PHYS 435

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Spring 2025

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Chapter 1

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This goddamn professor is half retired im so cooked

1.1 Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$
$$\vec{E}(\vec{r}) = \sum \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \rightarrow \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$$

1.2 Gauss's Law

The flux of $\vec{E}$ throuhg a closed surface equations to the enclosed charge C_0

$$\frac{1}{C_0}\int_V d^3r\rho(\vec{r}) = \int_{\partial V} da\rho(\vec{r})$$

1.3 Divergence Theorem

$$\vec{\nabla} E = \partial_x E_x + \partial_y E_y + \partial_z E_z$$
$$\int_V d^3 r \vec{\nabla} \vec{E}(\vec{r}) = \int_{\partial V} d\vec{a} \cdot \vec{E}(\vec{r})$$
$$\vec{\nabla} \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

1.4 Faraday's Law

The circulation of $\vec{E}$ around any closed path N is equal to $(-1)\times$ the time derivative of the magnetic flux through ANY surface bounded by the closed path.

$$\int_{\partial S} d\vec{l} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{a} \vec{B}$$

1.5 Stoke's Theorem

$$\int_{\partial S} d\vec{l} \cdot \vec{E} = \int_S d\vec{a} \vec{\nabla} \times \vec{E}$$

1.5.1 Differential Laws

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Gauss

$$\vec{\nabla} \cdot \vec{E} = -\frac{\rho(\vec{r})}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Ampere

$$\vec{\nabla} \times \vec{B} = \mu_0 J$$

1.6 Electric Potential

Start with Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -\frac{\partial}{\partial t}\vec{B}(\vec{r}, t)$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\vec{E}$$

remove the time dependent equations

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

the curl of $\vec{E}$ is 0 which means the electric field is conservative? A scalar potential function convenience.

consider the path integral

$$\int\limits_P d\vec{l} \cdot \vec{E}$$

We can show that the integral is path independent (because the curl is 0)

$$\int\limits_{P1} - \int\limits_{P2} = \oint\limits_{\partial S} d\vec{l} \cdot \vec{E}(\vec{r}) = \int\limits_S d\vec{a} \cdot \vec{\nabla} \times \vec{E} = 0$$

$$\int\limits_{P1} d\vec{l} \cdot \vec{E} = \int\limits_{P2} d\vec{l} \cdot \vec{E}$$

That's actually a really smart proof damn

Now we do actual potential stuff

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l} \cdot \vec{E}(\vec{r})$$

Where $\vec{0}_r$ is the vector where the potential is 0

$$U(\vec{a}) - U(\vec{b}) = -\int_a^b d\vec{l'} \cdot \vec{F}(\vec{r})$$

$$\vec{F}_{Lorentz} = q\vec{E}(\vec{r}) + q\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$$

$q\vec{E}(\vec{r})$ can do work, but $q\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$ cannot do any work (always in opposite direction of motion)

$$W = q\int_{\vec{0}_r}^{\vec{r}} d\vec{l}\cdot\vec{v}\times\vec{B} = q\int_{\vec{0}}^{\vec{r}} d\vec{l}\cdot\frac{d\vec{l}}{dt}\times\vec{B}(\vec{r}) =$$

$$q\int_{\vec{0}}^{\vec{r}} dt\frac{d\vec{l}}{dt}\cdot(\frac{d\vec{l}}{dt}\times\vec{B}(\vec{r})) = 0$$

That part cannot do any work

$$W_{other} = U(\vec{r}) - U(\vec{0})$$

$$\frac{U(\vec{r}) - U(\vec{0})}{q} = \Delta V$$

More Stuff

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l'}\cdot\vec{E}(\vec{r})$$

$$-\vec{\nabla}V(\vec{r}) = -\left[\hat{x}\frac{\partial}{\partial x}V(\vec{r}) + \hat{y}\frac{\partial}{\partial y}V(\vec{r}) + \hat{z}\frac{\partial}{\partial z}V(\vec{r})\right]$$

$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz = r + dx$$

consider the slightest motion dx in the $\hat{x}$ direction so that $\vec{r}\rightarrow\vec{r}+\hat{x}dx$

$$V(\vec{r}+\hat{x}dx) = V(\vec{r}) + dx\hat{x}\cdot\vec{E}(\vec{r}) \qquad E_x(\vec{r}) = \hat{x}\cdot\vec{E}(\vec{r})$$

$$\frac{V(\vec{r}+\hat{x}dx) - V(\vec{r})}{dx} = E_x(\vec{r})$$

That gives us

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

real important equation

1.6.1 Potential Equation

consider a point mass

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} q \frac{\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l} \cdot \vec{E}$$

$$E_y = \frac{1}{4\pi\epsilon_0}\frac{q}{y^2} \qquad -\int_{\infty}^{y} dy' \frac{1}{4\pi\epsilon_0}\frac{q}{y^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

Use the principle of superposition to get the general answer

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

1.6.2 Infinite Line Charge

Consider a straight line of infinite length and constant charge density.

Where should $\vec{0}_r$ be?

I think we just pick an arbitrary point

$$\vec{E}(s) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}$$

$$V(\vec{r}) = -\int d\vec{l}\vec{E}(s) = V(s) = -\int_{O_r}^{s} ds\frac{1}{2\pi\epsilon_0}\frac{\lambda}{s'}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln(s')\Big|_{\vec{O}_r}^{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{O_r}{s}\right)$$

What PDE governs $V(\vec{r})$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{E} = -\vec{\nabla}V$$

$$\vec{\nabla}\cdot\vec{\nabla}V(\vec{r}) = -\frac{\rho}{\epsilon_0}$$

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)V = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

1.7 Work

If you move 1 charge, there is no work done because there are no other fields.

If you bring in a 2nd charge, you get a total work of $\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$

If you bring in a third charge you just sum the things together

$$U_{1\to N} = \frac{1}{2}\sum_i^N \sum_{j>i}^N \frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\int d^3r d^3r' \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{r})\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$
$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\vec{r'})V(\vec{r'}) \qquad \rho(\vec{r'}) = -\epsilon_0\nabla^2 V(\vec{r'})$$
$$U = -\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\nabla^2 V(\vec{r})$$

1.7.1 X-component

Let's consider just the x-component for a little bit

$$-\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\partial_x\left[\partial_x V(\vec{r})\right]$$
$$\partial_x\left[V(\vec{r})\partial x V(\vec{r})\right] = \partial_x V\cdot\partial_x V + V\partial_x^2 V$$
$$\partial_x\left[V(\vec{r})\partial x V(\vec{r})\right] - \partial_x V\cdot\partial_x V = V\partial_x^2 V$$

So with that you get

$$-\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\partial_x\left[\partial_x V(\vec{r})\right] = -\frac{\epsilon_0}{2}\int d^3r\partial_x\left(V(\vec{r})\partial_x V(\vec{r})\right) - E_x^2$$

generalize

$$\frac{\epsilon_0}{2}\int d^3r\,\vec{\nabla}\cdot V(\vec{r})\vec{E}(\vec{r}) + \vec{E}\cdot\vec{E}$$

The main point of all of this is that it goes to 0 for large r

$$\int d^3r\vec{\nabla}\cdot V\vec{E} \to \int d^3r\vec{\nabla}\cdot\frac{C}{r}\frac{1}{r^2}\hat{r} \to$$
$$\int da\frac{C}{r^3} \to \frac{1}{r} \to \lim_{r\to\infty} = 0$$

So our potential somehow gets to

$$U = \int d^3r\frac{\epsilon_0}{2}E^2$$

1.8 Office Hours Math

So using Gauss's Law we can say that the net charge around the eye sockets is 0

$$\phi_E = \frac{Q_{enc}}{\epsilon_0} = \oiint EdA$$

If we consider

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

where r is the surface of the thing.

Basically the locations of all the things makes sense

1.9 More Energy

So we spent all of last lecture getting

$$U_E(\vec{r}) = \frac{\epsilon_0}{2}E^2$$
$$U_{b\to c} = \int r^2 dr \sin(\theta) d\theta d\phi \frac{\epsilon_0}{2}\frac{Q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}$$
$$= \frac{Q^2}{8\pi\epsilon_0}\int_b^c \frac{r^2}{r^4}dr = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{c}\right)$$

1.10 Dirac Delta

Consider a Gaussian curve given by

$$G(x) = Ge^{\frac{-x^2}{a^2}} = \frac{1}{a\sqrt{\pi}}e^{\frac{-x^2}{a^2}}$$
$$\int dx \frac{1}{a\sqrt{\pi}}e^{\frac{-x^2}{a^2}} = 1$$
$$\delta(X) = \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}e^{\frac{-x^2}{a^2}}$$

The integral is 1, but all of the area is located at the point $x = 0$

$$\delta(x - x_0) = \lim_{a\to 0}\frac{1}{a\sqrt{\pi}}e^{\frac{-(x-x_0)^2}{a^2}}$$

$$\int_{-\infty}^{\infty} dx\delta(x - x_0) = 1$$
$$\int_{-\infty}^{\infty} dx\delta(x - c)f(x) = f(c)$$
$$\rho(\vec{r}) = q\delta(x - x')\delta(y - y')\delta(z - z')$$

Chapter 2

Conductors and Surfaces

The electric field at the surface of a plane is

$$E_{\perp} = \frac{1}{2}\frac{\sigma}{\epsilon_0}$$

Because of the surface field, there is a pressure on the surface

$$P = E * \sigma = \frac{1}{2}\frac{\sigma^2}{\epsilon_0}$$

Given some set of conductors, there is a number that, given the potential at a point, can tell you the net charge in the set.

That number is the capacitance

$$C\Delta V = Q$$

2.0.1 Parallel Plate Capacitors

Consider 2 plates of charges $+Q$ and $-Q$.

We can integrate the electric field to get the potential difference

$$V = d * E = d * \frac{\sigma}{\epsilon_0} = \frac{d * Q}{\text{Area} * \epsilon_0} = \frac{Q}{C}$$

$$C = \frac{\epsilon_0 A}{d}$$

Where d is distance and not the differential operator

Now we can find the energy of the system

$$U = \frac{\epsilon_0}{2}\frac{\sigma^2}{\epsilon_0^2}\frac{A^2}{A} * d = \frac{1}{2}\frac{Q^2}{\epsilon_0 A/d} = \frac{1}{2}\frac{Q^2}{C}$$

That is the stored energy in a field.

There's another way to get the same derivation.

$$dW = dQ \cdot V$$

$$\int_0^W dW = U = \int_{Q=0}^{Q} dQV = \int_0^Q dQ\frac{Q}{C} = \frac{Q^2}{2C}$$

2.0.2 Constant Q

What happens if we pull part the plates of the capacitor?

d goes up, C goes down, V goes up, U goes up

2.0.3 Constant V

We have the parallel plate capacitor hooked up to a battery of constant voltage

d goes up, C does down, Q must go down for V to not change

The internal energy can also be written as $U = \frac{1}{2}CV$

$$dU = -\vec{F} \cdot dr$$

In a volume V without charges, $V(\vec{r})$ has a maximum and minimum on ∂V, the boundary, not inside.

Imagine a blob with all sorts of potential and voltage and whatever

This just gives us our definition $\partial V =$ the boundary of the object

We have the equation

$$\nabla^2 U(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

Let's try to figure out the average potential on the surface of the sphere

$$V(\vec{r}) = \frac{1}{4\pi R^2} \int da\, V$$

What we'll learn is that $V(\text{origin}) =$ the area of V on the sphere

$$V((\vec{r}')) = \frac{q}{4\pi\epsilon_0}\frac{1}{|\vec{r'} - z\hat{z}|}$$
$$\vec{r'} = R\sin(\theta)\cos(\phi)\hat{x} + R\sin(\theta)\sin(\phi)\hat{y} + R\cos(\theta)\hat{z}$$
$$d(\theta,\phi) = \sqrt{R^2\sin^2(\theta)\cos^2(\phi) + R^2\sin^2(\theta)\sin^2(\phi) + (R\cos(\theta) - z)^2}$$
$$= \sqrt{R^2 + z^2 - 2Rz\cos(\theta)}$$
$$V(\vec{r}) = \frac{1}{4\pi R^2}\int da \cdot V(\vec{r'}) \qquad (\sin(\theta)d\theta = -d\cos(\theta))$$
$$= \frac{q}{4\pi\epsilon_0}\frac{2\pi}{4\pi}\int -d\cos(\theta)$$

Do a U-sub

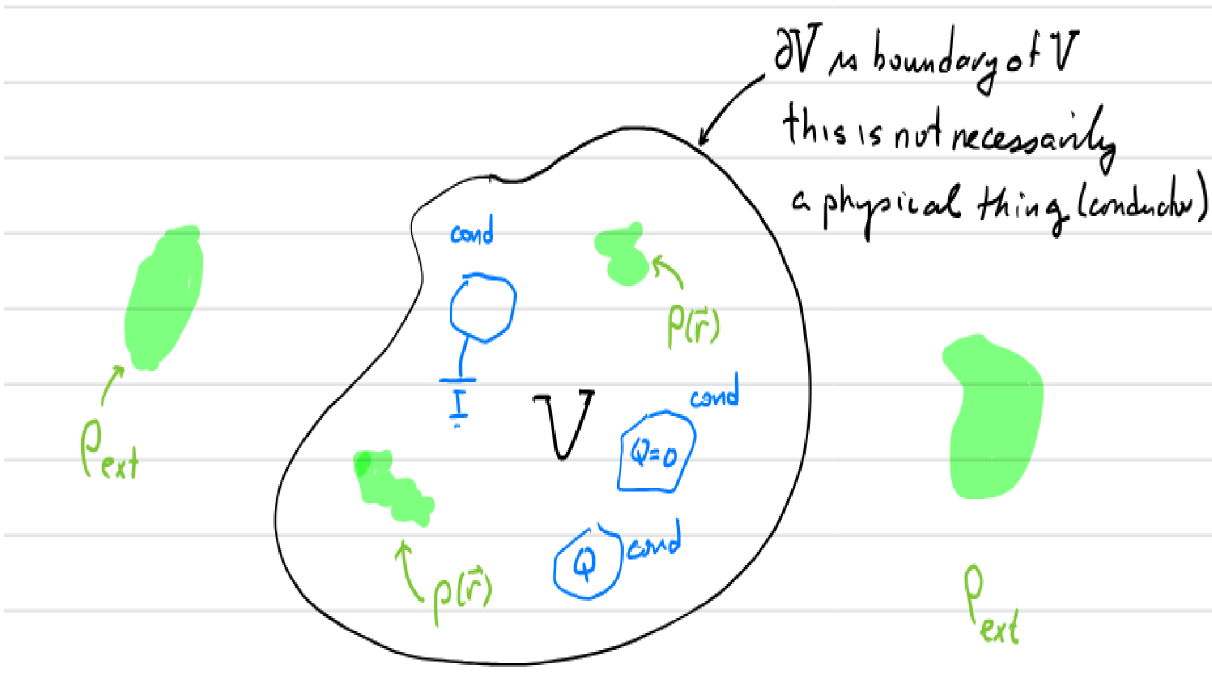
$$\frac{1}{2}\frac{q}{4\pi\epsilon_0}\int_{-1}^{1} du\,\frac{1}{\sqrt{R^2 + z^2 - 2Rzu}}$$
$$= \frac{1}{2Rz}\frac{q}{4\pi\epsilon_0}\sqrt{R^2 + z^2 - 2RzU}\Big|_{-1}^{1} = R^2 + z^2 - 2Rz$$
$$\frac{1}{2Rz}\frac{q}{4\pi\epsilon_0}[z - R - z - R]$$
$$= \frac{q}{4\pi\epsilon_0}\frac{1}{z}$$

2.1 Surface Potential

Where there is no charge, what is the max + min of $V(\vec{r})$ on ∂V the boundary

2.2 Dirichlet Problems

Consider an aribitrary blob with some potential



If given $V(\vec{r})$ on ∂V and the charge density $\rho(\vec{r})$, then $V(\vec{r})$ inside the blob is uniquely determined.

consider $V_1(\vec{r})$ and $V_2(\vec{r})$ such that

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon_0} \qquad V_1(\vec{r}) \text{ on } \partial V = B_s(\vec{r})$$
$$\nabla^2 V_2 = -\frac{\rho}{\epsilon_0} \qquad V_2(\vec{r}) \text{ on } \partial V = B_s(\vec{r})$$
$$V_3 = V_1 - V_2 \qquad \nabla^2 V_3(\vec{r}) = 0$$
$$V_3(\vec{r}) \text{ on } \partial V = 0$$
$$V_3 = 0 \text{ everywhere in } V$$

So $V_1 = V_2$ so they describe the same potential function so there's only 1 unique answer for each boundary potential and density

2.3 Neumann Problem

Where do the charges lie on a conductor?

Consider a volume with a conducting boundary

The electric field inside the boundary is unique

$$\vec{\nabla}\vec{E} = \frac{\rho}{\epsilon_0}$$

2.4 New Lecture

Find the solution for $V(\vec{r})$ inside a volume of V.

Utilize a point-source response function.

$$G(\vec{r}, \vec{r'}) \Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r'\, G(\vec{r}, \vec{r'})\rho(\vec{r})$$

So if we consider the normal potential function in free space (FS)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r\, \frac{\rho(\vec{r})}{|\vec{r} - \vec{r'}|}$$

We can see that

$$G_{FS}(\vec{r}) = \frac{1}{|\vec{r} - \vec{r'}}$$

Which is just the response to a single point charge.

Now let's consider the potential of a conducting grounded plane (GP)

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, G_{GP}(\vec{r}, \vec{r'})\rho(\vec{r'})$$

Image solution:

If we consider equidistant point charges on 2 sides of the non-conducting plane, Then we electric field lines are perpendicular to the plane at the plane. The conducting plane ACTS equivalently to the equidistant point charge

Back to math

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\, G_{GP}(\vec{r},\vec{r'})\rho(\vec{r'})$$

$$= \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\rho(\vec{r'})\left[\frac{1}{|\vec{r}-\vec{r'}|} - \frac{1}{|\vec{r}-\vec{r''}|}\right] \qquad \vec{r''} = \vec{r} - 2\hat{z}(\vec{r'}, z)$$

$$= \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\rho(\vec{r'})\left[\frac{1}{|\vec{r}-\vec{r'}|} - \frac{1}{|\vec{r}-\vec{r'}+2\hat{z}(\vec{r},\hat{z})|}\right]$$

$$V(\vec{r}) =$$

$$\frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\rho(\vec{r'})$$

$$\Big[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\Big]$$

All of those are equivalent

There is not an actual image charge, but the plane acts like the image charge because the potential at the plane is 0 and all the electric field lines are perpendicular to the plane at the plane.

Consider when $z = 0$

$$\vec{E}(x, y, z = 0) = \hat{z}\frac{\sigma}{\epsilon_0}$$

$$= -\hat{z}\frac{q}{4\pi\epsilon_0}\partial_z\Big(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\Big)$$

$$\sigma = \frac{q}{4\pi}\frac{\partial}{\partial z}\Big(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\Big)$$

$$\sigma = \frac{-q}{2\pi}\frac{z'}{(x^2+y^2+(z')^2)^{3/2}}$$

Because the coordinates are squared, you can swap $\vec{r}$ and $\vec{r'}$ and nothing changes.

2.5 Grounded Conducting Sphere

2.5.1 Exterior

Use the law of superposition

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\rho(\vec{r'})G_{GS}(\vec{r}, \vec{r'})$$

Consider a point a outside the sphere and a point b inside the sphere and a mapping from the inside to the outside

$$a * b = R^2 \to b = R\frac{R}{a}$$

This is basically just a method guess without any real backing, but we'll see that this mapping does work mathematically.

$$q'' = -q' * \frac{R}{a}$$

$$V(\vec{r}) = \frac{q'}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{x^2+y^2+(z+a)^2}} - \frac{R/a}{\sqrt{x^2+y^2+(z-\frac{r^2}{a})^2}}\right)$$

The point on the sphere have a potential of 0

$$\sqrt{x^2+y^2+(z+a)^2} = \frac{R}{a}\sqrt{x^2+y^2+(z-\frac{r^2}{a})^2} \Rightarrow$$

$$\Rightarrow\Rightarrow = (x^2+y^2+z^2) = R^2$$

That means that we have in fact found our response function for a grounded sphere.

$$\frac{q'}{4\pi\epsilon_0}\left(\frac{1}{|\vec{r}-\vec{r'}|} - \frac{R/r'}{|\vec{r}-\frac{R^2}{r'}\frac{\vec{r'}}{r'}}\right)$$

$$V(\vec{r}) = \frac{q'}{4\pi\epsilon_0}\int d^3r'\,\rho(\vec{r'})\left(\frac{1}{|\vec{r}-\vec{r'}|} - \frac{R/r'}{|\vec{r}-\frac{R^2}{r'}\frac{\vec{r'}}{r'}}\right)$$

That last bit is our Green's function.

2.5.2 Interior

Because the sphere is grounded, there is no real potential on the outside of the sphere if the charge is on the sindie of the sphere

The previously calculated solution works for both the inside and the outside of the sphere.

The point source response function is useful because it allows you to consider any charged shape as a superposition of points.

$$V = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r' G(\vec{r}, \vec{r'}) \rho(\vec{r'})$$

2.6 Boundary Value Problems

You have a conducting shape such that $\nabla^2 V = 0$ on the inside and you're trying to find the potential on the boundary.

2.6.1 Box

Consider a box. We have to think of the potential on each side of the box

$$V_T(x, y, z = c) \qquad V_B(x, y, z = 0)$$

Specify V_B

Find

$$V_T = V_L = V_R + V_F + V_B = 0$$

We know for sure that $\nabla^2 V_{inside} = 0$
Let $V(x,y,z) = X(x)Y(y)Z(z)$

$$\nabla^2 V = (\partial_x^2 + \partial_y^2 + \partial_z^2)(XYZ) = 0$$

$$\frac{YZ\partial_x^2 X}{XYZ} + \frac{XZ\partial_y^2 Y}{XYZ} + \frac{XY\partial_z^2 Z}{XYZ} = 0$$

$$\frac{1}{X(x)}\partial_x^2 X(x) + \frac{1}{Y(y)}\partial_y^2 Y(y) + \frac{1}{Z(z)}\partial_z^2 Z(z) = 0$$

$$\frac{d^2}{dx^2}X(x) = \pm M^2 X(x)$$

The diff eq can be solved pretty easily and it applies to Y and Z as well

$$\frac{d^2}{dx^2}X(x) = -M^2 X(x) \qquad \frac{d^2}{dy^2}Y(y) = -N^2 Y(y)$$

$$\frac{d^2}{dz^2}Z(z) = (M^2 + N^2)Z(z)$$

$$X(x) = A\cos(Mx) + B\sin(Mx)$$

Because of of the symmetry in X

$$X(x) = B\sin(Mx)$$

Because the potential is 0 at the edge of the wall

$$X(x) = B\sin\left(\frac{m\pi}{a}x\right) \qquad m \in \mathbb{N}$$

$$Y(y) = C\sin\left(\frac{n\pi}{b}y\right) \qquad n \in \mathbb{N}$$

$$\partial_z Z(z) = \left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right] Z(z)$$

$$Z(z) =$$

$$C\exp\left(\left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right]^{1/2} z\right) + D\exp\left(-\left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right]^{1/2} z\right)$$

Let $\Gamma_{min} = \left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right]$

$$Z(z) = \frac{\sinh(\Gamma_{min})(c-z)}{\sinh(\Gamma_{min})c}$$

So now that we have all our equations, we plug them all back in

$$V_B(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{min}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$I_{mm'} = \int_b^a dx\,\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m'\pi x}{a}\right) = \delta_{mm'}\frac{a}{2}$$

$$= \int_b^a dx\,\frac{1}{2i}\,(e^{i\frac{m\pi x}{a}} - e^{-i\frac{m\pi x}{a}})$$

$$\int_0^a dx\,\int_0^a dy\,\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)$$

$$\int dx\,dy\,\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{n'\pi y}{b}\right)V_b(x,y,0)$$

$$= \sum A_{mn}\int_0^a dx\,\sin\left(\frac{m'\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)$$

$$*\int_0^b dx\,\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$= \frac{ab}{4}\sum A_{mn}\delta_{mm'}\delta_{nn'}$$

$$= \frac{ab}{4}A_{m'\,n'}$$

Now to consider the Z plane

$$Z_{mn}(z) = \frac{\sinh(\Gamma_{min})(c-z)}{\sinh(\Gamma_{min})c} \qquad \Gamma_{min} = \left[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2\right]$$

$$\sinh(\Gamma) = \frac{e^{2\Gamma} - e^{-2\Gamma}}{2} \qquad V_B = V_0$$

$$A_{mn} = \sqrt{\frac{4}{ab}} \int_0^a dx \int_0^a dy \, \sin\left(\frac{m'\pi x}{a}\right) \sin\left(\frac{n'\pi y}{b}\right) V_0$$

$$= \sqrt{\frac{4}{ab}} \frac{a}{m\pi} \left(-\cos(\theta)\Big|_0^{m\pi} \right) \frac{b}{m\pi} \left(-\cos(\theta)\Big|_0^{n\pi} \right)$$

$$= \frac{8V_0}{\pi^2 mn} \sqrt{ab}$$

We're gonna make up some units

$$\hat{mn} = \frac{4}{ab} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \qquad \hat{mn} \cdot \hat{mn} = 1$$

$$\hat{mn} * \hat{m'n'} = \delta_{mm'}\delta_{nn'}$$

We need to try to get a square function from these unit vectors

$$A_{mn} = \sqrt{\frac{4}{ab}} \int_0^a dx \int_0^a dy \, \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \delta(x - x_0)\delta(y - y_0)$$

$$= \frac{4}{ab} \sin\left(\frac{m\pi x_0}{a}\right) \sin\left(\frac{n\pi y_0}{b}\right)$$

2.7 Conducting Sphere

Imagine a conducting sphere

We want to find the potential on the boundary and then we want to find the laplacian of that potential, but we want to do that in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}V(\vec{r})\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}V(\vec{r})\right)\dots$$

$$\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V(\vec{r}) + \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial V(\vec{r})}{\partial\theta} + \frac{1}{sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}V(\vec{r}) = 0$$

Let's consider $V(\vec{r}) = R(r)Y(\theta,\phi)$

Note that $Y(\theta,\phi)$ must be periodic in ϕ and we should expect to wiggle in θ

Plug all the things in the thing

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) + \frac{1}{Y\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) + \frac{1}{Y\sin(\theta)}\frac{\partial^2}{\partial\phi^2}Y(\theta,$$
$$= 0$$

Now let's get

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) = l(l+1)R(r)$$
$$\frac{1}{Y\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y(\theta,\phi)}{\partial\theta}\right)$$

2.7.1 Associated Legendre Function

this is the thing you have to google because this lecture is too damn fast.

$$P_l^m(\theta)$$

2.7.2 Azimuthal Symmetry

We assume that most things have azimuthal symmetry

So our diff eq becomes

$$\frac{d}{\sin(\theta)\,d\theta}\sin^2(\theta)\frac{d}{\sin(\theta)\,\theta}\Theta(\theta) = -l(l+1)\Theta(\theta)$$

Use substitution to solve the diff eq

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}\Theta(\theta) = -l(l+1)\Theta(\theta) \qquad -1 < x < 1$$

These are called Legendre Polynomials

$$P_l(x) = c_0 + c_2x^2 + \dots$$

for even l and then the same thingy but odd for odd

let $x = \cos(\theta)$

$$\Theta_l(\theta) = P_l(\cos(\theta))$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \qquad \dots$$

2.7.3 Example(?)

Let's say we have an insulating sphere

The only P_l that matches the behavior of the sphere is $l = 1$

$$V(r,\theta) = \left[A_1 R + \frac{B_1}{R^2}\right]\cos(\theta)$$

However, we can see that the potential diverges as we go inside the sphere for non-zero B, so $B = 0$ inside the sphere.

$$V_{in} = \frac{V_0}{r}\cos(\theta)$$

If we take the negative gradient we see that

$$\vec{E} = -\frac{V_0}{R}\hat{z}$$

So all the field lines are pointing straight down no matter what

Now let's consider the outside of the sphere

$$V_0\cos(\theta) = \frac{B_1}{r^2}\cos(\theta) \rightarrow$$
$$B_1 = V_0 R^2$$

So our potential is

$$V_{out} = V_0\frac{R^2}{r^2}\cos(\theta)$$

If we use the spherical gradient operator

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \dots$$
$$\vec{E}_{out} = \vec{\nabla}V = \hat{r}\frac{2V_0}{r}\frac{R^2}{r^2}\cos(\theta) + \hat{\theta}\frac{V_0}{r}\frac{R^2}{r^2}\sin(\theta)$$

2.8 Homework 4

If we specify the potential on the cylinder, we want to find the potential on the inside and outside of the surface.

problem 2 is find the potential on a rectangle (we did it in lecture and it's obnoxious)

2.9 Surface Charges

remember that P_l is your Legendre polynomial because it shows up a bunch.

You can relate the surface charge density to the electric field with a pretty simple use of Gauss's Law

2.9.1 COME BACK HERE

2.9.2 New Boundary Value Problem

Consider a conducting sphere in an electric field that is completely in the vertical direction.

We have radial symmetry because the E field is only in the vertical direction.

What is the electric field at the north pole? Does the strength of that electric field depend on the diameter of the sphere?

2.10 Green's Function

$$V(\vec{r}) = \int\limits_{V} d^3r' G_{GB}(\vec{r}, \vec{r'})\rho(\vec{r'}) + \frac{1}{4\pi}\int\limits_{\partial V}$$

2.11 Discussion Notes

Green's function solves Poisson's Equation

Look at Jackson textbook chapters 1.9-10

$$\nabla^2 V = \frac{\rho}{\epsilon_0} \qquad \nabla^2 U = 0$$

Consider separation of variables

$$V(x, y, z) = X(x)Y(y)Z(z) \qquad V(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Cartesian Coordinates

$$\frac{d}{dx^2}X_1 = k^2 x \qquad \frac{d}{dx^2}X_2 = -k^2 x$$
$$X_1(0) = A_1 + B_1$$

You either get exponential or oscillating answers.

ohhhhh because $Y(y)$ and $Z(z)$ are going to be constants when computing only the x derivative, so that's where the sinusoidal stuff comes from.

You then do a wacky sum stuff.

$$f(x) = \sum_{k \in \mathrm{Z}} c_k f_k(x)$$

$$\int_{-\infty}^{\infty} dx\, f_x(x) = \delta_{kk'}$$

You can do the same thing in spherical coordinates

It's a bunch of math shenanigans

but you get Legendre polynomials out of some stuff

I took pictures but the TA said that none of this will need to be remembered.

Legendre polynomials are just a solution to a form of differential equations.

2.12 Multipoles

This lecture was online on February 19th, 2025

$$\frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(r')}{|vecr - \vec{r'}|}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_{tot}}{r} + \frac{\vec{\rho} \cdot \vec{r}}{r^3} + \frac{r_i r_j Q_{ij}}{r^5} \right)$$

$$Q_{ij} = \int d^3r' \, \rho(\vec{r'}) \left(\frac{3}{2} r_i r_j - \frac{1}{2} {r'}^2 \delta_{ij} \right)$$

If the first N multipole moments are 0, then the Nth multipole moment is the same in all coordinate systems.

2.13 Homework 5

We got an equation from somewhere that looked like

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\, G(\vec{r},\vec{r'})\rho(\vec{r'}) + \frac{1}{4\pi}\int\limits_{\partial V} da'\, \hat{m'} \cdot \vec{\nabla}_r G(\vec{r},\vec{r'})V(\vec{r'})$$

I wouldn't fully trust that ending $V(\vec{r'})$ but maybe it is true because it's something about the potential on the boundary Dirichlet problem.

2.13.1 Dipole Moment Stuff

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{\rho}\vec{r}}{r^3}$$

$$\vec{E} = -\vec{\nabla}V(\vec{r}) =$$

$$\frac{1}{4\pi\epsilon_0}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(\frac{P_x x + P_y y + P_z z+}{r^3}\right) =$$

$$\frac{1}{4\pi\epsilon_0}\left(\frac{\vec{P}}{r^3} + \vec{P}\cdot\vec{r}\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\frac{1}{r^3}\right)$$

interjection

$$\hat{x}\vec{P}\cdot r\frac{3x}{(x^2+y^2+z^2)^{3/2}}$$

end of interjection

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{3(\vec{p}\cdot\vec{r})}{r^5} - \frac{\vec{p}}{r^3}\right)$$

If we want to calculate the energy we can do some more math

$$U_{ext} = \int\limits_V d^3r\,\rho(\vec{r})V_a(\vec{r})$$

We can make a taylor series expansion of the potential

$$U_{TS}(\vec{r}) = \int d^3r' \left(V(0) + \vec{r} \cdot \vec{\nabla} V(\vec{r}) \Big|_{\vec{r}=0} + \frac{1}{2} \sum_{ij} r_i r_j \frac{\partial^2 V(\vec{r})}{\partial r_i \partial r_j} \Big|_{\vec{r}=0} \right) \rho(\vec{r})$$

$$= q_{tot} V(0) - \vec{E} \cdot \vec{p} + \frac{1}{2} \partial_i \vec{E}_j \int d^3r' \, r'_i r'_j \rho(\vec{r}')$$

$$Q_{ij} = \int d^3r' \left(\frac{3}{2} r'_i r'_j - \frac{1}{2} r'^2 \partial_{ij} \right) \rho(\vec{r'}) \qquad \partial_i \vec{E}_i = \vec{\nabla} \cdot \vec{E}$$

$$+ \frac{1}{3} \partial_i E_j Q_{ij}$$

Chapter 3

Dielectrics

consider an upwards constant E field in a grid of conducting spheres.

Each sphere can be written with polarization equation

$$\vec{p} = q\vec{d} = \alpha\vec{E}$$

Where α is the approximate polarizability of the sphere. It has units Fm^2 where F is a farad, or the unit of capacitance.

$$hd = q\vec{E} \qquad d = \frac{q}{h}\vec{E} \qquad \vec{p} = \frac{q^2}{h}\vec{E} \qquad \alpha = \frac{q^2}{h}$$

You can then get a polarization density function

$$\vec{P}(r) = \vec{p} \cdot n = \epsilon_0 \frac{nq^2}{h\epsilon_0}\vec{E} = \epsilon_0 X\vec{E}$$

X is the electroc susceptibility and is a dimensionless unit (you can check)

3.1 Bound Charge

Imagine a single dipole

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}$$

Now consider a bunch of dipoles

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\vec{P}(\vec{r})\cdot(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}$$

Focus on the fraction and remember the formulas for potential and E field for a point charge and we get

$$-\vec{\nabla}_r\frac{1}{|\vec{r}-\vec{r'}|} = \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} = \vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r'}|}$$

Now we put that back in the original polarization equation

$$\frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\vec{P}(\vec{r})\cdot(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3} = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\vec{P}(\vec{r})\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r'}|}$$

Now you can do integration by parts specifically because ∇ is with respect to r' and not r

$$\vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} = \vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} + \vec{P}(\vec{r'})\cdot\vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

$$\vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} - \vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} = \vec{P}(\vec{r'})\cdot\vec{\nabla}_{r'}\cdot\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

So we can plug that back into our integral

$$\frac{1}{4\pi\epsilon_0}\int d^3r'\,\vec{P}(\vec{r})\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r'}|} =$$
$$\frac{1}{4\pi\epsilon_0}\int d^3r'\,\vec{\nabla}_{r'}\frac{\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} - \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$
$$= \frac{1}{4\pi\epsilon_0}\int_{\partial V} da\,\frac{\hat{n}\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} - \frac{1}{4\pi\epsilon_0}\int_{V} d^3r'\,\frac{\vec{\nabla}_{r'}\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} = V$$

This shows us that the induced charge on a conductor plus the charges that do the inducing are always going to be less than just the original inducing charge(?) I think.

A volume bound charge $-\vec{\nabla}\cdot\vec{P}(\vec{r})$ has surface charge $\hat{n}\cdot\vec{P}$

3.1.1 Polarized Dielectric

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\frac{\vec{P}(\vec{r'})(\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3} = \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\vec{P}(\vec{r'})\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r'}|}$$

Where P is the dipole moment I think

We can do integration by parts to get

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{\partial V} da\,\frac{\hat{n}\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|} - \frac{1}{4\pi\epsilon_0}\int_V d^3r'\,\frac{\vec{\nabla}\cdot\vec{P}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

THat was all just review

Now we look at the polarizing susceptibility

$$\vec{p} = \epsilon_0 X_e \vec{E}$$

For a capacitor filled with a linear dielectric, there is a free charge density on the conductor, and a bound charge density on the surface of the dielectric induced by $\vec{E}$. But $\sigma_B = P$ is determined by $\vec{E}$ which in turn induced $\vec{P}$

for a linear dielectric

$$\vec{E} + \frac{\sigma_F}{(1+X)\epsilon_0}$$

It's called a dielectric because the di- prefix means reduced so its a reduced electric field effect from the original field.

$$\rho_{net} = \rho_F + \rho_b$$

The total charge density is caused by both the bulk and the induced charge.

Gauss's Law

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\left(p_f(\vec{r}) + p_b(\vec{r})\right) = \frac{1}{\epsilon_0}\left(p_f(\vec{r}) - \vec{\nabla} \cdot \vec{P}(\vec{r})\right)$$

$$\vec{\nabla} \cdot \left(\vec{E}(\vec{r}) + \frac{1}{\epsilon_0}\vec{P}(\vec{r})\right) = \frac{\rho_f(\vec{r})}{\epsilon_0}$$

We're gonna make up a number for the "Displacement Field"

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r})$$
$$\vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho_{free}(\vec{r})$$
$$\vec{\nabla} \times \left(\epsilon_0 \vec{E}(\vec{r}) + \vec{P}(\vec{r})\right) = \vec{\nabla} \times \vec{P}(\vec{r}) \neq 0$$

3.1.2 Helmholtz Theorem

If you know $\vec{\nabla} \cdot \vec{F}$ and $\vec{\nabla} \times \vec{F}$, and charges dont $\to \infty$, then you know enough to determine the unique function $\vec{F}(\vec{r})$.

units of D are $coulombs/meters^2$

$$\vec{p} = \epsilon_0 X \vec{E}$$
$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 X \vec{E} = \epsilon_0 (1 + X) \vec{E} = \epsilon_0 \kappa \vec{E}$$

D is a construct and a helper field, its not a real thing.

If we consider a bound charge on a surface, one side will be positive and the other side will be negative, so there are two difference helper functions D_1 and D_2

$$\hat{n} \cdot \vec{D}_1 + \hat{n} \cdot \vec{D}_2 = \sigma_{free}$$
$$\hat{n} \cdot \vec{D}_1 + \hat{n} \cdot \vec{D}_2 \to k_1 \hat{n} \cdot \vec{E}_1 + k_2 \hat{n} \cdot \vec{E}_2$$
$$k_1 E_{n,1} = k_2 E_{n,2}$$

where E_n denotes the electric field in the normal direction.

We can use all of these equations to solve for something.

The lecture goes over finding the electric field inside a slab given $\vec{\nabla} \times \vec{E} = 0$

3.2 New Lecture

Imagine a linear dielectric

$$\vec{p}(\vec{r}) = \epsilon_0 X(r)\vec{E}(\vec{r})$$

On dielectrics there is a bound charge density

$$\rho_b(\vec{r}) = -\nabla \cdot \vec{p}(\vec{r})$$

and we can consider the fake physics tool "displacement field" $\vec{D}$

3.2.1 Boundary Matching Equations

If we have an interface between two dielectrics

$$\hat{n} \cdot \vec{D}_1 - \hat{n} \cdot \vec{D}_2 = \sigma_F$$

We can also use ampere's law to show that the transverse electric fields on two sides of a boundary are equal to each other.

When $\vec{D} = \epsilon_0 \kappa \vec{E}$, or its a linear dielectric, you get

$$\hat{n}\epsilon_0\kappa_1\vec{E}_1 - \hat{n} \cdot \epsilon_0\kappa_2\vec{E}_2 = \sigma_F$$

We consider a slab with an electric field.

3.2.2 Example

We will be going between spherical and cartesian coordinates

Consider a parallel electric field and a dielectric with

$$\kappa = 1 + \epsilon$$

If κ is very small, then the dielectric will not affect the field all that much.

Now if we consider a larger dielectric

$$V(\vec{r}) = \frac{V_0}{d}\left(r \cdot \frac{R^3}{r^2}\right)\cos(\theta)$$

Where V_0/d is the electric field

We are going to try to find the potential inside and outside of the dielectric.

We've done separation of variables to get the outside

$$V(\vec{r}) = V(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l \cos(\theta)$$

We know that the external electric field is given by

$$V(\vec{r}) = -E_0 z = -E_0 r\cos(\theta) = \frac{V_0}{d}\hat{z}$$

P_l is a legendre polynomial which has some silly properties we looked over before. Now we find the coefficients

$$\sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l \cos(\theta) \qquad l = 1 \rightarrow$$

$$\left(Ar + \frac{B}{r^2}\right)\cos(\theta) \rightarrow$$

$$\left(\frac{-V_0}{d}r + \frac{B_l}{r^2}\right)\cos(\theta) = \left(-E_0 r + \frac{B_{1,out}}{r^2}\right)\cos(\theta)$$

$$V_{in} = rA_{in}\cos(\theta)$$

That is a decent start but what else do we know do figure out the thing?

We have a very important boundary condition

$$E_{||in} = E_{||out}$$

The parallel electric fields are going to equal each other

This is equivalent to saying the potential is continuous at the dielectric boundary.

$$\hat{n}\cdot\vec{D}_{in} = \hat{n}\cdot\vec{D}_{out} \qquad \vec{D} = \epsilon_0\kappa\vec{E} \qquad \vec{E} = \hat{n}\cdot -\nabla V$$

$$\hat{\vec{D}}_{out} = -\epsilon_0 * 1 * \hat{r}\cdot\vec{\nabla}V_{out}(r,\theta) = -E_0 - 2\frac{B_{1,out}}{r^3}$$

$$\hat{n}\cdot\vec{E}_{in} = -\epsilon_0\kappa\hat{r}\cdot\vec{\nabla}V_{in}(r,\theta) = -\kappa A_{in} = -E_0 - 2\frac{B_{1,out}}{R^3}$$

So now we do some more math

$$RA_{1,in} = -E_0 R + \frac{B_{1,out}}{R^2}$$

$$V_{out}(r,\theta) = -E_0 \left(R - \frac{\kappa - 1}{\kappa + 2}\frac{R^3}{r^2} \right) \cos(\theta)$$

$$V_{in} = -\frac{3E_0}{\kappa + 2} r\cos(\theta) = -\frac{3E_0}{\kappa + 2} z$$

if κ is very large, then you will get potentials similar to if the dielectric is a conductor.

consider a shape with unknown dielectric constant

$$\kappa = 1 + X$$

We are trying to find the macroscopic dipole moment of the shape

$$\frac{\vec{P}_{mac}}{V} = \vec{P}$$

Now the wrong answer that you can get is

$$X = \frac{\vec{P}}{\epsilon_0 E'_{applied}}$$

The issue with this expression is that is fails to consider the induced electric field on the inside of the dielectric

$$X = \frac{\vec{P}}{\epsilon_0 E_{in}}$$

3.2.3 High κ Dielectrics

You have a dielectric and you apply some voltage to something in some way and something happens.

$$Q_{deplete} = pd \cdot A \qquad C = \frac{\epsilon_0 \kappa A}{d} \qquad V_G = \frac{Q_{depl}}{C}$$
$$\text{energy } = Q_{depl} \cdot V_G$$

3.2.4 Parallel Plate Capacitor

Consider two flat planes of charge next to each other

$$\hat{n} \cdot D = \text{const} \qquad D_{out} = \epsilon_0 \vec{E}_{applied} = D_{in} = \epsilon_0 \kappa E_{in}$$
$$\epsilon_0 \vec{E}_{in} = \frac{\epsilon_0 E_{applied}}{1 + X} = \epsilon_0 E_{app} - \vec{p}$$

3.2.5 Sphere

Consider a dielectric sphere

$$V_{out}(r,\theta) = -E_0\left(r - \frac{\kappa-1}{\kappa+2}\frac{R^3}{r^2}\right)\cos(\theta)$$
$$V_{in}(r,\theta) = -\frac{3E_0}{\kappa+2}r\cos(\theta) = \frac{3E_0}{\kappa+2}z \qquad \kappa = 1+X$$
$$E_{in} = -\nabla\left(\frac{3E_0}{\kappa+2}z\right) = \hat{z}\frac{\epsilon_0 3E_0}{3+X}$$
$$\epsilon_0(3+X)E_{in} = 3\epsilon_0 E_{in} + P = 3\epsilon_0 E_0$$
$$\epsilon_0 E_{in} = \epsilon_0 E_0 - \frac{1}{3}p$$

The neat little subtract-y thing at the end there is geometry dependent (you can see that it is $1P$ and not $P/3$ for the parallel plates)

Our geometry independent thing is

$$X = \frac{P}{\epsilon_0 E_{applied} - NP}$$

3.2.6 Optical Tweezers

Consider a laser than traps some small dielectric particle

$$\vec{F} = -\vec{\nabla}UhpU_D = -\int d^3r\,\vec{P}(\vec{r})\vec{E}_{in}(\vec{r})$$
$$U = -\epsilon_0 X\int d^3r\,E^2$$

consider a parallel plate capacitor

$$U_D(x) = -Ldx\epsilon_0 xE^2$$

Where Ldx is the volume of the plate

$$F = -\vec{\nabla}U_D(x) = Ld\epsilon_0 XE^2$$

In the left direction.

$$E = \hat{z}\left(E_0 + \frac{\partial E}{\partial z}x\right)$$

$$U_D = -\epsilon_0 X \int d^3r\, E^2(\vec{r})$$

$$= -\epsilon_0 XWh \int_{x-a/2}^{x+a/2} dx' \left(E_0^2 + 2E_0\frac{\partial E}{\partial x} + \frac{\partial^2 E}{\partial^2 x}x^2\right)$$

and the last term goes to 0

$$U_D(x) =$$

$$-\epsilon_0 Xwh\left(E_0^2(x - \frac{a}{2}) - (x + \frac{a}{2}) + E_0\frac{\partial E}{\partial x}\left((x + \frac{a}{2})^2 - (x - \frac{a}{2})^2\right)\right)$$

$$= -\epsilon_0 Xwh\left(E_0^2 a + E_0\frac{\partial E}{\partial x}(2ax)\right)$$

$$F = -2\epsilon_0 XwhaE_0\frac{\partial E}{\partial x}\hat{x}$$

And there's a fancy graph to go with it but that's how optical tweezers work.

$$\text{Intensity } = \epsilon_0 E^2 c \qquad \frac{dI}{dz} = 2\epsilon_0 E \frac{dE}{dz} =$$

Chapter 4

Magnetostatics

I'm just writing stuff down

the Equation of Continuity is. . . something.

Use superposition to get something

$$Q_{box} = \int d^3r\, \rho(\vec{r})$$

$$\frac{d}{dt} Q_{box} = -\int da\, J(\vec{r}, t) = \int d^3r\, \vec{\nabla} \cdot J(\vec{r}, t)$$

$$\frac{d}{dt} \rho(\vec{r}, t) = -\vec{\nabla} \cdot J(\vec{r}, t)$$

The change in charge is related to the flux of charge in and out of the box.

The units of J is current per area.

That is our equation of continuity.

Now we do some magnet math

$$\int_{\partial S} d\vec{l} \cdot \vec{B} = \mu_0 = \int_S da\, \hat{n} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \int_S da\, \hat{n} \cdot \vec{J}$$

We do more math

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(r)}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r})$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$$

Now include time dependence

$$\vec{\nabla} \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0} \qquad \vec{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}(\vec{r}) + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r}, t) \qquad \vec{\nabla} \cdot \vec{B}(\vec{r}, t) = 0$$

4.1 Magnetic Field of a Straight Wire

You have a line with a current in that direction

$$\int da\, \vec{\nabla} \times \vec{B} = \mu_0 \int da\, \vec{J} \rightarrow \int_{\partial S} dl\, \vec{B} = \mu_0 I$$

$$2\pi R B(R) = \mu_0 I \qquad B(r) = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$

4.2 $\vec{B}$ on an Axis

A current of a wobbly wire can be written as $\vec{I} = I\vec{l}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\oint I dl \times \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3} \rightarrow$$
$$\int d^3r\, \vec{J}(\vec{r}) \times \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}$$

If we consider a distance away from the loop perpendicular to it

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I d\vec{l} \times (\vec{r}-\vec{r'})}{|\vec{r}-\vec{r'}|^3}$$
$$\vec{r'} = \hat{x}R\cos(\phi) + \hat{y}R\sin(\phi) \qquad d\vec{l} = -\hat{x}R\sin(\phi)d\phi + \hat{y}R\cos(\phi)d\phi$$

You do cross product vector math and a bunch of stuff cancels out to get

$$\vec{B}(z_0) = \hat{z}$$

I didn't write it down in time

4.3 Cross Product Math

The Levi-Civita symbol is back

4.4 New Lecture

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\vec{J}(\vec{r'}) \times \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial}{\partial t}\vec{E}$$

Where the second part is 0 because we're considering static fields.

Our main equation for magnetostatics is

$$\vec{\nabla} \cdot \vec{B} = 0$$

Because something

$$\vec{B} \equiv \vec{\nabla} \times \vec{A}(\vec{r}) \to \vec{\nabla} \cdot \vec{B} =$$
$$\vec{\nabla} \cdot (\hat{x}(\partial_z A_z - \partial_z A_y) - \hat{y}(\partial_z A_x - \partial_x A_z) + \hat{z}(\partial_z A_y - \partial_y A_x))$$
$$\vec{\nabla} \times \vec{B} = \mu_0 J = \vec{\nabla} \times (\vec{\nabla} \times A) \qquad \vec{u} \times \vec{v} = \epsilon_{ijk} u_j v_k$$
$$\vec{U} \times \vec{V} \times \vec{W} = \epsilon_{ijk}\, U_j\, \epsilon_{klm}\, V_l\, W_m$$

There's only a single shared index between the two Levi-Civita symbols

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

The proof was given and it works but it's real tedious.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \epsilon_{ijk}\,\partial_j\,\epsilon_{klm}\,\partial_l\,A_m \to \epsilon_{ijk}\,\epsilon_{klm}\,\partial_j\,\partial_l\,A_m \to$$
$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,\partial_l\,A_m = \partial_i\partial_j A_j - \partial_j\partial_j A_l$$
$$= \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 A = -\mu_0 J$$

Now we show that the middle term disappears and we get a comfy familiar differential equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Which would just get you a Green's Function.

Now I have to show that the middle thing actually disappears

$$\vec{\nabla} \times \vec{A} = \vec{B} \qquad \vec{A} + \vec{C} = \vec{A}' \qquad \vec{\nabla} \times \vec{C} = 0$$

If we consider any scalar function $g(\vec{r})$

$$\vec{\nabla} g(\vec{r}) = C \qquad \vec{A} + \vec{\nabla} g(\vec{r}) = \vec{A}' \qquad \nabla^2 g = -\vec{\nabla}\vec{A}$$
$$g = \frac{1}{4\pi} \int d^3r'\, \frac{\vec{\nabla} \cdot \vec{A}}{|\vec{r} - \vec{r'}|} \qquad \vec{\nabla} \cdot \vec{A} = 0$$
$$\nabla^2 \vec{A} = \mu_0 \vec{J}$$

This is called the Coulomb gauge

We have the Ampere's Law PDF

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 A = -\mu_0 J$$

and then the middle thing cancels out

$$\nabla^2 A = \mu_0 J$$

And then you can also multiply it by a scalar function g

$$A + g = A' \qquad \nabla \times A = \nabla \times A'$$

Becauses translational scalar functions don't affect the curl of anything

4.4.1 Units

$B = 1$ Tesla $= 1$ Volt-second / meter^2
other stuff that I didn't write down

4.5 Loop of Wire

Consider a loop of wire on the $x - y$ plane

$$A(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r \, \frac{J_x(\vec{r'})}{|\vec{r} - \vec{r'}|} \qquad \vec{J}(\vec{r}) = \vec{\nabla} \times \vec{B}(\vec{r})$$

If we consider a line-directional current with directions

$$I_x = -I \sin(\theta) \qquad I_y = +I \cos(\theta)$$

What we want to do is turn the thing into a 1d path integral

$$A_x(x,y,z) = \frac{\mu_0}{4\pi}\int Rd\theta\,\frac{I\sin(\theta)}{\sqrt{(x-R\cos(\theta))^2+(y-R\sin(\theta))^2+z^2}}$$

$$= \frac{\mu_0 R}{4\pi}\int d\theta\,\frac{I\sin(\theta)}{\sqrt{r^2+R^2-2R(x\cos(\theta)+y\sin(\theta))}}$$

So now we choose another plane and hopefully get something

$$A_x(0,y,z) = \frac{\mu_0 R}{4\pi}\int d\theta\,\frac{-I\sin(\theta)}{\sqrt{y^2+R^2-2Ry\sin(\theta)}}$$

If we consider a point inside the loop and a point very far away from the loop then we get

$$A(0,y<<R,0) = -\frac{\mu_0 R}{4\pi}\int d\theta\,\frac{-I\sin(\theta)}{R\sqrt{\frac{y^2}{R^2}+1-2\frac{y}{R}\sin(\theta)}} \to$$

$$\frac{\mu_0 RI}{4\pi R}\int d\theta\,\frac{\sin(\theta)}{\sqrt{1+\frac{y}{R}\sin(\theta)}} \to$$

$$\frac{\mu_0 I}{4\pi}\int d\theta\,\sin(\theta)\left(1+\frac{y}{R}\sin(\theta)\right) =$$

$$\frac{\mu_0 I}{8\pi}\frac{y}{R}$$

And we consider something else

$$A(0, y >> R, 0) = \frac{\mu_0 R}{4\pi} \int d\theta \, \frac{-I \sin(\theta)}{y\sqrt{1 + \frac{R^2}{y^2} - 2\frac{R}{y}\sin(\theta)}} \Rightarrow\Rightarrow$$

$$\frac{\mu_0 I R}{4\pi y} \int d\theta \, \sin(\theta) \left(1 + \frac{R}{y}\sin(\theta)\right) =$$

$$\frac{\mu_0 I}{8\pi} \left(\frac{R}{y}\right)^2$$

4.5.1 Solenoid

If we want the magnetic field of a solenoid we get

$$\vec{A} = \frac{A_0}{2}\left(-\frac{y}{R}\hat{x} + \frac{x}{R}\hat{y}\right)$$

4.6 Magnetic Multipoles

There are magnetic dipole moments and stuff

If we consider an electric dipole Then we have multiplication.

If we have a magnetic dipole, we use the cross product

The vector potential of a magnetic dipole is

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3}$$

The magnetic field of a dipole is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m} \cdot \vec{r})}{r^5} - \frac{\vec{m}}{r^3}\right)$$

We can do this with all sorts of nice math

$$\frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \, \frac{\vec{J}(\vec{r})}{\sqrt{1-\frac{2\vec{r}\cdot\vec{r'}}{r^2}+\frac{r'^2}{r^2}}}$$

We consider to first order and take a taylor series expansion

$$\approx \frac{\mu_0}{4\pi}\frac{1}{r}\int d^3r' \, \vec{J}(\vec{r})\left(1-\frac{\vec{r}\cdot\vec{r'}}{r^2}\right) \qquad \frac{1}{\sqrt{1-\epsilon}} \approx 1+\frac{1}{2}\epsilon+\ldots$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{1}{r^3}\int d^3r' \, r_j \cdot r'_j \vec{J}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{r_j}{r^3}\int d^3r' \, r'_j \vec{J}(\vec{r})$$

Now we have to show that that expression is anti-symmetric, meaning

$$T_{ij} = -T_{ji}$$

And we do this via some math

$$T_{ij} = \frac{1}{2}\left(S_{ij}+A_{ij}\right) \qquad S_{ij} = T_{ij}+T_{ji} = \int d^3r' \, \left(r_j J_i(\vec{r'}) + r_i J_j(\vec{r'})\right)$$

$$A_{ij} = T_{ij}-T_{ji} = \int d^3r' \, \left(r_j J_i(\vec{r'}) - r_i J_j(\vec{r'})\right)$$

$$A_i(\vec{r'}) \approx \frac{\mu_0}{4\pi}\frac{r_j}{r^3}\frac{1}{2}\int_{V<R} d^3r' \, \left(r_j J_i(\vec{r'}) - r_i J_j(\vec{r'})\right)$$

$$\vec{A}(\vec{r'}) \approx \frac{\mu_0}{4\pi}\frac{1}{r^3}\frac{1}{2}\int_{V<R} d^3r' \, \left(J(\vec{r'})\cdot(\vec{r}\cdot\vec{r'}) - \vec{r'}(\vec{r}\cdot J(\vec{r'}))\right)$$

This is all based off of some wacky vector identity

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

So we go back to our magnetic dipole equation

$$\frac{\mu_0}{4\pi}\frac{1}{r^3}\frac{1}{2}\int_{V<R} d^3r' \left(J(\vec{r'}) \cdot (\vec{r} \cdot \vec{r'}) - \vec{r'}(\vec{r} \cdot J(\vec{r'}))\right) =$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{r}}{r^3} \times \frac{1}{2}\int d^3r' \left(\vec{J}(\vec{r}) \times \vec{r'}\right) \qquad \int d^3r' \, \vec{J}(\vec{r}) \times \vec{r'} = \vec{m}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3}$$

4.7 Fancy Math Trick

Consider

$$\vec{J}_l(\vec{r'})\partial'_l r'_i r'_j = J_l \delta_{il} r'_j + J_l \delta_{lj} r'_i = J_i r'_j + J_j r'_i$$

$$= \frac{1}{2}\int d^3r' \, J_l(\vec{r'})\partial'_l r'_i r'_j$$

So now that we have that, we get

$$\partial'_l \left(J_l(\vec{r'}) r'_i r'_j\right) = r'_i r'_j \vec{\nabla} \cdot \vec{J} - \vec{J}_l(\vec{r'})\partial_l r'_i r'_j \qquad \vec{\nabla} \cdot \vec{J} = 0$$

So with our 2 tricks, we can do

$$\frac{1}{2}\int d^3r' \, J_l(\vec{r'})\partial'_l r'_i r'_j = \frac{1}{2}\int d^3r' \, \partial_l J_l(\vec{r'}) r'_i r'_j$$

I'll look back at this cuz I don't fully understand what happened

4.8 Magnetic Field Jargon

$$\vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} \Rightarrow\Rightarrow\Rightarrow$$
$$\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$

You do some levi-civita symbol shenanigans to get

4.9 Post-Spring Break

There is no magnetic multipole

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r \, \frac{\vec{J}(\vec{r})}{|\vec{r} - \vec{r'}|}$$

If we think about magnetic dipoles, we can calculate the dipole moment as

$$\vec{m} = \int_V d\vec{m} = \int_V d^3r \, \frac{\vec{r} \times \vec{J}(\vec{r})}{2}$$

If we consider a loop of charge, we need to find the magnetic moment

$$\vec{m} = \frac{1}{2}I\int \vec{r} \times d\vec{l}$$

If we consider a circle of charge, then our magnetic moment is

$$\vec{m} = \hat{n}\frac{1}{2}I2\pi R^2 = I\pi R^2$$

Which matches what we know from highschool E & M
What if the loop is not planar?
Then we still do a line integral

$$\vec{m} = \int_S d\vec{m} = I\oint \frac{\vec{r} \times d\vec{l}}{2} \approx I\int_S d\vec{a}$$

Now we can use $\vec{A}(\vec{r})$ to find $\vec{B}(\vec{r})$ from a magnetic dipole

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} \qquad \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

You do a bunch of obnoxious math kronecker deltas and get something

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\vec{m}(\vec{\nabla} \cdot \frac{\vec{r}}{r^3}) - (\vec{m} \cdot \vec{\nabla})\frac{\vec{r}}{r^3}\right)$$

Now we consider the magnetic field caused by a point charge
The first bit of our magnetic field equation is a dirac delta, so we don't really have to think about it.

$$\vec{m}(\vec{\nabla} \cdot \frac{\vec{r}}{r^3}) = \mu_0 \vec{m} \delta^3(\vec{r})$$

The second half of the magnetic field equation is going to yield something that's non-zero

So our magnetic field is given by

$$\frac{\mu_0}{4\pi}\left(\frac{3(\vec{m} \cdot \vec{r})\vec{r}}{r^5} - \frac{\vec{m}}{r^3}\right)$$

It's basically the same as the electric field caused by an electric dipole.

4.10 Force from $\vec{B}$

$$\vec{F} = q\vec{v} \times \vec{B}$$

So you find net forces and net works, you do a line integral

$$\vec{F} = \oint_{\partial S} d\vec{F} = \lambda v \oint_{\partial S} dvecl \times \vec{B}_{external}$$

4.10.1 Torque

There's also a torque that can be caused by magnetic fields

$$\vec{N} = \vec{m} \times \vec{B}$$

4.10.2 Potential Energy

$$U = -\vec{m} \cdot \vec{B}$$

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$$\nabla \cdot \vec{B} = 0 \qquad \vec{B} = \nabla \times \vec{A}$$

The latter equation is called the vector/gauge potential

You can transform A

$$\vec{A} \rightarrow \vec{A'} = \vec{A} + \nabla\psi$$

Our ability to pick A is called gauge fixing.

$\vec{\nabla} \cdot \vec{A} = 0$ is called the Coulomb Gauge

We have all sorts of silly equations

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \oint_{\partial S} \vec{A} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int dl' \, \frac{\vec{I} \times (\hat{r} - \hat{r'})}{|\vec{r} - \vec{r'}|^2} \qquad \vec{E} = k \int d^3r \, \frac{\rho}{r^2}$$

And we have two real important Laplacians in magnetostatics

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \qquad \nabla^2 \vec{V} = -\frac{\rho}{\epsilon_0}$$

$$V = k \int d^3r \, \frac{\rho}{r}$$

So we do some math and get

$$\vec{A} = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

This is all coming from the Coulomb Gauge solution, but there is other math that comes from other gauges.

There are also Lorentz forces given by

$$\vec{F} = q\vec{v} \times \vec{B}$$

And we know for a fact that there are no magnetic monopoles, so all magnetic point charges are dipoles.

$$\vec{A}_{above} = \vec{A}_{below} \qquad \vec{B}^{||}{}_{above} - \vec{B}^{||}{}_{below} = \mu_0 \vec{K}$$

Where $\vec{K}$ is the surface current density

And that can also be written as

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = \mu_0 \vec{K}$$

And we can call $\vec{A}$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint dl' \, (r')^n P_n(\cos\alpha)$$

Where P_n is the Legendre Polynomial that we dealt with earlier.

And now we have to deal with the fact that $\vec{A}$ will always be caused by a dipole

$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2}$$

4.11.1 IMPORTANT

$\vec{A}$ is the vector potential of a magnet

$$\vec{\nabla} \times \vec{A} \equiv \vec{B}$$

It's the same as the electric potential except its the potential derived from the magnetic field.

4.11.2 Problem 1

You have an infinite sheet all with a current $\vec{K} = K\hat{x}$ on that surface with a surface current $\vec{K} = K\hat{x}$

What is the magnetic field caused by these moving charges?

I have a suspicion that I can pull some shenanigans with superposition.

4.12 New Lecture

4.12.1 Energy

$$U(\theta) = \vec{m} \cdot \vec{B}$$

The force of a magnetic dipole can be given by

$$\vec{F} = \vec{\nabla}\left(\vec{m} \cdot \vec{B}\right)$$

4.13 Magnetic Materials

You have a magnetically polarized material

4.13.1 Paramagnetism

Apply a magnetic field to a randomly oriented material $\vec{B} = \hat{z}B$

Each atom has two directions of magnetic moment, and the probability of each direction is random (determined by the Boltzmann Factor)

$$e^{-E/kT} \qquad U(up) = -mb \qquad U(down) = +mB$$

$$P(up) = \frac{e^{-(-mb/kT)}}{e^{mb/kT} + e^{-mb/kT}} = \frac{e^{mb/kT}}{e^{mb/kT} + e^{-mb/kT}}$$

$$P(down) = \frac{-e^{mb/kT}}{e^{mb/kT} + e^{-mb/kT}}$$

And the magnetization density is given by a superposition of the ups and downs

$$\vec{M} = n\left(m\hat{z}\frac{e^{mb/kT}}{e^{mb/kT} + e^{-mb/kT}} + (-m\hat{z})\frac{-e^{mb/kT}}{e^{mb/kT} + e^{-mb/kT}}\right) = nm\hat{z}\tanh\Big($$

For large fields, the magnetization density saturates.
For small values

$$\vec{M} = \frac{nm^2\vec{B}}{kT}$$

4.14 Magnetization Density

consider a magnetic density vector field

$$\vec{M}(\vec{r}) = n(\vec{r})\vec{m}(\vec{r})$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V d^3r'\,\vec{M}(\vec{r'}) \times \frac{\vec{r}-\vec{r'}}{|\vec{r}-\vec{r'}|^3}$$

4.15 Boundary Math?

There's alot of stuff in the lecture that I'm not fully getting

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V d^3r'\,\left(\frac{\vec{\nabla}_r \times \vec{M}(\vec{r'})}{\vec{r}-\vec{r'}} - \vec{\nabla}_{r'} \times \frac{\vec{M}(\vec{r'})}{\vec{r}-\vec{r'}}\right) =$$

$$\frac{\mu_0}{4\pi}\int_V d^3r'\,(A_1 - A_2)$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d^3r' \, \frac{\vec{J}(\vec{r'})}{|\vec{r} - \vec{r'}|} \qquad \vec{J}_{bound}(\vec{r'}) = \vec{\nabla}_{r'} \times \vec{M}(\vec{r'})$$

Which is found from A_1

4.16 Magnetic Particles

Electrons have a "spin", that means that they act as a magnetic dipole.

If we consider a dielectric in a field, there are free charges that we place, and bound charges that are induced by the free charges.

Consider the displacement field

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

where P is the polarizing susceptibility or whatever its called. It's basically the dipole density.

$$\vec{\nabla} \cdot \vec{D} = \epsilon_0 \left(\frac{\rho_{free}}{\epsilon_0} + \frac{\rho_{bound}}{\epsilon_0} \right) - \rho_{bound} = \rho_{free}$$

Consider two dielectrics on top of each other.

$$\nabla \times \vec{E} = 0 = -\frac{\partial}{\partial t}\vec{B}$$
$$\int_S d^3r\, \nabla \times \vec{E} = \int_{\partial S} d^3r\, d\vec{l} * \vec{E} = 0$$
$$\hat{n} \cdot \vec{D}_1 - \hat{n} \cdot \vec{D}_2 = \sigma_{free}$$

If we consider some silly field equations

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J}$$

Helmholtz Theorem basically tells you that you can solve for a function F given the curl and divergence.

Consider a bound surface with a 2D current K

$$\vec{J}_{bound} = \nabla \times \vec{M}$$

If we make a loop perpendicular to the surface, then we can use Ampere's Law to show that a magnetic field is generated.

$$\int_S da\, \nabla \times \vec{B} = \int_{\partial S} d\vec{l} * \vec{B} = \int_S da\, \mu_0 \vec{J} = \int_0^L dy\, \mu_0 \vec{K} = \mu_0 K L$$
$$\Rightarrow 2BL \Rightarrow \mu_0 K L$$

So we can get a magnetic field equation

$$\vec{B}_{2||} - \vec{B}_{1||} = \mu_0 \vec{K} \times \hat{n}$$

Where B_2 and B_1 are the fields on either side of the surface.
And you figure out the direction of the field using the right hand rule.
Consider a cube with a uniform magnetic dipole moment.
K_b is $\nabla \times \vec{M}$ applied to the surface

$$\vec{K}_b = \vec{M} \times \hat{n}$$

The surfaces where the magnetic moment is parallel to the normal vector will have a net current of 0

$$\nabla \times B = \mu_0 \vec{J}_{free} - \mu_0 \vec{J}_{bound} \rightarrow \nabla \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \mu_0 \vec{J}_{free}$$

So we make a helper field

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \qquad \vec{B} = \mu_0 (\vec{H} + \vec{M}) \qquad \nabla \times \vec{H} = \vec{J}_{free}$$
$$\vec{H}_{1,||} - \vec{H}_{2,||} = \vec{K}_{free} \times \hat{n}$$

If $J_{free} = 0$, then

$$\vec{H}_{1,||} = \vec{H}_{2,||}$$

If we consider a surface of charge and make a Gaussian pillbox around it, then

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 = \int_S d\vec{a} \cdot \vec{B}$$

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Lecture stuff

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2} \qquad \vec{m} = \gamma\vec{c}A \qquad \vec{M} = \frac{\vec{m}}{V}$$
$$\vec{U} = 0\vec{m} \cdot \vec{B} \qquad \vec{\tau} = \vec{m} \times \vec{B}$$
$$P^i = \alpha^{ij} E^j_{external}$$

Apparently we will be talking about paramagnetism and diamagnetism today.

ferromagnets and anti-ferromagnets are the normal magnets that we think about when we think magnets.

$$vecA(\vec{r}) = \frac{\mu_0}{4\pi}\int\limits_{\Sigma} d^3x'\, \frac{\vec{M}(\vec{r'}) \times (\hat{r} - \hat{r'})}{|\vec{r} - \vec{r'}|^2} =$$

$$\frac{\mu_0}{4\pi}\int\limits_{\Sigma} d^3x'\, \vec{M}(\vec{r'}) \times \vec{\nabla}'\frac{1}{|\vec{r} - \vec{r'}|}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{\Sigma} d^3x'\, \vec{\nabla}' \times \frac{\vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|} - \frac{\vec{\nabla}' \times \vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{\Sigma} d^3x'\, \frac{\vec{\nabla}' \times \vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|} - \frac{\mu_0}{4\pi}\oint\limits_{\partial\Sigma} d^2a'\, \frac{\vec{M}(\vec{r'}) \times \hat{n}}{|\vec{r} - \vec{r'}|}$$

$$= \frac{\mu_0}{4\pi}\int\limits_{\Sigma} d^3x'\, \frac{\vec{J}_b(\vec{r'})}{|\vec{r} - \vec{r'}|} - \frac{\mu_0}{4\pi}\oint\limits_{\partial\Sigma} d^2a'\, \frac{\vec{K}_b}{|\vec{r} - \vec{r'}|}$$

$$\vec{J}_b(\vec{r'}) = \vec{\nabla}' \times \vec{M}(\vec{r'}) \qquad \vec{K}_b = \vec{M}(\vec{r'}) \times \hat{n}$$

That's the derivation
And then there's some more math

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J}_b + \vec{J}_f = \nabla \times \vec{M} + \vec{J}_f \Rightarrow \nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f$$

$$\frac{\vec{B}}{\mu_0} - \vec{M} = \vec{H}$$

So now we have the silly new helper function $\vec{H}$

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$
$$\vec{H^{||}_{above}} - \vec{H^{||}_{below}} = \vec{K}_{free} \times \hat{n}$$

4.18 $\vec{A}$ From a Magnetized Object

We have fundamental results and computational tools

Consider the fundamental results

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{M}(\vec{r'}) \times (\vec{r} - \vec{r'})}{|\vec{r} - \vec{r'}|^3}$$

That is a physics based something or other.

Then we have the computational convenience

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{\nabla} \times \vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|} \qquad \vec{J}_B = \vec{\nabla} \times \vec{M} \qquad \vec{K}_b = \vec{M} \times \hat{n}$$

And you can pull some Ampere's Law shenanigans to get

$$\boldsymbol{\nabla} \times \vec{B} = \vec{J}_f + \vec{J}_b \qquad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \Rightarrow \vec{\nabla} \times \vec{H} = \vec{J}_{free}$$

There's an equivalence in electrics and magnetostatics

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad \vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) \qquad \vec{M} = X_m \vec{H}$$

If we consider the magnetic field from a spherical surface we can use the Laplacian and some differential equations to get

$$A_x(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{\infty}\left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)$$

But that's going to take forever to solve even if technically correct.

Let's define a magnetic scalar potential

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \qquad \vec{H} = -\vec{\nabla}\phi_m(\vec{r})$$

So with that object $\vec{H}$, and no free currents, we get the properties

$$\vec{\nabla}\cdot\vec{B} - 0 = \vec{\nabla}\cdot mu_0(\vec{H}+\vec{M}) \qquad \vec{\nabla}\cdot\vec{H} + \vec{\nabla}\cdot\vec{M} \rightarrow$$
$$\nabla^2\phi_m = \vec{\nabla}\cdot\vec{M}$$

now we get some nice equivalent electricity vs magnetism properties

$$(\vec{D}_1 - \vec{D}_2)\hat{n} = \sigma_{free} \qquad \hat{n}\cdot(\vec{B}_1 - \vec{B}_2) = 0 \qquad \vec{H}_{||,1} = \vec{H}_{||,2}$$

If we consider a uniformly magnetized sphere, the way we find the field is by constructing a parameterized solution region by region

$$B_{in} = \mu_0(1+X_m)H_0 \qquad B_{out} = \mu_0 H_0$$

Consider a spherical paramagnetic in an external magnetic field with azimuthal symmetry.

$$\phi_m(\vec{r}) = \sum_l \left(\alpha_l r^l + \frac{\beta_l}{r^{l+1}}\right) P_l(\cos(\theta))$$

$$\phi_{m,out}(\vec{r}) = \alpha_{1,out} r\cos(\theta) + \beta_{1,out}\frac{1}{r^2}\cos(\theta)$$

$$\phi = Hz \qquad -\vec{\nabla}\phi = \vec{H}$$

$$\alpha_{1,out} = -\frac{B}{\mu_0} \qquad \phi_{in} = \alpha_{in} r\cos(\theta) \qquad \vec{B}_{out}\cdot\hat{n} = \vec{B}_{in}\cdot\hat{n}$$

$$-\frac{B_0}{\mu_0} - \frac{2B_{out}}{R^3} = (1+X_m)\alpha_{in} \to -\frac{B_0}{\mu_0} + \frac{B_{out}}{R^3} = \alpha_{in}$$

$$\phi_{in} = \alpha_{in} r\cos(\theta) \qquad \vec{H}_{in} = -\vec{\nabla}\phi_{in}$$

$$B_{in} = \mu_0(1+X_m)H_{in} = \hat{z}B_{out}\frac{1+X_m}{1+\frac{1}{3}X_m}$$

I think I got down all the equations, but you have to figure out how to put them together.

4.19 Dielectric Boundary Problems

We have our helper functions and scalar potentials $\vec{H}$ and ϕ.

$$\nabla^2\phi_m(\vec{r}) = +\vec{\nabla}\cdot\vec{M}$$

A vibrating sample magnetometer is a thing.

As two plates get farther apart, $\vec{H}$ goes to 0 so $\vec{B}$ is determined by $\vec{M}$

4.20 Magnetic Circuits

idk

4.21 Magnetic Materials

idk

4.22 Helmholtz Theorem

$$\boldsymbol{\nabla} \cdot \vec{F}(\vec{r}) = D(\vec{r}) \qquad \boldsymbol{\nabla} \times \vec{F}(\vec{r}) = C(\vec{r}) \qquad \vec{F} = -\vec{\nabla} U + \boldsymbol{\nabla} \times \vec{W}$$

$$U = \int d^3\, r' \, \frac{D(\vec{r})}{|\vec{r} - \vec{r'}|} \qquad \vec{W} = \frac{C(\vec{r})}{|\vec{r} - \vec{r'}|}$$

It just shows that there's math that gives you a vector field given its curl and divergence

Chapter 5

Electrodynamics

Time-dependent math

We're first gonna work with Faraday's Law

Consider a cylinder with some resistance $R = \rho L/A$ and a current I such that $V_1 - V_2 = IR = V = EL$ and an energy $E = J \cdot \rho$ and $\vec{J} = \sigma\vec{E}$ and $\sigma = 1/\rho$

$$m\dot{v} = F = qE$$

$\int dl \cdot E$ for some closed circuit is called the electromotor force (EMF).

I think we're just going over all of the stuff that comes with electrons moving around in a circuit.

$$q \cdot EMF = qVBW$$

So we get a big string of things

$$\oint d\vec{l} \cdot q\vec{v} \times \vec{B} = qvB \cdot w = q \oint_{\partial S} d\vec{l} \cdot \vec{E} = q \int da\, \boldsymbol{\nabla} \times \vec{E}$$

So we get

$$\frac{d}{dl}\phi_m = \frac{d}{dl}\int d\vec{a}\,\vec{B}$$

So Faraday's Law is

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{d}{dt}\vec{B}$$

5.1 Discussion 11

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\left(\int_\Sigma \frac{\vec{J}_b(\vec{r'})}{|\vec{r}-\vec{r'}|}\,d^3r' + \oint \frac{\vec{K}_b(\vec{r'})}{|\vec{r}-\vec{r'}|}\,da'\right)$$

$$\vec{J}_b = \boldsymbol{\nabla} \times \vec{M} \qquad \vec{K}_b = \vec{M} \times \hat{n} \qquad \boldsymbol{\nabla} \times \vec{H} = \vec{J}_f$$

And we consider a linear polarization

$$\vec{M} = X_m\vec{H} = \mu_0(1+X_m)\vec{B} = \mu\vec{B}$$

And with some math we get some nice identities

$$H_{above}H^{\perp}_{above} = H_{below}H^{\perp}_{below} \qquad \vec{H}^{\perp}_{above} - \vec{H}^{\perp}_{below} = \vec{K}_f \times \hat{n}$$

5.1.1 Problem 1

Find the magnetic field between two hollow tubes.

5.2 EMF

In statics, Voltage was defined as

$$V = -\int_{\vec{0}}^{\vec{r}} d\vec{l} \cdot \vec{E}$$

Now we have a new equation

$$\nabla \times \vec{E} = -\frac{d}{dt}\vec{B}$$

$$\int_{\vec{0}}^{\vec{r}} d\vec{l} \cdot \vec{E} = \int\limits_S d\vec{a} \cdot \nabla \times \vec{E} = -\int\limits_S d\vec{a} \cdot \frac{d}{dt}\vec{B} = \frac{d}{dt}\vec{\Phi}_m$$

$$-\int_{\vec{0}}^{\vec{r}} d\vec{l} \cdot \vec{E} = \frac{d}{dt}\vec{\Phi}_m = V_0$$

Given a constant EMF, we can see that the flux increases linearly.

The units of $\frac{d}{dt}\vec{\Phi}_m$ is $Tm^2/s = \frac{V\,s}{m^2}\frac{m^2}{s} = V$

5.3 More Circuit Stuff

$$\Phi_m = I \cdot L \qquad V = L \cdot \frac{dI}{dt}$$

Where L is the inductance.

$$\frac{d}{dt}V = \frac{d}{dt}Q/C = \frac{I}{C}$$

Let's consider if both the current and voltage are both driven by something that's sinusoidal.

$$I = I_0 e^{i\omega t} \qquad V = V_0 e^{i\omega t}$$

Then when we take our time derivatives we get

$$\frac{d}{dt} V_0 e^{i\omega t} = \frac{I_0 e^{i\omega t}}{C} \to i\omega V_0 = \frac{I_0}{C} \to V_0 = \frac{I_0}{i\omega C} \to V_0 = \mathbb{Z}_C I_0$$

So we can see that the voltage and current are phase shifted from each other.

$$V_0 e^{i\omega t} = L \cdot i\omega I_0 e^{i\omega t} \to V_0 = i\omega L I_0 \qquad \mathbb{Z}_L = i\omega L$$

We put some stuff together to get an impedance

$$Z = \frac{1}{i\omega C} + i\omega L = \frac{1 - \omega^2 LC}{i\omega C}$$

We have a resonance at $\omega = 1/\sqrt{LC}$

Z is either positive imaginary or negative imaginary.

$$\frac{1}{Z} = i\omega C + \frac{1 - \omega^2 LC}{i\omega L} \qquad Z_{//} = \frac{i\omega L}{1 - \omega^2 LC}$$

5.4 Magnet in a Copper Tube

So there's a gravational force and there's a magnet that gives off a magnetic field.

We have a changing magnetic field which generates an EMF in the tube.

The flux above the magnet is increasing

The flux below the magnet is decreasing

The magnetic field makes a current in the tube, and that tube current then induces its own magnetic field. That induced magnetic field pushes against gravity and thus the magnet falls slower than expected.

5.5 Field Energy Density

We know that magnetic fields do no work on objects,

$$U = QV \qquad U(\vec{r}_[1]) + U(\vec{r}_[2]) = Q\left(V(\vec{r}_1) + V(\vec{r}_2)\right)$$

$$I = qn\vec{v} = \frac{dQ}{dt} \qquad \frac{dU}{dt} = \frac{dQ}{dt}V = IV$$

$$\Delta U = \int_{t1}^{t2} dt\, \frac{dU}{dt}$$

Construct circuit elements that have $\vec{E}$ and/or $\vec{B}$ in them.

$$Q = V * C \qquad \frac{dU}{dt} = I * V = \int_0^t dt\, \frac{dQ}{dt}V = \int_0^t dt\, \frac{dQ}{dt}\frac{Qd}{\epsilon_0 A}$$

$$= \frac{1}{C}\int_0^Q dQ\, Q = \frac{1}{C}\frac{Q^2}{2} = \frac{1}{2}CV^2$$

That's our math stuff

$$\Delta U = \frac{1}{2}\frac{\epsilon_0 AL(?)}{d^2} = \frac{\epsilon_0}{2}AdE^2 = U_E * \text{volume} \qquad U_E = \frac{\epsilon_0 E^2}{2}$$

If we consider a solenoid, where $\vec{B} = \mu_0 n I$

$$U(t) = \int_0^t dt\, V(t) * I(t) = \int_0^t dt\, L * \frac{dI}{dt} * I = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\phi_B^2}{L}$$

$$\phi_B = L * I \qquad L = \frac{\phi_{net}}{I} = \frac{1}{I} \mu n I * \pi R^2 * nh = \mu_0 \pi n^2 R^2 h$$

$$\phi_B = B \pi R^2 nh \qquad U = \frac{1}{2} \frac{Bnh\pi R^2}{\mu_0 n^2 \pi R^2 h} = \frac{1}{2\mu_0} B^2 * h\pi R^2$$

ta da

5.6 Repairing Ampere's Law

Imagine a circuit with just a resistor and capacitor (RC circuit)

$$V_{Battery} = \frac{Q}{C} + I(t) R \qquad RC \frac{dQ}{dt} + Q = \frac{V_0}{C}$$

$$Q = CV_0 (1 - e^{-t/RC}) \qquad I(t) = \frac{V}{R} e^{-t/RC}$$

$$I_0 = \frac{V}{R} \qquad \int_{\partial S} d\vec{l}\, \vec{B} = \mu_0 I(t) \qquad \vec{B} = \hat{\phi} \frac{\mu_0 I(t)}{2\pi s}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

Ampere's Law contradicts Stoke's Theorem because if the amperian loop changes, the answer changes, when Stoke's Theorem says that shouldn't be the case.

Let's imagine a capacitor

$$\vec{J} * A = \vec{I} \qquad J = \frac{d\sigma}{dt} = \epsilon_0 \frac{d}{dt} E \qquad E = \frac{\sigma}{\epsilon_0}$$

$$\int_S d\vec{a}\, \nabla \times \vec{B} = \int_{\partial S} d\vec{l} \cdot \vec{B}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}(\vec{r}, t) + \epsilon_0 \frac{d}{dt} \vec{E}(\vec{r}, t) \right)$$

The divergence of the magnetic field is always 0 no matter what.

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{d}{dt} \vec{B} \qquad \nabla \cdot \vec{E}(\vec{r}, t) = \frac{\rho(r, t)}{\epsilon_0}$$

$$U_E(\vec{r}) = \frac{\epsilon_0}{2} E^2 \qquad U_B = \frac{1}{2\mu_0} B^2$$

We know for a fact that fields have energy. Do field conserve their energy though?

$$U_E(\vec{r}, t) = \frac{\epsilon_0}{2} E^2 \qquad U_B(\vec{r}, t) = \frac{1}{2\mu_0} B^2$$

$$\boldsymbol{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0 \vec{J}_{total}(\vec{r}, t) + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \vec{E}(\vec{r}, t)$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \qquad \boldsymbol{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{d}{dt} \vec{B} \qquad \boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r}, t)}{\epsilon_0}$$

$$\frac{d}{dt} U_Q(\vec{r}, t) = -\frac{d}{dt} \int\limits_V d^3r' \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) =$$

$$-\frac{1}{\mu_0} \int\limits_{\partial V} d^2a' \, \vec{E} \times \vec{B}$$

That's basically us showing how conservation of energy is kept for fields

Now let's consider the work done by a field

$$W = \Delta U = \int_{ti}^{tf} dt\, \frac{d\vec{l}}{dt}\vec{F}(\vec{r}, 0) =$$
$$\int_{ti}^{tf} dt\, \vec{V}(t) \cdot \left(q\vec{E}(\vec{r}, t) + q\vec{V}(t) \times \vec{B}(\vec{r}, t)\right) = \int_{ti}^{tf} dt\, \vec{V}(t) \cdot q\vec{E}(\vec{r}, t)$$
$$\rightarrow \frac{dU_Q}{dt} = \int d^3r\, qn(\vec{r}, t) \cdot \vec{V}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)$$
$$= \int d^3r\, \vec{J}(\vec{r}, t) \cdot \vec{E}(\vec{r}, t)$$
$$= \int_V d^3r\, \frac{1}{\mu_0}\left(\boldsymbol{\nabla} \times \vec{B}(\vec{r}, t) - \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r}, t)\right) \cdot \vec{E}(\vec{r}, t)$$

Now we look at both parts of the $\vec{J}$ equation and see how they work in fields

$$-\mu_0\epsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = -\epsilon_0\int d^3r\,\vec{E}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{E}(\vec{r},t)$$

$$= -\frac{\epsilon_0}{2}\frac{\partial}{\partial t}\int_V d^3r\,E^2(\vec{r},t)$$

$$\boldsymbol{\nabla}\times\vec{B}(\vec{r},t) = \int_V d^3r\,\vec{E}\cdot\boldsymbol{\nabla}\times\vec{B} = \int_V d^3r\,\varepsilon_{ijk}E_i\partial_j B_k =$$

$$\partial_j E_i B_k = B_k\partial_j E_i + E_i\partial_j B_k$$

$$\boldsymbol{\nabla}\times\vec{B}(\vec{r},t) = \frac{1}{\mu_0}\int_V d^3r\,\varepsilon_{ijk}\left(\partial_j E_i B_k - B_k\partial_j E_i\right)$$

$$= -\frac{1}{\mu_0}\int_V d^3r\,\varepsilon_{jik}\partial_j E_i B_k + \varepsilon_{kji}B_k\partial_j E_i$$

$$= \frac{1}{\mu_0}\int_V d^3r\,\left(-\boldsymbol{\nabla}\times\left(\vec{E}\times\vec{B}\right) + \vec{B}\cdot\boldsymbol{\nabla}\times\vec{E}\right)$$

We have our two parts of the integral, now we can put it all together

$$-\frac{1}{\mu_0}\int_V d\vec{a}\cdot\vec{E}\times\vec{B}-\frac{1}{\mu_0}\int d^3r\,\vec{B}(\vec{r},t)\cdot\frac{\partial}{\partial t}\vec{B}(\vec{r},t)$$

$$=-\frac{1}{\mu_0}\int_V d\vec{a}\cdot\vec{E}\times\vec{B}-\frac{1}{2\mu_0}\int d^3r\,\frac{\partial}{\partial t}B^2(\vec{r},t)$$

$$\frac{dU_Q}{dt}=\int_V d^3r\,\vec{J}_{Total}(\vec{r},t)\cdot\vec{E}(\vec{r},t)$$

$$=-\frac{\partial}{\partial t}\int_V d^3r\,\left(U_E(\vec{r},t)+U_B(\vec{r},t)\right)-\frac{1}{\mu_0}\int_{\partial V} d\vec{a}\cdot\vec{E}\times\vec{B}$$

That's a thing and there are other things

$$\vec{S}(\vec{r},t)=-\frac{1}{\mu_0}\vec{E}\times\vec{B}$$

That is called Pointing's Vector

5.7 MIDTERM 2

2d current density K_B has units amps per meter because the amps are through a cross section of the 2d space.

Magnetic boundary value problems have like 2 important equations

$$\vec{H}_{1,||}-\vec{H}_{2,||}=\vec{K}_{free}\times\hat{n}\qquad\vec{B}_{\perp,1}=\vec{B}_{\perp,2}$$

We also do have a kind of Green's function approach with the magnetic pseudo-potential

$$\boldsymbol{\nabla} \cdot \vec{H} = -\boldsymbol{\nabla} \cdot \vec{M} \qquad \vec{H} = -\vec{\nabla}\phi_M(\vec{r})$$
$$\nabla^2\phi_m(\vec{r}) = -\rho_m(\vec{r}) = +\boldsymbol{\nabla} \cdot \vec{M}$$

REMEMBER YOU MAGNETIC DIPOLE EQUATIONS

and you magnetic field from a current equations

If you have both a current and a magnetization density, you can just put them together and try to solve.

All you do is solve both parts independently.

Solve the perma-magnetic with the magnetic pseudopotential and solve the moving current with the other math.

5.8 Conservation of Energy

The rate at which kinetic energy in a volume V increases + rate at which EM field energy in V increases = rate at which field energy enters V

$$\int\limits_V d^3r\, \vec{J} \cdot \vec{E} + \frac{d}{dt}\int\limits_V d^3r\, \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) = \frac{1}{\mu_0}\int d^2a\, \vec{E} \times \vec{B}$$

Where the last object is Pointing's Vector

Given an increasing polarization in a dielectric, part of the current driving energy is a polarization current

$$\boldsymbol{\nabla} \times \frac{\vec{B}}{\mu_0} - \boldsymbol{\nabla} \times \vec{M} = \vec{J}_F \ldots\ldots\ldots\ldots$$

$$\boldsymbol{\nabla} \times \vec{H} = \vec{J}_F + \frac{\partial}{\partial t}\vec{D}$$

You do a whole bunch of math to get

$$\frac{dU_Q}{dt} = \int_V d^3r\, \vec{J}_F \cdot \vec{E}$$

$$= -\int d\vec{a}\, \vec{E} \times \vec{H} - \left(\int d^3r\, \vec{E} \cdot \frac{d}{dt}\vec{D} + \int d^3r\, \vec{H} \cdot \frac{d}{dt}\vec{B}\right)$$

the rate at which fields do work on matter plus the rate which field energy goes up = the rate energy is added to the field.

Consider all of Maxwell's equations

$$\boldsymbol{\nabla} \cdot \vec{E}(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0} \qquad \boldsymbol{\nabla} \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r}, t)$$

$$\boldsymbol{\nabla} \cdot \vec{B}(\vec{r}, t) = 0 \qquad \boldsymbol{\nabla} \times \vec{B}(\vec{r}, t) = \mu_0\vec{J}(\vec{r}, t) + \mu_0\epsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r}, t)$$

You can do some math to uncouple the E and B fields

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \vec{E}\right) = -\frac{\partial}{\partial t}\boldsymbol{\nabla} \times \vec{B} \qquad \boldsymbol{\nabla} \times \vec{B} = \mu_0 \frac{\partial}{\partial t}\vec{J}_F + \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

$$-\vec{\nabla}(\boldsymbol{\nabla} \cdot \vec{E}) + \nabla^2\vec{E} = \mu_0 \frac{\partial}{\partial t}\vec{J} + \mu_0\epsilon_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

$$- --$$

$$\left(\nabla^2 - \epsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0 \frac{\partial}{\partial t}\vec{J} + \vec{\nabla}\frac{\rho}{\epsilon_0}$$

$$\left(\nabla^2 - \epsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right)\vec{B} = -\mu_0 \boldsymbol{\nabla} \times \vec{J}$$

The two inhomogeneous wave equations for E and B

The homogeneous wave equation would be

$$\left(\nabla^2 - \epsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r}, t) = 0$$

Let's consider the x component of an E field that only accelerates in the z direction

$$\left(\frac{\partial^2}{\partial z^2} - \epsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\right)E_x(z, t) = 0$$

There's some witchcraft

$$\sqrt{\frac{1}{\mu_0\epsilon_0}} = c$$

That's the speed of light

So our equation as actually

$$\left(\partial_z - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}$$

So if our field only has a z component, then we should get that there is not change in the z component for the wave

plane waves are transverse

If we use a lens to make a focused wave we get

$$B_y = \frac{E_x}{c}$$

volts per meter are cheap and easy to get

teslas are hard to get

5.9 HW Problems

5.9.1 Magnet Falling Through Pipe

We have to know

$$U = -\vec{m}\cdot\vec{B} \qquad \vec{F} = -\vec{\nabla}U = -\vec{m}\cdot\frac{d}{dt}B_z$$

And then you can treat the outside copper pipe as a current loop.

5.10 Plane Waves

Consider the equations

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

And the plane wave equations

$$\vec{E}(\vec{z}, t) = \hat{x}E_x(z - ct) = \hat{x}E(s)$$

You take the cross product of that E field equation to get

$$E_z = cB_y$$

c is a big number so its easy to remember that electric fields are much easier to make than magnetic fields.

We can consider wave propogation with the unit vectors

$$\hat{e} \times \hat{b} = \hat{n}$$

We can find the energy densities

$$U_B = \frac{1}{2\mu_0}B^2 \qquad U_E = \frac{\epsilon_0}{2}E^2 = \frac{\epsilon_0}{2}\frac{1}{\epsilon_0\mu_0}B^2 = \frac{1}{2\mu_0}B^2$$

The magnetic and electric fields carry half of the energy of the total system each (despite the magnetic field having less strength)

Now we can find S

$$S = \frac{1}{\mu_0}\vec{E} \times \vec{B} = \hat{n}\frac{1}{\mu_0}E \cdot \frac{E}{c} === \hat{n}c\epsilon_0 E^2$$

Which is $\hat{n} \cdot c \cdot$ energy density of lightwave

There's a picture of the light wave fields propogating through space.

Consider an Amperian loop through the light fields as it propogates through Δz.

Doing that and calculating gives us

$$\frac{\partial E_x}{\partial z} = c\frac{\partial B_y}{\partial z} = \frac{\partial B_y}{\partial t}$$

$\vec{B}/dt$ induces $\vec{E}$ via Faraday's Law

$\vec{E}/dt$ induces $\vec{B}$ via Ampere's Law

5.11 Polarizing Light

$hate$ and $\hat{b}$ can be 360 deg worth of directions with $\hat{e} \times \hat{b}$ being in the same spot.

5.12 Sinusoidal Waves

5.13 Complex Harmonic Functions

5.14 Discussion 12

Consider all the same electrodynamics equations we've been talking about forever

$$\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \boldsymbol{\nabla} \times \vec{E} = 0 \qquad \boldsymbol{\nabla} \cdot \vec{B} = 0 \qquad \boldsymbol{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

The continuity equation is

$$\nabla \cdot \vec{J} = -\partial_t \rho$$

Faraday's Law can be written as

$$\nabla \times \vec{E} = -\partial_t \vec{B} \qquad \oint \vec{D} \cdot d\vec{l} \equiv \varepsilon \qquad \varepsilon = -\frac{d\phi_B}{dt}$$
$$\nabla \cdot \left(\nabla \times \vec{B}\right) = \mu_0 \nabla \cdot \vec{J} = -\mu_0 \partial_t \rho$$

Consider a displacement current

$$\nabla \cdot \vec{J}_d = \partial_t \rho$$

This has an actual physical meaning and isn't just a mathematical model

$$\partial_t \rho = \partial_t (\frac{1}{\epsilon_0}) \nabla \cdot \vec{E} \Rightarrow \vec{J}_d \equiv \partial_t \vec{E} \Rightarrow \nabla \times \vec{B} = \mu_0 (\vec{J} + \epsilon_0 \vec{J}_d)$$

All of magnets is equivalent to all of electrics because its all just fields.

$$\nabla \cdot \vec{D} = 0 \qquad \nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J}_f$$

So we do some new stuff to get

$$-\partial_t \rho_b \equiv \nabla \cdot \vec{J}_p \qquad \vec{J} = \vec{J}_b + \vec{J}_f \qquad \vec{J}_b = \nabla \times \vec{M}$$
$$-\partial_t \rho = -\partial_t (\rho_b + \rho_f) \equiv \vec{P} \cdot \hat{n}$$

So now we get the last of Maxwell's equations

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{J}_b + \vec{J}_p + \epsilon_0 \partial_t \vec{E} \right)$$

Know that

$$\vec{J}_b = \nabla \times \vec{M} \qquad \vec{J}_p = \partial_t \vec{P} \qquad \vec{J}_d = \epsilon_0 \vec{E}$$

Move some stuff around and use the rest of our equations to get

$$\nabla \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_f + \partial_t (\vec{P} + \epsilon_0 \vec{E})$$

So now we have all our electrostatic but magnet equivalences

$$\nabla \times \vec{D} = -\partial_t \vec{B} \qquad \nabla \times \vec{H} = \vec{J}_f + \partial_t \vec{D}$$

Now we can use these to get our boundary conditions equations

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad B_1^{\perp} = B_2^{\perp}$$
$$\vec{E_1^{//}} - \vec{E_2^{//}} = 0 = \frac{\partial \phi_B}{\partial t} \qquad \vec{H_1^{//}} - \vec{H_1^{//}} = \vec{K}_f \times \hat{n}$$