangle|^2 = \frac{1}{2} |1 + 0|^2 = \frac{1}{2}$$

4.2.2 Expected Values

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle \quad \hat{A} = \sum_k \lambda_k |k\rangle \langle k| \quad \lambda_k = \text{kth eigenvalue}$$

You can use the Spectral Theorem to get

$$\hat{A} = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|$$

So to find the expected value, we use

$$\langle \hat{A} \rangle = \langle \Psi | (|\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|) | \Psi \rangle = \dots = 0$$

4.3 Double Slit and the Wave Function

we need $|\Psi\rangle$ in the position basis.

Imagine if the space was discrete.

$$|\Psi\rangle = \sum_k c_k |x\rangle$$

But our space is continuous, so instead of taking a discrete sum, we take an integral.

$$\langle x_k | x_j \rangle = \delta(k - j) \quad \int dx \langle x_k | x_j \rangle = 1$$

$$|\Psi\rangle = \int dx |x\rangle \langle x | \Psi \rangle = \int dx \Psi(x) |x\rangle$$

That's the wave function
implement Born's Rule to get

$$P(x) = |\langle x | \Psi \rangle|^2 = |\Psi(x)|^2$$

4.3.1 Bra-Ket Notation

$$\langle x | y \rangle$$

Is an inner product that yields just 1 number.

$$|y\rangle \langle x|$$

is an outer product that yields a matrix.

4.3.2 Stationary State

A stationary state is an eigenstate of the Hamiltonian. It has a perfectly well-defined energy.

4.4 Interpreting the Double Slit

You have a quantum state dependent probability density

$$P(x) = |\Psi(x)|^2$$

That's a probability density function that needs to be normalized

$$\int_{-\infty}^{\infty} dx |\Psi(x)|^2 = 1$$

So to find the expected value, we get

$$\langle x \rangle = \int_{-\infty}^{\infty} dx x |\Psi(x)|^2 \quad \langle f(x) \rangle = \int_{-\infty}^{\infty} dx f(x) |\Psi(x)|^2$$

And the variance of that probability distribution is given by

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Given a quantum state, if we can measure the position, we can **only** estimate $|\Psi(x)|^2$ with finite accuracy. Each position sampled is just a single point of a probability distribution that cannot be directly measured.

Chapter 5

Wave Mechanics

Let's start with

$$|\Psi(x)\rangle = \Psi(x, t) \quad H = -\frac{\hbar^2}{2m} \left(\frac{\partial}{\partial x} \right) + V(x)$$

Where $V(x)$ is the potential. This is our wave function and our Hamiltonian. The Schrodinger Wave equation is

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi(x, t) + V(x) \Psi(x, t)$$

5.1 Important Properties of the S.W.E.

5.1.1 Unitary

Conservation of probability. Found is Griffiths Textbook chapter 1.4

$$\begin{aligned} 0 &= \frac{d}{dt} \int_{-\infty}^{\infty} dx |\Psi(x, t)|^2 = \int dx \frac{\partial}{\partial t} |\Psi(x, t)|^2 = \\ &\int dx \left(\Psi^* \frac{\partial}{\partial t} \Psi + \Psi \frac{\partial}{\partial t} \Psi^* \right) \rightarrow \int dx \left(\Psi^* \frac{\partial^2}{\partial x^2} \Psi - \Psi \frac{\partial}{\partial x^2} \Psi^* \right) \frac{i\hbar}{2m} \\ &= \int dx \frac{d}{dx} \left(\Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) \frac{i\hbar}{2m} \end{aligned}$$

So now we have to

$$\left(\Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right)_{-\infty}^{\infty} \frac{i\hbar}{2m} = 0 \quad \Psi, \Psi' \rightarrow 0 \quad |x| \rightarrow \infty$$

To first try and figure out the wave function, we start with a bunch of plane waves.

A single plane wave will have equation

$$\Psi_k = A e^{i(kx - \omega t)}$$

Plug that into the S.W.E. to get

$$\begin{aligned} \frac{\partial}{\partial t} \Psi_k &= -i\omega \Psi_k & \frac{\partial}{\partial x} \Psi_k &= ik \Psi_k & \frac{\partial^2}{\partial x^2} \Psi_k &= -k^2 \Psi_k \\ \hbar\omega \Psi_k &= \frac{\hbar^2 k^2}{2m} \Psi_k \Rightarrow E = \frac{p^2}{2m} \end{aligned}$$

I can express all wave functions as plane waves

$$f(x) = \int \frac{dk}{2\pi} e^{ikx} \tilde{f}(x)$$

Add time and

$$\omega = \frac{\hbar k^2}{2m}$$

And get equation

$$\Psi(x, t) = \int \frac{dk}{2\pi} e^{ikx - i\omega t} \dots \dots \dots$$

Unbounded plane waves are not normalizable.

$$\int_{-\infty}^{\infty} dx |A e^{i(kx - \omega t)}|^2 = |A|^2 \int_{-\infty}^{\infty} dx = \infty$$

The way to fix this is by putting the plane wave in a finite box that is much larger than the bounds of the problem

$$1 = \int_{-L}^L dx |A|^2 \rightarrow A = \frac{1}{\sqrt{2L}}$$

So the SWE can be given as

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + V(x) \Psi$$

In classical mechanics, we would have

$$E = \frac{p^2}{2m} + V(x) \rightarrow p \iff \frac{\partial}{\partial x} \Psi$$

So what is \hat{p} ?

For plane waves

$$p\Psi_k = \hbar k \Psi_k$$

From De Broglie. We put that into our wave equation

$$p\Psi_k = \hbar k \Psi_k = -i\hbar \frac{\partial}{\partial x} \Psi_k$$

Classically, we know that

$$V = \dot{x} \quad p = mV$$

The expected value is the value over many many quantum mechanical states

$$\frac{d\langle x \rangle}{dt} = \frac{d}{dt} \int dx x |\Psi(x, t)|^2 = \int dx x \left(\left(\frac{\partial}{\partial t} \Psi^* \right) + \Psi^* \left(\frac{\partial}{\partial t} \Psi \right) \right)$$

This derivation is in Griffiths 1.5

$$\frac{i\hbar}{2m} \int dx x \frac{\partial}{\partial x} \left(-\Psi \frac{\partial}{\partial x} \Psi^* + \Psi^* \frac{\partial}{\partial x} \Psi \right)$$

We do integration by parts

$$\frac{\partial}{\partial x} \left(-\Psi \frac{\partial}{\partial x} \Psi^* + \Psi^* \frac{\partial}{\partial x} \Psi \right) = g' \quad x = f$$

$$\begin{aligned} -\frac{i\hbar}{2m} \int dx \left(\Psi^* \frac{\partial}{\partial x} \Psi - \Psi \frac{\partial}{\partial x} \Psi^* \right) &= \frac{i\hbar}{m} \int dx \Psi^* \frac{\partial}{\partial x} \Psi \\ &= \frac{1}{m} \int dx \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi = \frac{\partial \langle x \rangle}{\partial t} \end{aligned}$$

Momentum should be

$$p \sim m \frac{d}{dt} \langle x(t) \rangle$$

$$\langle p \rangle = \int dx \Psi^* \left(-i\hbar \frac{\partial}{\partial x} \right) \Psi$$

The momentum operator (expressed in position), known as \hat{p} , is

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

5.2 Recap

Given a state vector $|\Psi\rangle$, you can make a probability distribution to describe the position of the particle at time t

$$P(x, t) = |\Psi(x, t)|^2$$

Plane waves cannot be normalized, so they are bad wave functions.

Solve for $\Psi \rightarrow$ Plane waves (only for math)

The momentum of the particle can be written as

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

That is specifically in the position basis.

The ep

$$\langle \hat{p} \rangle = \int dx \Psi^* \hat{p} \Psi$$

That is true for any

$$\hat{A} \equiv \hat{A}(\hat{x}, \hat{p}) \quad \langle \hat{A} \rangle = \Psi^* \hat{A} \Psi$$

5.3 Uncertainty Relation

Given a plane wave wavefunction. Because I have a single wave with a well defined wavelength, the momentum is well-defined.

$$\sigma_p \rightarrow 0$$

However, because it's a plane wave, we cannot say anything about the position of the particle.

$$\sigma_x \rightarrow \infty$$

If given a wave function that looks like a dirac delta with a single spike, then the position is well-defined but the momentum is not.

If you perfectly measure position, the momentum is then a superposition of all possible momenta, so the particle will spread out over time because the particle has to be moving due the momentum uncertainty.

Generally, the bound for momentum and position variance is

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

There are wave functions that minimize the uncertainty relation (Gaussian Wave Packets).

5.4 Solving the SWE

Given $V(x)$, how do we get $\Psi(x, t)$

The Schrodinger wave equation is given as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Here we assume that the state is **not** time-dependent

$$\frac{\partial V}{\partial t} = 0$$

Because of our time-independence, we can just separate the variables.

$$\Psi(x, t) = \Psi(x)\varphi(t)$$

Initial Condition: know what $\Psi(x, 0)$ is

$$i\hbar \frac{1}{\varphi} \frac{d\varphi}{dt} = -\frac{\hbar^2}{2m} \frac{1}{\Psi} \frac{d^2\Psi}{dx^2} + V$$

The separation constant is the energy E

$$i\hbar \frac{1}{\varphi} \frac{d\phi}{dt} = E \rightarrow \frac{d\varphi}{dt} = \frac{-iE}{\hbar} \varphi \rightarrow \\ \phi = e^{\frac{-iE}{\hbar} t}$$

Now we consider the position from the state space

$$-\frac{\hbar^2}{2m} \frac{d^2\Psi}{dx^2} + V\Psi = E\Psi$$

The left side is the Hamiltonian operator on Ψ and the right side is the energy.

The stationary solution is of the form

$$\Psi(x, t) = \psi(x) \exp\left(\frac{-iEt}{\hbar}\right)$$

The time independent and time dependent states are equal
For any \hat{A}

$$\langle \hat{A} \rangle = \int \psi^* \hat{A} \psi \, dx = \text{const}$$

We can define the energy as

$$H = \frac{p}{2m} + V(x)$$

This is the total energy in classical mechanics (kinetic + potential energy)

Our quantum Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \quad \hat{H}\psi = E\psi \quad \langle \hat{H} \rangle = E \\ H^2\psi = E^2\psi \rightarrow \langle H^2 \rangle = E^2 \rightarrow \sigma_H^2 = 0$$

So the general solution is a linear combination of the separated solutions.

$$V(x) \rightarrow \{\psi_n\} \Rightarrow \{E_n\}$$

With an Initial Condition:

$$\Psi(x, 0) = \sum_n c_n \psi_n(x)$$

And then we evolve that system

$$\Psi(x, t) = \sum_n c_n \psi_n(x) \exp\left(\frac{-iE_n t}{\hbar}\right)$$

make sure that everything is normalized

$$\sum_n |c_n|^2 = 1 \quad \langle H \rangle = \sum_n |c_n|^2 E_n$$

5.4.1 Question

Given a stationary state $\{\psi_n(x)\}$ for some \hat{H} , how can we explain motion?
To explain motion, we need a superposition of $\psi_n(x)$.

$$\begin{aligned} \Psi(x, 0) &= c_1 \psi_1(x) + c_2 \psi_2(x) \rightarrow \Psi(x, t) \\ &= c_1 \psi_1(x) \exp\left(\frac{-iE_1 t}{\hbar}\right) + c_2 \psi_2(x) \exp\left(\frac{-iE_2 t}{\hbar}\right) \\ |\Psi(x, t)|^2 &= \psi^* \psi \end{aligned}$$

Use Euler's equation

$$c_1^2 \psi_1^2 + c_2^2 \psi_2^2 + 2c_1 c_2 \psi_1 \psi_2 \cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)$$

You have two stationary states in position. They interfere and something happens in time.

5.5 Infinite Square Well

Horrendously artificial, but one of the few problems we can actually solve.

Consider a particle with mass m and velocity v in a valley of height h .
The particle is trapped in the well because $mv^2/2 < mgh$.

We can consider the valley as $h \rightarrow \infty$, because the particle can quantum tunnel out of it.

The width of the well goes from $0 \rightarrow a$. The potential of the particle can be stated as

$$V(x) = \begin{cases} 0 : 0 \leq x \leq a \\ \infty : \text{elsewhere} \end{cases}$$

At $V = \infty$, we can say that the wave function $\psi = 0$

5.5.1 Boundary Conditions

- $\psi(x)$ is always continuous.
- $\psi'(x)$ is always continuous but not if $|V(x)| = \infty$.

So now all we have to do is solve the SWE inside the well

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi = E\psi \iff \psi''(x) = -k^2\psi \quad k = \frac{\sqrt{2mE}}{\hbar}$$

Everything is a simple harmonic oscillator

$$\psi = A e^{ikx} + B e^{-ikx}$$

Now we have to consider the boundary conditions.

$$\psi(x=0) = \psi(x=a) = 0$$

$$A + B = 0 \rightarrow A = -B$$

$$\psi(x) = A(e^{ikx} - e^{-ikx}) = A \sin(kx)$$

$$\psi(x=0) = 0 \rightarrow \sin(ka) = 0$$

shit

5.5.2 Energies

$$k^2 = \frac{2mE}{\hbar^2} \rightarrow E_n = \frac{\hbar^2 k_n^2}{2ma^2}$$

5.5.3 Normalization

$$\int_0^a A^2 \sin^2(kx) dx = 1 \rightarrow A = \sqrt{\frac{2}{a}}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \quad E_n = \frac{k^2 \pi^2 \hbar^2}{2ma^2}$$

5.6 Discussion 1

5.6.1 The Classical Recipe

$$F = ma \quad \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial x} \frac{\partial L}{\partial x}$$

5.6.2 Quantum Reality

Input is ket. Output is bra.

Ket to bra is input to output based off of the inner product.

The wavefunction gives the probability density of finding a particle in a volume element dx

$$\rho(x) = \Psi^*(x, t)\Psi(x, t)$$

$$P(a < x < b) = \int_a^b \Psi^*(x, t)\Psi(x, t) dx$$

The wave function is also normalized

$$\int_{-\infty}^{\infty} \Psi^*(x, t)\Psi(x, t) dx = 1$$

Given an input state $|\Psi_1\rangle$ and an output state $\langle\Psi_2|$. The probability amplitude is given in the form

$$\langle\Psi_2|\Psi_1\rangle = \Psi_2^*(x, t)\Psi_1(x, t)$$

The output state comes first, and it is the one that's the conjugate.

Given a superposition

$$\Psi_{in} = \frac{1}{\sqrt{3}}\Psi_1 + \frac{\sqrt{2}}{\sqrt{3}}\Psi_2$$

So to find the probability of the output being 2

$$P(2) = |\langle\Psi_{out}|Psi_{in}\rangle|^2 = \Psi_2 \frac{1}{\sqrt{3}}\Psi_1 + \Psi_2 \frac{\sqrt{2}}{\sqrt{3}}\Psi_2$$

something something I'll look at the lecture notes later

5.6.3 Questions

Consider a wave function

$$\psi(x) = A(a^2 - x^2)$$

inside interval $\{-a, a\}$

Determine the normalization constant A

$$\begin{aligned} \int \psi(x)^2 dx &= 1 \\ A^2 \int_{-a}^a (a^2 - x^2)^2 dx &\quad A^2 (a^4 x - (2a^2 x^3)/3 + x^5/5) \Big|_{-a}^a = \\ A^2 (a^4 a - (2a^2 a^3)/3 + a^5/5) &- (-a^4 a + (2a^2 a^3)/3 - a^5/5) \\ = 2A^2 \left(a^5 - \frac{2}{3}a^5 + a^5/5 \right) &= \frac{16}{15} A^2 a^5 = 1 \rightarrow A = \sqrt{\frac{15}{16a^5}} \\ \frac{15}{15} - \frac{10}{15} + \frac{3}{15} &= \frac{8}{15} \end{aligned}$$

What is the probability of finding the particle at $x = a/2$. It is 0 because that's a minuscule-ly small point. The range $-a/2 \rightarrow a/2$ is different.

$$\begin{aligned} P(-a/2 < x < a/2) &= \int_{-a/2}^{a/2} \Psi^*(x) \Psi(x) \\ &= 2 \frac{15}{16a^5} (a^4 x - (2a^2 x^3)/3 + x^5/5) \Big|_{-a/2}^{a/2} \\ &= \frac{15}{4a^5} a^4 \left(\frac{a}{2}\right) - (2a^2 \left(\frac{a}{2}\right)^3)/3 + \left(\frac{a}{2}\right)^5/5 = \frac{15}{4} \left(\left(\frac{1}{2}\right) - (2\left(\frac{1}{8}\right))/3 + \left(\frac{1}{32}\right)/5 \right) \\ &= \frac{15}{4} \left(\frac{1}{2} - \frac{1}{12} + \frac{1}{160} \right) \end{aligned}$$

For what potential is the state an eigenfunction?

The energy eigenfunctions are determined from the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi + V(x) = E\psi$$

So I need to take the double derivative

$$\frac{\partial^2}{\partial x^2} Aa^2 - Ax^2 = -2A$$

$$\frac{\hbar^2}{2m} 2A + V(x)(Aa^2 - Ax^2) = EA(a^2 - x^2)$$

Set energy to 0

$$-\frac{\hbar^2}{2m}2\sqrt{\frac{15}{16a^5}}$$

5.6.4 bra-ket

Consider a three dimension vector space spanned by an orthonormal basis

$$|1\rangle \quad |2\rangle \quad |3\rangle$$

$|\alpha\rangle$ and $|\beta\rangle$ are defined as

$$|\alpha\rangle = i|1\rangle - 2|2\rangle - i|3\rangle \quad |\beta\rangle = i|1\rangle + 2|3\rangle$$

Construct $|\alpha\rangle$ and $|\beta\rangle$ in terms of the dual basis $\langle 1|, \langle 2|, \langle 3|$
Something that might be useful

$$A_{ij} = \langle i| \hat{A} |j\rangle = \langle i| (|\alpha\rangle \langle \beta|) |j\rangle = \langle i|\alpha\rangle \cdot \langle \beta|j\rangle$$

Chapter 6

Solving the SWE

6.1 Recap

Our starting point is

$$\hat{H} = -\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} + V(x)$$

with initial conditions.

The time-indepedent SWE can be written as

$$\hat{H}\psi(x) = E_n\psi(x) \quad \psi(x, 0) = \sum_n c_n \psi_n(x) \quad \psi(x, t) = \sum_n c_n \psi_n(x) e^{\frac{-i}{\hbar} E_n t}$$

And infinite square well can be written as

$$\psi(x \leq 0) = \psi(x \geq a) = 0$$

6.1.1 Stationary State

If we consider a probability distribution

$$|\psi(x, t)| = \psi^*(x, t)\psi(x, t)$$

So for an infinite square well.

Consider the general statement

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V(x) - E)\psi$$

if $E < V(x)$, then ψ'' and ψ always have the same sign. The issue with this is the wave will not be normalizable (because for positive x, the concavity has to be positive, so it has to increase).

The eigenstates are given by

$$E_1 = \frac{\pi^2 \hbar^2}{2ma^2} \quad E_2 = 4E_1 \quad E_3 = 9E_2$$

In a stationary state, $\{\psi_n\}$ forms a complete orthonormal eigenbasis.

$$\int dx \psi_m^*(x) \psi_n(x) = \delta_{mn} (m = n)$$

$$f(x) = \sum_n c_n \psi_n(x) = \sqrt{\frac{2}{a}} \sum_n c_n \sin\left(\frac{n\pi x}{a}\right) \quad c_n = \langle \psi_n | f(x) \rangle$$

Question

Given a wave function $\psi(x, t = 0)$ and a stationary state, how can we find c_n such that $\psi = \sum_n c_n \psi_n(x)$?

$$\begin{aligned} \int dx \psi_m^*(x) f(x) &= \int dx \psi_m^*(x) f(x) \sum_n c_n \psi_n(x) \\ &= \sum_n c_n \int \psi_m^*(x) \psi_n(x) = \sum_n c_n f_{mn} = c_m \end{aligned}$$

6.2 Time-Dependence

$$\psi(x, t = 0) = \sum_n c_n \psi_n(x) \rightarrow \psi(x, t) = \sum_n c_n \psi_n e^{\frac{-i}{\hbar} E_n t}$$

This is just summing over all of the possible standing waves within an infinite square well.

6.2.1 THIS WILL BE ON THE MIDTERM1

The thing above

6.2.2 Quantum Number

It's a fake number that is just an index to show the different eigenvectors

$$E_n \quad n = \text{quantum number}$$

Question

How can we describe a "ball" bouncing between the walls of the infinite square well?

$$\psi(x, t = 0) = \sum_{\text{odd}} c_n \psi_n(x)$$

sum odd means that the function that we're summing over has to be an odd function.

An even function will just have an expected value of 0.

6.2.3 Free Particle

$$V = 0 \rightarrow \hat{H} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$$

and the time-indepedent SWE is

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi = E\psi \rightarrow \psi = Ae^{ikx} + Be^{-ikx} \quad E = \frac{\hbar^2 k^2}{2m} \quad kc = \sqrt{\frac{2mE}{\hbar^2}}$$

and the time-**dependent** SWE is

$$\psi(x, t) Ae^{ikx - \frac{i}{\hbar} Et} + Be^{-ikx - \frac{i}{\hbar} Et} = Ae^{i(kx - \omega t)} + Be^{-i(kx + \omega t)}$$

The wave function is a superposition of a left-propagating and a right-propagating wave both with constant kintetic energy.

allow k to be

$$k = \pm \sqrt{\frac{2mE}{\hbar^2}} \rightarrow Ae^{i(kx - \omega t)}$$

The velocity can be written as

$$e^{-i(kx - \omega t)} = e^{-ik(x - \frac{\omega}{k}t)} \rightarrow v = \frac{\hbar|k|}{2m} = \sqrt{\frac{E}{2m}}$$

This is different from the classical definition of velocity by a factor of 2

$$\frac{1}{2}mv^2 = E \rightarrow v = \sqrt{\frac{2E}{m}}$$

The reason this is so is because the quantum definition of velocity is **not** for a particle.

Consider a Gaussian wave packet

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{i(kx - \omega t)} dk$$

For some initial conditions $\psi(x, t = 0)$, we find the weight function with

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x, t = 0) e^{-ikx} dx$$

6.3 MISSED LECTURE

6.4 Discussion

Time-Evolving superpositions. We use a Hamiltonian basis

$$\hat{H}\psi_n = E_n\psi_n \quad \text{Basis } = \{\psi_1, \psi_2, \dots\}$$

This basis is orthonormal

$$\langle \psi_m | \psi_n \rangle = \delta_{mn}$$

And the basis is complete

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \dots$$

And we have a time-evolution operator

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \dots$$

$$\Psi(x, t) = e^{\frac{-i\hat{H}t}{\hbar}} (c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \dots)$$

And this effects each individual state differently

$$e^{\frac{-i\hat{H}t}{\hbar}} c_3\psi_3(x) = e^{\frac{-iE_3t}{\hbar}} c_3\psi_3(x)$$

6.4.1 Questions

Find the normalized wave function at time t for a particle in an infinite square well with a wave function given by

$$\psi(x, 0) = A (\psi_1(x) + e^{i\theta} \psi_2(x))$$

$$P(x) = |\psi(x)|^2 = A^2 |(\psi_1(x) + e^{i\theta} \psi_2(x))|^2$$

The time dependent portion of the wave function is

$$e^{-i\frac{E}{\hbar}t}$$

So our equation will be

$$P(x) = e^{-i\frac{E}{\hbar}t} |\psi(x)|^2 = A^2 |(\psi_1(x) + e^{i\theta} \psi_2(x))|^2$$

For an infinite square well, the solution can be written as

$$\psi(x, 0) = \sin\left(\frac{n\pi x}{L}\right)$$

$$|\psi(x, 0)\rangle = A (|\psi_1\rangle + e^{i\theta} |\psi_2\rangle)$$

$$|\psi(x)|^2 = A^2 (\langle \psi_1 | + e^{-i\theta} \langle \psi_2 |) (|\psi_1\rangle + e^{i\theta} |\psi_2\rangle) = \\ A^2 (\langle \psi_1 | \psi_1 \rangle + |\psi_1\rangle e^{i\theta} |\psi_2\rangle + \langle \psi_2 | \psi_2 \rangle) = 2A^2 = 1$$

The probability density is just something

6.4.2 Infinite Square Well

Given a unique wave function that's put in the paper and on the website, expand the wave function in terms of its eigenvectors using a Fourier series.

Go to chapter 11 of Townsend for bracket notation or chapter 4 for time-evolution.

6.5 Finite Square Well

Consider a well with potential

$$V(x) = \begin{cases} -V_0 & : -a < x < a \\ 0 & : x < -a, x > a \end{cases}$$

The wave function can extend out of the well now, and the wave function **must be continuous**. Exact solutions are typically numerical bound states (≥ 1)

6.5.1 Scattering Stationary States ($E > 0$)

Consider that we're starting with a plane wave (coming from the left)

$$\psi_I(x, t = 0) = Ae^{ikx} + Be^{-ikx}$$

In the middle, we have 2 boundary conditions, and both the wavefunction itself and its derivative have to be continuous.

$$\psi_{II}(x, t = 0) = C \sin(lx) + D \cos(lx)$$

On the right side, there is still 1 boundary condition, but the other side is unbounded

$$\psi_{III}(x, t = 0) = Fe^{ikx}$$

You do some math (Griffith's 2.6). Solve for Boundary Conditions and make sure x is continuous at the important parts

$$\begin{aligned} \frac{|F|^2}{|A|^2} &= T & R = 1 - T & k = \frac{\sqrt{2mE}\hbar}{\hbar} l = \frac{\sqrt{2m(E + V_0)}}{\hbar} \\ Ae^{-ika} + Be^{ika} &= -C \sin(la) + D \cos(la) \\ \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow \Rightarrow & \\ B &= i \frac{\sin(2la)}{2kl} (l^2 - k^2) F & F &= Ae^{-2ika} \left(\cos(2la) - i \frac{(k^2 + l^2)}{2kl} \sin(2la) \right)^{-1} \end{aligned}$$

Those are the answer given

$$\psi(x, 0) \propto \int dk \psi(k) e^{-ikx}$$

An interesting case is $T = 1$. This can be fulfilled for

$$\frac{2a}{\hbar} \sqrt{2m(E_n + V_0)} = n\pi \quad (E_n + V_0) = \frac{n^2 \pi^2 \hbar^2}{2m(2a)^2}$$

We are matching k to the "infinite well".

$$B = i \frac{\sin(2la)}{2kl} (l^2 - k^2) F$$

$$F = (e^{-2ika} A) * \left(\cos(2la) - i \frac{k^2 + l^2}{2kl} \sin(2la) \right)^{-1}$$

6.6 Quantum Tunneling

Consider the opposite of a finite square well

$$V = \begin{cases} V_0 & : 0 < x < a \\ 0 & : x < 0, x > a \end{cases}$$

The stationary state consists of a plane wave on the left side.

$$\psi_I = A e^{ikx} + B e^{-ikx}$$

We have exponential decay inside the "well". The particle can also be reflected, so the wave function is given by

$$\psi_{II} = C e^{-kx} + D e^{kx}$$

And then we have another plane wave on the 2nd side

$$\psi_{III}(x, 0) = F e^{ikx}$$

6.7 Recap

Given a function in the position basis

$$|\psi(x)\rangle = \int dx \psi(x) |x\rangle$$

We are able to turn it into the energy basis

$$|\psi\rangle = \sum c_n |n\rangle \quad \langle x|\psi\rangle \rightarrow \psi_n(x)$$

6.8 Basis Vectors

Consider the notation $|1\rangle, |2\rangle, |3\rangle$, where each of those corresponds to the $\vec{i}, \vec{j}, \vec{k} = x, y, z$ unit vectors.

6.8.1 Hilbert Space

A complex space for $N = \dots? (\rightarrow \infty)$.

Pick a basis $|n\rangle$ in that space. The column vectors are written in the form.

$$\vec{\alpha} = |\alpha\rangle = \sum_n \alpha_n |n\rangle$$

This is just saying that $\vec{\alpha}$ can be written as a sum of the orthonormal basis vectors. The row vectors are written in the form

$$\vec{\alpha} = \langle \alpha | = \sum_n \langle n | \alpha_n \rangle$$

And the inner product is defined as

$$\langle \alpha | \beta \rangle = \sum_n \alpha^* \beta = \langle \beta | \alpha \rangle^*$$

It's just a dot product.

The norm is defined as

$$\langle \alpha | \alpha \rangle = \sum_n \alpha_n^* \alpha_n = \sum_n |\alpha_n|^2$$

If the basis vectors are normalized, then

$$\langle k | n \rangle = \delta_{kn} \quad \langle k | k \rangle = 1$$

6.8.2 Development in a Basis

$$|\alpha\rangle = \sum_k |k\rangle \langle k | \alpha \rangle = \sum_k \alpha_k |k\rangle \quad \alpha_k = \langle k | \alpha \rangle$$

$$|f\rangle = \sum_x |x\rangle \langle x | f \rangle = \sum_x \langle x | f \rangle |x\rangle \quad f_x = \langle x | f \rangle$$

$$\sum_k c_k \rightarrow \int dx \rho \quad \rho = \text{density} \Rightarrow \int dx |x\rangle \langle x | f \rangle = \int dx f(x) |x\rangle$$

$$\langle f | g \rangle = \int dx \int dy \langle x | f^*(x) g(y) | y \rangle \quad \langle x | y \rangle = \delta(x - y)$$

$$\langle f | g \rangle = \int dx f^*(x) g(x) \langle g | f \rangle^*$$

The norm of a function can be written as

$$\langle f | f \rangle = \int dx f^*(x) f(x)$$

And in a Hilbert Space,

$$|\langle f | f \rangle|^2 = 1$$

6.8.3 Operators

An operator acts on a state

$$\hat{A} |\psi\rangle$$

In the discrete case, operators can be written as matrices

$$\hat{A} |\alpha\rangle = |\alpha'\rangle \quad \langle \alpha'| = (\hat{A} |\alpha\rangle)^{*T} = \langle \alpha| \hat{A}^\dagger \quad x^\dagger = (x^*)^T$$

The basis of matrix elements can be written as

$$\langle 1| \hat{A} |3\rangle = \alpha_{13} \quad \text{1st row, 3rd column}$$

6.8.4 Projections

$$\langle k|\alpha\rangle = \langle k|(\alpha_1|1\rangle + \dots) = \alpha_k$$

The projection operator can be written as

$$\hat{P}_i = |i\rangle \langle i|$$

everything is 0 except for the ith row and ith column

$$\hat{P}_1 |\alpha\rangle = |1\rangle \langle 1| (\alpha_1|1\rangle + \alpha_2|2\rangle + \dots) = \alpha_1|1\rangle$$

The sum of all projections can be written as

$$\sum_i P_i = \sum_i |i\rangle \langle i| = 1$$

Generally

$$|k\rangle \langle i| = \quad \text{Matrix with 1 at kth row, ith column}$$

$$\hat{A} = \sum_k \sum_n \alpha_{kn} |k\rangle \langle n|$$

That is just a fancy way to fill a matrix

6.9 Observables

Observables are operators that are "measurable" (have a real value)

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \rightarrow \langle \hat{A} \rangle^* = \langle \hat{A} \rangle \rightarrow \langle \hat{A} \psi | \psi \rangle = \langle \psi | \hat{A} \psi \rangle$$

This is equivalent to the Hermitian Conjugate $\hat{A}^\dagger = (\hat{A}^*)^T$. This means that Observables are Hermitian Operators $\hat{A}^\dagger = \hat{A}$. Eigenvalues of Hermitian Operators are real.

$$\begin{aligned}\hat{A} |v\rangle &= \alpha |v\rangle \rightarrow \langle v | \hat{A} |v\rangle = \alpha \langle v | v \rangle = \alpha \quad \alpha \in \mathbb{R} \rightarrow \alpha = \alpha^* \\ \langle v | A |v\rangle &= \langle v | Av \rangle = \langle v | Av \rangle^* = \langle Av | v \rangle = \langle v | A^\dagger |v\rangle\end{aligned}$$

We can also solve for the expectation values of the operator

$$\begin{aligned}\langle \hat{A} \rangle &= \langle \psi | \hat{A} | \psi \rangle = \left(\sum_k c_k^* \langle \alpha_k | \right) \hat{A} \left(\sum_n c_n | \alpha_n \rangle \right) = \sum_{k,n} c_k^* c_n \langle \alpha_k | \hat{A} | \alpha_n \rangle \\ &= \sum_{k,n} c_k^* c_n \alpha_k \langle \alpha_k | \alpha_n \rangle = \sum_k c_k^* c_n \alpha_n \delta_{kn} = \sum_k |c_k|^2 \alpha_k\end{aligned}$$

The kronecker delta is important because that is the operator that removes all of the n terms in the expression.

6.9.1 is \hat{p} Hermitian?

$$\langle f | \hat{p} g \rangle = \int f^*(-i\hbar \frac{d}{dx}) g dx = -i\hbar f^* g|_{-\infty}^{\infty} + \int (-i\hbar \frac{df^*}{dx}) g dx = \langle \hat{p} f | g \rangle$$

6.9.2 Relationships Between Operators

The commutator is an operator that is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -[\hat{B}, \hat{A}] \quad (\hat{A}, \hat{B})^T = B^T A^T \quad (\hat{A}\hat{B})^\dagger = B^\dagger A^\dagger$$

For \hat{A}, \hat{B} , Hermitian Operators. $[\hat{A}, \hat{B}] = 0$ iff there is a basis in which **both** \hat{A} and \hat{B} are diagonal.

$$\langle k | \hat{A} | k' \rangle = 0 \quad \langle k | \hat{B} | k' \rangle = 0 \quad \forall k \neq k'$$

6.10 Discussion (The Quantum Recipe)

1. Identify the Hamiltonian
2. Establish the basis (eigenstates)
3. Is your state in the basis?
4. Decompose your state to the basis (Often a fourier transform)
5. Time evolve each element in the decomposition
6. Compute whatever you want

6.10.1 Example

We have a wave function of a free particle given by

$$\Psi(x, 0) = A \cos(2kx) + B \sin(kx)$$

Identify the Hamiltonian, idk what this means. Because it's a free particle, it has a Hamiltonian given by

$$-\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} \psi(x, 0) = \hat{H} \psi(x, 0)$$

Now establish the basis. The particle is a bunch of waves, so it can be written in the basis

$$\psi_n(x, 0) = A e^{inx} \quad k \in \mathbb{Z}$$

Is your state in the basis? NO because it is not a sum of eigenvectors. Decompose your state to the basis

$$\Psi(x, 0) = \frac{A}{2} (e^{i2kx} + e^{-i2kx}) + \frac{B}{2i} (e^{ikx} - e^{-ikx})$$

Time-evolve your solution

$$\Psi(x, 0) = \frac{A}{2} \left(e^{\frac{-iE_k t}{\hbar}} e^{i2kx} + e^{\frac{-iE_{-k} t}{\hbar}} e^{-i2kx} \right) + \frac{B}{2i} \left(e^{\frac{-iE_k t}{\hbar}} e^{ikx} - e^{\frac{-iE_{-k} t}{\hbar}} e^{-ikx} \right)$$

6.10.2 Example 2

You have a Hamiltonian and a particle of the form

$$\hat{H} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \Psi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Identify the Hamiltonian (given)

Identify the basis (take the eigenvalues of the matrix)

$$\Psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Is your state in the basis? no (it is not one of the eigenvectors)

Decompose the state as a form of eigenvectors

$$\Psi = \Psi_1 + \Psi_2$$

Time evolve each element in the decomposition. Because the two states have different energy values (eigenvalues), the decomposed portions will have different frequencies

$$\Psi(t) = e^{\frac{-iE_1 t}{\hbar}} \Psi_1 + e^{\frac{-iE_2 t}{\hbar}} \Psi_2$$

6.10.3 DO YOU NEED TO MEMORIZE THE BASES (YES)

Study properties, behaviors, and form.

6.10.4 Questions (2)

You have a Hamiltonian and a system defined by

$$\hat{H} = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix} \quad |S(0)\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find the time-dependent system

$$\begin{bmatrix} a - \lambda & 0 & b \\ 0 & c - \lambda & 0 \\ b & 0 & a - \lambda \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & (a - \lambda) - b^2/(a - \lambda) \end{bmatrix}$$

$$(a - \lambda)(c - \lambda)((a - \lambda) - b^2/(a - \lambda)) = 0$$

$$((a - \lambda) - b^2/(a - \lambda)) = 0 \rightarrow (a - \lambda)^2 - b^2 = 0 \rightarrow a - \lambda = b \rightarrow$$

eigenvalues are $c, (a - b), (a + b)$. Plug the eigenvalues back into the matrix to get the eigenvectors

$$\begin{aligned} & \begin{bmatrix} a - c & 0 & b \\ 0 & c - c & 0 \\ b & 0 & a - c \end{bmatrix} \rightarrow \begin{bmatrix} a - c & 0 & b \\ 0 & 0 & 0 \\ b & 0 & a - c \end{bmatrix} \\ & = \begin{bmatrix} a - c & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & a - c - \frac{b^2}{a-c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \\ & \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

I'm so unbelievably stupid

$$\begin{aligned} & \begin{bmatrix} a - (a + b) & 0 & b \\ 0 & c - (a + b) & 0 \\ b & 0 & a - (a + b) \end{bmatrix} = \begin{bmatrix} -b & 0 & b \\ 0 & c - a - b & 0 \\ b & 0 & -b \end{bmatrix} \\ & = \begin{bmatrix} -b & 0 & b \\ 0 & c - a - b & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

Im so sos os oso so so so stupid oh my god

$$-1v_1 + v_3 = 0 \quad v_2 = 0 \rightarrow v_1 - v_3$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

ajgfilagjiladfngjilasdnfjasdj;fkasdjfiasdfnasdfnsadklfnasd;fnasdfs

6.10.5 3 (Diagonalizing Matrices)

6.11 Formalism

This is a discrete basis

$$|\psi\rangle = \sum_k c_k |\alpha_k\rangle \quad c_k = \langle \alpha_k | \psi \rangle \quad \langle \alpha_k | \alpha_j \rangle = \delta_{kj}$$

This is a continuous basis

$$|\psi\rangle = \int dx \psi(x) |x\rangle \quad \psi(x) = \langle x | \psi \rangle \quad \langle x' | x \rangle = \delta(x - x')$$

An observable is a Hermitian operator with real eigenvalues

$$\hat{A} = \hat{A}^\dagger \rightarrow \text{real eigenvalues}$$

$$A_{jk} = \langle k | \hat{A} | j \rangle$$

Consider matrix elements of \hat{x} in the position basis

$$\langle x' | \hat{x} | x \rangle = \langle x' | x | x \rangle = x \langle x' | x \rangle =====$$

A commutator is an operator such that

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

If $[\hat{A}, \hat{B}] = 0$, then a simultaneous eigenbasis exists. This means that

$$\exists \{|j\rangle\} : \hat{A}|j\rangle = \alpha_j|j\rangle \quad \hat{B}|j\rangle = \beta_j|j\rangle$$

6.11.1 Eigenvalues/vectors

Consider the momentum operator

$$\hat{p} = -i\hbar \frac{\partial}{\partial x} \rightarrow -i\hbar \frac{\partial}{\partial x} \psi = p\psi \rightarrow \psi_p = A e^{ip/\hbar x}$$

6.11.2 Energy Eigenstates

consider the time independent schrodinger equation

$$\begin{aligned} \hat{H}\psi &= E\psi & \hat{H} &= \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \\ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi &= E\psi \rightarrow \left(\frac{\partial}{\partial x} - \frac{i\sqrt{2mE}}{\hbar} \right) \left(\frac{\partial}{\partial x} + \frac{i\sqrt{2mE}}{\hbar} \right) \psi &= 0 \end{aligned}$$

This gives you two degenerate solutions.

$$\psi_1 = A e^{ikx} \quad \psi_2 = B e^{-ikx} \quad \psi_k = A e^{ikx} + B e^{-ikx}$$

6.12 Basis Transformations

Suppose we know $|\psi\rangle$ in some basis $\{|\alpha_k\rangle\}$, and we want to know it in some different basis $\{|\beta_k\rangle\}$. Know that $c_k = \langle \alpha_k | \psi \rangle$ and $d_k = \langle \beta_k | \psi \rangle$

$$d_n = \langle \beta_n | \psi \rangle = \langle \beta_n | \sum_k c_k |\alpha_k \rangle = \sum_k c_k \langle \beta_n | \alpha_k \rangle$$

=====

What about in the continuous case?

$$\hat{p}|p\rangle = p|p\rangle \rightarrow \text{position basis?}$$

$$\int dx |x\rangle \langle x| \hat{p} |p\rangle = p \int dx |x\rangle \langle x| p \rangle = p \int dx |x\rangle f_p(x)$$

$$\int dx dx' |x\rangle \langle x| \hat{p} |x'\rangle \langle x'| p \rangle$$

$$\langle x| \hat{p} |\psi\rangle = \int dx' \langle x| \hat{p} |\psi\rangle \langle x'| \psi\rangle = \int dx' \langle x| \hat{p} |x'\rangle \psi(x')$$

$$= \int dx' \hbar \frac{\partial}{\partial x} f(x - x') \psi(x') \Rightarrow -i\hbar \frac{\partial \psi}{\partial x}$$

$$\int dx |x\rangle i\hbar \frac{\partial f_p(x)}{\partial x} = \hat{p} \int dx |x\rangle f_p(x)$$

$\langle x| \hat{p} |x'\rangle$ is the matrix elements of \hat{p} in the x-basis.

6.12.1 What is $\psi(p)$?

$$\begin{aligned} |\psi\rangle &= \int dp \langle p|\psi\rangle |p\rangle = \int dp dx \langle p|x\rangle \langle x|\psi\rangle |p\rangle \\ &= \int dx dp A \cdot e^{-ipx/\hbar} \psi(x) |p\rangle = \int dp \psi(p) |p\rangle dp \end{aligned}$$

6.12.2 Infinite Square Well

The energy basis is described as

$$\langle n|\psi\rangle = c_n \quad |\psi\rangle = \sum_n c_n |n\rangle$$

and the position basis is described as

$$\langle x|\psi_n\rangle = \psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$$

And the momentum basis is

$$\langle p|\psi_n\rangle = \psi_n(p)$$

In the position basis, our wave function is made by a superposition of planar waves. The higher the energy of the planar wave, the more defined is the

corresponding momentum. Each eigenstate is described by

$$\psi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{2}{a}} e^{-iE_nt/\hbar} \int_0^a e^{-ipx/\hbar} \sin\left(\frac{n\pi x}{a}\right) dx$$

This is solved with a calculator because why would you go through the trouble.

$$\frac{4\pi a}{\hbar} \frac{n^2}{\left[(n\pi)^2 - \left(\frac{ap}{\hbar}\right)^2\right]^2} * \begin{cases} \cos^2\left(\frac{ap}{2\hbar}\right) : n = odd \\ \sin^2\left(\frac{ap}{2\hbar}\right) : n = even \end{cases}$$

6.12.3 Know the theory, but don't worry about the calculations

6.13 Important Example: 2 Level System

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \hbar\omega\sigma_z$$

σ_z is the pauli matrix operator with eigenvectors

$$+1, |0\rangle \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad -1, |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{Energy Basis}$$

The general state is of the form

$$\alpha|0\rangle + \beta|1\rangle \quad p(\psi = 0) = |\alpha|^2$$

Consider an operator

$$\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \equiv \hbar\sigma_x = \hbar(|0\rangle\langle 1| + |1\rangle\langle 0|)$$

This operator **is observable** because it is Hermitian. The eigenvectors are

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with an energy value of $+\hbar$

6.13.1 Double Well Potential

Imagine two wells from $-b \rightarrow -a$ and $b \rightarrow a$ with potential $-V_0$ and then potential 0 everywhere else (in between them is $-b \rightarrow b$).

In this system, we imagine measuring the left well vs the right well. Consider the tunneling operator

$$\begin{aligned}\hat{T}|0\rangle &= |1\rangle & \hat{T}|1\rangle &= |0\rangle & \langle 1|\hat{T}|1\rangle &= 0 & \langle 0|\hat{T}|0\rangle &= 1 \\ \langle 0|\hat{T}|1\rangle &= \langle 1|\hat{T}|0\rangle = 1 & \hat{T} = \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & |\pm\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\end{aligned}$$

Starting with quantum mechanics with waves and with two levels are functionally the same.

6.13.2 The Uncertainty Principle

Incompatible Observables: measuring \hat{A} disturbs the possible results of \hat{B} .

$$\sigma_A \sigma_B \geq \frac{\hbar}{2}$$

The variance of an observable can be written in the form

$$\begin{aligned}\sigma_A^2 &= \langle (\hat{A} - \langle \hat{A} \rangle)^2 \rangle = \langle \psi | (\hat{A} - \langle \hat{A} \rangle)^2 | \psi \rangle = \langle (\hat{A} - \langle \hat{A} \rangle) \psi | (\hat{A} - \langle \hat{A} \rangle) | \psi \rangle \\ &\equiv \langle f | f \rangle\end{aligned}$$

6.13.3 Schwarz Inequality

$$\begin{aligned}|\langle f | g \rangle|^2 &= \langle f | f \rangle \langle g | g \rangle \\ \sigma_A^2 \sigma_B^2 &= \langle f | f \rangle \langle g | g \rangle \geq |\langle f | g \rangle|^2\end{aligned}$$

For some complex number $z = z' + iz''$, we know that

$$\frac{1}{2i} (z - z^*)^2$$

So now we have to find the inner products of f and g

$$\begin{aligned}\langle f | g \rangle &= \langle \psi | (\hat{A} - \langle \hat{A} \rangle)(\hat{B} - \langle \hat{B} \rangle) | \psi \rangle \\ &= \langle \psi | \hat{A} \hat{B} - \hat{A} \langle B \rangle - \hat{B} \langle A \rangle + \langle A \rangle \langle B \rangle | \psi \rangle\end{aligned}$$

Something with commutativity

$$= \langle \psi | \langle \hat{A} \hat{B} \rangle - \langle A \rangle \langle B \rangle | \psi \rangle = \langle g | f \rangle$$

Now we put everything together

$$\langle f | g \rangle - \langle g | f \rangle = \langle [\hat{A}, \hat{B}] \rangle \Rightarrow \sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

6.13.4 Back to 2 Layer System (TLS)

The Pauli matrices are known as

$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \quad \sigma_x = |0\rangle\langle 1| - |1\rangle\langle 0| \quad \sigma_y = |1\rangle\langle 0| - |0\rangle\langle 1|$$

Permutations of these matrices are given in the form

$$[\sigma_i, \sigma_j] = 2i * asdfasdfasdfasdfadfasdfasdfasdfdasfasdf$$

6.14 Block Sphere

This is a graphical representation of all the possible states that a wave function can hold. The 6 axes are given by

$$\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$$

6.15 Discussion 1

A couple midterm sanity checks.

Express in wavefunction notation the dirac inner product between a ket and a bra.

$$\langle \psi_i | \psi \rangle = \int_{-\infty}^{\infty} dx \psi_i^* \psi$$

It's just a convolution integral.

Express the orthonormal condition of a basis in both dirac and wavefunction notation

$$\langle e_1 | e_2 \rangle = \delta_{ij} \quad \int_{-\infty}^{\infty} dx e_1^* e_2 = \delta_{ij}$$

That is the kronecker delta. Basically, when $i \neq j$, its just 0.

You should be very keen on the equivalence between Dirac notation and integral notation.

6.15.1 Adjoint Operators

The hermitian adjoint is the complex conjugate transpose of the original operator. An operator is hermitian (or self-adjoint) if its hermitian adjoint is the same operator.

$$\langle \psi_i | \hat{L} \psi_j \rangle = \langle L^\dagger \psi_i | \psi_j \rangle$$

All observables are hermitian (This is because the conjugate of a real eigenvalue is just itself).

6.15.2 SPOILER

Show that \hat{p} is self-adjoint (hermitian)

6.15.3 The Quantum Recipe

1. identify your Hamiltonian
2. Establish your basis (eigenstates of the Hamiltonian)
3. Is your state a basis vector (in the basis)
4. If not, decompose your state to basis vectors
5. Time evolve your state

6.15.4 Spin

Spin is like the zodiac of particles. Spin is an intrinsic property that determines the behavior of the particle.

Up and Down are spins that determine how they go through a Stern-Gerlach filter.

6.15.5 Reorienting Spin

Particles don't by themselves have definite spin. Spin up and spin down are only determined by the orientation of the Stern-Gerlach filter. The Hamiltonian of the spin experiment changes basis.

The **Spin Hamiltonian** gives you the orientation of the Stern Gerlach filters. The eigenstates of the Hamiltonian will be the two states parallel and anti-parallel to the spcial orientation described by the Hamiltonian.

The visual for the direction of the orientation is called the **Bloch Sphere**.

6.15.6 Spin Decomposition

We start off in the z -basis, and then we decompose as states in the z -basis. So, given the Hamiltonian

$$\hat{A} = \hat{\sigma}_z$$

Looking at it, the Hamiltonian is in the z -direction. The eigenstates are

$$|\uparrow_H\rangle = |\uparrow_x\rangle \quad |\downarrow_H\rangle = |\downarrow_x\rangle$$

If the Hamiltonian changes from $\hat{\sigma}_z$ to $\hat{\sigma}_x$, then our eigenstates change, and then they need to be written in terms of the z -basis.

Now, we get

$$\hat{H} = \hat{\sigma}_z + \hat{\sigma}_x$$

The direction of the Hamiltonian is 45 degrees in the xz plane, and the eigenstates are

$$|\uparrow_H\rangle = \frac{1}{\sqrt{2}} |\uparrow_z\rangle + \frac{1}{\sqrt{2}} |\uparrow_x\rangle$$

But then x needs to be decomposed to the z -basis.

6.15.7 Show \hat{p} is Hermitian

Show that $\hat{p}^\dagger = \hat{p}$

$$\begin{aligned} \langle g | \hat{p} f \rangle &= \int_{-\infty}^{\infty} dx g^*(\hat{p}f) \rightarrow \int_0^a dx g^* \hat{p} f = \int_0^a dx g^* (-i\hbar \frac{\partial}{\partial x}) f \\ dv = \frac{\partial}{\partial x} f &\quad v = f \quad u = (-i\hbar)g^* \quad du = \frac{d}{dx} (-i\hbar)g^* \rightarrow \\ g^*(-i\hbar)f \Big|_0^a dx - \int_0^a dx \frac{d}{dx} (-i\hbar)g^* f &= \int_0^a dx (i\hbar) \frac{d}{dx} g^* f = \int p^* g^* f \\ &= \int (\hat{p}g)^* f = \langle \hat{p}g | f \rangle \end{aligned}$$

6.15.8 b)

$$\begin{aligned} g^*(-i\hbar)f \Big|_0^a dx - \int_0^a dx \frac{d}{dx} (-i\hbar)g^* f &\rightarrow \\ g^*(-i\hbar)f \Big|_0^a dx &= g^*(-i\hbar)f(0) - g^*(-i\hbar)\lambda f(0) = g^*(-i\hbar)(1 - \lambda) \end{aligned}$$

6.15.9 Skip 2 go to 3

$$\hat{H} = \hbar\omega (\alpha\hat{\sigma}_z + \beta\hat{\sigma}_x)$$

Write H as both a matrix and bra-ket form.

$$\hbar\omega \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} = \hbar\omega (\alpha |1\rangle\langle 1| + \beta |0\rangle\langle 1| + \beta |1\rangle\langle 0| - \alpha |1\rangle\langle 1|)$$

All observables are **Hermitian**.

Draw the quantization axis in a Bloch Sphere Picture.

Chapter 7

Harmonic Oscillator

7.1 MISSED LECTURE

Now our Hamiltonian is of the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2$$

And we have a new operator

$$a_{\pm} = \frac{1}{\sqrt{2\hbar\omega}}$$

a_+ and a_- are known as the ladder operators.

$$\hat{H} |0\rangle = \frac{\hbar\omega}{2} |0\rangle$$

$$a_+ a_- = \frac{H}{\hbar\omega} - \frac{1}{2} = \hat{n}$$

This is known as the number operator.

$$\langle n | \hat{H} | n \rangle = \frac{1}{2}\hbar\omega + n * \hbar\omega$$

So we do some shenanigans $\langle n | a_+$ is a stationary state and $a_- |n\rangle$ is a stationary state, so

$$\langle n | a_+ * a_- | n \rangle \neq 0 \rightarrow (a_-)^\dagger = a_+$$

both those operators are Hermitian

7.2 Stationary States in the Position Basis

$$\hat{a}_-\psi_0(x) = (2\hbar m\omega)^{-1/2}(\hbar \frac{\partial}{\partial x} + m\omega x)\psi_0(x) = 0$$

$$\frac{\partial}{\partial x}\psi_0(x) = -\frac{m\omega x}{\hbar}\psi_0(x) \rightarrow \int dx \frac{1}{\psi_0(x)} \frac{\partial}{\partial x}\psi_0(x) = \int dx -\frac{m\omega x}{\hbar} = = =$$

$$\psi_0(x) = Ae^{-m\omega x^2/2\hbar}$$

It is a Gaussian distribution. The Fourier transform of a Gaussian is also a Gaussian. Normalize the function to get

$$1 = \langle \psi | \psi \rangle = \int dx \psi_0^*(x)\psi_0(x) = [-] \Rightarrow A = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4}$$

And now we try to find $\psi_1(x)$.

7.2.1 I SKIPPED WHOOPS

$$\psi_n(x) = \left(\prod_{k=1}^n \frac{a_+}{\sqrt{k}} \right) \psi_0 = \frac{1}{\sqrt{n!}} a_+^n \psi_0$$

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(x) e^{-x^2/2}$$

$$H_n(\zeta) = (-1)^n e^{\zeta^2} \frac{\partial^n}{\partial \zeta^n} e^{-\zeta^2}$$

7.3 Discussion

is a physically valid wavefunction of a free particle

If energy is definite, it cannot bounce :(

If you put enough unphysical states together, you get a physical state.

7.3.1 Recap

$$V(x) = \frac{1}{2}m\omega^2x^2 \Rightarrow \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2 \right) \psi(x) = E\psi(x) \rightarrow$$

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}} (\pm i\hat{p} + m\omega\hat{x}) \quad \hat{a}_+ = (\hat{a}^\dagger) \quad \hat{a}_- = (\hat{a})$$

$$a_+a_- = \hat{n} \quad \hat{a}_+ |n\rangle = \sqrt{n+1} |n+1\rangle$$

$$\hat{a}_- |n\rangle = \sqrt{n} |n-1\rangle \quad [a_-, a_+] = 1 \frac{\hat{H}}{\hbar\omega} = a_+a_- + \frac{1}{2}$$

$$a_-(0) = |0\rangle \quad a_- |\psi\rangle = \psi_0(x) = A e^{\frac{-m\omega x^2}{2\hbar}}$$

Stationary States/Eigenfunctions are Gaussians centered at 0. $a_+ |0\rangle = |1\rangle$ has energy $\hbar\omega(1 + 1/2)$, but it contains noise carrying an energy of $1/2\hbar\omega$.

7.4 Relation to Classical S.H.O.

You have a mass oscillating around with $x \propto \cos(\omega t)$ and $p(t) \propto \sin(\omega t)$. The relationship between x and p looks like a circle.

What do the stationary states of \hat{H} look like in the phase space? The wavefunctions of the QSHO are of the form

$$\psi_0(x) = A_0 e^{\frac{-m\omega x^2}{2\hbar}} \quad \psi_1(x) = A_1 x e^{\frac{-m\omega x^2}{2\hbar}}$$

Take a fourier transform to get the momentum space

$$\psi_0(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar} \psi_0(x) dx = B_0 e^{\frac{-p^2}{2\hbar m\omega^2}} \quad \psi_1(p) = B_1 p e^{\frac{-p^2}{2\hbar m\omega^2}}$$

You can Guassians of some width and height that I didn't write down :(For a classical SHO, you have

$$x = x_0 \cos(\omega t) \quad E_{tot} = \frac{1}{2}m\omega^2x_0^2 \quad p(x)dx = 2\frac{dt}{T} \quad T = \frac{2\pi}{\omega}$$

$$dx = -x_0\omega \sin(\omega t)dt = -x_0\omega \sqrt{1 - \left(\frac{x}{x_0}\right)^2} dt \Rightarrow p(x) = \frac{1}{\pi x_0 \sqrt{1 - \left(\frac{x}{x_0}\right)^2}}$$

for the quantum world, we solve for x and p in terms of a_+ and a_-

$$a_{\pm} = \frac{1}{\sqrt{2\hbar m\omega}}(\pm i\hat{p} + m\omega\hat{x}) \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m}}\frac{\hat{a}_- + \hat{a}_+}{\sqrt{2}} \quad \hat{p} = \sqrt{\hbar m\omega}\frac{-i(\hat{a}_- - \hat{a}_+)}{\sqrt{2}}$$

$$\langle x \rangle = \langle n | \hat{x} | n \rangle \propto \langle n | (a_- + a_+) | n \rangle = 0$$

$$\langle \Delta x \rangle^2 = \langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle n | a_-^2 + a_- a_+ + a_+ a_- + a_+^2 | n \rangle$$

$$= \langle n | a_+ a_- + \frac{1}{2} | n \rangle = \frac{\hbar}{m\omega}(n + \frac{1}{2})$$

$$\langle p \rangle = 0 \quad \langle p^2 \rangle = \hbar m\omega(n + \frac{1}{2}) \quad \Delta x \Delta p = (n + \frac{1}{2})\hbar$$

We graph $\tilde{x} \equiv \frac{a_- + a_+}{\sqrt{2}}$ over $\tilde{p} \equiv \frac{-i(a_- - a_+)}{\sqrt{2}}$. We get concentric circles

$$\tilde{x}^2 + \tilde{p}^2 = 2\hat{n}$$

7.5 Coherent State

Let's attempt n such that $\langle n | x | n \rangle = 0$.

$$a_- |\psi_\alpha\rangle = \alpha |\psi_\alpha\rangle \quad \alpha \in \mathbb{C}$$

$$\langle \psi_\alpha | a_- + a_+ | \psi_\alpha \rangle = \alpha^* + \alpha = 2Re(\alpha)$$

$$\langle n | \psi_\alpha \rangle = \frac{1}{\sqrt{n!}} \langle 0 | a_-^n | \psi_\alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} \langle 0 | \psi_\alpha \rangle = \frac{\alpha^n}{\sqrt{n!}} A$$

$$|\psi_\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n | \psi_{alpha} \rangle = A \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \equiv A \sum_{n=0}^{\infty} \frac{(\alpha a_+)^n}{\sqrt{n!}} |0\rangle$$

$$A = e^{-|\alpha|^2/2}$$

A is the normalization constant from $\langle \psi_\alpha | \psi_\alpha \rangle = 1$. This is the most coherent quantum harmonic oscillator that we can get and this is the wave function of lasers. We can do all sorts

7.5.1 Time Evolution

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-i\omega tn} |n\rangle$$

Chapter 8

3D Quantum Mechanics

Trying to figure out the wave equation with the Coulomb Potential.

8.0.1 Assumptions

Only observe the wave function of the electron (because the nucleus is heavy so we can fix the position). Maybe something else I didn't pay attention.

8.1 Quantum Numbers

Because we're now in 3d space, we have 4 quantum numbers (x, y, z, spin).

Because we're working with a central potential, angular momentum is conserved, and angular momentum is **quantized**.

If an operator is conserved, it commutes with the Hamiltonian, which means it is **simultaneously diagonalizable**.

The magnitude of angular momentum is conserved, but the direction is not necessarily.

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \quad \hat{H} = \frac{\hat{p}^2}{2m} + V = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + V(x, y, z)$$
$$\hat{p} = -i\hbar\nabla \rightarrow \frac{\vec{p}^2}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$
$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}, t) + V\psi(\vec{r}, t)$$

8.2 Spherical Coordinates

Just google it every time you need to use it.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2}{\partial \phi^2}$$

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{r}, t) + V \psi(\vec{r}, t) = E \psi$$

Use separation of variables to find solutions

$$\psi(r, t) = R(r)Y(\theta, \phi)$$

$$-\frac{\hbar^2}{2m} \left(\frac{Y}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} + \frac{R}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial Y}{\partial \theta} \right) + \frac{R}{r^2 \sin^2(\theta)} \frac{\partial^2 Y}{\partial \phi^2} \right)$$

$$+ V(\vec{r}, t) RY = E * RY$$

$$\left(\frac{1}{R} \frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r} - \frac{2mr^2}{\hbar^2} [V(r) - E] \right)$$

$$+ \frac{1}{Y} \left(\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2 Y}{\partial \phi^2} \right) = 0$$

We now have an angular equation and a radial equation, with a separation constant $l(l + 1)$.

8.3 Angular Momentum

$$\hat{L} = \hat{r} \times \hat{p} = -i\hbar (\hat{r} \times \vec{\nabla}) \Rightarrow$$

$$L^2 = -\hbar^2 \left[\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \phi^2} \right]$$

The eigenstates are going to be $R(r)*Y(\theta, \phi)$ = an eigenfunction of angular momentum.

Because angular momentum is conserved, $[\hat{H}, \hat{L}^2] = 0$.

$$\hat{L} = \hat{r} \times \hat{p} \Rightarrow L_x = yp_z - zp_y \quad L_y = zp_x - xp_z \quad L_z = xp_y - yp_x$$

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

$$= [yp_z, zp_x] - 0 - 0 + [zp_y, xp_z] = yp_x[p_z, z] + xp_y[z, p_z]$$

$$= i\hbar (xp_y - yp_z) = i\hbar L_z$$

This looks awfully similar to the pauli matrices.

$$[L_x, L_y] = i\hbar L_z, [L_y, L_y] = i\hbar L_x, [L_z, L_x] = i\hbar L_y \quad [L_j, L_k] = i\hbar\epsilon_{jkm}L_m$$

The magnitude squared of angular momentum can be written as

$$\begin{aligned} L^2 &= L_x^2 + L_y^2 + L_z^2 & [L^2, L_x] &= [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x] \\ &= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z = 0 \end{aligned}$$

8.4 Discussion 6: Navigating the Gaussian (HALF MISSED)

The ground state is a Gaussian.

The raising operator has a $\sqrt{n+1}$ and the lowering operator has \sqrt{n} . If you start at state 12 and go to state 14, you'll get change of

$$\hat{a}^\dagger \hat{a}^\dagger \psi_{12} = \hat{a}^\dagger \sqrt{13} \psi_{13} = \sqrt{14} \sqrt{13} \psi_{14}$$

The harmonic oscillator gas **linear** energy dependence.

8.4.1 Questions

Find x^2 in terms of a_+ and a_-

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}} (+i\hat{p} + m\omega\hat{x}) \quad a_- = \frac{1}{\sqrt{2\hbar m\omega}} (-i\hat{p} + m\omega\hat{x})$$

$$a_+ + a_- = \frac{1}{\sqrt{2\hbar m\omega}} 2\hat{x} \Rightarrow x = \sqrt{2\hbar m\omega} \frac{a_+ + a_-}{2}$$

$$x^2 = \hbar m\omega \frac{(a_+ + a_-)^2}{2}$$

$$\langle n | x^2 | n \rangle = \frac{\hbar m\omega}{2} \langle n | (a_+ + a_-)^2 | n \rangle = \frac{\hbar m\omega}{2} \langle n | a_+^2 + a_-^2 + a_+ a_- | n \rangle$$

$$\langle n | a_+^2 | n \rangle + \langle n | a_-^2 | n \rangle + \langle n | a_+ a_- | n \rangle =$$

$$\langle n | \sqrt{(n+2)(n+1)} | n+2 \rangle + \langle n | \sqrt{(n)(n-1)} | n-2 \rangle$$

$$+ n \langle n | n \rangle + (n+1) \langle n | n \rangle = 0 + 0 + n \langle n | n \rangle + (n+1) \langle n | n \rangle = 2n + 1$$

8.4.2

Given a state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2} \right)$$

And we know the average position

$$\langle \hat{x} \rangle \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t)$$

do shenanigans to find $\langle p \rangle$

$$\hat{p} = \frac{\sqrt{2\hbar m\omega}}{2} (a_+ - a_-)$$

$$\langle \psi | \hat{p} | \psi \rangle = \frac{\sqrt{2\hbar m\omega}}{2} \langle \psi | (a_+ |\psi\rangle - a_- |\psi\rangle)$$

$$a_+ |\psi\rangle = \frac{1}{\sqrt{2}} \left(1 |1\rangle e^{-i\omega t/2} + \sqrt{2} |2\rangle e^{-3i\omega t/2} \right)$$

$$a_- |\psi\rangle = \frac{1}{\sqrt{2}} \left(0 + 1 |0\rangle e^{-3i\omega t/2} \right)$$

$$\begin{aligned} & \frac{1}{\sqrt{2}} \left(\langle 0 | e^{i\omega t/2} + \langle 1 | e^{3i\omega t/2} \right) \frac{1}{\sqrt{2}} \left(1 |1\rangle e^{-i\omega t/2} + \sqrt{2} |2\rangle e^{-3i\omega t/2} \right. \\ & \left. 1 |0\rangle e^{-3i\omega t/2} \right) \\ &= \frac{1}{2} \left(e^{-3i\omega t/2} e^{i\omega t/2} + e^{3i\omega t/2} e^{-i\omega t/2} \right) = \frac{1}{2i} \left(e^{-i\omega t/2} + e^{i\omega t/2} \right) = \sin(\omega t/2) \end{aligned}$$

there's an i is the thing for $\langle p \rangle$ trust me bro

8.4.3 3D harmonic oscillator

$$\frac{-\hbar}{2m} \nabla^2 |\psi\rangle + \frac{m\omega^2}{2} (x^2 + y^2 + z^2) |\psi\rangle = E |\psi\rangle$$

$$\frac{\hbar}{2m} \nabla^2 |\psi\rangle = \frac{m\omega^2}{2} (x^2 + y^2 + z^2 - E) |\psi\rangle$$

Consider just x

$$\frac{\hbar}{2m} \frac{\partial^2}{\partial x^2} |\psi\rangle = \frac{m\omega^2}{2} (x^2 - E) |\psi\rangle \Rightarrow \psi = e^{x^2} \quad \frac{\partial}{\partial x} e^{x^2} = 2x e^{x^2}$$

$$\frac{\partial}{\partial x} 2x e^{x^2} = 2e^{x^2} + 4x^2 e^{x^2} \quad \psi = X(x)Y(y)Z(z)$$

$$\frac{-\hbar}{2m} \left(2e^{x^2} + 4x^2 e^{x^2} \right) + \frac{m\omega^2}{2} e^{x^2} = E e^{x^2} \Rightarrow \psi_0 = A_0 e^{\frac{-m\omega x^2}{2\hbar}} \quad \psi_1 = A_1 x e^{\frac{-m\omega x^2}{2\hbar}}$$