PHYS 435

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Contents

Chapter 1

PHYS435

This goddamn professor is half retired im so cooked

1.1 Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$
$$\vec{E}(\vec{r}) = \sum \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} \to \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}')$$

1.2 Gauss's Law

The flux of $\vec{E}$ throuhg a closed surface equations to the enclosed charge C_0

$$\frac{1}{C_0}\int_V d^3r\rho(\vec{r}) = \int_{\partial V} da\rho(\vec{r})$$

1.3 Divergence Theorem

$$\vec{\nabla} E = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

$$\int_V d^3 r \vec{\nabla} \vec{E}(\vec{r}) = \int_{\partial V} d\vec{a} \cdot \vec{E}(\vec{r})$$

$$\vec{\nabla} \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

1.4 Faraday's Law

The circulation of $\vec{E}$ around any closed path N is equal to $(-1)\times$ the time derivative of the magnetic flux through ANY surface bounded by the closed path.

$$\int_{\partial S} d\vec{l} \cdot \vec{E} = -\frac{d}{dt} \int_S d\vec{a} \vec{B}$$

1.5 Stoke's Theorem

$$\int_{\partial S} d\vec{l} \cdot \vec{E} = \int_S d\vec{a} \vec{\nabla} \times \vec{E}$$

1.5.1 Differential Laws

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

Gauss

$$\vec{\nabla} \cdot \vec{E} = -\frac{\rho(\vec{r})}{\epsilon_0} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Ampere

$$\vec{\nabla} \times \vec{B} = \mu_0 J$$

1.6 Electric Potential

Start with Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = -\frac{\partial}{\partial t}\vec{B}(\vec{r}, t)$$

$$\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\vec{E}$$

remove the time dependent equations

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

$$\vec{\nabla} \times \vec{E}(\vec{r}) = 0$$

the curl of $\vec{E}$ is 0 which means the electric field is conservative? A scalar potential function convenience.

consider the path integral

$$\int_P d\vec{l} \cdot \vec{E}$$

We can show that the integral is path independent (because the curl is 0)

$$\int_{P1} - \int_{P2} = \oint_{\partial S} d\vec{l} \cdot \vec{E}(\vec{r}) = \int_S d\vec{a} \cdot \vec{\nabla} \times \vec{E} = 0$$

$$\int_{P1} d\vec{l} \cdot \vec{E} = \int_{P2} d\vec{l} \cdot \vec{E}$$

That's actually a really smart proof damn

Now we do actual potential stuff

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l} \cdot \vec{E}(\vec{r})$$

Where $\vec{0}_r$ is the vector where the potential is 0

$$U(\vec{a}) - U(\vec{b}) = -\int_a^b d\vec{l'} \cdot \vec{F}(\vec{r})$$

$$\vec{F}_{Lorentz} = q\vec{E}(\vec{r}) + q\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$$

$q\vec{E}(\vec{r})$ can do work, but $q\vec{v}(\vec{r}) \times \vec{B}(\vec{r})$ cannot do any work (always in opposite direction of motion)

$$W = q\int_{\vec{0}_r}^{\vec{r}} d\vec{l}\cdot\vec{v}\times\vec{B} = q\int_{\vec{0}}^{\vec{r}} d\vec{l}\cdot\frac{d\vec{l}}{dt}\times\vec{B}(\vec{r}) =$$

$$q\int_{\vec{0}}^{\vec{r}} dt\frac{d\vec{l}}{dt}\cdot(\frac{d\vec{l}}{dt}\times\vec{B}(\vec{r})) = 0$$

That part cannot do any work

$$W_{other} = U(\vec{r}) - U(\vec{0})$$

$$\frac{U(\vec{r}) - U(\vec{0})}{q} = \Delta V$$

More Stuff

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l'}\cdot\vec{E}(\vec{r})$$

$$-\vec{\nabla}V(\vec{r}) = -\left[\hat{x}\frac{\partial}{\partial x}V(\vec{r}) + \hat{y}\frac{\partial}{\partial y}V(\vec{r}) + \hat{z}\frac{\partial}{\partial z}V(\vec{r})\right]$$

$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz = r + dx$$

consider the slightest motion dx in the $\hat{x}$ direction so that $\vec{r}\rightarrow\vec{r}+\hat{x}dx$

$$V(\vec{r}+\hat{x}dx) = V(\vec{r}) + dx\hat{x}\cdot\vec{E}(\vec{r}) \qquad E_x(\vec{r}) = \hat{x}\cdot\vec{E}(\vec{r})$$

$$\frac{V(\vec{r}+\hat{x}dx) - V(\vec{r})}{dx} = E_x(\vec{r})$$

That gives us

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

real important equation

1.6.1 Potential Equation

consider a point mass

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} q \frac{\vec{r}}{r^3} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$$V(\vec{r}) = -\int_{\vec{0}_r}^{\vec{r}} d\vec{l} \cdot \vec{E}$$

$$E_y = \frac{1}{4\pi\epsilon_0} \frac{q}{y^2} \qquad -\int_{\infty}^{y} dy' \frac{1}{4\pi\epsilon_0} \frac{q}{y^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r}$$

Use the principle of superposition to get the general answer

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

1.6.2 Infinite Line Charge

Consider a straight line of infinite length and constant charge density.

Where should $\vec{0}_r$ be?

I think we just pick an arbitrary point

$$\vec{E}(s) = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}$$

$$V(\vec{r}) = -\int d\vec{l}\vec{E}(s) = V(s) = -\int_{O_r}^{s} ds\frac{1}{2\pi\epsilon_0}\frac{\lambda}{s'}$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\ln(s')\Big|_{\vec{O}_r}^{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{O_r}{s}\right)$$

What PDE governs $V(\vec{r})$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{E} = -\vec{\nabla}V$$

$$\vec{\nabla}\cdot\vec{\nabla}V(\vec{r}) = -\frac{\rho}{\epsilon_0}$$

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)V = -\frac{\rho}{\epsilon_0}$$

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\epsilon_0}$$

1.7 Work

If you move 1 charge, there is no work done because there are no other fields.

If you bring in a 2nd charge, you get a total work of $\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$

If you bring in a third charge you just sum the things together

$$U_{1\to N} = \frac{1}{2}\sum_{i}^{N}\sum_{j>i}^{N}\frac{1}{4\pi\epsilon_0}\frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\int d^3r d^3r' \frac{1}{4\pi\epsilon_0}\frac{\rho(\vec{r})\rho(\vec{r'})}{|\vec{r} - \vec{r'}|}$$

$$= \frac{1}{2}\frac{1}{4\pi\epsilon_0}\int d^3r'\rho(\vec{r'})V(\vec{r'}) \qquad \rho(\vec{r'}) = -\epsilon_0\nabla^2 V(\vec{r'})$$

$$U = -\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\nabla^2 V(\vec{r})$$

1.7.1 X-component

Let's consider just the x-component for a little bit

$$-\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\partial_x\left[\partial_x V(\vec{r})\right]$$

$$\partial_x\left[V(\vec{r})\partial x V(\vec{r})\right] = \partial_x V\cdot\partial_x V + V\partial_x^2 V$$

$$\partial_x\left[V(\vec{r})\partial x V(\vec{r})\right] - \partial_x V\cdot\partial_x V = V\partial_x^2 V$$

So with that you get

$$-\frac{\epsilon_0}{2}\int d^3r V(\vec{r})\partial_x\left[\partial_x V(\vec{r})\right] = -\frac{\epsilon_0}{2}\int d^3r\partial_x\left(V(\vec{r})\partial_x V(\vec{r})\right) - E_x^2$$

generalize

$$\frac{\epsilon_0}{2}\int d^3r\,\vec{\nabla}\cdot V(\vec{r})\vec{E}(\vec{r}) + \vec{E}\cdot\vec{E}$$

The main point of all of this is that it goes to 0 for large r

$$\int d^3r\vec{\nabla}\cdot V\vec{E} \rightarrow \int d^3r\vec{\nabla}\cdot\frac{C}{r}\frac{1}{r^2}\hat{r} \rightarrow$$

$$\int da\frac{C}{r^3} \rightarrow \frac{1}{r} \rightarrow \lim_{r\rightarrow\infty} = 0$$

So our potential somehow gets to

$$U = \int d^3r\frac{\epsilon_0}{2}E^2$$