PHYS 325

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Contents

Chapter 1

PHYS325

I missed everything from the first 2 weeks because my laptop exploded whoopsies

Chapter 2

Equations of motion

derive the equations of motion to solve for a trajectory $\vec{r}(t)$ which is a position vector with respect to time

2.1 Newton's Second Law

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}(t)}{dt^2} \qquad \vec{F} = \frac{d\vec{p}(t)}{dt}$$

2.2 Strategy

1. choose a reference frame and coordinates
2. identify all the relevant forces (external forces)
   make a force diagram lol
3. integrate N2 for a given force $\vec{F}(\vec{r}, \dot{\vec{r}}, t)$ to find $\vec{r}(t)$

4. fix integration constants from inital or boundary conditions. (e.g. $\vec{v}_0 = \vec{v}(t = 0) = 0$)

5.

$$\vec{F} = 0$$
$$fromN2L0 = \vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$$
$$\rightarrow \vec{v} = const = \vec{v}_0$$
$$\vec{r}(t) = \int v_0 dt = v_0 t + r_0$$

6. if F is constant

$$\dot{\vec{v}} = \ddot{\vec{r}} = \vec{a} = \frac{\vec{F}_0}{m}$$
$$\int dv = \int \frac{F_0}{m} dt \rightarrow \vec{v}(t) = \frac{F_0}{m} t + \vec{v}_0$$
$$\vec{r}(t) = \int dr = \int \vec{v} dt =$$
$$\frac{1}{2}\frac{\vec{F}_0}{m} t^2 + \vec{v}_0 t + \vec{r}_0$$

only valid for constant force

2.2.1 Time Dependent Force

$$\frac{d\vec{v}}{dt} = \vec{a} = \frac{\vec{F}}{m}$$

separation of variables

$$d\vec{v} = \frac{\vec{F}(t)}{m}dt$$

$$\int d\vec{v} = \int \frac{\vec{F}(t)}{m}dt$$

$$\vec{v}(t) = \frac{\vec{F}}{m} + \vec{C} \qquad \vec{F} = \int F(t)dt$$

C is the integration constant

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{r} = \int \vec{v}dt$$

2.2.2 Forced Harmonic Oscillator

A particle m moves along $-\infty < x < \infty$. It is subjected to a force $F = F_0 \cos(\alpha t)$. It starts at time $t = 0, x_0 = x(t = 0) = 0, v_0 = v(t = 0) = 0$

1. coordinate system is just 1D
2. force is $F = F_0 \cos(\alpha t)$

3. equation of motion from N2L is $\frac{dv}{dt} = a = \frac{F}{m}$
4. separation of variables

$$dv = \frac{F}{m}dt = \frac{F_0}{m}dt = \frac{F_0}{m}\cos\alpha t dt$$

$$v(t) = \int dv = \frac{F_0}{m}\int \cos(\alpha t)dt = \frac{F_0}{m}\frac{1}{2}\sin\alpha t + C_1$$

$$x(t) = int dx = \int v dt =$$

$$\int \frac{F_0}{\alpha m}\sin(\alpha t) + C_1 dt =$$

$$-\frac{F_0}{\alpha^2 m}\cos(\alpha t) + C_1(t) + C_2$$

5. find initial conditions

$$0 = v_0 = \frac{F_0}{\alpha m}\sin\alpha 0 + C_1 \to C_1 = 0$$

$$0 = x_0 = -\frac{F_0}{\alpha^2 m}\cos(\alpha 0) + 0 + C_2 \to$$

$$C_2 = \frac{F_0}{\alpha^2 m}$$

$$x(t) = \frac{F_0}{\alpha^2 m}(1 - \cos(\alpha t))$$

$$v(t) = \frac{F_0}{\alpha m}\sin(\alpha t)$$

2.3 Position Dependent Force

focusing on 1 dimension for simplicity

get the equation of motion from Newton's 2nd Law

$F(x) = ma = m\frac{dv}{dt}$

chain rule

$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$

$mv\frac{dv}{dx} = F(x)$

separation of variables

$mv\,dv = F(x)dx$

Use definite integrals. relabel v = v' and x = x' (not derivatives)

$m\int_{v_0}^{v} v'\,dv' = \int_{x_0}^{x} F(x')\,dx'$

$\frac{1}{2}m(v^2 - v_0^2) = \int_{x_0}^{x} F(x')\,dx'$

solve for v

It looks like change in kinetic energy and work

$\Delta T = \frac{1}{2}m(v^2 - v_0^2) \qquad W = \int F(x)\,dx$

if force is conservative, it is path independent,

and it can be written as a gradient of a potential $\vec{F} = -\nabla U$

$F = -\frac{dU}{dx}$

$T - T_0 = \int_{x_0}^{x} -\frac{dU}{dx}\,dx' = -\left(U(x) - U(x_0)\right)$

$E = T + U(x) = T_0 + U(x_0)$

Conservation of Mechanical Energy

$E = T + U(x) = \frac{1}{2}mv^2 + U(x)$

$v = \pm\sqrt{\frac{2}{m}(E - U(x))}$

use $v = \frac{dx}{dt}$ to find $x(t)$

2.4 Analyzing the Potential

infer velocity and position

given energy, what is motion

$E = E_0 = E(x_0) = T(x_0) + U(x_0)$

$U(x_0) = E_0 = \text{const} \rightarrow T(x_0) = 0 \rightarrow v(x_0) = C$

2.4.1 case 2: energy of particle is _

$E = E_1 = T(x) + U(x)$

particle can move between x_{1a} and x_{1b}

$x_0 : E_1 = E = T + U(x_0)$

potential energy is at minimum, kinetic at max

$v(x_0)$ is maximal

2.4.2 Case 3: $E = E_2 = U(x_2)$

$T(x_2) = 0$ so $v = 0$

2.5 Analyzing Extrema of Potential

To find the extrema points, find the derivative of the potential and set it equal to 0.

To find the type of extrema point, take the second derivative and then check the sign

2.5.1 Push Particle Towards Extrema

1. $x = x_1$ is a minimum

   If the particle is slightly moved from the minimum, it will return to the minimum because that is where the lowest energy point is.

   This point is called stable for the particle.

2. $x = x_2$ is a maximum

   The particle will move farther away from the equilibrium point.

   This point is unstable

3. $x = x_3$ is a saddle point

   The particle is stable in one direction and unstable in the other marginally stable point

2.6 Simple Harmonic Oscillator

Simple Harmonic Oscillator

Figure out motion near the equilibrium point

To approximate a function only near a specific point, use a Taylor Series

Taylor expand potential around x_0

$$U(x) \approx U(x_0) + U'|_{x=x_0}(x - x_0) + \frac{1}{2}U''|_{x=x_0}(x - x_0)^2 + \dots$$

choose $U(x_0) = 0$ extremum

$$U(x) \approx \frac{1}{2}kx^2$$

Can also taylor expand force near x_0

$$F(x) \approx F(x_0) + F'|_{x=x_0}(x - x_0) + \frac{1}{2}F''|_{x_0}(x - x_0)^2$$

$$-U'|_{x_0} = 0 \qquad -U''|_{x_0} = -k \qquad \text{rest is small}$$

insert equation of motion

$$m\ddot{x} = F(x) \approx F'|_{x_0}(x - x_0) = -kx$$
$$m\ddot{x} + kx = 0$$

ansatz: guess the form of the solution (trig or exponent)

$$x(t) = A\sin(\omega t + \varphi)$$
$$\dot{x}(t) = \frac{dx}{dt} = A\omega\cos(\omega t + \varphi)$$
$$\ddot{x}(t) = -A\omega^2\sin(\omega t + \varphi)$$

insert ansatz into equation of motion

$$m(-A)\omega^2\sin(\omega t + \varphi) + kA\sin(\omega t + \varphi) = 0 \rightarrow$$
$$\omega = \sqrt{\frac{k}{m}} \qquad \text{period} = T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

2.7 velocity Dependent Force

$\vec{F} = \vec{F}(\vec{v})$

drag forces, friction, air resistance

2.7.1 Types of drag forces

1. Stoke's drag (linear)

$$\vec{F} = -c\vec{v}$$

laminar flow, valid for small velocities and viscous fluids

2. Newtonian Drag (nonlinear)

$$\vec{F}(\vec{v}) = -k\vec{v}^2$$

valid for larger velocities and less viscous fluids.

The type of drag is applicable based of Reynold's Number

$$R = \frac{\rho v L}{\mu} \qquad \text{density, fluid flow, size, viscosity}$$

2.8 Linear Drag

particle of mass m and initial velocity v_0 in laminar flow
goal, derive velocity after a long time (should be 0)
choose coordinate system: 1D along x direction $v = \dot{x}$
force $F(v) = -cv$
introduce new constant $\kappa = \frac{c}{m}$

$$[k] = [\frac{F}{m * v}] = [\frac{1}{t}]$$

$$F = -cv = -m\kappa v$$

$$F = m\ddot{x} = m\frac{dv}{dt} = -m\kappa v$$

separation of variables

$$\frac{1}{v}dv = -\kappa dt$$

use definite integrals, relabel variables

$$\int_{v_0}^{v} \frac{1}{v'}dv' = -\kappa \int_{t_0}^{t} dt'$$

$$-\kappa t = \ln(v')\Big|_{v_0}^{v} = \ln\left(\frac{v}{v_0}\right)$$

$$v(t) = v_0 \exp(-\kappa t)$$

exponential decay with a rate of κ

For some sanity checks, you can try $t = \infty$ and $t = 0$ to make sure they make sense

2.9 non-linear drag with gravity

motion is 1D along the z axis

drag force $F = -cv^2$ sgn(v)

(sgn(v) is ± 1 depending on if v is greater or less than 0)

new constant

$$\sigma = \frac{c}{m}$$

$$F = -mg + cv^2 \to -mg + \frac{C}{m}mv^2 \to m(g - \sigma v^2) \to$$

$$\frac{dv}{dt} = -g + \sigma v^2 \to \int_0^v \frac{1}{-g + \sigma v^2} dv = \int_0^t dt$$

$$t = \frac{-1}{\sqrt{g\sigma}} \tanh^{-1}(v\sqrt{\frac{\sigma}{g}})$$

$$v = -\sqrt{\frac{g}{\sigma}} \tanh\Big(\sqrt{gh}t\Big)$$

$$t \to \infty \qquad v \to -\sqrt{\frac{g}{\sigma}}$$

You can see that v asymptotically approaches a constant velocity with large time which you can kind of see from the initial equation.

For most cases $m\frac{dv}{dt} = \vec{F}(\vec{r}, \vec{v}, t)$ is unsolvable. However, for some cases $F = f(v)g(t)$, you get $\int \frac{m}{f(v)} dv = \int g(t)dt$

For another special case $F = f(v)h(x)$, you get

$$m\frac{dv}{dt}\frac{1}{f(v)} = h(x)$$

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$mv\frac{dv}{dx}\frac{1}{f(v)} = h(x) \to \int \frac{mv}{f(v)} dv = \int h(x)dx$$

2.10 Time Varying Mass $M(t)$

u is the speed of the mass being projected out.

$$P_i = M_i v_i$$

$$P_f = dm(v-u) \to t(m_f v_f) = fm(v-u)$$

$$P_i = P_f \Rightarrow Mv_i = -udm + m_i v_i - dMdv$$

$dMdv \approx 0$ and cancel some stuff

$$Mdv = tudm \Longrightarrow dv = \frac{-u}{M}dm \to \int dv = \int \frac{-u}{M}dm \to$$

$$v - v_0 = -u\ln\left(\frac{M}{M_0}\right) \to v = v_0 - u\ln\left(\frac{M}{M_0}\right)$$

Because its logarithmic, its really hard to boost something by chucking mass out the back.

Conservation of momentum is not true in gravity $P_i \neq P_f$

$$dp = P_f - P_i \neq 0$$

$$dp = -Mgdt - Mg\,dt = Mdv + udM$$

$$\frac{dv}{dt} = -g - \frac{u}{M}\frac{dM}{dt} \qquad \text{THIS IS THE IMPORTANT EQUATION}$$

u, M, and dM/dt are all controllable by the rocket

2.10.1 velocity from acceleration

$$v(t) = v_0 - gt - u\ln\left(\frac{M_0 - dt}{M_0}\right)$$

2.11 2D Rocket

$$\vec{F}(\vec{r}, \dot{\vec{r}}, t) = m\vec{a} = m\ddot{\vec{r}}$$

Cartesian Coordinates

$$F_x(\vec{r}, \dot{\vec{r}}, t) = m\ddot{x} \qquad F_y(\vec{r}, \dot{\vec{r}}, t) = m\ddot{y} \qquad F_z(\vec{r}, \dot{\vec{r}}, t) = m\ddot{z}$$

Consider a magnetic field Be_z

$$\vec{F} = q\vec{v} \times \vec{B} = q(\vec{v} \times \vec{B}) \rightarrow$$
$$qB(v_y e_x - v_x e_y) \qquad F_x = qBv_y = m\ddot{x} \qquad F_y = -qBv_x = m\ddot{y}$$
$$m\ddot{z} = 0 \Longrightarrow \dot{z} = \text{const}$$
$$qB\dot{y} = m\ddot{x} \qquad qB\dot{x} = m\ddot{y}$$
$$qB\ddot{y} = m\dddot{x} \rightarrow qB(\frac{-qB}{m})\dot{x} = m\dddot{x} \rightarrow \ddot{v}_x = -\omega^2 v_x$$

this is the spring equation

$$\ddot{x} = \frac{k}{m}x \longrightarrow v_x = A\sin(\omega t + \phi) \qquad v_y = A\cos(\omega t + \phi)$$

it goes in a circle

2.11.1 something

$$\ddot{v}_x = \omega \dot{v}_y$$
$$\ddot{v}_x = -\omega^2 v_x \qquad \ddot{v}_y = -\omega^2 v_y$$
$$\dot{\nu} = \dot{v}_x + i\dot{v}_y$$
$$= \omega v_y - i\omega v_x$$
$$\dot{\nu} = -i\omega\nu$$

and some stuff I missed

2.11.2 Particle in Field

Motion of a charged particle in a homogeneous magnetic field

$$\dot{v}_x = \omega v_y \qquad \dot{v}_y = -\omega v_x \qquad w/\omega = \frac{qB}{m}\mu = v_x + iv_y$$
$$\dot{m}u = \dot{v}_x + i\dot{v}_y \longrightarrow \omega v_y - i\omega v_x \to -i\omega(v_x + iv_y)$$
$$\dot{\mu} = -i\omega\mu$$

solve with separation of variables

$$\frac{1}{\mu}\,d\mu = -i\omega\,dt \to \ln(\mu) + \tilde{C} = \ln\left(\frac{\mu}{C}\right)$$
$$\mu = C\exp(-i\omega t) \qquad C = Ae^{i\delta}$$
$$\mu = A\exp(i(\delta - \omega t))$$

Use the euler identity to keep doing math

$\mu = A(\cos(\omega t - \delta) - i\sin(\omega t - \delta))$

the signs reversed and im not entirely sure why or how

$v_x = \mathrm{Re}(\mu) = A\cos(\omega t - \delta) \qquad v_y = \mathrm{Im}(\mu) = -A\sin(\omega t - \delta)$

$|v^2| = A^2\cos(\omega t - \delta) + A_2\sin(\omega t - \delta) = A^2$

double check this is true by showing derivative of v is 0

$\frac{1}{2}\frac{d}{dt}|v|^2 = \frac{1}{2}m2v \cdot \frac{d}{dt}v$

$= v \cdot q(v \times B) = 0$

$= 0 \qquad A = \text{constant}$

Now you figure out the trajectory

$\vec{v} = \frac{dr}{dt} \rightarrow v\,dt = dr$

$x(t) = \int v_x\,dt = \int A\cos(\omega t - \delta)\,dt = \frac{A}{\omega}\sin(\omega t - \delta) + C_y$

$z(t) = \int v_z\,dt = \int v_{z0}\,dt = v_{z0}t + z_0$

$A = \sqrt{v_x^2 + v_y^2}$

Chapter 3

Curvilinear Coordinates

non-Cartesian coordinates

3.0.1 Examples

2d polar coordinates (r, θ)

$$x = r\cos(\theta(t)) \qquad y = r\sin(\theta(t)) \qquad r = \sqrt{x^2 + y^2}$$

3d cylindrical coordinates (r, ϕ, z)

$$x = r\cos(\phi) \qquad y = r\sin(\phi) \qquad z = z$$

3d spherical coordinates (r, θ, ϕ)

$$x = r\cos(\phi)\sin(\theta) \qquad y = r\sin(\phi)\sin(\theta) \qquad z = r\cos(\theta)$$

3.1 Whirling Stick

A rigid stick whriling with fixed ω

use polar coordinates to have the easiest math

$\omega = \varphi t = \text{const}$

$e_r = e_r(t) \qquad e_\varphi = e_\varphi(t)$

relate r and φ to x and y basis vectors

$e_r(t) = \cos(\varphi(t))e_x + \sin(\varphi(t))e_y$

$e_\varphi(t) = -\sin(\varphi(t))e_x + \cos(\varphi(t))e_y$

position vector $r(t) = r(t)e_r(t)$

Determine the first derivatives

$\dot{e}_r = \frac{d}{dt}\cos(\varphi(t))e_x + \sin(\varphi(t))e_y \rightarrow$

$\frac{d}{dt}\cos(\varphi(t))e_x + \cos(\varphi)\frac{d}{dt}e_x + \frac{d}{dt}\sin(\varphi(t))e_y + \sin(\varphi(t))\frac{d}{dt}e_y$

$\frac{d}{dt}\cos(\varphi(t))e_x + \frac{d}{dt}\sin(\varphi(t))e_y$

$\dot{e}_r = -\dot{\varphi}\sin(\varphi)e_x + \dot{\varphi}\cos(\varphi)e_y = \dot{\varphi}e_\varphi$

$\cdot e_\varphi = -\dot{\varphi}e_\varphi$

$v(t) = \dot{r}(t)e_r + r\dot{\varphi}e_\varphi = v_r e_r + v_\varphi e_\varphi$

now acceleration lol

$a = \frac{dv}{dt} = (\ddot{r} - r\dot{\varphi}^2)e_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})e_\varphi$

$\varphi = \omega t \qquad r\dot{\varphi}^2 = r\omega^2$

centripetal force

3.2 Bead on a Whirling Rod

Use polar coordinates for sake of convenience. Use basis vectors e_r and e_ϕ

Rod whirls at a rate of ω, so $\phi(t) = \omega t$

Our general strategy is draw a sketch and then figure out your coordinates and then write out position, velocity, and acceleration vectors.

Used Newton's 2nd law to to get a differential equation.

Solve the differential equation.

$a(t) = [\ddot{r} - r\dot{\phi}^2]\vec{e}_r + [r\ddot{\phi} + 2\dot{r}\dot{\phi}]\vec{e}_\phi$

$F = ma$

no force in the radial direction because it's all normal force

$F_{net} = F_n = F_n(t)e_\phi(t) + 0e_r(t)$

$F_n e_\phi(t) = m[\ddot{r} - r\dot{\phi}^2]\vec{e}_r + m[r\ddot{\phi} + 2\dot{r}\dot{\phi}]\vec{e}_\phi$

$e_r = 0 = m(\ddot{r} - r\dot{\phi}^2) \qquad e_\phi = F_n = m(r\ddot{\phi} + 2\dot{r}\dot{\phi})$

$0 = \ddot{r} - r\dot{\phi}^2 \rightarrow \ddot{r} = r\dot{\phi}^2 \qquad \phi(t) = \omega t, \dot{\phi}(t) = \omega$

$\ddot{r} = \omega^2 r$ use an ansatz/guess

$r = e^{\lambda t} \qquad \dot{r} = \lambda e^{\lambda t} \qquad \ddot{r} = \lambda^2 e^{\lambda t}$

$\lambda^2 e^{\lambda t} = \omega^2 e^{\lambda t} \qquad \lambda^2 = \omega^2 \qquad \lambda = \pm\omega$

$r(t) = Ae^{\omega t} + Be^{-\omega t}$ solve with initial condition

$\dot{r}(t) = \omega Ae^{\omega t} - \omega Be^{\omega t}$

$r(t = 0) = r_0 \qquad v(t = 0) = v_0$

$A + B = r_0 \qquad A - B = 0 \rightarrow A = B \qquad A = B = \frac{r_0}{2}$

$r(t) = \frac{r_0}{2}(e^{\omega t} + e^{-\omega t}) = \frac{r_0}{2}\cosh(\omega t)$

$F_n = m(r\ddot{\phi} + 2\dot{r}\dot{\phi}) \qquad \phi = \omega t$

$F_n = m2\dot{r}\omega = m\omega^2 r_0 \sinh(\omega t)$

3.2.1 Bead on Spinning Loop

Loop of fixed radius R spinning about vertical axis at fixed rate Ω. Bead of mass m, free to move along the loop. Everything is in a gravitational field.

Use spherical coordinates for math convenience.

$x = r\sin(\theta)\cos(\phi) \qquad y = r\sin(\theta)\sin(\phi) \qquad z = r\cos(\theta)$

$r^2 = x^2 + y^2 + z^2 \qquad \tan(\theta) = \dfrac{\sqrt{x^2+y^2}}{x^2}$

the coordinate vectors are real difficult to find

coords, position, velocity, acceleration

$e_r = \sin(\theta)\cos(\phi)e_x + \sin(\theta)\sin(\phi)e_y + \cos(\theta)e_z$

$e_\theta = \cos(\theta)\cos(\phi)e_x + \cos(\theta)\sin(\phi)e_y - \sin(\theta)e_z$

$e_\phi = -\sin\phi e_x + \cos(\phi)e_y$

$r(t) = R = \text{const} \qquad \phi(t) = \Omega t \qquad \Omega = \text{const} \qquad \theta(t)$

$\dot{e}_r = \text{blah}\dot{e}_\theta = \text{blah}\dot{e}_\phi = \text{blah}$

$r(t) = r(t)e_r(t) \qquad v(t) = \dfrac{d}{dt}\left(r(t)e_r(t)\right) = \dot{r}e_r + r\dot{e}_r =$

$\dot{r}e_r + r\left(\dot{\theta}e_\theta + \dot{\phi}\sin(\theta)e_\phi\right) = \dot{r}e_r + r\dot{\theta}e_\theta + r\sin(\theta)\dot{\phi}e_\phi$

$a = \dfrac{dv}{dt} = \dfrac{d}{dt}\dot{r}e_r + r\dot{\theta}e_\theta + r\sin(\theta)\dot{\phi}e_\phi =$

$\ddot{r}e_r + \dot{r}\dot{e}_r + \dot{r}(\dot{\theta}e_\theta) + r(\ddot{\theta}e_\theta + \dot{\theta}\dot{e}_\theta) +$

$\dot{r}(\sin(\theta)\dot{\phi}e_\phi) + r(\cos(\theta)\dot{\phi}e_\phi + \sin(\theta)(\ddot{\phi}e_\phi + \dot{\phi}\dot{e}_\phi))$

what insanityplug in all the coordinate vectors to the thingy

$\vec{a}(t) = [\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2]e_r + r[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2]e_\theta +$

$[2\dot{r}\sin(\theta)\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}]e_\phi$

now use $F = ma$ to do some bullshit

$F = ma \qquad F_g = -mge_z = -mg(\cos\theta e_r - \sin(\theta)e_\theta)$

$F_n = N_r e_r + N_\phi e_\phi$

$F_{net} = F_G + F_n = F_n = N_r e_r + N_\phi e_\phi - mg(\cos\theta e_r - \sin(\theta)e_\theta) =$

$[N_r - mg\cos(\theta)]e_r + mg\sin(\theta)e_\theta + N_\phi e_\phi$

$F_\theta = ma_\theta$

$mg\sin(\theta) =$

$[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2]e_r + r[2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin(\theta)\cos(\theta)\dot{\phi}^2]e_\theta +$

$[2\dot{r}\sin(\theta)\dot{\phi} + 2r\cos(\theta)\dot{\theta}\dot{\phi} + r\sin\theta\ddot{\phi}]e_\phi$

$g\sin(\theta) = r\ddot{\theta} - R\sin(\theta)\cos(\theta)\Omega^2$

Chapter 4

Types of Forces

4.0.1 Conservative Forces in 2d and 3d

4.0.2 Conservative Force

a force $\vec{F}(\vec{r})$ is conservative if it can be written as the gradient of a potential $\vec{U}(\vec{r})$.

$$\vec{F}(\vec{r}) = -\vec{\nabla} U(\vec{r})$$

4.0.3 ∇ operator

$$\vec{\nabla} = \vec{e}_x \frac{d}{dx} + \vec{e}_y \frac{d}{dy} + \vec{e}_z \frac{d}{dz}$$

if you have another coordinate system

$$\vec{\nabla} = \vec{e}_r \frac{d}{dr} + \vec{e}_\phi \frac{d}{d\phi} + \vec{e}_z \frac{d}{dz}$$

4.0.4 Curl of Conservative Force

$\nabla \times \vec{F}(\vec{r}) = \vec{\nabla} \times (-\nabla U) = 0$

NOT every force is conservative.

$$F = F_{cons} + F_{diss}$$
$$\nabla \times F = \nabla \times F_{cons} + \nabla \times F_{diss} \neq 0$$

If $\nabla \times F = 0$, then the forces can be written as $F = -\nabla U$ (has a potential) and $\vec{F}$ is converative. (useful to check)

4.0.5 Work along a path

$$w = \int_{p_2}^{p_1} \vec{F} \cdot d\vec{r} = -\int_{p_2}^{p_1} \vec{\nabla} U \, dr \qquad \nabla \approx \frac{d}{dr}$$
$$-\int_{p_2}^{p_1} dU = -(U(p_2) - U(p_1))$$

If path is closed $-(U(p_1) - U(p_1)) = 0$
A conservative force does NO work along a closed path.

4.0.6 Energy Conservation

If a force is conservative, $F = -\nabla U$, then the total energy $\vec{E} = T + U = \text{const.}$ (constant of motion)

How to show that this is true?

$$\frac{d}{dt}E = \frac{d}{dt}(T+U) = 0 = E = \text{const}$$
$$\frac{d}{dt}T = \frac{d}{dt}\frac{1}{2}mv^2 = mv \cdot \frac{dv}{dt} = \vec{v} \cdot \vec{F}$$
2nd term
$$\frac{d}{dt}U(t, \vec{r}(t)) = \frac{d}{dt}U + \frac{dx}{dt}\frac{dU}{dx} + \frac{dy}{dt}\frac{dU}{dy} + \frac{dz}{dt}\frac{dU}{dz}$$
$$\frac{d}{dt}U(t, r(t)) = \frac{dU}{dt} + \frac{dr}{dt} \cdot \nabla U = -\vec{v} \cdot \vec{F} + \frac{dU}{dt}$$
$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \vec{v} \cdot \vec{F} - \vec{v} \cdot \vec{F} + \frac{dU}{dt} = 0$$

$E = T + U = \text{const}$ is conserved, if $\vec{F}$ is conservative, and $U = U(\vec{r})$ (ie, no explicit time dependence)

4.0.7 How to Find Potential

given some force F

option a: show this with indefinite integrals

$$\vec{F} = -\vec{\nabla}U$$

option b: use indefinite integrals

$$F = -\nabla U \rightarrow U = \int_{p_1}^{p_2} = -\int_{p_1}^{p_2} \vec{F} \cdot dr$$

4.0.8 example

$$\vec{F} = (2xy + 1)\vec{e}_x + (x^2 + 2)\vec{e}_y$$

find U

$$-U = \int \vec{F} \cdot d\vec{r} = \int (F_x dx + F_y dy + F_z dz)$$
$$\int_0^x F_x(x', 0, 0)dx + \int_0^y F_y(x, y', 0)dy + \int_0^z F_z(x, y, z')dz$$
$$\int_0^x 1dx' + \int_0^y (x^2 + 2)dy' + \int_0^z 0dz' \rightarrow$$
$$-U = x + (x^2 + 2)y + 0 = -x - x^2y - 2y$$

4.1 Central Forces

force that points towards or away from a point radially and is dependent on distance r

$$F = f(r, \theta, \phi)\vec{e}_r$$

4.1.1 Conservation of Angular Momentum

Define angular momentum $\vec{L}$ with respect to a reference point o

4.1.2 YOUR LAPTOP DIED FIGURE OUT LATER

I still don't get anything about orbits or angular momentum of anything like that.

You missed the last half of the friday lecture on like september 20th ish

4.2 Conservative Central Forces

$\vec{L} = \vec{r} \times m\vec{v}$

If you have a conservative central force, you have conservation of angular momentum.

$L = \text{const}$

to prove, take the derivative with respect to time

$$\frac{dL}{dt} = \frac{d}{dt}\left(\vec{r} \times m\vec{v}\right) = \dot{\vec{r}} \times m\vec{v} + \vec{r} \times m\frac{dv}{dt}$$

$$\left(m\frac{dv}{dt} = \vec{F}\right) \qquad \dot{\vec{r}} \times m\vec{v} = \vec{v} \times m\vec{v} = 0$$

$$F = f(r,\theta,\phi)\vec{e}_r \qquad \vec{r} = r\vec{e}_r$$

$= r\vec{e}_r \times mf(r,\theta,\phi)\vec{e}_r = 0$ because the unit vectors are the same

4.2.1 useful math

$$f(r,\theta,\phi) \qquad e_r \times e_\phi = -\vec{e}_\theta \qquad e_\theta = -e_z$$

4.2.2 Consequences

- Motion remains in $\vec{r} - \vec{v}$ plane
- can rotate coords s.t. $\theta = \pi/2 \qquad \dot{\theta} = 0$
- consider

$$\vec{r} = r(t)\vec{e}_r \qquad v = \dot{r}\vec{e}_r + r\dot{\theta}e_\theta + r\dot{\phi}\sin(\theta)e_\phi = \dot{r}\vec{e}_r + r\dot{\phi}e_\phi$$
$$\vec{L} = \vec{r} \times m\vec{v} = mr\vec{e}_r \times (\dot{r}\vec{e}_r + r\dot{\phi}e_\phi) =$$
$$mr\vec{e}_r \times \dot{r}\vec{e}_r \approx \vec{e}_r \times e_r = 0$$
$$\vec{L} = -mr^2\dot{\phi}\vec{e}_\theta \qquad L\vec{e}_z = +mr^2\dot{\phi}\vec{e}_z$$

magnitude $L = mr^2\dot{\phi}$

direction in e_z

transformed our 3d equation to 2d because of conservation of angular momentum

4.2.3 Conservative, Central Force

$$\vec{F} = f\vec{e}_r = -\nabla U$$

no dependence in $\vec{e}_\phi, \vec{e}_\theta$

$$\frac{dU}{d\theta} = 0 \qquad \frac{dU}{d\phi} = 0$$
$$\vec{F} = -\frac{dU}{dr}\vec{e}_r \to U = -\int f\,dr$$

Examples of Conservative, Central Forces: gravity, coulomb potential, Yukawa potential (nuclear force)

4.2.4 IMPORTANT

$L = mr^2\dot{\phi}$

4.2.5 back to example

$$\vec{F} = -\frac{K}{r^2}\vec{e}_r \qquad K > 0$$

$$U = -\int -\frac{K}{r^2}\,dr = \frac{-K}{r}(\lim_{r\to\infty} = 0)$$

$$L = \text{const}$$

choose $L = Le_z, \theta = \pi/2, \dot{\theta} = 0$

$$\dot{\phi} = \frac{L}{mr^2} \qquad E = T + U = \text{const}$$

$$T = \frac{1}{2}m|\vec{v}|^2 \qquad \vec{v} = \dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi$$

$$T = \frac{1}{2}m(\dot{r}\vec{e}_r + r\dot{\phi}\vec{e}_\phi) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\left(\frac{r^2L^2}{m^2r^4}\right) = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2}$$

$$E = T + U = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{K}{r}$$

$$T = \frac{1}{2}m\dot{r}^2 \qquad U_{eff} = \frac{L^2}{2mr^2} - \frac{K}{r}$$

EFFECTIVE POTENTIAL IS IMPORTANT

$$U_{eff} = \frac{L^2}{2mr^2} + U$$

4.3 Analyze Effective Potential

$$U_{eff} = \frac{L^2}{2mr^2} - \frac{K}{r} \qquad K > 0$$

You can circular, elliptical, parabolic, and hyperbolic orbits.

4.3.1 Case: $E > 0$

$\frac{L^2}{2mr^2}$

particle moves towards $r = 0$ and then has a turning point where $\frac{dr}{dt} = 0$. The particle then returns back to $r \to \infty$

This is a hyperbolic trajectory.

4.3.2 Case: $E < 0$

$\frac{K}{r} > (\frac{L^2}{2mr^2} + T_r)$

E is stuck in a potential well and will oscillate between turning points in the potential.

elliptical orbit.

4.3.3 Case: $E < 0$ and $U'_{eff}\Big|_{r=r_c} = 0$

$U'_{eff} = \frac{dU_{eff}}{dr}$

orbit is circular with radius r_c

$$r = r_c = \text{const} \qquad T_r = \frac{1}{2}m\dot{r}^2 = 0$$

because the radius never changes.

$E = 0$ is a parabolic orbit but eh.

Chapter 5

Gravity

It's a pretty important thing in our day to day life.

Need to think about both gravitational force and potential.

2 masses a distance r apart experience a force in the $\vec{e}_r$ direction

$$\vec{F}_G = -\frac{Gm_1m_2}{r^2}\vec{e}_r \qquad G = 6.67 * 10^{-11}\frac{Nm^2}{kg^2}$$

consider extended mass m_1

$$m_1 = \int dm_1 = \int^{V_1} \rho_1(\vec{r}_1) d\vec{r}_1$$

$$F_G = -Gm_2 \int \frac{\rho_1(r_1)}{|r_2 - r_1|^3}(r_2 - r_1) dr_1$$

$$\vec{g} = \frac{\vec{F}_G}{m_2} = -G \int \rho(\vec{r}_1) \frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} d^3r$$

$$\frac{1}{|r_2 - r_1|^2} \cdot \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$$

$$\frac{1}{|r_2 - r_1|^2} \equiv \text{"}\frac{1}{r^2}\text{"} \qquad \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} = \vec{e}_r$$

5.1 Gravitational Potential

$$\vec{g} = -\nabla\Phi$$

analogous to $U = m_2\Phi$, $\vec{F}_G = -\nabla U$

$$U = \frac{Gm_1m_2}{r} \qquad \Phi = \frac{U}{m_2} = \frac{-Gm_1}{r}$$

$$\Phi = -G \int_V \frac{\rho(\vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} d^3r$$

5.1.1 Φ of a Spherical Shell

Setup: given a uniform spherical shell with mass M, radius R, thickness h, mass density $\phi = H/V = M/(4\pi R^2 h)$.

Compute Φ

What we do is make an arbitrary point p in the mass such that the distance from that point to m_2 is $\vec{r}$ and for some point R away from p, we can see the distance from that point to m_2 is $\vec{r} - \vec{r}_1$. We also make a function s such that $s(\theta) = |\vec{r} - \vec{r}_1|$

the thickness is very small so $h = dr$ Strategy:

$$\Phi = \int d\Phi \qquad d\Phi = \frac{-G}{|\vec{r} - \vec{r}_1|} dm$$

$$dm(\vec{r}_1) = \rho(\vec{r}_1) d^3\vec{r}_1 = \rho R^2 dr \sin(\theta) d\theta d\phi$$

$$R^2 dr = \text{radius} \qquad \sin(\theta) d\theta d\phi = \text{Surface of sphere}$$

$$d\Phi = \frac{Gdm}{|\vec{r} - \vec{r}_1|} = \frac{-G}{s(\theta)} dm$$

$$\Phi = \int d\Phi = -G \int \frac{dm}{s(\theta)} = -G\rho h R^2 \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\sin(\theta)}{s(\theta)} d\theta d\phi =$$

$$-2\pi G h \rho R^2 \int_0^{\pi} \frac{\sin(\theta)}{s(\theta)} d\theta$$

solve integral (substitute $\int d\theta \to \int ds$)

$$s(\theta) = |\vec{r} - \vec{r}_1| \qquad s^2(\theta) = |\vec{r} - \vec{r}_1|^2 = r^2 - 2\vec{r} \cdot \vec{r}_1 + \vec{r}_1^2 =$$

okay so how s works is $\vec{r}$ is a vector at the radius of the sphere that moves constantly in order for you to integrate over the whole sphere and $\vec{r}_1$ is a vector that point solely to x which is the point that you are trying to find the potential at.

$$x^2 + R^2 - 2Rx\cos(\theta)$$

why is that θ the same as spherical coords θ (angle from $\vec{z}$ line)

$$\frac{d(s^2)}{d\theta} = 2s\frac{ds}{d\theta} = 2Rx\sin(\theta)$$

$$\frac{\sin(\theta)}{s}\,d\theta = \frac{1}{Rx}ds$$

$$\int_0^{\pi}\frac{\sin(\theta)}{s}\,d\theta = \int_{s_{min}}^{s_{max}}\frac{1}{Rx}ds$$

$$s_{min} = s(\theta = 0) = \pm(R - x) > 0$$

$$s_{max} = s(\theta = \pi) = (R + x) > 0$$

$$\Phi = -2\pi G\frac{\rho hR}{x}(s_{max} - s_{min})$$

5.1.2 case: m_2 outside of mass

$$s_{min} = -(R - x) = x - R \qquad s_{max} = (R + x)$$

$$\Phi = -4\pi G\frac{\rho hR^2}{x} = \frac{-GM}{x}$$

Outside of an extended mass, the gravitational potential is the same as Φ of a point mass with the same M

5.1.3 case: m_2 inside of mass

$$s_{min} = (R - x) \qquad s_{max} = (R + x)$$
$$\Phi = -4\pi G\rho h R = \frac{-GM}{R} = \text{const}$$

5.2 Φ of a uniform sphere

Radius R, mass M, constant mass density $\rho = \begin{cases} \rho^*, r < R \\ 0, r > R \end{cases}$

Find $\Phi = \Phi(x)$
Strategy: Use infinitely many thin shells

$$d\phi = \begin{cases} -4\pi G\frac{\rho r^2}{x}dr & x > r \\ -4\pi G\rho r dr & x < r \end{cases}$$
$$\Phi = \int d\phi = -4\pi G\int_0^R \frac{\rho r^2}{x}dr = -4\pi G\int_0^R \frac{\rho^* r^2}{x}dr = \frac{-GM}{x}$$

5.2.1 inside sphere

$$\Phi = \int_0^x d\Phi_{inner} + \int_x^R d\Phi_{out} =$$
$$-4\pi G\frac{\rho^*}{x}\int_0^x r^2 dr - -4\pi G\rho^* \int_x^R r dr =$$
$$\frac{1}{2}GM\frac{3R^2 - x^2}{R^3}$$

5.3 Gravitational Orbits

5.3.1 Kepler's Laws

- Everything orbits in planar ellipses
- the line between the sun and a planet sweeps equal areas in equal times
- The orbital period around the sun is proportional to the 3/2 power of the semimajor axis

$L = mr^2\dot{\varphi}$ conservation of angular momentum

per unit mass $l = \frac{L}{m} = r^2\dot{\varphi} = \text{const}$

energy (per unit mass) conservation

$\varepsilon = \frac{E}{m} = \frac{1}{m}(T + U_{grav}) \qquad U_{grav} = -\frac{GMm}{r}$

$\varepsilon = \frac{1}{2}\vec{v}^2 - \frac{GM}{r}$

$\vec{v} = \dot{r}\vec{e}_r + r\dot{\varphi}e_\varphi \qquad v^2 = \vec{v} \cdot \vec{v} = \dot{r}^2 + r^2\dot{\varphi}^2$

$\varepsilon = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{GM}{r} = T + U_{eff}$

Now I should be able to write out the orbit as a 1d equation.
perigee r_p is the minimal distance from M
apogee r_a is the maximal distance from M
velocity vectors v_p and v_a perpendicular to the eerigee and apogee
$v_p \cdot v_a = 0$ and the angle between r_p and r_a is $\pi/2$
angular momentum

$\vec{l} = \vec{r} \times \vec{v} \qquad l = |\vec{r}||\vec{v}|\sin(\alpha)$

in perigee and apogee, $\sin(\alpha) = 1$

$l = r_p v_p = r_a v_a = \text{const}$

5.3.2 Energy (per unit mass)

$$\varepsilon = \frac{l^2}{2r_{p,a}^2} - \frac{GM}{r_{p,a}}$$
$$= \frac{1}{2}v_{p,a} - \frac{GM}{r}$$
$$\varepsilon = \frac{1}{2}v_{p,a}^2 - \frac{GM}{l}v_{p,a}$$
$$v_{p,a} = \frac{GM}{l} \pm \sqrt{\left(\frac{GM}{l}\right)^2 + 2\varepsilon}$$

$+$ is perigee and $-$ is apogee

$$r_{p,a} = \frac{l}{v_{p,a}} = l\left(\frac{GM}{l} \pm \sqrt{\left(\frac{GM}{l}\right)^2 + 2\varepsilon}\right)^{-1}$$

5.4 Effective Potential and Orbits

type of orbit depends on the sign of the total energy of the particle

$$\varepsilon = \frac{1}{2}\dot{r}^2 + U_{eff} \qquad U_{eff} = \frac{l^2}{2r^2} - \frac{GM}{r}$$

5.4.1 case: $\varepsilon < 0$ and $U'_{eff} = 0$

Circular orbit. Absolute minimum in effective potential if $U''_{eff} > 0$ Find speed

$$-m\frac{v^2}{r}\vec{e}_r = -ma_c\vec{e}_r = m\vec{a} = -\frac{GMm}{r^2}\vec{e}_r$$

$$v = \sqrt{\frac{GM}{r}}$$

Find orbital period

$$p = \frac{2\pi}{\omega} = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{r^3}{GM}}$$

Angular momentum per unit mass

$$l = |\vec{r}||\vec{v}|\sin(\alpha) = rv$$

$$v = \frac{GM}{l} \qquad r = \frac{l^2}{GM}$$

energy

$$\varepsilon = \frac{1}{2}v^2 - \frac{GM}{r} \qquad v^2 = \frac{GM}{r}$$

$$\varepsilon = -\frac{1}{2}v^2 = -\frac{1}{2}\left(\frac{GM}{l^2}\right)^2 < 0$$

escape velocity (velocity such that $E = 0$)

$$v_{esc} = \sqrt{\frac{2GM}{r}}$$

5.4.2 case: Elliptic Orbit

$$-\frac{1}{2}\left(\frac{GM}{l}\right)^2 = \varepsilon_c < \varepsilon < 0$$

particle moves between perigee and apogee determined by (ε, l)

5.4.3 case: $\varepsilon = 0$

parabolic orbit

perigee $v_p = \frac{2GM}{l}$

$r_p = \frac{l}{v_p} = \frac{l^2}{2GM}$

at the "apogee", $v_a \to 0$, $r_a = \frac{l}{v_a} \to \infty$

marginally unbound orbit

5.4.4 case: $\varepsilon > 0$

hypoerbolic orbit

$$v_{p,a} = \frac{GM}{l} \pm \sqrt{\left(\frac{GM}{l}\right)^2 + 2\varepsilon}$$

$v > 0, r > 0$ hyperbola $v < 0$ unphysical speed

$$v_{p,a} = \frac{GM}{l} \pm \sqrt{\left(\frac{GM}{l}\right)^2 + 2\varepsilon}$$

if $\varepsilon > 0$ then it has to be plus
if $\varepsilon = 0$ then $v_p = \frac{2GM}{l}, v_a = 0$
if $\varepsilon < 0$ then you can have $+$ or $-$

5.5 2 body problem

$$F = m_1\ddot{r}_1 = -\frac{Gm_1m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$
$$F = m_2\ddot{r}_1 = -\frac{Gm_1m_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|}$$

where $(\vec{r}_1 - \vec{r}_2)$ is the distance between the two masses.

Instead of dealing with 6 differential equations. We can use the center of mass frame to treat the 2 body problem as a 1 body problem.

1 mass of total mass $M = m_1 + m_2$
1 orbiting object of reduced mass $\mu = \frac{m_1 * m_2}{M}$
Center of mass

$$\vec{C} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{M}$$

Check that there is not external force so this reference frame is inertial

$$F_{net} = ma + ma = m_1\ddot{r}_1 + m_2\ddot{r}_2 =$$
$$\frac{Gm_1m_2}{|r_1 - r_2|^2}\frac{\vec{r}_2 - \vec{r}_1}{|r_2 - r_1|} + \frac{Gm_1m_2}{|r_1 - r_2|^2}\frac{\vec{r}_1 - \vec{r}_2}{|r_2 - r_1|} = 0$$

5.5.1 Equations of Motion

$$\ddot{\vec{r}} = \frac{GM}{|r|^2}\frac{\vec{r}}{|r|}$$

The relative position vector $\vec{r}$ is governed by the same equations of motion as a test mass μ in the gravitational field of M.

5.5.2 Energy

$$E = T + U = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - \frac{Gm_1m_2}{|r_2 - r_1|}$$

$$\rightarrow$$

$$E = \frac{1}{2}\mu\dot{r}^2 - \frac{GM\mu}{|\vec{r}|} + \frac{1}{2}M\dot{\vec{C}}^2$$

$$\frac{1}{2}M\dot{\vec{C}}^2 = \text{const} \qquad \frac{1}{2}\mu\dot{r}^2 - \frac{GM\mu}{|\vec{r}|} = \text{const}$$

5.5.3 Trajectory

Choose Spherical coordinates (r, θ, φ)

For a central force, angular momentum is conserved. $\dot{L} = 0$
Choose $\vec{L} = L\vec{e}_z$ so that $\theta = \frac{\pi}{2}, \dot{\theta} = 0$
$\vec{v} = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$

$$
\begin{aligned}
&l = \frac{L}{\mu} = |\frac{1}{\mu}r \times \mu\vec{v}| = \text{const} \ldots = r^2\dot{\varphi} \\
&\dot{\varphi} = \frac{l}{r^2} \\
&T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}) = \frac{1}{2}\mu(\dot{r}^2 + \frac{l^2}{r^2}) \\
&\varepsilon = \frac{E}{\mu} = \frac{1}{2}\dot{r}^2 + U_{eff} = \frac{1}{2}\dot{r}^2 + \frac{l^2}{2r^2} - \frac{GM}{r} \\
&\ldots \\
&-\dot{r} = \sqrt{2(\varepsilon - U_{eff})} = \frac{dr}{dt} \to \int dt = \int \frac{dr}{\sqrt{2(\varepsilon - U_{eff})}} \\
&\text{calculate and invert to get } r(t)
\end{aligned}
$$

5.5.4 Kepler's Orbits $r(\varphi)$

Goal: find orbits parameterized by $r(\varphi)$.

Introduce a new variable that's the inverse of r and then write down Newton's Second Law (N2L) to get a new differential equation and then solve it.

$$u = \frac{1}{r}$$
$$e_r : \ddot{r} - \frac{l^2}{r^3} = -\frac{GM}{r^2}$$
$$e_\varphi : \frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}) = 0$$

Derivation is left as an exercise to the reader (fucking killing myself)

Now we have to relate the time derivatives of r and u

$$\dot{r} = \frac{d}{dt}\frac{1}{u} = -u^{-2}\dot{u} = -u^{-2}\frac{du}{d\varphi}\frac{d\varphi}{dt} = \ldots = -lu'$$
$$\ddot{r} = -l^2u^2u''$$

insert e_r

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{l^2}$$

solution

$$u(\varphi) = D\cos(\varphi - \varphi_0) + \frac{GM}{l^2}$$

choose coords such that $\varphi_0 = 0$
Get D from $\varepsilon = \text{const}$

$$D = \sqrt{\frac{2\varepsilon}{l^2} + \left(\frac{GM}{l^2}\right)^2}$$

put D into u and then invert to get $1/u = r$
ϵ and ε are different things

$$r = \frac{\alpha}{1 + \epsilon\cos(\varphi)}$$

$$\alpha = \frac{l^2}{GM} \text{ latus rectum} \qquad \epsilon = \sqrt{1 + \frac{2\varepsilon l^2}{G^2M^2}} \text{ eccentricity}$$

$$r_{perigee} = \frac{\alpha}{1+\epsilon} \qquad r_{apogee} = \frac{\alpha}{1-\epsilon}$$

5.5.5 Cases: $\epsilon > 1$

the eccentricity is greater than 1 so we get a hyperbola

the critical angle is that such $1 + \epsilon\cos\varphi = 0$ because thats where the particle asymptotically reaches

5.5.6 Case: $\epsilon = 1$

parabolic tragectory

perigee of $\alpha/2$ at the turning point

the critical angle is π because the object reaches that largest angle asymptotically.

If you write r in cartesian coordinates you get

$$x = \pm\left(-\frac{y^3}{2\alpha} + \frac{\alpha}{2}\right)$$

5.5.7 Case: $\epsilon < 1$

Elliptic Orbit

$$\frac{-1}{2}\left(\frac{GM}{l}\right)^2 = \epsilon_{min} < \epsilon < 0$$

$$0 < \varepsilon < 1$$

$$r = \frac{\alpha}{1+\varepsilon\cos(\varphi)}$$

$$a = \text{ semimajor axis } = \frac{r_{min}+r_{max}}{2}$$

$$b = \text{ semiminor axis } = a\sqrt{1-\varepsilon^2} = \frac{\alpha}{\sqrt{1-\varepsilon^2}} = \frac{l}{\sqrt{|2\varepsilon|}}$$

$$\frac{\alpha}{1-\varepsilon^2} = \frac{GM}{2|\epsilon|}$$

5.5.8 Case: Circulat Orbit

$\varepsilon < 0$ and $U'_{eff}\Big|_{r=r_c} = 0$

$$r_c = \frac{l^2}{GM} \qquad \epsilon = \frac{-1}{2}\left(\frac{GM}{l}\right)^2 \qquad \varepsilon = 0$$

5.6 More with Kepler's Laws

5.6.1 2nd Law

$$dA = \frac{1}{2} r d\varphi$$
$$\frac{dA}{dt} = \frac{1}{2} r \frac{d\varphi}{dt} = \frac{1}{2} r^2 \frac{l}{r^2} = \frac{l}{2} = \text{const}$$

5.6.2 3rd Law

$$a^3 \equiv p^2 \text{ (a is semimajor axis and p is period)}$$
$$\text{area of ellipse } A = \pi ab$$
$$\int dA = \frac{l}{2} \int dt \rightarrow \pi ab = \frac{l}{2} p$$
$$p^2 = \frac{4\pi^2 a^2 b^2}{l^2} = \frac{4\pi^2}{GM} a^3$$
$$\frac{p^2}{a^3} = \frac{4\pi^2}{GM}$$

Chapter 6

Harmonic and Damped Motion

6.1 Equations of Motion

carm of mass m attached to a spring with spring constant k and length l.

$x(t)$ is the time dependent position

$x = 0$ wall, $x_0 = L$ equilibrium.

$$\text{spring force } \vec{F}_s = -k(x(t) - L)$$

$$m\vec{a} = \vec{F}_s \qquad m\ddot{x} = -k(x(t) - L) \rightarrow m\ddot{x} + k(x - L) = 0$$

You can also derive it from energy conservation

$$E = T + U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x-L)^2$$

derive with respect to time

$$\frac{dE}{dt} = 0 = \frac{1}{2}m * 2\dot{x} * \ddot{x} + \frac{1}{2}k * 2(x-L)\dot{x} \rightarrow$$

$$\dot{x}(m\ddot{x} + k(x-L)) = 0 \qquad \dot{x} \neq 0$$

$$m\ddot{x} + k(x-L) = 0$$

6.1.1 Solving EoM

Choose reference frame such that the equilibrium point is at $x = 0$

$x = x_0 = L \to y = 0 = x - L, \qquad \ddot{y} = \ddot{x}$

make an ansatz for the solution

$y(t) \approx e^{\lambda t} \qquad \dot{y}(t) = \lambda e^{\lambda t} \qquad \ddot{y}(t) = \lambda^2 e^{\lambda t}$

insert

$m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0$

$$\lambda^2 = -\frac{k}{m} \to y = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

$y(t) = D_1 e^{i\omega t} + D_2 e^{-i\omega t} \qquad D_1, D_1 \in \mathbb{C}$

use Euler's Identity

$y(t) = A\cos(\omega t) + B\sin(\omega t) \qquad A = D_1 + D_2, B = i(D_1 - D_2)$

$y(t) = C\cos(\omega t + \phi)$

period $= \frac{2\pi}{\omega}$ and $\omega = \sqrt{\frac{k}{m}}$

6.1.2 Energy

$$U = \frac{1}{2}ky^2 = \frac{1}{2}kC^2\cos^2(\omega t - \phi)$$
$$U \geq 0 \qquad U_{max} = \frac{1}{2}kC^2$$
$$T = \frac{1}{2}m\dot{y}^2 = \frac{m}{2}C^2\omega^2\sin^2(\omega t - \phi)$$
$$T \geq 0 \qquad T_{max} = \frac{m}{2}C^2\omega^2$$
$$E = T + U = \frac{1}{2}mC^2\omega^2\sin^2(\omega t - \phi) + \frac{1}{2}kC^2\cos^2(\omega t - \phi) = \text{const}$$
$$E = T_{max} = U_{max} = \text{const}$$

6.2 Harmonic Oscillator

simple harmonic oscillator

$$m_{eff}\ddot{x} + k_{eff}x = 0 \qquad \ddot{x} + \omega^2 x = 0 \qquad \omega = \sqrt{\frac{k_{eff}}{m_{eff}}}$$

The spring constant and mass are now effective because we're in a dampened oscillator instead of an ideal one.

6.2.1 EOM

Consider a vertical spring with height equation $h = \alpha(s - x)^2$

Use conservation of energy

$$U = U_{spring} + U_g \qquad U_{spring} = \frac{1}{2}k(x-L)^2$$
$$U_g = mgh = mg\alpha(s-x)^2$$
$$\frac{dU}{dx} = 0 \to x_{eq} = \frac{kL + 2\alpha mgs}{k + 2\alpha mg} \qquad \text{do yourself :(}$$

Now find the effective spring constant
Taylor expand U near x_{eq}

$$U = U(x_{eq}) + U'\Big|_{xeq}(x - x_{eq}) + \frac{1}{2}U''\Big|_{xeq}(x - x_{eq})^2 \qquad U'(x_{eq}) = 0$$
$$= U(x_{eq}) + \frac{1}{2}U''(x_{eq})(x - x_{eq})^2 \qquad k_{eff} = U''(x_{eq})$$
$$k_{eff} = U''(x_{eq}) = k + 2\alpha mg$$

Now to find kinetic energy

$$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$$

if h is small you can Taylor expand around $\frac{\dot{y}^2}{\dot{x}^2} = 4\alpha^2(x - s)^2 << 1$

$$T \approx \frac{1}{2}m\dot{x}^2$$
$$E = T + U = \frac{1}{2}m\dot{x}^2 + U_{x_{eq}} + \frac{1}{2}k_{eff}(x - x_{eq})^2$$

Shift the coordinates for convenience

$$\bar{x} = x - x_{eq} \qquad \dot{\bar{x}} = \dot{x}$$

back to the equation of motion

$$\frac{1}{2}m\dot{\bar{x}}^2 + \frac{1}{2}k_{eff}\bar{x}^2 = E - U(x_{eq}) = \text{const}$$

take the derivative

$$m\ddot{\bar{x}} + k_{eff}\bar{x} = 0 \rightarrow \ddot{\bar{x}} + \omega^2\bar{x} = 0$$

$$\omega = \sqrt{\frac{k_{eff}}{m}} = \sqrt{\frac{k + 2\alpha mg}{m}}$$

some stuff with $\dot{y}$ and $\dot{x}$

$$y = h = \alpha(s - x)^2 \qquad \dot{y} = 2\alpha(s - x)(-x)\dot{x}$$

take derivative

$$\frac{d}{dt}(\frac{1}{2}m\dot{\bar{x}}^2 + \frac{1}{2}k_{eff}\bar{x}^2) \rightarrow \frac{1}{2}m2\dot{\bar{x}}\ddot{\bar{x}} + \frac{1}{2}k_{eff}2\bar{x}\dot{\bar{x}} = 0$$

6.3 Simple Pendulum

choose polar coordinates

$$r = L = \text{const} \qquad \vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}e_\theta = L\dot{\theta}\vec{e}_\theta$$

Use energy consevation to find equation of motion

$$U = mgh = mgL(1 - \cos(\theta))$$
$$T = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}mL^2\dot{\theta}^2$$
$$\frac{dE}{dt} = \frac{d}{dt}(\frac{1}{2}mL^2\dot{\theta}^2 + mgL(1 - \cos(\theta))) \rightarrow$$
$$L\ddot{\theta} + g\sin(\theta) = 0$$

small angle approximation

$$\ddot{\theta} + \omega^2\theta = 0 \qquad \omega = \sqrt{\frac{g}{L}}$$
$$\theta(t) = A\cos(\omega t - \phi)$$

A is amplitude and ϕ is the phase which are both derived from initial conditions

6.4 Physical Pendulum

Let the pendulum have like an actual mass distribution with a center of mass and a pivot that it rotates around.

choose coordinate system (polar coordinates)

$$r = L \qquad \theta = \theta(t) \qquad \vec{v} = L\dot{\theta}\vec{e}_\theta$$

use kinetic energy (but you have to use moment of inertia and whatnot instead of just regular kinetic energy)

$$T = \frac{1}{2} I_p \dot{\theta}^2 \qquad U = mgh = mgL(1 - \cos(\theta))$$

$$\frac{dE}{dt} = \frac{d}{dt}(T + U) = I_p \ddot{\theta} + mgL\sin(\theta) = 0$$

small angle

$$I_p \ddot{\theta} + mgL\theta = 0 \to \ddot{\theta} + \omega^2 \theta = 0 \qquad \omega = \sqrt{\frac{mgL}{I_p}}$$

6.5 Dampened

oscillator that's damped due to friction. E is no longer conserved

Imagine a spring connected to a cart with friction force $\vec{F} = -C\vec{v}$

6.5.1 EOM

$$\frac{dE}{dt} = P_{lost} = \frac{dE_{lost}}{dt} = \vec{F}\frac{dx}{dt} = \vec{F} \cdot \vec{v} = -c\vec{v}^2$$

mechanical energy

$$E = T + U_s = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(x-L)^2$$

something?

$$-c\dot{x}^2 = P_{lost} = \frac{dE}{dt} = \frac{m}{2}2\dot{x}\ddot{x} + k(x-L)\dot{x} \rightarrow$$

$$m\ddot{x} + c\dot{x} + k(x-L) = 0$$

Translate the coordinates such that $y = x - L$

$$m\ddot{y} + c\dot{y} + ky = 0 \qquad \ddot{y} + 2\zeta\omega_N\dot{y} + \omega_N^2 y = 0$$

That's the equation for a damped harmonic oscillator $\omega_N^2 = \frac{k_{eff}}{m_{eff}}$ and a damping parameter $\zeta = \frac{c}{2m\omega_N}$

Take an ansatz $y = e^{\lambda t}, \dot{y} = \lambda e^{\lambda t}, \ddot{y} = \lambda^2 e^{\lambda t}$

$$\lambda^2 + 2\zeta\omega_N\lambda + \omega_N^2 = 0 \rightarrow \lambda_\pm = -\omega_N(\zeta \pm \sqrt{\zeta^2 - \lambda})$$

$$y = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

6.5.2 $\zeta = 0$

undamped

$$\ddot{y} + \omega_N^2 y = 0 \qquad \lambda = \pm i\omega_N$$
$$y(t) = Ae^{i\omega_N t} + Be^{-i\omega_N t}$$

6.5.3 $\zeta > 1$

overdamped

$$\lambda_\pm = \omega_N(\zeta \pm \sqrt{\zeta^2 - 1}) < 0$$
$$y(t) = Ae^{-|\lambda_+|t} + Be^{-|\lambda_-|t}$$

6.5.4 $\zeta = 1$

critically damped

$$\lambda_\pm = -\omega_N$$
$$y(t) = Ae^{-\omega_N t} + Bte^{-\omega_N t}$$

6.5.5 $0 < \zeta < 1$

Underdamped

$$\lambda_\pm = -\omega_N(\zeta \pm \sqrt{\zeta^2 - 1}) = \omega_N\zeta \pm i\omega_N\sqrt{1-\zeta^2} = -\zeta\omega_N \pm i\omega_d$$
$$y(t) = e^{-\zeta\omega_N t}(D_1 e^{i\omega_d t} + D_2 e^{-i\omega_d t}) \qquad \text{or}$$
$$y(t) = e^{-\zeta\omega_N t}(A\cos(\omega_d t) + B\sin(\omega_d t)) \qquad \text{or}$$
$$y(t) = Ce^{-\zeta\omega_N t}\cos(\omega_d t - \phi)$$

ω_d is the damping frequency

6.5.6 $\zeta < 0$

accelerated oscillator?

$$\lambda_\pm = +\omega_N|\zeta| \pm \sqrt{\dots}$$
$$y(t) \approx e^{\omega_N|\zeta|t}\dots$$

It's gonna blow up exponentially

6.6 Forced Oscillator

oscillator with external driving force

Energy is NOT conserved

$$\frac{dE}{dt} + P_{diss} = (\frac{dW}{t})_{ext} = \dots = \vec{F}_{ext} \cdot \vec{v}$$
$$m_{eff}\ddot{\vec{x}} + c_{eff}\dot{\vec{x}} + k_{eff}\vec{x} = F_{ext}$$

That's the equation for a damped harmonic oscillator in 3d

6.6.1 EOM

no damping

use coordinates (r, θ)

use modified energy method

$$\dot{E} + P_{diss} = \vec{F}_{ext} \cdot \vec{v}$$

$$P_{diss} = 0 \qquad T = \frac{1}{2}m\vec{v}^2 \qquad v = L\dot{\theta}\vec{e}_\theta$$

$$U = mgh = mgL(1 - \cos(\theta))$$

$$\vec{F}_{ext} \cdot \vec{v} = L\dot{\theta}\vec{F} \cdot e_\theta = L\dot{\theta}F\cos(\theta)$$

$$\dot{E} = \frac{1}{2}mL^2 2\dot{\theta}\ddot{\theta} + mgL\dot{\theta}\sin(\theta) = FL\dot{\theta}\cos(\theta) \rightarrow$$

$$mL\ddot{\theta} + mg\sin(\theta) = F(t)\cos(\theta)$$

Figure out the eom for small θ so $\sin(\theta) = \theta$ and $\cos(\theta) \approx 1$

$$mL\ddot{\theta} + mg\theta = F(t) \rightarrow m_{eff}\ddot{\theta} + k_{eff}\theta = F(t)$$

6.6.2 example

cart on a cart

spring of constant k and length L is connected to cart of mass m with position $x(t)$ and that whole thing is on a bigger cart with position $y(t)$

net displacemen of cart $= x(t) + y(t)$. Spring force $\vec{F}_s = -k\vec{x}$

use N2L

$$\vec{F} = m\vec{a} \rightarrow -kx = m(\ddot{x} + \ddot{y}) \rightarrow m\ddot{x} + kx = -m\ddot{y}$$

6.7 Forced Harmonic Oscillator

$$m_{eff}\frac{d^2\vec{x}}{dt^2} + c_{eff}\frac{d\vec{x}}{dt} + k_{eff}\vec{x} = \vec{F}_{ext}(t)$$

How to solve: introduce linear operator $\mathcal{L}[x] = F$
The linear operator is a map of a function into a function

$$\mathcal{L}f(t,x) \rightarrow g(t,x) = \mathcal{L}[f(t,x)]$$

Follows linear property

$$\mathcal{L}[af(t,x) + bf_2(t,x)] = a\mathcal{L}[f(t,x)] + b\mathcal{L}[f_2(t,x)]$$

for example

$$\mathcal{L} = A\frac{d}{dx} \rightarrow L[g] = A\frac{dg}{dx}$$

$$\mathcal{L} = m_{eff}\frac{d^2}{dt^2} + c_{eff}\frac{d}{dt} + k_{eff}$$

$$\mathcal{L}[\vec{x}(t)] = m_{eff}\frac{d^2\vec{x}}{dt^2} + c_{eff}\frac{d\vec{x}}{dt} + k_{eff}\vec{x}$$

so for a forced oscillator

$$\mathcal{L}[\vec{x}(t)] = \vec{F}_{ext}(t)$$

6.7.1 General Solution

$x(t) = x_h(t) + x_p(t)$

where $x_n(t)$ is the homogeneous solution without the external force and $x_p(t)$ is the particular solution of just the external force

homogeneous solution solves $\mathcal{L}[x(t)] = 0$

particular solution needs just a single solution to $\mathcal{L}[x(t)] = F_{ext}$

$$\omega_n = \sqrt{\frac{k_{eff}}{m_{eff}}} \qquad \zeta = \frac{c_{eff}}{2m_{eff}\omega_n} \qquad \omega_N\sqrt{1-\zeta^2}$$

ω_n is natural frequency. ζ is dimensionless damping parameter. ω_d is dampened frequency.

6.7.2 Force as a power series

$F_{ext} = a + bt + ct^2 + \dots$

$m_{eff}\ddot{x} + c_{eff}\dot{x} + k_{eff}x = a + bt + ct^2 + \dots$

Strategy

- solve for $x_h(t)$
- determine 1 possible $x_p(t)$
- find initial conditions

ansatz for $x_p(t)$

$x_p(t) = \alpha + \beta t + \gamma t^2$

and then just solve

6.7.3 Example

$F_{ext} = t^2 \qquad$ no damping

$c_{eff} = 0 \qquad k_{eff} = 1 \qquad m_{eff} = 1 \qquad \ddot{x} + \omega_n x = 0$

$x(0) = 0 \qquad \dot{x}(0) = 0$

solve particular solution

$x(t) = \alpha + \beta t + \gamma t^2 \qquad \dot{x} = \beta + 2\gamma t \qquad \ddot{x} = 2\gamma$

$2\gamma + \alpha + \beta t + \gamma t^2 = t^2 \qquad \beta = 0 \qquad \gamma = 1 \qquad \alpha = -2$

$x_p = -2 + t^2$

find general solution (simple harmonic oscillator)

$x_h(t) = A\cos(\omega_n t) + B\sin(\omega_n t)$

$x = x_h(t) + x_p(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + t^2 - 2$

initial conitions

$x(0) = 0 \rightarrow A = 2$

$\dot{x}(0) = 0 \rightarrow B = B = 0$

$x(t) = 2\cos(\omega_n t) + t^2 - 2$

6.7.4 Exponential Force

$$m_{eff}\ddot{x} + c_{eff}\dot{x} + k_{eff}x = F_0e^{\alpha t}$$
$$x_p \text{ ansatz } x_p = Ae^{\alpha t}$$
$$m_{eff}\alpha^2Ae^{\alpha t} + c_{eff}A\alpha e^{\alpha t} + k_{eff}Ae^{\alpha t} = F_0e^{\alpha t} \rightarrow$$
$$m_{eff}\alpha^2A + c_{eff}A\alpha + k_{eff}A = F_0 \rightarrow A = \frac{F_0}{m_{eff}\alpha^2 + c_{eff}\alpha + k_{eff}}$$
$$x = x_h + x_0 =$$
$$e^{-\zeta\omega_n t}(A\cos(\omega_d t) + B\sin(\omega_d t)) + \frac{F_0}{m_{eff}\alpha^2 + c_{eff}\alpha + k_{eff}}e^{\alpha t}$$

6.7.5 Harmonic Force

$$F = F_0\cos(\omega t - \theta)$$

F_0 is amplitude, ω is angular speed, θ is phase

$$m\ddot{x} + x\dot{x} + kx = F_0\cos(\omega t - \theta)$$
$$\text{ansatz } x_p = D\cos(\omega t - \psi)$$

That does work but theres a trick

$e^{i\Omega t} = \cos(\Omega t) + i\sin(\Omega t)$

$F(x) = \mathrm{Re}(F_0 e^{i(\omega t - \theta)})$

ansatz $x_p = \mathrm{Re}(X e^{i\omega t}) X \in \mathbb{C}$

$\mathrm{Re}(-m\omega^2 X e^{i\omega t} + i\omega c X e^{i\omega t} + kXe^{i\omega t}) = \mathrm{Re}(F_0(e^{i\omega t}e^{-i\theta})) \rightarrow$

$$X = \frac{F_0 e^{-i\theta}}{-m\omega^2 + i\omega c + k} = \frac{F_0}{k} e^{-i\theta}\left[1 + 2i\zeta\frac{\omega}{\omega_N} - \left(\frac{\omega}{\omega_N}\right)^2\right]^{-1}$$

$= \overset{N}{G}(\omega) F_0 e^{-i\theta}$

$$\overset{N}{G}(\omega) = \frac{1}{k}\left(1 + 2i\zeta\frac{\omega}{\omega_N} - \left(\frac{\omega}{\omega_N}\right)^2\right)^{-1} = G(\omega)e^{i\phi(\omega)}$$

$$G(\omega) = |\overset{N}{G}(\omega)| = \frac{1}{k}\left[\left(1 - \frac{\omega^2}{\omega_N^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_N}\right)^2\right]^{-1/2}$$

$$\phi(\omega) = Arg(\overset{N}{G}(\omega)) = \arctan\left(\frac{2\zeta\omega\omega_N}{\omega_N^2 - \omega^2}\right) \qquad 0 \leq \phi < \pi$$

$x_p(t) = \mathrm{Re}$

trying this again because it's a new lecture

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(\omega t - \theta)$$
$$x_p(t) = F_0 G(\omega) \cos(\omega t - \theta - \phi(t))$$
$$G(\omega) = \frac{1}{k}\left[\left(1 - (\frac{\omega}{\omega_N})^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_N}\right)^2\right]^{-1/2}$$
$$\phi(t) = \arctan\left(\frac{2\zeta\omega\omega_N}{\omega_N^2 - \omega^2}\right)$$

I get to derive this at home oh boy such fun

$x_p(t)$ is oscillating with a frequency $\omega \neq \omega_N$ that is the same as the driving force.

The amplitude $|x_p(t)| = F_0 G(\omega)$ is frequency dependent.

The phase $\phi(\omega)$ also depends on the frequency.

At late times, the oscillation will be just the driven force because the regular oscillator will have been dampened to nothing. $x_p(t)$ is the "steady state" solution.

$$e^{-\zeta\omega_N t} = e^{-t/\tau} \qquad \tau = \frac{1}{\zeta\omega_N}$$

6.8 Analyzing $x_p(t)$

6.8.1 peak (resonance)

at the peak, the system is in resonance (critical point)

$$\frac{dG(\omega)}{d\omega}\bigg|_{\omega=\omega_R} = 0 \qquad \omega_R = \pm\omega_N\sqrt{1 - 2\zeta^2}$$

magnitude of the peak

$$G(\omega = \omega_R) = \frac{1}{2k\zeta\sqrt{1-\zeta^2}}$$

for a small damping parameter, the peak magnitude is very large. $\omega_R \approx \omega_N$

6.8.2 small ω

$\omega/\omega_N << 1$ and $\omega < \omega_R$

$$G(\omega) \approx \frac{1}{k} + \frac{1-2\zeta^2}{k}\left(\frac{\omega}{\omega_N}\right)^2 \approx \frac{1}{k} + O(\frac{\omega}{\omega_N})^2$$
$$\phi(t) \simeq 2\zeta\frac{\omega}{\omega_N} \approx 0$$

6.8.3 very large ω

$$G(\omega) \simeq \frac{1}{k}\frac{\omega_N}{\omega} + O(\frac{\omega_N}{\omega})^3$$
$$\phi(\omega) \approx -2\zeta\frac{\omega_N}{\omega} + n\pi \simeq \pi$$
$$x_p(t) = \frac{F_0}{k}(\frac{\omega_N}{\omega})^2\cos(\omega t - \theta) \qquad \omega_N^2 = \frac{k}{m}$$
$$x_p \approx \frac{F_0}{m\omega^2}\cos(\omega t - \theta)$$

for the different phases

- at resonance

  $\phi(\omega_R) \simeq \pi/2$ so x is $\pi/2$ behind force

- $\omega << \omega_N$

  $\phi \simeq 0$ so x and F are in phase

- $\omega >> \omega_N$

  $\phi \simeq \pi$ so x is π behind the force

Chapter 7

Periodic Forces and Fourier Series

The force can be all sorts of wacky as long as it's periodic along a period T

$F(t+T) = F(t)$ and $T = \frac{2\pi}{\Omega}$ where Ω is the frequency of the periodic force.

Describe the solution for the periodic force as a superposition of harmonic functions (sin and cos). We can do this because the differential operator $\mathcal{L} = m\frac{d^2}{dt^2} + c\frac{d}{dt} + k$ is linear

7.1 Fourier Series

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_N \cos(n\Omega t) + \sum_{p=1}^{\infty} b_p \sin(p\Omega t)$$

$$n, p \in \mathbb{N} \qquad a_n, b_p = \text{ Fourier Coefficients}$$

An even function is a function such that $f(t) = f(-t)$ (like cos)

$f_{even}(t) = \frac{1}{2}(F(t) + F(-t))$
odd function $f(-t) = -f(t)$ (like sin)
$f_{odd} = \frac{1}{2}(F(t) - F(-t))$
if your function is even, the fourier series will have 0 for all sin coefficients.

If it's odd, then all the cos coefficients will be 0

7.1.1 Coefficients

$$a_0 = \frac{2}{T}\int_0^T F(t)dt$$
$$a_N = \frac{2}{T}\int_T^0 F(t)\cos(n\Omega t)dt$$
$$b_p = \frac{2}{T}\int_T^0 F(t)\sin(p\Omega t)dt$$

The way the integral bounds work is you just need to integrate for 1 period of T.

Verify the coeffients

$$\int \cos(p\Omega t)\cos(n\Omega t)dt =$$
$$\frac{1}{2}\int \cos((n+p)\Omega t)dt + \int \cos((n-p)\Omega t)dt$$

it vanishes so I think something's good

7.2 Fouier Series

This is just a new lecture so I'm writing all the stuff

$$
\begin{aligned}
F(t) &= \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos(p\Omega t) + \sum_{p=1}^{\infty} b_p \sin(p\Omega t) \\
a_0 &= \frac{2}{T} F(t) dt \\
a_p &= \frac{2}{t} \int_T F(t) \cos(p\Omega t)\, dt \qquad b_p = \frac{2}{t} \int_T F(t) \sin(p\Omega t)\, dt
\end{aligned}
$$

In order to verify a_0, a_p, b_p, we need to show that the thing is orthogonal

$$
\begin{aligned}
\int\limits_T \cos(m\Omega t) \cos(p\Omega t)\, dt &= \begin{cases} 0 & m \neq p \\ \frac{T}{2} & m = p \end{cases} \\
\int\limits_T \sin(m\Omega t) \sin(p\Omega t)\, dt &= \begin{cases} 0 & m \neq p \\ \frac{T}{2} & m = p \end{cases} \\
\int\limits_T \cos(m\Omega t) \sin(p\Omega t)\, dt &= 0
\end{aligned}
$$

insert $F(t)$ to get something

$$a_m = \frac{2}{T}\int\limits_T F(t)\cos(m\Omega t)\,dt =$$

$$\frac{2}{T}\int\limits_T dt\cos(m\Omega t)\left[\frac{a_0}{2} + \sum_{p=1}^{\infty} a_p\cos(p\Omega t) + \sum_{p=1}^{\infty} b_p\sin(p\Omega t)\right]$$

$$a_m = \frac{2}{T}\int\limits_T dt\left[\frac{a_0}{2}\cos(m\Omega t) + \cos(m\Omega t)\sum_{p=1}^{\infty} a_p\cos(p\Omega t)\right]$$

$$\text{if } m = 0 : a_0 = \frac{2}{T}\int\limits_T dt \rightarrow \frac{a_0}{2} = a_0$$

$$\text{if } m \neq 0 : a_m = \frac{2}{T}\int\limits_T dt\cos(m\Omega t)\sum_p a_p\cos(p\Omega t) = a_p = m$$

7.2.1 Example: Square Wave

it's 1 from $\frac{-T}{2}$ to 0 and -1 from 0 to $\frac{T}{2}$

because the function is odd, $a_n = 0$

Also because the function is odd,

$a_0 = \langle F(t)\rangle = \frac{1}{T}\int\limits_T F(t)dt = 0$

b_p is the only one that involves actual calculation

$$b_p = \frac{2}{T}\int\limits_T F(t)\sin(p\Omega t)\,dt$$

integrate from $-\frac{T}{2}$ to $\frac{T}{2}$

use integration by parts as well

$$u = F(t) \to u' = 0 \qquad v' = \sin(p\Omega t) \to v = \frac{-1}{p\Omega}\cos(p\Omega t)$$

$$b_p = \frac{2}{T}\left[F(t)(\frac{-1}{p\Omega})\cos(p\Omega t)\right]\Big|_{-T/2}^{T/2} - \int_T 0 * \frac{-1}{p\Omega}\cos(p\Omega t)dt \to$$

$$\frac{2}{T}\left[\frac{-1}{p\Omega}F(t)\cos(p\Omega t)\Big|_{-T/2}^{0} + \frac{-1}{p\Omega}F(t)\cos(p\Omega t)\Big|_{0}^{T/2}\right]$$

You're going to get

$$b_p = \begin{cases} 0 & \text{p is even} \\ \frac{-4}{p\pi} & \text{p is odd} \end{cases}$$

$$F(t) = \sum_{p=1}^{\infty} b_p \sin(p\Omega t)$$

That is the truncated series

If you take only the first n terms of a fourier series, youre going to get a noticeably wave-y function but it will look closer to your original function that just a sine or cosine wave.

7.2.2 Gibb's Phenomenon

When doing the fourier series of a square wave, the edges of the square are going to have little horns becauseidk math.

This is characteristic of a truncated series.

7.2.3 Summary

equation of motion

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

with periodic force $F(t)$

Fourier series

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos(p\Omega t) + \sum_{p=1}^{\infty} b_p \sin(p\Omega t)$$

with coefficients

$$a_p = \frac{2}{T}\int_T F(t)\cos(p\Omega t)\,dt \qquad b_p = \frac{2}{T}\int_T F(t)\sin(p\Omega t)\,dt$$

$$a_0 = \frac{2}{T}\int_T F(t)\,dt \qquad \text{time average}$$

Some shenanigans
equation of motion with linear operator $\mathcal{L} = F(t)$
with homogeneous solution $x_h(t)$ for $\mathcal{L}(x) = 0$
and a general solution $x(t) = x_h(t) + x_p(t)$
How to find the particular solution?

$$\mathcal{L}(x) = \left(m\frac{d^2}{dt^2} + c\frac{d}{dt} + k\right)x = \sum_q F_q(t)$$

$$x(t) = x_1(t) + x_2(t) + \ldots = \sum_{f=1}^{\infty} x_f(t) \qquad m\ddot{x}_1 + c\dot{x}_1 + kx_1 = F_1(t)$$

$$m\ddot{x}_q + c\dot{x}_q + kx_q = F_q(t)$$

each $F_q(t)$ is a harmonic function

$$x_q(t) = f(q)\cos(q\Omega t - \theta - \phi(q\Omega)) \qquad q\Omega = \omega$$

$$x_p(t) =$$

$$\frac{a_0}{2k} + \sum_{q=1}^{\infty} a_q G(q\Omega)\cos(q\Omega t - \phi(q\Omega)) + \sum_{q=1}^{\infty} b_q G(q\Omega)\sin(q\Omega t - \phi(q\Omega))$$

$$G(q\Omega) = \frac{1}{k}\left[\left(1 - \left(\frac{q\Omega}{\omega_N}\right)^2\right)^2 + \left(2\zeta\frac{2\Omega}{\omega_n}\right)^2\right]^{-1/2}$$

$$\phi(q\Omega) = \arctan\left(\frac{2\zeta q\Omega\omega_N}{\omega_n^2 - (q\Omega)^2}\right)$$

7.3 Steady State Solution

$x(t)$ at late times ($t \to \infty$)

for an undamped oscillator

$$x(t) = A\cos(\omega_n t) + B\sin(\omega_n t) + x_p(t)$$

for a damped oscillator

$$x(t) = x_p(t)$$

Can there be resonance? Yes. if $q\Omega = \omega_R = \omega_n\sqrt{1-2\zeta^2} \approx \omega_n($ if $\zeta << 1)$

$q \simeq \omega_R/\Omega$

$G = \sum G(q\Omega)$

You can have resonance for each q

Chapter 8

Impulse Response and Green's Function

Consider a force that exerts an impulse

The function is 0 everywhere except for the time of impulse in which it is instantaneously non-zero and then it goes back to 0

8.1 Math Break: Delta Distribution

Describe this with a **Delta Distribution** or "delta function".

$$F(t) = \delta(t-a) = \begin{cases} \infty & t = a \\ 0 & \text{otherwise} \end{cases}$$

consider a rectangle with width ϵ and height $1/\epsilon$ at time $t = a$. and take the limit as $\epsilon \to 0$

This gets you the delta distribution $\delta(t-a)$

$$f(t=a) = \int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt$$

for functions continuous at $t = a$ and with limits

$\delta(\infty), \delta(-\infty) = 0.$

8.1.1 example

$$\int_{-\infty}^{\infty} \sin(t)\delta(t - \frac{3}{2}\pi) = sin(3/2)$$

You can do a u sub to get to the right form of the thing you want.

There's also a thing you can do with integration by parts

$$\int_{-\infty}^{\infty} f(t)\frac{d}{dt}\delta(t-a)dt \to f(t)\delta(t-a)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} (\frac{df}{dt})\delta(t-a)dt$$

$$= -\frac{df}{dt}\Big|_{t=a}$$

8.1.2 Impulse Forces

Consider a force acting for $\Delta t << \omega_N$

It can be described using a delta distribution.

$$F(t) = I\delta(t) \qquad I = \Delta p = \Delta(F * t)$$

Response to impulse force

8.1.3 Green's Function

Let force be at $t = 0$ with impulse of 1 and $x(t = 0) = 0$ and $\dot{x}(t = 0) = \frac{1}{m}$.

$$G(t) = \begin{cases} 0 & t < 0 \\ \exp\left(-\frac{C}{2m}t\right)\frac{\sin(\omega_d t)}{m\omega_d} \end{cases}$$

That is the response for

$$m\ddot{x}+c\dot{x}+kx = \delta(t-a) \qquad \omega_N = \sqrt{\frac{k}{m}}, \omega_d = \omega_N\sqrt{1-\zeta^2}, \zeta = \frac{c}{2m\omega_N}$$

8.1.4 Math Break: Heaviside

It's just a step function

$$H(y - y_0) = \begin{cases} 1 & y > y_0 \\ 0 & y < y_0 \end{cases}$$

8.1.5 Back to Green's Function

$$G(t) = \begin{cases} 0 & t < 0 \\ \exp\left(-\frac{C}{2m}t\right)\frac{\sin(\omega_d t)}{m\omega_d} \end{cases} = H(t)\exp\left(-\frac{C}{2m}t\right)\frac{\sin(\omega_d t)}{m\omega_d}$$

at late times

$$x(t) = \begin{cases} 0 & t < a \\ \exp\left(-\frac{C}{2m}(t-a)\right)\frac{\sin(\omega_d (t-a))}{m\omega_d} \end{cases} =$$

$$= H(t-a)\exp\left(-\frac{C}{2m}(t-a)\right)\frac{\sin(\omega_d (t-a))}{m\omega_d}$$

$$= G(t-a)$$

recognize response to impulse as Green's Function $G(t-a)$ satisfies equation of motion

$$\mathcal{L}(G(t-a)) = (m\frac{d^2}{d^2t} + c\frac{d}{dt} + k)G(t-a) = \delta(t-a)$$

8.2 Arbitrary Forcing and Convolution

Imagine a force with 2 consecutive impulses $I_1(t = t_1)$ and $I_2(t = t_2)$

Goal: figure out the forces and the particular solution

$$F(t) = I_1\delta(t-t_1) + I_2\delta(t-t_2)$$

$$x_p(t) = I_1 G(t-t_1) + I_2 G(t-t_2)$$

to generalize

$$F(t) = \sum_{q=1}^{N} I_q \delta(t - t_q) \rightarrow$$

$$x_p(t) = \sum_{q=1}^{N} I_q G(t - t_q)$$

$$= \sum_{q=1}^{N} I_q H(t - t_q) \exp\left(\frac{c}{2m}(t - t_q)\right) \frac{\sin(\omega_d(t - t_q))}{m\omega_d}$$

consider a continuous force

$$F(t) = \sum_{q=1}^{N} I_q \delta(t - t_q) = \sum_{q=1}^{N} F(t_q)\delta(t - t_q)\Delta t$$

if we consider the limits $\Delta t \rightarrow 0$

$$F(t) = \int_{-\infty}^{\infty} F(t')\delta(t - t')dt'$$

the response $x_p(t)$ is

$$x_p(t) = \int_{-\infty}^{\infty} F(t')G(t - t')dt'$$

solution for a damped oscillator with an arbitrary driving force.

verify

$$\mathcal{L}[x] = \mathcal{L}_t\left[\int_{-\infty}^{\infty} F(t')G(t - t')dt'\right] = \int_{-\infty}^{\infty} F(t')\mathcal{L}_t\left[G(t - t')dt'\right] =$$

$$\int_{-\infty}^{\infty} F(t')\delta(t - t') = F(t)$$

G(t) is called Green's Function
x(t) is called convolution

$$x = F \otimes G = \int_{-\infty}^{\infty} F(t')G(t-t')dt' = \int_{-\infty}^{\infty} F(t'')G(t'')dt'' = G \otimes F$$

Its commutative

8.2.1 Integration Bounds

if F is off for $t - t' < 0$

$$G(t-t') = \begin{cases} = 0 & t - t' < 0 \\ \neq 0 & t - t' > 0 \end{cases}$$

this means that

$$x(T) = \int_{-\infty}^{\infty} F(t')G(t-t')dt'$$

integrate to where $t' = t$

If the force starts at $t = t_0$ then $F(t) = 0$ for $t < t_0$ so our lower integration boundary can be t_0

so our new integral is

$$x(T) = \int_{t=t_0}^{t'=t} F(t')G(t-t')dt'$$

motion depends on the entire past.

new lecture

Green's function is the solution to

$$\mathcal{L}[G(t-t')] = \left(m\frac{d^2}{dt^2} + c\frac{d}{dt} + k\right)[G(t-t')] = \delta(t-t')$$

$$x(t) = F \otimes G = \int_{-\infty}^{\infty} F(t')G(t-t')dt' =$$

$$\int_{-\infty}^{\infty} F(t')H(t-t')\exp\left(-\frac{c(t-t')}{2m}\right) * \frac{\sin(\omega_d(t-t'))}{m\omega_d}dt'$$

$$H = \begin{cases} 1 & t > t' \\ 0 & t < t' \end{cases}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2m\omega_n}$$

8.3 Undamped Oscillator

constant force $F = F_0$ switched on at $t = 0$
$x(0) = 0$ and $\dot{x}(0) = 0$
find $x(t)$
Green's Function Approach

$$x(t) = \int_{-\infty}^{\infty} F_0 * H(t-t') * \exp\left(\frac{-c(t-t')}{2m}\right) * \frac{\sin(\omega_d(t-t'))}{m\omega_d}$$

undamped oscillator means $c = 0$ plus fix bounds

$$\int_0^t F_0(t)\frac{\sin(\omega_d(t-t'))}{m\omega_d}\,dt = \frac{F_0}{m\omega_n^2}(1-\cos(\omega_n t))$$

8.4 Fourier Transform

Tranform a function of time into a function of frequency

rewrite differential equation of t into algebraic equation of ω

fourier transform = trafo lmao

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}$$

Inverse trafo

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(t) e^{+i\omega t}\, d\omega$$

$*$ and $**$ are the trafo pair
convention:
$*$ = prefactor = 1
$** = \frac{1}{2\pi}$
or
$* = \frac{1}{\sqrt{2\pi}}$
$** = \frac{1}{\sqrt{2\pi}}$

8.4.1 Example

$f(t) = \delta(t)$

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} \delta(t-0) e^{-i\omega t} = 1$$

inverse

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 e^{i\omega t} d\omega$$

8.4.2 Square pulse

$f(t) = 1$ for $-T < t < T$ and 0 everywhere else

$$f(t) = H(T+t) * H(T-t)$$

$$\delta f(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \int_{-\infty}^{\infty} H(T+t) * H(T-t) e^{-i\omega t}\, dt =$$

$$\int_{-T}^{T} e^{-i\omega t}\, dt = \frac{2}{\omega} \sin(\omega t)$$

That function has zeroes at $\omega = n\frac{\pi}{T}$

8.4.3 Differential vs Algebraic

$$\mathcal{L} = m\ddot{x} + c\dot{x} + kx = F(t)$$

$$\tilde{f}(\omega) =$$

$$m\int_{-\infty}^{\infty} \ddot{x}e^{-i\omega t} + c\int_{-\infty}^{\infty} \dot{x}e^{-i\omega t} + k\int_{-\infty}^{\infty} xe^{-i\omega t} = \int_{-\infty}^{\infty} F(T)e^{-i\omega t}dt$$

$$FT[x] = \int_{-\infty}^{\infty} e^{-i\omega t}dt$$

integration by parts

$$xe^{-\omega t}\Big|_{-\infty}^{\infty} - \int^{u} x(-i\omega)e^{-i\omega t}dt = \int^{u} x(i\omega)e^{-i\omega t}dt = \tilde{x}(\omega)$$

$$-m\omega^2\tilde{x}(\omega) + ic\omega\tilde{x}(\omega) + k\tilde{x}(\omega) = \tilde{F}(\omega)$$

if $F(t) = \delta(t)$ then $x(t) = G(t)$

$$F(\omega) = \int \delta(t)e^{-i\omega t} = 1$$

$$\mathcal{L}_\omega[\tilde{G}(\omega)] = (-m\omega^2 + i\omega c + k)\tilde{G}(\omega) = 1$$

$$\tilde{G}(\omega) = \frac{1}{-m\omega^2 + i\omega c + k} = \frac{1}{k}\left(1 - (\frac{\omega}{\omega_n})^2 + 2i\zeta\frac{\omega}{\omega_n}\right)^{-1}$$

$$\mathcal{L}[x(t)] = F(t) \to x(t) = \int_{-\infty}^{\infty} F(t')G(t-t')dt$$

$$-m\omega^2\tilde{x}(\omega) + ic\omega\tilde{x}(\omega) + k\tilde{x}(\omega) = \tilde{F}(\omega) \to \tilde{x}(\omega) = \tilde{F}(\omega)\tilde{G}(\omega)$$

If you want to find x given the fourier transform, you just do the inverse fourier transform.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{x}(\omega)e^{i\omega t}\,d\omega$$

8.5 Interpretation

FT (fourier tranform) variable ω is frequency,

In acoustics, ω is vibration frequency.

$\tilde{x}$ is a spectrum, or distribution of frequency.

in quantum mechanics, ω is related to energy $E = \hbar\omega$ so $\tilde{x}(\omega)$ is an energy spectrum

8.5.1 Example

$x(t) = \cos(3t)$

recall euler identity and do FT

$$\cos(\omega_0 t) = \frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t}) \qquad \omega_0 = 3$$

$$FT : \tilde{x}(\omega) = \int_{-\infty}^{\infty} \cos(3t)e^{-i\omega t} = \int_{-\infty}^{\infty} e^{-i\omega t}\frac{1}{2}(e^{i3t} + e^{-i3t}) =$$

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{i(3-\omega)t} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(3+\omega)t} = \tilde{x}(\omega) = \pi(\delta(\omega-3) + \delta(\omega+3))$$

2 spikes of size π at $\omega = 3$ and $\omega = -3$

if $x(t)$ has a long duration then its FT $\tilde{x}(\omega)$ is narrow.

if $x(t)$ is of finite short length then the spikes will be wider.

8.5.2 Gaussian Wave Packet

$$f(t) = e^{i\alpha t} \exp\left(-\frac{(t-t_0)^2}{T^2}\right)$$

It's just an oscillation times a Gaussian (normal distribution) centered around $t = t_0$ with a width of T

$$FT \approx \exp\left(\frac{(\omega-\alpha)^2}{\Omega^2}\right) \qquad \Omega = \frac{2\pi}{T}$$

Chapter 9

Noninertial Reference Frames

In a regular inertial frame, Newton's 2nd Law works, $F = ma$ and $F = \sum F_{real}$

non-inertial reference frames have either rotation or linear acceleration or both.

Acceleration can be described as "fictitious" force

The mass in $F = ma$ is an inertial mass

$$\vec{F} = m_I \vec{a} \qquad \vec{F}_a = m_g \vec{g} \text{ gravitational mass}$$

$$m_I \approx m_g \qquad |\frac{m_I}{m_g} - 1| \leq 10^{-15}$$

some relativity shenanigans

9.1 Equivalence Principle

inertial and gravitational mass are the same.

consider 2 uniformly accelerating frames (1 in a gravitational field, 1 not near any field)

The 1st elevator is in NO gravity, so the acceleration is just upwards, so the force is downwards. with a force F_N

The 2nd elevator both accelerates upwards AND has a gravitational force,

$$F_{net} = \vec{F}_N - m\vec{g} = 0 \rightarrow \vec{F}_N = m\vec{g}$$

The physical descriptions are equivalent. A force and being in an accelerated reference frame are the same.

Locally, an observer cannot distinguish an experiment done in a uniform gravitational field compared to a uniformly accelerated reference frame.

9.2 Non-rotating Kinematics

Consider 2 frames accelerated relative to each other

O is the rest frame $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ with $\dot{e} = 0$

O' is the accelerated frame $(\vec{e}_{x'}, \vec{e}_{y'}, \vec{e}_{z'})$

position of A in the rest frame O is given by $r_{OA} = r_{OO'} + r_{O'A}$

the position of O' in O and the position of A in O'

The velocity of A in the rest frame O, also known as the "true" velocity, is $v_{OA} = v_{OO'} + v_{O'A}$

$a_{OA} = a_{OO'} + a_{O'A}$

In rest frame, $F = ma_{OA} = m(a_{OO'} + a_{O'A})$

In accelerated frame

$$F_{true} - ma_{OO'} = ma_{O'A}$$

N2L in accelerated frame

$$F_{net} = F_{true} + F_{fict} = ma_{O'A}$$

9.3 Rotating Frame

rest frame O and rotating frame O' with a frequency ω

position vector L appears fixed in O' is seen in O with $\frac{dL}{dt} = \omega \times L$

$$\omega = \frac{d\theta}{dt}$$
$$r_{OA} = r_{OO'} + r_{O'A}$$
$$v_{OA} = v_{OO'} + v_{O'A}$$
$$v_{OO'} = \frac{d}{dt} r_{OO'} = \frac{d}{dt}\left(r_{OO'x}e_x + r_{OO'y}e_y + r_{OO'z}e_z\right)$$
$$v_{OA} = \frac{d}{dt} r_{OA} = \frac{d}{dt}\left(x'e_{x'} + y'e_{y'} + z'e_{z'}\right)$$
$$= \left(\dot{x}'e_{x'} + \dot{y}'e_{y'} + \dot{z}'e_{z'}\right) + \left(x'\dot{e}_{x'} + y'\dot{e}_{y'} + z'\dot{e}_{z'}\right)$$
$$= v_{apparent} + \omega \times r_{O'A} \qquad \dot{e}_{x'} = \omega \times e_{x'}$$
$$V_{OA} = V_{OO'} + v_{apparent} + \omega \times r_{O'A}$$

the velocity in a rest frame equals the velocity of the rotating frame in the rest frame + the apparent velocity of A in O' + $\omega \times r_{O'A}$

9.3.1 Example

person is a radius b away from the center of a turntable of frequency ω inside a moving train with linear velocity v_{train}

$$v_{OA} = v_{train} + u + \omega \times b$$

u is the velocity of the person in the rotating frame (same-ish frame as person)

9.3.2 Acceleration

$$a_{OA} = \frac{d}{dt}v_{OO'} + \frac{d}{dt}v_{app} + \frac{d}{dt}(\omega \times r_{O'A})$$
$$\frac{d}{dt}v_{app} = \frac{d}{dt}(v_{appx'}e_{x'} + \ldots) = a_{app} + \omega \times v_{app}$$
$$\frac{d}{dt}(\omega \times r_{O'A}) = \dot{\omega} \times r_{O'A} + \omega \times \dot{r}_{O'A} =$$
$$\dot{\omega} \times r_{O'A} + \omega \times (v_{app} + \omega \times r_{O'A})$$
$$a_{OA} = a_{OO'} + a_{app} + \omega \times (\omega \times r_{o'A}) + 2\omega \times v_{app} + \dot{\omega} \times r_{O'A}$$

$a_{OO'}$ acceleration between O and O'
a_{app} apparent acceleration of object as seen in O'
$\omega \times (\omega \times r_{O'A})$ is centripetal acceleration. for example, if $\omega \perp r_{O'A}$, then $\omega \times r_{O'A} = \omega r_{O'A} e_{\perp}$. $\omega \times (\omega \times r_{O'A}) = -\omega^2 r_{O'A} e_{O'A}$
$2\omega \times v_{app}$ is the coriolis effect
$\dot{\omega} \times r_{O'A}$ is the Euler Term

9.3.3 Example (cont.)

recall $v_{OA} = v_{train} + u + \omega \times b$
let $u, v_{train}, \omega = \text{const}$

$$u = \text{const} \rightarrow a_{app} = 0$$
$$v_{train} = \text{const} \rightarrow a_{OO'} = 0$$
$$\omega = \text{const} \rightarrow \dot{\omega} = 0$$
$$a_{OA} = a_{OO'} + a_{app} + \omega \times (\omega \times r_{o'A}) + 2\omega \times v_{app} + \dot{\omega} \times r_{O'A} =$$
$$a_{OA} = 0 + 0 + \omega \times (\omega \times r_{o'A}) + 2\omega \times v_{app} + 0$$
$$v_{app} = u = \text{const} \sim e_x$$
$$r_{O'A} = b \sim e_x$$
$$a_{OA} = -\omega^2 b\vec{e}_x + 2\omega u\vec{e}_y$$

centripetal + coriolis effects

9.4 Dynamics

N2L in non-inertial reference frames works by adding fictitious forces.

$$F_{true} = ma_{OA} - F_{fict}$$

insert what you find and rearrange

$$F_{true} + F_{fict} = ma_{app}$$

$$ma_{app} =$$
$$F_{true} - ma_{OO'} - m\omega \times (\omega \times r_{O'A}) - 2m\omega \times v_{app} - m\dot{\omega} \times r_{O'A}$$
elevator, centrifugal, coriolis, azimuthal

9.4.1 Puck on Icy Disk

Rotating disk moving counterclockwise with speed ω. Puck launched from perimeter towards B. Initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $y(0) = -R$, $\dot{y}(0) = u$

$$F_{true} = 0 \qquad a_{OO'} = 0 \qquad \dot{\omega} \times r_{O'A} = 0$$
centrifugal force $\omega \times (\omega \times r_{O'A}) = -\omega^2(xe_x + ye_y)$
$$\vec{\omega} = \omega\vec{e}_z \qquad r_{O'A} = xe_x + ye_y$$
coriolis force $\omega \times v_{app} = \omega \times (\dot{x}e_x + \dot{y}e_y) = \omega\dot{x}e_y - \omega\dot{y}e_x$
$$m(\ddot{x}e_x + \ddot{y}e_y) = ma_{app} = m\omega^2(xe_x + ye_y) - 2m\omega(\dot{x}e_y - \dot{y}e_x)$$
$$\ddot{x} = \omega^2 x + 2\omega\dot{y} \qquad \ddot{y} = \omega^2 y + 2\omega\dot{x}$$

coupled 2nd order diff eq to solve

use a complex function and it somehow works

$\ddot{\psi} + 2i\omega\dot{\psi} - \omega^2\psi = 0 \qquad \psi = x(t) + iy(t) \in \mathbb{C}$

ansatz $e^{\lambda t}$

$\lambda^2 e^{\lambda t} + 2i\omega\lambda e^{\lambda t} - \omega^2 e^{\lambda t} = 0$

$\lambda = -\dfrac{2i\omega}{2} \pm \sqrt{-\omega^2 + \omega^2} = -i\omega$

$\psi(t) = Ae^{-i\omega t} + Bte^{-i\omega t}$

A, B from initial conditions

$A \to \psi(0) = 0 = x(0) + iy(0) = 0 - iR$

$B \to \dot{\psi}(0) = 0 = \dot{x}(0) + i\dot{y}(0) = 0 = R\omega + iu$

$\phi(t) = -iRe^{-i\omega t} + (R\omega + iu)e^{-i\omega t}$

$x = \mathrm{Re}(\psi(t)) = Rt\omega\cos(\omega t) + (ut - R)\sin(\omega t)$

$y = \mathbb{C}(\psi(t)) = (ut - R)\sin(\omega t) - Rt\omega\cos(\omega t)$

9.4.2 Motion on Earth

Let the Earth be a sphere of radius R spinning with radius ω. Someone is an angle θ above the equator

$$\vec{\omega} = \omega\vec{p} \qquad v_{lab} = \omega R\cos(\theta) \qquad \dot{\omega} = \frac{1.7ms}{\text{century}} \approx 0$$

relative acceleration $\vec{a}_{OO'} = \dfrac{d^2R}{dt^2} = \omega \times (\omega \times R)$

$$|\vec{a}_{OO'}| = \omega^2 R\cos(\theta) = 0.034\frac{m}{s^2} \approx 0$$

centripetal force $\omega \times (\omega \times r_{O'A}) = \omega^2 r$

$r << R\omega^2 r << \omega^2 R << g$ neglect

only need to consider coriolis force

$$m\vec{a}_{app} = F_{true} - 2m\omega \times v_{app}$$

9.4.3 Coriolis Force on Earth

use coordinates (e, n, u) where e is east, n is north, u is upwards

$$\hat{p} = \cos(\theta)\hat{n} + \sin(\theta)\hat{u}$$

$$m\ddot{r}_{app} = \sum F_{true} + F_{fict}$$

$$m\ddot{r} = F_{true} - 2m\vec{\omega} \times \dot{r} \qquad F_{true} = -mg\hat{u}$$

$$m\ddot{r} = -mg\hat{u} - 2m\omega\hat{p} \times \dot{r} \qquad \dot{r} = (\dot{x}\hat{e} + \dot{y}\hat{n} + \dot{z}\hat{u})$$

$$m\ddot{r} = m(\ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}) =$$

$$-mg\hat{u} - 2m\omega\left[(\dot{z}\cos(\theta) - \dot{y}\sin(\theta))\hat{e} + \dot{x}\sin(\theta)\hat{n} - \dot{x}\cos(\theta)\hat{u}\right]$$

$$\ddot{x} = 2\omega\dot{z}\cos(\theta) + 2\omega\dot{y}\sin(\theta)$$

$$\ddot{y} = -2\omega\dot{x}\sin(\theta)$$

$$\ddot{z} = -g + 2\omega\dot{x}\cos(\theta)$$

9.5 Perturbation Theory

Consider something dropping down on the direction with intitial conditions, $\dot{x}(0) = \dot{y}(0) = \dot{z}(0) = x(0) = y(0) = 0$ and $z(0) = h$

because $F_{cor} << F_g$, we can treat it as a small perturbation ϵ

$$F_{total} = F_g + \epsilon F_{cor} \qquad x(t) = x_0(t) + \epsilon x_1(t)$$
$$y(t) = y_0(t) + \epsilon y_1(t) \qquad y(t) = y_0(t) + \epsilon y_1(t)$$
$$\text{eom for } F_g + \epsilon F_{cor}$$
$$\ddot{x} = 2\epsilon\omega\dot{z}\cos(\theta) + 2\epsilon\omega\dot{y}\sin(\theta)$$
$$\ddot{y} = -2\epsilon\omega\dot{x}\sin(\theta)$$
$$\ddot{z} = -g + 2\epsilon\omega\dot{x}\cos(\theta)$$
$$\ddot{x}_0 + \epsilon\ddot{x}_1 = 2\epsilon\omega(\dot{z}_0 + \epsilon\dot{z}_1)\cos(\theta) + 2\epsilon\omega(\dot{y}_0 + \epsilon\dot{y}_1)\sin(\theta)$$
$$\ddot{y}_0 + \epsilon\ddot{y}_1 = -2\epsilon\omega(\dot{x}_0 + \epsilon\dot{x}_1)\sin(\theta)$$
$$\ddot{z}_0 + \epsilon\ddot{z}_1 = -g + 2\epsilon\omega(\dot{x}_0 + \epsilon\dot{x}_1)\cos(\theta)$$

treat everything order-by-order in ϵ

in ϵ^0

$$\ddot{x}_0 + \epsilon\ddot{x}_1 = 0 \qquad \ddot{y}_0 + \epsilon\ddot{y}_1 = 0 \qquad \ddot{z}_0 + \epsilon\ddot{z}_1 = -g$$
$$x_0(t) = 0 \qquad y_0(t) = 0 \qquad z_0(t) = h - \frac{1}{2}gt^2$$

in ϵ^1

$$\ddot{x}_1 = -2\omega\dot{z}_0\cos(\theta) + 2\omega\dot{y}_0\sin(\theta)$$
$$\ddot{y}_1 = -2\omega\dot{x}_0\sin(\theta)$$
$$\ddot{z}_1 = 2\omega\dot{x}_0\cos(\theta)$$
$$\dot{x}_0 = 0 \qquad \dot{y}_0 = 0 \qquad \dot{z}_0 = -gt \qquad \text{from } e^0$$
$$\ddot{x}_1 = 2\omega gt\cos(\theta) \qquad \ddot{y}_1 = 0 \qquad \ddot{z}_1 = 0$$
$$\dot{x}_1 = \int 2\omega gt\cos(\theta)dt = \omega gt^2\cos(\theta) + C_1$$
$$x_1(t) = \frac{1}{2}\omega gt^3\cos(\theta) + C_2$$

insert into initial equation (9.58 - 9.59) in ϵ^1

$$x(t) = x_0(t) + \epsilon x_1(t) = 0 + \frac{1}{2}\omega gt^3\cos(\theta)$$
$$y(t) = 0$$
$$z(t) = h - \frac{1}{2}gt^2$$

The thingy moves eastward.

This is because when the object is "up", its actually closer to the equator, so as it falls to the ground, it goes closer to the axis of rotation while keeping its initially faster eastward inertia.

9.5.1 Foucault Pendulum

If you have a pendulum that's not affected by the rotation of the earth, then the earth will rotate under the pendulum and it will go in a circle.

Goal: solve

Consider a pendulum of length L with an angle β and height h.

However, this pendulum is 3d so we have to consider every movement direction d

$$d = \sqrt{x^2 + y^2} \qquad \cos(\beta) = \frac{h}{L} \qquad \sin(\beta) = \frac{d}{L}$$

use fictitious forces

$ma_{app} = \sum F_{true} + F_{fict}$

get $F_{net} = \sum F_{true}$ from $F_{net} = -\nabla U$

$U = -mgh = -mgL\cos(\beta)$

assume β is small

$U = \text{const} + \frac{mg}{2L}(x^2 + y^2)$

$F_{net} = -\nabla U = -\frac{mg}{L}(x\hat{e} + y\hat{n})$

fictitious forces

$F_{cor} = -2m\vec{\omega} \times \vec{v}_{app} \qquad \vec{\omega} = \omega\hat{p} = \omega(\cos(\theta)\hat{n} + \sin(\theta)\hat{u})$

$v_{app} = \dot{x}\hat{e} + \dot{y}\hat{n}$

$F_{cor} = -2m\omega\left(-\dot{x}\cos(\theta)\hat{u} + \dot{x}\sin(\theta)\hat{n} - \dot{y}\sin(\theta)\hat{e}\right)$

N2L

$\ddot{x} = -\frac{g}{L}x + 2\omega\dot{y}\sin(\theta)$

$\ddot{y} = -\frac{g}{L}y - 2\omega\dot{x}\sin(\theta)$

$\ddot{z} = 0 + 2\omega\dot{x}\cos(\theta)$

introduce $\omega_z = \omega\sin(\theta)$

$\ddot{x} = -\frac{g}{L}x + 2\omega_z\dot{y}$

$\ddot{y} = -\frac{g}{L}y - 2\omega_z\dot{x}$

$\ddot{z} = 0 + 2\omega\dot{x}\cos(\theta)$

Case 1: $\theta = 0$ (equator) meaning $\omega_z = 0$
simple harmonic oscillator

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{\frac{g}{L}}$$

Case not 1: do some solving junk

$$\psi = x + iy \qquad \dot{\psi} = \dot{x} + i\dot{y} \qquad \ddot{\psi} = \ddot{x} + i\ddot{y}$$
$$\ddot{\psi} = \left(-\frac{g}{L}x + 2\omega_z\dot{y}\right) + i\left(-\frac{g}{L}y - 2\omega_z\dot{x}\right) = -\frac{g}{L}\psi - 2i\omega_z\dot{\psi}$$
$$\ddot{\psi} + \frac{g}{L}\psi + 2i\omega_z\dot{\psi} = \ddot{\psi} + 2i\omega_z\dot{\psi} + \omega_n^2\psi = 0$$

ansatz $\psi = e^{\lambda t}$

$$\lambda^2 + 2i\omega_z\lambda + \omega_n^2 = 0 \qquad \lambda = \frac{-2i\omega_z \pm \sqrt{-4\omega_z^2 - 4\omega_n^2}}{2} =$$
$$-i\omega_z \pm \sqrt{\omega_z^2 + \omega_n^2}$$
$$\lambda = i\omega_z \pm i\omega_n$$
$$\psi(t) = e^{i\omega_z \pm i\omega_n} = e^{-i\omega_z t}(C_1 e^{i\omega_n} + C_2 e^{-i\omega_n})$$
$$x = iy = e^{-i\omega_z t}(x_{NR}(t) + iy_{NR}(t))$$
$$e^{i\omega_z t} = \cos(\omega_z t) + i\sin(\omega_z t)$$
$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_z t) & \sin(\omega_z t) \\ -\sin(\omega_z t) & \cos(\omega_z t) \end{pmatrix} \begin{pmatrix} x_{NR}(t) \\ y_{NR}(t) \end{pmatrix}$$

rotates counterclockwise by $\phi = \omega_z$

Chapter 10

Analytical Mechanics

Lagrangian and Hamiltonian shenanigans

based on "least action" principle

introduce configuration for "phase space" with a smooth function $\mathcal{L}$, the Lagrangian.

Define an action that is a functional- integral of the Lagrangian.

Find the extrema of the Lagrangian "stationary points" of the action ("least action")

Get equations of motion from the stationary point.

10.0.1 Snell's Law and Fermat's Principle

Snell's Law is how light refracts through a medium.

Setup: consider 2 media with refraction indices from $A \rightarrow B$

Determine the path of light that minimizes the time from $A \rightarrow B$. Special case of Fermat's Principle where light will take the path between 2 points that takes the elast time.

We need the relationship between refraction indices and the angles of incidence.

Speed of light in a medium: $v = c/n$ where n is the refraction index. and $c = 3 * 10^8$

$$t = t_1 + t_2 = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{l_1 n_1}{c} + \frac{l_2 n_2}{c}$$
$$l_1^2 = d^2 + h^2 \qquad l_2^2 = (x-d)^2 + (y-h)^2$$

Find h that minimizes t

$$t = \frac{l_1}{v_1} + \frac{l_2}{v_2} = \frac{l_1 n_1}{c} + \frac{l_2 n_2}{c} = \frac{\sqrt{d^2+h^2}n_1}{c} + \frac{\sqrt{(x-d)^2+(y-h)^2}n_2}{c}$$
$$t' = \frac{2hn_1}{c\sqrt{d^2+h^2}} + \frac{2(y-h)n_2}{c\sqrt{(x-d)^2+(y-h)^2}} = 0$$
$$2hn_1 + 2(y-h)n_2 = 0 \rightarrow 2hn_1 + 2yn_2 - 2hn_2 = 0 \rightarrow$$
$$2hn_1 + -2hn_2 = -2yn_2$$
$$h = \frac{yn_2}{n_2 - n_1}$$

plus some trig stuff that I took pictures of but did not put in my notes

10.1 More Stuff

introduce generalized coordinates $\{q_i\} = \{q_1, q_2, q_3, \ldots\}$ and a generalized velocity $\dot{q}_i = \frac{d}{dt} q_i$

denote path $\vec{q}(t)$ with fixed endpoints $q(t_1)$ and $q(t_2)$.
introduce action

$$S[q(t)] = \int_{t_1}^{t_2} \mathcal{L}(q_i(t), \dot{q}_i(t), t)$$

10.1.1 total vs partial derivatives

∂ has explicit dependence.
total derivative of a function $f(q(t), \dot{q}(t), t)$

$$\frac{d}{dt} f = \frac{\partial}{\partial t} f(q, \dot{q}, t) + \frac{\partial q}{\partial t}\frac{\partial f}{\partial q} + \frac{\partial \dot{q}}{\partial t}\frac{\partial f}{\partial \dot{q}}$$

10.2 Finding the Least Path

let $\{\vec{q}(t), \dot{\vec{q}}(t)\}$ be the extremal path. Consider small deviations $\delta\vec{q}(t)$.
$S[\vec{q}]$ should be stationary- $\delta S[\vec{q}] = 0$

$$\delta S[\vec{q}] = S[\vec{q} + \delta\vec{q}] - S[\vec{q}] = \int_{t_1}^{t_2} dt \mathcal{L}(q_i + \delta q_i, \dot{q}_i + \delta q_i, t) - \int_{t_1}^{t_2} dt \mathcal{L}(q_i, \dot{q}_i, t)$$

taylor expand term 1

$$\mathcal{L}(q_i + \delta q_i, \dot{q}_i + \delta q_i, t) =$$
$$\mathcal{L}(q_i, \dot{q}_i, t) + \sum_i \frac{\partial \mathcal{L}(q_i, \dot{q}_i, t)}{\partial q_i} \delta q_i + \sum_i \frac{\partial \mathcal{L}(q_i, \dot{q}_i, t)}{\partial q_i} \delta \dot{q}_i + O(\delta q^2)$$
$$\delta S[\vec{q}] = \int_{t_1}^{t_2} dt \{ \sum_i \frac{\partial \mathcal{L}}{\partial q} \delta q + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} \} = 0$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q\right) = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q\right) + \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q}$$
$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q\right) = \int_{t_1}^{t_2} dt \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q\right) - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q\right) =$$
$$\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \delta q\right] \qquad \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \Big|_{t_1}^{t_2} = 0$$
$$\int_{t_1}^{t_2} dt \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \delta q\right] \rightarrow$$

idk im missing the derivation but

10.2.1 Euler-Lagrange Euquation

$$\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q} - \frac{d}{dt} \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} = 0$$

10.3 New Lecture

$$S[\vec{q}_i(t)] = \int_{t_1}^{t_2} dt\mathcal{L}(q_i(t), \dot{q}_i(t); t)$$

Where S is defined as the path integral of the system, or the action of the system.

- Find the Lagrangian and set up the action/path integral
- Identify quantities $(q, \dot{q}, t, \mathcal{L})$
- Find and solve the E.L. equation

Euler-Lagrange Euquation (again lol)

$$\frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial q} - \frac{d}{dt}\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} = 0$$

10.3.1 Path between 2 points

Setup:2 fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$ in $\mathbb{R}^2$

Goal: Find a curve with the shortest length sonnecting A and B

Consider an infinitesimally small path $dl = dx^2 + dy^2$

The length of the curve is an integral

$$L = \int_A^B \sqrt{dx^2 + dy^2} \to \int_A^B \sqrt{1 + (\frac{dy}{dx})^2} dx \to \int_A^B \sqrt{1 + (y')^2} dx$$

$$q_i = y, \qquad \dot{q}_i = \frac{dy}{dx}, \qquad t = x \qquad \mathcal{L} = \sqrt{1 + (y')^2}$$

now we plug that into the E-L equation

$$\frac{\partial \mathcal{L}(q,\dot{q},t)}{\partial q} - \frac{d}{dt}\frac{\partial L(q,\dot{q},t)}{\partial \dot{q}} = 0 \to \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} =$$
$$\frac{\partial}{\partial y}\sqrt{1+(y')^2} - \frac{d}{dx}\frac{\partial}{\partial y'}\sqrt{1+(y')^2} = -\frac{d}{dx}\frac{\partial}{\partial y'}\sqrt{1+(y')^2} =$$
$$-\frac{d}{dx}\frac{y'}{\sqrt{1+(y')^2}} = 0 \to \frac{y'}{\sqrt{1+(y')^2}} = c \to$$
$$y' = c\sqrt{1+(y')^2} \to y'^2 = c^2 + c^2(y')^2 \to$$
$$y'^2 - c^2y'^2 = c^2 \to y' = \frac{c}{\sqrt{1-c^2}} = a = \text{const}$$
$$y = ax + b$$

a straight line woo

10.4 More Calculus of Variations

functional

$$S[\vec{q_i}] = \int_{t_1}^{t_2} \mathcal{L}(q_i, \dot{q}_i, t)dt$$

fix the end points $q(t_1), q(t_2)$

extremize $s[\vec{q_i}]$ by the path $\vec{q}$ that satisfies the EL equation $\to$ leads you to a differential equation for the system $q(t)$ with boundary conditions.

if $q(t_1)$ is bounded but $q(t_2)$ is undefined, then the term not involving

$$" \int " : \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2} \neq 0$$

thus we need to require that $\frac{\partial}{\partial \dot{q}_i} \mathcal{L} = 0$

10.4.1 Higher Derivatives

what if $\mathcal{L}$ is $\mathcal{L}(q, \dot{q}, \ddot{q}, t)$?

then some math happens

$$\frac{\partial}{\partial q} \mathcal{L} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2} \frac{\partial L}{\partial \ddot{q}} = 0$$

10.4.2 Brachisochrone

A particle is sliding down a curve without friction but with gravity.

Goal: Find the path that takes the least time

First: set up the action/path integral (action)

Our function will be T because we're trying to minimize T

$$dt = \frac{dl}{v} \qquad dl^2 = dx^2 + dy^2$$

energy conservation

$$\frac{1}{2}mv^2 = mgy \to v = \sqrt{2gy}$$

$$T = \int \frac{dl}{v} = \int_A^B \frac{\sqrt{dx^2 + dy^2}}{\sqrt{2gy}} = \int_A^B \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2gy}} dx$$

$$\mathcal{L} = \frac{\sqrt{1 + (\frac{dy}{dx})^2}}{\sqrt{2gy}} \qquad q = y, \qquad \dot{q} = \frac{dy}{dx}, \qquad t = x$$

ig that's actually wrong so im just gonna follow lecture

$$\mathcal{L} = \frac{\sqrt{1 + x'^2}}{\sqrt{2gy}}$$

$$\frac{\partial}{\partial x}\frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} - \frac{d}{dy}\frac{\partial}{\partial x'}\frac{\sqrt{1 + x'^2}}{\sqrt{2gy}} = 0 \to$$

$$\frac{d}{dy}\frac{1}{\sqrt{2gy}}\frac{x'}{\sqrt{1 + x'^2}} = 0 \to \frac{1}{\sqrt{2gy}}\frac{x'}{\sqrt{1 + x'^2}} = c$$

$$\frac{x'}{\sqrt{2gy}} = c\sqrt{1 + x'^2} \to \frac{x'^2}{2gy} = c^2 + c^2x'^2 \to x' = \frac{\sqrt{2gy}c}{\sqrt{1 - 2gyc^2}}$$

$$x(y) = \frac{1}{2c}\sqrt{\frac{2}{gy} - 4c^2} + \frac{1}{2gc^2}\arctan\left(\sqrt{\frac{1}{2gyc^2} - 1}\right)$$

10.5 Lagrange Multipliers

What is there is a constraint on your functional?

given a functional $S_1[q_i(t)]$ subject to a global (integral) restraint $S_2[q_i(t)]$

functional S_1 depending on $x(t), y(t)$ that have a local relationship.

Goal: minimize S_1 while holding S_2 fixed

$$S_1[q_i(t)] = \int_{t1}^{t2} \mathcal{L}_1(q_i(t), \dot{q}_i(t), t)dt \qquad \text{is minimized}$$
$$S_2[q_i(t)] = \int_{t1}^{t2} \mathcal{L}_2(q_i(t), \dot{q}_i(t), t)dt = \gamma = \text{const}$$

Use these along with a constant to construct a compound functional

$$C[q_i(t)] = S_1[q_i(t)] - \lambda S_2[q_i(t)] =$$
$$\int_{t1}^{t2} (\mathcal{L}_1(q_i(t), \dot{q}_i(t), t) - \lambda \mathcal{L}_2(q_i(t), \dot{q}_i(t), t))dt$$
$$\delta C = \delta S_1 - \lambda \delta S_2 = 0 \qquad \text{make an E-L equation}$$
$$\frac{\partial}{\partial q}(\mathcal{L}_1 - \lambda \mathcal{L}_2) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}(\mathcal{L}_1 - \lambda \mathcal{L}_2) = 0$$

Solve the equation with 2 integration constants c and λ.

10.5.1 Chain Example

Consider a chain hanging between 2 points. How does the chain lie to minimize potential energy?

consider a linear mass density $\rho = \frac{dm}{dl}$

The chain has start and endpoints $(0,0)$ and (a,b)

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{(\frac{dx}{dy})^2 + 1dy} = \sqrt{x'^2 + 1}dy$$

$$m = \int \rho dl \rightarrow$$

$$U = -mgl = -\int gy\rho dl = -\int g\rho y\sqrt{x'^2 + 1}dy$$

now make the compount function considering the length of the thingy

$$\mathcal{L}[x(y)] = \int dl = \int_0^b \sqrt{x'^2 + 1}dy$$

$$C[x(y)] = U - \tilde{\lambda}L = U + g\rho\lambda\sqrt{x'^2 + 1} = -g\rho(y - \lambda)\sqrt{x'^2 + 1}$$

plug it into the Euler-Lagrange equation and hope for the best

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dy}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \rightarrow$$

$$0 - \frac{d}{dy}\frac{\partial}{\partial \dot{x}} - g\rho(y - \lambda)\sqrt{x'^2 + 1}$$

$$-g\rho(y - \lambda)\frac{x'}{\sqrt{x'^2 + 1}} = \text{const} = k \rightarrow x' = \pm\frac{k}{\sqrt{(y - \lambda^2)} - k^2}$$

$$x = \pm k \ln\left(\lambda - y + \sqrt{(y - \lambda)^2 - k^2}\right) \rightarrow y = k\cosh\left(\frac{x - k_2}{k}\right) + \lambda$$

solve and you'll get hyperbolic cosine

10.6 Local Constraints

consider functionals of 2 or more functions that are related to each other

$$
\begin{aligned}
&F[x(z), y(z); z] = \int f(x(z), x'(z)y(z), y'(z), z)dz \\
&g(x(z), y(z); z) = 0 \\
&\mathcal{L} = f - \lambda g \\
&C = \int (f - \lambda g)dz
\end{aligned}
$$

that's your new lagrangian, so plug that into the E-L equation and you're good

10.7 Lagrangian Mechanics

consider a particle in a gravitational field

$$
\begin{aligned}
&\vec{g} = g\vec{e}_y \qquad m\ddot{y} = -mg \\
&\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \\
&\mathcal{L} = \frac{1}{2}m\dot{y}^2 - mgy \rightarrow \\
&\mathcal{L} = T - U
\end{aligned}
$$

The Lagrangian for mechanical systems with conservative forces that determine the potential

10.7.1 Caution

$T - U \neq T + U$

THE SIGN MATTERS ALOT

$T + U = E = \text{const}$

$T - U = \mathcal{L}$

10.7.2 Spring

Consider a mass m on a spring with k sliding along the x axis
Goal: Find the equation of motion using the Lagrangian

$$F = m\ddot{x} = -kx$$

$$L = T - U \qquad T = \frac{1}{2}m\dot{x}^2 \qquad U = \frac{1}{2}kx^2$$

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

Now just get the Euler-Lagrange equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -kx - \frac{d}{dt}m\dot{x} = 0 \rightarrow m\ddot{x} + kx = 0$$

Look at that we got N2L from the Lagrangian

You get the equation of motion from the Euler-Lagrange equation