Chapter 2

Mathematics has beauty and romance. It's not a boring place to be, the mathematical world. It's an extraordinary place; it's worth spending time there.

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Contents

Functions

1.1 Relations

A relation is a set of ordered pairs.

- The set of first components in the ordered pairs is called the **domain** of the relation.
- The set of second components in the ordered pairs is called the **range** of the relation.

The following are examples of relations:

$$A = \{(-1,\, 3),\, (2,\, 0),\, (2,\, 5),\, (-3,\, 2)\}$$

The Domain and range of this relation are:

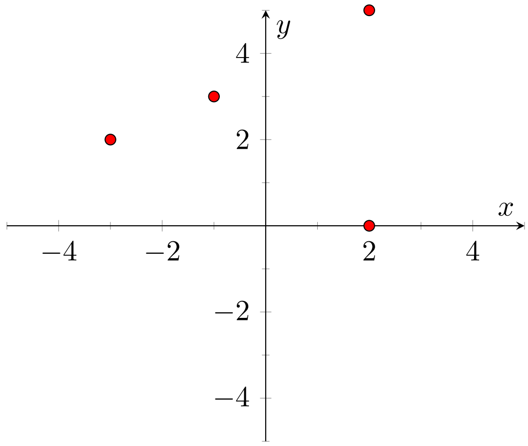
$$\begin{aligned} Domain &= \{-1,\, 2,\, -3\} \\ Range &= \{3,\, 0,\, 5,\, 2\} \end{aligned}$$

When listing the elements of both domain and range, get rid of duplicates and write them in increasing order.
However, aside from set notation, there are other ways to write this same relation. We can show it in a table, plot it on the xy-plane, and express it using a mapping diagram.

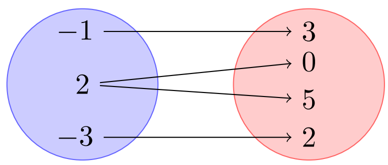
Relation A in table

x	y
-1	3
2	0
2	5
-3	2

Relation A in graph



Relation A in mapping diagram



$$A : X \to Y$$

1.2 Functions

A function is actually a special kind of relation because it follows an extra rule. Just like a relation, a function is also a set of ordered pairs; however, *every x-value must be associated to only one y-value.*

For instance, consider the relation A. This relation is not a function since we have repetitions or duplicates of x-values with different y-values, then this relation ceases to be a function.

$$A = \{(-1,\ 3),\ (2,\ 0),\ (2,\ 5),\ (-3,\ 2)\} \quad ✗$$

Example 1.1. Determine whether each relation is a function.

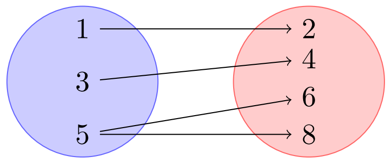
(a) $S = \{(1, 2), (3, 4), (5, 6), (5, 8)\}$

(b) $T = \{(1, 2), (3, 4), (6, 5), (8, 5)\}$

(a) As you can see number 5 in domain corresponds to both number 6 and 8. Since number 5 in the domain corresponds to more than one element in range, the relation S is not a function.

$$S = \{(1, 2), (3, 4), (5, 6), (5, 8)\} \quad ✗$$

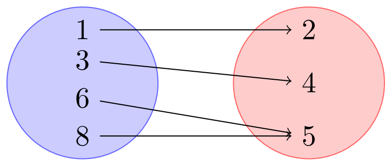
This relation is also shown in Figure (1.1).



$$S : X \to Y$$

Figure 1.1: Mapping diagram of relation S

(b) In this relations, every element in the domain corresponds to exactly one element in the range. Therefore, the relation T is a function. The mapping diagram of relation T is shown below.



$$T : X \to Y$$

Figure 1.2: Mapping diagram of relation T

1.2.1 Functions as equations

Functions are usually given in terms of equations rather than as sets of of ordered pairs. Examples of several functions are below:

$$\begin{aligned}
y &= 0.01x^2 - 0.2x + 8.7 \\
y &= -2.9x + 286 \\
y &= \sqrt{x-3} \\
y &= \frac{x-1}{x+1} \\
&\vdots
\end{aligned}$$

A function is much like a machine, where you drop some random stuff in and the machine provides you with something else-much like a candy or chip machine where you input coins and you get out a tasty snack.

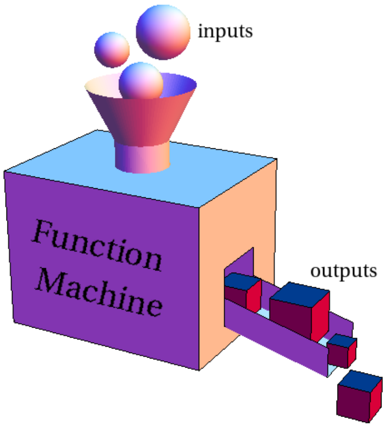


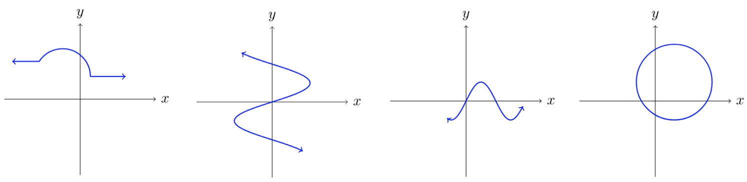
Figure 1.3: Function as a machine

The function has has an input, x-value and an output, y-value. For each value of x, there is one and only one value of y. For this reason x is considered as **an independent variable** and y is considered as **dependent variable**.

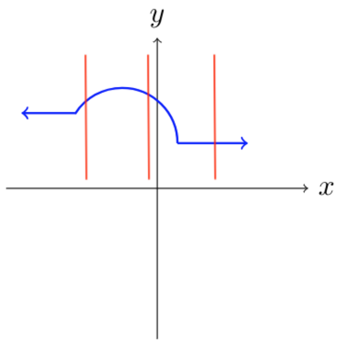
1.2.2 Vertical line test

A great way to determine whether an equation is a function is looking at its graph. Because x values are vertical lines we will draw a vertical line through the graph. If the vertical line crosses the graph more than once, that means we have too many possible y values. If the graph crosses the graph only once, then we say the equation represent a function.

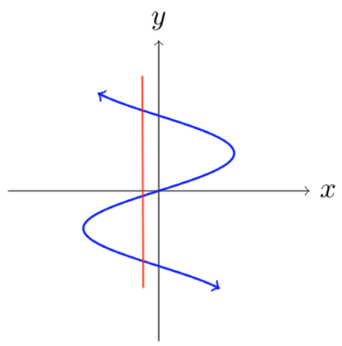
Example 1.2. Use the vertical line test to identify graphs in which y is a function of x.



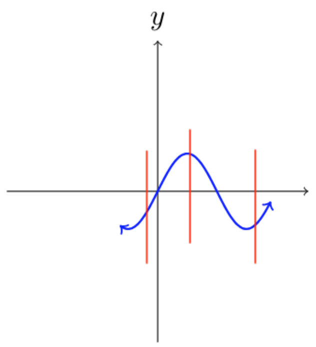
(a) **(b)** **(c)** **(d)**



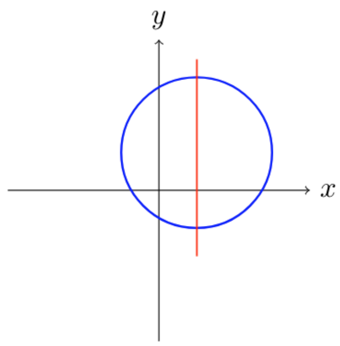
(a) y is a **function** of x.



(b) y **is not a function** of x. Because a vertical line hit the graphs more than once.



(c) y is a **function** of x.



(d) y **is not a function** of x. Because a vertical line hit the graphs more than once.

1.2.3 Function Notation

Often we will change the notation used to emphasis the fact that it is a function. Instead of writing $y =$, we will use function notation which can be written $f(x) =$. We read this notation "f of x". Consider the function $y = x^2 - 4$. In function notation, we write it as $f(x) = x^2 - 4$.

Note 1.1. It is important to point out that $f(x)$ does not mean f times x, it is merely a notation that names the function with the first letter (function f) and then in parenthesis we are given information about what variables are in the function (variable x). The first letter can be anything we want it to be, often you will see $g(x)$ (read g of x).

1.2.4 Evaluate a function

To evaluate a function, $f(x)$, we need to substitute the specified input values for x in the function's equation. This is shown in the following examples.

Example 1.3. If $f(x) = x^2 - 2x + 7$, evaluate each of the following:

a. $f(-5)$ b. $f(x+4)$ c. $f(-x)$

$$\begin{aligned} f(-5) &=? && \text{Replace } x \text{ with -5} \\ (-5)^2 - 2(-5) + 7 & && \text{Simplify} \\ 25 + 10 + 7 & && \text{Add} \\ 42 & && \text{Our solution to part a} \end{aligned}$$

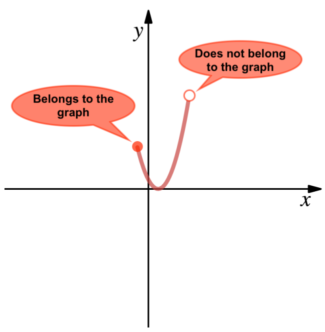
$$\begin{aligned} f(x+4) &=? && \text{Replace } x \text{ with } x+1 \\ (x+4)^2 - 2(x+4) + 7 & && \text{Simplify} \\ x^2 + 16 + 8x - 2x - 8 + 7 & && \text{Combine like term} \\ x^2 + 6x + 17 & && \text{Our solution to part b} \end{aligned}$$

$$\begin{aligned} f(-x) &=? && \text{Replace } x \text{ with } -x \\ (-x)^2 - 2(-x) + 7 & && \text{Simplify} \\ x^2 + 2x + 7 & && \text{Our solution to part c} \end{aligned}$$

1.3 Obtaining information from a graph

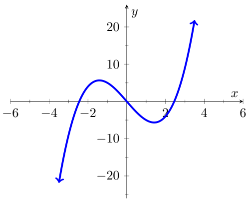
At the right or left of a graph you will see closed dots, open dots or arrows:

- A closed dot indicate that the graph does not extend beyond this point and the point itself **belongs** to the graph.
- An open dot indicate that the graph does not extend beyond this point and the point itself **does not belong** to the graph.
- An arrow indicate that the graph extend indefinitely in the direction which the arrows points.

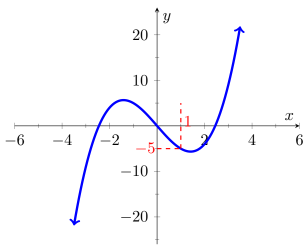


To evaluate a function at a given point, we must locate the point on the x-axis first. Then we draw a vertical line, to find the corresponding y-coordinate on the graph.

Example 1.4. Given the graph, find $f(1)$.



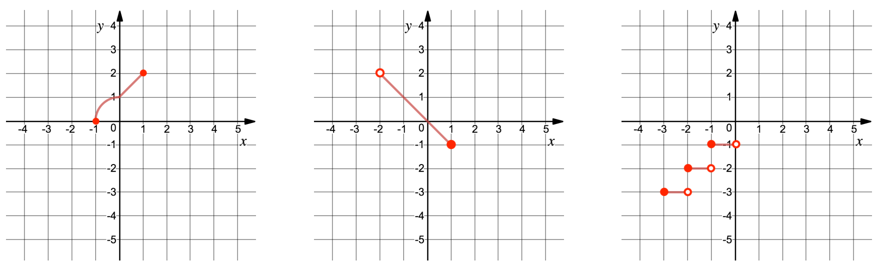
To find $f(1)$ locate $x = 1$ on the x-axis. Then we draw a vertical line to hit the graph to find the corresponding y-coordinate. We see that y-coordinate is -5. Therefore, $f(1) = -5$.



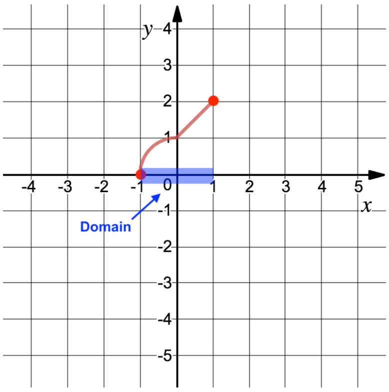
1.4 Identifying domain and range from a graph

To find the domain from a graph, compress (projecting) the graph onto the x-axis and see which values will be covered. Similarly, to find the range using the graph of a function, compress (project) the graph onto the y-axis. We can use set-builder notation or interval notation to express the domain or range of a function.

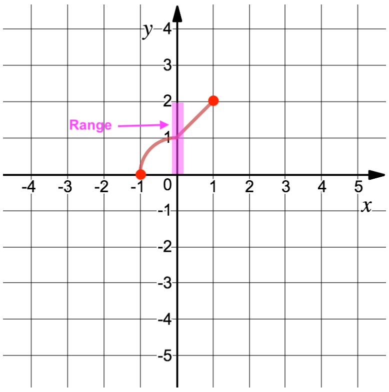
Example 1.5. Use the graph of each function to identify its domain and its range.



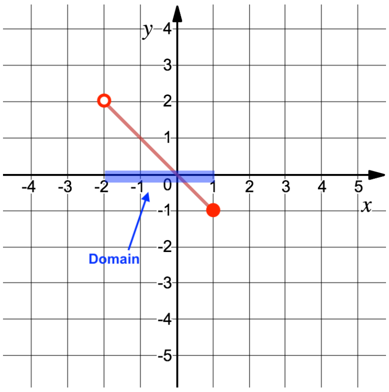
(a) **(b)** **(c)**



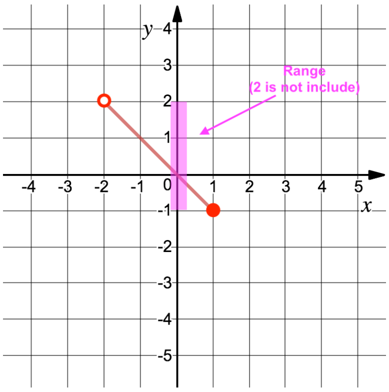
(a) Domain :
$[-1, 1]$ or $\{x \mid -1 \leq x \leq 1\}$



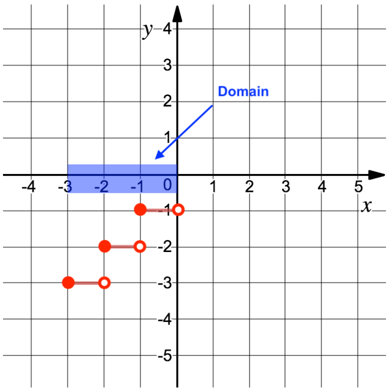
Range:
$[0, 2]$ or $\{y \mid 0 \leq y \leq 2\}$



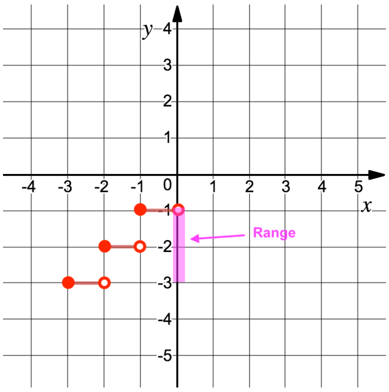
(b) Domain :
$(-2, 1]$ or $\{x \mid -2 < x \leq 1\}$



Range:
$[-1, 2)$ or $\{y \mid -1 \leq y < 2\}$



(c) Domain :
$[-3, 0)$ or $\{x \mid -3 \leq x < 0\}$

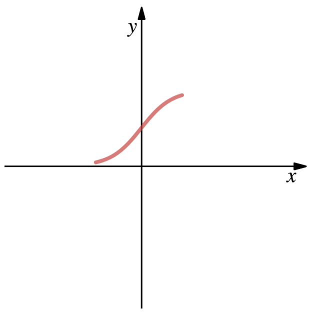


Range:
$[-3, -1]$ or $\{y \mid -3 \leq y \leq -1\}$

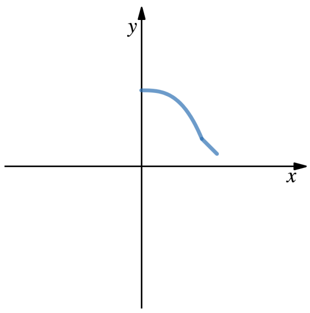
More on Functions

2.1 Increasing and decreasing functions

If the function *rises from left to right*, the function is called **increasing**. On the other hand, if the function *falls from left to right*, the function is called **decreasing**.



(a) Increasing function



(b) Decreasing function

Figure 2.1: Increasing and decreasing function

If the function *neither rise nor fall* and remains constant, it is called **constant**. Constant function is a horizontal line.

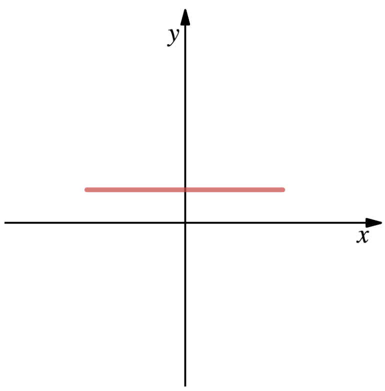
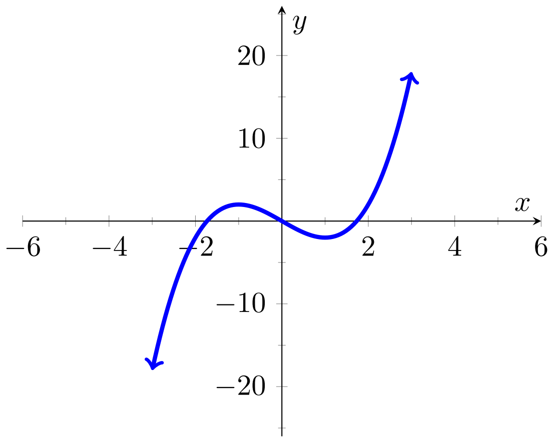


Figure 2.2: Constant function is a horizontal line.

Example 2.1. State the intervals on which the given function is increasing, decreasing, or constant.



As you can see, the function rises from $-\infty$ to -1. Then start falling from -1 to 1. Finally, it rises again from 1 to $+\infty$. So

$(-\infty, -1)$	Increasing
$(-1, 1)$	Decreasing
$(1, +\infty)$	Increasing

2.2 Difference Quotient

The difference quotient gives us the slope of line between the two points of x and $x+h$:

$$\text{Difference quotient} = \frac{f(x+h)-f(x)}{h} \tag{2.1}$$

This value is actually the average rate of change of a function. Later in calculus, once you learn limit, you will find out that this expression becomes the slope of a tangent line at any point of the graph when h approaches to 0. In this case, this expression is called derivative of a function.

Example 2.2. If $f(x) = -2x^2 + 4x - 1$, find and simplify each expression.

(a) $f(x+h)$ (b) $\dfrac{f(x+h)-f(x)}{h}$

(a) To find $f(x+h)$ replace x with $x+h$:

$$\begin{aligned} f(x+h) &= -2(x+h)^2 + 4(x+h) - 1 \\ &= -2(x^2 + h^2 + 2xh) + 4(x+h) - 1 \\ &= -2x^2 - 2h^2 - 4xh + 4x + 4h - 1 \quad \checkmark \end{aligned}$$

(b) We already found $f(x+h)$ from previous part. Substitute both $f(x+h)$ and $f(x)$. Then simplify

$$\frac{f(x+h)-f(x)}{h}$$

$$\frac{(-2x^2-2h^2-4xh+4x+4h-1)-(-2x^2+4x-1)}{h}$$

$$\frac{\cancel{-2x^2}-2h^2-4xh+\cancel{4x}+4h-\cancel{1}+\cancel{2x^2}-\cancel{4x}+\cancel{1})}{h}$$

$$\frac{-2h^2-4xh+4h}{h}$$

$$\frac{\cancel{h}(-2h-4x+4)}{\cancel{h}}$$

$$-2h-4x+4 \quad \checkmark$$

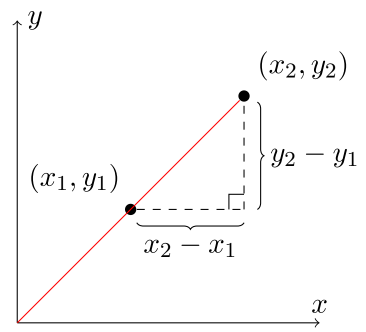
Linear functions and slope

3.1 Slope of a line

The steepness or slope of a line in the xy-coordinate system is the ratio of the rise (the change in y-coordinates) to the run (the change in x-coordinates) between two points on the line. The slope is denoted as m.

$$m = \frac{\textit{Change in } y}{\textit{Change in } x} = \frac{\textit{Rise}}{\textit{Run}} = \frac{y_2 - y_1}{x_2 - x_1} \tag{3.1}$$

Notice the subscript 1 and 2 indicate first and second point, respectively.



In order to avoid any confusions, always label your points before using the (3.1) formula. That way you wouldn't mix up their x- or y-coordinates.

Example 3.1. Find the slope of the line passing through $(-3, 3)$ and $(-5, 7)$.

First, Label our points.

$(-3, 3)$	Our first point, so $x_1 = -3$ and $y_1 = 3$
$(5, 7)$	Our second point, so $x_2 = -5$ and $y_2 = 7$

Plug them into slope formula (3.1) to find slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{7-3}{-5-(-3)}$$
$$m = \frac{4}{-2}$$
$$m = -2 \quad \text{Our solution}$$

3.1.1 Slope of vertical and horizontal lines

Imagine you are outside the classroom and walking on the ground. Since the ground doesn't have any steepness, you'd spend a minimum energy to walk; In fact, you prefer walking on the ground forever. On the other hand, you know as a mathematician, how hard would it to climb a precipitous cliff-almost impossible!
The ground is a horizontal line and since there is no steepness, the slope of horizontal line is 0. Likewise, the precipitous cliff is like a vertical line. Because it is impossible to climb the cliff-at least for mathematicians-the slope of a vertical line is undefined. We can also prove this mathematically. Considering the vertical line, we know it passes through all points with the same x coordinate. Using slope formula (3.1), we will get

$$m = \frac{y_2 - y_1}{0}$$

Since the denominator become zero and the slope is undefined. Let's consider horizontal line. The y values of all points on a horizontal line are equal. Thus, using slope formula (3.1), we will get

$$m = \frac{0}{x_2 - x_1} = 0$$

zero divided by anything yields to zero. So, m is zero.
We can also use "*rise over run*" formula. In a vertical line, run is zero, so

$$m = \frac{Rise}{0}$$

which is undefined. However, in a horizontal line, there is no rise, so

$$m = \frac{0}{Run}$$

which gives us a zero slope.

Following graph summarize what we learned about the slope of the horizontal and vertical lines.

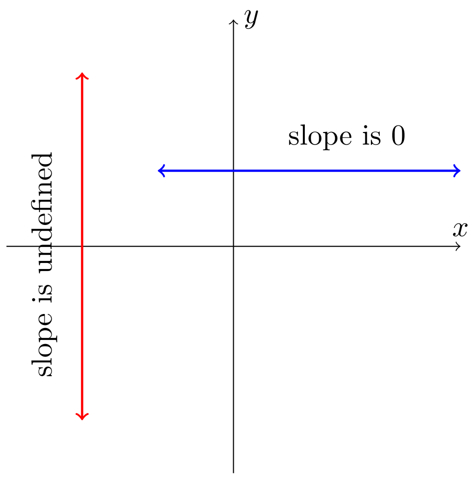


Figure 3.1: Slope of a horizontal and vertical lines

3.1.2 Positive and negative slopes

Lines sloping upward to the right have *positive* slopes; In fact, if one end of line passes through quadrant I and the other end passes through quadrant III then slope is always positive.
On the other hand, lines sloping downward to the left have *negative* slopes. In other words, a line with negative slope has one end at quadrant II and the other end at quadrant IV.

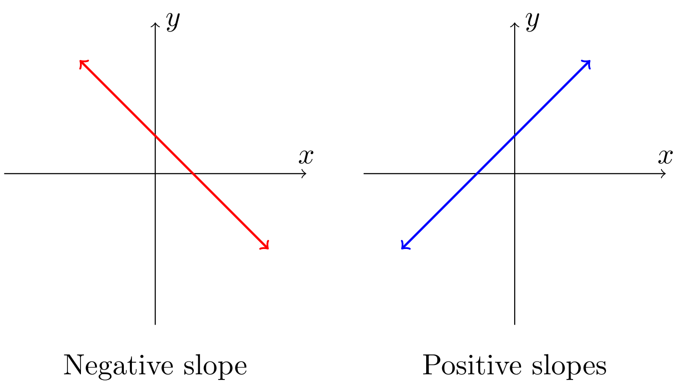


Figure 3.2: Negative and positive slopes

3.2 Point-slope formula

To find the equation of a line, we need to find a relationship between any x and y values on the line. Let's assume, we know the slope of a line, m, and we have one point on the line, (x_1, y_1). Slope between any points on the line such as (x, y) and our point (x_1, y_1) should be constant and equal to m. So using definition of slope, (3.1).

$$\begin{aligned} \frac{y-y_1}{x-x_1} &= m && \text{Multiply both sides by } (x-x_1) \\ \cancel{(x-x_1)}\frac{y-y_1}{\cancel{x-x_1}} &= m(x-x_1) && \text{Simplify} \\ y-y_1 &= m(x-x_1) && \text{Equation of the line} \end{aligned}$$

This formula is called point-slope formula which is a general formula. We will use this formula all the time to find the equation of any lines.

Point-Slope formula

If a line has a slope of m and is passing through (x_1, y_1) then the equation of the line is

$$y - y_1 = m(x - x_1) \tag{3.2}$$

Note 3.1. Since we need to find the linear function, we often need to solve for y and then replace it with $f(x)$. This form is also called, **slop-intercept form** which we will discuss about it later in this section.

Example 3.2. Write the equation of line through the point $(3, -4)$ with a slope of 6. Then solve for the equations for y.

We are given $m = 6$ and $x_1 = 3, y_1 = -4$. so

$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y+4 &= 6(x-3) && \text{Our solution} \end{aligned}$$

Then we solve it for y

$$\begin{aligned} y+4 &= 6(x-3) && \text{Distribute} \\ y+4 &= 6x-18 && \text{Subtract 4 from both sides} \\ y &= 6x-18-4 && \text{Simplify} \\ y &= 6x-22 && \text{Our linear function} \end{aligned}$$

Note 3.2. Sometimes we want to find the equation of a line passing through **two points**. In this case, we first need to find the slope of the line. Then we use one of the point, and plug it into the point-slope formula to find the equation of our line.

Example 3.3. Find the equation of the line through the points $(2, 5)$ and $(6, -3)$. Then solve the equation for y.

Label points and find the slope.

$$\begin{aligned} (-2, 1) & \qquad \text{First point, } x_1 = 2 \text{ and } y_1 = 5 \\ (4, 5) & \qquad \text{Second point, } x_2 = 6 \text{ and } y_2 = -3 \\ m = \frac{y_2 - y_1}{x_2 - x_1} & \qquad \text{Slope formula} \\ m = \frac{(-3) - (5)}{(6) - (2)} & \qquad \text{Simplify} \\ m = \frac{-8}{4} & \qquad \text{Reduce} \\ m = -2 & \qquad \text{Our slope} \end{aligned}$$

Then I choose the first point, and plug it into the point-slope formula

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= -2(x - 2) \qquad \text{Our solution} \end{aligned}$$

Now, solve the equation for y

$$\begin{aligned} y - 5 &= -2(x - 2) & \text{Distribute} \\ y - 5 &= -2x + 4 & \text{Add 5 to both sides} \\ y &= -2x + 4 + 5 & \text{Simplify} \\ y &= -2x + 9 & \text{Our linear function} \end{aligned}$$

3.3 Slope-intercept form

Sometimes we are given the slope m but our given point is not any ordinary point; it is y-intercept. Let's say the y-intercept is $(0, b)$. By plugging them into point-slope formula (3.2), we'll get $y - b = m(x - 0)$. Adding b to both sides yields

$$y = mx + b \qquad \textit{or} \qquad f(x) = mx + b \tag{3.3}$$

This form is called *the slope-intercept form.* There is only y variable on the left-hand (with coefficient 1). On the other side, we have x and a constant.

Example 3.4. Write an equation of the line with slope $m = \dfrac{5}{6}$ and y-intercept $(0, -3)$.

Substitute into the slope-intercept form (3.3)

$$
\begin{aligned}
y &= mx + b && \text{The slope-intercept form} \\
y &= \frac{5}{6}x - 3 && \text{Our solution}
\end{aligned}
$$

Note 3.3. Slope-intercept form is the simplest and most important form. If the equation of a line is written in this form, then

- the coefficient of x is our slope;
- the constant is the y-intercept.

For example, if we have $y = -\frac{4}{5}x - 5$ we realize quickly that the slope of this line is $-\frac{4}{5}$ and its y-intercept is $(0, -5)$—without using any other formulas or graphs.

3.4 General form

Every line has a equation that can be expressed in the form

$$
Ax + By + C = 0 \tag{3.4}
$$

Where $A, B, C \in \mathbb{R}$ and $A, B \neq 0$. This form is called the **general form**. To find the slope and y-intercept of an equation of a line, we convert the general form to the slope-intercept form; This can be achieved by solving the general form for y. Once we found the slope-intercept form, the coefficient of x is our slope and the constant is the y-intercept.

Example 3.5. Find the slope and y-intercept of the equation:

$$
-3x - 4y + 8 = 0
$$

We begin by solving for y, so

$$
\begin{aligned}
-3x - 4y + 8 &= 0 && \text{Add } 3x \text{ to both sides} \\
-4y + 8 &= -3x && \text{Subtract 8 from both sides} \\
-4y &= 3x - 8 && \text{Divide both sides by } -4 \\
y &= \frac{3}{4}x - \frac{8}{4} && \text{Simplify} \\
y &= \frac{3}{4}x - 2 && \text{Slope-intercept form}
\end{aligned}
$$

The slope is the coefficient of x, so slope is $\frac{3}{4}$ and the constant is our y-intercept which is $(0, -2)$.

3.5 Graphing a linear function

To graph a linear function, we at least need two points on the line. The simplest points, yet useful, are x- and y-intercept. To find the x-intercept, we set $y = 0$ and solve for x. Similarly, to get the y-intercept, we set $x = 0$ and find the y-value.
Once we found our intercepts, we plot them on xy-plane, and then connect them to get our line.
In short, draw a T-table like the following table and find x- and y-intercepts.

x	y	
0	?	$\longrightarrow$ *y-intercept*
?	0	$\longrightarrow$ *x-intercept*

Example 3.6. Graph the equation $y = -4x + 2$.

To find the y-intercept, we set $x = 0$ thus

$$\begin{aligned} y &= -4(0) + 2 && \text{Plug } x = 0 \\ y &= 2 && y\text{-intercept} \end{aligned}$$

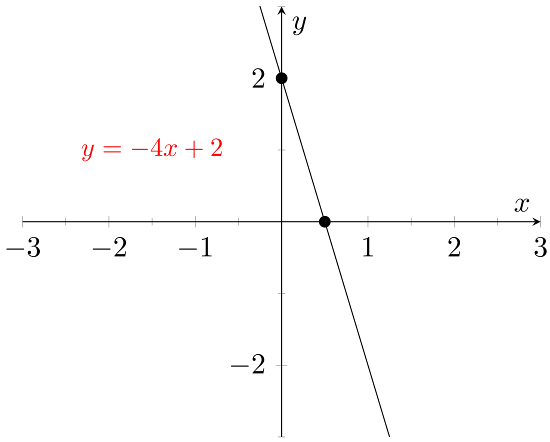
For x-intercept, set $y = 0$ and solve for x

$$\begin{aligned} 0 &= -4x + 2 && \text{Plug } y = 0 \\ -2 &= -4x && \\ \frac{1}{2} &= x && x\text{-intercept} \end{aligned}$$

Our table of values becomes like this

x	y
0	2
$\frac{1}{2}$	0

Therefore our first point on the line is $(0, 2)$ and the second one is $\left(\frac{1}{2}, 0\right)$.
Plot these points and connect them to get the line.



3.5.1 Vertical and horizontal lines

The graph of the equation $x = h$, where h is any real number, is a vertical line through the point $(h, 0)$.
To graph a vertical line, locate $x = h$ on x-axis and then draw a vertical line passing through it. The following graph, for example, shows the line $x = -2$.

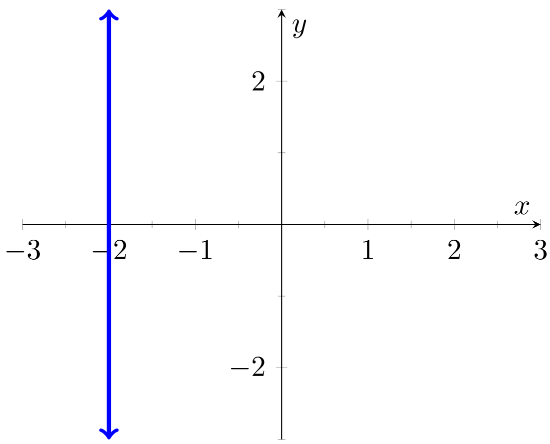


Figure 3.3: Graph of $x = -2$.

On contrary, the graph of the equation $y = k$, where k is any real number, represents a horizontal line through the point $(0, k)$.
To graph a horizontal line, locate $y = k$ on the y-axis and then draw a horizontal line passing though it. For instance, the following graph shows the horizontal line $y = 1$.

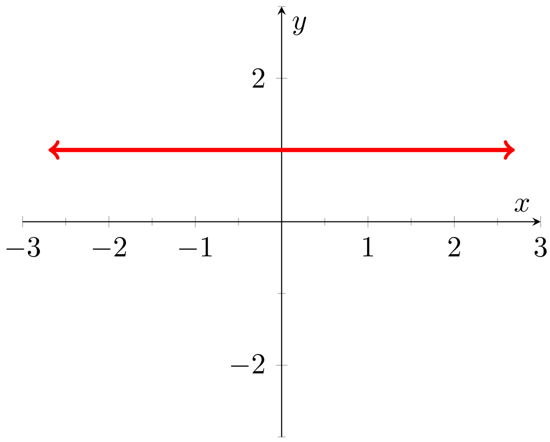


Figure 3.4: Graph of $y = 1$

As you might remember, the slope of a vertical line is not defined and the slope of a horizontal line is always 0.

More on slope

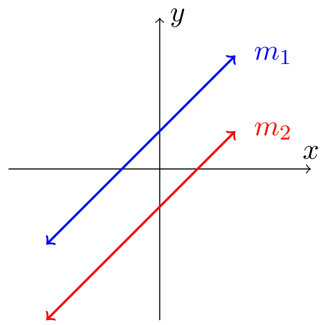
4.1 Parallel lines

When you graph two or more linear equations in a coordinate plane, they generally cross at a point. However, when two lines in a coordinate plane never cross, they are called parallel lines. **When two lines are parallel, their slopes are equal to each other.**

colframeParallel lines

Let's consider two parallel lines. If the slope of first line is m_1 and the slope of other line is m_2, then

$$m_1 = m_2 \tag{4.1}$$



Parallel lines

Example 4.1. Write an equation of line passing through $(-2, 5)$ and parallel to the line whose equation is $y = 3x - 1$. Express the equation in slope-intercept form.

To find the equation of a line, we need its slope and one point on the line. The point on the line is given $(-2, 5)$.
Since the line is parallel to $y = 3x - 1$, then their slopes are equal to each other. The slope of $y = 3x - 1$ is 3, thus the slope of our line is also $m = 3$.

Using the point-slope formula:

$$
\begin{aligned}
y - y_1 &= m(x - x_1) && \text{Substitute slope and given point}\\
y - (5) &= 2(x - (-2)) && \text{Simplify}\\
y - 5 &= 2(x + 2) && \text{Distribute 2 on LHS}\\
y - 5 &= 2x + 4 && \text{Add 5 to both sides}\\
y &= 2x + 9 && \text{Our answer in slope-intercept form}
\end{aligned}
$$

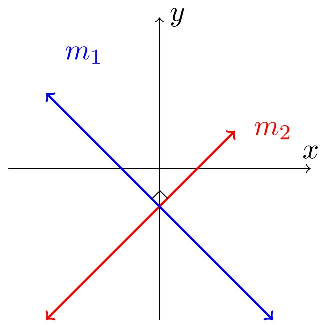
4.2 Perpendicular lines

We also might have a case where two lines in the coordinate plane cross at a right angle. These are called perpendicular lines. **In this case, the slope of one line is negative reciprocal of slope of the other line.**

Perpendicular lines

Consider two perpendicular lines with slope of m_1 and m_2. The relationship between their slope is

$$m_1 m_2 = -1 \tag{4.2}$$



Perpendicular lines

Example 4.2.

a. Find the slope of any line that is perpendicular to the line whose equation is $x + 3y - 12 = 0$.

b. Write the equation of line passing through $(-2, -6)$ and perpendicular to the line whose equation is $x + 3y - 12 = 0$. Express the equation in general form.

a. Let' call the given line $x + y - 12 = 0$, l_1 and the line we are looking for l_2. We begin by converting l_1 into slope-intercept form

$$\begin{aligned} l_1 : x + 3y - 12 &= 0 && \text{Isolate } y\text{, subtract } x \\ 3y - 12 &= -x && \text{Add 12} \\ 2y &= -x + 12 && \text{Divide by 2} \\ y &= -\frac{1}{2}x + 6 && \text{Slope-intercept form} \end{aligned}$$

The slope of l_1 is $-1/2$. Since l_2 is perpendicular to l_1, the slope of l_2 will be

$$\begin{aligned} m_2 &= -\frac{1}{m_1} && \text{Negative reciprocal} \\ m_2 &= -\frac{1}{(-1/2)} && \text{Simplify} \\ m_2 &= 2 && \text{Slope of } l_2 \end{aligned}$$

b. From previous part, we know the slope of l_2 is 2. We also know that our line, l_2, is passing through $(-2, -6)$. Using the point-slope formula, we'll get

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Substitute } m_2 = 2 \text{ and given point} \\ y - (-6) &= 2(x - (-2)) && \text{Simplify} \\ y + 6 &= 2(x + 2) && \text{Distribute LHS} \\ y + 6 &= 2x + 4 && \text{Subtract 6} \\ y &= 2x - 2 && \text{Our solution in the slope-intercept form} \end{aligned}$$

To express it in general form, move all terms to one sides, so

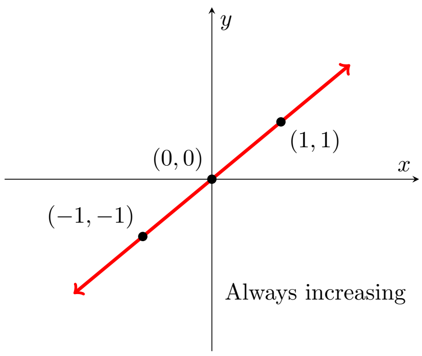
$$\begin{aligned} y &= 2x - 2 && \text{Subtract } 2x \\ y - 2x &= -2 && \text{Add 2} \\ y - 2x + 2 &= 0 && \text{Our solution in the general form} \end{aligned}$$

Transformations of functions

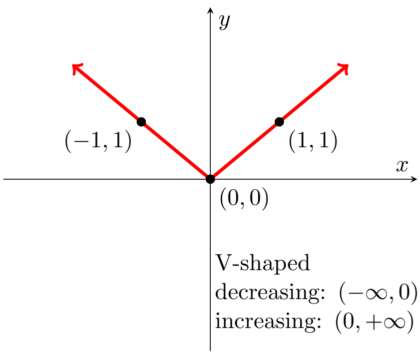
5.1 Graph of common functions

We begin this section by introducing 6 common graphs in algebra. Knowing these graphs is essential for analyzing their transformations into more complicated graphs.

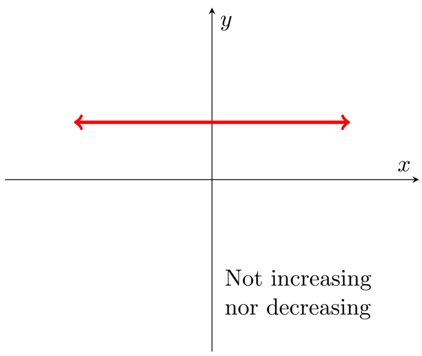
Identity function: $f(x) = x$



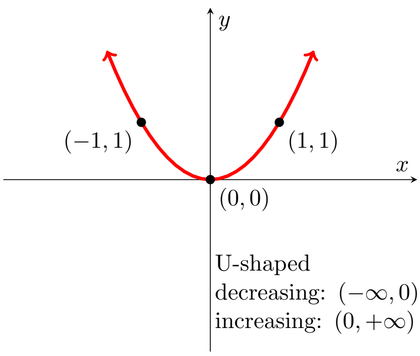
Absolute value function: $f(x) = |x|$

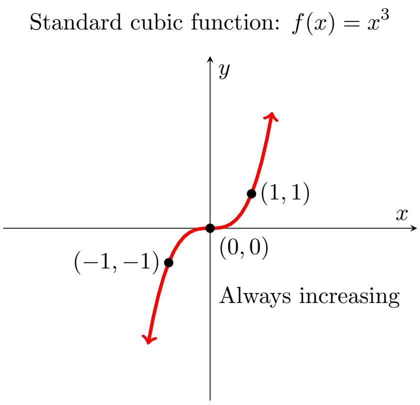


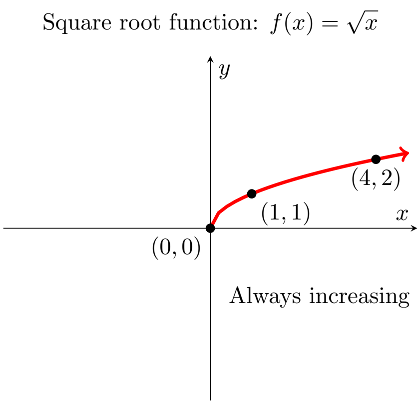
Constant function: $f(x) = k$



Standard quadratic function: $f(x) = x^2$







5.2 Vertical shifts

Consider $f(x) = |x|$ and let's say we want to graph $g(x) = |x| + 2$. Here what we can say about $g(x)$:

- ◇ Number 2 is added to $|x|$; In other words, it is added to f. Thus, this added number is directly affecting the y-coordinates of f.
- ◇ 2 units will be added to all y-coordinates of f.
- ◇ So, the graph of $g(x)$ is actually the graph of $f(x)$ shifted 2 units vertically upward.

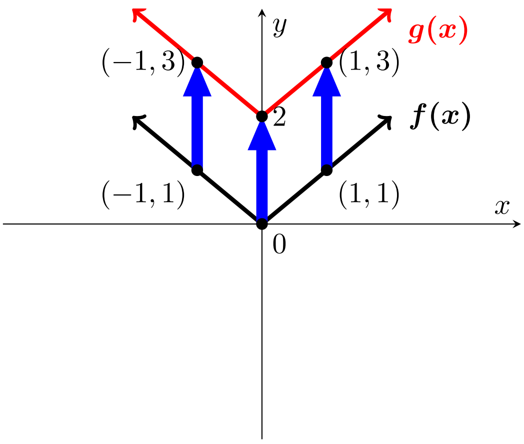


Figure 5.1: The graph of $f(x) = |x|$ and $g(x) = |x| + 2$

Similarly, if we instead of $g(x) = |x| + 2$, we had $h(x) = |x| - 2$ then the graph of $h(x)$ is the graph of $f(x)$ shifted 2 units vertically downward.

Vertical shifts

Let f be a function and c a positive real number:

- $y = f(x) + c$ is the graph of $y = f(x)$ shifted c units vertically upward by adding c to the y-coordinates of the points on the graph of f.
- $y = f(x) - c$ is the graph of $y = f(x)$ shifted c units vertically downward by subtracting k from the y-coordinates of the points on the graph of f.

5.3 Horizontal shifts

Consider $f(x) = \sqrt{x}$ and we want to graph $g(x) = \sqrt{x-1}$:

- ⋄ Number -1 is grouped with x. Thus, this number is affecting the x-values.
- ⋄ We expect the graph shift 1 unit to the left.
- ⋄ However, 1 unit is adding to x-coordinates of f and shifting the graph to the right **(an opposite effect)**.

Notice, when a number is grouped with x, it has a **opposite effect** from what initially looks like. We anticipate a horizontal shift 1 unit to the left. Instead, the graph moves to the right.

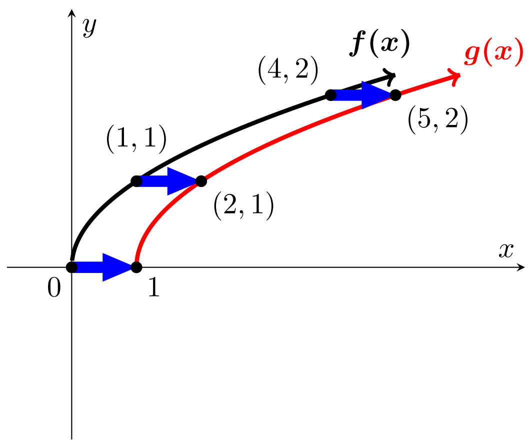


Figure 5.2: The graph of $f(x) = \sqrt{x}$ and $g(x) = \sqrt{x-1}$

Similarly, if we had $h(x) = \sqrt{x+2}$ then the graph of $h(x)$ is the graph of f shifted 2 units to the left.

Horizontal shifts

Let f be a function and c a positive real number:

- $y = f(x + c)$ is the graph of $y = f(x)$ shifted c units to the left by subtracting c from the x-coordinates of the points on the graph of f.
- $y = f(x - c)$ is the graph of $y = f(x)$ shifted c units to the right by adding c to the x-coordinates of the points on the graph of f.

5.4 Vertical stretching or shrinking

Consider the graph of function $f(x) = |x|$. We are interest in graphing $g(x) = 2|x|$:

◇ Number 2 is multiplied by $|x|$ which is our f. Thus, this number is affecting y-coordinates of f.

◇ All y-coordinates of f are multiplied by number 2.

◇ So, the graph of g is actually the graph of f stretched along y-axis by factor of 2.

Likewise, When the graph is multiplied by a number between 0 and 1, like $h(x) = \frac{1}{3}|x|$, then the graph will shrink along y-axis by factor of 3.

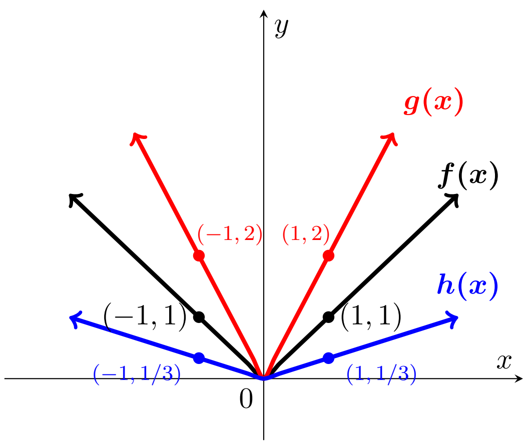


Figure 5.3: The graph of $f(x) = |x|$, $g(x) = 2|x|$ and $h(x) = \frac{1}{3}|x|$

Vertical stretching or shrinking

Let f be a function and c a positive real number:

- If $c > 1$, then $y = cf(x)$ is the graph of $y = f(x)$ stretched along y-axis by multiplying c by its y-coordinate.
- If $0 < c < 1$, then $y = cf(x)$ is the graph of $y = f(x)$ shrunk along y-axis by multiplying c by its y-coordinate.

5.5 Horizontal stretching or shrinking

Consider the graph of function $f(x) = |x|$. We are looking for the graph of $g(x) = |4x|$:

◇ Number 4 is grouped with x; So, this number will change the x-coordinates of f.

◇ Since it is grouped with x, it has a **opposite effect** and all x-coordinates of points on f will be divide by 4.

◇ Thus, the graph of $g(x)$ is actually the graph of f shrunk along x-axis by factor of 4.

Likewise, if the multiplier is between 0 and 1, such as $h(x) = |\frac{1}{2}x|$, then the graph will stretch along x-axis by factor of 2.

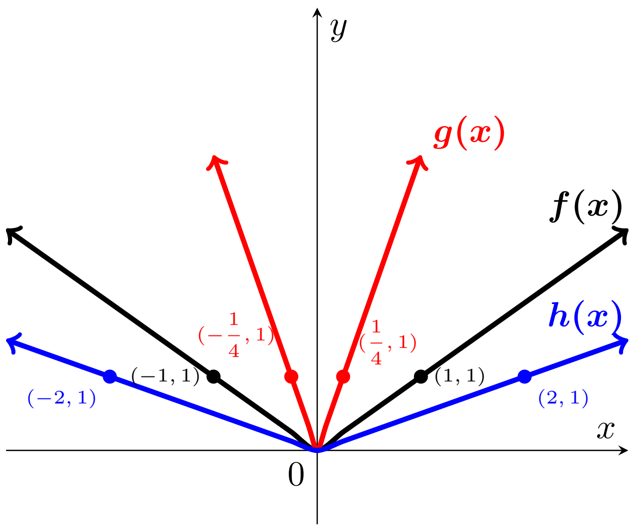


Figure 5.4: The graph of $f(x) = |x|$, $g(x) = |4x|$ and $h(x) = |\frac{1}{2}x|$

As you can see in Figure 5.4, the point $(1, 1)$ on $f(x)$ will become $(1/3, 1)$ on $g(x)$ and will change to $(2, 1)$ on $h(x)$.

Horizontal stretching or shrinking

Let f be a function and c a positive real number:

- If $c > 1$, then $y = f(cx)$ is the graph of $y = f(x)$ shrunk along x-axis by dividing c by its x-coordinate.
- If $0 < c < 1$, then $y = f(cx)$ is the graph of $y = f(x)$ stretched along x-axis by dividing c by its x-coordinate.

5.6 Reflection about x-axis

Let's consider $f(x) = \sqrt{x}$. The graph of $g(x) = -\sqrt{x}$ is the graph of $f(x)$ in which its y-coordinate is multiplied by -1. Notice the graph of $g(x)$, Figure 5.5 , is actually $y = f(x)$ reflected about x-axis.

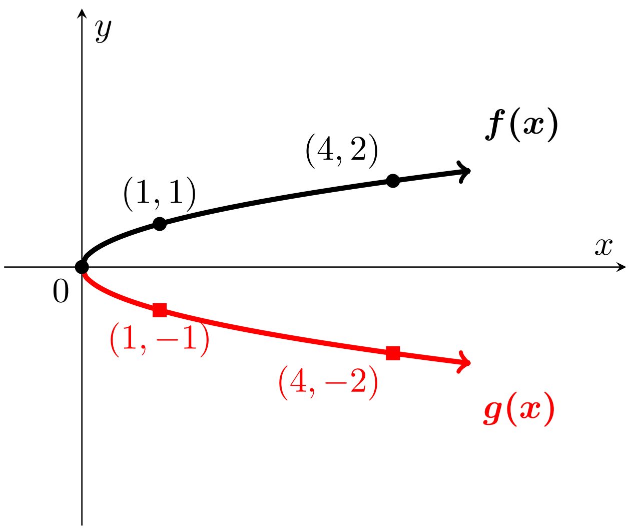


Figure 5.5: The graph of $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$.

Reflection about x-axis

Let f be a function:
The graph of $y = -f(x)$ is the graph of $y = f(x)$ reflected about x-axis by multiplying the y-coordinates of the points on the graph of f by -1.

5.7 Reflection about y-axis

Here I will also consider $f(x) = \sqrt{x}$. To graph $h(x) = \sqrt{-x}$, since the -1 is grouped with x, we need to divide all x-coordinate by -1. Notice the graph of $h(x)$, Figure 5.6, is actually $y = f(x)$ reflected about y-axis.

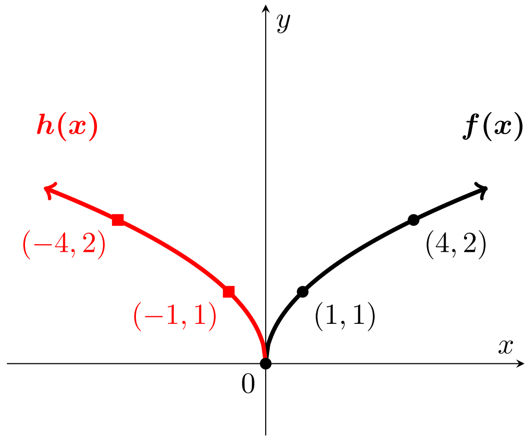


Figure 5.6: The graph of $f(x) = \sqrt{x}$ and $h(x) = \sqrt{-x}$

Reflection about y-axis

Let f be a function:
The graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected about y-axis by multiplying the x-coordinates of the points on the graph of f by -1.

5.8 Order of transformations

A function involving more than one transformation can be graphed by performing transformations in the following order:

1. Horizontal shifting
2. Stretching or shrinking
3. Reflecting
4. Vertical shifting

In other words, we start from inside of function, numbers that are affecting our x-coordinates and we gradually go outside, numbers that are affecting our y-coordinates.

Order of transformations

Let f be a function. If $A \neq 0$ and $B \neq 0$, then to graph

$$g(x) = Af(Bx + H) + K$$

1. If $H > 0$ then Subtract H from each of the x-coordinates of the points on the graph of f. However, if $H < 0$ then Add H to x-coordinates of f. *This results in a horizontal shift to the left for $H > 0$ or right for $H < 0$.*

2. Divide the x-coordinates of the points on the graph, obtained in Step 1, by B. *This results in a horizontal stretching or shrinking, but may also include a reflection about the y-axis if $B < 0$.*

3. Multiply the y-coordinates of the points on the graph, obtained in Step 2, by A. *This results in a vertical stretching or shrinking, but may also include a reflection about the x-axis if $A < 0$.*

4. Add K to each of the y-coordinates of the points on the graph obtained in Step 3 if $k > 0$. Otherwise, if $K < 0$, subtract it from each of the y-coordinates of f. *This results in a vertical shift up for $K > 0$ or down for $K < 0$.*

Example 5.1. Use the graph of $y = f(x)$ given in Figure 5.7 to graph $y = -2f(x-2) - 1$.

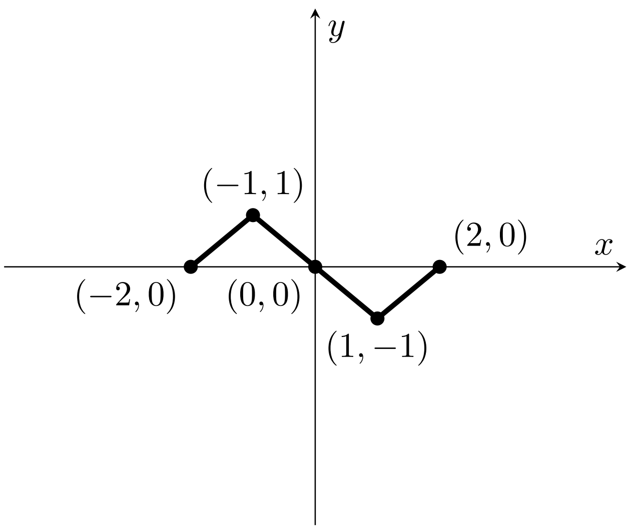
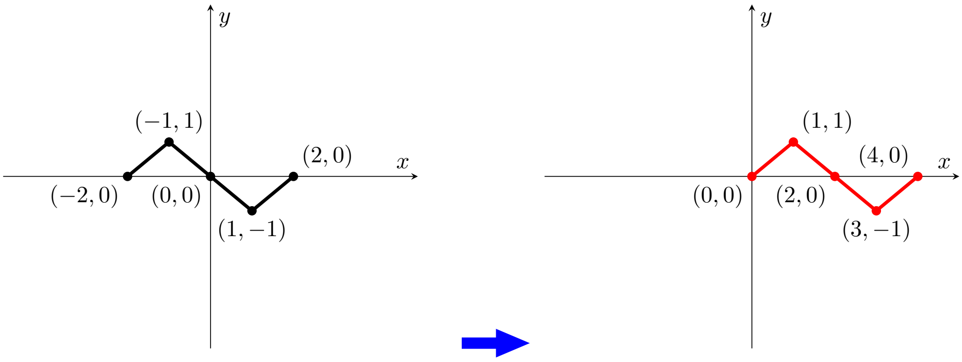
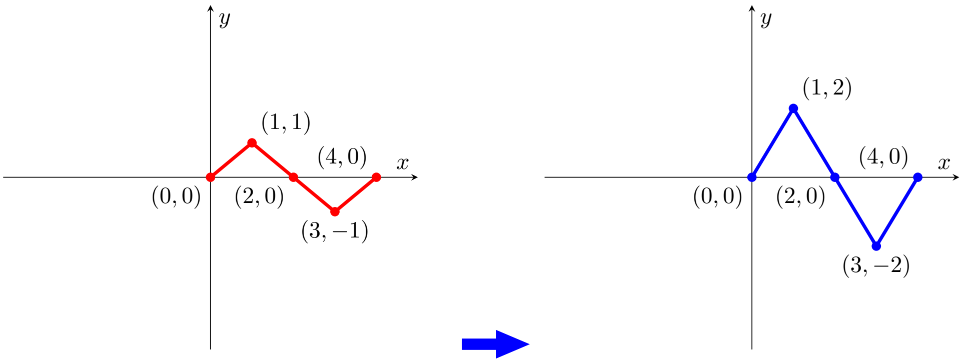


Figure 5.7: The graph of $y = f(x)$

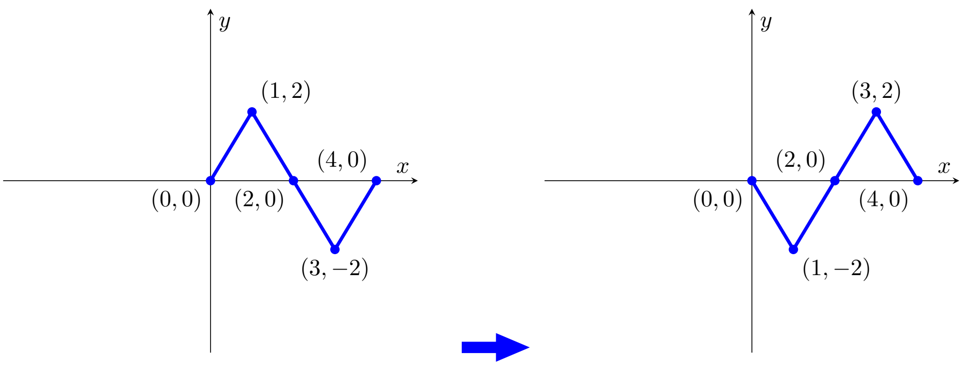
We first begin from inside. Number -2 grouped with x is telling us that we need to shift the graph of f 2 units to the right. This gives us



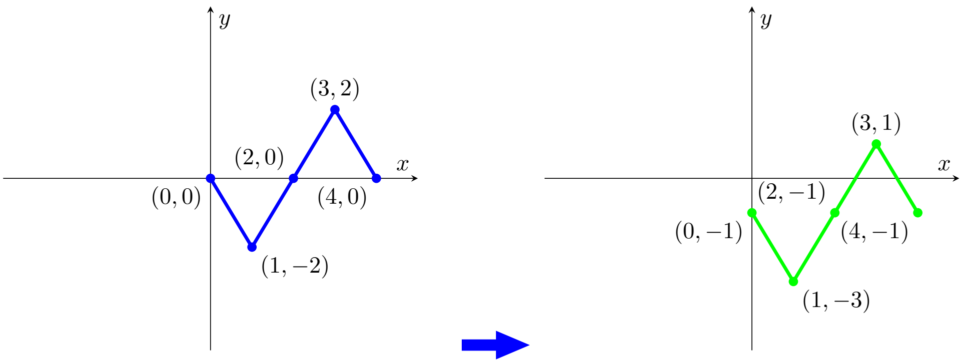
Then we need to multiply y-coordinate by 2 (horizontal stretching).



Next, we should reflect the graph about y-axis; meaning to multiply its y-coordinate by -1.



Finally, we shift the graph 1 unit vertically downward.



The green graph is our solution.

5.9 Summary of All Rules

Table 5.1 shows the summary of all transformations:

Table 5.1: Summary of Transformations

New function	Transformation of points	Visual effect
$f(x)+d$	$(a,b)\mapsto(a,b+d)$	shift up by d
$f(x)-d$	$(a,b)\mapsto(a,b-d)$	shift down by d
$f(x+h)$	$(a,b)\mapsto(a-h,b)$	shift left by h
$f(x-h)$	$(a,b)\mapsto(a+h,b)$	shift right by h
$cf(x)$	$(a,b)\mapsto(a,cb)$	stretch vertically by c
$\frac{1}{c}f(x)$	$(a,b)\mapsto(a,\frac{1}{c}b)$	shrink vertically by $\frac{1}{c}$
$f(kx)$	$(a,b)\mapsto(\frac{1}{k}a,b)$	shrink horizontally by $\frac{1}{k}$
$f(\frac{1}{k}x)$	$(a,b)\mapsto(ka,b)$	stretch horizontally by k
$-f(x)$	$(a,b)\mapsto(a,-b)$	reflect about x-axis
$f(-x)$	$(a,b)\mapsto(-a,b)$	reflect about y-axis

† d, h, c and k are all real positive numbers.
†† c and k are greater than 1.

Combination of Functions

6.1 Domains of a function

The domain of a function is the set of all real numbers you can substitute into the function; In other words, all x-values.
Consider the standard quadratic function, $f(x) = x^2$. As you can see, we can substitute any real number we want into this function; We can use $x = 3$ or -10 or 10000 or 3.14159 or whatever and we still get an answer. So because there is no restriction on x, the domain is all real numbers. We can express it both in set-builder and interval notation as follows:

$$\begin{aligned} \text{Set-builder: } & \text{Domain} : \{x \mid x \ \in \mathbb{R}\ \} \\ \text{Interval: } & \text{Domain} : (-\infty,\, +\infty) \end{aligned}$$

Now let's consider $f(x) = \dfrac{1}{x-6}$. It looks like any number will work. But wait! What happens if we use $x = 6$? The denominator becomes zero and we divide by zero, which is not allowed. Since any other number is fine, the domain of this function is all real number except 6. Therefore,

$$\begin{aligned} \text{Set-builder: } & \text{Domain} = \{x \mid x \neq 6\ \} \\ \text{Interval: } & \text{Domain} = (-\infty,\, 6) \cup (6,\, +\infty) \end{aligned}$$

As you can see, there are some situation that impose a restriction on our domain. When determining the domain of a function, we really only have to look out for four situations:

1. **Rational Expressions**: Division by zero is not allowed. So we must omit any values of x which make the denominator 0; In other words, the domain will be all real number except roots of denominator.

$$f(x) = \frac{u}{v} \quad \longrightarrow \quad \text{Domain} : \{x \mid v \neq 0\ \}$$

2. **Even roots**: For even roots such as square roots, the radicand can not be negative. Thus, to find the domain find all x that makes the radicand non-negative:

$$f(x) = \sqrt{u} \quad \longrightarrow \quad \text{Domain} : \{x \mid u \geq 0\ \}$$

③ **Even roots in the denominator**: In this case, since we have a even root, the radicand should be non-negative. However, it cannot be zero because it is located in denominator. Therefore, to find domain:

$$f(x) = \frac{u}{\sqrt{v}} \quad \longrightarrow \quad \text{Domain} : \{x \mid v > 0\ \}$$

④ **Logarithmic function**: In chapter 4, we'll see that logarithms can only be take of positive values.

Example 6.1. Find the domain of each functions.

a. $f(x) = \dfrac{5x}{x^2 - 49}$

b. $g(x) = x^2 - 4x + 2$

c. $h(x) = \sqrt{9x - 27}$

d. $k(x) = \dfrac{5x}{\sqrt{24 - 3x}}$

a. We have a rational expression, so we must exclude roots of denominators.

$$\begin{aligned} x^2 - 49 &= 0 && \text{Set the denominator to 0, add 49} \\ x^2 &= 49 && \text{Take square root, don't forget } \pm \\ x &= \pm\sqrt{49} && \text{49 is a perfect square} \\ x &= \pm 7 && \text{These roots are excluded} \end{aligned}$$

So the domain is all real numbers except -7 and 7 so domain is

$$(-\infty, -7) \cup (-7, 7) \cup (7, +\infty)$$

b. Since we don't have any type of restrictions, the domain is all real numbers or $(-\infty, +\infty)$.

c. The radicand of square roots should be non-negative, so

$$\begin{aligned} 9x - 27 &\geq 0 && \text{Add 27} \\ 9x &\geq 27 && \text{Divide both sides by 9} \\ x &\geq 3 && \text{Our domain} \end{aligned}$$

The domain is $x \geq 3$ or in interval notation $[3, +\infty)$.

d. The radicand should be non-negative but cannot be zero so

$$\begin{aligned} 24 - 3x &> 0 && \text{Subtract 24} \\ -3x &> -24 && \text{Divide both sides by } -3\text{, flip the sign!} \\ x &< 8 && \text{Our domain} \end{aligned}$$

Thus, domain is $x < 8$ or using interval notation $(-\infty, 8)$.

6.2 The algebra of functions

We can combine the functions using addition, subtraction, multiplication and division.

Algebra of functions

Let f and g be two functions:

1. **Sum:** $(f+g)(x) = f(x) + g(x)$
2. **Difference:** $(f-g)(x) = f(x) - g(x)$
3. **Product:** $(fg)(x) = f(x) \cdot g(x)$
4. **Quotient:** $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$

Example 6.2. Let $f(x) = x - 5$ and $g(x) = x^2 - 1$. Find each of the following functions:

a. $(f+g)(x)$

b. $(f-g)(x)$

c. $(fg)(x)$

d. $\left(\dfrac{f}{g}\right)(x)$

a. $(f+g)(x)$ is $f(x) + g(x)$ so,

$f(x) + g(x)$	Substitute
$(x-5) + (x^2-1)$	Combine like terms
$x + x^2 - 6$	Our solution

b. $(f+g)(x)$ is $f(x) - g(x)$ so,

$f(x) - g(x)$	Substitute
$(x-5) - (x^2-1)$	Distribute -1
$x - 5 - x^2 + 1$	Combine like terms
$x - x^2 - 4$	Our solution

c. $(fg)(x)$ is $f(x) \cdot g(x)$ so,

$f(x) \cdot g(x)$	Substitute
$(x-5) \cdot (x^2-1)$	FOIL method

$$x^3 - x - 5x^2 + 5 \qquad \text{Our solution}$$

d. $\left(\dfrac{f}{g}\right)(x)$ is $\dfrac{f(x)}{g(x)}$ so,

$$\frac{f(x)}{g(x)} \qquad \text{Substitute}$$

$$\frac{x-5}{x^2-1} \qquad \text{Our solution}$$

6.2.1 Domain of combining functions

Domain of each of these functions consists of all real numbers that are common to domains of f and g. If D_f represent domain of f and D_g represent domain of g, therefore the domain for each function is $D_f \cap D_g$. For the quotient case, we have to add the condition that denominator cannot be zero.

Domain of combining functions

Let f and g be two functions:

1. **Sum:** Domain: $D_f \cap D_g$
2. **Difference:** Domain: $D_f \cap D_g$
3. **Product:** Domain: $D_f \cap D_g$
4. **Quotient:** Domain: $D_f \cap D_g$ and $g(x) \neq 0$

Example 6.3. If $f(x) = 5x + 1$ and $g(x) = x^2 - 3x - 4$, determine the domain of each of the following functions:

a. $(f + g)(x)$

b. $(f - g)(x)$

c. $(fg)(x)$

d. $\left(\dfrac{f}{g}\right)(x)$

Since the equations for f and g do not involve division or contain even roots, the domain of both of them is the set of all real numbers. Therefore, the domain of $f + g$, $f - g$, and fg is the set of all real numbers or $(-\infty, +\infty)$. The function $\dfrac{f}{g}$ contains division. We must exclude the roots of denominator, $x^2 - 3x - 4$. Let's find them

$$x^2 - 3x - 4 = 0 \qquad \text{Factor out}$$

$$\begin{aligned} (x+1)(x-4) &= 0 && \text{Set each factor equal to 0} \\ x = -1 \quad x &= 4 && \text{roots of denominators} \end{aligned}$$

We must exclude -1 and 4 from the domain of $\frac{f}{g}$.

$$\text{Domain of } \frac{f}{g} = (\infty, -1) \cup (-1, 4) \cup (4, +\infty)$$

6.3 Composite functions

Another way to combine two functions is composition. A composition of functions is a function inside of a function. The notation used for composition of functions is:

$$(f \circ g)(x) = f(g(x)) \tag{6.1}$$

To find a composite function, we will take the inside function and substitute into the outside function.

Example 6.4. If $f(x) = x^2 - 3$ and $g(x) = x + 2$, find each composite function:

a. $(f \circ g)(x)$

b. $(g \circ f)(x)$

a. $(f \circ g)(x)$ means $f(g(x))$:

$$\begin{aligned} f(g(x)) & && \text{Substitute } g(x) \text{ into } f(x) \\ (g(x))^2 - 3 & && g(x) \text{ is } x+2 \\ (x+2)^2 - 3 & && \text{Simplify} \\ x^2 + 2^2 + 2(x)(2) - 3 & && \text{Combine like terms, 4 and } -3 \\ x^2 + 4x + 1 & && \text{Our solution} \end{aligned}$$

b. $(g \circ f)(x)$ means $g(f(x))$:

$$\begin{aligned} g(f(x)) & && \text{Substitute } f(x) \text{ into } g(x) \\ f(x) + 2 & && f(x) \text{ is } x^2 - 3 \\ x^2 - 3 + 2 & && \text{Combine like terms} \\ x^2 - 1 & && \text{Our solution} \end{aligned}$$

Note 6.1. It is important to note that very rarely is $(f \circ g)(x)$ the same as $(g \circ f)(x)$ as the previous example showed.

Note 6.2. To calculate a composition of function we will evaluate the inner function and substitute the answer into the outer function. This is shown in the following example.

Example 6.5. If $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$, evaluate each composite function:

a. $(f \circ g)(3)$

b. $(g \circ f)(-1)$

a. $(f \circ g)(3)$ is actually $f(g(3))$. Evaluate the inner function first, $g(3)$:

$$\begin{aligned} g(x) &= x - 5 && \text{This is } g(x) \\ g(3) &= 3 - 5 && \text{Subtract} \\ &= -2 && \text{This is } g(3) \end{aligned}$$

Now -2 goes into f,

$$\begin{aligned} f(-2) &= (-2)^2 - 2(-2) + 1 && \text{Simplify} \\ &= 4 + 4 + 1 && \text{Add} \\ &= 9 && \text{Our solution} \end{aligned}$$

b. $(g \circ f)(-1)$ is actually $g(f(-1))$. Evaluate the inner function first, $f(-1)$:

$$\begin{aligned} f(x) &= x^2 - 2x + 1 && \text{This is } f(x) \\ f(-1) &= (-1)^2 - 2(-1) + 1 && \text{Simplify and add} \\ &= 4 && \text{This is } f(-1) \end{aligned}$$

Now 4 goes into g,

$$\begin{aligned} g(4) &= 4 - 5 && \text{Simplify} \\ &= -1 && \text{Our solution} \end{aligned}$$

6.3.1 Domain of a composite function

Finding the domain of a composite function consists of two steps:

1. Find the domain of the "inside" (input) function. If there are any restrictions on the domain, keep them.

2. Construct the composite function. Find the domain of this new function. If there are restrictions on this domain, add them to the restrictions from Step 1.

Example 6.6. Given $f(x) = x^2 + 2$ and $g(x) = \sqrt{7 - x}$, find each of the following:

a. $(f \circ g)(x)$ **b.** Domain of $(f \circ g)(x)$

a. $(f \circ g)(x)$ is $f(g(x))$ so

$$\begin{aligned}
f(g(x)) &\quad g(x) \text{ goes into } f(x) \\
(g(x))^2 + 2 &\quad \text{Replace } g(x) \text{ with } \sqrt{7-x} \\
(\sqrt{7-x})^2 + 2 &\quad \text{Simplify} \\
7 - x + 2 &\quad \text{Combine like terms} \\
9 - x &\quad \text{Our solution}
\end{aligned}$$

b. Domain of $(f \circ g)(x)$:

1. We must find domain of inside, $g(x)$. Since $g(x)$ contains even roots, therefore

$$\begin{aligned}
7 - x \geq 0 &\quad \text{Solve for } x \\
x \leq 7 &\quad \text{Keep this!}
\end{aligned}$$

2. In next step, we must look at the composite function itself. Since $(f \circ g)(x) = 9 - x$, This function puts no additional restrictions on the domain. So the composite domain is $x \leq 7$.

Example 6.7. Given $f(x) = \dfrac{3x}{x-1}$ and $g(x) = \dfrac{2}{x}$, find each of the following:

a. $(f \circ g)(x)$ **b.** Domain of $(f \circ g)(x)$

a. $(f \circ g)(x)$ is $f(g(x))$ so

$$\begin{aligned}
f(g(x)) &\quad g(x) \text{ goes into } f(x) \\
\frac{3\,g(x)}{g(x) - 1} &\quad \text{Replace } g(x) \text{ with } \frac{2}{x} \\
\frac{3(\frac{2}{x})}{(\frac{2}{x}) - 1} &\quad \text{Simplify by multiply by } \frac{x}{x} \\
\frac{(\frac{6}{x})}{(\frac{2}{x}) - 1} \cdot \frac{\boldsymbol{x}}{\boldsymbol{x}} &\quad \text{Multiply} \\
\frac{6}{2 - x} &\quad \text{Our solution}
\end{aligned}$$

b. To find domain of $(f \circ g)(x)$, follow these steps:

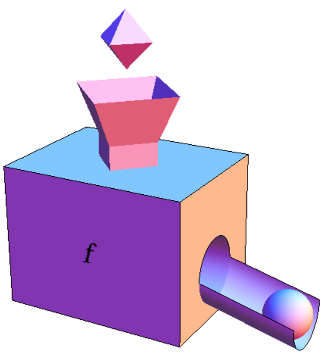
1. We must find domain of inside, $g(x)$. Since $g(x)$ contains division, therefore $x \neq 0$. Keep this!
2. We must find the domain of $(f \circ g)(x) = \dfrac{6}{2-x}$. Since this function also contains division, $2 - x \neq 0$, gives us $x \neq 2$.

Combine this domain with the domain from Step 1: the composite domain is $\{x \mid x \neq 0 \text{ and } x \neq 2\}$ or in interval notation $(-\infty, 0) \cup (0, 2) \cup (2, +\infty)$.

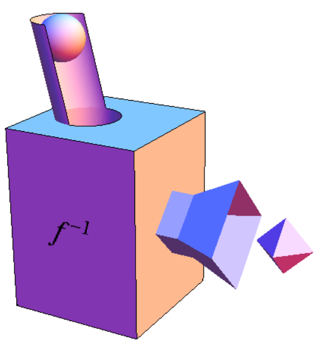
Inverse Functions

7.1 Inverse functions

When a value goes into a function it is called the input. The result that we get when we evaluate the function is called the output. Figure 7.1a shows the function f, where an octahedron, representing input, enters through the function machine's input funnel, and the function machine spits out a particular sphere, representing output.
When working with functions sometimes we will know the output and be interested in what input gave us the output. To model this behavior, we use an inverse function. As the name suggests an inverse function undoes whatever the function did. If a function is named $f(x)$, the inverse function will be named $f^{-1}(x)$ (read "f inverse of x"). Figure 7.1b illustrates the inverse of f. The inverse works backward meaning if the input is a sphere, it will return an octahedron–undoing f.



(a) Function $f(x)$



(b) Inverse function $f^{-1}(x)$

Figure 7.1: Function, $f(x)$, and its inverse, $f^{-1}(x)$, as machines.

Note 7.1. $f^{-1}(x) \neq \dfrac{1}{f(x)}$. It is very important not to confuse function notation with negative exponents. But only a symbol to let us know that this function is the inverse of f.

Domain and range of inverse function

- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.

In order to test if two functions, $f(x)$ and $g(x)$ are inverses we will calculate the composition of the two functions at x. If f and g are inverse of each other, then their composition of them will do nothing and return x itself.

Test if two functions are inverse

Functions $f(x)$ and $g(x)$ are inverse of each other, if and only if

$$f(g(x)) = x \quad \text{or} \quad g(f(x)) = x \tag{7.1}$$

Example 7.1. Show that each function is the inverse of the other:

$$f(x) = 4x - 7 \quad \text{and} \quad g(x) = \frac{x+7}{4}$$

$f\left(g(x)\right)$	$g(x)$ goes into $f(x)$
$4g(x) - 7$	Replace $g(x)$ with $\frac{x+7}{4}$
$\cancel{4}\left(\frac{x+7}{\cancel{4}}\right) - 7$	Cancel out 4
$x + 7 - 7$	Combine like terms
x	Simplified to x, so they are inverse!

7.2 Finding inverse of a function

If we think of x as our input and y as our output from a function, then the inverse will take y as an input and give x as the output. This means if we switch x and y in our function we will find the inverse! This process is called the switch and solve strategy.

1. Replace $f(x)$ with y
2. Switch x and y
3. Solve for y
4. Replace y with $f^{-1}(x)$

Example 7.2. Find the inverse of $f(x) = 4x^3 - 1$.

$$\begin{aligned}
f(x) &= 4x^3 - 1 && \text{Replace } f(x) \text{ with } y \\
y &= 4x^3 - 1 && \text{Switch } x \text{ and } y \\
x &= 4y^3 - 1 && \text{Solve for } y\text{, add 1} \\
x + 1 &= 4y^3 && \text{Divide by 4} \\
\frac{x+1}{4} &= y^3 && \text{Cube root both sides} \\
\sqrt[3]{\frac{x+1}{4}} &= y && \text{Replace } y \text{ with } f^{-1}(x) \\
\sqrt[3]{\frac{x+1}{4}} &= f^{-1}(x) && \text{Our solution}
\end{aligned}$$

Example 7.3. Find the inverse of $f(x) = \dfrac{2x-1}{x+3}$.

$$\begin{aligned}
f(x) &= \frac{2x-1}{x+3} && \text{Replace } f(x) \text{ with } y \\
y &= \frac{2x-1}{x+3} && \text{Switch } x \text{ and } y \\
x &= \frac{2y-1}{y+3} && \text{Solve for } y\text{, Multiply by } y+3 \\
x(y+3) &= 2y - 1 && \text{Distribute } x \text{ on LHS} \\
xy + 3x &= 2y - 1 && \text{Subtract } 2y \\
xy - 2y + 3x &= -1 && \text{Subtract } 3x \\
xy - 2y &= -3x - 1 && \text{Factor out } y \text{ on LHS} \\
y(x-2) &= -3x - 1 && \text{Divide both sides by } x - 2 \\
y &= \frac{-3x-1}{x-2} && \text{Replace } y \text{ with } f^{-1}(x) \\
f^{-1}(x) &= \frac{-3x-1}{x-2} && \text{Our solution}
\end{aligned}$$

7.3 Graph of f^{-1} and f

There is an fascinating relationship between the graph of f and its inverse f^{-1}, which can be helpful to better understand how f^{-1} looks like.
The graph of f^{-1} is a reflection of the graph of f about the line $y = x$; In other words, if we folded the plane along $y = x$, the two graph will coincide. This is illustrate in Figure 7.2.
Inverse function have ordered-pairs with coordinate switched, if the point

(a, b) is on the graph of f, then point (b, a) is on the graph of f^{-1}.That's why f and f^{-1} have a symmetry around $y = x$.

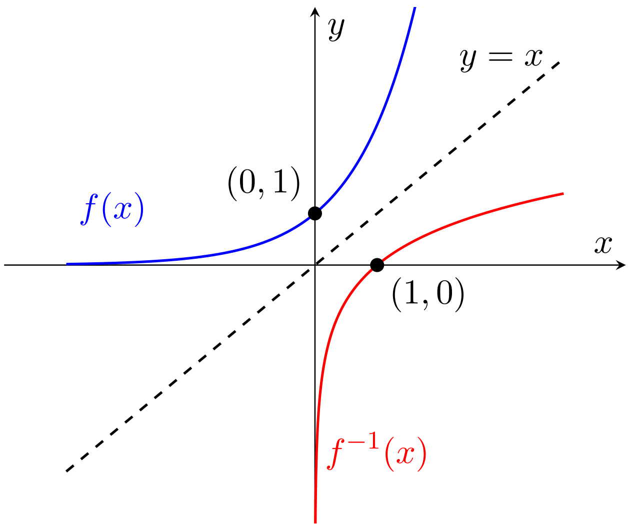


Figure 7.2: The graph of $f(x)$ and $f^{-1}(x)$. Notice point $(0, 1)$ on $f(x)$ became $(1, 0)$ on $f^{-1}(x)$.

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