**1. Rigid Body Dynamics: Static Equilibrium**

**Core Principles**

- **Newton's 1st Law (Equilibrium)**
- **Equilibrium Conditions (2D):**
  $\Sigma Fx = 0$ (Sum of horizontal forces is zero)
  $\Sigma Fy = 0$ (Sum of vertical forces is zero)
  $\Sigma M\_O = 0$ (Sum of moments about *any* point O is zero)
- **Moment:** $M = F * d$ (Force × perpendicular distance).
  3D: $M_{vec} = r_{vec} \times F_{vec}$.
- **Couple:** Two equal, opposite, non-collinear forces. Produces pure rotation.

**Free Body Diagram (FBD) Steps**

1. **Isolate:** Choose the rigid body to analyse.
2. **Draw:** Sketch the complete boundary of the isolated body.
3. **Show Forces:** Draw *all* external forces acting *on* the body:
   Applied loads (e.g., weight in hand).
   Gravitational force (Weight = mg), acting at the segment's Centre of Mass (COM).
   Reaction forces at joints (e.g., $J_x, J_y$).
   Muscle forces (modelled as cables, tension T).
4. **Add Coordinates:** Add a coordinate system and label all distances and angles.

**Lever Classes**

- **Mechanical Advantage (MA):** $MA = \frac{Effort\ Arm}{Resistance\ Arm}$
- **1st Class (Seesaw):** Fulcrum in middle. MA varies.
- **2nd Class (Wheelbarrow):** Resistance in middle. MA>1 (force efficient).
- **3rd Class (Elbow):** Effort in middle. MA<1 (speed/ROM efficient). Most common in the human body.

**Example 1: Elbow Statics (3rd Class Lever)**

- **Problem:** Holding 50 N weight ($F_H$) at $d_H$ = 0.20 m. Biceps ($F_M$) inserts at $d_M$ = 0.02 m. Find muscle force $F_M$ and joint force $F_J$.
- **Step 1: Find F_M (Sum moments about Elbow Joint J)**
  $\Sigma M_J = 0 = (F_M * d_M) - (F_H * d_H)$
  $(F_M \times 0.02) = (50 * 0.20) = F_M = \mathbf{500\ N}$
- **Step 2: Find F_J (Sum vertical forces)**
  $$\Sigma F_y = 0 = F_M - F_H - F_J$$
  $$F_J = F_M - F_H = 500 - 50 = \mathbf{450\ N}\ (compressive)$$

**Example 2: 2D Hip Statics (Single-Leg Stance)**

- **Problem:** Find Abductor Muscle Force (F) and Hip Joint Reaction Force (H) during slow walking (neglect inertia).
- **Forces:**
  **W:** Effective body weight (e.g., 0.8 * BW), acting at distance D from hip centre.
  **F:** Abductor muscle force, acting at moment arm L1.
  **H:** Hip joint reaction force.
- **Step 1: Find F (Sum moments about Hip Joint H)**
  $$\Sigma M\_H = 0 = (Fy * L1) - (Wy * D)$$
  $$Fy = Wy * (D/L1)$$
  Common simplification: D ≈ 2 * L1
  $$Fy \approx Wy * ((2 * L1) / L1) = \mathbf{2 * Wy}$$
- **Step 2: Find H (Sum vertical forces)**
  $$\Sigma Fy = 0 = Hy - Fy - Wy$$
  $$Hy = Fy + Wy = (2 * Wy) + Wy = \mathbf{3 * Wy}$$
- **Result:** Hy = 3 * (0.8 * BW) = **2.4 * Total Body Weight**.

**Strategies to Reduce Hip Load**

- **Duchenne Limp:** Lean trunk over stance leg. This **reduces moment arm D** of body weight, which reduces the required muscle force F, and thus reduces the final joint force H.
- **Cane (Contralateral):** Use cane in the hand *opposite* the affected hip. The cane force creates an assistive moment, reducing the required muscle force F.

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**2. Rigid Body Dynamics: Gait Analysis (Inverse Dynamics)**

**Definition:** A computational method to find unknown *internal* net joint forces and moments from *known* motion (kinematics) and *measured* external forces (kinetics).

**Required Inputs:**

1. **3D Kinematics:** 3D marker trajectories from motion capture (e.g., Vicon).
2. **External Forces:** Ground Reaction Forces (GRF) and Centre of Pressure (COP) from force plates.
3. **Anthropometry:** Segment mass, COM location, and mass moment of inertia (I), often from a **Link Segment Model**.

**Gait Analysis Workflow**

1. **Input:** Capture motion (markers) and forces (GRF).
2. **Inverse Kinematics (IK):** Optimization process. Finds the joint angles (q) that best match the experimental marker positions.
3. **Inverse Dynamics (ID):** Uses q and GRF to solve the equations of motion for the **Net Joint Moments ($\boldsymbol{\tau_m}$)**.
4. **Static Optimisation (SO):** Solves the "Muscle Redundancy Problem" to find individual **Muscle Forces ($\boldsymbol{f_m}$)** that sum to $\tau_m$.
5. **Joint Reaction Analysis (JRA):** Calculates the final **Joint Reaction Forces (JRF)** by summing muscle forces and joint moments.

**Newton-Euler Equations (Equations of Motion)**

- **Translation:** $\Sigma F = m * a_{COM}$
- **Rotation:** $\Sigma M_{COM} = I_G * \alpha$

**The Muscle Redundancy Problem**

- **Problem:** There are **more muscles crossing a joint than there are mechanical degrees of freedom (DOFs)**.
- **Consequence:** The equation $\tau_m = \Sigma(f_m * L_{MA})$ is an **indeterminate system**. There are infinite mathematical combinations of muscle forces ($f_m$) that could produce the same net moment $\tau_m$.
- **Solution (Static Optimisation):** An algorithm finds one "optimal" solution by minimizing a physiological **cost function**, e.g., **min** $\boldsymbol{\Sigma(Muscle\ Activation)^2}$ or **min** $\boldsymbol{\Sigma(Muscle\ Stress)^2}$ .

**Key Kinematic Calculations**

- **Absolute Segment Angle:** $\theta = atan2(y2 - y1, x2 - x1)$.
- **Joint Angle:** $\theta_{Knee} = \theta_{Thigh} - \theta_{Shank}$
- **Finite Difference (Central Difference):** Used to get velocity/acceleration from position data (x) with timestep ($\Delta t$).
  - Velocity: $v_i = \frac{(x_i+1 - x_i-1)}{(2\Delta t)}$
  - Acceleration: $a_i = \frac{(x_i+1-2x_i+x_i-1)}{\Delta t^2}$

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**3. Deformable Bodies: Bone Bending (Beam Theory)**

**Core Assumptions**

- **Bernoulli's Hypothesis:** A cross-section that is flat before bending remains flat after bending.
- **Consequence:** Axial strain ($\varepsilon_x$) and axial stress ($\sigma_x$) vary *linearly* across the cross-section.

**Key Geometric Properties (Moments of Area)**

- **First Moment of Area (S):** Measures distribution of area relative to an axis. Used to find the **centroid**.
  **Centroid ($\overline{y}$):** The geometric centre of the shape. The neutral axis passes through the centroid in pure bending.
  Found by: $\overline{\boldsymbol{y}} = \frac{\boldsymbol{\Sigma(A_i * y_i)}}{\boldsymbol{\Sigma A_i}}$
- **Second Moment of Area (I):** A geometric property describing a cross-section's resistance to bending. A larger I = more resistance. Units: $m^4$ or $mm^4$.
  For a rectangle: $I_{centroid} = \frac{b \times h^3}{12}$
- **Parallel Axis Theorem (Steiner's Theorem):** Crucial for finding the total I of a complex shape made of simple parts. It "moves" the I of a part from its own local centroid ($I_c$) to the main neutral axis of the *entire* composite shape.
  $$\boldsymbol{I_{new\_axis} = I_{centroid} + A * d^2}$$
  Where A is the area of the part and d is the perpendicular distance between the part's centroid and the main neutral axis.

**Key Stress Formulas**

- **Bending Stress (Flexure Formula):** $\sigma_x = \frac{(M*y)}{I}$
  $\sigma_x$= Stress at a point
  $M$ = Bending moment
  $I$ = Second moment of area (about the neutral axis)
  $y$ = Perpendicular distance from the neutral axis to the point of interest.
  Stress is **zero** at the neutral axis and **maximum** at the furthest points (top/bottom fibres).
- **General Axial Stress:** Combines axial load (N) and bending moments (M) about the principal axes (eta, zeta).

$$\sigma_x = \left(\frac{N}{A}\right) - \left(\frac{M_\zeta}{I_\zeta}\right)\eta + \left(\frac{M_\eta}{I_\eta}\right)\zeta$$

**Composite Beam Theory (e.g., Bone + Fixation Plate)**

- **Goal:** Analyse a beam made of two materials (e.g., $E_{Bone}$, $E_{Plate}$).
- **Step 1: Transform.** Convert to an "equivalent beam" of one material (e.g., all bone).
  **Transformation Factor ($n$):** $n = \frac{E_{stiff}}{E_{Less\ Stiff}}$
  **Transform Width:** Multiply the width of the *stiffer* material by n. $w_{transformed} = n * w_{original}$
- **Step 2: Analyse Geometry.** Find the centroid ($\bar{y}$) and total I (using Parallel Axis Theorem) of the *new, transformed* composite shape.
- **Step 3: Calculate Equivalent Stress ($\sigma_{eq}$).**
  Apply the flexure formula: $\sigma_{eq} = \frac{(M*y)}{I_{Total}}$
  **Step 4: Correct for Real Stress.**
  The stress in the *un-transformed* material (bone) is correct: $\boldsymbol{\sigma_{real\ bone} = \sigma_{eq}}$
  The stress in the *transformed* material (plate) must be scaled by n: $\boldsymbol{\sigma_{real\ plate} = n * \sigma_{eq}}$
- **Tension Band Principle:** A fixation plate is most effective when placed on the **tension side** of the bone, converting tensile forces into compression at the fracture site.

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**4. Deformable Bodies: Bone Adaptation**

**Wolff's Law:** The foundational principle of bone adaptation. "Every change in the form and function of a bone is followed by definite changes in its internal architecture and external conformation...".

- **Key Principles:**
  Bone optimizes strength while minimizing mass.
  Trabecular architecture aligns with principal stress trajectories.
  A biological regulatory system senses load and directs adaptation.

**Frost's Mechanostat Theory:** Quantifies Wolff's Law using mechanical strain thresholds (measured in micro strain, $\mu\varepsilon$, where 1000 $\mu\varepsilon$ = 0.1% change).

**Four Strain Zones:**

1. **Disuse Zone ($< \sim 200\ \mu\varepsilon$):**
   **Stimulus:** Strain is too low to maintain bone mass.
   **Response:** Net **bone resorption** (bone loss, Osteoclasts > Osteoblasts). Seen in bedrest or spaceflight.
2. **Physiological Zone ($\sim 200 - 2500\ \mu\varepsilon$):**
   **Stimulus:** The "lazy zone" or normal daily strain.
   **Response: Homeostasis**. Remodelling is balanced (resorption = formation). No net change in mass.
3. **Overload Zone ($> \sim 2500\ \mu\varepsilon$):**
   **Stimulus:** Strain exceeds the normal threshold; a "good" overload.
   **Response:** Net **bone formation** (bone gain, Osteoblasts > Osteoclasts). This is the target for osteogenic exercise.
4. **Pathological Zone ($> \sim 4000\ \mu\varepsilon$):**
   **Stimulus:** Strain is excessively high and damaging.
   **Response:** Microdamage accumulates faster than repair, leading to **stress fractures**.

**Turner's 3 Rules of Adaptation**

1. **Dynamic, Not Static:** Adaptation is driven by *changes* in strain (dynamic, high-impact loads, high strain rate), not static loads.
2. **Saturation:** The adaptive response saturates quickly. Only a small number of loading cycles (e.g., 40-100) are needed per day for a maximal response.
3. **Accommodation & Rest:** Cells become less responsive to familiar loads (stimulus must be novel). Inserting *rest periods* (hours) between loading bouts significantly enhances the bone formation response.

**Computational Bone Adaptation Models**

- **Iterative Algorithm:**
  1. $\boldsymbol{t = 0}$**:** Define initial bone geometry/density in a Finite Element (FE) model.
  2. **Apply Loads:** Apply realistic loads (e.g., from gait analysis).
  3. **Calculate Stimulus:** Solve FE model for stress/strain and calculate the stimulus (psi) at every point (e.g., Carter's $\psi = (\Sigma(n_i * \sigma_i{}^m))^{(\frac{1}{m})}$.
  4. **Compare to Target:** Compare stimulus ($\psi$) to the target "attractor state" stimulus ($\psi_{AS}$), which represents the physiological zone.
  5. **Adapt Density:** Use a rate law to change local bone density:
     - If $\psi > \psi_{AS}$ (overload) --> Add bone density.
     - If $\psi < \psi_{AS}$ (disuse) --> Remove bone density.
     - If $\psi \approx \psi_{AS}$ ("lazy zone") --> No change.
  6. **Update & Repeat:** Update the FE model's properties. Advance time ($t = t + \Delta t$) and repeat from Step 2

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**5. Key Definitions and Concepts**

- **Stress (σ):** Internal force per unit area (Units: N/m² or Pa). It is a measure of the internal forces acting within a body.
- **Strain (ε):** A dimensionless measure of deformation, representing the relative change in size or shape (e.g., change in length / original length).
- **Anisotropic:** A material whose mechanical properties (like stiffness $E$) are **direction-dependent**. Bone is a key example, as it is strongest along its longitudinal axis.
- **Viscoelastic:** A material whose properties depend on the **rate of loading**. Bone is viscoelastic; it is stiffer and stronger when loaded *quickly* (e.g., during an impact) than when loaded slowly.
- **Principal Stresses (σ1, σ2, σ3):** The maximum and minimum normal stresses at a point. They act on planes where all **shear stresses are zero**.
- **Trabecular Bone Scaling:** The Apparent Modulus ($E^*$) of trabecular bone is extremely sensitive to its relative density $\left({\rho^*}/{\rho_s}\right)$. The relationship is: $E^* = C(\rho^*/\rho_s)^2$. This means a small decrease in density (e.g., osteoporosis) causes a *dramatic* loss in stiffness.
- **Forward vs. Inverse Dynamics:**
  **Forward Dynamics:** *Predictive*. Uses neural commands/forces to solve for the resulting motion (acceleration, $\ddot{q}$). Answers: "What motion will occur?"
  **Inverse Dynamics:** *Analytical*. Uses *measured motion* (kinematics) to calculate the unknown Net Joint Moments ($\tau_m$) that must have caused it. Answers: "What forces caused this motion?"

**Fracture Healing Phases:** A 4-stage process.

1. **Inflammation:** A haematoma (blood clot) forms immediately.
2. **Soft Callus:** Fibrocartilage replaces the clot for initial stability.
3. **Hard Callus:** Woven bone replaces the soft callus via endochondral ossification.
4. **Bone Remodelling:** Woven bone is slowly remodelled into stronger, lamellar bone.

**Fracture Patterns by Load:** The pattern indicates the load type.

- **Tension** → **Transverse**
- **Compression** → **Crushing** (e.g., vertebrae)
- **Bending** → **Transverse** (often with a **"butterfly"** fragment)
- **Torsion** (twisting) → **Spiral** fracture