MATH241

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Contents

Chapter 1

Calc I Review

- derivative $\rightarrow$ rate of change
- $\int_b^a g(x)dx =$ area under $g(x)$ from a to b
- **FUNDAMENTAL THEOREM OF CALCULUS**

$$\int_b^a f'(x)dx = f(b) - f(a)$$

Chapter 2

3D Coordinates

- x, y, AND z
  crazy, I know

12.2 Vectors

A vector is a quantity that has magnitude and direction.
A scalar is a quantity that has only magnitude.

Chapter 3

Reviewing all of Calc 3

Chapter 4

11 Basic Vectors

3D coordinates are goofy but I know just about everything for em

4.0.1 cross and dot product

goofy properties

$$u \cdot (v \times w) = (u \times w) \cdot v$$

$$u \times u = 0$$

$$u \times v = -(v \times u)$$

$$u \times v \text{ is orthogonal to both } u \text{ and } v$$

$$u \times v = 0 \text{ if u and v are scalar multiples}$$

$$u \cdot v = ||u||||v|| \cos(\theta) = 0 \text{ if u and v are orthogonal}$$

$$\text{proj}_v u = \frac{|v \cdot u|}{||v||}$$

4.1 11.5 planes

Given 3 points A, B, C, $AB \times BC$ = normal vector for the plane

given normal vector $n = \langle a, b, c \rangle$ and points $P(x_1, y_1, z_1)$

$$x = x_1 + at, y = y_1 + bt, z = z_1 + zt$$

$$a(x - x_1) + b(y - y_1) + z(z - z_1) = 0 \text{ or}$$
$$ax + by + zc + d = 0$$

Distance Formula

Let Q be a point and P any point on a plane and n the normal vector of said plane.

$$d = \frac{|\vec{PQ} \cdot \vec{n}|}{||n||} = \text{proj}_n PQ$$

Angle between two planes given their normal vectors n_1 and n_2

$$\cos(\theta) = \frac{n_1 \cdot n_2}{||n_1||||n_2||}$$

other distance shit that I need to probably look at later

plane and point, plane and plane, line and point

Chapter 5

12 3D Shenanigans

5.1 12.1

I remember how to graph shit in 3D

Practice Question:

$x^2 + z^2 = 9$

It is a cylinder of radius 3 parallel to the y-axis

Surfaces in space eh you understand

5.2 More Surfaces

Look at revolutions?

5.3 12.2 Vectors

Goofy parallelogram addition

vectors are built of components

$$\vec{v} = \langle x, y, z \rangle$$

$$\text{Proj}_v u = \left(\frac{\vec{u} \cdot \vec{v}}{||v||^2} \right) \vec{v} = \text{projection of v onto u} = \vec{v}\text{cos}(\theta)$$

$$cos(\theta) = \left|\left| \frac{\text{proj}_v u}{||u||} \right|\right| = \frac{\vec{u} \cdot \vec{v}}{||u||||v||}$$

Chapter 6

13 Vector Functions

6.1 Unit Tangent Vector

13.2

$$\mathbf{T}(t) = \frac{r'(t)}{|r'(t)|} \text{ where r is a vector function}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \text{ Normal unit vector}$$

6.2 Arc Length and Curvature

13.3

$$L = \int_a^b \sqrt{(\frac{dx}{dt}) + (\frac{dy}{dt}) + (\frac{dz}{dt})}\text{dt} = \text{arclength from a to b}$$

$$L(t) = \int_a^t \sqrt{(\frac{dx}{du}) + (\frac{dy}{du}) + (\frac{dz}{du})}du = \text{arc length parameter}$$

$$\frac{ds}{dt} = |r'(t)|$$

if $||r'(t)|| = 1$, then t is the arc length parameter. That is, $t = s(t)$.

6.2.1 Curvature

$$K = \frac{|y|}{(1+(y')^2)^{3/2}} = |\frac{d\mathbf{T}}{ds}| = \frac{d\mathbf{T}/ds}{ds/dt} = \frac{|\mathbf{T}'(t)|}{r'(t)} = \frac{|r' \times r''|}{|r'|^3}$$

$$a(t) = \frac{d^2(x)}{dt^2}T + K\left(\frac{ds}{dt}\right)^2 N \text{ where } \frac{ds}{dt} = \text{speed}$$

6.2.2 Binormal Vector

perpendicular to both T and N

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

6.2.3 Torsion

$$\tau(t) = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|r'(t)|} = \frac{[r'(t) \times r''(t)] \cdot r'''(t)}{|r'(t) \times r''(t)|^2}$$

Chapter 7

14 Partial Derivatives

Literally just think of the numbers you're not deriving as a constant Let $f(x, y) = 3x - x^2y^2 + 2x^3y$

$$f_x(x, y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = 3 - 2xy^2 + 6x^2y$$

$$f_y(x, y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = 2x^2y + 2x^3$$

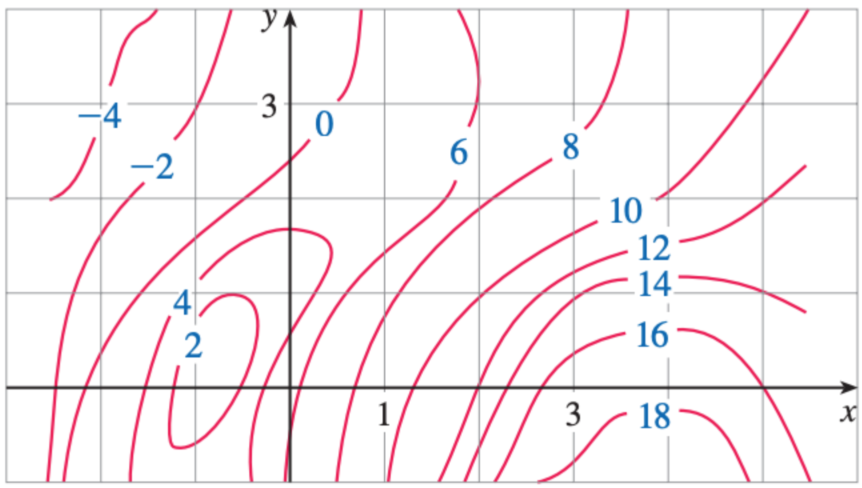
7.0.1 higher order notation

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y)$ notice the subscript order

7.0.2 Partial Derivative Problem

14.3 #6

Estimate $f_x(2,1)$ and $f_y(2,1)$ in the following contour map.



(2, 1) is on the contour line $z = 10$. Because the 12 contour line is as about $(2.66, 1)$, $f_x(2,1)$ will probably have a value of about 3. $f_y(2,1)$ will be about -2 because of where the 8 contour line is located.

$$f_x(2,1) \approx 3 \qquad f_y(2,1) \approx 2$$

Quizlet+ give or take agrees phew

7.1 Limits

Basically set one of the numbers to a constant or the other variable and see what happens.

Try y=0, x=0, y=x, x=y, y=mx, x=my, idfk

7.1.1 Limit Practice Problem

2012 practice midterm 1 problem

Exactly one of the two limits exists, show which and why.

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{\sqrt{x^2+y^2}} \qquad \lim_{(x,y)\to(0,0)} \frac{xy}{(x^2+y^2)^2}$$

I can factor the first one

$$\frac{\sqrt{x^2+y^2}\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}} = \sqrt{x^2-y^2} = 0 \text{ definitely exists}$$

Now I'll prove the other one doesn't exist just for funsies.

line $y=0$, $\lim = 0$

line $y=x$, lim DNE

7.2 Tangent Planes and Linear Approximations

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Calculate the differential and then replace dx and dy with Δx and Δy

Yea I was basically right. Use the tangent plane as an approximation

$$f(x_0,y_0) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

7.3 Differentials

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy$$

Theorem

If f_x and f_y are are continuous, then f is differentiable.

7.3.1 Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b).
If f_{xy} and f_{yx} are both continuous on D, then
$f_{xy}(a, b) = f_{yx}(a, b)$

7.4 Chain Rule for Partial Derivatives

14.5

page 1020

Let $w = f(x, y)$ where $x. = g(t)$ and $y = h(t)$. Then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

7.4.1 Chain Rule Problem

2012 Practice Midterm 1 #11

An exceptionally tiny spaceship positioned as shown is travelling so that its x-coordinate increases at a rate of 1/2 m/s and y-coordinate increases at a rate of 1/3 m/s. Use the Chain Rule to calculate the rate at which the distance between the spaceship and the point (0, 0) is increasing.

$$\frac{\partial x}{\partial t} = \frac{1}{2}t, \frac{\partial y}{\partial t} = \frac{1}{3}t, w = \sqrt{x^2 + y^2}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial w}{\partial x} = \frac{1}{2}(x^2 + y^2)^{-1/2}2x \qquad \frac{\partial w}{\partial y} = \frac{1}{2}(x^2 + y^2)^{-1/2}2y$$

Just plug the numbers in, I don't have the actual answer for this one but this looks correct.

7.4.2 Implicit Differentiation

Let $F(x, y) = 0$ and let $y = f(x)$

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

7.5 Directional Derivatives and Gradients

If f is a differentiable function of x and y, and f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b \rangle$, then

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Gives the slope of the function in the direction of $\vec{u}$

$$\nabla f(x,y) = f_x(x,y)\mathbf{i} + f_y(x,y)\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

The gradient exists such that $\nabla f(x,y) \cdot \mathbf{u} = D_u f(x,y)$

7.6 Extrema

The maximum possible directional derivative is $||\nabla f(\vec{x})||$, and it occurs when $\vec{u}$ is in the direction of $\nabla f(\vec{x})$

$$D_u f = \nabla f \cos(\theta)$$

7.7 Tangent Planes to Level Surfaces

Let S be a surface with the equation $F(x,y,z) = k$. Let $P = (x_0, y_0, z_0)$ be a point on S. Let C be a curve on S that passes through P. C has the equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$. Let t_0 correspond to P, meaning $r(t_0) = P$.

We can derive F to get

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0 = \nabla F \cdot \mathbf{r}'(t)$$

We can use this and dot product properties to show that the gradient of F is orthogonal to the tangent vector of C.

Therefore, the gradient can be the normal vector for a plane tangent to S. So our tangent plane equation will be

$$F_x(x_0, y_0, z_0)(x-x_0)+F_y(x_0, y_0, z_0)(y-y_0)+F_z(x_0, y_0, z_0)(z-z_0) = 0$$

7.8 Extrema

14.7

If $f(a, b)$ has a local extrema at (a, b) and the first order partial derivatives of $f(a, b)$ exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

7.8.1 Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that (a, b) is a critical point of f. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$ then $f(a, b)$ is a local minimum

2. If $D > 0$ and $f_{xx}(a, b) < 0$ then $f(a, b)$ is a local maximum
3. If $D < 0$, them $f(a, b)$ is a saddle point

7.8.2 Extrema Problem

14.7 #5

Find the extrema of $f(x, y) = x^2 + xy + y^2 + y$

$$f_x = 2x + y | f_y = 2y + x + 1 | f_{xx} = 2 | f_{yy} = 2 | f_{xy} = 1$$

Critical points uhh somewhere

$2x = -y$

$-4x + x + 1 = 0 \rightarrow x = 1/3, y = -2/3$

$$D = f_{xx} f_{yy} - (f_{xy})^2 = 2 * 2 - 1 = 3$$

$D < 0 =$ saddle point else
$f_{xx} > 0 =$ minimum
$f_{xx} < 0 =$ maximum

Minimum at $(1/3, -2/3)$

7.9 Lagrange Multipliers

How to find all the maximum and minimum values of $f(x, y, z)$ under the constraints that $g(x, y, z) = k$ for some equation g.

Step 1: Find all values such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, yz) \text{ and } g(x, y, z) = k \text{ for some scalar } \lambda$$

Step 2: Evaluate at all points, the biggest is a maximum, the smallest is a minimum.

7.9.1 Two contraints

Let $g(x, y, z) = k$ and $h(x, y, z) = c$

$$\nabla f(x, y, z) = \lambda g(x, y, z) + \mu h(x, y, z)$$

Find the components to get enough equations to solve for the 7 billion variables.

7.10 Lagrange Multipliers Example Problems

pg 1061 #6

$$f(x,y) = xe^y \qquad g(x,y) = x^2 + y^2 = 2$$

$$\nabla g(x,y) = (2x)\vec{i} + (2y)\vec{j} \qquad \nabla f(x,y) = (e^y)\vec{i} + (xe^y)\vec{j}$$

$2x = \lambda e^y$
$2y = \lambda xe^y$
$x^2 + y^2 = 2$

$2x = \lambda e^y$
$y = x^2$

$$x^4 + x^2 - 2 = 0 \to x = \frac{-1 \pm 3}{2} \to x^2 = -2, 1, x = \pm 1$$

$x = -2, y = 4 nopenotactuallyallowed$
$-4 = \lambda e^4 \to \lambda = -4/e^4$ I dont think this is necessary

$(1,1)$ is a maximum this was a waste of my time $(-1,1)$ is a minimum lets fucking go I actually did it right omg

7.10.1 pg 1061 # 7

$$f(x,y) = 2x^2 + 6y^2, \qquad g(x,y) = x^4 + 3y^4 = 1$$

$$\nabla f(x,y) = 4x\mathbf{i} + 12y\mathbf{j}, \qquad \nabla g(x,y) = 4x^3\mathbf{i} + 12y^3\mathbf{j}$$

$4x = \lambda 4x^3$
$12y = \lambda 12y^3$
$x^4 + 3y^4 = 1$

$\lambda = 1/x^2$
$1 = y^2/x^2 \rightarrow \pm x = \pm y$
$4x^4 = 1 \rightarrow x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{2}$

time to figure out all the sets of points that work
$(1/\sqrt{2}, 1/\sqrt{2})$
$(-1/\sqrt{2}, 1/\sqrt{2})$
$(1/\sqrt{2}, -1/\sqrt{2})$
$(-1/\sqrt{2}, -1/\sqrt{2})$

All these points have the exact same f(x, y) values so theyre all maximums?

Ah damn I missed one. the minimums are $(\pm 1, 0)$ but I give or take understand

pg 1062 # 33

$$f(x,y,z) = yz + xy \qquad xy = 1 \qquad y^2 + z^2 = 1$$

$f_x = y = \lambda y$
$f_y = z + x = \lambda x + \mu 2y$
$f_z = y = \mu 2z$
$xy = \lambda$
$y^2 + z^2 = 1$

$\lambda = 1$
$z = \mu 2y$
$y = \mu 2z$
$z = y/(2\mu)$
$4\mu^2 = 1 \to \mu = \pm 1/2$ **IMPORTANT**
$y^2 = z^2 = \pm 1/\sqrt{2}$
$x = \pm\sqrt{2}$

$(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$
$(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
$(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$f(x,y,z) = 3/2 =$ maximum
$f(x,y,z) = 1/2 =$ minimum

Okay I think I understand this shit assuming I'm given a g(x, y, z)

Chapter 8

15 Multiple Integrals

$$\iint\limits_R f(x,y)dA = V$$

8.1 Iterated Integrals

Just consider whatever you aren't integrating as constant

$$\int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx$$

Example

$$\int_1^2\int_0^3 x^2y\,dx\,dy \to \int_1^2 \left.\frac{x^3y}{3}\right|_0^3 dy \to \int_1^2 9y\,dy$$

$$\left.4.5y^2\right|_1^2 \to 18 - 4.5 = 13.5$$

Fubini's Theorem

If $f(x, y)$ is continuous on a rectangle R, then

$$\iint\limits_R f(x,y)\,dA = \int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y)\,dx\,dy = \int_{x_0}^{x_1}\int_{y_0}^{y_1} f(x,y)\,dy\,dx$$

If $f(x, y)$ can be factored into 2 functions multiplying each other, meaning $f(x, y) = g(x)h(y)$, then

$$\iint\limits_R f(x,y)dA = \int_{y_0}^{y_1}\int_{x_0}^{x_1} g(x)h(y)\,dx\,dy = \int_{x_0}^{x_1} g(x)\,dx \int_{y_0}^{y_1} h(y)\,dy$$

8.2 Average Value

$$f_{avg} = \frac{1}{A(R)}\iint\limits_R f(x,y)dA$$

8.3 Integrating Over General Regions

$$\iint\limits_D f(x,y) = \int_{x_0}^{x_1}\int_{g(x)_0}^{g(x)_1} f(x,y)dydx$$

8.3.1 Changing Order of Integration

Just fucking floop the shit

$$\int_0^1 \int_x^1 f(x,y)dydx \longrightarrow \int_0^1 \int_0^y f(x,y)dxdy$$

$y = x$ so $x = y$ and $y = 1$ where $x = 0$

Think of it in picture, thats like literally the only way to do it

8.4 15.5 Surface Area

$$A(S) = \iint\limits_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$$

8.5 15.6 Triple Integrals

It's like a double integral but another one.

I skipped a bunch of shit but I also dont give a shit I can figure it out fuck you

Chapter 9

16 Vector Calculus

Vector Fields

$$\mathbf{F}(a,b) = P(a,b)\mathbf{i} + Q(a,b)\mathbf{j} = \langle P(a,b), Q(a,b)\rangle$$

Gradient is a vector field

9.0.1 Dfn: Conservative

A vector field $\mathbf{F}$ is conservative if it acts as the gradiant for some scalar function. That is, there exists a function $f(x,y)$ such that

$$\mathbf{F} = \nabla f(x,y)$$

9.1 16.2 Line Integrals

Integrate over a line instead of a regular region

$$\text{arclength } L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)} dt$$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right) + \left(\frac{dy}{dt}\right)} dt$$

Integrating over x and y

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t)) x'(t) dt$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t)) y'(t) dt$$

9.1.1 Line Integral Problem

16.2 # 9 $\int_c x^2 y\, ds \qquad C = \langle \cos(t), \sin(t), t \rangle (0 < t < \pi/2)$

$$\int_0^{\pi/2} \cos^2(t) \sin(t) \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\, dt$$

$$\int_0^{\pi/2} \cos^2(t) \sin(t) \sqrt{(-\sin(t))^2 + (\cos(t))^2 + 1}\, dt \rightarrow$$

$$\sqrt{2} \int_0^{\pi/2} \cos^2(t) \sin(t)\, dt \qquad u = \cos(t), -du = \sin(t) dt$$

$$-\sqrt{2}\int_1^0 u^2\,du = -\sqrt{2}u^3/3\Big|_1^0 = \sqrt{2}/3$$

9.1.2 Integrating Over a Vector Field

Let $\mathbf{F}$ be integrated over a smooth curve C.
Let $F = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$

$$\int_C \mathbf{F}\cdot dr = \int_a^b \mathbf{F}(r(t))\cdot r'(t)dt = \int_C \mathbf{F}\cdot\mathbf{T}ds = \int_C Pdx + Qdy + Rdz$$

9.2 The Fundamental Theorem For Line Integrals

$$\int_C \nabla f(x, y) \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

also shows how conservative fields get the same shenanigans independent of path

9.2.1 Independence of Path Theorem

$\int_C \mathbf{F} \cdot dr$ is independent of the path taken iff $\int_C \mathbf{F} \cdot dr = 0$ for every closed path C (every loop)

9.3 16.4 Green's Theorem

relationship between a double integral of a region and a line integral over the border of that region.

Let C be a positively oriented (*meaning counterclockwise*), piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

9.4 Curl

Curl is associated with rotation around a point. The magnitude of curl is the speed of rotation, and the direction of curl is the axis of rotation.

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Imagine it as a cross product

$$\nabla \times \mathbf{F} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$\text{curl}(\nabla f) = 0$$

Theorem

If $\mathbf{F}$ is function whose components have continuous partial derivatives and $\text{curl}(\mathbf{F}) = 0$, then $\mathbf{F}$ is a conservative vector field.

9.5 Divergence

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \boldsymbol{\nabla} \cdot \mathbf{F}$$

$$\text{div curl } \mathbf{F} = 0$$

9.6 Vector Forms of Green's Theorem

$$\oint_C \mathbf{F} \cdot dr = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{k}\, dA$$

also

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D (\boldsymbol{\nabla} \cdot 2\mathbf{F}(x, y))\, dA$$

9.7 Parametric Surfaces

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

9.7.1 Tangent Planes

$$r_u \times r_v = n$$

9.7.2 Surface Area

$$A(S) = |r_u \times r_v|\, dA$$

Surface Area of Graphs of Functions

x=x, y=y, z = f(x, y)

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

9.8 Surface Integrals

$$\iint_S f(x,y,z)d\mathbf{S} = \iint_D f(r(u,v))|r_u \times r_v|dA$$

9.8.1 Graphs of Functions

x=x, y=y, z=f(x, y)

$$\iint_S f(x,y,z)d\mathbf{S} = \iint_D f(x,y,f(x,y))\sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dA$$

Similar vibes as line integrals using arclength

9.9 Oriented Surfaces

the unit vector for certain surfaces can be either n or -n. Let S be a surface given by the vector function $r(u,v)$

$$n = \frac{r_u \times r_v}{|r_u \times r_v|}$$

Yea I don't actually entirely know how the orientation changes things I'll be honest

9.10 Flux

if $\mathbf{F}$ is a continuous vector field over a surface S with a normal vector $\mathbf{n}$, then the Surface Integral of $\mathbf{F}$ over S, or the Flux of

$\mathbf{F}$ over S, is:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r_u} \times \mathbf{r_v}) dA$$

Where D is the parameter domain.

if S is given by $g(x, y) = z$

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle P, Q, R \rangle \cdot \langle -\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1 \rangle \, dA$$

9.11 Stoke's Theorem

let $\mathbf{F}$ be a piecewise smooth surface bounded by S, a region with a boundary C with positive (counterclockwise) orientation

$$\int_C \mathbf{F} d\mathbf{r} = \iint_S \text{ curl } \mathbf{F} \cdot d\mathbf{S}$$

Literally just generalized Green's Theorem for higher dimensions.

9.12 Divergence Theorem

Green's Theorem Extended to Vector Fields

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let F be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \boldsymbol{\nabla} \cdot \mathbf{F} \, dV$$

Chapter 10

Extras: Cylindrical and Spherical Coordinates

10.1 Polar Coordinates

$$\iint_S f(x, y)dA = \iint_D f(r\cos(\theta), r\sin(\theta)r\, dr\, d\theta)$$

10.2 Cylindrical Coordinates

instead of (x, y, z), you got (r, θ, z), where θ is counter-clockwise relative to the $+x$ line

$$\iiint_A f(x, y, z)dA = \iiint_A f(r\cos(\theta), r\sin(\theta), z)r\, dzdrd\theta$$

10.3 Spherical Coordinates

instead of (x, y, z), you got (r, θ, ϕ), where ϕ goes down from the $+z$ line

$$\iiint_A f(x, y, z)dV =$$

$$\iiint_A f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\psi))\rho^2\sin(\phi)d\rho d\theta d\phi$$

10.4 Jacobians

A 1D Jacobian is just a u-sub.

Let $x = g(u, v)$ and $y = h(u, v)$. The Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Change of Variables

$$\iint\limits_R f(x, y)dA = \iint\limits_S f(g(u, v), h(u, v))\left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, qdv$$

3 variables

$$\iiint\limits_R f(x, y, z)dV = \iiint\limits_S f(x(), y(), z())\left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw$$

(3d determinant)

$$a\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

Chapter 11

Test Problems

11.1 Midterm 2 2012 #4

Let C be the curve in $\mathbb{R}^3$ parameterized by $r(t) = \langle \sin(t), 2t, \cos(t) \rangle$
Compute the length of C over $0 < t < \pi/2$.

$$\text{arclength}(C) = \int_0^{\pi/2} \sqrt{\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}}\, dt$$