MATH213 Final Study Guide

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Chapter 1

Lets FUcking Go EZ Final Ace

1.0.1 Pigeonhole Principle

if you have N objects in k boxes, then one box contains at least the ceiling of N/k objects.

WHAT THE FUCK IS A PERMUTATION RAAAAAAHHHHHHHHHH

https://www.youtube.com/watch?v=zWy77XbkRF8

1.0.2 Binomial Theorem

$$(x+y)^n = \sum_{k=1}^{n} \binom{n}{k} x^{n-k} y^k$$

1.0.3 Another useful theorem

$$\sum_{k=1}^{n} \binom{n}{k} = 2^n \text{ power set identity basically}$$

1.1 Combinatorial Identities

1.1.1 Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial proof: youre choosing $n+1$ objects $\{1, 2, 3, \ldots n, \alpha\}$. $\binom{n}{k-1}$ is choosing from the rest of the items because you've picked α and $\binom{n}{k}$ is picking from the rest of the items because you definitely have not picked α

1.1.2 Vandermonde's Identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Combinatorial proof: Just imagine the two different combinatorics things as different subsets of the same set $m + n$ and you're picking the same amount of stuff from the same set, just at two different times.

1.1.3 Hockey Stick Identity

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$

Ive proved this before but $\binom{n}{r}$ is you definitely picked the $n+1$st item and $\binom{n-1}{r}$ is you definitely picked the nth item and not the $n+1$st and so on and so forth.

1.1.4 Stars and Bars

if given you have to pick n total objects of r different types of objects, then the total way is

$$\binom{n+r-1}{r-1} = \binom{n+r-1}{n}$$

1.1.5 Distinguishable People in Undistinguishable Boxes

if you have n total objects with a of type 1, b of type 2, etc. the formula is $\frac{n!}{a!b!c!...}$

Just fucking figure it out permutations are not that difficult.

1.1.6 Bernoulli Trials

Probability of exactly k successes for a given independent thingy and n total trials with success chance p and fail chance $1-p$ is

$$\binom{n}{k} p^k (1-p)^{n-k}$$

1.1.7 Expected Value and Variance

$$E(x) = \sum_{s \in S} p(s)X(s)$$

Expected values have some nice properties

$$V(x) = \sum_{s \in S} (X(s) - E(X))^2 p(s)$$

If X is a random variable on a sample space, then

$$V(X) = E(X^2) - E(X)^2 = E(X - E(X))^2$$

Both follow the law of superposition if the probabilities are independent

1.1.8 Example Problems

The amount of ways to permute the letters in SUCCESS is

$$C(7,3)C(4,2)C(2,1)C(1,1)$$

The (7, 3) is 7 open spots for 3 letters (S), then 4 spots for 2 (C), then 2 for 1 (E) and 1 for 1 (U)

1.2 Linear Homogeneous Recurrence Relations

Solve the things, get the solutions, put the solutions as things that go to the nth power. Solve for coefficients based off of initial conditions.

1.3 Solving Recurrence Relations with Generating Functions

1.4 Power Series

$$1 + x + x^2 + \ldots = \frac{1}{1-x}$$

Trust the process, or remember the youtube video where the guy put x^n in each recurrence relation for each level n and developed generating functions that way.

$$a_n = 3a_{n-1} \to G(x) - 3xG(x) = a_0 \to G(x) = \frac{a_0}{1-3x}$$

and just solve

1.4.1 MORE ON GENERATING FUNCTIONS

$$(1 + x + x^2 + \ldots + x^n) = \frac{1-x^{n+1}}{1-x}$$

useful for permutation stuff and then you can just use the stars n bars

$$\left(\frac{1}{1-x}\right)^n = \sum_{k=1}^{\infty} \binom{n+k-1}{k-1} x^k$$

$$1 + x^2 + x^4 \ldots = \frac{1}{1-x^2}$$

the derivative of a generating function is $\sum nx^n$

1.5 Fixing Midterm 2

The last proof

Prove that

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

2^{n-1} is the amount of subsets of a set of cardinality $n-1$. Let N have cardinality n. $n2^{n-1}$ is the total amount of ways to pick an element $n \in N$ and then a subset of N that does NOT contain n.

Now we abuse the identity

$$\binom{n}{r} = \binom{n}{n-r}$$

Let $1\binom{n}{1}$ be the amount of ways you can pick a subset of size $n-1$ and then an element outside of that subset. $2\binom{n}{2}$ is the amount of ways you can pick a subset with cardinality $n-2$ and an element outside of that subset. So on and so forth until you reach $n\binom{n}{n}$ which is picking a subset of size 0 and an element in N

1.5.1 The Generating Function One

$$x_1 \leq 2, x_2 \leq 2, x_3 \geq 3, x_4 \geq 3 \rightarrow x^6(1-x^3)^2\left(\frac{1}{1-x}\right)^4 \rightarrow$$

$$x^6(1-2x^3+x^6)\left(\frac{1}{1-x}\right)^4 \rightarrow \binom{14+4-1}{4-1} - 2\binom{11+4-1}{4-1} + \binom{8+4-1}{4-1}$$

1.5.2 3d

Find the coefficient of x^{16} in $(2x - 5/x^3)^{100}$

$$(2x - 5x^{-3})^{100} =$$

The binomial theorem is

$$(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-k} y^j$$

I need to figure out for what j does

$x^{100-j} + x^{-3j} = x^{16} \to 100 - j - 3j = 16 \to -4j = -84 \to j = 21$

$$\binom{100}{21} 2^{79} (-5)^{21}$$

1.6 Review Stuff

Figure out how to prove an isomorphism using a bijection between vertices

For 10.3.26 just use the exact same bijection that makes the isomorphism between regular G and H.

Chapter 2

Graphs

2.0.1 Handshaking Lemma

$$\sum \deg(V) = 2 \times \text{edges}$$

for a degree sequence to be graphic the sum of degrees needs to be even and nothing needs to break math.

2.1 The Shit with Euler and the bridges

$$\text{path} \subset \text{trail} \subset \text{walk}$$

A trail is a walk with no edges repeated.

A path is a walk with no vertices or edges repeated

- A circuit is a path that starts and ends at the same vertex.
- An Eulerian Circuit is a circuit that uses every edge exactly once (can repeat vertices)

- A graph has an Eulerian Circuit iff all its vertices have even degrees.

  It has an Eulerian Path, but not an Eulerian Circuit iff 2 of its vertices have odd degrees.

- A Hamiltonian Circuit is a circuit that goes through every vertex exactly once.

- a graph is connected if there exists a path from any vertex to any other vertex.

- Strong connected and weakly connected is only important in di-graphs

- vertex and edge cuts are kind of self explanatory

2.1.1 Dirac's Theorem

Let G be a simples graph with vertices $n \geq 3$. if each vertex has a degree greater than $n/2$ then there exists a hamiltonian circuit.

2.1.2 Dijkstra's Algorithm

Just know it to find the shortest path between 2 nodes in a weighted graph.

2.2 Graph Colorings

bipartite graphs are 2-colorable. A matching is the subset of edges that connect the 2 sides of a bipartite graph

COLORS ARE VERTICES NOT FACES I SWEAR TO FUCKING GOD IF YOU MAKE THIS MISTAKE AGAIN

2.3 Graph Representations

2.3.1 Adjacency Matrices

an adjacency matrix has 0 if there isnt an edge between two vertices and 1 if there is.

2.3.2 Incidence Matrices

rows are vertices, columns are edges. 1 if the edge is connected to a vertex, 0 if othewise.

2.4 Isomorphism

If there exists a 1 to 1 function between each vertex such that vertices in F are adjacent iff theyre adjacent in G.

2.5 Planar Graphs

Graphs that can lay on a plane without any overlaps

2.5.1 Euler's Formula

Let G be a planar graph.

$$\text{vertices} + \text{faces - edges} = 2$$

2.5.2 Kuratowski's Theorem

A graph is non-planar if it contains a subgraph of either $K_{3,3}$ or K_5

Chapter 3

Trees

I don't actually think there was much we went over in like any of the lectures that really covered trees but to be honest I don't give a shit im learning it all anyways.

A full m-ary tree with i internal vertices has mi + 1 total vertices.

Oh SHit figure out minimum weighted spanning trees.

3.0.1 Full M-ary Tree

- if n vertices, then $(n-1)/m$ interval vertices and $((m-1)n+1)/m$ leaves.
- if i interval vertices then $mi+1$ regular vertices and $(m-1)i+1$ leaves
- if l leaves then $(ml-1)/(m-1)$ vertices and $(l-1)/(m-1)$ internal leaves

3.1 Spanning Trees

A spanning tree of a graph G is a subgraph of G that is a tree and contains every vertex in G.

A simple graph is connected iff it has a spanning tree.

if spanning tree is unweighted just use depth first or breadth first search to find a spanning tree.

3.1.1 Minimum Spanning Trees

in a connected weighted graph, it is a spanning tree that has the smallest possible sum of edge weights.

3.1.2 Prim's Algorithm

Start at one node. add the smallest possible edge every time.

3.1.3 Kruskal's Algorithm

Add the smallest weighted edges that do NOT contain a simple circuit. Stop after $n - 1$ edges have been selectied (for n vertices)

3.2 Miscellaneous

3.2.1 Bubble Sort

you know it

3.2.2 Selection Sort

find smallest, put at beginning, repeat.

3.2.3 Merge Sort

split up a bunch. merge function actually sort.

3.3 Time Complexity of Divide n Conquer

Let

$$f(n) = af(n/b) + cn^d$$

for $a \geq 1$, $n = b^k$ for integer b, c, d are real numbers.

$$f(n) = \begin{cases} O(n^d) \to a < b^d \\ O(n^d \log n) \to a = b^d \\ O(n^{\log_b a}) \to a > b^d \end{cases}$$

3.3.1 Relations

Equivalence Classes are basically the range of the function

S is totally ordered if the partial ordering of S contains all elements in S. The partial ordering is well-ordered

3.3.2 Hasse Diagrams

turn a relation into a di-graph

3.3.3 Well-Ordering Property

Every non-empty set of integers has a least element.

3.3.4 Extra thing abt Generating Function

$$1 + x^2 + x^4 \ldots = \frac{1}{1 - x^2}$$