PHYS 435 Midterm 1

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Chapter 1

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This midterm is open book open note so if I just write down absolutely every topic ever then I can look back at this notebook and be absolutely set

1.1 Calculus Junk

Know my Divergence and curls and junk

1.1.1 Divergence Theorem

Connects a volume to a surface

$$\iiint\limits_V d^3V \, \nabla \cdot F = \iint\limits_S d^2S \, \hat{n} \cdot F$$

1.1.2 Poisson's Equation

$$\vec{\nabla}\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0} \rightarrow -\nabla^2 V(\vec{r}) = \frac{\rho(\vec{r})}{\epsilon_0}$$

1.1.3 Green's Theorem

Connects a surface to a loop

$$\oint\limits_{\partial S} dS\, F = \iint\limits_{V} d^3V\, \nabla \times F$$

There's probability some derivative identities I can use

1.2 Electric Fields

Everything can be solve with either Gauss's Law, superposition, or both.

$$\vec{E}(r) = \frac{Q}{r^2}\hat{r}$$

1.3 Electric Potential

Everything can be solved with either superposition or integrating the electric field

$$V = -\int_O^r dr'\, E(r') \qquad E = -\nabla V$$

1.4 Conductors

conductors are pretty neat.

1.4.1 Method of Images

Given a conducting object and a point charge on the outside, the potential on the outside of the conductor is equivalent to the superposition of the real charge and an "image" point charge.

We've found the image charges for both a plane and a sphere.

1.5 Capacitance

$$C = \frac{Q}{\Delta V}$$

1.6 Boundary Value Problems

This was a big part of the class we had, and luckily because of the law of superposition,

1.6.1 Potential Everywhere Given Boundary Potential (Don't Know Green's Function)

This works because inside a conductor, there is no charge, so we can use Poisson's equation.

$$\nabla^2 V = 0 \rightarrow V(x, y, z) = X(x)Y(y)Z(z)$$

Solve the diff EQ from there to get some fourier series than can be solved for any surface potential.

This works if there is no charge on either the inside or outside of the conductor.

1.6.2 Potential Given Charge Density

It's just the superposition of the green's function and the charge density. This one is fine

1.7 Green's Function

It' s a little wacky but its essentially the response function given a single point charge.

$G = 0$ on the boundary of a conductor

The green's function changes for conductors, but is

$$G(r, r') = \frac{1}{|\vec{r} - \vec{r'}|}$$

for insulators

So our boundary value problem becomes

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r'\, G(\vec{r}, \vec{r'})\rho(\vec{r'}) + \frac{1}{4\pi} \int\limits_{\partial V} da'\, \hat{m}' \cdot \vec{\nabla}_r G(\vec{r}, \vec{r'}) V(\vec{r'})$$

The first part makes sense because it's just the superposition of the inside charges, but the second part is harder to derive

THIS ONLY WORKS FOR CONDUCTORS

This derivation is obnoxious

$$\nabla^2 V(\vec{r}) = -\frac{\rho}{\epsilon_0} \qquad \nabla^2 G(\vec{r}) = -4\pi\delta(\vec{r} - \vec{r'})$$

$$\vec{\nabla} \cdot \left(G\vec{\nabla}V - V\vec{\nabla}G\right) = \vec{\nabla}G\vec{\nabla}V + G\vec{\nabla}^2 V - \vec{\nabla}G\vec{\nabla}V - V\vec{\nabla}^2 G$$

$$= G\vec{\nabla}^2 V - V\vec{\nabla}^2 G \to$$

$$\int\limits_V d^3r'\, G\vec{\nabla}^2 V - V\vec{\nabla}^2 G = \int\limits_V d^3r'\, \vec{\nabla} \cdot \left(G\vec{\nabla}V - V\vec{\nabla}G\right) \to$$

$$\int\limits_V d^3r'\, G\vec{\nabla}^2 V - V\vec{\nabla}^2 G = \int\limits_{\partial V} da\, G\hat{n} \cdot \vec{\nabla}V - V\hat{n} \cdot \vec{\nabla}G \to$$

$$-\int\limits_V d^3r'\, G\frac{\rho}{\epsilon_0} + \int\limits_V d^3r'\, V\vec{4}\pi\delta(\vec{r} - \vec{r'}) = \int\limits_{\partial V} da\, G\hat{n} \cdot \vec{\nabla}V - V\hat{n} \cdot \vec{\nabla}G \to$$

$$-\int\limits_V d^3r'\, G\frac{\rho}{\epsilon_0} + V(\vec{r'})4\pi = \int\limits_{\partial V} da\, G\hat{n} \cdot \vec{\nabla}V - V\hat{n} \cdot \vec{\nabla}G \to$$

$$-\int\limits_V d^3r'\, G\frac{\rho}{\epsilon_0} + V(\vec{r'})4\pi = \int\limits_{\partial V} da\, G * -\vec{E} \cdot \hat{m} - V\hat{n} \cdot \vec{\nabla}G \to$$

$$V(\vec{r'}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\, G\rho - \frac{1}{4\pi}\int\limits_{\partial V} da\, V\hat{n} \cdot \vec{\nabla}G$$

$$+\frac{1}{4\pi}\int\limits_{\partial V} da\, G * -\vec{E} \cdot \hat{m}$$

And the last term is 0 because the potential caused by Green's function on the surface of a conductor should be 0

That's a kind of need derivation that gives you a pretty comfy equation.

THIS ONLY WORKS FOR CONDUCTORS

Because the Green's Function is specifically for a conducting sphere.

For a non-conducting surface you use the separation of variables.

1.8 Legendre Polynomials (FIGURE OUT WHAT'S HAPPENING)

I think its just the separation of variables that happens with a sphere

It's a consequence of the divergence of spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}V + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}V$$

so now we say that the laplacian is 0 and the potential is separable

$$V = R(r)Y(\theta,\phi) \qquad \nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}V + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}V = 0$$

$$\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}V + \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}V + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}V = 0$$

$$Y\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) + R(r)\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}Y + R(r)\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}Y = 0$$

$$\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) + \frac{1}{Y}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}Y + \frac{1}{Y}\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}Y = 0$$

This is our main diff eq that we now solve by taking various partial derivatives.

$$\frac{\partial}{\partial r}\left(\frac{1}{R}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r)\right) = 0 \to$$

$$\frac{1}{r}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) = l(l+1)$$

$$\frac{1}{Y}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}Y + \frac{1}{Y}\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}Y = -l(l+1)$$

With these, we get the equation

$$\frac{1}{r(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) - l(l+1) = 0 \to \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R(r) = l(l+1)R(r)$$

$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}Y + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial \phi^2}Y + l(l+1)Y = 0 \to$$

$$\sin(\theta)\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}Y + \frac{\partial^2}{\partial \phi^2}Y + l(l+1)\sin^2(\theta)Y = 0$$

You can separate Y into $\Theta(\theta)\Phi(\phi)$

$$\sin(\theta)\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\Theta\Phi + \frac{\partial^2}{\partial \phi^2}\Theta\Phi + l(l+1)\sin^2(\theta)\Theta\Phi \to$$

$$\frac{1}{\Theta(\theta)}\sin(\theta)\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\Theta + \frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial \phi^2}\Phi + l(l+1)\sin^2(\theta) = 0$$

Take our partial derivatives

$$\frac{\partial}{\partial \theta}\left(\frac{1}{\Theta(\theta)}\sin(\theta)\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\Theta + l(l+1)\sin^2(\theta)\right) = 0 \to$$

$$\frac{1}{\Theta(\theta)}\sin(\theta)\frac{\partial}{\partial \theta}\sin(\theta)\frac{\partial}{\partial \theta}\Theta + l(l+1)\sin^2(\theta) = +m^2$$

and on the other side

$$\frac{\partial}{\partial \phi}\left(\frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial \phi^2}\Phi\right) = 0 \rightarrow$$
$$\frac{1}{\Phi(\phi)}\frac{\partial^2}{\partial \phi^2}\Phi = -m^2 \rightarrow \Phi(\phi) = A\sin(m\phi) + B\cos(m\phi)$$

The solution for Θ is where we get the obnoxious legendre function

$$\frac{1}{\Theta(\theta)}\sin(\theta)\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}\Theta + l(l+1)\sin^2(\theta) = +m^2 \rightarrow$$
$$\sin(\theta)\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}\Theta = \left(m^2 - l(l+1)\sin^2(\theta)\right)\Theta$$

The solution to this is the legendre function

$$P_l^m(\theta)$$

So then you multiply the things together, and in spherical harmonics you get

$$Y(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\theta)e^{im\phi} \qquad -l \leq m \leq l$$

Let's assume that we have azimuthal symmetry, so no ϕ dependence.

It basically just means that $m = 0$ so our equation becomes

$$\sin(\theta)\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}\Theta = -l(l+1)\sin^2(\theta)\Theta \rightarrow$$
$$\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta}\Theta = -l(l+1)\sin(\theta)\Theta \rightarrow$$
$$\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\sin^2(\theta)\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\Theta = -l(l+1)\Theta$$

You do a substitution to make solving easier

$$\frac{dx}{d\theta} = -\sin(\theta) \rightarrow dx = -\sin(\theta)d\theta$$
$$\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}\Theta = -l(l+1)\Theta$$

The legendre polynomials come back in the solution
Legendre polynomials are

$$P_l(x) = c_0 + c_2x^2 + c_4x^4 + \ldots + c_lx^l \qquad l \text{ even}$$
$$P_l(x) = c_1x + c_3x^3 + c_5x^5 + \ldots + c_lx^l \qquad l \text{ odd}$$

Now we put back in $x = \cos(\theta)$ and we get

$$\Theta_l(\theta) = P_l(\theta)$$
$$P_0 = 1 \qquad P_1 = x \qquad P_2 = \frac{1}{2}(3x^2-1) \qquad P_3 = \frac{1}{2}(5x^3-3x) \qquad \ldots$$
$$P_0 = 1 \qquad P_1 = \cos(\theta) \qquad P_2 = \frac{1}{2}(3\cos^2(\theta)-1)$$
$$P_3 = \frac{1}{2}(5\cos^3(\theta)-3\cos(\theta)) \qquad \ldots$$

You just take a superposition of a bajillion legendre polynomials to get the θ dependence of the potential

Now we need to consider the R potential

$$\frac{1}{R(r)}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R_l(r) - l(l+1) = 0 \rightarrow \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}R_l(r) = l(l+1)R(r)$$
$$\rightarrow R_l(r) = A_l r^l + B_l\frac{1}{r^{l+1}}$$

We put everything together and get the potential for an azimuthally symmetric sphere

$$V(r,\theta) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos(\theta))$$

1.8.1 Conducting Sphere in a Capacitor

1.9 Multipoles

They're like a just a heuristic thing to figure out roughly what you distance dependence is.

It's based off of a taylor series that I'm sure you can derive if you really want to.

monopole is 1/r, dipole is $1/r^2$, quadrupole is $1/r^3$

1.10 Dielectrics

I genuinely have no idea but there's some indentities and some polarizability stuff.

Dielectrics are not conductors, but are made of atoms that can conduct electricity.

Each atom has a symmetric charge distribution

Now apply an electric field so that the dielectric has a dipole moment.

$$\vec{p} = \alpha \vec{E}$$

Where α is the polarizability coefficient.

1.10.1 Spring Model of Atoms

The dipole moment is proportional to the displacement of the positive and negative charges relative to each other.

1.10.2 Polarization Density

Because dielectrics are like continuous

$$\vec{P} = \vec{p} \cdot \text{atomic density} = pn$$
$$\vec{P} = n\alpha \vec{E}$$

Consider an electronic suscebtilibility factor X

$$\vec{P} = \epsilon_0 X \vec{E} \qquad X = \frac{nq^2}{\epsilon_0 \kappa}$$

We find the polarizability constants of single atoms but I doubt that's super necessary

1.10.3 Bound Charge Density

I think this is finding the new charge density from a polarized dielectric

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \vec{r}}{|\vec{r} - \vec{r'}|^3}$$

consider polarization density and $\vec{p} = \int d^3r \, \vec{P}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r' \, \frac{\vec{P} \cdot \vec{r}}{|\vec{r} - \vec{r'}|^3}$$

we know that the derivative of $1/r$ is $-1/r^2 = -r/r^3$, so

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r' \, \vec{P} \cdot -\vec{\nabla}_r \frac{1}{|\vec{r} - \vec{r'}|} = \frac{1}{4\pi\epsilon_0} \int\limits_V d^3r' \, \vec{P} \cdot \vec{\nabla}_{r'} \frac{1}{|\vec{r} - \vec{r'}|}$$

You do some math

$$\vec{\nabla}_{r'} \cdot \frac{\vec{P}}{|\vec{r} - \vec{r'}|} = \frac{\vec{\nabla} \cdot \vec{P}}{|\vec{r} - \vec{r'}|} + \vec{P} \cdot \vec{\nabla}_{r'} \cdot \frac{1}{|\vec{r} - \vec{r'}|}$$

and then get

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\vec{P}\cdot\vec{\nabla}_{r'}\frac{1}{|\vec{r}-\vec{r'}|} =$$

$$\frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\vec{\nabla}_{r'}\cdot\frac{\vec{P}}{|\vec{r}-\vec{r'}|} - \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r'}|}$$

Then do divergence theorem

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int\limits_{\partial V} da\,\frac{\hat{n}\cdot\vec{P}}{|\vec{r}-\vec{r'}|} - \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\frac{\vec{\nabla}\cdot\vec{P}}{|\vec{r}-\vec{r'}|}$$

With all of this shenanigans, we can say that

$$V = \frac{1}{4\pi\epsilon_0}\int\limits_V d^3r'\,\frac{p_b(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

Where the bound charge density is given by

$$p_b(r) = -\vec{\nabla}\cdot\vec{P}(\vec{r})$$

and the surface bound charge density is given by

$$\sigma_b(\vec{r}) = \hat{n}\cdot\vec{p}(\vec{r})$$

So the total potential is given by integrating over the bulk bound charge and the surface bound charge.

1.10.4 $\vec{E}$ inside Dielectric

Consider a linear dielectric

$$\vec{p} = \alpha\vec{E} \qquad \vec{P} = n\alpha\vec{E} = \frac{\alpha}{a^3}\vec{E}$$

Define the susceptibility to polarization X

$$\vec{P} = \epsilon_0 X_E \vec{E} \qquad X_E = \frac{n\alpha}{\epsilon_0} = \frac{q^2 n}{\epsilon_0 \kappa}$$

A capacitor with a dielectric has both an external charge and a bound charge and a bound surface charge, the latter two are induced by both itself and the external charge.

$$E = \frac{\sigma_{total}}{\epsilon_0} \qquad \sigma_{tot} = \sigma_{Free} - p$$

$$E = \frac{1}{\epsilon_0}\left(\sigma_{free} - p\right) = \frac{1}{\epsilon_0}\left(\sigma_{free} - \epsilon_0 X\vec{E}\right)$$

$$\vec{E} = \frac{\sigma_{Free}}{\epsilon_0(1+X)} = \frac{\sigma_{Free}}{\epsilon_0\kappa}$$

There's some capacitance shenanigans but I don't really care at all

Consider a 3d bulk charge density

$$p_b(\vec{r}) = -\vec{\nabla}\cdot\vec{p}$$

This happens if X, the polarize susceptibility function, is position dependent.

The bound charges come from the dipole moment $\vec{p} = \epsilon_0 X \vec{E}$, which comes from the external electric field.
consider a total charge density

$$p_{tot} = p_{free} + p_{bound}$$

use Gauss's Law

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \left(p_{free}(\vec{r}) + p_{bound}(\vec{r}) \right) = \frac{1}{\epsilon_0} \left(\vec{p}_{free}(\vec{r}) + \vec{\nabla} \cdot \vec{P}(\vec{r}) \right)$$

$$\nabla \cdot \left(\vec{E}(\vec{r}) + \frac{1}{\epsilon_0} \vec{P}(\vec{r}) \right) = \frac{p_f(\vec{r})}{\epsilon_0}$$

This equation is why we make the "Displacement field"

$$\vec{D}(\vec{r}) = \epsilon_0 \vec{E} + \vec{P}(\vec{r})$$

I think that's the polarization density and not the dipole moment

$$\vec{\nabla} \cdot \vec{D}(\vec{r}) = \rho_{free}$$

In general nothing here should equal 0

1.10.5 Helmholtz Theorem

If you know the curl and divergence of a function, and the function does not diverge anywhere, then you know the function.

Consider a linear dielectric

$$\vec{p} = \epsilon_0 X \vec{E} \qquad \kappa = 1 + X$$
$$\vec{D} = \epsilon_0 \vec{E} + \epsilon_0 X \vec{E} = \epsilon_0 \vec{E}(1 + X) = \epsilon_0 \vec{E} \kappa$$

$\vec{D}$ doesn't actually do anything, it just lets us do math easier.

If we have a parallel plate capacitor, and we consider the boundary between the bound charge density and the free charge density, we get

$$\hat{n} \cdot \vec{D}_1 - \hat{n} \cdot \vec{D}_2 = \sigma_{free}$$

If we have no external potential and thus $\sigma_{free} = 0$, then

$$\hat{n} \cdot \vec{D}_1 = \hat{n} \cdot \vec{D}_2 \rightarrow \kappa_1 \hat{n} \cdot \vec{E}_1 = \kappa_2 \hat{n} \cdot \vec{E}_2$$

This lets you determine the electric fields of dielectrics given both of their spring constants and some other stuff.

That is the end of lecture 16.