PHYS 486

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Contents

Chapter 1

PHYS486

Quantum Mechanics lol. This is apparently a math language more than actual physics.

1.1 What is QM

- Classical Mechanics - $F = ma$, solve for x and p, position and momentum.
- Classical E & M - $\ddot{\vec{E}}(\vec{r}, t) - \nabla^2 \vec{E}(\vec{r}, t) = 0$

You have observable entities and you work with just the observable entities to figure out the exact numbers.

1.1.1 The Quantum State $|\psi\rangle$

This is a state vector.

The quantum state by definition is **not** observable.

An observable operator is something like position that describes the system that we can actually see.

$$\hat{A}\,|\psi\rangle = a\,|\psi\rangle$$

where $\hat{A}$ is an observable and a is the measurement out.

This is just an eigenvalue equation.

1.1.2 Schrodinger Equation

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

$\hat{H}$ is a matrix and $|\psi(t)\rangle$ is a vector.

1.2 Summary of "Central Phenomena"

Imagine an electron orbiting around a proton.

The electron **cannot** take arbitrary orbits.

The electron can only have **certain discrete** orbits.

1.2.1 Superpositions

Consider two discrete states in which an electron can orbit around a proton.

Because 0 and 1 are solutions for the electron, a superposition of 0 and 1 is also a solution for the electron.

$$\alpha |0\rangle + \beta |1\rangle$$

This is the electron being in both discrete states at the same time.

$$\sim \left[|0\rangle \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, SP \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right]$$

1.3 Probabilistic Interpretation

Consider a system with the solution

$$\alpha |0\rangle + \beta |1\rangle$$

If we have a thing that measures the energy, we can get **either** 0 **or** 1 with probabilities depending on the values α and β.

1.4 Entanglement

Consider atom A with state $\{|0\rangle_A, |1\rangle_A\}$ and atom B with state $\{|0\rangle_B, |1\rangle_B\}$.

It is completely legal to have an entangled state with the form

$$\alpha |0\rangle_A |0\rangle_B + \beta |1\rangle_A |1\rangle_B$$

1.5 Course Outline

1. Basic Rules
2. Wave Mechanics (”toy models”)
3. Formalism (Midterm 1)
4. Simplest Real System (Hydrogen Atom) (Midterm 2)
5. Intro to Multiparticle Descriptions

Chapter 2

Actual Physics Now

2.1 Black Body Radiation

Consider an object that perfectly absorbs radiation and turns it into heat.

Place it in thermal equilibrium (finite T, constant E, absorption $\rightarrow$ emission).

2.1.1 Classical Description

Consider the system as a harmonic oscillator.

$$\langle E\rangle = k_B T \Rightarrow I(\omega) \sim \frac{\omega^2 k_B T}{\pi^2 c^3} \qquad \text{Rayleigh-Jeans Law}$$

This **does not** match experimental data.

While the classical description says that temperature will go up forever, the experiments show that the temperature stops increasing for large frequencies.

2.1.2 Plack, 1900

Complete guess that maybe energy is discretized. Maybe the harmonic oscillator comes in chunks of size $\hbar\omega$.

Now, energy can be written in the form

$$P(E) = \alpha e^{-\frac{-E}{k_B T}} \longrightarrow \langle E\rangle = \frac{\hbar\omega}{\exp((\hbar\omega/k_B T) - 1)}$$

$$I(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3 \exp((\hbar\omega/k_B T) - 1)}$$

This perfectly matches experimental data.

2.2 Photoelectric Effect

Consider a circuit that absorbs light with frequency ω and intensity J to create a potential.

As light intensity J increases, current I increases, but V_0 stays the same.

V_0 is dependent on ω and the properties of the material.

Einstein figured out that there is an energy threshold such that light will not be absorbed if it does not have a high enough energy.

The kinetic energy of the electron can be given in the form:

$$kE = \hbar\omega - W > -eV_0$$

2.3 Wave Particle Duality

Consider a photon with energy and momentum

$$E = \hbar\omega \qquad p = \hbar k \left(= \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda}\right)$$

The photon has no mass ($m = 0$), so consider the relativistic mass of the photon

$$E^2 = m^2c^4 + p^2c^2 \rightarrow p^2c^2 \rightarrow$$
$$E = pc \qquad \omega = kc \qquad f\lambda = c$$

2.3.1 De Broglie, 1924

Everything has a wavelength and frequency

$$\lambda = \frac{h}{p}$$

The wavelength for even electrons is extremely small, so this feature is very hard to observe even if its true for everything.

2.3.2 Davisson and Germer, 1927

Electrons act in the same wave-like nature as light when shot through a small slit (double slit experiment but for electrons).

This proved the De Broglie wavelength theory for particles.

2.4 Beginning of QM

The issue with quantum mechanics is that the math is derived only from experimental data. There is no classical physics basis or derivation behind quantum mechanics.

Multiple textbooks will bring up quantum mechanics in different ways. The professor recommends the Townsend QM textbook

Chapter 3

Ruleset

Motivated from experimental observations. These are base postulates with no derivations.

3.1 System State

The state of the physical system is a vector $|\Psi\rangle$ in Hilbert Space.

A Hilbert space ($\equiv H^2$) is a complex vector space with a well-behaved inner product.

(bra + ket = braket). You have to take the complex conjugate

$$\langle\Psi|\Psi\rangle \qquad \text{"bra"} = (|\Psi\rangle^*)^T \qquad \langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^*$$

3.2 Observables

Observables are operators in Hilbert Space, and they have to have real eigenvalues.

The measurement outcomes of the observable are the eigenvalues.

$$\hat{A}\,|\alpha_n\rangle = a_n\,|\alpha\rangle_n$$

I think $\hat{A}$ is the observable of the state $|\alpha_n\rangle$ and a_n is the n-th eigenvalue of the observable.

3.2.1 Born's Rule

The probability for an outcome a_n is given by

$$P(a_n) = |\,\langle\alpha_n|\Psi\rangle\,|^2$$

Where α_n is an eigenvector of the operator.

3.2.2 Expected Value

Given an observable operator $\hat{A}$, the expected value is given by

$$\langle \hat{A} \rangle = \langle \Psi | \hat{A} | \Psi \rangle$$

3.2.3 Waveform "Collapse"

Given a quantum state $|\Psi\rangle$, when you measure an observable $\hat{A}$, you return only a single result a_n

$$|\Psi\rangle \longrightarrow |\alpha_n\rangle$$

The system state collapses to just the individual eigenvector state.

3.3 Time Evolution (Schrodinger Equation)

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle \qquad \hat{H} = \text{Hamiltonian (Energy Operator)}$$

The Eigenvectors of $\hat{H}$ are stationary states (do not change with time). This is the time-independent Schrodinger Equation

$$\hat{H} |\Psi\rangle = E_h |\Psi\rangle$$

Now you know the entirety of quantum mechanics. Everything else can be derived.

Chapter 4

Experiments

4.1 Double Slit

If you shine light through a small slit, you will see a diffraction pattern.

Imagine shining light at 2 slits and an attennuator at the other side that **measures** the location of each photon at the screen.

If you just shoot a single photon, the location will be random, but if you shoot multiple photons, there is a very distinct diffraction pattern that demonstrate the probability distribution of the list.

4.2 Stern-Gerlach

Consider a beam of atoms with varying spins. This beam of atoms is put into a magnetic field gradient.

If the magnetic moment is up, the atom will go up. If the magnetic moment is down, the atom will go down.

If we put the beam through this field, then half the atoms will go up, and half the atoms will go down (probabilistically).

This makes sense.

The quantum state of each atom is given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle + |\downarrow\rangle\right) \hat{z}$$

Consider a basis

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let's call spin up $+1$ and spin down -1.

Because of those numbers, our observable operator is

$$\hat{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \hat{A}\left|\uparrow\right\rangle = +1\left|\uparrow\right\rangle \qquad \hat{A}\left|\downarrow\right\rangle = -1\left|\downarrow\right\rangle$$

Now we have our quantum state and our observable operator matrix.

Because we have two possible eigenvectors, the superposition of our quantum state can be written as

$$\left|\Psi\right\rangle = \left(\sum_k \left|k\right\rangle \left\langle k\right|\right) \left|\Psi\right\rangle \equiv \lambda_i, \left|k\right\rangle \qquad \text{kth basis vector}$$

$$= \sum_k c_k \left|k\right\rangle \qquad c_k = \left\langle k|\Psi\right\rangle$$

4.2.1 Born's Rule

$$P(+1) = |\left\langle \uparrow|\Psi\right\rangle|^2 = \frac{1}{2}|\left\langle \uparrow|\uparrow\right\rangle + \left\langle \uparrow|\downarrow\right\rangle|^2$$

Because we're using Dirac notation, we know exactly what inner products are parallel and what inner products are orthonormal.

$$P(+1) = |\left\langle \uparrow|\Psi\right\rangle|^2 = \frac{1}{2}|1+0|^2 = \frac{1}{2}$$

4.2.2 Expected Values

$$\langle \hat{A} \rangle = \left\langle \Psi\right| \hat{A} \left|\Psi\right\rangle \qquad \hat{A} = \sum_k \lambda_k \left|k\right\rangle \left\langle k\right| \qquad \lambda_k = \text{kth eigenvalue}$$

You can use the Spectral Theorem to get

$$\hat{A} = \left|\uparrow\right\rangle\left\langle\uparrow\right| - \left|\downarrow\right\rangle\left\langle\downarrow\right|$$

So to find the expected value, we use

$$\langle \hat{A} \rangle = \left\langle \Psi\right| \left(\left|\uparrow\right\rangle\left\langle\uparrow\right| - \left|\downarrow\right\rangle\left\langle\downarrow\right|\right) \left|\Psi\right\rangle = \ldots = 0$$

4.3 Double Slit and the Wave Function

we need $|\Psi\rangle$ in the position basis.

Imagine if the space was discrete.

$$|\Psi\rangle = \sum_k c_k |x\rangle$$

But our space is continuous, so instead of taking a discrete sum, we take an integral.

$$\langle x_k|x_j\rangle = \delta(k-j) \qquad \int dx \, \langle x_k|x_j\rangle = 1$$

$$|\Psi\rangle = \int dx \; |x\rangle \langle x|\Psi\rangle = \int dx \, \Psi(x) \, |x\rangle$$

That's the wave function
implement Born's Rule to get

$$P(x) = |\langle x|\Psi\rangle|^2 = |\Psi(x)|^2$$

4.3.1 Bra-Ket Notation

$$\langle x|y\rangle$$

Is an inner product that yields just 1 number.

$$|y\rangle \langle x|$$

is an outer product that yields a matrix.

4.3.2 Stationary State

A stationary state is an eigenstate of the Hamiltonian. It has a perfectly well-defined energy.

4.4 Interpreting the Double Slit

You have a quantum state dependent probability density

$$P(x) = |\Psi(x)|^2$$

That's a probability density function that needs to be normalized

$$\int_{-\infty}^{\infty} dx |\Psi(x)|^2 = 1$$

So to find the expected value, we get

$$\langle x \rangle = \int_{-\infty}^{\infty} dx\, x|\Psi(x)|^2 \qquad \langle f(x) \rangle = \int_{-\infty}^{\infty} dx\, f(x)|\Psi(x)|^2$$

And the variance of that probability distribution is given by

$$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2$$

Given a quantum state, if we can measure the position, we can **only** estimate $|\Psi(x)|^2$ with finite accuracy. Each position sampled is just a single point of a probability distribution that cannot be directly measured.

Chapter 5

Wave Mechanics

Let's start with

$$|\Psi(x)\rangle = \Psi(x,t) \qquad H = -\frac{\hbar^2}{2m}\left(\frac{\partial}{\partial x}\right) + V(x)$$

Where $V(x)$ is the potential. This is our wave function and our Hamiltonian. The Schrodinger Wave equation is

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t)$$

5.1 Important Properties of the S.W.E.

5.1.1 Unitary

Conservation of probability. Found is Griffiths Textbook chapter 1.4

$$0 = \frac{d}{dt}\int_{-\infty}^{\infty} dx\,|\Psi(x,t)|^2 = \int dx\,\frac{\partial}{\partial t}|\Psi(x,t)|^2 =$$
$$\int dx\,\left(\Psi^*\frac{\partial}{\partial t}\Psi + \Psi\frac{\partial}{\partial t}\Psi^*\right) \to \int dx\,\left(\Psi^*\frac{\partial^2}{\partial x^2}\Psi - \Psi\frac{\partial}{\partial x^2}\Psi^*\right)\frac{i\hbar}{2m}$$
$$= \int dx\,\frac{d}{dx}\left(\Psi^*\frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right)\frac{i\hbar}{2m}$$

So now we have to

$$\left(\Psi^*\frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right)_{-\infty}^{\infty}\frac{i\hbar}{2m} = 0 \qquad \Psi, \Psi' \to 0 \qquad |x| \to \infty$$

To first try and figure out the wave function, we start with a bunch of plane waves.

A single plane wave will have equation

$$\Psi_k = Ae^{i(kx-\omega t)}$$

Plug that into the S.W.E. to get

$$\frac{\partial}{\partial t}\Psi_k = -i\omega\Psi_k \qquad \frac{\partial}{\partial x}\Psi_k = ik\Psi_k \qquad \frac{\partial^2}{\partial x^2}\Psi_k = -k^2\Psi_k$$
$$\hbar\omega\Psi_k = \frac{\hbar^2k^2}{2m}\Psi_k \Rightarrow E = \frac{p^2}{2m}$$

I can express all wave functions as plane waves

$$f(x) = \int \frac{dk}{2\pi}e^{ikx}\tilde{f}(x)$$

Add time and

$$\omega = \frac{\hbar k^2}{2m}$$

And get equation

$$\Psi(x,t) = \int \frac{dk}{2\pi}e^{ikx-i\omega t}\ldots\ldots\ldots\ldots\ldots\ldots$$

Unbounded plane waves are not normalizable.

$$\int_{-\infty}^{\infty} dx\,|Ae^{i(kx-\omega t)}|^2 = |A|^2\int_{-\infty}^{\infty} dx = \infty$$

The way to fix this is by putting the plane wave in a finite box that is much larger than the bounds of the problem

$$1 = \int_{-L}^{L} dx|A|^2 \rightarrow A = \frac{1}{\sqrt{2}L}$$

So the SWE can be given as

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\Psi \qquad (+V(x)\Psi)$$

In classical mechanics, we would have

$$E = \frac{p^2}{2m} + V(x) \rightarrow p \iff \frac{\partial}{\partial x}\Psi$$

So what is $\hat{p}$?
For plane waves

$$p\Psi_k = \hbar k \Psi_k$$

From De Broglie. We put that into our wave equation

$$p\Psi_k = \hbar k \Psi_k = -i\hbar\frac{\partial}{\partial x}\Psi_k$$

Classically, we know that

$$V = \dot{x} \qquad p = mV$$

The expected value is the value over many many quantum mechanical states

$$\frac{d\langle x\rangle}{dt} = \frac{d}{dt}\int dx\, x|\Psi(x,t)|^2 = \int dx\, x\left(\left(\frac{\partial}{\partial t}\Psi^*\right) + \Psi^*\left(\frac{\partial}{\partial t}\Psi\right)\right)$$

This derivation is in Griffiths 1.5

$$\frac{i\hbar}{2m}\int dx\, x\frac{\partial}{\partial x}\left(-\Psi\frac{\partial}{\partial x}\Psi^* + \Psi^*\frac{\partial}{\partial x}\Psi\right)$$

We do integration by parts

$$\frac{\partial}{\partial x}\left(-\Psi\frac{\partial}{\partial x}\Psi^* + \Psi^*\frac{\partial}{\partial x}\Psi\right) = g' \qquad x = f$$

$$-\frac{i\hbar}{2m}\int dx\,\left(\Psi^*\frac{\partial}{\partial x}\Psi - \Psi\frac{\partial}{\partial x}\Psi^*\right) = \frac{i\hbar}{m}\int dx\,\Psi^*\frac{\partial}{\partial x}\Psi$$

$$= \frac{1}{m}\int dx\,\Psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\Psi = \frac{\partial\langle x\rangle}{\partial t}$$

Momentum should be

$$p \sim m\frac{d}{dt}\langle x(t)\rangle$$

$$\langle p\rangle = \int dx\, \Psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)\Psi$$

The momentum operator (expressed in position), known as $\hat{p}$, is

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

5.2 Recap

Given a state vector $|\Psi\rangle$, you can make a probability distribution to describe the position of the particle at time t

$$P(x,t) = |\Psi(x,t)|^2$$

Plane waves cannot be normalized, so they are bad wave functions.
Solve for $\Psi \to$ Plane waves (only for math)
The momentum of the particle can be written as

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}$$

That is specifically in the position basis.
The ep

$$\langle\hat{p}\rangle = \int dx\, \Psi^*\hat{p}\Psi$$

That is true for any

$$\hat{A} \equiv \hat{A}(\hat{x},\hat{p}) \qquad \langle\hat{A}\rangle = \Psi^*\hat{A}\Psi$$

5.3 Uncertainty Relation

Given a plane wave wavefunction. Because I have a single wave with a well defined wavelength, the momentum is well-defined.

$$\sigma_p \to 0$$

However, because it's a plane wave, we cannot say anything about the position of the particle.

$$\sigma_x \to \infty$$

If given a wave function that looks like a dirac delta with a single spike, then the position is well-defined but the momentum is not.

If you perfectly measure position, the momentum is then a superposition of all possible momenta, so the particle will spread out over time because the particle has to be moving due the momentum uncertainty.

Generally, the bound for momentum and position variance is

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

There are wave functions that miimize the uncertainty relation (Gaussian Wave Packets).

5.4 Solving the SWE

Given $V(x)$, how do we get $\Psi(x,t)$

The Schrodinger wave equation is given as

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

Here we assume that the state is **not** time-dependent

$$\frac{\partial V}{\partial t} = 0$$

Because of our time-independence, we can just separate the variables.

$$\Psi(x,t) = \Psi(x)\varphi(t)$$

Initial Condition: know what $\Psi(x,0)$ is

$$i\hbar \frac{1}{\varphi}\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\frac{1}{\Psi}\frac{d^2\Psi}{dx^2} + V$$

The separation constant is the energy E

$$i\hbar\frac{1}{\varphi}\frac{d\phi}{dt} = E \to \frac{d\varphi}{dt} = \frac{-iE}{\hbar}\varphi \to$$
$$\phi = e^{\frac{-iE}{\hbar}}$$

Now we consider the position from the state space

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V\Psi = E\Psi$$

The left side is the Hamiltonian operator on Ψ and the right side is the energy.

The stationary solution is of the form

$$\Psi(x,t) = \psi(x)\exp\left(\frac{-iEt}{\hbar}\right)$$

The time independent and time dependent states are equal

For any $\hat{A}$

$$\langle\hat{A}\rangle = \int \psi^*\hat{A}\psi\,dx = \text{const}$$

We can define the energy as

$$H = \frac{p}{2m} + V(x)$$

This is the total energy in classical mechanics (kinetic + potential energy)

Our quantum Hamiltonian operator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \qquad \hat{H}\psi = E\psi \qquad \langle\hat{H}\rangle = E$$
$$H^2\psi = E^2\psi \to \langle H^2\rangle = E^2 \to \sigma_H^2 = 0$$

So the general solution is a linear combination of the separated solutions.

$$V(x) \to \{\psi_n\} \Rightarrow \{E_n\}$$

With an Initial Condition:

$$\Psi(x,0) = \sum_n c_n\psi_n(x)$$

And then we evolve that system

$$\Psi(x,t) = \sum_n c_n \psi_n(x) \exp\left(\frac{-iE_n t}{\hbar}\right)$$

make sure that everything is normalized

$$\sum_n |c_n|^2 = 1 \qquad \langle H \rangle = \sum_n |c_n|^2 E_n$$

5.4.1 Question

Given a stationary state $\{\psi_n(x)\}$ for some $\hat{H}$, how can we explain motion?

To explain motion, we need a superposition of $\psi_n(x)$.

$$\Psi(x,0) = c_1\psi_1(x) + c_2\psi_2(x) \rightarrow \Psi(x,t)$$
$$= c_1\psi_1(x)\exp\left(\frac{-iE_1 t}{\hbar}\right) + c_2\psi_2(x)\exp\left(\frac{-iE_2 t}{\hbar}\right)$$
$$|\Psi(x,t)|^2 = \psi^*\psi$$

Use Euler's equation

$$c_1^2\psi_1^2 + c_2^2\psi_2^2 + 2c_1c_2\psi_1\psi_2\cos\left(\frac{(E_2 - E_1)t}{\hbar}\right)$$

You have two stationary states in position. They interfere and something happens in time.

5.5 Infinite Square Well

Horrendously artificial, but one of the few problems we can actually solve.

Consider a particle with mass m and velocity v in a valley of height h.

The particle is trapped in the well because $mv^2/2 < mgh$.

We can consider the valley as $h \rightarrow \infty$, because the particle can quantum tunnel out of it.

The width of the well goes from $0 \rightarrow a$. The potential of the particle can be stated as

$$V(x) = \begin{cases} 0 : 0 \leq x \leq a \\ \infty : \text{elsewhere} \end{cases}$$

At $V = \infty$, we can say that the wave function $\psi = 0$

5.5.1 Boundary Conditions

- $\psi(x)$ is always continuous.
- $\psi'(x)$ is always continuous but not if $|V(x)| = \infty$.

So now all we have to do is solve the SWE inside the well

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi \iff \psi''(x) = -k^2\psi \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

Everything is a simple harmonic oscillator

$$\psi = Ae^{ikx} + Be^{-ikx}$$

Now we have to consider the boundary conditions.

$$\psi(x=0) = \psi(x=k) = 0$$
$$A + B = 0 \rightarrow A = -B$$
$$\psi(x) = A(e^{ikx} - e^{-ikx}) = A\sin(kx)$$
$$psi(x=0) = 0 \rightarrow \sin(kx) = 0$$

shit

5.5.2 Energies

$$k^2 = \frac{2mE}{\hbar^2} \rightarrow E_n = \frac{\hbar^2 k_n^2}{2ma^2}$$

5.5.3 Normalization

$$\int_0^a A^2\sin^2(kx)\,dx = 1 \rightarrow A = \sqrt{\frac{2}{a}}$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad E_h = \frac{k^2\pi^2\hbar^2}{2ma^2}$$

5.6 Discussion 1

5.6.1 The Classical Recipe

$$F = ma \qquad \frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial x}\frac{\partial L}{\partial x}$$

5.6.2 Quantum Reality

Input is ket. Output is bra.

Ket to bra is input to output based off of the inner product.

The wavefunction gives the probability density of finding a particle in a volume element dx

$$\rho(x) = \Psi^*(x,t)\Psi(x,t)$$
$$P(a < x < b) = \int_a^b \Psi^*(x,t)\Psi(x,t)\,dx$$

The wave function is also normalized

$$\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\,dx = 1$$

Given an input state $|\Psi_1\rangle$ and an output state $\langle\Psi_2|$. The probability amplitude is given in the form

$$\langle\Psi_2|\Psi_1\rangle = \Psi_2^*(x,t)\Psi_1(x,t)$$

The output state comes first, and it is the one that's the conjugate.

Given a superposition

$$\Psi_{in} = \frac{1}{\sqrt{3}}\Psi_1 + \frac{\sqrt{2}}{\sqrt{3}}\Psi_2$$

So to find the probability of the output being 2

$$P(2) = |\langle\Psi_{out}|Psi_{in}\rangle|^2 = \Psi_2\frac{1}{\sqrt{3}}\Psi_1 + \Psi_2\frac{\sqrt{2}}{\sqrt{3}}\Psi_2$$

something something I'll look at the lecture notes later

5.6.3 Questions

Consider a wave function

$$\psi(x) = A(a^2 - x^2)$$

inside interval $\{-a, a\}$
Determine the normalization constant A

$$\int \psi(x)^2 \, dx \; = 1$$

$$A^2 \int_{-a}^{a} (a^2 - x^2)^2 \, dx \qquad A^2 \left(a^4 x - (2a^2x^3)/3 + x^5/5\right) \Big|_{-a}^{a} =$$

$$A^2 \left(a^4 a - (2a^2a^3)/3 + a^5/5\right) - \left(-a^4 a + (2a^2a^3)/3 - a^5/5\right)$$

$$= 2A^2 \left(a^5 - \frac{2}{3}a^5 + a^5/5\right) = \frac{16}{15}A^2 a^5 = 1 \rightarrow A = \sqrt{\frac{15}{16a^5}}$$

$$\frac{15}{15} - \frac{10}{15} + \frac{3}{15} = \frac{8}{15}$$

What is the probability of finding the particle at $x = a/2$. It is 0 because that's a miniscule-ly small point. The range $-a/2 \rightarrow a/2$ is different.

$$P(-a/2 < x < a/2) = \int_{-a/2}^{a/2} \Psi^*(x)\Psi(x)$$

$$= 2\frac{15}{16a^5} \left(a^4 x - (2a^2x^3)/3 + x^5/5\right) \Big|_{-a/2}^{a/2}$$

$$= \frac{15}{4a^5} a^4 (\frac{a}{2}) - (2a^2(\frac{a}{2})^3)/3 + (\frac{a}{2})^5/5 = \frac{15}{4}\left((\frac{1}{2}) - (2(\frac{1}{8}))/3 + (\frac{1}{32})/5\right)$$

$$\frac{15}{4}\left(\frac{1}{2} - \frac{1}{12} + \frac{1}{160}\right)$$

For what potential is the state an eigenfunction?
The energy eigenfunctions are determined from the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + V(x) = E\psi$$

So I need to take the double derivative

$$\frac{\partial^2}{\partial x^2} Aa^2 - Ax^2 = -2A$$

$$\frac{\hbar^2}{2m} 2A + V(x)(Aa^2 - Ax^2) = EA(a^2 - x^2)$$

Set energy to 0

$$-\frac{\hbar^2}{2m}2\sqrt{\frac{15}{16a^5}}$$

5.6.4 bra-ket

Consider a three dimension vector space spanned by an orthonormal basis

$$|1\rangle \qquad |2\rangle \qquad |3\rangle$$

$|\alpha\rangle$ and $|\beta\rangle$ are defined as

$$|\alpha\rangle = i\,|1\rangle - 2\,|2\rangle - i\,|3\rangle \qquad |\beta\rangle = i\,|1\rangle + 2\,|3\rangle$$

Construct $|\alpha\rangle$ and $|\beta\rangle$ in terms of the dual basis $\langle 1|$, $\langle 2|$, $\langle 3|$
Something that might be useful

$$A_{ij} = \langle i|\,\hat{A}\,|j\rangle = \langle i|\,(|\alpha\rangle\,\langle\beta|)\,|j\rangle = \langle i|\alpha\rangle \cdot \langle\beta|j\rangle$$

Chapter 6

Solving the SWE

6.1 Recap

Our starting point is

$$\hat{H} = -\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2} + V(x)$$

with initial conditions.

The time-indepedent SWE can be written as

$$\hat{H}\psi(x) = E_n\psi(x) \qquad \psi(x,0) = \sum_n c_n\psi_n(x) \qquad \psi(x,t) = \sum_n c_n\psi_n(x)e^{\frac{-i}{\hbar}E_nt}$$

And infinite square well can be written as

$$\psi(x \leq 0) = \psi(x \geq a) = 0$$

6.1.1 Stationary State

If we consider a probability distribution

$$|\psi(x,t)| = \psi^*(x,t)\psi(x,t)$$

So for an infinite square well.

Consider the general statement

$$\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}(V(x) - E)\psi$$

if $E < V(x)$, then ψ'' and ψ always have the same sign. The issue with this is the wave will not be normalizable (because for positive x, the concavity has to be positive, so it has to increase).

The eigenstates are given by

$$E_1 = \frac{\pi^2\hbar^2}{2ma^2} \qquad E_2 = 4E_1 \qquad E_3 = 9E_2$$

In a stationary state, $\{\psi_n\}$ forms a complete orthonormal eigenbasis.

$$\int dx\, \psi_m^*(x)\psi_n(x) = \delta_{mn}(m = n)$$

$$f(x) = \sum_n c_n\psi_n(x) = \sqrt{\frac{2}{a}}\sum_n c_n \sin\left(\frac{n\pi x}{a}\right) \qquad c_n = \langle\psi_n|f(x)\rangle$$

Question

Given a wave function $\psi(x, t = 0)$ and a stationary state, how can we find c_n such that $\psi = \sum_n c_n\psi_n(x)$?

$$\int dx\, \psi_m^*(x)f(x) = \int dx\, \psi_m^*(x)f(x)\sum_n c_n\psi_n(x)$$

$$= \sum_n c_n \int \psi_m^*(x)\psi_n(x) = \sum_n c_n f_{mn} = c_m$$

6.2 Time-Dependence

$$\psi(x, t = 0) = \sum_n c_n\psi_n(x) \to \psi(x, t) = \sum_n c_n\psi_n e^{\frac{-i}{\hbar}E_n t}$$

This is just summing over all of the possible standing waves within an infinite square well.

6.2.1 THIS WILL BE ON THE MIDTERN1

The thing above

6.2.2 Quantum Number

It's a fake number that is just an index to show the different eigenvectors

$$E_n \qquad n = \text{quantum number}$$

Question

How can we describe a "ball" bouncing between the walls of the infinite square well?

$$\psi(x, t=0) = \sum_{odd} c_n \psi_n(x)$$

sum odd means that the function that we're summing over has to be an odd function.

An even function will just have an expected value of 0.

6.2.3 Free Particle

$$V = 0 \rightarrow \hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

and the time-indepedent SWE is

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi \rightarrow \psi = Ae^{ikx} + Be^{-ikx} \qquad E = \frac{\hbar^2 k^2}{2m} \qquad kc = \sqrt{\frac{2mE}{\hbar^2}}$$

and the time-**dependent** SWE is

$$\psi(x,t)Ae^{ikx-\frac{i}{\hbar}Et} + Be^{-ikx-\frac{i}{\hbar}Et} = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

The wave function is a superposition of a left-propagating and a right-propagating wave both with constant kintetic energy.

allow k to be

$$k = \pm\sqrt{\frac{2mE}{\hbar^2}} \rightarrow Ae^{i(kx-\omega t)}$$

The velocity can be written as

$$e^{-i(kx-\omega t)} = e^{-ik(x-\frac{\omega}{k}t)} \rightarrow v = \frac{\hbar|k|}{2m} = \sqrt{\frac{E}{2m}}$$

This is different from the classical definition of velocity by a factor of 2

$$\frac{1}{2}mv^2 = E \rightarrow v = \sqrt{\frac{2E}{m}}$$

The reason this is so is because the quantum definition of velocity is **not** for a particle.

Consider a Guassian wave packet

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int \Phi(k) e^{i(kx-\omega t)}\, dk$$

For some initial conditions $\psi(x, t = 0)$, we find the weight function with

$$\Phi(k) = \frac{1}{\sqrt{2\pi}} \int \psi(x, t = 0) e^{-ikx}\, dx$$

6.3 MISSED LECTURE

6.4 Discussion

Time-Evolving superpositions. We use a Hamiltonian basis

$$\hat{H}\psi_n = E_n\psi_n \qquad \text{Basis } = \{\psi_1, \psi_2, \ldots\}$$

This basis is orthonormal

$$\langle \psi_m | \psi_n \rangle = \delta_{mn}$$

And the basis is complete

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \ldots$$

And we have a time-evolution operator

$$\Psi(x, 0) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \ldots$$
$$\Psi(x, t) = e^{\frac{-i\hat{H}t}{\hbar}} (c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \ldots)$$

And this effects each individual state differently

$$e^{\frac{-i\hat{H}t}{\hbar}} c_3\psi_3(x) = e^{\frac{-iE_3t}{\hbar}} c_3\psi_3(x)$$

6.4.1 Questions

Find the normalized wave function at time t for a particle in an infinite square with a wave function given by

$$\psi(x,0) = A\left(\psi_1(x) + e^{i\theta}\psi_2(x)\right)$$
$$P(x) = |\psi(x)|^2 = A^2|\left(\psi_1(x) + e^{i\theta}\psi_2(x)\right)|^2$$

The time dependent portion of the wave function is

$$e^{-i\frac{E}{\hbar}}$$

So our equation will be

$$P(x) = e^{-i\frac{E}{\hbar}}|\psi(x)|^2 = A^2|\left(\psi_1(x) + e^{i\theta}\psi_2(x)\right)|^2$$

For an infinite square well, the solution can be written as

$$\psi(x,0) = \sin\left(\frac{n\pi x}{L}\right)$$
$$|\psi(x,0)\rangle = A\left(|\psi_1\rangle + e^{i\theta}|\psi_2\rangle\right)$$
$$|\psi(x)|^2 = A^2\left(\langle\psi_1| + e^{-i\theta}\langle\psi_2|\right)\left(|\psi_1\rangle + e^{i\theta}|\psi_2\rangle\right) =$$
$$A^2\left(\langle\psi_1|\psi_1\rangle + |\psi_1\rangle e^{i\theta}|\psi_2\rangle + \langle\psi_2|\psi_2\rangle\right) = 2A^2 = 1$$

The probability density is just something

6.4.2 Infinite Square Well

Given a unique wave function that's put in the paper and on the website, expand the wave function in terms of its eigenvectors using a Fourier series.

Go to chapter 11 of townsend for braket notation or chapter 4 for time-evolution.

6.5 Finite Square Well

Consider a well with potential

$$V(x) = \begin{cases} -V_0 : -a < x < a \\ 0 : x < -a, x > a \end{cases}$$

The wave function can extend out of the well now, and the wave function **must be continuous**. Exact solutions are typically numerical bound states (≥ 1)

6.5.1 Scattering Stationary States $(E > 0)$

Consider that we're starting with a plane wave (coming from the left)

$$\psi_I(x, t = 0) = Ae^{ikx} + Be^{-ikx}$$

In the middle, we have 2 boundary conditions, and both the wavefunction itself and its derivative have to be continuous.

$$\psi_{II}(x, t = 0) = C\sin(lx) + D\cos(lx)$$

On the right side, there is still 1 boundary condition, but the other side is unbounded

$$\psi_{III}(x, t = 0) = Fe^{ikx}$$

You do some math (Griffith's 2.6). Solve for Boundary Conditions and make sure x is continuous at the important parts

$$\frac{|F|^2}{|A|^2} = T \qquad R = 1 - T \qquad k = \frac{\sqrt{2mE}\hbar}{} l = \frac{\sqrt{2m(E + V_0)}}{\hbar}$$

$$Ae^{-ika} + Be^{ika} = -C\sin(la) + D\cos(la)$$

$$\Rightarrow\Rightarrow\Rightarrow\Rightarrow\Rightarrow\Rightarrow$$

$$B = i\frac{\sin(2la)}{2kl}(l^2 - k^2)F \qquad F = Ae^{-2ika}\left(\cos(2la) - i\frac{(k^2 + l^2)}{2kl}\sin(2la)\right)^{-1}$$

Those are the answer given

$$\psi(x, 0) \propto \int dk\, \psi(k)e^{-ikx}$$

An interesting case is $T = 1$. This can be fulfilled for

$$\frac{2a}{\hbar}\sqrt{2m(E_n + V_0)} = n\pi \qquad (E_n + V_0) = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}$$

We are matching k to the "infinite well".

$$B = i\frac{\sin(2la)}{2kl}(l^2 - k^2)F$$

$$F = (e^{-2ika}A) * \left(\cos(2la) - i\frac{k^2 + l^2}{2kl}\sin(2la)\right)^{-1}$$

6.6 Quantum Tunneling

Consider the opposite of a finite square well

$$V = \begin{cases} V_0 : 0 < x < a \\ 0 : x < 0, x > a \end{cases}$$

The stationary state consists of a plane wave on the left side.

$$\psi_I = A^{ikx} + B^{-ikx}$$

We have exponential decay inside the "well". The particle can also be reflected, so the wave function is given by

$$\psi_{II} = Ce^{-kx} + De^{kx}$$

And then we have another plane wave on the 2nd side

$$\psi_{III}(x,0) = Fe^{ikx}$$

6.7 Recap

Given a function in the position basis

$$|\psi(x)\rangle = \int dx\, \psi(x) \, |x\rangle$$

We are able to turn it into the energy basis

$$|\psi\rangle = \sum c_n \, |n\rangle \qquad \langle x|\psi\rangle \to \psi_n(x)$$

6.8 Basis Vectors

Consider the notation $|1\rangle, |2\rangle, |3\rangle$, where each of those corresponds to the $\vec{i}, \vec{j}, \vec{k} = x, y, z$ unit vectors.

6.8.1 Hilbert Space

A complex space for $N = \ldots?(\to \infty)$.

Pick a basis $|n\rangle$ in that space. The column vectors are written in the form.

$$\vec{\alpha} = |\alpha\rangle = \sum_n \alpha_n \, |n\rangle$$

This is just saying that $\vec{\alpha}$ can be written as a sum of the orthonormal basis vectors. The row vectors are written in the form

$$\vec{\alpha} = \langle\alpha| = \sum_n \langle n|\,\alpha_n ????????????????????????$$

And the inner product is defined as

$$\langle\alpha|\beta\rangle = \sum_n \alpha^*\beta = \langle\beta|\alpha\rangle^*$$

It's just a dot product.

The norm is defined as

$$\langle\alpha|\alpha\rangle = \sum_n \alpha_n^*\alpha_n = \sum_n |\alpha_n|^2$$

If the basis vectors are normalized, then

$$\langle k|n\rangle = \delta_{kn} \qquad \langle k|k\rangle = 1$$

6.8.2 Development in a Basis

$$|\alpha\rangle = \sum_k |k\rangle\langle k|\alpha\rangle = \sum_k \alpha_k\,|k\rangle \qquad \alpha_k = \langle k|\alpha\rangle$$

$$|f\rangle = \sum_x |x\rangle\langle x|f\rangle = \sum_x \langle x|f\rangle\,|x\rangle \qquad f_x = \langle x|f\rangle$$

$$\sum_k c_k \to \int dx\,\rho \qquad \rho = \text{ density} \Rightarrow \int dx\ |x\rangle\langle x|f\rangle = \int dx\, f(x)\,|x\rangle$$

$$\langle f|g\rangle = \int dx \int dy\ \langle x|\, f^*(x)g(y)\,|y\rangle \qquad \langle x|y\rangle = \delta(x-y)$$

$$\langle f|g\rangle = \int dx\, f^*(x)g(x)\,\langle g|f\rangle^*$$

The norm of a function can be written as

$$\langle f|f\rangle = \int dx\, f^*(x)f(x)$$

And in a Hilbert Space,

$$|\,\langle f|f\rangle\,|^2 = 1$$

6.8.3 Operators

An operator acts on a state

$$\hat{A}\,|\psi\rangle$$

In the discrete case, operators can be written as matrices

$$\hat{A}\,|\alpha\rangle = |\alpha'\rangle \qquad \langle\alpha'| = (\hat{A}\,|\alpha\rangle)^{*^T} = \langle\alpha|\,\hat{A}^\dagger \qquad x^\dagger = (x^*)^T$$

The basis of matrix elements can be written as

$$\langle 1|\,\hat{A}\,|3\rangle = \alpha_{13} \qquad \text{1st row, 3rd column}$$

6.8.4 Projections

$$\langle k|\alpha\rangle = \langle k|\,(\alpha_1\,|1\rangle + \ldots) = \alpha_k$$

The projection operator can be written as

$$\hat{P}_i = |i\rangle\,\langle i|$$

everything is 0 except for the ith row and ith column

$$\hat{P}_1\,|\alpha\rangle = |1\rangle\,\langle 1|\,(\alpha_1\,|1\rangle + \alpha_2\,|2\rangle + \ldots) = \alpha_1\,|1\rangle$$

The sum of all projections can be written as

$$\sum_i P_i = \sum_i |i\rangle\,\langle i| = 1$$

Generally

$$|k\rangle\,\langle i| = \qquad \text{Matrix with 1 at kth row, ith column}$$

$$\hat{A} = \sum_k \sum_n \alpha_{kn}\,|k\rangle\,\langle n|$$

That is just a fancy way to fill a matrix

6.9 Observables

Observables are operators that are "measurable" (have a real value)

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle \to \langle \hat{A} \rangle^* = \langle \hat{A} \rangle \to \left\langle \hat{A}\psi \middle| \psi \right\rangle = \left\langle \psi \middle| \hat{A}\psi \right\rangle$$

This is equivalent to the Hermitian Conjugate $\hat{A}^\dagger = (\hat{A}^*)^T$. This means that Observables are Hermitian Operators $\hat{A}^\dagger = \hat{A}$. Eigenvalues of Hermitian Operators are real.

$$\hat{A} |v\rangle = \alpha |v\rangle \to \langle v| \hat{A} |v\rangle = \alpha \langle v|v\rangle = \alpha \qquad \alpha \in \mathbb{R} \to \alpha = \alpha^*$$
$$\langle v| A |v\rangle = \langle v|Av\rangle = \langle v|Av\rangle^* = \langle Av|v\rangle = \langle v| A^\dagger |v\rangle$$

We can also solve for the expectation values of the operator

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \left(\sum_k c_k^* \langle \alpha_k | \right) \hat{A} \left(\sum_n c_n |\alpha_n\rangle \right) = \sum_{k,n} c_k^* c_n \langle \alpha_k | \hat{A} | \alpha_n \rangle$$
$$= \sum_{k,n} c_k^* c_n \alpha_k \langle \alpha_k | \alpha_n \rangle = \sum_k c_k^* c_n \alpha_n \delta_{kn} = \sum_k |c_k|^2 \alpha_k$$

The kronecker delta is important because that is the operator that removes all of the n terms in the expression.

6.9.1 is $\hat{p}$ Hermitian?

$$\langle f|\hat{p}g\rangle = \int f^*(-ih\frac{d}{dx})gdx = -i\hbar f^* g|_{-\infty}^{\infty} + \int(-ih\frac{df^*}{dx})gdx = \langle \hat{p}f|g\rangle$$

6.9.2 Relationships Between Operators

The commutator is an operator that is

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = -[\hat{B}, \hat{A}] \qquad (\hat{A}, \hat{B})^T = B^T A^T \qquad (\hat{A}\hat{B})^\dagger = B^\dagger A^\dagger$$

For $\hat{A}, \hat{B}$, Hermitian Operators. $[\hat{A}, \hat{B}] = 0$ iff there is a basis in which **both** $\hat{A}$ and $\hat{B}$ are diagonal.

$$\langle k| \hat{A} |k'\rangle = 0 \qquad \langle k| \hat{B} |k'\rangle = 0 \qquad \forall k \neq k'$$

6.10 Discussion (The Quantum Recipe)

1. Identify the Hamiltonian
2. Establish the basis (eigenstates)
3. Is your state in the basis?
4. Decompose your state to the basis (Often a fourier transform)
5. Time evolve each element in the decomposition
6. Compute whatever you want

6.10.1 Example

We have a wave function of a free particle given by

$$\Psi(x,0) = A\cos(2kx) + B\sin(kx)$$

Identify the Hamiltonian, idk what this means. Because it's a free particle, it has a Hamiltonian given by

$$-\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,0) = \hat{H}\psi(x,0)$$

Now establish the basis. The particle is a bunch of waves, so it can be written in the basis

$$\psi_n(x,0) = Ae^{inx} \qquad k \in \mathbb{Z}$$

Is your state in the basis? NO because it is not a sum of eigenvectors. Decompose your state to the basis

$$\Psi(x,0) = \frac{A}{2}\left(e^{i2kx} + e^{-i2kx}\right) + \frac{B}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

Time-evolve your solution

$$\Psi(x,0) = \frac{A}{2}\left(e^{\frac{-iE_kt}{\hbar}}e^{i2kx} + e^{\frac{-iE_{-k}t}{\hbar}}e^{-i2kx}\right) + \frac{B}{2i}\left(e^{\frac{-iE_kt}{\hbar}}e^{ikx} - e^{\frac{-iE_{-k}t}{\hbar}}e^{-ikx}\right)$$

6.10.2 Example 2

You have a Hamiltonian and a particle of the form

$$\hat{H} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \Psi = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Identify the Hamiltonian (given)
Identify the basis (take the eigenvalues of the matrix)

$$\Psi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \Psi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Is your state in the basis? no (it is not one of the eigenvectors)
Decompose the state as a form of eigenvectors

$$\Psi = \Psi_1 + \Psi_2$$

Time evolve each element in the decomposition. Because the two states have different energy values (eigenvalues), the decomposed portions will have different frequencies

$$\Psi(t) = e^{\frac{-iE_1 t}{\hbar}}\Psi_1 + e^{\frac{-iE_2 t}{\hbar}}\Psi_2$$

6.10.3 DO YOU NEED TO MEMORIZE THE BASES (YES)

Study properties, behaviors, and form.

6.10.4 Questions (2)

You have a Hamiltonian and a system defined by

$$\hat{H} = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{bmatrix} \qquad |S(0)\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Find the time-dependent system

$$\begin{bmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & (a-\lambda) - b^2/(a-\lambda) \end{bmatrix}$$

$$(a-\lambda)(c-\lambda)((a-\lambda) - b^2/(a-\lambda)) = 0$$

$$((a-\lambda) - b^2/(a-\lambda)) = 0 \rightarrow (a-\lambda)^2 - b^2 = 0 \rightarrow a - \lambda = b \rightarrow$$

eigenvalues are $c, (a-b), (a+b)$. Plug the eigenvalues back into the matrix to get the eigenvectors

$$\begin{bmatrix} a-c & 0 & b \\ 0 & c-c & 0 \\ b & 0 & a-c \end{bmatrix} \to \begin{bmatrix} a-c & 0 & b \\ 0 & 0 & 0 \\ b & 0 & a-c \end{bmatrix}$$

$$= \begin{bmatrix} a-c & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & a-c-\frac{b^2}{a-c} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

I'm so unbelievably stupid

$$\begin{bmatrix} a-(a+b) & 0 & b \\ 0 & c-(a+b) & 0 \\ b & 0 & a-(a+b) \end{bmatrix} = \begin{bmatrix} -b & 0 & b \\ 0 & c-a-b & 0 \\ b & 0 & -b \end{bmatrix}$$

$$= \begin{bmatrix} -b & 0 & b \\ 0 & c-a-b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Im so sos os oso so so so stupid oh my god

$$-1v_1 + v_3 = 0 \qquad v_2 = 0 \to v_1 - v_3$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

ajgfilagjiladfngjilasdnfjasdj;fkasdjfasdfnasdfnsadklfnasd;fnasdfs

6.10.5 3 (Diagonalizing Matrices)

6.11 Formalism

This is a discrete basis

$$|\psi\rangle = \sum_k c_k |\alpha_k\rangle \qquad c_k = \langle \alpha_k|\psi\rangle \qquad \langle \alpha_k|\alpha_j\rangle = \delta_{kj}$$

This is a continuous basis

$$|\psi\rangle = \int dx\, \psi(x)\, |x\rangle \qquad \psi(x) = \langle x|\psi\rangle \qquad \langle x'|x\rangle = \delta(x-x')$$

An observable is a Hermitian operator with real eigenvalues

$\hat{A} = \hat{A}^\dagger \rightarrow$ real eigenvalues

$A_{jk} = \langle k| \hat{A} \langle j|$

Consider matrix elements of $\hat{x}$ in the position basis

$\langle x'| \hat{x} |x\rangle = \langle x'| x |x\rangle = x \langle x'|x\rangle$ ======================

A commutator is an operator such that

$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

If $[\hat{A}, \hat{B}] = 0$, then a simultaneous eigenbasis exists. This means that

$\exists\{|j\rangle\} : \hat{A} |j\rangle = \alpha_j |j\rangle \qquad \hat{B} |j\rangle = \beta_j |j\rangle$

6.11.1 Eigenvalues/vectors

Consider the momentum operator

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \rightarrow -i\hbar\frac{\partial}{\partial x}\psi = p\psi \rightarrow \psi_p = Ae^{i\frac{p}{\hbar}x}$$

6.11.2 Energy Eigenstates

consider the time independent schrodinger equation

$$\hat{H}\psi = E\psi \qquad \hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi = E\psi \rightarrow \left(\frac{\partial}{\partial x} - \frac{i\sqrt{2mE}}{\hbar}\right)\left(\frac{\partial}{\partial x} + \frac{i\sqrt{2mE}}{\hbar}\right)\psi = 0$$

This gives you two degenerate solutions.

$\psi_1 = Ae^{ikx} \qquad \psi_2 = Be^{-ikx} \qquad \psi_k = Ae^{ikx} + Be^{-ikx}$

6.12 Basis Transformations

Suppose we know $|\psi\rangle$ in some basis $\{|\alpha_k\rangle\}$, and we want to know it in some different basis $\{|\beta_k\rangle\}$. Know that $c_k = \langle\alpha_k|\psi\rangle$ and $d_k = \langle\beta_k|\psi\rangle$

$$d_n = \langle\beta_n|\psi\rangle = \langle\beta_n| \sum_k c_k |\alpha_k\rangle = \sum_k c_k \langle\beta_n|\alpha_k\rangle$$

==============================

What about in the continuous case?

$$\hat{p}\,|p\rangle = p\,|p\rangle \rightarrow \text{position basis?}$$

$$\int dx\ |x\rangle \langle x|\,\hat{p}\,|p\rangle = p\int dx\ |x\rangle \langle x|p\rangle = p\int dx\ |x\rangle\, f_p(x)$$

$$\int dx\,dx'\ |x\rangle \langle x|\,\hat{p}\,|x'\rangle \langle x'|p\rangle$$

$$\langle x|\,\hat{p}\,|\psi\rangle = \int dx'\ \langle x|\,\hat{p}\,|\psi\rangle \langle x'|\psi\rangle = \int dx'\ \langle x|\,\hat{p}\,|x'\rangle\,\psi(x')$$

$$= \int dx'\,\hbar\frac{\partial}{\partial x} f(x-x')\psi(x') \Rightarrow -i\hbar\frac{\partial\psi}{\partial x}$$

$$\int dx\ |x\rangle\, i\hbar\frac{\partial f_p(x)}{\partial x} = \hat{p}\int dx\ |x\rangle\, f_p(x)$$

$\langle x|\,\hat{p}\,|x'\rangle$ is the matrix elements of $\hat{p}$ in the x-basis.

6.12.1 What is $\psi(p)$?

$$|\psi\rangle = \int dp\ \langle p|\psi\rangle\,|p\rangle = \int dp\,dx\ \langle p|x\rangle \langle x|\psi\rangle\,|p\rangle$$

$$= \int dx\,dp\,A\cdot e^{-ipx/\hbar}\psi(x)\,|p\rangle = \int dp\,\psi(p)\,|p\rangle\ dp$$

6.12.2 Infinite Square Well

The energy basis is described as

$$\langle n|\psi\rangle = c_n \qquad |\psi\rangle = \sum_n c_n\,|n\rangle$$

and the position basis is described as

$$\langle x|\psi_n\rangle = \psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

And the momentum basis is

$$\langle p|\psi_n\rangle = \psi_n(p)$$

In the position basis, our wave function is made by a superposition of planar waves. The higher the energy of the planar wave, the more defined is the corresponding momentum. Each eigenstate is described by

$$\psi_n(p) = \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{a}}e^{-iE_nt/\hbar}\int_0^a e^{-ipx/\hbar}\sin\left(\frac{n\pi x}{a}\right)dx$$

This is solved with a calculator because why would you go through the trouble.

$$\frac{4\pi a}{\hbar}\frac{n^2}{\left[(n\pi)^2-\left(\frac{ap}{\hbar}\right)^2\right]^2} * \begin{cases}\cos^2(\frac{ap}{2\hbar}) : n = odd\\ \sin^2(\frac{ap}{2\hbar}) : n = even\end{cases}$$

6.12.3 Know the theory, but don't worry about the calculations

6.13 Important Example: 2 Level System

$$H = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} = \hbar\omega\sigma_z$$

σ_z is the pauli matrix operator with eigevectors

$$+1, |0\rangle \equiv \begin{bmatrix}1\\0\end{bmatrix} \qquad -1, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \qquad \text{Energy Basis}$$

The general state is of the form

$$\alpha|0\rangle + \beta|1\rangle \qquad p(\psi = 0) = |\alpha|^2$$

Consider an operator

$$\hbar\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \equiv \hbar\sigma_x = \hbar\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

This operator **is observable** because it is Hermitian. The eigenvectors are

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

with an energy value of $+\hbar$

6.13.1 Double Well Potential

Imagine two wells from $-b \to -a$ and $b \to a$ with potential $-V_0$ and then potential 0 everywhere else (in between them is $-b \to b$).

In this system, we imagine measuring the left well vs the right well. Consider the tunneling operator

$$\hat{T}\,|0\rangle = |1\rangle \qquad \hat{T}\,|1\rangle = |0\rangle \qquad \langle 1|\,\hat{T}\,|1\rangle = 0 \qquad \langle 0|\,\hat{T}\,|0\rangle = 1$$

$$\langle 0|\,\hat{T}\,|1\rangle = \langle 1|\,\hat{T}\,|0\rangle = 1 \qquad \hat{T} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

Starting with quantum mechanics with waves and with two levels are functionally the same.

6.13.2 The Uncertainty Principle

Incompatible Observables: measuring $\hat{A}$ disturbs the possible results of $\hat{B}$.

$$\sigma_A \sigma_B \geq \frac{\hbar}{2}$$

The variance of an observable can be written in the form

$$\sigma_A^2 = \langle \left(\hat{A} - \langle \hat{A} \rangle\right)^2 \rangle = \langle \psi|\left(\hat{A} - \langle \hat{A} \rangle\right)^2 |\psi\rangle = \left\langle (\hat{A} - \langle \hat{A}\rangle)\psi \middle| (\hat{A} - \langle \hat{A} \rangle)\,|\psi\rangle\right.$$
$$\equiv \langle f|f\rangle =================================$$

6.13.3 Schwarz Inequality

$$|\,\langle f|g\rangle\,|^2 = \langle f|f\rangle\,\langle g|g\rangle$$
$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\,\langle g|g\rangle \geq |\,\langle f|g\rangle\,|^2$$

For some complex number $z = z' + iz''$, we know that

$$\frac{1}{2i}\left(z - z^*\right)^2$$

So now we have to find the inner products of f and g

$$\langle f|g\rangle = \langle \psi|\,(\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\,|\psi\rangle$$
$$= \langle \psi|\,\hat{A}\hat{B} - \hat{A}\langle B\rangle - \hat{B}\langle A\rangle + \langle A\rangle\langle B\rangle\,|\psi\rangle$$

Something with commutativity

$= \langle\psi| \langle\hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle |\psi\rangle = \langle g|f\rangle$

Now we put everything together

$$\langle f|g\rangle - \langle g|f\rangle = \langle[\hat{A}, \hat{B}]\rangle \Rightarrow \sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$$

6.13.4 Back to 2 Layer System (TLS)

The Pauli matrices are known as

$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \qquad \sigma_x = |0\rangle\langle 1| - |1\rangle\langle 0| \qquad \sigma_y = |1\rangle\langle 0| - |0\rangle\langle 1|$

Permutations of these matrices are given in the form

$[\sigma_i, \sigma_j] = 2i * asdfasdfasdfasdfasdfadfasdfasdfasdfasdfdasfasdf$

6.14 Block Sphere

This is a graphical representation of all the possible states that a wave function can hold. The 6 axes are given by

$\{|0\rangle, |1\rangle\}, \{|+\rangle, |-\rangle\}, \{|+i\rangle, |-i\rangle\}$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$

6.15 Discussion 1

A couple midterm sanity checks.

Express in wavefunction notation the dirac inner product between a ket and a bra.

$$\langle\psi_i|\psi\rangle = \int_{-\infty}^{\infty} dx\, \psi_i^*\psi$$

It's just a convolution integral.

Express the orthonormal condition of a basis in both dirac and wavefunction notation

$$\langle e_1|e_2\rangle = \delta_{ij} \qquad \int_{-\infty}^{\infty} dx\, e_1^* e_2 = \delta_{ij}$$

That is the kronecker delta. Basically, when $i \neq j$, its just 0.

You should be very keen on the equivalence between Dirac notation and integral notation.

6.15.1 Adjoint Operators

The hermitian adjoint is the complex conjugate transpose of the original operator. An operator is hermitian (or self-adjoint) if its hermitian adjoint is the same operator.

$$\left\langle \psi_i \middle| \hat{L}\psi_j \right\rangle = \langle L^{\dagger}\psi_i | \psi_j \rangle$$

All observables are hermitian (This is because the conjugate of a real eigenvalue is just itself).

6.15.2 SPOILER

Show that $\hat{p}$ is self-adjoint (hermitian)

6.15.3 The Quantum Recipe

1. identify your Hamiltonian
2. Establish your basis (eigenstates of the Hamiltonian)
3. Is your state a basis vector (in the basis)
4. If not, decompose your state to basis vectors
5. Time evolve your state

6.15.4 Spin

Spin is like the zodiac of particles. Spin is an intrinsic property that determines the behavior of the particle.

Up and Down are spins that determine how they go through a Stern-Gerlach filter.

6.15.5 Reorienting Spin

Particles don't by themselves have definite spin. Spin up and spin down are only determined by the orientation of the Stern-Gerlach filter. The Hamiltonian of the spin experiment changes basis.

The **Spin Hamiltonian** gives you the orientation of the Stern Gerlach filters. The eigenstates of the Hamiltonian will be the two states parallel and anti-parallel to the sptial orientation described by the Hamiltonian.

The visual for the direction of the orientation is called the **Bloch Sphere**.

6.15.6 Spin Decomposition

We start off in the z-basis, and then we decompose as states in the z-basis. So, given the Hamiltonian

$$\hat{A} = \hat{\sigma}_z$$

Looking at it, the Hamiltonian is in the z-direction. The eigenstates are

$$|\uparrow_H\rangle = |\uparrow_x\rangle \qquad |\downarrow_H\rangle = |\downarrow_x\rangle$$

If the Hamiltonian changes from $\hat{\sigma}_z$ to $\hat{\sigma}_x$, then our eigenstates change, and then they need to be written in terms of the z-basis.

Now, we get

$$\hat{H} = \hat{\sigma}_z + \hat{\sigma}_x$$

The direction of the Hamiltonian is 45 degrees in the xz plane, and the eigenstates are

$$|\uparrow_H\rangle = \frac{1}{\sqrt{2}}|\uparrow_z\rangle + \frac{1}{\sqrt{2}}|\uparrow_x\rangle$$

But then x needs to be decomposed to the z-basis.

6.15.7 Show $\hat{p}$ is Hermitian

Show that $\hat{p}^\dagger = \hat{p}$

$$\langle g|\hat{p}f\rangle = \int_{-\infty}^{\infty} dx\, g^*(\hat{p}f) \rightarrow \int_0^a dx\, g^*\hat{p}f = \int_0^a dx\, g^*(-i\hbar\frac{\partial}{\partial x})f$$

$$dv = \frac{\partial}{\partial x}f \qquad v = f \qquad u = (-i\hbar)g^* \qquad du = \frac{d}{dx}(-i\hbar)g^* \rightarrow$$

$$g^*(-i\hbar)f\Big|_0^a dx - \int_0^a dx\,\frac{d}{dx}(-i\hbar)g^*f = \int_0^a dx\,(i\hbar)\frac{d}{dx}g^*f = \int p^*g^*f$$

$$= \int(\hat{p}g)^*f = \langle\hat{p}g|f\rangle$$

6.15.8 b)

$$g^*(-i\hbar)f\Big|_0^a dx - \int_0^a dx\,\frac{d}{dx}(-i\hbar)g^*f \rightarrow$$

$$g^*(-i\hbar)f\Big|_0^a dx = g^*(-i\hbar)f(0) - g^*(-i\hbar)\lambda f(0) = g^*(-i\hbar)(1-\lambda)$$

6.15.9 Skip 2 go to 3

$$\hat{H} = \hbar\omega\left(\alpha\hat{\sigma}_z + \beta\hat{\sigma}_x\right)$$

Write H as both a matrix and bra-ket form.

$$\hbar\omega\begin{pmatrix}\alpha & \beta\\ \beta & -\alpha\end{pmatrix} = \hbar\omega\left(\alpha\left|1\right\rangle\left\langle 1\right| + \beta\left|0\right\rangle\left\langle 1\right| + \beta\left|1\right\rangle\left\langle 0\right| - \alpha\left|1\right\rangle\left\langle 1\right|\right)$$

All observables are **Hermitian**.
Draw the quantization axis in a Bloch Sphere Picture.

Chapter 7

Harmonic Oscillator

7.1 MISSED LECTURE

Now our Hamiltonian is of the form

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

And we have a new operator

$$a_\pm = \frac{1}{\sqrt{2\hbar\omega}}$$

a_+ and a_- are known as the ladder operators.

$$\hat{H}\left|0\right\rangle = \frac{\hbar\omega}{2}\left|0\right\rangle$$

$$a_+a_- = \frac{H}{\hbar\omega} - \frac{1}{2} = \hat{n}$$

This is known as the number operator.

$$\left\langle n\right|\hat{H}\left|n\right\rangle = \frac{1}{2}\hbar\omega + n * \hbar\omega$$

So we do some shenanigans $\left\langle n\right| a_+$ is a stationary state and $a_-\left|n\right\rangle$ is a stationary state, so

$$\left\langle n\right| a_+ * a_-\left|n\right\rangle \neq 0 \rightarrow (a_-)^\dagger = a_+$$

both those operators are Hermitian

7.2 Stationary States in the Position Basis

$$\hat{a}_-\psi_0(x) = (2\hbar m\omega)^{-1/2}(\hbar\frac{\partial}{\partial x} + m\omega x)\psi_0(x) = 0$$
$$\frac{\partial}{\partial x}\psi_0(x) = -\frac{m\omega x}{\hbar}\psi_0(x) \to \int dx\,\frac{1}{\psi_0(x)}\frac{\partial}{\partial x}\psi_0(x) = \int dx\, -\frac{m\omega x}{\hbar} ===$$
$$\psi_0(x) = Ae^{-m\omega x^2/2\hbar}$$

It is a Gaussian distribution. The fourier transform of a Gaussian is also a Gaussian. Normalize the function to get

$$1 = \langle\psi|\psi\rangle = \int dx\,\psi_0^*(x)\psi_0(x) = [-] \Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

And now we try to find $\psi_1(x)$.

7.2.1 I SKIPPED WHOOPS

$$\psi_n(x) = \left(\prod_{k=1}^{n}\frac{a_+}{\sqrt{k}}\right)\psi_0 = \frac{1}{\sqrt{n!}}a_+^n\psi_0$$
$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(x)e^{-x^2/2}$$
$$H_n(\zeta) = (-1)^n e^{\zeta^2}\frac{\partial^n}{\partial\zeta^n}e^{-\zeta^2}$$

7.3 Discussion

is a physically valid wavefunction of a free particle

If energy is definite, it cannot bounce :(

If you put enough unphysical states together, you get a physical state.

7.3.1 Recap

$$V(x) = \frac{1}{2}m\omega^2x^2 \Rightarrow \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2x^2\right)\psi(x) = E\psi(x) \rightarrow$$

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}\left(\pm i\hat{p} + m\omega\hat{x}\right) \qquad \hat{a}_+ = (\hat{a}^\dagger) \qquad \hat{a}_- = (\hat{a})$$

$$a_+a_- = \hat{n} \qquad \hat{a}_+\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

$$\hat{a}_-\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle \qquad [a_-, a_+] = 1\frac{\hat{H}}{\hbar\omega} = a_+a_- + \frac{1}{2}$$

$$a_-(0) = \left|0\right\rangle \qquad a_-\left|\psi\right\rangle = \psi_0(x) = Ae^{\frac{-m\omega x^2}{2\hbar}}$$

Stationary States/Eigenfunctions are Gaussians centered at 0. $a_+\left|0\right\rangle = \left|1\right\rangle$ has energy $\hbar\omega(1 + 1/2)$, but it contains noise carrying an energy of $1/2\hbar\omega$.

7.4 Relation to Classical S.H.O.

You have a mass oscillating around with $x \propto \cos(\omega t)$ and $p(t) \propto \sin(\omega t)$. The relationship between x and p looks like a circle.

What do the stationary states of $\hat{H}$ look like in the phase space? The wavefunctions of the QSHO are of the form

$$\psi_0(x) = A_0e^{\frac{-m\omega x^2}{2\hbar}} \qquad \psi_1(x) = A_1xe^{\frac{-m\omega x^2}{2\hbar}}$$

Take a fourier transform to get the momentum space

$$\psi_0(p) = \frac{1}{\sqrt{2\pi\hbar}}\int e^{-ipx/\hbar}\psi_0(x)\,dx \; = B_0e^{\frac{-p^2}{2\hbar m\omega^2}} \qquad \psi_1(p) = B_1pe^{\frac{-p^2}{2\hbar m\omega^2}}$$

You can Guassians of some width and height that I didn't write down :(
For a classical SHO, you have

$$x = x_0\cos(\omega t) \qquad E_{tot} = \frac{1}{2}m\omega^2x_0^2 \qquad p(x)dx = 2\frac{dt}{T} \qquad T = \frac{2\pi}{\omega}$$

$$dx = -x_0\omega\sin(\omega t)dt = -x_0\omega\sqrt{1 - \left(\frac{x}{x_0}\right)^2}dt \Rightarrow p(x) = \frac{1}{\pi x_0\sqrt{1 - \left(\frac{x}{x_0}\right)^2}}$$

for the quantum world, we solve for x and p in terms of a_+ and a_-

$$a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\pm i\hat{p} + m\omega\hat{x}) \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m}}\frac{\hat{a}_- + \hat{a}_+}{\sqrt{2}} \qquad \hat{p} = \sqrt{\hbar m\omega}\frac{-i(\hat{a}_- - \hat{a}_+}{\sqrt{2}}$$

$$\langle x\rangle = \langle n|\,\hat{x}\,|n\rangle \propto \langle n|\,(a_- + a_+)\,|n\rangle = 0$$

$$\langle \Delta x\rangle^2 = \langle x^2\rangle = \frac{\hbar}{2m\omega}\langle n|\, a_-^2 + a_-a_+ + a_+a_- + a_+^2\,|n\rangle$$

$$= \langle n|\, a_+a_- + \frac{1}{2}\,|n\rangle = \frac{\hbar}{m\omega}(n + \frac{1}{2})$$

$$\langle p\rangle = 0 \qquad \langle p^2\rangle = \hbar m\omega(n + \frac{1}{2}) \qquad \Delta x\Delta p = (n + \frac{1}{2})\hbar$$

We graph $\tilde{x} \equiv \frac{a_- + a_+}{\sqrt{2}}$ over $\tilde{p} \equiv \frac{-i(a_- - a_+)}{\sqrt{2}}$. We get concentric circles

$$\tilde{x}^2 + \tilde{p}^2 = 2\hat{n}$$

7.5 Coherent State

Let's attempt n such that $\langle n|\,x\,|n\rangle = 0$.

$$a_-\,|\psi_\alpha\rangle = \alpha\,|\psi_\alpha\rangle \qquad \alpha \in \mathbb{C}$$

$$\langle\psi_\alpha|\, a_- + a_+\,|\psi_\alpha\rangle = \alpha^* + \alpha = 2Re(\alpha)$$

$$\langle n|\psi_\alpha\rangle = \frac{1}{\sqrt{n!}}\langle 0|\, a_-^n\,|\psi_\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\psi_\alpha\rangle = \frac{\alpha^n}{\sqrt{n!}}A$$

$$|\psi_\alpha\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|\psi_{alpha}\rangle = A\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \equiv A\sum_{n=0}^{\infty}\frac{(\alpha a_+)^n}{\sqrt{n!}}|0\rangle$$

$$A = e^{-|\alpha|^2/2}$$

A is the normalization constant form $\langle\psi_\alpha|\psi_\alpha\rangle = 1$. This is the most coherent quantum harmonic oscillator that we can get and this is the wave function of lasers. We can do all sorts

7.5.1 Time Evolution

$$|\alpha(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega tn}\,|n\rangle$$

Chapter 8

3D Quantum Mechanics

Trying to figure out the wave equation with the Coulomb Potential.

8.0.1 Assumptions

Only observe the wave function of the electron (because the nucleus is heavy so we can fix the position). Maybe something else I didnt pay attention.

8.1 Quantum Numbers

Because we're now in 3d space, we have 4 quantum numbers (x, y, z, spin).

Because we're working with a central potential, angular momentum is conserved, and angular momentum is **quantized**.

If an operator is conserved, it commutes with the Hamiltonian, which means it is **simultaneously diagonalizable**.

The magnitude of angular momentum is conserved, but the direction is not necessarily.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \qquad \hat{H} = \frac{\hat{p}^2}{2m} + V = \frac{1}{2m}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + V(x, y, z)$$

$$\hat{p} = -i\hbar\nabla \to \frac{\vec{p^2}}{2m} = -\frac{\hbar^2}{2m}\nabla^2$$

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}, t) + V\psi(\vec{r}, t)$$

8.2 Spherical Coordinates

Just google it every time to need to use it.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) + V\psi(\vec{r},t) = E\psi$$

Use separation of variables to find solutions

$$\psi(r,t) = R(r)Y(\theta,\phi)$$

$$-\frac{\hbar^2}{2m}\left(\frac{Y}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} + \frac{R}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right)$$

$$+V(\vec{r},t)RY = E * RY$$

$$\left(\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} - \frac{2mr^2}{\hbar^2}\left[V(r)-E\right]\right)$$

$$+\frac{1}{Y}\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right) = 0$$

We now have an angular equation and a radial equation, with a separation constant $l(l+1)$.

8.3 Angular Momentum

$$\hat{L} = \hat{r}\times\hat{p} = -i\hbar\left(\hat{r}\times\vec{\nabla}\right) \Rightarrow$$

$$L^2 = -\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]$$

The eigenstates are going to be $R(r)*Y(\theta,\phi)$ = an eigenfunction of angular momentum.

Because angular momentum is conserved, $[\hat{H},\hat{L}^2] = 0$.

$$\hat{L} = \hat{\vec{r}}\times\hat{\vec{p}} \Rightarrow L_x = yp_z - zp_y \qquad L_y = zp_x - xp_z \qquad L_z = xp_y - yp_x$$

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$

$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$

$$= [yp_z, zp_x] - 0 - 0 + [zp_y, xp_z] = yp_x[p_z, z] + xp_y[z, p_z]$$

$$= i\hbar\left(xp_y - yp_z\right) = i\hbar L_z$$

This looks awfully similar to the pauli matrices.

$$[L_x, L_y] = i\hbar L_z, [L_y, L_y] = i\hbar L_x, [L_z, L_x] = i\hbar L_y \qquad [L_j, L_k] = i\hbar\epsilon_{jkm}L_m$$

The magnitude squared of angular momentum can be written as

$$L^2 = L_x^2 + L_y^2 + L_z^2 \qquad [L^2, L_x] = [L_x^2, L_x] + [L_y^2, L_x] + [L_z^2, L_x]$$
$$= L_y[L_y, L_x] + [L_y, L_x]L_y + L_z[L_z, L_x] + [L_z, L_x]L_z = 0$$

8.4 Discussion 6: Navigating the Gaussian (HALF MISSED)

The ground state is a Gaussian.

The raising operator has a $\sqrt{n+1}$ and the lowering operator has $\sqrt{n}$.
If you start at state 12 and go to state 14, you'll get change of

$$\hat{a}^\dagger\hat{a}^\dagger\psi_{12} = \hat{a}^\dagger\sqrt{13}\psi_{13} = \sqrt{14}\sqrt{13}\psi_{14}$$

The harmonic oscillator gas **linear** energy dependence.

8.4.1 Questions

Find x^2 in terms of a_+ and a_-

$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(+i\hat{p} + m\omega\hat{x}) \qquad a_- = \frac{1}{\sqrt{2\hbar m\omega}}(-i\hat{p} + m\omega\hat{x})$$
$$a_+ + a_- = \frac{1}{\sqrt{2\hbar m\omega}}2\hat{x} \Rightarrow x = \sqrt{2\hbar m\omega}\frac{a_+ + a_-}{2}$$
$$x^2 = \hbar m\omega\frac{(a_+ + a_-)^2}{2}$$
$$\langle n| x^2 |n\rangle = \frac{\hbar m\omega}{2}\langle n| (a_+ + a_-)^2 |n\rangle = \frac{\hbar m\omega}{2}\langle n| a_+^2 + a_-^2 + a_+a_- |n\rangle$$
$$\langle n| a_+^2 |n\rangle + \langle n| a_-^2 |n\rangle + \langle n| a_+a_- |n\rangle =$$
$$\langle n| \sqrt{(n+2)(n+1)}\, |n+2\rangle + \langle n| \sqrt{(n)(n-1)}\, |n-2\rangle$$
$$+n\langle n|n\rangle + (n+1)\langle n|n\rangle = 0 + 0 + n\langle n|n\rangle + (n+1)\langle n|n\rangle = 2n+1$$

8.4.2

Given a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle e^{-i\omega t/2} + |1\rangle e^{-3i\omega t/2}\right)$$

And we know the average position

$$\langle \hat{x}\rangle \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)$$

do shenanigans to find $\langle p\rangle$

$$\hat{p} = \frac{\sqrt{2\hbar m\omega}}{2}(a_+ - a_-)$$

$$\langle\psi|\,\hat{p}\,|\psi\rangle = \frac{\sqrt{2\hbar m\omega}}{2}\langle\psi|\left(a_+|\psi\rangle - a_-|\psi\rangle\right)$$

$$a_+|\psi\rangle = \frac{1}{\sqrt{2}}\left(1\,|1\rangle e^{-i\omega t/2} + \sqrt{2}\,|2\rangle e^{-3i\omega t/2}\right)$$

$$a_-|\psi\rangle = \frac{1}{\sqrt{2}}\left(0 + 1\,|0\rangle e^{-3i\omega t/2}\right)$$

$$\frac{1}{\sqrt{2}}\left(\langle 0|\, e^{i\omega t/2} + \langle 1|\, e^{3i\omega t/2}\right)\frac{1}{\sqrt{2}}\left(1\,|1\rangle e^{-i\omega t/2} + +\sqrt{2}\,|2\rangle e^{-3i\omega t/2} 1\,|0\rangle e^{-3i\omega t/2}\right)$$

$$= \frac{1}{2}\left(e^{-3i\omega t/2}e^{i\omega t/2} + e^{3i\omega t/2}e^{-i\omega t/2}\right) = \frac{1}{2i}\left(e^{-i\omega t/2} + e^{i\omega t/2}\right) = \sin(\omega t/2)$$

there's an i is the thing for $\langle p\rangle$ trust me bro

8.4.3 3D harmonic oscillator

$$\frac{-\hbar}{2m}\nabla^2|\psi\rangle + \frac{m\omega^2}{2}\left(x^2+y^2+z^2\right)|\psi\rangle = E|\psi\rangle$$

$$\frac{\hbar}{2m}\nabla^2|\psi\rangle = \frac{m\omega^2}{2}\left(x^2+y^2+z^2-E\right)|\psi\rangle$$

Consider just x

$$\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}|\psi\rangle = \frac{m\omega^2}{2}\left(x^2-E\right)|\psi\rangle \Rightarrow \psi = e^{x^2} \qquad \frac{\partial}{\partial x}e^{x^2} = 2xe^{x^2}$$

$$\frac{\partial}{\partial x}2xe^{x^2} = 2e^{x^2} + 4x^2e^{x^2} \qquad \psi = X(x)Y(y)Z(z)$$

$$\frac{-\hbar}{2m}\left(2e^{x^2} + 4x^2e^{x^2}\right) + \frac{m\omega^2}{2}e^{x^2} = Ee^{x^2} \Rightarrow \psi_0 = A_0e^{\frac{-m\omega x^2}{2\hbar}} \qquad \psi_1 = A_1xe^{\frac{-m\omega x^2}{2\hbar}}$$

8.5 3D Stationary States

8.5.1 Recap

Imagine we have 2 particles, one of which with a central potential $V(r)$. Because it is central, we can say that angular momentum is conserved $\vec{L} = \vec{r} \times \vec{p} = -i\hbar\left(\vec{r} \times \vec{\nabla}\right)$.

The SWE can be written as

$$i\hbar\frac{\partial\psi(r,\theta,\phi)}{\partial t} = -\frac{\hbar}{2m}\nabla^2\psi(r,\theta,\phi) + V(r)\psi(r,\theta,\phi) \qquad \psi = R(r)Y(\theta,\phi) \Rightarrow$$
$$\left(\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial R}{\partial r} - \frac{2mr^2}{\hbar^2}\left[V(r) - E\right]\right)$$
$$+\frac{1}{Y}\left(\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right) = 0$$

That is both the radial and angular equation that solve for our wave function.

The expected value of angular momentum to be conserved is

$$[\hat{H}, \hat{L}^2] = 0$$

Eigenstates and values of angular momentum can be written as

$$[L_j, L_k] = i\hbar\epsilon_{jkl}L_l \qquad j,k,l = \{x,y,z\}$$
$$[L^2, L_j] = 0 \Rightarrow L^2\psi = \lambda\psi \qquad L_z\psi = \mu\psi$$

Let's define a thing and solve some stuff

$$L_\pm = L_x \pm iLy$$
$$[L_z, L_\pm] = \pm\hbar L_\pm \qquad [L^2, L_\pm] = 0$$

If ψ is an eigenfunction of L^2 and L_z, then $L_\pm\psi$ is also an eigenfunction of those.

$$L^2(L_\pm\psi) = L_\pm L^2\psi = L_{pm}\lambda\psi = \lambda(L_{pm}\psi)$$
$$L_z(L_{pm}\psi) = \left(L_zL_\pm - L_\pm L_z\right)\psi + L_\pm L_z\psi = \left(\mu \pm \hbar\right)\psi$$

$L_+(L_-)$ raises (or lowers) the Eigenvalue of L_z by $\hbar$, but **do not change** L^2.

$$L_\pm\psi = \psi' \qquad L^2\psi = \lambda\psi$$

8.6 Bounds of Angular Momentum

(z-component, given fixed λ)

The angular momentum looks like a ladder with rungs $\hbar$ apart. The top of ψ_t and the bottom is ψ_b

$$\psi = \mu \Rightarrow L_+\psi = \mu + \hbar \Rightarrow L_-\psi = \mu - \hbar \qquad L_-\psi_b = 0 \qquad L_+\psi_t = 0$$
$$L_z\psi_t = \hbar l\psi_t \qquad L_z\psi_b = \hbar\bar{l}\psi_b \qquad L^2\psi_{t/b} = \lambda\psi_{t/b}$$

l and $\bar{l}$ are just the maximum and minimum values of μ.

I can do some identity shenanigans

$$L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \mp (L_xL_y - L_yL_x)$$
$$= L_2 - L_z^2 \mp i(i\hbar L_z) \Rightarrow L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$
$$L^2\psi_t = \left(L_-L_+ + L_z^2 + \hbar L_z\right)\psi_t = \hbar^2 l(l+1)\psi_t = \lambda\psi_t$$

These look suspiciously like Legendre Polynomials? eh we'll find out. That is how the top of the ladder looks, but the bottom is different.

$$\lambda \equiv \hbar^2\bar{l}(\bar{l}-1)$$

8.7 Index Wavefunctions

Some idiot decided to make the index m so now we have another m that isn't mass.

$$L^2\psi_l^m = \hbar^2 l(l+1)\psi_l^m = \lambda\psi_l^m \qquad L_z\psi_l^m = \hbar m\psi_l^m = \mu\psi_l^m$$

$L_\pm$ raises/lowers the z-projection by $\pm\hbar$ $m_{min} = -l(-l+1)$ and it increments by $\hbar$ until to get $m_{max} = l(l+1)$. l is an integer or an integer $+1/2$.

8.7.1 Griffiths Fig. 4.12

If we imagine a sphere, a quantized value L_z could be a circular trajectory that lies near the top of the sphere. The largest value of L_z is the trajectory that lies on the diameter of the sphere, which is $\sqrt{L^2}$.

8.8 Recap

Consider the SWE in 3D with a central potential $V = V(r)$. You write it all out in spherical coordinates and do separation of variables and then you get a radial equaation plus an angular equation $= 0$. The angular equation became equal to $l(l+1)$ and the radial $= -l(l+1)$.

Angular momentum is conserved. We should look for eigenfunctions of L^2 and L_z ($[L^2, L_z] = 0$). Use the structure of orthogonal projections.

Introduce raising and lowering functions L_+ and L_-, that raise and lower the z-projection of angular momentum.

$$L^2\psi_l^m = \hbar^2 l(l+1)\psi_l^m \qquad L_z\psi_l^m = \hbar m\psi_l^m \qquad l = \text{integer or integer} + 1/2$$
$$m - -l, -l+1, \ldots, l+1, l$$

It's like a ladder with $\#m = zl + 1$ values. The magnitude of angular momentum is conserved **and** the z-projection ~~are conserved~~ is quantized.

8.9 Connecting L to the angular equation

$$\vec{L} = -i\hbar\left(\vec{r}\times\vec{\nabla}\right) \qquad \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin(\theta)}\frac{\partial}{\partial\phi} \qquad \vec{r} = r\hat{r}$$

We cannot use just spherical coordinates because we know that only the z-projection of the angular momentum is quantized.

$$\vec{L} = -i\hbar\left[r\left(\hat{r}\times\hat{r}\right)\frac{\partial}{\partial r} + \left(\hat{r}\times\hat{\theta}\right)\frac{\partial}{\partial\theta} + \left(\hat{r}+\hat{\phi}\right)\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right]$$
$$= -i\hbar\left[\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\phi}\right] \qquad \hat{\theta} = \cos(\theta)\cos(\phi)\hat{i} \qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$L_x = -i\hbar\left[-\sin(\phi)\frac{\partial}{\partial\theta} - \cos(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right]$$
$$L_y = -i\hbar\left[\cos(\phi)\frac{\partial}{\partial\theta} - \sin(\phi)\cot(\theta)\frac{\partial}{\partial\phi}\right] \qquad L_z = -i\hbar\frac{\partial}{\partial\phi}$$
$$L_+ = L_x \pm iL_y ==== \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot(\theta)\frac{\partial}{\partial\phi}\right)$$
$$L^2 = L_+L_- + L_z^2 - \hbar L_z = -\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]$$
$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot(\theta)\frac{\partial}{\partial\theta} + \cot^2(\theta)\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$

With all this shenanigans, no derivations are given, our eigenvalue equation is

$\hbar^2 l(l+1)\gamma = L^2\gamma$

8.9.1 Angular Equation (Y)

$x\sin^2(\theta) * Y \qquad Y = \Omega(\theta)\Lambda(\phi)$

$\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -l(l+1)\sin^2(\theta)Y$

$= \left(\frac{1}{\Omega}\left[\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Omega}{\partial\theta}\right)\right] + l(l+1)\sin^2(\theta)\right) + \frac{1}{\Lambda}\frac{\partial^2}{\partial\phi^2}\Lambda = 0$

Say the first part equals m^2 and the second part equals $-m^2$.

$\frac{1}{\Lambda}\frac{\partial^2\Lambda}{\partial\phi^2} = -m^2 \Rightarrow \Lambda(\phi) = e^{im\phi} \Rightarrow \Lambda(\phi) \stackrel{!}{=} \Lambda(\phi + 2\pi) \Rightarrow m \in \mathbb{Z}!!!!$

We also figured out earlier than $m = -l, -l+1, \ldots, l-1, l$ and l doesn't have to be an integer (can be integer + 1/2).

in principle, Angular momentum allows half integer. but for **orbital** angular momentum, only integers are allowed.

8.9.2 $\Omega(\theta)$ equation

$$\sin(\theta)\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Omega}{\partial\theta}\right) + \left(l(l+1)\sin^2(\theta) - m^2\right)\Omega = 0$$

You get the Legendre Polynomials (remember from PHYS435). You will not have to derive it thank god.

$\Omega(\theta) = AP_l^m(\cos(\theta))$

$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\left(\frac{\partial}{\partial x}\right)^m P_l(x) \qquad P_l(x) = \frac{1}{2^l l!}\left(\frac{\partial}{\partial x}\right)^l (x^2-1)^l$

With normalization, you get

$Y_l^m(\theta,\phi) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}e^{im\phi}P_l^m(\cos(\theta))$

This is known as "Spherical Harmonics"

$\left\langle Y_l^m \middle| Y_{l'}^{m'}\right\rangle = \delta_{ll'}\delta_{mm'}$

l and m have to equal each other.

8.10 Summary

$V = V(r)$ leads to

$$L^2 Y(\theta, \phi) = \hbar^2 l(l+1) Y(\theta, \phi) \qquad L_z Y(\theta, \phi) = \hbar m Y(\theta, \phi)$$

Where l is the total angular momentum and m is something. Imagine the Griffiths sphere.

8.11 Discussion

8.11.1 MIDTERM 1 KNOW EVERYTHING ABOUT THE QUANTUM HARMONIC OSCILLATOR

8.11.2 3D QM

a 1D operator is now a 3D operator.

$$\hat{p}\psi = \begin{bmatrix} \partial_x \psi \\ \partial_y \psi \\ \partial_z \psi \end{bmatrix} = 3D \qquad p^2 = \hat{p} \cdot \hat{p} = 1D \qquad \hat{p} \cdot \hat{p} \neq \hat{p}\hat{p}$$

8.11.3 Degeneracy

Different Quantum States can have the same energy.

The degeneracy of E_1 is 3 because each of $n_{x,y,z}$ can have $n = 1$. The degeneracy of E_2 is 6 because of permutation shenanigans.

8.11.4 New Quantum Numbers

$n =$ principle quantum number (energy level)

$l =$ angular momentum number. $0 < l < n$, so n bounds l

$m =$ angular momentum z-component. $m \in [-l, l]$ because it is essentially just the z projection of $\vec{L}$.

$|L| = \sqrt{l(l+1)}\hbar$

8.11.5 new notation

$|l, m\rangle$ is a vector that tells you about your conserved angular momentum properties.

$$L_+ |l, m\rangle = C_+ |l, m+1\rangle \qquad L_- |l, m\rangle = C_- |l, m-1\rangle$$

When the angular momentum gets "raised" or "lowered", the energy level **does not change**, which means angular momentum magnitude also doesn't change.

8.11.6 Commutator Trick

Taking the commutator of 2 operators.

$$[\hat{A}, \hat{B}]f = (\ldots)\, f \qquad \ldots = [A, \hat{B}]$$

ALWAYS USE A TEST FUNCTION

8.11.7 Questions

Confirm the 3D Ehrenfest's Theorem

$$\frac{d}{dt}\langle \hat{p} \rangle = \langle -\nabla V \rangle$$

Where V is potential is not velocity.
I think we solved the expected value of momentum and

$$\nabla V(r) = \frac{1}{r}\sin(\theta) asdfjaisfjasdfjasfadsfadksp[fkasdjfasdfnasdfnsadkl$$

Show that it's all 0 and you're chilling.

8.11.8 Uncertainty Principle

Formula Heisenberg's uncertainty principle, showing that there actually is not a correlation between dimensions.

You jus yoi jsut do ou tuju standing

8.11.9 $|l, m\rangle$

Show that $L_+ |l, m\rangle$ is an eigenstate of L_z with eigenvalue $\hbar(m+1)$.

$$L^2 L_+ |l, m\rangle = L^2 (L_x + iL_y) |l, m\rangle$$

8.12 Recap

You do a bunch of absolutely horrendous spherical coordinates math to get a radial equation and an angular equation.

$$L^2 Y(\theta,\phi) = \hbar l(l+1) Y(\theta,\phi)$$

8.13 Solving the Radial Equation

Let $u(r) = rR(r)$, so then our equation can be written as

$$\frac{dR}{dr} = \left(r\left(\frac{du}{dr}\right) - u\right)\frac{1}{r^2} \Rightarrow$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

We can reduce the whole potential to just a $V_{\text{effective}}$

8.13.1 Spherical Infinite Square Well

$$V = \begin{cases} 0 : r \leq a \\ \infty : r > a \end{cases}$$

Inside the well, the SWE can be written as

$$\frac{d^2u}{dr^2} = \left[\frac{l(l+1)}{r^2} - k^2\right]u \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

With boundary condition $u(a) = 0$
for $l = 0$

$$\frac{d^2u}{dr^2} = k^2 u \Rightarrow u = A\sin(kr) + B\cos(kr) \Rightarrow A\sin(kr) \qquad ka = \pi N$$
$$R = A\frac{\sin(kr)}{r} + B\frac{\cos(kr)}{r} \Rightarrow A\frac{\sin(kr)}{r}$$

The cosine disappears because the limit of cos(r)/r means that $B = 0$

8.13.2 $l > 0$

Look in the big book of functions and see that spherical harmonic functions solve this equation

$$u(r) = A \cdot rj_l(kr) + Bru_l(kr)$$

$$j_l(x) = (-x)^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \frac{\sin(x)}{x} = \text{Bessel Function}$$

$$u_l(x) = \text{Spherical Neumann Function}$$

Bessel functions just look like a damped oscillator
The entire wave function looks like

$$\psi(r, \theta, \phi) = A_{nl} j_l \left(B\right) asfasdkjlfjadfghadflkhadsj$$

It looks like a bunch of step functions but with slopes between them.
n is the principle quantum number (indexes energy)

8.14 MIDTERM

Everything up to HW9 - Discussion 8

Don't worry about spherical harmonics that much. Know what's up with the angular momentum operator. Know $\hat{L}$, but not $Y(\theta, \phi)$.

The harmonic oscillator will be the most important part of the midterm.

8.15 The Hydrogen Atom

You have a coulomb potential $V(r) = \frac{-e^2}{4\pi\epsilon_0}\frac{1}{r}$

The radial equation is of the form ($R = u/r$)

$$-\frac{\hbar^2}{2m}\frac{\partial u^2}{\partial r^2} + \left[-\frac{e^2}{4\pi\epsilon_0}\frac{1}{r} + \frac{\hbar^2}{2m_e}\frac{l(l+1)}{r^2}\right] u = Eu \qquad \psi = R(r)Y_l^m(\theta, \phi)$$

You get a nice central potential with a central potential well like we saw in classical mechanics and E&M a million times over. Everything in [] is the effective potential, or V_{eff}

Let's clean up the function

$$\kappa \equiv \frac{\sqrt{-2m_e E}}{\hbar} \Rightarrow \frac{1}{\kappa^2}\frac{d^2u}{dr^2} = \left[1 - \frac{m_e e^2}{2\pi\epsilon_0\hbar^2\kappa}\frac{1}{\kappa r} + \frac{l(l+1)}{(\kappa r^2)}\right]\kappa$$

$$\rho = \kappa \cdot r \qquad \rho_0 = \frac{m_e e^2}{2\pi\epsilon_0\hbar^2\kappa} \Rightarrow \frac{d^2u}{d\rho^2} = \left[1 - \frac{\rho_0}{\rho} + \frac{l(l+1)}{\rho^2}\right] u$$

This is absurd to solve analytically so we can instead observe the behavior at the edges.

8.15.1 Asymptotic Behavior

$$\lim_{\rho\to\infty} \Rightarrow \frac{d^2u}{d\rho^2} = u \Rightarrow u(\rho) = Ae^{-\rho} + Be^{\rho} = Ae^{-\rho}$$
$$\lim_{\rho\to 0} \Rightarrow \frac{d^2u}{d\rho^2} = \frac{l(l+1)}{\rho^2}u \Rightarrow u(\rho) = C\rho^{l+1} + D\rho^{-l} = C\rho^{l+1}$$

Remove terms that lead to the equation being non-normalizable.

This means that our ansatz given the asymptotes can be written as

$$u(\rho) = \rho^{l+1}e^{-\rho}v(\rho) \qquad \frac{du}{d\rho} = \rho^l e^{-\rho}\left((l+1-\rho)v(\rho) + \rho\frac{dv}{d\rho}\right)$$
$$\frac{d^2u}{d\rho^2} = \rho^l e^{-\rho}\left[\left(-2l-2+\rho+\frac{l(l+1)}{\rho}\right)v(\rho) + 2(l+1-\rho)\frac{dv}{d\rho} + \rho\frac{d^2v}{d\rho^2}\right]$$

The way to solve this is by finding a recursive formula such that

$$\rho\frac{d^2v}{d\rho^2} + \rho^l e^{-\rho}\left[\left(-2l-2+\rho+\frac{l(l+1)}{\rho}\right)v(\rho) + 2(l+1-\rho)\frac{dv}{d\rho}\right] = 0$$
$$jfaklsdjfksadlfasklfklasdjflska;jfaldskjflksda$$

Develop $v(\rho)$ as a power series

$$v(\rho) = \sum_{j=0}^{\infty} c_j\rho^j \qquad \frac{dv(\rho)}{d\rho} = \sum_{j=0}^{\infty} jc_j\rho^{j-1} = \sum_{j=0}^{\infty}(j+1)c_{j+1}\rho^j$$
$$\frac{d^2v}{d\rho^2} = \sum_{j=0}^{\infty} j(j+1)c_{j+1}\rho^{j-1}$$

Insert that into the original equation to get

$$\sum_{j=0}^{\infty} j(j+1)c_{j+1}\rho^j + 2(l+1)\sum_{j=0}^{\infty}(j+1)c_{j+1}\rho^j$$
$$-2\sum_{j=0}^{\infty} jc_j\rho^j + [\rho_0 - 2(l+1)]\sum_{j=0}^{\infty} c_j\rho^j = 0$$

Now that we have this giant gross equation, it has to be true that we get an equation for **each** power of ρ

$$j(j+1)c_{j+1} + 2(l+1)(j+1)c_{j+1} - 2jc_j + [\rho_0 - 2(l+1)] \sum_{j=0}^{\infty} c_j = 0$$

We now have a relation of c_{j+1} to c_j

$$c_{j+1} = \frac{2(j+l+1) - \rho_0}{(j+1)(j+2l+2)} c_j$$

Now we have to find some end to the series.

$$\text{large } \rho \to u(\rho) = e^{-\rho} \qquad \text{large } j \to c_{j+1} \approx \frac{2}{j+1} c_j \Rightarrow c_j = \frac{2j}{j!} c_0$$

But we know this cant be true for all j because it leads to an incorrect answer

$$v(\rho) = c_0 \sum_j \frac{2j}{j!} \rho^j = c_0 e^{2\rho} \Rightarrow u(\rho) = c_0 \rho^{l+1} e^{\rho} \neq e^{-\rho}$$

8.15.2 I zoned out whoops

The series has to terminate somewhere

$$c_{N-1} \neq 0 : c_N = 0$$
$$2(N+l) - \rho_0 = 0 \qquad N = n + l$$
$$\rho_0 = 2n \Rightarrow E = -\frac{\hbar^2 \kappa^2}{2m_e} = -\frac{m_e e^4}{8\pi^2 \epsilon_0^2 \hbar^2 \rho_0^2} \qquad E_n = -\left[\frac{m_e}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2} = \frac{E_1}{n^2}$$
$$E_1 = -13.6meV$$

Bohr actually found this result (Bohr Model 1913) without quantum mechanics which is kind of insane if you think about it.

N control the series termination and it controls the amount of nodes in the radial equation (similar to in other models with n).

If we observe earlier definitions

$$\kappa = \left(\frac{m_e e^2}{4\pi\epsilon_0 \hbar^2}\right)\frac{1}{n} = \frac{1}{an} \qquad a \equiv \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.53 \cdot 10^{-10} m$$
$$\rho = \frac{r}{an} \qquad \psi_{nlm}(r,\theta,\phi) = R_{nl}(r) Y_l^m(\theta,\phi) \qquad R_{nl} = \frac{1}{r}\rho^{l+1} e^{-\rho} v(\rho)$$

Where v is a polynomial of degree $(n-l-1)$ and c_j is known from the recursion formula.

The wave function now fits in the neat potential well who would have guessed

$$\psi_{1,0,0} = R_{1,0}(r)Y_0^0(\theta,\phi) \qquad R_{1,0}(r) = \frac{c_0}{a}e^{-r/a} \qquad v(\rho) = c_0 = \text{const}$$

$$\int_0^\infty dr\,|R_{1,0}|^2 r^2 = 1 \to c_0 = \frac{2}{\sqrt{a}} Y_0^0 = \frac{1}{\sqrt{4\pi}}$$

$$\psi_{1,0,0}(r,\theta,\phi) = \frac{1}{\sqrt{4a^3}}e^{-r/a}$$

$v(\rho) = c_0$ is found from the recursive formula.

8.15.3 Importance

We now have a nice graph of energy levels related to quantum number that I have definitely seen before.

As N increases, degeneracy also goes up.

$$d(n) = \sum_{l=0}^{n-1}(2l+1) = n^2$$

If I add anything more than 13.6meV when its in the lowest state the hydrogen atom will just be ionized.

This is as far as you can get with analytical math in quantum mechanics. Everything else involves approximations and computation.

8.16 Discussion

8.16.1 Potentials You Know

- free particle $= 0$
- infinite square is 0 or infinite
- oscillator is $\frac{1}{2}m\omega^2x^2$
- Now we study the hydrogen atom $V = \frac{1}{r}$

The SWE can be solved for the H atom and nothing else

$$\hat{H} = -\frac{\hbar^2}{2\mu}\nabla^2 - \frac{Ze^2}{4\pi\epsilon_0 r}$$

And it has solution that can be derived from separation of variables

$$\psi_{nlm}(r,\theta,\phi) = R_n(r)Y_{l,m}(\theta,\phi)$$

m gives us the z-projection of the angular momentum quantum number which is given to us by l. n gives us energy, and n is independent from l and m.

8.16.2 Normalizing in Space

$$1 = \int_0^\infty dr\,|R(r)|^2 r^2 \int_0^{2\pi}\int_0^{\pi} d\theta\, d\phi |Y(\theta,\phi)|^2 \sin(\theta)$$

8.16.3 Spherical Harmonics

These are a class of well behaved functions that define our eigenstates

$$\psi_{nlm} = |n,m,l\rangle$$

And l and m are coupled. m has bounds $\pm l$ in unit steps.

8.16.4 Chemical Orbitals

This is the consequence of spherical harmonics in quantum shenanigans.

The orbital images are surfaces of equal probability density. (Technically, electrons are in a superposition of a bunch of the lowest states, but if we observe all of them, we get the chemists POV where they take up the lowest states in a neat ordered shelf).

8.16.5 New Eigenvalues

We have $\hat{L}_z$ operator, which projects the angular momentum to the z-axis.

$$\hat{L}_z\,|l,m\rangle = \hbar m\,|l,m\rangle \qquad L^2\,|l,m\rangle = l(l+1)\,|l,m\rangle \qquad Y_l^m$$

L is on the bottom and m is on the top.

8.16.6 Questions

$$-\frac{\hbar^2}{2m}\nabla^2\psi+\frac{1}{2}m\omega^2r^2\psi=E\psi\Rightarrow$$
$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+\frac{1}{2}m\omega^2(x^2+y^2+z^2)\psi=E\psi\Rightarrow$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}$$
$$=\frac{1}{2}m\omega^2(x^2+E_x)\psi+\frac{1}{2}m\omega^2(y^2+E_y)\psi+\frac{1}{2}m\omega^2(z^2+E_z)\psi$$
$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}=\frac{1}{2}m\omega^2(x^2+E_x)\Rightarrow\frac{\partial^2}{\partial x^2}=-\frac{1}{2}m\omega^2\frac{2m}{\hbar^2}(x^2+E_x)$$

This is just the 1d harmonic oscillator 3 times.
Find the degeneracy

$$E=\hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right)$$

The degeneracy you just get from stars and bars.
Show that L_x is conserved.

$$[\hat{H},L_x]=[\hat{H},yp_x]+[\hat{H},xp_x]$$

The previous part implied that the eigenstates of the 3d SHO.
L_x and L_y are different and not gonna also be eigenstates of L_z

8.16.7 Other Question

You have a rigid rotor in a magnetic field
If energy is measured, what are the possible results?

$$\frac{L^2}{2I}\psi+\omega_0L_z\psi=E\phi \qquad L^2\psi=\hbar^2l(l+1)\psi \qquad L_z\psi=\hbar m\psi$$
$$\frac{\hbar^2l(l+1)}{2I}\psi+\omega_0\hbar m\psi=E\psi$$
$$\sqrt{\frac{3}{4\pi}}\sin(\theta)\sin(\phi)=\sqrt{\frac{3}{4\pi}}\sin(\theta)\left(\frac{e^{i\psi}-e^{-i\psi}}{2i}\right)$$

You turn the thing into 2 eigenstates.

8.17 MIDTERM 2 REVIEW

Commutators $[\hat{A}, \hat{B}] = 0$ can be simultaneously diagonalized.

But there is a certain amount of uncertainty in that.

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{\langle [\hat{A}, \hat{B}] \rangle}{2i} \right)^2$$

A quantity is conserved if $[\hat{H}, \hat{A}] = 0$

The generalized Ehrenfest Theorem is

$$\frac{d}{dt}\langle A \rangle = [\hat{H}, \hat{A}] + \langle \frac{dA}{dt} \rangle$$

Where the 2nd term is typically 0. The G.E.T. also leads to energy-time uncertainty ($\Delta E \Delta t$).

Basically, given some observable, you'll probably have to figure out the uncertainty and if the observable is conserved.

8.17.1 Simple Harmonic Oscillator

Just a wave function with a potential $V(x) = \frac{1}{2}m\omega^2 x^2$

$$H = \hbar\omega \left(a_+ a_- + \frac{1}{2} \right) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \qquad [a_-, a_+] = 1$$

$$a\_0 = 0 \qquad a_+ \left| n \right\rangle = \sqrt{n+1} \left| n+1 \right\rangle \qquad a_- \left| n \right\rangle = \sqrt{n} \left| n-1 \right\rangle \qquad \left\langle n \right| \hat{A} \left| n \right\rangle =$$

The stationary states are some sort of special polynomial with a fancy name, but that won't come up on the midterm.

You give us an observable made of +'s and -'s and a wave function made of stationary states and you have to calculate something. There is also time evolution but the time evolution is very simple.

8.17.2 Angular Momentum

Given some angular momentum l, there is basically a ladder for the quantum number m that spans from $+l$ to $-l$.

$$L_\pm = L_x + iL_y$$

I'm not 100% sure that this is true, double check the angulular momentum math.

For the radial equation/3D harmonic oscillator, just solve for each dimension.

8.17.3 Know 3D QM

8.17.4 Know Coherent States

8.17.5 Know Pauli Matrices

Chapter 9

Spin

Classically, orbiting things have angular momentum $\vec{L} = \vec{r} \times \vec{p}$ and spinning things also have that $\vec{s} = \vec{I} \cdot \vec{\omega}$

Quantum mechanical particles have integer spin

- Boson: integer spin (photons have s=1)
- Fermions: Integer + 1/2

Spin is comparable to orbital angular momentum

$$\vec{s} = \begin{bmatrix} s_x \\ s_y \\ s_z \end{bmatrix} \qquad [\hat{s}_i, \hat{s}_j] = i\hbar\hat{s}_k$$

And we have ladder operators for spin the same way as angular momentum

$$\hat{s}^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle \qquad \hat{s}_z = asfjklagjkoa; djfklsdajf;$$

9.1 Spin 1/2

The eigenstates of $\hat{s}_z$ are known as spinors

$$|0\rangle \equiv \chi_+ \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle \equiv |\uparrow\rangle \qquad |1\rangle \equiv \chi_- \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \equiv \left|\frac{1}{2}, \frac{1}{2}\right\rangle \equiv |\downarrow\rangle$$

And the eigenvalues of these states can be written as

$$\hat{s}^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \qquad \hat{s}^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

$$\hat{s}^2 = \frac{3}{4}\hbar^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{3}{4}\hbar^2 \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$\hat{s}_z\chi_+ = \frac{\hbar}{2}\chi_+ \qquad \hat{s}_z\chi_- = -\frac{\hbar}{2}\chi_-$$

9.1.1 Time Evolution

μ rotates under an external magnetic field $\vec{B}$

There's a gyromagnetic ratio $\vec{\mu} = \gamma \cdot \vec{L}$. So spin in a magnetic field experiences:

$$\dot{\vec{L}} = \vec{\mu} \times \vec{B} \Rightarrow \dot{\vec{\mu}} = \gamma \vec{\mu} \times \vec{B}$$

9.2 Hamiltonian

This can be derived with some shenanigans

$$H = \vec{\mu} \cdot \vec{B} \qquad \vec{\mu} = \gamma \vec{s}$$

9.2.1 Example

$$\vec{B} = B_0 \hat{z} \qquad \hat{H} = -\gamma \vec{B} \cdot \vec{s} = -\frac{\gamma B_0 \hbar}{2} \hat{\sigma}_z$$

σ is a pauli matrix

idk something something I'm texting my family

At $t = 0$

$$\alpha \left|0\right\rangle + \beta \left|1\right\rangle = \cos\left(\frac{\theta}{2}\right) \left|0\right\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi} \left|1\right\rangle$$

$$\chi(t) = \alpha \chi_+ e^{-i\frac{E_+}{\hbar}t} + \beta \chi_- e^{-i\frac{E_-}{\hbar}t}$$

$$\langle s_x \rangle = \left\langle \chi(t) \right| s_x \left| \chi(t) \right\rangle = \frac{\hbar}{2} \sin(\theta) \cos(\gamma B_0 t)$$

$\omega = \gamma B_0$ is the Larmor frequency or something.

9.2.2 Bloch Spheres

They can be used to represent spin but I can't draw in this notebook.

Magnetic resonance is just making spin states go from $\left|\uparrow\right\rangle$ to $\left|\downarrow\right\rangle$

9.3 Recap

Spin is a vector with a commutator that acts like Pauli matrices and there's also a ladder function that goes with it

$$\hat{s}^2 \left|s, m\right\rangle = \hbar^2 s(s+1) \left|s, m\right\rangle \qquad \hat{s}_z \left|s, m\right\rangle = \hbar m \left|s, m\right\rangle$$
$$s_\pm \left|s, m\right\rangle = \hbar\sqrt{s(s+1) - m(m \pm 1)} \left|s, m \pm 1\right\rangle$$

For spin 1/2 particles, the operator is

$$\vec{s} = \frac{\hbar}{2}\vec{\sigma} = \frac{\hbar}{2}\begin{bmatrix}\sigma_x \\ \sigma_y \\ \sigma_z\end{bmatrix}$$

The general state of a spin particle can be written as

$$\chi = \alpha\chi_+ + \beta\chi_-$$

9.4 1/2 Spin in B Field

$$\hat{H} = -\gamma\vec{B} \cdot \vec{s} = -\frac{\gamma B_0 \hbar}{2}\hat{\sigma}_z$$

9.5 Stern-Gerlach

$$\vec{F} = \vec{\nabla}(\vec{\mu} \cdot \vec{B}) \qquad \chi = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle + \left|\downarrow\right\rangle)$$

You just use a magnetic field, and when the particle is the wrong spin it gets pushed away.

$$\left|\pm\right\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle \pm \left|\downarrow\right\rangle)$$

$\left|x, +\right\rangle$ means that 1st one is direction, and the 2nd is spin.

9.5.1 Angular Momentum

Consider 2 separate spin 1/2 particles. We have new operators

$$\hat{S}^{(j)}_{x,y,z} \qquad \hat{S}^{(j)^2}$$

You can take the composite of 2 states

$$|s_1, s_2, m_1, m_2\rangle \equiv |s_1, m_1\rangle \otimes |s_2, m_2\rangle$$

Where $\otimes$ is the kronecker product.

You take all the operators of all the things and get either $s(s+1)$ or $\hbar m$.

To measure the total angular momentum, you use a new operator

$$\hat{s} = \hat{s}^{(1)} + \hat{s}^{(2)}$$

What happens if we measure $\hat{s}_z$?

$$\hat{s}_z |s_1, s_2, m_1, m_2\rangle = s^{(1)}_z |s_1, s_2, m_1, m_2\rangle + s^{(2)}_z |s_1, s_2, m_1, m_2\rangle$$
$$= \hbar(m_1 + m_2) |s_1, s_2, m_1, m_2\rangle \qquad |\uparrow\uparrow\rangle \left|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle \to m = 1$$
$$|\uparrow\downarrow\rangle \to m = 0 \qquad |\downarrow\uparrow\rangle \to m = 0 \qquad |\downarrow\downarrow\rangle \to m = -1$$

This doesn't work because we now have 2 separate states with the same quantum number.

9.5.2 Joint Ladder Operators

the top state is $|\uparrow\uparrow\rangle$, and if we apply the lowering operator to it, we get

$$s_- |\uparrow\uparrow\rangle = \left(s^{(1)}_- |\uparrow\rangle\right) |\uparrow\rangle + |\uparrow\rangle \left(s^{(2)}_- |\uparrow\rangle\right) = \hbar |\downarrow\rangle |\uparrow\rangle + \hbar |\uparrow\rangle |\downarrow\rangle$$
$$s_- |10\rangle = 2\hbar |\downarrow\downarrow\rangle \equiv |1, -1\rangle$$

And then you normalize it

$$s = 1 : \Big\{ dfhdaf; jfjkadsljflsdj$$

We have a missing basis vector that is

$$\frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) \qquad s=0 \qquad m=0$$

$$\hat{s}^2\frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = ?$$

$$\hat{s}^2 = s^{(1)^2} + s^{(2)^2} + 2s^{(1)}s^{(2)}$$

$$s^{(1)^2}\left|\psi_[-]\right\rangle = nadfjkladjfkadsnfjkls$$

$$s^{(1)}\cdot s^{(2)}\left|\psi_-\right\rangle = -\frac{3\hbar^2}{\psi}\left|\psi_-\right\rangle$$

$$\hat{s}^2\left|\psi\right\rangle = 0 \Rightarrow \frac{1}{\sqrt{2}}\left(\left|\uparrow\downarrow\right\rangle - \left|\downarrow\uparrow\right\rangle\right) = \left|00\right\rangle$$

9.6 Discussion

Write down the matrices for S_+ and S_-. Write

9.7 Identical Particles

a 2 electron system is not the state as 2 combined 1-electron systems.

The hamiltonian of the system is

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2)$$

We can use separation of variables

$$V(\vec{r}_1, \vec{r}_2) = V_1(\vec{r}_1) + V_2(\vec{r}_2)$$

Now we can make an ansatz for the wave function

$$\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \qquad H_{1,2} = -\frac{\hbar^2}{2m_{1/2}}\nabla_{1/2}^2 + V_{1/2}(\vec{r}_{1/2})$$

$$H_{1/2}\psi_{a//b} = E_{a/b}\psi_{a/b}H_{tot} = H_1 + H_2 \qquad E_{tot} = E_a + E_b$$

$$\psi = \psi_a\psi_b = e^{-iE_a/\hbar t}e^{-iE_b/\hbar t}$$

We can have solutions that are linear combinations

$$\psi = A\psi_a(\vec{r}_1)\psi_c(\vec{r}_2) + B\psi_b(\vec{r}_1)\psi_d(\vec{r}_2)$$

9.8 Bosons are Fermions

There are two ways to do something in 3D

$$\psi_{\pm}(\vec{r}_1, \vec{r}_2) = A\left(\psi_a(\vec{r}_1)\psi_b(\vec{r}_2) \pm \psi_a(\vec{r}_2)\psi_b(\vec{r}_1)\right)$$

+ is Bosons and - is Fermions.

9.8.1 Particle Exchange

Bosons are symmetric and need exchange and Fermions are anti-symmetric

$$\psi_+(\vec{r}_1, \vec{r}_2) = \psi_+(\vec{r}_2, \vec{r}_1) \qquad \psi_-(\vec{r}_1, \vec{r}_2) = -\psi_-(\vec{r}_2, \vec{r}_1)$$

Consider 2 particles in the same state

$$\psi_+(\vec{r}_1, \vec{r}_2) = (\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) + \psi_a(\vec{r}_2)\psi_a(\vec{r}_1)) = 2A\psi_a(\vec{r}_2)\psi_a(\vec{r}_1)$$
$$\psi_-(\vec{r}_1, \vec{r}_2) = (\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_2)\psi_a(\vec{r}_1)) = 0$$

Pauli exclusion principle! Fermions can't be in the same quantum state.

9.9 2 Particles in a Well

If we imagine indistinguishable bosons, we take the sum of the energies from each particle being in each state.

If we imagine 2 indistinguishable Fermions.

$$\psi_{12} = \psi_a(x_1)\psi_b(x_2)$$

But if the fermions/bosons are distinguishable, then

$$\psi_{12} = A\left(\psi_a(x_1)\psi_b(x_2) \pm \psi_b(x_1)\psi_a(x_2)\right)$$

Consider spin, the total wave is

$$\psi = \psi(x)\chi \qquad \chi = \text{spinor}$$

For bosons:

$$\psi_+(\vec{r}_1, \vec{r}_2) = \psi_+(\vec{r}_2, \vec{r}_1)$$

For Fermions

$$\psi_-(\vec{r}_1, \vec{r}_2) = -\psi_-(\vec{r}_2, \vec{r}_1)$$

We can do some math shenanigans with Bosons

$$\psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2) = |a\rangle |b\rangle \Rightarrow \psi_B = \frac{1}{\sqrt{2}} (|a\rangle |b\rangle + |a\rangle |b\rangle)$$

and Fermions

$$\psi_F = \frac{1}{\sqrt{2}} (|a\rangle |b\rangle + |a\rangle |b\rangle)$$

We can consider a distance operator for distinguishable particles

$$\langle d^2\rangle = \langle (x_1 - x_2)^2\rangle = \langle x_1^2\rangle + \langle x_2^2\rangle + \langle x_1 x_2\rangle$$
$$\langle x_1^2\rangle = \langle a| x_1^2 |a\rangle \langle b|b\rangle = \langle x^2\rangle_a \qquad \langle x_2^2\rangle === \langle x^2\rangle_b$$
$$\langle x_1 x_2\rangle = \langle a| x_1 |a\rangle \langle b| x_2 |b\rangle = \langle x\rangle_a \langle x\rangle_b$$

And for Bosons/Fermions, we can calculate the same thing

$$\langle x_1^2\rangle = \frac{1}{2} [\langle a| x_1^2 |a\rangle \langle b|b\rangle + \langle b| x_1^2 |b\rangle \langle a|a\rangle \pm \langle a| x_1^2 |b\rangle \langle b|a\rangle \pm \langle b| x_1^2 |a\rangle \langle a|b\rangle]$$
$$= \frac{1}{2} [\langle x^2\rangle_a + \langle x^2\rangle_b] = \langle x_2^2\rangle$$
$$\langle x_1 x_2\rangle === \langle x\rangle_a \langle x\rangle_b \pm |\langle x\rangle_{ab}|^2 \qquad \langle x\rangle_{ab} = \langle a| x |b\rangle$$
$$\langle d^2\rangle = \langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a \langle x\rangle_b \mp 2|\langle x\rangle_{ab}|^2$$

This whole phenomenon is known as the Exchange Force

9.9.1 that's the end of final material, only 1 lecture left

9.10 DISCUSSION

Imagine a particle with spin. If spin=1, then there are 3 possible spin states.

Spin 1/2 is up and spin -1/2 is down.

For each spin value S, we have 2S + 1 spin states.

Consider an in state with up and down and out state with up and down. This gives us 4 possible input and output combinations, which gives us a spin matrix with 4 entries.

Consider 2 particles, each with an input and output, each with 2 possible states up and down.

Now we work in a new basis where

$$|1\rangle = |\uparrow\uparrow\rangle \qquad |2\rangle = |\uparrow\downarrow\rangle \qquad |3\rangle = |\downarrow\uparrow\rangle \qquad |4\rangle = |\downarrow\downarrow\rangle$$
$$S_{ij} = \langle i|\, S \,|j\rangle$$

This then means that S is a 4x4 matrix with 16 entries.

We can do some goofy tensor product shenanigans

$$S = S_1 \otimes S_2 \qquad S_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$
$$S = \begin{bmatrix} a_1 S_2 & b_1 S_2 \\ c_1 S_2 & d_1 S_2 \end{bmatrix}$$

This shows how S is a 4x4 because it's the tensor product of each particle which are 2x2 matrices.

We can't add two spins together because they live in different vector spaces for each particle

$$\vec{S}_1 + \vec{S}_2 = S_1 \otimes 1_2 + 1_2 \otimes S_2$$

Where 1_2 is the identity for the particle. Think of addition as 2 particles communicating with each other in different vector spaces.

9.10.1 Degeneracy

given a 2 particle system, the maximum total spin is $S_1 + S_2$. The minimum total spin magnitude is $|S_1 - S_2|$.

For example, if both spins are 1, then $0 < S < 2$, so there are 3 possible spin magnitudes. The degeneracy, however, is 2S+1 for each S, so its 1, 3, 5 for 0, 1, 2. This means the total degeneracy is 9.

9.10.2 Questions

Consider 2 spin 1/2 particles (2 possible states up and down)

Find the total spin S^2 and z component S_z in terms of the spin m operators for the individual particles (S_1, S_2).

$$\vec{S}_x = \frac{\hbar}{2}\sigma_z$$

The only possible values for total spin are 0 and 1. Because the state itself has spin states 0 and 1, the eigenvalues of that vector must also be 0 or 1.

$$\hat{S}\left|S_{1+2}\right\rangle = \lambda\left|S_{1+2}\right\rangle = 1\left|S_{1+2}\right\rangle \qquad 0\left|S_{1+2}\right\rangle$$
$$\hat{S}^2\left|S_{1+2}\right\rangle = \lambda^2\left|S_{1+2}\right\rangle = 1\left|S_{1+2}\right\rangle \qquad 0\left|S_{1+2}\right\rangle$$

The degeneracy comes from the total spin values. For $S^2 = 0$ there is only degeneracy 1 which leads to $S_z = 0$. For $S^2 = 1$ we have $2s+1 = 3$ degeneracy, which means $S_z = -1, 0, 1$. -1 is connected to $\left|\downarrow\downarrow\right\rangle$, you can correspond $S_z = 0$ to $\frac{1}{\sqrt{2}}(\left|\downarrow\uparrow\right\rangle + \left|\uparrow\downarrow\right\rangle)$.

Consider the hamiltonian for 2 interactive spin

$$\hat{H} = \alpha\hat{S}_1 \cdot \hat{S}_2$$

Use your answers from question 1 to show that

$$\hat{S}_1 \cdot \hat{S}_2 = \frac{1}{2}\left(\hat{S}^2 - \hat{S}_1^2 - \hat{S}_2^2\right)$$

Chapter 10

Extra Homeworks

10.1 Hw10

10.1.1 1

Let

$$L_+ f_l^m = A_l^m f_l^{m+1} \qquad L_- f_l^m = A_l^m f_l^{m-1}$$

What are A and B if the eigenfunctions f are normalized? I think we learned about this and it's just $m+1$ or whatever.

First show that $L_\pm$ is the Hermitian Conjugate of $L_\mp$ and then uh something. Since L_x and L_y are observables, you may assume they are Hermitian, but you can prove it if you want.

$$L_\pm = L_x \pm i L_y$$

Proof by obvious that it's Hermitian ($-i^* = i$). We use a specific identity.

$$\begin{aligned}
&L_\pm L_\mp = L^2 - L_z^2 \pm \hbar L_z \\
&\langle f_l^m | L^2 - L_z^2 \pm \hbar L_z | f_l^m \rangle = \langle f_l^m | L^2 | f_l^m \rangle - \langle f_l^m | L_z^2 | f_l^m \rangle \pm \langle f_l^m | \hbar L_z | f_l^m \rangle \\
&= \hbar^2 l(l+1) \langle f_l^m | f_l^m \rangle - \hbar^2 m^2 \langle f_l^m | f_l^m \rangle \pm \hbar^2 m \langle f_l^m | f_l^m \rangle \Rightarrow \\
&\langle L_\mp f_l^m | L_\mp f_l^m \rangle = \left(\hbar^2 l(l+1) - \hbar^2 m^2 \pm \hbar^2 m \right) \langle f_l^m | f_l^m \rangle \\
&= \hbar^2 \left(l(l+1) - m^2 \pm m \right) \langle f_l^m | f_l^m \rangle \\
&B = \hbar^2 \sqrt{l(l+1) - m^2 + m} \qquad A = \hbar^2 \sqrt{l(l+1) - m^2 - m}
\end{aligned}$$

The $\pm$ and $\mp$ are important, I fucked that up on last midterm as well lol.

At the top and bottom of the ladder you get 0 which makes sense but we actually see the math math out because $l = m$.

10.1.2 2

Do a hundred million commutation relations lmao.

$$L = \vec{r} \times \vec{p} = \hat{x}(yp_z - zp_y) - \hat{y}(xp_z - zp_x) + \hat{z}(xp_y - yp_x) \qquad L_z = L\hat{z}$$
$$[L_z, x] = (xp_y - yp_x)x - x(xp_y - yp_x) \qquad L_z = xp_y - yp_x =$$
$$[L_z, p_x]$$

Okay I can figure out the rest but I gotta know my commutation relations.

10.1.3 3

Get a rotational analog of Ehrenfest's theorem

$$\frac{d}{dt}\langle L\rangle = \langle N\rangle = \langle r \times (-\nabla V)\rangle$$

I have no clue how to solve this right off the bat

$$\frac{d}{dt}\langle L\rangle = \frac{d}{dt}\langle r \times p\rangle = \langle r \times \frac{d}{dt}p\rangle = \langle r \times -\nabla V\rangle$$

Show that angular momentum is conserved for any spherical potential (idk)

10.1.4 4

What is $L_+Y_+^+$? It's 0 lmao

Do some other stuff. Okay I feel better and I'll definitely be better after writing everything on a cheat sheet

10.2 Homework 12

10.2.1 1

Compute the expected values of $\langle S_{x,y,z}\rangle$ for a normalized spinor χ. Spinors are eigenfunctions of spin I think.

$$\chi = \begin{bmatrix} a \\ b \end{bmatrix} = a\chi_+ + b\chi_- = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$S_z\chi_+ = \frac{\hbar}{2}\chi$$

$$S_{xyz} i\hbar\sigma_{xyz}$$

This can be calculated by just knowing all of the matrices

10.2.2 2

An electron is in a magnetic field

$$B = B_0 \cos(\omega t)\hat{k}$$

Construct the Hamiltonian

$$H = -\gamma \vec{B} \cdot \vec{S} = -\gamma B_0 \cos(\omega t) S_z = E$$

Find $\chi(t)$
This can be done by solving the time dependent SWE directly.

$$\chi(t) = \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$$

$$H\chi = i\hbar \frac{\partial}{\partial t}\chi \Rightarrow \frac{-\gamma B_0 \hbar}{2} \cos(\omega(t)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix} = i\hbar \begin{pmatrix} \alpha'(t) \\ \beta'(t) \end{pmatrix}$$

Just solve the diff eq by separating the variables.
Find $p(S_x = \hbar/2)$
The two states $S_x = \pm\hbar/2$ correspond to eigenstates $|\pm\rangle$, so we specifically need to find the probability that the state collapses to state $|-\rangle$.
Find the necessary B to guarantee a spin flip. This is just math, if you solved the last part correctly then this is easy.

10.3 Hw 13

Write the Hamiltonian for 2 non-interacting particles in an infinite square well. Verify that the fermion ground state is an eigenfunction of H, and find the eigenvalue.
The fermion ground state is

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \qquad E_n = n^2 K$$

$$\psi_{n1,n2} = \psi_{n1}(x_1)\psi_{n2}(x_2) \qquad E_{n1,n2} = (n_1^2 + n_2^2)K$$

It's not actually that bad you can solve it with separation of variables.

10.3.1 1b

Find the next two excited states (beyond the ground state). Wave functions, energies, and degeneracies. For each of the three cases (distinguishable, identical bosons, identical fermions).

you just like go up in energy and keep thinking about the particle.

distinguishable particles will have non-zero degeneracy if the energies are different because you just swap the particles. This cannot be done for identical particles. Identical fermions (same spin) must follow the Pauli exclusion principle, and thus can't have the same energy.

You do a buncha bullshit

Chapter 11

FINAL PREP

l can only be an integer and S can be either an integer or integer + 1/2

- Know the SWE for time-independent and dependent lmao
- Know/derive every ensemble (square wells, free particle, oscillators, 2-level, radial, spin? etc)
- Know the quantum formula (energy eigenstates, time operator, etc)
- KNOW MATH (commutators, operators, observables, eigenvalues, eigenstates, pauli matrices, permutation symbol ϵ
- know eigenbasis transformations (you saw on homework)
- Ehrenfest theorem and inequalities and Heisenberg Uncertainty Principle
- Hermitian stuff
- OPERATORS (momentum, S, L, S_{xyz}, L_{xyz}, L^2, S^2, etc)
- Separation of variables
- degeneracy
- angular momentum
- Bosons vs Fermions lmao
- multiple particles? (yea just in case)

11.1 Multi-Particle Stuff

Electrons are indistinguishable, so we do some bullshit to show this off

$$\psi_{\pm}(r_1, r_2) = A\left(\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\right)$$

The plus sign is bosons and the minus sign is fermions. You can clearly see the pauli exclusion principle because if 2 fermions are identical, your wave function is just 0.

Bosons are symmetric under interchange, and fermions are antisymmetric under interchange. So bosons are integer spin and fermions are half integer spin.