MATH 285

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Contents

Chapter 1

Intro to Differential Equations

It's an equation that contains derivatives of some functions instead of just regular functions

1.0.1 Ordinary Differential Equations (ODE's)

Does not contain directional derivatives

1.0.2 Partial Differential Equations (PDE's)

contains partial derivatives.

1.0.3 System of Differential Equations

self explanatory

1.0.4 Order of a Differential Equation

Number of the highest derivative

$y'' + y' = y$ is a 2nd order differential equation because of the y"

1.0.5 linear

the unknown and its derivatives appear linearly

$$y'' + ty' = y$$

1.0.6 non-linear

not linear

$$y'' + \sqrt{y} = t^2 \qquad y'' + yy' = t$$

1.1 Linear Nth Order ODE

$$a_n(t)y^n a_{n-1}(t)y^{n-1} + \ldots + a_1(t)y' + a_0(t)y = f(t)$$

If $f(t) = 0$, then the equation is homogeneous, and it is inhomogeneous if otherwise.

$y(t)$ is a solution to the equation if all the necessary derivatives of y exist and satisfy the equation.

$$y' = 3y \Longrightarrow y = Ce^{3t} \qquad \text{C is called the free parameter}$$

$$\begin{cases} y' = 3y \\ y(0) = 5 \end{cases} \quad \text{this is called an initial condition}$$

1.1.1 test

$$y' = ay \qquad y'' + a^2y = 0 \qquad y'' - a^2y = 0$$

have the solutions

$$y = A\cos(at) + B\sin(at)$$

I didn't write it in time but im sure something interesting was there

1.2 Integration

The easiest way to solve some differential equations is just integrate it and that'll give you some variety of answer.

$$\frac{\partial y}{\partial t} = t^2 \Longrightarrow y(t) = \frac{t^3}{3} + A$$

1.2.1 Initial Conditions

$$y''(t) + 2y'(t) + y(t) + 3 \Rightarrow y''(t) = -2y'(t) - y(t) - 3 \Rightarrow$$
$$y''(0) = -2y'(0) - y(t) - 3$$

1.2.2 Separation of Variables

$$\frac{\partial y}{\partial t} = \frac{t}{y} \Rightarrow y\partial y = t\partial t \Rightarrow \int y\,\partial y = \int t\,\partial t \Rightarrow \frac{y^2}{2} = \frac{t^2}{2} \Rightarrow y^2 = t^2 + A$$

implicit solution

$$y = \pm\sqrt{t^2 + A} \qquad \text{explicit solution}$$

1.2.3 $\frac{dy}{dt} = f(t, y) = g(t) \cdot h(y)$

$$\int \frac{1}{h(y)}\, dy = \int g(t)\, dt$$

$$\frac{dy}{dt} = \frac{3y^2}{t^4} \Rightarrow \int \frac{1}{y^2}\, dy = \int \frac{3}{t^4}\, dt \Rightarrow \frac{-1}{y} = \frac{-1}{t^3} + A \Rightarrow y = \frac{t^3}{1 + At^3}$$

Thats the general solution. You also have to consider the singular solution $y(t) = 0$

1.2.4 Partial Fraction something

$$\frac{dp}{dt} = 2P - P^2 = P(2-P) \Rightarrow \frac{dP}{P(2-P)} = dt$$

$$\frac{1}{P(2-P)} = \frac{A}{P} + \frac{B}{2-P} \Rightarrow \frac{2A + (B-A)P}{P(2-P)}$$

you can solve for A and B

$$\int \left(\frac{0.5}{P} + \frac{0.5}{2-P} \right) dP = \int dt$$

solve??????

1.3 General and Explicit Solutions

idk just solve the way you normally do it seems to work well enough

1.3.1 Autonomous Differential Equation

In the form $y' = f(y)$

1.4 Slope Fields

You know what a slope field is.

1.5 Existence and Uniqueness Theorem

Let

$$y' = f(y,t) \qquad y(t_0) = y_0$$

The theorem states that if $f(yt)$ is continuous around the neighborhood of (y_0, t_0), then there exists a solution to the differential equation around (y_0, t_0), and if $\frac{d}{dy} f(y,t)$ is continuous around the neighborhood of (y_0, t_0), then there exists a unique solution to the differential equation.

Also known as the Picard Theorem

$$\exists S : \exists [a_0, b_0] \in S : \lim_{a \to a_0, b \to b_0} f(a,b) = f(a_0, b_0)$$

$$\implies \text{there exists a solution to the differential equation}$$

$$\lim_{a \to a_0, b \to b_0} \frac{df}{dt}(a,b) = \frac{df}{dt}(a_0, b_0) \implies \text{there exists a unique solution}$$

1.6 Picard's Iterations

$$\int_a^t \frac{dy}{dt} = \int_a^t f(t, y(t))\, dt$$

$$y(t) - y(a) = \int_a^t f(s, y(s))\, ds \qquad a = h$$

$$y(t) = b + \int_a^t f(s, y(s))\, ds$$

Use successive approximation to find an answer

$$y_0(s) = b \qquad y_1(t) = b+\int_a^t f(s, y_0(t))\, ds \qquad y_2(t) = b+\int_a^t f(s, y_1(t))\, ds$$

take the limit and get

$$y_\infty(t) = b + \int_a^t f(s, y_\infty(t))\, ds$$

if y converges, then it converges to the original equation.

1.6.1 Example

$$\frac{dy}{dt} = 2(y+1) \qquad y(0) = 0$$

$$y(t) = 0 + \int_0^t 2(y+1)\, ds$$

$$y_1 = 2(0+1)\, ds = 2t \qquad y_2 = 2(2s+1)\, ds = 2t^2 + 2t$$

$$y_3 = 2(2t^2 + 2t + 1)\,ds = \frac{4t^3}{3} + 2t^2 + 2t$$

$$y_N(t) = 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \ldots + \frac{(2t)^N}{N!} =$$

$$\sum_{k=1}^{N} \frac{(2t)^k}{k!} = -1 + 1 + \sum_{k=1}^{N} \frac{(2t)^k}{k!} = -1 + \sum_{k=0}^{N} \frac{(2t)^k}{k!} = -1 + e^{2t}$$

will maybe need to find first 2 y's on a test.

1.7 Solving First Order Linear Equations

$$y' + P(t)y = Q(t) \qquad y(t_0) = 0$$

1.7.1 Theorem

If $\exists$ an interval I around $t = t_0$ where P and Q are continuous, then $\exists$ 1 and only 1 solution to the I.V.P. over all of I.

1.7.2 Integrating Factor

$$\mu(t) = e^{\int P(t)\,dt}$$

Then get

$$(ye^{\int P\,dt})' = Qe^{\int P\,dt}$$

then divide by $e^{\int P(t) dt}$ to get

$$y(t) = Ce^{-\int P\,dt} + e^{-\int P\,dt} \int Qe^{\int P\,dt}$$

1.7.3 Example Problem

$$t^2 y' - y = 3 \qquad y(1) = 0$$

$$y' + (\frac{-1}{t^2})y = \frac{3}{t^2}$$

First find integrating factor

$$e^{\int \frac{-1}{t^2}\,dt} = e^{1/t}$$

$$(ye^{1/t})' = \frac{3}{t^2}e^{1/t}$$

$$ye^{1/t} = \int \frac{3}{t^2}e^{1/t}\,dt = -3e^{1/t} + C$$

$$y = -3 + Ce^{-1/t}$$

Chapter 2

Exact Equations

$$\Psi(x, y) = C$$

On the curve y(x), $\frac{d\Psi}{dx}dx + \frac{d\Psi}{dy}dy = 0$

$$\frac{d\Psi}{dy}\frac{dy}{dx} + \frac{d\Psi}{dx} = 0$$

$$N(x, y)\frac{dy}{dx} + M(x, y) = 0$$

If $\exists\Psi(x, y) : N = \frac{d\Psi}{dy}, M = \frac{d\Psi}{dx}$, then the differential equation has an exact solution.

another helping thing

$$\frac{dM}{dy} = \frac{dN}{dx}$$

if thats true, then $\Psi(x, y)$ exists and there's an exact equation.

2.0.1 Example Problem

$$(x^2 + 3y^2)\frac{dy}{dx} + 2xy = 0$$

$$2xy\,dx + (x^2 + 3y^2)\,dy = 0$$

$$N = (x^2 + y^2) \qquad M = 2xy$$

$$\frac{dN}{dx} = 2x \qquad \frac{dM}{dy} = 2x$$

they equal each other so Ψ exists. Now we have to find it

$$\int \frac{\partial}{\partial \Psi} x\,dx = \int 2xy\,dx \qquad \Psi(x, y) = x^2y + f(y)$$

$$x^2 + f'(y) = x^2 + 3y^2 \longrightarrow f'(y) = 3y^2 \rightarrow f(y) = y^3 + A$$

$$\Psi(x, y) = c \longrightarrow x^2y + y^2 + A = C \rightarrow x^2y + y^2 = C$$

if given an initial condition, you can use that to solve for C.

2.0.2 Another Problem

$$\frac{dy}{dx} = \frac{2xy^2}{\sin(y) - 2x^2y}$$

$$(\sin(y) - 2x^2y)\,dy - 2xy^2dx \qquad 2xy^2dx + (2x^2y - \sin(y))\,dy = 0$$

$$N = (2x^2y - \sin(y)) \qquad M = 2xy^2$$

$$\frac{dM}{dy} = 4xy \qquad \frac{dN}{dx} = 4xy$$

The equation exists

$$\frac{d\Psi}{dx} = M = 2xy^2 \to \Psi = x^2y^2 + f(y)$$

$$\frac{d\Psi}{dy} = 2x^2y + f'(y)$$

$$\frac{d\Psi}{dy} = N = 2x^2y - \sin(y) \longrightarrow 2x^2y + f'(y) = 2x^2y - \sin(y) \to$$

$$f'(y) = -\sin(y) \to f(y) = \cos(y)$$

$$\Psi(x, y) = x^2y^2 + \cos(y) = C$$

2.1 Substitutions

2.1.1 Scale Invariant

$$\frac{dy}{dt} = F\left(\frac{y}{t}\right)$$

Use a substitution $v(t) = \frac{y}{t}$

$$y\frac{dy}{dt} = t + \frac{y^2}{t} \qquad \frac{dy}{dt} = \frac{t}{y} + \frac{y}{t}$$

$$v(t) = \frac{y(t)}{t} \qquad y(t) = v(t) \cdot t$$

$$\frac{dy}{dt} = \frac{d}{dt}(v \cdot t)) = v't + v$$

Now our original equation becomes

$$v't + v = \frac{1}{v} + v \qquad t\frac{dv}{dt} = \left(\frac{1}{v} + v - v\right) \rightarrow \frac{dv}{\frac{1}{v} + v - v} = \frac{dt}{t}$$

solve like regular separable equation

$$tv' + v = F(v) \longrightarrow t\frac{dv}{dt} = (F(v) - v)$$

Chapter 3

Tricks with V substitution

$$y' + P(t)y = Q(t)y^\alpha \qquad \alpha \neq 0, 1$$

use $v(t) = y^{1-\alpha}$

3.0.1 Example

$$y' = \frac{y^2}{t^2} + ty \rightarrow y' - ty = \frac{1}{t^2}y^2$$

use $v(t) = y^{1-2} \rightarrow v(t) = 1/y$

$$y = 1/v \qquad y' = \frac{-1}{v^2}v'$$

$$\frac{-1}{v^2}v' - t\frac{1}{v} = \frac{1}{t^2} \cdot \frac{1}{v^2}$$

$$v' + tv = \frac{1}{t^2}$$

first order linear equation. Solve with Integrating Factor.

3.1 Reducible Equation

$F(t, y, y', y'') = 0$ but y is missing

use $v(t) = y'$, $y'' = \frac{d}{dt}y' = v'$

$F(t, v, v') = 0$

3.1.1 Example

$$y'' + \frac{2}{t}y' = t^2$$

use $v = y', v' = y''$

$$y' + \frac{2}{t}v = t^2 \qquad \mu = e^{\int \frac{2}{t}\,dt} = e^{2\ln t}$$

3.1.2 Another One

$$v't^2 + 2tv = t^4$$

$$(vt^2)' = t^4$$

Integrate

$$vt^2 = \frac{t^5}{5} + A$$

$$y' = \frac{t^3}{5} + \frac{A}{t^2}$$

Integrate

$$y = \frac{t^4}{20} - \frac{A}{t} + B$$

3.2 If t is missing (F(y, y', t"))

$$y'' = \frac{d}{dt}(y') = \frac{d}{dt}v(y(t)) = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dt}v$$

3.2.1 Example

$$y'' + 2yy' = 0$$

Use $v(y) = y'$

$$y'' = \frac{dv}{dy}v$$

$$\frac{dv}{dy}v + 2yv = 0$$

$$v = 0 \longrightarrow y' = 0 \longrightarrow y = C$$

$$v \neq 0 \longrightarrow \frac{dv}{dy} + 2y = 0 \longrightarrow \frac{dv}{dy} - 2y \rightarrow v = A - y^2$$

$$\frac{dy}{dt} = A - y^2 \longrightarrow \frac{dy}{A - y^2} = dt$$

Chapter 4

More Slope FIeld Stuff

4.1 Autonomous Equations

$$\frac{dP}{dt} = 3P - P^2 = P(3-P)$$

$P = 0$ and $P = 3$ are critical points aka also equilibrium solutions.

toward point = stable equilibrium solution. away from point = unstable equilibrium solution.

The other one is semi-stable

4.2 Add Harvesting

$$\frac{dP}{dt} = 6P - P^2 \Longrightarrow \frac{dP}{dt} = 6P - P^2 - H$$

if we make H = 5 then

$$\frac{dP}{dt} = (P-1)(P-5)$$

if H = 9 then

$$\frac{dP}{dt} = (P-3)(P-3)$$

semistable equilibrium

If H = 10

$$\frac{dP}{dt} = (P-3)(P-3) - 1$$

no equilibriums

just set equal to 0

$$6P - P^2 - H = 0 \qquad P = \pm\sqrt{9-H}$$

Chapter 5

Approximations

5.1 Euler's Method???

$$\frac{dy}{dt} = f(t, y)$$

Integrate both sides to get

$$y(t_B) - y(t_0) = \int_{t_0}^{t_B} f(t, y)$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)\, dt$$

$$y_{i+1} = y_i + f(t_i, y_i)\, \Delta t$$

5.2 Midpoint Rule

$$y_{i+1} = y_i + f(t_i + \frac{\Delta t}{2}, y_i + f(t_i, y_i))\frac{\Delta t}{2}\Delta t$$

5.3 Improved Euler Method

$$y_{i+1} = y_i + f(t_i, y_i)\frac{\Delta t}{2} + f(t_i + \Delta t, y_i + f(t_i, y_i)\Delta t)$$

Chapter 6

Higher Order Linear Equations

$$y^N + a_{n-1}(t)y^{n-1} + a_{n-2}(t)y^{n-2} + a_{n-3}(t)y^{n-3} + \ldots a_1(t)y^1 = f(t)$$

plus a bunch of initial conditions

6.1 Existence and Uniqueness Theorem

If there exists an interval I around t_0 such that $a_{N-1}, a_{N-2} \ldots f(n)$ are all continuous, then there exists one and only one solution for the differential equation

6.1.1 Example

$$y'' + \frac{1}{t}y' - \frac{9}{t^2}y = 0 \qquad y(1) = 0, y'(1) = 1$$

Consider the interval $I = (0, \infty)$ and see a solution

6.2 Superposition

if $y_1, y_2, \dots y_n$ are solutions to a differential equation, then any linear combination of those are also a solution to the differential equation.

6.2.1 example

$$y'' - 3y' + 2y = 0$$

$y_1 = e^t, y_2 = e^{2t}$ are solutions

6.2.2 linear non-homogeneous

$$y'' - y = 4$$

$$y_1 = 4 \qquad y_2 = e^t + 4$$

are solutions, but there sum is not

6.3 Equidimentional

$$\alpha(\alpha-1)t^{\alpha-2} + \frac{1}{t}\alpha t^{\alpha-1} - \frac{9}{t^2}t^\alpha = 0 \qquad y(1) = 0, y'(1) = 1$$

$$y = t^\alpha \qquad y' = \alpha t^{\alpha-1} \qquad y'' = \alpha(\alpha-1)t^{\alpha-2}$$

$$\alpha(\alpha-1) + \alpha - 9 = 0 \to \alpha = \pm 3$$

$$y_1 = t^3 \qquad y_2 = t^{-3}$$

I wasnt paying attention

but you can use superposition to figure out a solution that works with the initial conditions of the problem

$$y = c_1 t^3 + \frac{c^2}{t^3} \qquad y' = 3c_1 t^2 - \frac{3c_2}{t^4}$$

Apply initial conditions

$$y = c_1 + c_2 = 0 \qquad y' = 3c_1 - 3c_2 = 1$$

6.3.1 another example

$$y'' + 4y = 0$$

solutions $y_1 = \sin(2t), y_2 = \sin(t)\cos(t)$

dont get nice answer because those two solutions are equivalent via double angle formula

I.C. $y(0) = 1, y'(0) = \ldots$

$$t = c_1 \sin(2t) + c_2 \sin(t)\cos(t)$$

apply initial conditions

$$y(0) = 0 + 0 \neq 1$$

6.4 Linear Independence

Given n solutions $y_1, y_2, \ldots y_n(t)$, write

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) + \ldots \alpha_n y_n(t) = 0$$

if $\alpha_1, \alpha_2, \ldots \alpha_n = 0$, then the y's don't matter, so the y's are linearly independent.

For all other values of α, y is linearly dependent.

6.4.1 Example

$$[1, e^t, e^{-t}] \rightarrow \alpha_1 1 + \alpha_2 e^t + \alpha_3 e^{-t} = 0 \qquad \alpha = 0 \text{ only choice}$$

Therefore its linearly independent

However

$$[\cos(t), e^{it}, e^{-it}] \rightarrow \alpha_1 \cos(t) + \alpha_2 e^{it} + \alpha_3 e^{-it} = 0$$

$$([1, \frac{-1}{2}, \frac{-1}{2}], [2, -1, -1], \ldots) \text{ many choices}$$

$$e^{it} = \cos(t) + i\sin(t) \qquad e^{-it} = \cos(t) - i\sin(t)$$

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \qquad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

6.4.2 What's dependent what's independent

$[1, t]$ is linearly independent

$[t - 1, 2 - 2t]$ dependent

e^t, e^{2t} independent

$[1, t, t^2]$ INdependent

$[1, t, 2t - 1]$ DEpendent

6.5 Wronskian

how to make system of equations

take $n-1$ derivatives over the whole thingy

$\alpha_1 y_1^{n-1} + \ldots + \alpha_n y_n^{n-1} = 0$

make a big ass matrix

$$\begin{bmatrix} y_1 & y_2 & \ldots & y_n \\ y_1' & y_2' & \ldots & y_n' \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots & y_n^{n-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

If the determinant $= 0$, then the only solution is making $\alpha = 0$

If the determinant is NOT 0, then it is linearly dependent so there are ∞ solutions.

The big stupid y matrix determinant is called the Wronskian.

6.5.1 Use

Imagine you have an th linear homogeneous something something with n solutions and n initial conditions of the form $y^n(t_0) = b_n$

You can use the Wronskian to find something or other

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) & \ldots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \ldots & y_n'(t_0) \\ \ldots & \ldots & \ldots & \ldots \\ y_1^{n-1}(t_0) & y_2^{n-1}(t_0) & \ldots & y_n^{n-1}(t_0) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

This matrix sure means something

6.5.2 Abel's Theorem

The Wronskian satisfies

$$W' + \alpha_{n-1}(t)W = 0$$

This is a separable equation

$$W' + \alpha_{n-1}(t)W = 0 \longrightarrow \begin{cases} W(t) = W(t_0)e^{-\int_{t_0}^{t} a_{n-1}(s)\,ds} \\ W(t) = 0 \end{cases}$$

6.6 MIDTERM INFO

Last Lecture with midterm info was February 2nd

Find independent solutions of nth order linear homogeneous equations.

Chapter 7

Constant Coefficient Equations

$$y^N + p_{N-1}y^{N-1} + \ldots + p_1 y = 0$$

use $y = e^{rt}$

7.0.1 Example Problem

$y(0) = 0, y'(0) = 1$

$$y'' + y' - 2y = 0 \rightarrow r^2 e^{rt} + re^{rt} - 2e^{rt} = 0$$

$$r^2 + r - 2 = 0 \rightarrow (r+2)(r-1) = 0$$

$$r = 1, -2 \rightarrow y_1 = e^{1t} \qquad y_2 = e^{-2t}$$

$$y = C_1 e^t + C_2 e^{-2t}$$

Apply initial conditions

$$y(0) = C_1 + C_2 = 0 \qquad y'(0) = C_1 - 2C_2 = 1$$

$$C_1 = -C_2 \qquad C_1 + 2C_1 = 1 \rightarrow C_1 = 1/3, C_2 = -1/3$$

$$y = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$$

7.1 Another Example

$$y''' - 3y'' + 2y' = 0 \to r^3e^{rt} - 3r^2e^{rt} + 2re^{rt} = 0$$
$$r^3 - 3r^2 + 2r = 0 \to r(r-1)(r-2) = 0 \qquad r = 0, 1, 2$$
$$y_1 = e^0, y_2 = e^t, y_3 = e^{2t}$$
$$y = C_1 + C_2e^t + C_3e^{2t}$$

Solve via initial conditions

7.2 When you Get Higher Multitude Zeros

$$y'' - 4y' + 4y = 0 \to r^2e^{rt} - 4re^{rt} + 4e^{rt} = 0$$
$$r^2 - 4r + 4 = 0 \qquad (r-2)^2 = 0 \to y_1 = e^{2t}, y_2 = t2^{2t}$$

Try $y(t) = v(t)e^{2t}$

$$y = Ate^{2t} + Be^{2t}$$

7.2.1 Example

$$y''' - 3y'' + 3y' - y = 0 = r^3e^{rt} - 3r^2e^{rt} + 3re^{rt} - e^{rt} = 0$$
$$r^3 - 3r^2 + 3r - 1 = 0 \to (r-1)^3 = 0 \to y_1 = e^t, y_2 = te^t, y_3 = t^2e^t$$

7.2.2 Complex Numbers

$$y'' + y = 0 \to r^2 e^{rt} + e^{rt} = 0 \to (r^2 + 1) = 0 \to r = \pm i$$
$$y = C_1 e^{it} + C_2 e^{-it} = C_1(\cos(t) + i\sin(t)) + C_2(\cos(t) - i\sin(t))$$
$$(C_1 + C_2)\cos(t) + i(C_1 - C_2)\sin(t) \to A\cos(t) + B\sin(t)$$

7.2.3 Example

$$y'' + 6y' + 13y = 0 \to r^2 e^{rt} + 6re^{rt} + 13e^{rt} = 0 \to r^2 + 6r + 13 = 0$$
$$r = -3 \pm 2i \to y = C_1 e^{(-3+2i)t} + C_2 e^{(-3-2i)t}$$
$$y = e^{-3t}\left(C_1 e^{2it} + C_2 e^{-2it}\right) \to e^{-3t}\left(A\cos(2t) + B\sin(2t)\right)$$

7.3 Operator Notation

$$Dy = \frac{dy}{dt} \qquad D^2 y = \frac{d^2 y}{dt^2} = y''$$

you can factor D operators

$$y'' - 3y' + 2y = (D^2 - 3D + 2)y = 0 \to (D - 2)(D - 1)y = 0$$
$$(D - 1)y = 0 \to D = 1 \to y = e^t$$

solutions to a factor is a solution to the whole equation.

7.3.1 Complex Examples

$$(D^2 - 2D + 5)y = 0 \to r^2 - 2r - 5 = 0$$
$$\to r = 1 \pm \sqrt{1-5} = 1 \pm 2i$$

You can use Euler trig nonsense to get

$$y = e^t \left(A\cos(2t) + B\sin(2t)\right) + te^t \left(C\cos(2t) + D\sin(2t)\right)$$

The 2t is from the 2i
other than that I'm lost

7.3.2 Another Thing

$$L = D^2 + \frac{1}{t}D - \frac{9}{t^2} \to y'' + \frac{1}{t}y' = \frac{9}{t^2}y = 0$$

7.4 Linear Operator

$$L\left(\alpha y_1 + \beta y_2\right) = \alpha L y_1 + \beta L y_2$$

If $Ly = 0$, then you have a linear homogeneous equation

$$y'' - 3y' + 2y = 0$$

If $Ly = f(t)$, then you have a linear INhomogeneous equation

$$y'' - 3y' + 2y = 4$$

Let $Ly = f(t)$

Call y_c (complementary) a general solution to the homogeneous equation $L(y_c) = 0$

Call y_p (particular) a solution to the inhomogeneous equation $L(y_p) = f$

$$L(y_c + y_p) = L(y_c) + L(y_p) = f$$

By adding together solutions, you get new solutions.

Say y is a solution of the inhomogeneous equation $L(y) = f$

$$L(y - y_p) = L(y) - L(y_p) = f - f = 0$$

subtracting to inhomogeneous solutions gets you a general solution

7.4.1 IMPORTANT

Any general solution of the inhomogeneous can be written as a particular solution to the inhomogeneous plus a general solution to the homogeneous

7.4.2 Example

$$y'' - 3y' + 2y = 4 \to (r-1)(r-2) = 0 \to r = 1, 2$$
$$y_c = c_1 e^t + c^2 e^{2t}$$

This is a general solution to the homogeneous equation

Just guess n check a particular solution

$$y_p = 1$$
$$y = y_c + y_p = c_1 e^t + c_2 e^{2t} + 2$$

Let initial conditions be $y(0) = 1$ and $y'(0) = 5$

$$y = c_1 e^t + c_2 e^{2t} + 2 = 1 \qquad y' = c_1 e^t + 2c_2 e^{2t} = 5$$
$$c_2 - 2 = 4 \rightarrow c_2 = 6 \rightarrow c_1 = -7$$
$$y = -7e^t + 6e^{2t} + 2$$

Solution to the initial value problem

7.4.3 Solve a Linear Inhomogeneous I.V.P

Find y_c: General solution to homogeneous equation
Find y_p: Particular solution to inhomogeneous equation
$y = y_c + y_p$
apply initial conditions
c = complementary

7.4.4 Example

$$y'' + 4y = 8t \qquad \begin{cases} y(0) = 5 \\ y'(0) = 2 \end{cases}$$

$$y_c : y'' + 4y = 0 \rightarrow c_1 \cos(2t) + c_2 \sin(2t)$$

$$y_p : y'' + 4y = 8t \rightarrow y_p = 2t$$

$$y = y_c + y_p = c_1 \cos(2t) + c_2 \sin(2t) + 2t$$

$$c_1 \cos(2t) + c_2 \sin(2t) + 2t = 5 = c_1 \qquad y'(0) = 2c_2 + 2 = 2$$

$$c_1 = 5, c_2 = 0 \Rightarrow y = 5\cos(2t) + 2t$$

7.5 Undetermined Coefficients

If the Linear Homogeneous Solution is onstant coeffiients and linear, and the right hand side is an exponent, sin, cos, polynomial, or product of these, try a y_p of the same form as a right hand side plus all possible derivatives, each [something] a coefficient, multiply by t until you have no duplication and find the coefficients.

7.5.1 Examples

$$y'' + y = e^t$$
$$y_p = Ae^t \qquad y_p'' = Ae^t$$
$$Ae^t + Ae^t = e^t \to 2Ae^t = e^t \to A = \frac{1}{2}$$
$$y = \frac{1}{2}e^t$$

$$y' + y = t^2$$
$$y_p = At^2 + Bt + C \qquad y_p'' = 2At + B$$
$$2At + At^2 = t^2 \qquad 2At + B + At^2 + Bt + C = t^2$$
$$t^2 : A = 1 \qquad t : 2A + B = 0 \qquad t^0 : B + C = 0$$
$$A = 1 \qquad B = -2 \qquad C = 2$$
$$y_p = t^2 - 2t + 2$$

$$y'' + y' + y = \cos(2t)$$
$$y = A\cos(2t) + B\sin(2t) \qquad y' = -2A\sin(2t) + 2B\cos(2t)$$
$$y'' = -4A\cos(2t) - 4B\sin(2t)$$
$$-4A\cos(2t) - 4B\sin(2t) - 2A\sin(2t) + 2B\cos(2t)$$
$$+A\cos(2t) + B\sin(2t) = \cos(2t)$$
$$\cos(2t)(-3A + 2B) = \cos 2t \qquad \sin(2t)(-3B - 2A) = 0$$
$$A = \frac{-3}{13} \qquad B = \frac{2}{13}$$

Some goofy math that I dont understand but I took a picture of it

$$y'' - y = e^t$$
$$y_p = Ae^t = y_p' = y_p''$$

I get $0 = e^t$ which breaks math so clearly I used the wrong y_p
find y_c complementary solution to homogeneous equation

$$y'' - y = 0 \to r = \pm 1 \to y_c = c_1e^t + c_2e^{-t}$$
$$y_p = Ae^t t \qquad y_p' = Ae^t + Ate^t$$
$$y_p'' = 2Ae^t + Ate^t$$

plug everything back in and solve for y_p

$$y = y_c + y_p$$

$$y''' - y' = \cos(t) + t^2$$

$$y_c \longrightarrow y''' - y' = 0 \rightarrow r(r-1)(r+1) = 0$$

$$y_c = C_1 + C_2 e^t + c_3 e^{-t}$$

$$y_p = [A\cos(t) + B\sin(t)] + [ct^2 + Dt + E]t$$

Chapter 8

Annihilator Method

Left Hand Side is constant coefficients

Right Side is exponential / sin / cos / polynomial or a product of these

$$Ly = f(t)$$

8.0.1 Find y_c

$$Ly = 0$$
$$y_c = e^{rt} \qquad r = r_1, r_2, \ldots$$
$$y_c = c_1 e^{r_1 t} + c_2 e^{r_2 t} + c_3 e^{r_3 t} + \ldots c_N e^{r_N t}$$

8.0.2 Find the Annihilator of f(t)

some function such that

$$\overline{L} f = 0$$

$$\overline{L}Ly = \overline{L}f = 0$$
$$\overline{L}Ly = 0$$
$$y = e^{rt}$$

I think you then find the homogeneous solution to that equation and then cancel out a bunch of terms

$$y_p = Ae^{r_{N+1}t} + \ldots + ke^{r_{N+k}t}$$

Find all the coefficients

8.0.3 Example

$$y'' - 4y = 4e^{2t}$$
$$y_c \to y'' - 4y = 0 \qquad y_c = e^{rt}$$
$$(r+2)(r-2) = 0 \to r = \pm 2$$
$$y_c = c_1e^{2t} + c^2e^{-2t}$$

Now find the Annihilator

He finds $(D-2)$ via some voodoo witchcraft

chuck the annihilator into the original equation because it makes the other side 0

$$(D-2)(D^2-4)y = (D-2)4e^{rt} = 0$$
$$(D-2)(D^2-4)y = 0$$
$$y = e^{rt}$$
$$(r-2)(r+2)(r-2) = 0$$
$$r = -2, +2, +2 \to e^{2t}, te^{2t}, e^{-2t}$$

cancel out e^{2t} and e^{-2t} because theyre in the complementary solution

$$y_p = Ate^{2t}$$

solve for A by plugging back into original equation

$$y = y_c + y_p = c_1e^{2t} + c_2e^{-2t} + te^{2t}$$

8.0.4 another example problem

$$y'''' + y'' = 3te^t + 2t^2$$
$$y'''' + y'' = 0 \qquad (r^4 + r^2) = 0 \rightarrow$$
$$r^2(r^2 + 1) = 0 \rightarrow r = 0, 0, \pm i$$
$$y_c = c_1 + c_2t + c_3\cos(t) + c_4\sin(t)$$

now find Annihilator

$$(D - 1)^2te^t = 0$$

Thats the annihilator via more voodoo witchcraft

$$(D - 1)^2(D^4 + D^2)y = (D - 1)^2 3te^t = 0$$
$$(D - 1)^2(D^4 + D^2)y = 0$$
$$y = e^{rt}$$
$$(r - 1)^2(r^4 + r^2)y = 0 \rightarrow r = 1, 1, 0, 0, \pm i \rightarrow$$
$$e^t, te^t, 1, t, \cos(t), \sin(t)$$

get rid of the ones that already exist in the homogeneous solution

$$y_p = Ae^t + Bte^t$$

Annihilator is D^3

$$D^3(D^4 + D^2)y = 0$$
$$r = 0, 0, 0, 0, 0, \pm i$$
$$y_{p2} = Ct^2 + Dt^3 + Et^4$$

$$y_p = y_{p1} + y_{p2} = Ae^t + Bte^t + Ct^2 + Dt^3 + Et^4$$

8.1 Finding the Annihilator

Annihilator of t^2 is D^3 because derive that 3 times and get 0
example annihilators

$$(D - a)e^{at} = 0$$
$$(D - a)^{k+1}t^k e^{at} = 0$$
$$D^{k+1}t^k = 0$$

8.2 Variation of Parameters

$$
\begin{aligned}
&y'' + y = h(t) \\
&y_c \to y'' + y = 0 \to y_c = c_1 \cos(t) + c_2 \sin(t) \\
&y_p = u_1(t) \cos(t) + u_2(t) \sin(t) \\
&\text{product rule} \\
&y_p' = u_1'(t) \cos(t) - u_1(t) \sin(t) + u_2'(t) \sin(t) + u_2(t) \cos(t) \\
&y_p' = -u_1'(t) \sin(t) - u_1(t) \cos(t) + u_2'(t) \cos(t) - u_2(t) \sin(t) \\
&\text{do something to get} \\
&-u_1'(t) \sin(t) - u_1(t) \cos(t) + u_2'(t) \cos(t) - u_2(t) \sin(t) + u_1 \cos(t) \\
&+u_2 \sin(t) = \ln(t) \\
&\text{ cancel terms} \\
&-u_1' \sin(t) + u_2' \cos(t) = \ln(t) \to u_1' \sin(t) - u_2' \cos(t) = 0 \\
&u_2' = \frac{-\cos(t)}{\sin(t)} u_1' \qquad - u_1' \sin(t) - \frac{\cos^2(t)}{\sin(t)} u_1' = \ln(t) \\
&-\sin^2(t) u_1' - \cos^2(t) u_1' = \sin(t) \ln(t) \\
&\begin{cases} u_1' = -\sin(t) \ln(t) \\ u_2' = \cos(t) \ln(t) \end{cases} \qquad \begin{cases} u_1 = \int -\sin(t) \ln(t) \, dt \\ u_2 = \int \cos(t) \ln(t) \, dt \end{cases}
\end{aligned}
$$

same y_c and y_p stuff at the end I think

$$y'' + p(t)y' + q(t)y = f(t)$$
$$y_c \to y'' + p(t)y' + q(t)y = 0 \to y_1, y_2 : \text{ linearly independent}$$
$$y_1'' + py_1' + qy_1 = 0 \qquad y_2'' + py_2' + qy_2 = 0$$
$$y_c = c_1y_1 + c_2y_2$$
$$y_p = u_1(t)y_1 + u_2(t)y_2$$
$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$
$u_1'y_1$ and $u_2'y_2$ cancel each other out
$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$
$$+pu_1y_1' + pu_2y_2' + qu_1y_1 + qu_2y_2 = f$$
inside that big something is the homogeneous equations for y_1 and y_2 w
$$(u_1y_1'' + pu_1y_1' + qu_1y_1) + (+u_2y_2'' + pu_2y_2' + qu_2y_2)$$
$$+u_1'y_1' + u_2'y_2' = f$$
$$u_1'y_1' + u_2'y_2' = f \qquad u_1'y_1 + u_2'y_2 = 0$$
$$u_1' = \frac{y_2f}{y_1y_2' - y_1'y_2} \qquad u_2' = \frac{y_1f}{y_1y_2' - y_1'y_2}$$
$$u_1' = \int \frac{y_2f}{W(y_1, y_2)}\,dt \qquad u_2' = \int \frac{y_1f}{W(y_1, y_2)}\,dt$$

I think this was just a big ass derivation of the equations at the very end

8.2.1 Example

$$y'' - 4y' + 4y = 2e^{2t}$$

$$y_c : y'' - 4y' + 4y = 0 \rightarrow (r-2)^2 = 0 \rightarrow y_1 = e^{2t}, y_2 = te^{2t}$$

$$y_c = c_1 e^{2t} + c_2 te^{2t}$$

$$y_p = u_1 e^{2t} + u_2 te^{2t}$$

$$W = \begin{bmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{bmatrix} = e^{4t} + 2te^{4t} - 2te^{4t} = e^{4t}$$

$$u_1 = \int \frac{te^{2t} \cdot 2e^{2t}}{e^{4t}}\, dt = \int 2tdt = -t^2$$

$$u_2 = \int \frac{e^{2t} \cdot 2e^{2t}}{e^{4t}}\, dt = \int 2dt = 2t$$

$$y_p = -t^2 e^{2t} + 2t \cdot te^{2t} = t^2 e^{2t}$$

$$y = y_c + y_p = c_1 e^{2t} + c^2 te^{2t} + t^2 e^{2t}$$

Chapter 9

Laplace Transform

given $f(t)$

$$L(f) = \int_0^\infty e^{-st} f(t)\, dt$$
$$L(1) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s}$$
$$L(t) = \int_0^\infty e^{-st} t\, dt = \frac{-1}{s} e^{-st} t\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt = \frac{1}{s^2}$$
$$L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt = \int_0^\infty e^{-t(s-a)}\, dt = \frac{1}{s-a}$$
$$L(e^{ibt} = \frac{1}{s-ib}) = \frac{s+ib}{s^2+b^2} = \frac{s}{s^2+b^2} + i\frac{b}{s^2+b^2}$$
$$L(\cos(bt) + i\sin(bt)) = L(e^{ibt})$$
$$L(\cos(bt) + i\sin(bt)) = L(\cos(bt)) + iL(\sin(bt))$$

$$L(f') = \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= -f(0) + sL(f)$$

Its just integration by parts

$$L(f'') = -f'(0) + s(f(0) + sL(f)) = s^2 L(f) - sf(0) - f'(0)$$

$$L(y) = y_{(s)}$$

$$y' + 3y = 0 \qquad y(0) = 2$$
$$L(y') + 3L(y) = 0$$
$$sY - y(0) + 3Y = 0$$
$$(s+3)Y - 2 = 0$$
$$Y(s) = \frac{2}{s+3} \rightarrow y(t) = 2e^{-3t}$$

We get a table of important Laplace transforms

9.0.1 Example

$$y' + 3y = t \qquad y(0) = 2$$

$$sY - y(0) + 3Y = \frac{1}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{1}{s^2}$$

$$Y = \frac{1}{s^2(s+3)} + \frac{2}{s+3}$$

$$\frac{1}{s^2(s+3)} = \frac{A}{s+3} + \frac{B}{s} + \frac{C}{s^2} =$$

blah blah blah partial fractions

$$C = 1/3, B = -1/9, A = 1/9$$

$$Y = \frac{1}{9}\frac{1}{s+3} + \frac{-1}{9}\frac{1}{s} + \frac{1}{3}\frac{1}{s^2} + \frac{2}{s+3}$$

this sucks

$$y'' - 2y' - y = 2e^t \qquad y(0) = 3, y'(0) = 4$$
$$s^2Y - sy(0) - y'(0) - [sY - y(0)] - 2Y = \frac{2}{s-1}$$
$$(s^2 - s - 2)Y - 3s - 4 + 3 = \frac{2}{s-1}$$
$$(s-2)(s+1)Y = \frac{2}{s-1} + 3s + 1$$
$$Y = \frac{2}{(s-1)(s-2)(s+1)} + \frac{3s}{(s-2)(s+1)} + \frac{1}{(s-2)(s+1)}$$
$$Y = \frac{-1}{s-1} + \frac{3}{s-2} + \frac{1}{s+2}$$

9.0.2 Heaviside

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$

this has a use supposedly

$$L(H(t-a)) = \int_0^\infty e^{-st} H(t-a)\, dt = \int_0^\infty e^{-st}\, dt = \frac{-1}{s} e^{-st}\Big|_0^\infty = \frac{1}{s} s^{-sa}$$

Chapter 10

Oscillators

10.1 Undampened Oscillator

Hooke's Law and F = ma

$$-kx = m\frac{d^2x}{dt^2} \longrightarrow mx'' + kx = 0$$

If you solve this equation you get $x = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin\left(\sqrt{\frac{k}{m}}t\right)$

The frequency is $\omega = \sqrt{\frac{k}{m}}$ rads/second

If we add a periodic force to the thingy we get

$$-kx = m\frac{d^2x}{dt^2} + \sin(\omega t) \longrightarrow mx'' + kx = \frac{1}{m}\sin(\omega t)$$

Find a particular and a general solution and get

$$x = A\cos(\omega t) + B\sin(\omega t) + \frac{\sin(\omega t)}{k - m\omega^2}$$

particular solution has important properties such as small ω being positive and large negative

if $\omega_0 = \omega = \sqrt{\frac{k}{m}}$ then we get undefined properties which is resonance in the real world.

If thats the case then our original solution doesnt work and we use other particular solution

$$x_p = \frac{-t\cos(\omega_0 t)}{2m\omega_0}$$

10.2 Dampened Oscillator

regular Hooke's Law plus a dampening force given by $F = -\gamma x'$

$$mx'' = F_{spring} + F_{damp} = -kx - \gamma x'$$

$$F_{spring} + F_{damp} - F_{net} \rightarrow$$

$$mx'' + \gamma x' + kx = 0$$

find and solve characteristic equation

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}$$

the behavior of the equation depends on the sign of $\gamma^2 - 4mk$

if $\gamma^2 - 4mk < 0$ then the system is underdamped and you get sustained oscillations of decreasing amplitude

if $\gamma^2 - 4mk > 0$ then it's overdamped and the system never increases in distance, it just goes straight to 0. Roots are $r = \frac{1}{2m}(-\gamma \pm \sqrt{\gamma^2 - 4km})$

if $\gamma^2 - 4mk = 0$ then it's perfectly damped and you go up to a peak before going down to 0. Root is $r = \frac{-\gamma}{2m}$

10.2.1 External Oscillating Force

$mx'' + \gamma x' + kx = F_0 \sin(\omega t)$

find particular solution

$$x_p(t) = \frac{F_0}{(k - m\omega^2)^2 + \gamma^2\omega^2}(-\gamma\omega\cos(\omega t) + (k - m\omega^2)\sin(\omega t))$$

which is also

$$x_p(t) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}\sin(\omega t - \phi) \qquad \frac{\omega}{\phi} = \arctan\left(\frac{\gamma\omega}{k - m\omega^2}\right)$$

the behavior of the equation depends on the sign of $\gamma^2 - 4mk$

$$\text{amplitude}(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}}$$

Underdamped and critically damped oscillators look the same, and overdamped oscillators now look like critically damped oscillators.

When forced frequency equals natural frequency then the wave gets bigger and bigger and diverges.

10.3 RLC Circuits

Resistor - Inductor - Capacitor

Battery gives voltage V makes current I. Resistor makes resistance $V_R = RI$. Inductor has Inductance L with makes a voltage against the direction of resistance $V_L = L\frac{dI}{dt}$. Capacitor builds up charge and increases in resistance with charge gained $V_c = \frac{1}{C}Q$. Also $I = \frac{dQ}{dt}$. Kirchoff's Loop rule means all the forces sum up to 0.

$$V - V_R - V_L - V_C = 0$$

$$V - RI - L\frac{dI}{dt} - \frac{1}{C}Q = 0$$

derive or do another thing to get

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}$$

$$LI'' + RI' + \frac{1}{C}I = \frac{dV}{dt}$$

or

$$LQ'' + RQ' + \frac{1}{C}Q = V$$

You now have two useful differential equations to solve. It's basically the same as a mass spring with just different variables

10.4 Complexification

$$LI'' + RI' + \frac{1}{C}I = -\omega V_0 \sin(\omega t) \qquad (V = V_0 \cos(\omega t))$$

$$LI'' + RI' + \frac{1}{C}I = \mathrm{Re}(i\omega V_0 e^{i\omega t})$$

$$L\widetilde{I}'' + R\widetilde{I}' + \frac{1}{C}\widetilde{I} = i\omega V_0 e^{i\omega t} \qquad \widetilde{I} \in \mathbb{C}$$

take the real part of $\widetilde{I}$ to find I

Consider a regular RC circuit

$$R\widetilde{I}' + \frac{1}{C}\widetilde{I} = i\omega V_0 e^{i\omega t}$$

get particular solution with guess$I_p = I_0 e^{i\omega t}$

$$I_0 = \frac{i\omega V_0}{i\omega R + \frac{1}{C}} = \frac{V_0}{R - \frac{i}{\omega C}}$$

$$V_0 = (R - \frac{i}{\omega C})I_0$$

complex resistance is called impedance

Now consider an LR circuit

$$L\widetilde{I}'' + R\widetilde{I}' = i\omega V_0 e^{i\omega t} \rightarrow V_0 = (R + i\omega L)I_0$$

for a full RLC Circuit

$$V_0 = (R + i(\omega L - \frac{1}{\omega C}))I_0$$

10.5 FUCK THIS'LL BE ON THE MIDTERM

find the real current of something something

10.6 END OF MATERIAL FOR MIDTERM 2

Chapter 11

Linear Algebra

Ah yes I have never taken linear algebra before and definitely need this rerun

- multiply matrices
- invert matrices
- compute determinants

The determinant of an upper triangular matrix is the product of all of its diagonal entries

Remember / figure out Cramer's Rule

to find eigenvalues use $A - \lambda I$ and row operations to solve for lambda

The eigenvalues of A are the roots of its characteristic polynomial $\det(A - \lambda I)$ solve for lambda

Chapter 12

Systems of Equations

$$m_1 x_1'' = -k_1 x_1 - k x_1 + k x_2 + g_1(t)$$
$$m_2 x_2'' = k x_1 - k x_2 - k_2 x_2 + g_2(t)$$

substitute

$$x_3 = x_1' \qquad x_3' = x_1''$$
$$x_4 = x' \qquad x_4' = x_2''$$
$$x_3' = -\frac{(k_1 + k)x_1}{m_1} + \frac{k}{m_1} x_2 + \frac{g_1}{m_1}$$
$$x_4' = \frac{k}{m_2} x_1 - \frac{(k + k_2)x_2}{m_2} + \frac{g_2}{m_2}$$

12.1 Order Reduction Quiz Problem

$$y'' - 5y' + 6y = 0 \to x' - 5x + 6 = 0$$
$$x = y'(t) \qquad x' = 5x - 6y$$
$$\begin{bmatrix} y' \\ x' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{bmatrix} y \\ x \end{bmatrix}$$
$$(r-2)(r-3) = 0 \to r = 2, 3$$
$$y = C_1 e^{2t} + C_2 e^{3t}$$
$$4C_1 e^{2t} + 9C_2 e^{3t} - 10C_1 e^{2t} - 15C_2 e^{3t} + 6C_1 e^{2t} + 6C_2 e^{3t} = 0$$

12.2 Theorems

12.2.1 Existence and Uniqueness

$$\text{for } \begin{cases} y' = F(t,y) \\ y(t_0) = y_0 \end{cases}$$

if $\exists$ region around (t_0, y_0) where F is continuous $\Rightarrow$ a solution exists
if $\frac{dF_1}{dy_1}, \frac{dF_2}{dy_2}, \ldots$ are continuous, then the solution is unique

12.2.2 Another thing

$$\begin{cases} \underline{y}' = A(t)\underline{y} + \underline{y}(t) \\ \underline{y}(t_0) = \underline{y}_0 \end{cases}$$

if $A(t)$ and $\underline{y}(t)$ are continuous on an interval I around t_0, then $\exists$ 1 and only 1 solution to the Initial Value Problem around I.

12.3 Linear Homogeneous Solution

$$\underline{y}' = A(t)\underline{y}$$

12.3.1 Abel's Theorem

A(t) is an $n \times n$ matrix and we have n solutions $v_1 \ldots v_n$.
Take the Wronskian
If $W(t) \neq 0 \Rightarrow v_1 \ldots v_n$ are linearly independent

If $W(t) = 0 \Rightarrow v_1 \dots v_n$ are linearly dependent

$$\frac{dW}{dt} = Tr(A) \cdot W(t)$$

12.4 More Systems of Equations

$$y_1' = 2y_1 + 9y_2 \qquad y_2' = y_1 + 2y_2$$
$$y_1(0) = 1 \qquad y_2(0) = 1$$
$$\underline{y}' = A\underline{y} \qquad A = \begin{bmatrix} 2 & 9 \\ 1 & 2 \end{bmatrix}$$

Find n linearly independent solutions to the system of equations $v_1 \dots v_n$

$$\underline{v}_1' = A\underline{v}_1, \underline{v}_2' = A\underline{v}_2, \dots$$
$$W = \det(\underline{v}_1, \underline{v}_2, \dots) \neq 0$$
$$M(t) = \begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \dots \end{pmatrix} \text{ Fundamental Matrix}$$
$$M' = AM$$
$$\underline{y} = c_1\underline{v}_1 + c_2\underline{v}_2 + \dots c_n\underline{v}_n = M\underline{c}$$
$$\text{General Solution } \underline{y} = M\underline{c}$$

12.4.1 Example

$$y_1' = 2y_1 + 9y_2 \qquad y_2' = y_1 + 2y_2$$

$$y_1(0) = 1, y_2(0) = 1$$

$$A = \begin{pmatrix} 2 & 9 \\ 1 & 2 \end{pmatrix}$$

Find eigenvalues and eigenvectors to get v_1 and v_2

$$A = \begin{pmatrix} 2-\lambda & 9 \\ 1 & 2-\lambda \end{pmatrix} \to \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) = 0$$

$$\begin{pmatrix} 2-5 & 9 \\ 1 & 2-5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \to v_1 - 3v_3 = 0 \to \begin{bmatrix} 3 \\ 1 \end{bmatrix} v_2$$

$$\begin{pmatrix} 2+1 & 9 \\ 1 & 2+1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \to v_1 + 3v_3 = 0 \to \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\underline{v}_1 = \begin{pmatrix} 3e^{5t} \\ e^{5t} \end{pmatrix} \qquad \underline{v}_2 = \begin{pmatrix} 3e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$M = \begin{pmatrix} 3e^{5t} & 3e^{-t} \\ e^{5t} & -e^{-t} \end{pmatrix}$$

$$\underline{y} = M\underline{c} = c_1 \begin{pmatrix} 3e^{5t} \\ e^{5t} \end{pmatrix} + c_2 \begin{pmatrix} 3e^{-t} \\ -e^{-t} \end{pmatrix}$$

take inverse of M(0)

$$\underline{c} = M^{-1}(0) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$c_1 = \frac{2}{3}, c_2 = \frac{-1}{3}$$

$$\underline{y} = M(t)\underline{c} = M(t)M^{-1}(0)\underline{y}_0$$

12.4.2 Some Variation of Parameters Thing

$$\underline{y}' = A\underline{y} + f(t)$$

$$\underline{y} = M\underline{c} \longrightarrow \underline{y} = M\underline{u}(t)$$

$$(M\underline{u})' = AM\underline{u} + \underline{f}$$

$$\cancel{M'\underline{u}} + M\underline{u}' = \cancel{AM\underline{u}} + f$$

cancel out $\cancel{M'\underline{u}}$ and $AM\underline{u}$

$$M\underline{u}' = \underline{f}$$

$$u' = M^{-1}\underline{f}$$

$$\underline{u} = \int^t M^{-1}(s)\underline{f}(s)ds$$

work

$$\underline{y}(t) = M(t)\int_{t_0}^t M^{-1}(s)\underline{f}(s)ds + M(t)M^{-1}(t_0)\underline{y}_0$$

12.5 More Eigenvalue Determinant Something

$$y_1' = -2y_1 + y_2 \qquad y_2' = y_1 - 2y_2$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3)$$

$$(A - \lambda_1 I)x = 0 \to \xi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)x = 0 \to \xi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$v_1 = \xi_1 e^{\lambda_1 t} \qquad v_2 = \xi_2 e^{\lambda_2 t} \to M = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

$$\underline{y} = c_1 v_1 + c_2 v_2$$

12.5.1 Important Fundamental Matrix

$$\Phi(t) : \Phi(0) = I$$

12.6 Something

$$\underline{y}' = A\underline{y}$$

$$\underline{y}(t_0) = y_0$$

$$\begin{cases} \Phi(t) = A\Phi \\ \Phi(0) = I \end{cases}$$

$$\Phi(t) = e^{At}$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots$$

$$e^{At} = I + At + \frac{A^2t}{2!} + \frac{A^3t}{3!} + \dots$$

$$(e^{At})' = 0 + \sum_{k=1}^{\infty} \frac{A^k k t^{k-1}}{k!} = A + \sum_{k=2}^{\infty} \frac{A^k t^{k-1}}{(k-1)!} = A + A\sum_{k=2}^{\infty} \frac{A^{k-1}t^{k-1}}{(k-1)!} =$$

$$AI + A\sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!} = A\left(I + \sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!}\right) = e^{At}$$

$$\Phi(t) = e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!}$$

12.6.1 Matrix Exponential

if $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \qquad A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$

$e^{At} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}$

If A is not diagonal, but diagonalizable

A is diagonalizable $\iff$ A has n linearly independent eigenvectors

$A\underline{\xi}_1 = \lambda_1 \underline{\xi}_1$

$U = \begin{pmatrix} \xi_1 & \xi_2 & \dots \end{pmatrix} \qquad \Lambda = \begin{pmatrix} \lambda_1 & 0 & \dots \\ 0 & \lambda_2 & \dots \\ \dots & \dots & \lambda_3 \end{pmatrix}$

$AU = U\Lambda \Longrightarrow U^{-1}AU = \Lambda$

$A^2 = U\Lambda U U^{-1} \Lambda U^{-1} = U\Lambda^2 U^{-1} \qquad A^3 = U\Lambda^3 U^{-1}$

$$e^{At} = I + \sum_{k=1}^{\infty} \frac{A^k t^k}{(k)!} = I + \sum_{k=1}^{\infty} \frac{U\Lambda^k U^{-1} t^k}{(k)!} =$$

$$UIU^{-1} + U \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{(k)!} U^{-1} = U \left(I + \sum_{k=1}^{\infty} \frac{\Lambda^k t^k}{(k)!} \right) U^{-1}$$

$$\Phi(t) = e^{At} = U \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots \\ 0 & e^{\lambda_2 t} & \dots \\ \dots & \dots & e^{\lambda_3 t} \end{pmatrix} U^{-1}$$

$\underline{y}(t) = \Phi(t)\underline{y}_0 \qquad y(0) = y_0$

12.7 Putzer's Method

$$A_{N\times u}, \lambda_1, \lambda_2, \ldots$$

$$e^{At}$$

$$B_0 = I, B_1 = (A - \lambda_1 I)B_0 = A - \lambda_1 I,$$

$$B_2 = (A - \lambda_2 I)B_1 = (A - \lambda_1 I)(A - \lambda_1 I)$$

$$B_{N-1} = (A - I\lambda_{N-1} I)B_{N-2}$$

$$r_1(t) = \begin{cases} r_1' = \lambda_1 r_1 \\ r_1(0) = 1 \end{cases}$$

$$r_2 = \begin{cases} r_2' = \lambda_2 r_2 + r_1 \\ r_2(0) = 0 \end{cases}$$

$$r_N = \begin{cases} r_N' = \lambda_N r_N + r_{N-1} \\ r_N(0) = 0 \end{cases}$$

$$e^{At} = r_1 B_0 + r_2 B_1 + r_3 B_2 + \ldots + r_N B_{N-1}$$

12.7.1 Example

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \lambda_1 = -1, \lambda_2 = \lambda_3 = 3$$

$$B_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_1 = A - \lambda_1 I = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B_2 = (A - \lambda_2 I)B_1 = (\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}) \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

unexpected behavior but it means everything else is 0's

$$r_1(t) = \begin{cases} r_1' = -1r_1 \\ r_1(0) = 1 \end{cases} \to r_1 = e^{-t}$$

$$r_2(t) = \begin{cases} r_2' = 3r_2 + e^{-t} \\ r_2(0) = 0 \end{cases} \to r_2 = -\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t} \text{ integrating factor}$$

$$e^{At} = e^{-t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (-\frac{1}{4}e^{-t} + \frac{1}{4}e^{3t}) \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \text{something bad}$$

Chapter 13

Boundary Value Problem

Deflection on an Elastic Beam

L = load

$$Iy'''' = Iy^{IV} = -gx$$

$$y^{IV} = -1, y(0) = 0, y''(0) = 0, y(L) = 0, y'(L) = 0$$

$$y = \frac{-x^4}{24} + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D$$

Just plug everything into everything else

13.0.1 Examples

$$y'' + y = 0 \qquad y(0) = 0, y(\pi) = 1$$
$$y = c_1 \cos(x) + c_2 \sin(x)$$
apply boundary conditions
$$y(0) = c_1 = 0 \qquad y(\pi) = -c_1 = 1$$
no solutions
$$y'' + y = 0 \qquad y(0) = 1, y(\pi) = -1$$
$$y = c_1 \cos(x) + c_2 \sin(x)$$
apply boundary conditions
$$y(0) = c_1 = 1 \qquad y(\pi) = -c_1 = -1$$
$$y = \cos(x) + c_2 \sin(x)$$

$$y'' + y = \qquad y(0) = 0, y(1) = 0$$
$$y_c = c_1 \cos(x) + c_2 \sin(x)$$
$$y_p = Ax + B \qquad y_p' = A \qquad y_p'' = 0$$
$$y_p = x$$
$$y = c_1 \cos(x) + c_2 \sin(x) + x$$
apply boundary conditions
$$y(0) = c_1 = 0 \qquad y(1) = 0 \cos(x) + c_2 \sin(x) + 1 \rightarrow c_2 = \frac{-1}{\sin(x)}$$
$$y = \frac{-\sin(x)}{\sin(1)} + x$$

13.0.2 Homogeneous Boundary Problem

$$y'' + 4y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$
$$y_c = c_1 \cos(2x) + c_2 \sin(2x)$$
$$c_1 = 0$$
$$y = c_2 \sin(2x) \text{ infinite solutions}$$
$$y'' + 3y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$
$$y_c = c_1 \cos\left(\sqrt{3}x\right) + c_2 \sin\left(\sqrt{3}x\right)$$
$$c_1 = 0 \qquad c_2 \sin\left(\sqrt{3}\pi\right) = 0 \to c_2 = 0$$
$$y = 0 \text{ one solution}$$

initial parameters are important

13.0.3 Finding non-trivial solutions

$$y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(\pi) = 0$$

find λ such that solution is non-trivial

call λ an eigenvalue and the resulting solution an eigenfunction

$$\lambda = -\alpha^2 : \alpha \neq 0 \to y'' - \alpha^2 y = 0 \to y = Ae^{\alpha x} + Be^{-\alpha x}$$

apply boundary conditions

$$A + B = 0 \to A = -B$$
$$Ae^{\alpha\pi} + Be^{-\alpha\pi} = 0 \to Ae^{\alpha\pi} - Ae^{-\alpha\pi} \to A(e^{\alpha\pi} - e^{-\alpha\pi}) \to \alpha = 0$$

that's not allowed, so there's only the trivial solution

13.1 42 is a final answer

13.2 Eigenvalue Problems

just fuckin guess various things for γ and hope for the best

$$y'' + \lambda y = 0 \qquad y(0) = 0 \qquad y(L) = 0$$
$$\lambda_N = \left(\frac{N\pi}{L}\right)^2 : N \in \mathbb{Z}$$
$$\underline{y}_N = \sin\left(\frac{N\pi}{L}x\right)$$

memorize this thingy specifically

$$y'' + 2y' + \lambda y = 0 \qquad y(0) = 0 \qquad y(1) = 0$$
$$r^2 + 2r + \lambda = 0 \to r = -1 \pm \sqrt{1-\lambda}$$
$$\begin{cases} 1-\lambda < 0 \\ 1-\lambda = 0 \\ 1-\lambda > 0 \end{cases}$$

13.3 More Eigenvalue Shenanigans

13.3.1 Sturm-Lionille Theorem

idk what it is

$$p, p', q, r \in C[a, b] \& p, r > 0 \forall x \in [a, b] \Rightarrow$$

$\exists$ an infinite ordered sequence of eigenvalues and an eigenfunction for each eigenvalue

The eigenvalues will form an orthogonal basis

Chapter 14

Fourier Series

14.0.1 Sine Fourier Series Expansion

$$f(x) = \sum_{k=1}^{\infty} c_N y_N(x)$$

$$< f, g >= \int_a^b f(x)g(x)r(x)dx$$

r(x) is a weight function

$$f(x) = \sum_{N=1}^{\infty} B_N \sin\left(\frac{N\pi x}{L}\right)$$

$$B_N = \frac{\int_a^b f(x)\sin\left(\frac{N\pi x}{L}\right)dx}{\int_0^L \sin^2(\frac{N\pi x}{L})dx}$$

$$B_N = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{N\pi x}{L}\right)dx$$

14.1 FIGURE IT OUT LATER

Cosine Fourier Series Expansion

$$f(x) = A_0 + \sum_{N=1}^{\infty} A_N \cos\left(\frac{N\pi x}{L}\right)$$
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx$$
$$A_N = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{N\pi x}{L}\right)dx$$

14.1.1 Theorem

If f(x) and f'(x) are piecewise continuous in (a, b), then the fourier series expansion of f(x) converges to $(f(x)^+ + f(x)^-)/2$

14.1.2 Eigenfunction something

idk

14.2 Trig shenanigans

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2} \qquad \sin(x) = \frac{e^{ix} - e^{-ix}}{2}$$
$$\cosh(x) = \frac{e^{x} + e^{-x}}{2} \qquad \sinh(x) = \frac{e^{x} - e^{-x}}{2}$$
$$\frac{d}{dx}\sinh(x) = \cosh(x) \qquad \frac{d}{dx}\cosh(x) = \sinh(x)$$
$$y' - \alpha^2 y = 0 \rightarrow y = Ae^{\alpha x} + e^{-\alpha x} \rightarrow y = A\cosh(\alpha x) + B\sinh(\alpha x)$$

That's all you need to know about hyperbolic trig according to Manfroi

$$y'' + \lambda y = 0 \qquad y(-L) = y(L) \qquad y'(-L) = y'(L)$$
$$y = A\cosh(\alpha x) + B\sinh(\alpha x) \qquad y' = \alpha A\sinh(\alpha x) + \alpha B\cosh(\alpha x)$$

$$y_0 = 1$$

14.2.1 y_N is orthogonal

$$f(x) = \sum c_N y_N \qquad c_n = \frac{< f_1, y_N >}{< y_N, y_N >}$$
$$f(x) = A_0 + \sum (A_N \cos\left(\frac{N\pi x}{L}\right) + B_N \sin\left(\frac{N\pi x}{L}\right))$$

14.2.2 Example

$$
\begin{aligned}
&f(x) = x \qquad -10 < x < 10 : L = 10 \\
&A_0 = \frac{1}{20}\int_{-10}^{10} x dx = 0 \\
&A_N = \frac{1}{10}\int_{-10}^{10} x\cos\left(\frac{N\pi x}{10}\right)dx = 0 \\
&B_N = \frac{1}{10}\int_{-10}^{10} x\sin\left(\frac{N\pi x}{10}\right)dx = \frac{-20}{N\pi}\cos(N\pi) = \frac{-20}{N\pi}(-1)^N \\
&ff(x) = \frac{-20}{\pi}\sum\frac{(-1)^N}{N}\sin\left(\frac{N\pi x}{10}\right)
\end{aligned}
$$

14.2.3 Even and Odd Functions

$$
\begin{aligned}
&\int_{-L}^{L} \text{ odd } = 0 \\
&\int_{-L}^{L} \text{ even } = 2\int_{0}^{L} \text{ even} \\
&F = A_0 + \sum_{N=1}^{\infty} A_N\cos\left(\frac{n\pi x}{L}\right) + B_N\sin\left(\frac{n\pi x}{L}\right) \\
&A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx, A_N = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \\
&B_N = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx,
\end{aligned}
$$

knowing if a function is even or odd just saves a little bit of time

its all just fourier series if you can do one of em you can do all of em

Chapter 15

Partial Differential Equations

Finally something different holy shit

heat equation example

$$\mu(x,t) \qquad \text{flux} \propto \frac{\partial \mu}{\partial x}$$

$$\frac{\partial \mu}{\partial t} = \kappa \frac{\partial^2 \mu}{\partial x^2} \qquad \kappa \text{ is diffusion constant}$$

do some boundary condition osmethings

$\mu(0,t) = 0, \mu_x(0,t) = 0, \mu(L,t) = 0, \mu_x(0,t) = 0,$

15.0.1 Separation of Variables

$$\mu(x,t) = X(x)T(t)$$
$$\mu_t = XT' \qquad \mu_{xx} = X''T$$
$$XT' = \kappa X''T$$
$$\frac{T(t)'}{\kappa T(t)} = \frac{X(x)''}{X(x)} = -\lambda$$
$$\frac{X(x)''}{X(x)} = -\lambda \longrightarrow X_n = \sin\left(\frac{n\pi x}{L}\right)$$
$$T(t)_N = e^{\frac{-\kappa n^2\pi^2}{L^2}t}$$
$$\mu(x,t) = \sum_{k=1}^{\infty} c_N e^{\frac{-\kappa n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right)$$
$$c_N = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

15.0.2 Diffusion Equation in 1 Dimension

fuckin idk

$$T' = \frac{-\kappa n^2\pi^2}{L^2}T \rightarrow T_N(t) = e^{\frac{-\kappa n^2\pi^2}{L^2}t}$$
$$\mu = A_0X_0T_0 + A_1X_1T_1 \ldots$$
$$\mu = A_0 = \sum_{k=1}^{\infty} A_N e^{\frac{-\kappa n^2\pi^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right)$$

15.1 2D Laplace Equation

$u_{xx} + u_{yy} = 0$

im not paying attention at all fuck this shit