PHYS 325

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Contents

Chapter 1

PHYS325

I missed everything from the first 2 weeks because my laptop exploded whoopsies

Chapter 2

Equations of motion

derive the equations of motion to solve for a trajectory $\vec{r}(t)$ which is a position vector with respect to time

2.1 Newton's Second Law

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}(t)}{dt^2} \qquad \vec{F} = \frac{d\vec{p}(t)}{dt}$$

2.2 Strategy

1. choose a reference frame and coordinates
2. identify all the relevant forces (external forces)

   make a force diagram lol
3. integrate N2 for a given force $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ to find $\vec{r}(t)$

4. fix integration constants from inital or boundary conditions. (e.g. $\vec{v}_0 = \vec{v}(t=0) = 0$)

5.

$$\vec{F} = 0$$
$$from N2L 0 = \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$
$$\rightarrow \vec{v} = const = \vec{v}_0$$
$$\vec{r}(t) = \int v_0 dt = v_0 t + r_0$$

6. if F is constant

$$\dot{\vec{v}} = \ddot{\vec{r}} = \vec{a} = \frac{\vec{F}_0}{m}$$
$$\int dv = \int \frac{F_0}{m} dt \rightarrow \vec{v}(t) = \frac{F_0}{m} t + \vec{v}_0$$
$$\vec{r}(t) = \int dr = \int \vec{v} dt =$$
$$\frac{1}{2}\frac{\vec{F}_0}{m} t^2 + \vec{v}_0 t + \vec{r}_0$$

only valid for constant force

2.2.1 Time Dependent Force

$$\frac{d\vec{v}}{dt} = \vec{a} = \frac{\vec{F}}{m}$$

separation of variables

$$d\vec{v} = \frac{\vec{F}(t)}{m}dt$$

$$\int d\vec{v} = \int \frac{\vec{F}(t)}{m}dt$$

$$\vec{v}(t) = \frac{\vec{F}}{m} + \vec{C} \qquad \vec{F} = \int F(t)dt$$

C is the integration constant

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{r} = \int \vec{v}dt$$

2.2.2 Forced Harmonic Oscillator

A particle m moves along $-\infty < x < \infty$. It is subjected to a force $F = F_0 \cos(\alpha t)$. It starts at time $t = 0, x_0 = x(t = 0) = 0, v_0 = v(t = 0) = 0$

1. coordinate system is just 1D

2. force is $F = F_0 \cos(\alpha t)$

3. equation of motion from N2L is $\frac{dv}{dt} = a = \frac{F}{m}$

4. separation of variables

$$dv = \frac{F}{m}dt = \frac{F_0}{m}dt = \frac{F_0}{m}\cos\alpha t dt$$
$$v(t) = \int dv = \frac{F_0}{m}\int \cos(\alpha t)dt = \frac{F_0}{m}\frac{1}{2}\sin\alpha t + C_1$$
$$x(t) = intdx = \int vdt =$$
$$\int \frac{F_0}{\alpha m}\sin(\alpha t) + C_1 dt =$$
$$-\frac{F_0}{\alpha^2 m}\cos(\alpha t) + C_1(t) + C_2$$

5. find initial conditions

$$0 = v_0 = \frac{F_0}{\alpha m}\sin\alpha 0 + C_1 \rightarrow C_1 = 0$$
$$0 = x_0 = -\frac{F_0}{\alpha^2 m}\cos(\alpha 0) + 0 + C_2 \rightarrow$$
$$C_2 = \frac{F_0}{\alpha^2 m}$$
$$x(t) = \frac{F_0}{\alpha^2 m}(1 - \cos(\alpha t))$$
$$v(t) = \frac{F_0}{\alpha m}\sin(\alpha t)$$

2.3 Position Dependent Force

focusing on 1 dimension for simplicity

get the equation of motion from Newton's 2nd Law

$$F(x) = ma = m\frac{dv}{dt}$$

chain rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$mv\frac{dv}{dx} = F(x)$$

separation of variables

$$mv\,dv = F(x)dx$$

Use definite integrals. relabel v = v' and x = x' (not derivatives)

$$m\int_{v_0}^{v} v'\,dv' = \int_{x_0}^{x} F(x')\,dx'$$

$$\frac{1}{2}m(v^2 - v_0^2) = \int_{x_0}^{x} F(x')\,dx'$$

solve for v

It looks like change in kinetic energy and work

$$\Delta T = \frac{1}{2}m(v^2 - v_0^2) \qquad W = \int F(x)\,dx$$

if force is conservative, it is path independent,

and it can be written as a gradient of a potential $\vec{F} = -\nabla U$

$$F = -\frac{dU}{dx}$$

$$T - T_0 = \int_{x_0}^{x} -\frac{dU}{dx}\,dx' = -\left(U(x) - U(x_0)\right)$$

$E = T + U(x) = T_0 + U(x_0)$

Conservation of Mechanical Energy

$E = T + U(x) = \frac{1}{2}mv^2 + U(x)$

$v = \pm\sqrt{\frac{2}{m}(E - U(x))}$

use $v = \frac{dx}{dt}$ to find $x(t)$

2.4 Analyzing the Potential

infer velocity and position

given energy, what is motion

$E = E_0 = E(x_0) = T(x_0) + U(x_0)$

$U(x_0) = E_0 = \text{const} \rightarrow T(x_0) = 0 \rightarrow v(x_0) = C$

2.4.1 case 2: energy of particle is _

$E = E_1 = T(x) + U(x)$

particle can move between x_{1a} and x_{1b}

$x_0 : E_1 = E = T + U(x_0)$

potential energy is at minimum, kinetic at max

$v(x_0)$ is maximal

2.4.2 Case 3: $E = E_2 = U(x_2)$

$T(x_2) = 0$ so $v = 0$

2.5 Analyzing Extrema of Potential

To find the extrema points, find the derivative of the potential and set it equal to 0.

To find the type of extrema point, take the second derivative and then check the sign

2.5.1 Push Particle Towards Extrema

1. $x = x_1$ is a minimum

   If the particle is slightly moved from the minimum, it will return to the minimum because that is where the lowest energy point is.

   This point is called stable for the particle.

2. $x = x_2$ is a maximum

   The particle will move farther away from the equilibrium point.

   This point is unstable

3. $x = x_3$ is a saddle point

   The particle is stable in one direction and unstable in the other marginally stable point

2.6 Simple Harmonic Oscillator

Simple Harmonic Oscillator

Figure out motion near the equilibrium point

To approximate a function only near a specific point, use a Taylor Series

Taylor expand potential around x_0

$$U(x) \approx U(x_0) + U'|_{x=x_0}(x - x_0) + \frac{1}{2}U''|_{x=x_0}(x - x_0)^2 + \dots$$

choose $U(x_0) = 0$ extremum

$$U(x) \approx \frac{1}{2}kx^2$$

Can also taylor expand force near x_0

$$F(x) \approx F(x_0) + F'|_{x=x_0}(x - x_0) + \frac{1}{2}F''|_{x_0}(x - x_0)^2$$

$$-U'|_{x_0} = 0 \qquad -U''|_{x_0} = -k \qquad \text{rest is small}$$

insert equation of motion

$$m\ddot{x} = F(x) \approx F'|_{x_0}(x - x_0) = -kx$$
$$m\ddot{x} + kx = 0$$

ansatz: guess the form of the solution (trig or exponent)

$$x(t) = A\sin(\omega t + \varphi)$$
$$\dot{x}(t) = \frac{dx}{dt} = A\omega\cos(\omega t + \varphi)$$
$$\ddot{x}(t) = -A\omega^2\sin(\omega t + \varphi)$$

insert ansatz into equation of motion

$$m(-A)\omega^2\sin(\omega t + \varphi) + kA\sin(\omega t + \varphi) = 0 \rightarrow$$
$$\omega = \sqrt{\frac{k}{m}} \qquad \text{period} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

2.7 velocity Dependent Force

$\vec{F} = \vec{F}(\vec{v})$

drag forces, friction, air resistance

2.7.1 Types of drag forces

1. Stoke's drag (linear)

$$\vec{F} = -c\vec{v}$$

laminar flow, valid for small velocities and viscous fluids

2. Newtonian Drag (nonlinear)

$$\vec{F}(\vec{v}) = -k\vec{v}^2$$

valid for larger velocities and less viscous fluids.

The type of drag is applicable based of Reynold's Number

$$R = \frac{\rho v L}{\mu} \qquad \text{density, fluid flow, size, viscosity}$$

2.8 Linear Drag

particle of mass m and initial velocity v_0 in laminar flow
goal, derive velocity after a long time (should be 0)
choose coordinate system: 1D along x direction $v = \dot{x}$
force $F(v) = -cv$
introduce new constant $\kappa = \frac{c}{m}$

$$[k] = [\frac{F}{m * v}] = [\frac{1}{t}]$$

$$F = -cv = -m\kappa v$$

$$F = m\ddot{x} = m\frac{dv}{dt} = -m\kappa v$$

separation of variables

$$\frac{1}{v}dv = -\kappa dt$$

use definite integrals, relabel variables

$$\int_{v_0}^{v} \frac{1}{v'}dv' = -\kappa \int_{t_0}^{t} dt'$$

$$-\kappa t = \ln(v')\Big|_{v_0}^{v} = \ln\left(\frac{v}{v_0}\right)$$

$$v(t) = v_0 \exp(-\kappa t)$$

exponential decay with a rate of κ

For some sanity checks, you can try $t = \infty$ and $t = 0$ to make sure they make sense

2.9 non-linear drag with gravity

motion is 1D along the z axis

drag force $F = -cv^2$ sgn(v)

(sgn(v) is ± 1 depending on if v is greater or less than 0)

new constant

$$\sigma = \frac{c}{m}$$

$$F = -mg + cv^2 \rightarrow -mg + \frac{C}{m}mv^2 \rightarrow m(g - \sigma v^2) \rightarrow$$
$$\frac{dv}{dt} = -g + \sigma v^2 \rightarrow \int_0^v \frac{1}{-g + \sigma v^2} dv = \int_0^t dt$$
$$t = \frac{-1}{\sqrt{g\sigma}} \tanh^{-1}(v\sqrt{\frac{\sigma}{g}})$$
$$v = -\sqrt{\frac{g}{\sigma}} \tanh\left(\sqrt{gh}t\right)$$
$$t \rightarrow \infty \qquad v \rightarrow -\sqrt{\frac{g}{\sigma}}$$

You can see that v asymptotically approaches a constant velocity with large time which you can kind of see from the initial equation.

For most cases $m\frac{dv}{dt} = \vec{F}(\vec{r}, \vec{v}, t)$ is unsolvable. However, for some cases $F = f(v)g(t)$, you get $\int \frac{m}{f(v)} dv = \int g(t)dt$

For another special case $F = f(v)h(x)$, you get

$$m\frac{dv}{dt}\frac{1}{f(v)} = h(x)$$
$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$
$$mv\frac{dv}{dx}\frac{1}{f(v)} = h(x) \rightarrow \int \frac{mv}{f(v)} dv = \int h(x)dx$$

2.10 Time Varying Mass $M(t)$

u is the speed of the mass being projected out.

$$P_i = M_i v_i$$
$$P_f = dm(v-u) \to t(m_f v_f) = fm(v-u)$$
$$P_i = P_f \Rightarrow Mv_i = -udm + m_i v_i - dMdv$$
$$dMdv \approx 0 \text{ and cancel some stuff}$$
$$Mdv = tudm \Longrightarrow dv = \frac{-u}{M}dm \to \int dv = \int \frac{-u}{M}dm \to$$
$$v - v_0 = -u\ln\left(\frac{M}{M_0}\right) \to v = v_0 - u\ln\left(\frac{M}{M_0}\right)$$

Because its logarithmic, its really hard to boost something by chucking mass out the back.

Conservation of momentum is not true in gravity $P_i \neq P_f$

$$dp = P_f - P_i \neq 0$$
$$dp = -Mgdt - Mg\,dt = Mdv + udM$$
$$\frac{dv}{dt} = -g - \frac{u}{M}\frac{dM}{dt} \qquad \text{THIS IS THE IMPORTANT EQUATION}$$

u, M, and dM/dt are all controllable by the rocket

2.10.1 velocity from acceleration

$$v(t) = v_0 - gt - u\ln\left(\frac{M_0 - dt}{M_0}\right)$$

2.11 2D Rocket

$$\vec{F}(\vec{r},\dot{\vec{r}},t) = m\vec{a} = m\ddot{\vec{r}}$$

Cartesian Coordinates

$$F_x(\vec{r},\dot{\vec{r}},t) = m\ddot{x} \qquad F_y(\vec{r},\dot{\vec{r}},t) = m\ddot{y} \qquad F_z(\vec{r},\dot{\vec{r}},t) = m\ddot{z}$$

Consider a magnetic field Be_z

$$\vec{F} = q\vec{v}\times\vec{B} = q(\vec{v}\times\vec{B}) \rightarrow$$
$$qB(v_y e_x - v_x e_y) \qquad F_x = qBv_y = m\ddot{x} \qquad F_y = -qBv_x = m\ddot{y}$$
$$m\ddot{z} = 0 \implies \dot{z} = \text{const}$$
$$qB\dot{y} = m\ddot{x} \qquad qB\dot{x} = m\ddot{y}$$
$$qB\ddot{y} = m\dddot{x} \rightarrow qB(\frac{-qB}{m})\dot{x} = m\dddot{x} \rightarrow \ddot{v}_x = -\omega^2 v_x$$

this is the spring equation

$$\ddot{x} = \frac{k}{m}x \longrightarrow v_x = A\sin(\omega t+\phi) \qquad v_y = A\cos(\omega t+\phi)$$

it goes in a circle

2.11.1 something

$$\ddot{v}_x = \omega \dot{v}_y$$
$$\ddot{v}_x = -\omega^2 v_x \qquad \ddot{v}_y = -\omega^2 v_y$$
$$\dot{\nu} = \dot{v}_x + i\dot{v}_y$$
$$= \omega v_y - i\omega v_x$$
$$\dot{\nu} = -i\omega\nu$$

and some stuff I missed

2.11.2 Particle in Field

Motion of a charged particle in a homogeneous magnetic field

$$\dot{v}_x = \omega v_y \qquad \dot{v}_y = -\omega v_x \qquad w/\omega = \frac{qB}{m}\mu = v_x + iv_y$$
$$\dot{m}u = \dot{v}_x + i\dot{v}_y \longrightarrow \omega v_y - i\omega v_x \to -i\omega(v_x + iv_y)$$
$$\dot{\mu} = -i\omega\mu$$

solve with separation of variables

$$\frac{1}{\mu}\,d\mu = -i\omega\,dt \to \ln(\mu) + \tilde{C} = \ln\left(\frac{\mu}{C}\right)$$
$$\mu = C\exp(-i\omega t) \qquad C = Ae^{i\delta}$$
$$\mu = A\exp(i(\delta - \omega t))$$

Use the euler identity to keep doing math

$$\mu = A(\cos(\omega t - \delta) - i\sin(\omega t - \delta))$$

the signs reversed and im not entirely sure why or how

$$v_x = \mathrm{Re}(\mu) = A\cos(\omega t - \delta) \qquad v_y = \mathrm{Im}(\mu) = -A\sin(\omega t - \delta)$$

$$|v^2| = A^2\cos(\omega t - \delta) + A_2\sin(\omega t - \delta) = A^2$$

double check this is true by showing derivative of v is 0

$$\frac{1}{2}\frac{d}{dt}|v|^2 = \frac{1}{2}m2v \cdot \frac{d}{dt}v$$

$$= v \cdot q(v \times B) = 0$$

$$= 0 \qquad A = \text{constant}$$

Now you figure out the trajectory

$$\vec{v} = \frac{dr}{dt} \to v\,dt = dr$$

$$x(t) = \int v_x\,dt = \int A\cos(\omega t - \delta)\,dt = \frac{A}{\omega}\sin(\omega t - \delta) + C_y$$

$$z(t) = \int v_z\,dt = \int v_{z0}\,dt = v_{z0}t + z_0$$

$$A = \sqrt{v_x^2 + v_y^2}$$

2.12 Curvilinear Coordinates

non-Cartesian coordinates

2.12.1 Examples

2d polar coordinates (r, θ)

$$x = r\cos(\phi(t)) \qquad y = r\sin(\phi(t)) \qquad r = \sqrt{x^2 + y^2}$$

3d cylindrical coordinates (r, ϕ, z)

$$x = r\cos(\phi) \qquad y = r\sin(\phi) \qquad z = z$$

3d spherical coordinates (r, θ, ϕ)

$$x = r\cos(\phi)\sin(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\theta)$$

2.13 Whirling Stick

A rigid stick whriling with fixed ω
use polar coordinates to have the easiest math

$\omega = \varphi t = \text{const}$
$e_r = e_r(t) \qquad e_\varphi = e_\varphi(t)$
relate r and φ to x and y basis vectors
$e_r(t) = \cos(\varphi(t))e_x + \sin(\varphi(t))e_y$
$e_\varphi(t) = -\sin(\varphi(t))e_x + \cos(\varphi(t))e_y$
position vector $r(t) = r(t)e_r(t)$

Determine the first derivatives

$$\dot{e}_r = \frac{d}{dt}\cos(\varphi(t))e_x + \sin(\varphi(t))e_y \rightarrow$$
$$\frac{d}{dt}\cos(\varphi(t))e_x + \cos(\varphi)\frac{d}{dt}e_x + \frac{d}{dt}\sin(\varphi(t))e_y + \sin(\varphi(t))\frac{d}{dt}e_y$$
$$\frac{d}{dt}\cos(\varphi(t))e_x + \frac{d}{dt}\sin(\varphi(t))e_y$$
$$\dot{e}_r = -\dot{\varphi}\sin(\varphi)e_x + \dot{\varphi}\cos(\varphi)e_y = \dot{\varphi}e_\varphi$$
$$\cdot e_\varphi = -\dot{\varphi}e_\varphi$$
$$v(t) = \dot{r}(t)e_r + r\dot{\varphi}e_\varphi = v_r e_r + v_\varphi e_\varphi$$

now acceleration lol

$$a = \frac{dv}{dt} = (\ddot{r} - r\dot{\varphi}^2)e_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})e_\varphi$$
$$\varphi = \omega t \qquad r\dot{\varphi}^2 = r\omega^2$$

centripetal force

2.14 Bead on a Whirling Rod

Use polar coordinates for sake of convenience. Use basis vectors e_r and e_ϕ

Rod whirls at a rate of ω, so $\phi(t) = \omega t$

Our general strategy is draw a sketch and then figure out your coordinates and then write out position, velocity, and acceleration vectors.

Used Newton's 2nd law to to get a differential equation.

Solve the differential equation.

$a(t) = [\ddot{r} - r\dot{\phi}^2]\vec{e}_r + [r\ddot{\phi} + 2\dot{r}\dot{\phi}]\vec{e}_\phi$

$F = ma$

no force in the radial direction because it's all normal force

$F_{net} = F_n = F_n(t)e_\phi(t) + 0e_r(t)$

$F_n e_\phi(t) = m[\ddot{r} - r\dot{\phi}^2]\vec{e}_r + m[r\ddot{\phi} + 2\dot{r}\dot{\phi}]\vec{e}_\phi$

$e_r = 0 = m(\ddot{r} - r\dot{\phi}^2) \qquad e_\phi = F_n = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$

$0 = \ddot{r} - r\dot{\phi}^2 \rightarrow \ddot{r} = r\dot{\phi}^2 \qquad \phi(t) = \omega t, \dot{\phi}(t) = \omega$

$\ddot{r} = \omega^2 r$ use an ansatz/guess

$r = e^{\lambda t} \qquad \dot{r} = \lambda e^{\lambda t} \qquad \ddot{r} = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} = \omega^2 e^{\lambda t} \qquad \lambda^2 = \omega^2 \qquad \lambda = \pm\omega$

$r(t) = Ae^{\omega t} + Be^{-\omega t}$ solve with initial condition

$\dot{r}(t) = \omega Ae^{\omega t} - \omega Be^{\omega t}$

$r(t = 0) = r_0 \qquad v(t = 0) = v_0$

$A + B = r_0 \qquad A - B = 0 \rightarrow A = B \qquad A = B = \dfrac{r_0}{2}$

$r(t) = \dfrac{r_0}{2}(e^{\omega t} + e^{-\omega t}) = \dfrac{r_0}{2}\cosh(\omega t)$

$F_n = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \qquad \phi = \omega t$

$F_n = m2\dot{r}\omega = m\omega^2 r_0 \sinh(\omega t)$

2.14.1 Bead on Spinning Loop

Loop of fixed radius R spinning about vertical axis at fixed rate Ω. Bead of mass m, free to move along the loop. Everything is in a gravitational field.

Use spherical coordinates for math convenience.

$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta)$

$r^2 = x^2 + y^2 + z^2 \qquad \tan(\theta) = \dfrac{\sqrt{x^2+y^2}}{x^2}$

the coordinate vectors are real difficult to find

coords, position, velocity, acceleration

$e_r = \sin(\theta)\cos(\phi)e_x + \sin(\theta)\sin(\phi)e_y + \cos(\theta)e_z$

$e_\theta = \cos(\theta)\cos(\phi)e_x + \cos(\theta)\sin(\phi)e_y - \sin(\theta)e_z$

$e_\phi = -\sin\phi e_x + \cos(\phi)e_y$

$r(t) = R = \text{const} \qquad \phi(t) = \Omega t \qquad \Omega = \text{const} \qquad \theta(t)$

$\dot{e}_r = \text{blah}\dot{e}_\theta = \text{blah}\dot{e}_\phi = \text{blah}$

$r(t) = r(t)e_r(t) \qquad v(t) = \dfrac{d}{dt}\left(r(t)e_r(t)\right) = \dot{r}e_r + r\dot{e}_r =$

$\dot{r}e_r + r\left(\dot{\theta}e_\theta + \dot{\phi}\sin(\theta)e_\phi\right) = \dot{r}e_r + r\dot{\theta}e_\theta + r\sin(\theta)\dot{\phi}e_\phi$

$a = \dfrac{dv}{dt} = \dfrac{d}{dt}\dot{r}e_r + r\dot{\theta}e_\theta + r\sin(\theta)\dot{\phi}e_\phi =$

$\ddot{r}e_r + \dot{r}\dot{e}_r + \dot{r}(\dot{\theta}e_\theta) + r(\ddot{\theta}e_\theta + \dot{\theta}\dot{e}_\theta) +$

$\dot{r}(\sin(\theta)\dot{\phi}e_\phi) + r(\cos(\theta)\dot{\phi}e_\phi + \sin(\theta)(\ddot{\phi}e_\phi + \dot{\phi}\dot{e}_\phi))$

what insanityplug in all the coordinate vectors to the thingy

$\vec{a}(t) = [\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2]e_r + r[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2]e_\theta +$

$[2\dot{r}\sin(\theta)\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}]e_\phi$

now use $F = ma$ to do some bullshit

$F = ma \qquad F_g = -mge_z = -mg(\cos\theta e_r - \sin(\theta)e_\theta)$

$F_n = N_r e_r + N_\phi e_\phi$

$F_{net} = F_G + F_n = F_n = N_r e_r + N_\phi e_\phi - mg(\cos\theta e_r - \sin(\theta)e_\theta) =$

$[N_r - mg\cos(\theta)]e_r + mg\sin(\theta)e_\theta + N_\phi e_\phi$

$F_\theta = ma_\theta$

$mg\sin(\theta) =$

$[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2]e_r + r[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2]e_\theta +$

$[2\dot{r}\sin(\theta)\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}]e_\phi$

$g\sin(\theta) = r\ddot{\theta} - R\sin(\theta)\cos(\theta)\Omega^2$

2.15 Types of Forces

2.15.1 Conservative Forces in 2d and 3d

2.15.2 Conservative Force

a force $\vec{F}(\vec{r})$ is conservative if it can be written as the gradient of a potential $\vec{U}(\vec{r})$.

$$\vec{F}(\vec{r}) = -\vec{\nabla}U(\vec{r})$$

2.15.3 ∇ operator

$$\vec{\nabla} = \vec{e}_x \frac{d}{dx} + \vec{e}_y \frac{d}{dy} + \vec{e}_z \frac{d}{dz}$$

if you have another coordinate system

$$\vec{\nabla} = \vec{e}_r \frac{d}{dr} + \vec{e}_\phi \frac{d}{d\phi} + \vec{e}_z \frac{d}{dz}$$

2.15.4 Curl of Conservative Force

$\nabla \times \vec{F}(\vec{r}) = \vec{\nabla} \times (-\nabla U) = 0$

NOT every force is conservative.

$F = F_{cons} + F_{diss}$

$\nabla \times F = \nabla \times F_{cons} + \nabla \times F_{diss} \neq 0$

If $\nabla \times F = 0$, then the forces can be written as $F = -\nabla U$ (has a potential) and $\vec{F}$ is converative. (useful to check)

2.15.5 Work along a path

$$w = \int_{p_2}^{p_1} \vec{F} \cdot d\vec{r} = -\int_{p_2}^{p_1} \vec{\nabla} U \, dr \qquad \nabla \approx \frac{d}{dr}$$

$$-\int_{p_2}^{p_1} dU = -(U(p_2) - U(p_1))$$

If path is closed $-(U(p_1) - U(p_1)) = 0$

A conservative force does NO work along a closed path.

2.15.6 Energy Conservation

If a force is conservative, $F = -\nabla U$, then the total energy $\vec{E} = T + U =$ const. (constant of motion)

How to show that this is true?

$$\frac{d}{dt}E = \frac{d}{dt}(T+U) = 0 = E = \text{const}$$
$$\frac{d}{dt}T = \frac{d}{dt}\frac{1}{2}mv^2 = mv \cdot \frac{dv}{dt} = \vec{v} \cdot \vec{F}$$

2nd term

$$\frac{d}{dt}U(t, \vec{r}(t)) = \frac{d}{dt}U + \frac{dx}{dt}\frac{dU}{dx} + \frac{dy}{dt}\frac{dU}{dy} + \frac{dz}{dt}\frac{dU}{dz}$$
$$\frac{d}{dt}U(t, r(t)) = \frac{dU}{dt} + \frac{dr}{dt} \cdot \nabla U = -\vec{v} \cdot \vec{F} + \frac{dU}{dt}$$
$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \vec{v} \cdot \vec{F} - \vec{v} \cdot \vec{F} + \frac{dU}{dt} = 0$$

$E = T + U =$ const is conserved, if $\vec{F}$ is conservative, and $U = U(\vec{r})$ (ie, no explicit time dependence)

2.15.7 How to Find Potential

given some force F

option a: show this with indefinite integrals

$$\vec{F} = -\vec{\nabla}U$$

option b: use indefinite integrals

$$F = -\nabla U \rightarrow U = \int_{p_1}^{p_2} = -\int_{p_1}^{p_2} \vec{F} \cdot \, dr$$

2.15.8 example

$$\vec{F} = (2xy+1)\vec{e}_x + (x^2+2)\vec{e}_y$$

find U

$$-U = \int \vec{F} \cdot d\vec{r} = \int (F_x dx + F_y dy + F_z dz)$$

$$\int_0^x F_x(x', 0, 0)dx + \int_0^y F_y(x, y', 0)dy + \int_0^z F_z(x, y, z')dz$$

$$\int_0^x 1dx' + \int_0^y (x^2+2)dy' + \int_0^z 0dz' \rightarrow$$

$$-U = x + (x^2+2)y + 0 = -x - x^2y - 2y$$

2.15.9 Central Forces

force that points towards or away from a point radially and is dependent on distance r

$$F = f(r, \theta, \phi)\vec{e}_r$$

2.15.10 Conservation of Angular Momentum

Define angular momentum $\vec{L}$ with respect to a reference point o