MATH 257

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Contents

Chapter 1

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My laptop died and I skipped some lectures to go to a part time job fair but I know every basic thing about matrices and vectors so I should be fine

Chapter 2

Column Vectors and Basis Vectors

If you take the columns of a vector, then you get a couple vectors that span a space.

Solving a linear system is the same as finding the linear combinations that equal a certain result

2.1 Matrix Vector Multiplication

$$\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = ac_1 + bc_2 + cc_3$$

2.2 Transformations

You can multiply a vector by a matrix to transform it in a certain way

2.2.1 Rotation

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

2.3 Elementary Matrices

An elementary matrix is a matrix gotten by doing a single elementary row operation on the identity matrix.

To find the inverse of an elementary matrix, you just do the opposite of the row operation to an identity matrix.

2.4 Invertible Matrices

Suppose A and B are invertible. Then:

- A^{-1} is invertible then $(A^{-1})^{-1} = A$
- AB is invertible if $(AB)^{-1} = A^{-1}B^{-1}$
- A^T is invertible iff $(A^T)^{-1} = (A^{-1})^T$

2.5 LU Decomposition

idk what it is but it's probably important

It stands for lower upper decomposition.

You can find a upper and lower triangular matrices L and U such that $A = LU$

You know a matrix can be decomposed if you can put the matrix in echelon form with just row operations from a higher row to a lower row.

2.5.1 How To Steps

1. Row reduce
2. Find elementary matrices $E_1, E_2 \ldots$
3. L = $E_1^{-1}, E_2^{-1}, \ldots$
4. U = echelon form of original matrix that you already calculated

2.5.2 Solving a thingy

to solve $Ax = b$, you can solve $Ux = c$ such that $Lc = b$.

2.5.3 Inner Product

$$v \cdot w = v^T w$$

2.5.4 Norm

$$||v|| = \sqrt{v \cdot v}$$

2.5.5 Distance

$$\text{dist}(v, w) = ||v - w||$$

2.6 Orthogonality

if twe vectors are orthogonal or perpendicular to each other, then

$$v \cdot w = 0$$

2.6.1 Pairwise Orthogonal

A set of vectors is pairwise orthogonal if they are all orthogonal to each other.

2.6.2 Orthonormal Set

A set of unit vectors that are all orthogonal to each other.

2.7 Subsets/ Subspaces

A non-subset H of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if it satisfies the two following:

- if $u, v \in H$, then $u + v \in H$
  (closed under addition)
- if $u \in H$ and c is scalar, then $cu \in H$
  (closed under scalar multiplication)

subspaces are pretty useful

2.7.1 Column Space

The space created by spanning the columns of a matrix

It contains all the vectors b that can be written as $Ax = b$ for some x

If A and B are row equivalent, then $\text{col}(A) = \text{col}(B)$

2.7.2 Null Space

The space created by all the solutions of the equation $Ax = 0$.

The null space of an $m \times n$ matrix A is a subspace in $\mathbb{R}^n$

Let w and b be vectors such that $Aw = b$. Then $\{v \in \mathbb{R}^n : Av = b\} = w + \text{Nul}(A)$

Chapter 3

Coordinate Vectors

If you start with a basis $B = \{v_1, v_2, \ldots, v_m\}$ and want to go to a basis $D = \{w_1, w_2, \ldots, w_m\}$, then you can use a linear transformation.

to get from E_n which is the standard $\mathbb{R}^n$ basis to B you use the matrix I_{EB} such that $v_E = I_{EB}v_B$

if you have a linear transformation from E to E, the way to get it from B to D is

$$Tv = v_T \to T * I_{EB}v_b = I_{ED}v_D \to (I_{ED}^{-1} * T * I_{EB}) * v_b = v_{TD}$$

3.1 Determinants

For a 2×2 matrix its just $ad - bc$ but for larger matrices its wackier

the determinant has a couple special properties

- $\det I_N = 1$

- row replacement does not change the determinant
- row interchange changes the sign of the determinant
- scalar multiplication of a row scales the determinant by the same factor

The way to find the determinant of larger matrices is by taking the product of the diagonals of a triangular matrixes.

Just make sure to use only row replacement to create a triangular matrix and then take the product of the diagonal entries and boom you're golden.

if A is invertible, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Also, $\det(A^T) = \det(A)$

Probably some other stuff with determinants that I missed

Chapter 4

Eigen-Stuff

$$Ax = \lambda x$$

That's the whole thing.

if $\det(A - \lambda I) = 0$ then λ is an eigenvalue of A

4.1 Diagonal Matrices

if you have a Diagonal matrix D of all the eigenvalues of a matrix and a you have a matrix P of an eigenvector for each eigenvalue then

$$A = PDP^{-1}$$

4.1.1 Eigenbases

an eigenbasis is a basis of $\mathbb{R}^N$ made by all the possible eigenvectors of A

if A has an eigenbasis, then A is diagonalizable.

$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$ the you can just do the power of each eigenvalue in D for D^2

4.1.2 multiplicity

If there are 2 linearly independent eigenvectors of the same eigenvalue, then the geometric multiplicity of that eigenvalue is 2.

If the eigenvalue appears 2 times in the characteristic polynomial, then its algebraic multiplicity is 2.

4.2 Markov Matrices

It's an adjacency matrix except instead of 1 its a probability that a node will travel from 1 vertex to another.

4.3 Matrix Exponential

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!}$$

This is easy for diagonalizeable matrices because those can be taken to multiple powers very easily.

Chapter 5

Differential Equations

Trust we're still in lin alg

Let A be a matrix with an eigenbases $v_1, v_2 \dots v_n$. and a bunch of eigenvalues λ_n. if v is in the eigenbasis in the form $v = c_1v_1 + c_2v_2 + \dots$, then the unique solution to the differential equation $\frac{du}{dt} = Au$ with initial condition $u(0) = v$ is given by

$$e^{At}v = c_1e^{\lambda_1 t}v_1 + c_2e^{\lambda_2 t}v_2 + \dots$$

I haven't been paying attention at all but like I think thats the big thing

Chapter 6

Projections

They're just kinda the projections that you did in calc 3

$$proj_w(v) = \frac{v \cdot w}{w \cdot w}\vec{w}$$

the projection of v onto w

$v - proj_w(v)$ is called the error term

6.1 Least Squares Solution

given that $Ax = b$ is inconsistent, the least squares solution is a working vector $\hat{x}$ such that the distance between $A\hat{x}$ and b equals the minimum distance between Ax and b

You did least squared solutions on the lab theyre not bad

if you have l coordinates of x, y and a function $y = c_1x^2 + c_2x + c_3$ or whatever then you make an $l \times 3$ matrix of each x value as $x^2, x, 1$ and then you do some transpose stuff

$$y = AC$$
$$A^T Ax = A^T y$$

6.2 Gram-Schmidt Process

$$b_1 = a_1 \qquad b_2 = a_2 - proj_{span(q_2)}(a_2)$$
$$b_3 = a_3 - proj_{span(q_1 * q_2)}(a_3)$$

6.3 Midterm 3 Junk

6.3.1 Linear Transformations

- T(0) = 0
- distributive
- scalar multiplication holds

go re-find out all the bullshit with coordinate matrices

6.3.2 Determinants

cofactor expansion
its kinda like a cross product

$$C_{ij} = (-1)^{i+j} \det(A_{ij})$$

then the determinant is just the sum of that times the cross section of whatever row and column you're deleting

6.3.3 Diagonalizability

The algebraic and geometrix multiplicities of eigenvectors don't care about matrix multiplication

a matrix needs n linearly independent eigenvectors for it to be diagonalizable.

6.4 Least Squared

$$A(A^TA)^{-1}A^T \qquad Ax =$$

6.5 SVD Decomposition

let A be an $m \times n$ matrix with rank r

find the orthonormal eigenbasis of A^TA with eigenvalues $\lambda_1, \dots, \lambda_n$.

Set $\sigma_i = \sqrt{\lambda_i}$

let $u_r = \frac{1}{\sigma_r} A v_r$

find $u_{r+1} \to u_m$ such that $u_1 \to u_m$ is an orthonormal basis.

$$U = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \dots & \\ & & \sigma_{min(m,n)} \end{bmatrix}$$

$$V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}$$