Chapter 3

The ordinary operations of algebra suffice to resolve problems in the theory of curves.

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Contents

Quadratic Functions

1.1 Introduction

A function in form $f(x) = ax^2 + bx + c$, in which $a \neq 0$, is called a quadratic function. In other words, quadratic function is a polynomial with degree of 2.

1.2 Graph of quadratic function

The graph of quadratic function is called parabola. Based on the coefficient a, parabola may open upward or downward.

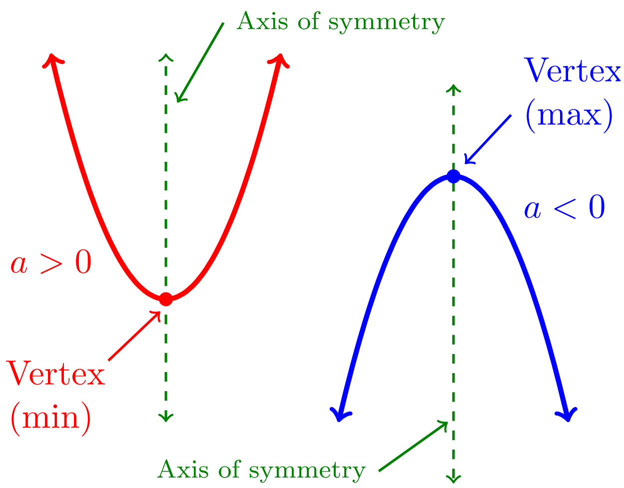

Figure 1.1: Parabola opens upward if $a > 0$ and open downward if $a < 0$.

As it is illustrated in Figure 1.1:

1. The graph opens upward (concave up) if $a > 0$ and opens downward (concave down) when $a < 0$.

2. **The vertex (or turning point)** of a parabola is the lowest point on the graph when it open upward and the highest point when it opens downward.

3. **The axis of symmetry** is a vertical line through the vertex divides the figure in half. Parabolas are symmetric with respect to this line.

1.3 Vertex

Vertex is a very important point on a parabola. Not only it helps us to graph the parabola perfectly, but also it is a key point in optimization problems. The quadratic functions can be written in either **standard form** or **general form**. Each form has its own way to find the vertex. Once we found the vertex, we can easily find the axis of symmetry.

1.3.1 Standard form of a quadratic function

The standard form of a quadratic function is

$$f(x) = a(x - h)^2 + k \tag{1.1}$$

The vertex of a standard quadratic function is located at (h, k). This can be easily proven using the graph of standard function $y = x^2$ and transformation method.
The axis of symmetry is a vertical line passes through the vertex. Therefore, the equation of axis of symmetry of standard quadratic function is $x = h$.

Example 1.1. Find the vertex and axis of symmetry of the following functions.

a. $f(x) = -(x - 4)^2 + 3$

b. $g(x) = 3(x + 2)^2 - 1$

a. Comparing the equation with (1.1) equation, we get $h = 4$ and $k = 3$. Therefore, the vertex is located at $(4, 3)$ and the axis of symmetry is $x = 4$.

b. First, rewrite the equation in standard form,

$$g(x) = 3\big(x - (-2)\big)^2 - 1$$

Thus, $h = -2$ and $k = -1$ which gives us vertex $(-2, -1)$ and the axis of symmetry $x = -2$.

1.3.2 General form of a quadratic function

We have already seen the general form of a quadratic function

$$f(x) = ax^2 + bx + c \tag{1.2}$$

The vertex of a quadratic function in this form is at $\left(-\frac{b}{2a}, f\big(-\frac{b}{2a}\big)\right)$. Therefore, the axis of symmetry is $x = -\frac{b}{2a}$.

Example 1.2. Find the vertex and the axis of symmetry of the following functions.

a. $f(x) = -x^2 + 4x + 2$

b. $g(x) = 3x^2 + x + 10$

a. Comparing with the general form, we got $a = -1$, $b = 4$ and $c = 2$, so the x-coordinate of vertex is

$$\begin{aligned} -\frac{b}{2a} &\qquad \text{Substitute} \\ -\frac{4}{2(-1)} &\qquad \text{Simplify} \\ 2 &\qquad x\text{-coordinate of the vertex} \end{aligned}$$

To find its y-coordinate, we need to plug $x = 2$ into $f(x)$, therefore

$$\begin{aligned} f(2) &\qquad \text{Plug it into function} \\ -(2)^2 + 4(2) + 2 &\qquad \text{Simplify} \\ 6 &\qquad y\text{-coordinate of the vertex} \end{aligned}$$

So the vertex is at $(2, 6)$ and the axis of symmetry is $x = 2$.

b.Here we have $a = 3$, $b = 1$ and $c = 10$, therefore the x-coordinate of the vertex is

$$-\frac{b}{2a} = -\frac{1}{2(3)} = -\frac{1}{6}$$

And its y-coordinate is

$$g\left(-\frac{1}{6}\right) = 3\left(-\frac{1}{6}\right)^2 + \left(-\frac{1}{6}\right) + 10 = \frac{119}{12}$$

So the vertex is located at $\left(-\frac{1}{6}, \frac{119}{12}\right)$. The axis of symmetry is simply $x = -\frac{1}{6}$.

Rational Functions

2.1 Introduction

$f(x)$ is a rational function if

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are both a polynomial and $q(x) \neq 0$.Here are some examples of rational functions:

$$\begin{aligned} f(x) &= \frac{1}{x+4} \\ g(x) &= \frac{x-3}{x+2} \\ h(x) &= \frac{x-12}{x^2-2x+3} \\ &\vdots \end{aligned}$$

2.2 Domain of a rational function

As we discussed in previous chapter, when we have a division the denominator cannot be equal to 0. Therefore, domain of the rational function consists of all real numbers except the zeros of the denominator.

Example 2.1. Find the domain of each rational functions:

a. $f(x) = \dfrac{x^2-25}{x+3}$

b. $g(x) = \dfrac{x^2-25}{x^2-25}$

c. $h(x) = \dfrac{x^2-25}{x^2+144}$

$$\begin{aligned} a. \quad x+3 &= 0 && \text{Set the denominator to 0} \\ x &= -3 && \text{Exclude this point} \\ \{x \mid x &\neq -3\} && \text{Domain} \end{aligned}$$

$$\begin{aligned} b. \quad x^2-25 &= 0 && \text{Set the denominator to 0} \\ x^2 &= 25 && \text{Use the square root property} \\ x &= \pm 5 && \text{Exclude these points} \\ \{x \mid x &\neq -5,\, 5\} && \text{Domain} \end{aligned}$$

$$\begin{aligned} c. \quad x^2+144 &= 0 && \text{Set the denominator to 0} \\ x^2 &= -144 && \text{No real solutions} \\ \{x \mid \text{all real numbers}\} & && \text{Domain} \end{aligned}$$

2.3 Arrow notation

Often we don't have access to a particular point, for instance maybe it's not included in our domain. However, we still want to know what will happen to our function around that point.
In mathematics, when it is said that x is approaching to a number b from left, it means $x < b$ but close to number b. On the other hand, when it is said x is approaching to the number b from right, it means $x > b$ but close to number b.
To indicate we are approaching to a particular number, we use superscripts: (i) "$-$" if we are approaching from left, (ii) "$+$" if we are approaching from right.
For example, if we are approaching to number b from left, we can express it as $x \to b^-$. Notice, arrow $\to$ here means approaching. Also, if we are approaching to number b from right, we can denote it as $x \to b^+$

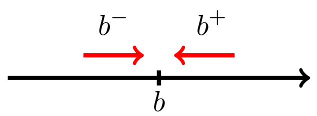

Figure 2.1: Approaching to number b, from left: b^- and from right: b^+

We might also have a situation when x is decreasing or increasing without bound, meaning we are moving toward $-\infty$ or $+\infty$. In mathematics, we can express them like this:

- Decreasing without bound: $x \to -\infty$
- Increasing without bound: $x \to +\infty$

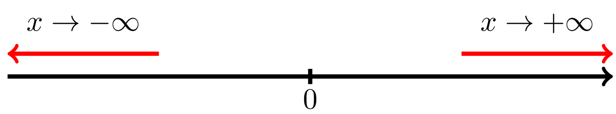

Figure 2.2: Decreasing and increasing without bound

Arrow notation is very important and useful, particularly in calculus when we need to use limit. You can find the summary of all arrow notations in the table below.

Table 2.1: Arrow Notations

Arrow Notation	Meaning
$x \to b^-$	x approaches to b from left ($x < b$ but close to b)
$x \to b^+$	x approaches to b from right ($x > b$ but close to b)
$x \to -\infty$	x approaches $-\infty$ (x decreases without bound)
$x \to +\infty$	x approaches $+\infty$ (x increases without bound)

2.4 Reciprocal function

Of many rational functions, reciprocal function defined as

$$f(x) = \frac{1}{x} \tag{2.1}$$

is the most noteworthy. The domain of this function is all real number, except 0.

$$\text{Domain of a reciprocal function} : \{x \mid x \neq 0\}$$

Let's plot the reciprocal function. Since $x = 0$ is not in our domain, we begin by observing the behaviour of the graph around $x = 0$.
If we move toward 0 from left, i.e. $x \to 0^-$, we will notice the function gets bigger and bigger but in negative direction. On the other hand, if we approach to 0 from right, i.e. $x \to 0^+$, function f get bigger in positive direction. Using mathematical notation, we can say

- If $x \to 0^-$, $f(x) \to -\infty$.

- If $x \to 0^+$, $f(x) \to +\infty$.

Table 2.2 is showing this behavior. As you can see, $x = 0$ is a pole that the f never reaches to it, but when we get close to this pole, the f goes to infinity.

Table 2.2: Function values around $x = 0$

x	$f(x) = \frac{1}{x}$
-1	-1
-0.5	-2
-0.1	-10
-0.01	-100
-0.001	-1000
$\vdots$	$\vdots$

x	$f(x) = \frac{1}{x}$
1	1
0.5	2
0.1	10
0.01	100
0.001	1000
$\vdots$	$\vdots$

Now let's see what will happen if x decreases or increases without bound. If x goes to negative infinity, we notice, the reciprocal function gets smaller and smaller. Likewise, if x goes to positive infinity, we'll see the same behavior. Table 2.3 is showing this behavior when x goes to infinity. As you can

Table 2.3: Function values around when x goes negative and positive infinity

x	$f(x) = \frac{1}{x}$
-1	-1
-2	-0.5
-10	-0.1
-100	-0.01
-1000	-0.001
$\vdots$	$\vdots$

x	$f(x) = \frac{1}{x}$
1	1
2	0.5
10	0.1
100	0.01
1000	0.001
$\vdots$	$\vdots$

observe, As x goes to infinity, whether positive or negative, f will get close to a horizontal line $y = 0$.
Using all of these information, we can graph the reciprocal function, shown in Figure 2.3. The pole $x = 0$ is called **vertical asymptote**. Whereas, the horizontal line $y = 0$ is called **horizontal asymptote**. In next session, we will discuss how to find these two types of asymptotes.

Note 2.1. There is also one more asymptote called **slant or oblique asymptote**. This type of asymptote appears in some rational functions.

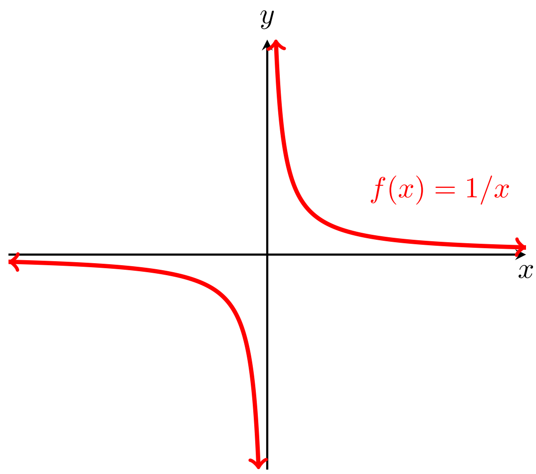

Figure 2.3: The graph of the reciprocal function.

In this case, when x goes to infinity, f approaches to a line, $y = mx + b$. We won't discuss this type of asymptote in this course.

Note 2.2. Using the transformations, we can graph any different types of reciprocal functions. If we have $f(x) = \dfrac{a}{bx+h} + k$, then

1. **Shift horizontally**: Shift h unit horizontally: right if $h < 0$; left if $h > 0$
2. **Shrink or stretch horizontally**: Multiply all x-coordinates of function by b. If $b > 1$, the graph will shrink and if $0 < b < 1$, the graph will stretch horizontally.
3. **Shrink or stretch vertically**: Multiply all y-coordinates of function by a. If $a > 1$, the graph will stretch and if $0 < a < 1$, the graph will stretch vertically.
4. **Shift vertically**: Shift k unit vertically: upward if $k > 0$; downward of $k < 0$.

Note 2.3. We use dashed line to show vertical and horizontal asymptotes when we graph a rational function.

2.5 Vertical asymptotes

The line $x = a$ is a vertical asymptote of the function $y = f(x)$ if y approaches $\pm\infty$ as x approaches a from the right or left.
To find vertical asymptote, it is first important to simplify the rational function, meaning to factor both numerator and denominator and cancel out the common factors. Then we set the denominator to 0 and solve for x.

Vertical asymptote

To find vertical asymptotes, follow these steps:

1. Simplify the rational expression
2. Set the denominator to 0 and solve for x

Example 2.2. Find the vertical asymptote(s) of following functions:

$$f(x) = \frac{x+3}{x-5}$$

$$\begin{aligned}
\frac{x+3}{x-5} &\qquad \text{Simplfy fraction} \\
\frac{1.(x+3)}{1.(x-5)} &\qquad \text{gcf of both numerator and denominator is 1} \\
&\qquad \text{No common factors!} \\
x-5=0 &\qquad \text{Set denominator to 0} \\
x=5 &\qquad \text{Our vertical asymptote}
\end{aligned}$$

The graph of this function is shown in Figure 2.4.

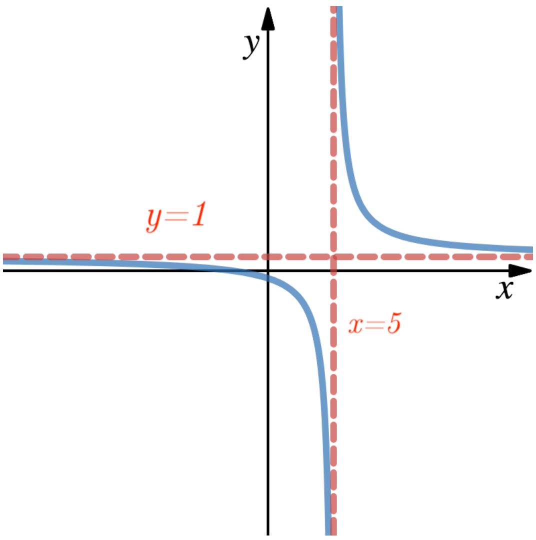

Figure 2.4: The graph of $f(x) = \dfrac{x+3}{x-5}$. There is a vertical asymptote which is $x = 5$. Also there is a horizontal one $y = 1$

Note 2.4. Once a factor in the denominator got canceled out, zeros of that factor is not a vertical asymptote instead there will be a hole at that point.

That's why it is very very important to simplify the rational function first and then set the denominator equal to 0.

Example 2.3. Find the vertical asymptote of the following function:

$$f(x) = \frac{x-5}{x^2-25}$$

$\dfrac{x-5}{x^2-25}$	Factor both numerator and denominator
$\dfrac{1.\cancel{(x-5)}}{\cancel{(x-5)}(x+5)}$	Cancel out $x-5$
$\dfrac{1}{x+5}$	Now set the denominator to 0
$x+5=0$	Solve for x
$x=-5$	Our vertical asymptote

Notice, factor $x-5$ got cancelled out during simplification step. Zero of this factor is $x=5$. At this point, we will have a hole in our graph instead of vertical asymptote, as shown in Figure 2.5

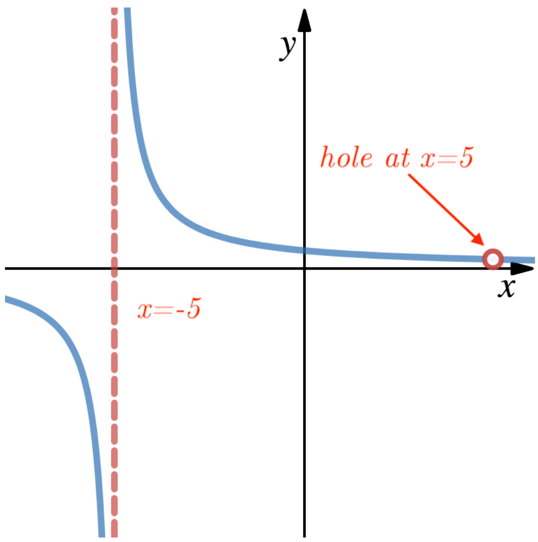

Figure 2.5: The graph of $f(x) = \dfrac{x-5}{x^2-25}$. There is only one vertical asymptote at $x=-5$. At $x=-5$, there is a hole in graph because it is not in our domain and it not a vertical asymptote as well.

Example 2.4. Find the vertical asymptote of the following function:

$$f(x) = \frac{x^2+1}{x^2+x-6}$$

$$\begin{aligned}
\frac{x^2+1}{x^2+x-6} &\quad \text{Factor out both numerator and denominator} \\
\frac{1.(x^2+1)}{(x-2)(x+3)} &\quad \text{No common factor!} \\
(x-2)(x+3)=0 &\quad \text{Set the denominator equal to 0} \\
x=2 &\quad \text{First vertical asymptote} \\
x=-3 &\quad \text{Second vertical asymptote}
\end{aligned}$$

As you can see, in this example we have more than 1 vertical asymptote. The graph of this function is shown in Figure 2.6.

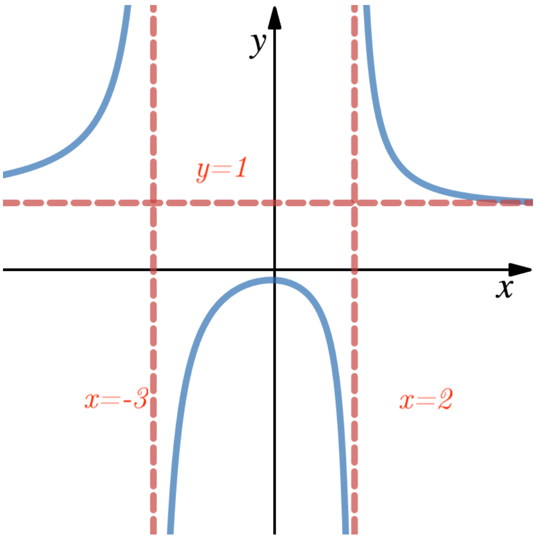

Figure 2.6: The graph of $f(x) = \dfrac{x^2+1}{x^2+x-6}$. There are two vertical asymptotes, $x=-3$ and $x=2$. There is also a horizontal asymptote $y=1$.

2.6 Horizontal asymptote

The line $y = b$ is a horizontal asymptote of the function $y = f(x)$ if y approaches b as x approaches $\pm\infty$.
To find the horizontal asymptote, degree of numerator and denominator play an important role. Let $f(x)$ be:

$$f(x) = \frac{a_1x^n + a_2x^{n-1} + \dots}{b_1x^m + b_2x^{m-2} + \dots}$$

The degree of numerator is n and degree of denominator is m. The leading coefficient of numerator is a_1 and the leading coefficient of denominator is b_1.

I. If $n = m$, then there is one horizontal asymptote at $y = \dfrac{a_1}{b_1}$.

II. If $n < m$, then there is one horizontal asymptote at $y = 0$.

III. If $n > m$, we don't have any horizontal asymptote.

Example 2.5. Find the horizontal asymptote of each functions:

a. $f(x) = \dfrac{4x^3 - 2x + 1}{2x^3 - 2x^2 + 10}$

b. $f(x) = \dfrac{3x - 4}{6x^2 + 2x - 1}$

c. $f(x) = \dfrac{7x^2 - 1}{x + 5}$

Let n degree of numerator and m degree of denominator.
a.

$$\begin{aligned}
&\frac{4x^3 - 2x + 1}{2x^3 - 2x^2 + 10} && n = 3 \text{ and } m = 3 \text{ so } n = m \\
&\frac{4x^3 - 2x + 1}{2x^3 - 2x^2 + 10} && \text{Case I, divide leading coefficients} \\
&y = \frac{4}{2} = 2 && \text{Our horizontal asymptote}
\end{aligned}$$

b.

$$\begin{aligned}
&\frac{3x - 4}{6x^2 + 2x - 1} && n = 1 \text{ and } m = 6 \text{ so } n < m \\
&\frac{3x - 4}{6x^2 + 2x - 1} && \text{Case II} \\
&y = 0 && \text{Our horizontal asymptote}
\end{aligned}$$

c.

$$\begin{aligned}
&\frac{7x^2 - 1}{x + 5} && n = 2 \text{ and } m = 1 \text{ so } n > m \\
&\frac{7x^2 - 1}{x + 5} && \text{Case III} \\
&\text{✗} && \text{No horizontal asymptote}
\end{aligned}$$

Non-linear inequalities

In chapter 1, we learned how to solve any linear inequalities plus absolute value inequalities. In this section, we will learn how to solve two types of non-linear inequalities:

1. Polynomial inequality
2. Rational inequality

Before jumping to solve these types of inequalities, we begin the section by creating a sign table first.

3.1 Sign table

Of all functions we learned, linear function is the easiest one. Consider the linear function, $P(x) = x - 3$. The first and easiest way to determine where the function is positive or negative, is **graphing**. The graph of this function is shown in Figure 3.1.

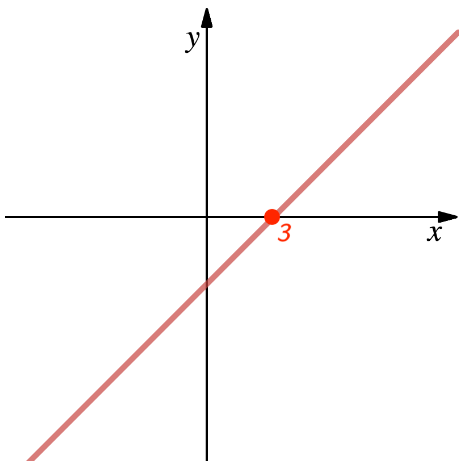

Figure 3.1: The graph of $P(x) = x - 3$. Notice the boundary point is 3

As you can see, point $x = 3$ plays an important role; After this point, the graph is above the x-axis, therefore, it is positive. Whereas, before this

point, the graph is below the x-axis and thus, it is negative. This point is called the *boundary point* or *critical point*.
The second method is the **algebraic method**. Since this method is much faster than first method, we will use this method instead of the first method. First, we need to find the boundary point by setting the function equal to 0, and then solving for x. In our example,

$$\begin{aligned} P(x) &= 0 \\ x-3 &= 0 \\ x &= 3 \qquad \text{Boundary point} \end{aligned}$$

Next, we draw a number line. The boundary point divide the whole number line into two intervals: $(-\infty, 3)$ and $(3, +\infty)$. To find out the sign of $P(x)$, we choose one number within each intervals. You can choose any point you want, however, always try to choose a small number to make your calculation easy. Finally, plug each chosen number into $P(x)$ and see if you get a positive or negative value. If you get a positive value, that means the function $P(x)$ is always positive in that interval; Otherwise, it will be always negative.

Intervals	$(-\infty, 3)$	$(3, +\infty)$
Chosen number	0	4
$P(x)$	$P(0) = -3$	$P(-3) = 1$
Sign	$-$	$+$

We can summarize all information we got in a number line shown in Figure 3.2 which is called the **sign table**.

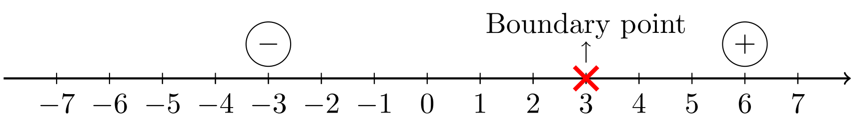

Figure 3.2: The sign table of $P(x) = x - 3$.

Note 3.1. In case of rational functions, we need to set both numerator and denominator equal to 0. Then solve them separately to find the boundary point(s). This is shown in the following example.

Example 3.1. Create a sign table for the function $R(x) = \dfrac{x-5}{x+4}$.

To find the boundary points, set numerator and denominator equal to 0.

$$\text{Numerator} = 0 \qquad\qquad \text{Denominator} = 0$$

$$x - 5 = 0 \qquad\qquad x + 4 = 0$$
$$x = 5 \qquad\qquad x = -4$$

So, we have two boundary points, -4 and 5. These two points divide the whole number line into three sections: $(-\infty, -4)$, $(-4, 5)$ and $(5, +\infty)$. We pick a number within each interval and plug them into $P(x)$ to find the sign of function in each interval.

Intervals	$(-\infty, -4)$	$(-4, 5)$	$(5, +\infty)$
Chosen number	-5	-3	6
$R(x)$	$R(-5) = 10$	$R(-3) = -8$	$R(6) = -1.25$
Sign	$+$	$-$	$+$

The sign table of $R(x)$ is shown in Figure 3.3 .

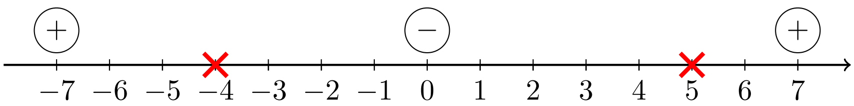

Figure 3.3: The sign table of $R(x) = \dfrac{x-5}{x+4}$.

Creating a sign table

Consider the function $F(x)$:

1. **Boundary point(s)**:
   - If $F(x)$ is a polynomial, set it equal to 0 and solve for x.
   - If $F(x)$ is a rational function, set both numerator and denominator equal to 0 and solve them separately.
2. **Intervals:** The boundary point(s) divide(s) the number line into several intervals.
3. **Pick a number**: choose a number withing each interval.Plug each chosen number into $F(x)$.
   - If $F(x)$ is positive, then the function is always positive within that interval.
   - If $F(x)$ is negative, then the function is always negative within that interval.
4. **Sign**: Indicate the sign of each function withing each interval on the number line.

3.2 Polynomial inequalities

The first type of non-linear inequalities is polynomial inequalities. The first step to solve this inequality is to express it in the standard form, i.e. we must have a zero on one side :

$$P(x) > 0 \qquad \text{or} \qquad P(x) < 0$$
$$\text{(same when we have } \leq \text{ or } \geq \text{)}$$

The next step, we find the boundary point(s) by solving $P(x) = 0$. Then, we have to decide about boundary point(s): if we have $>$ or $<$, then the boundary points are not included in our solution and we use open circle in our sign table. However, if we have $\geq$ or $\leq$, then the boundary points are included and we use shaded circle in the sign table.
Once we completed our sign table, we look at the original inequalities and based on that we choose positive or negative intervals as our solution.

Solving a polynomial inequality

1. **Standard form**: Express the equation in the standard form.

$$P(x) < 0 \quad \textit{or} \quad P(x) > 0 \quad \text{(same for } \leq \text{ or } \geq \text{)}$$

2. **Sign table**: Create a sign table
3. **Shaded or open circle**: We must decided whether the boundary points are included or excluded.
   - For $<$ or $>$ $\longrightarrow$ excluded and use open circles.
   - For $\leq$ or $\geq$ $\longrightarrow$ included and use shaded circles.
4. **Choosing intervals**: Based on the original inequality, we can choose appropriate intervals:
   - If $P(x) < 0$ or $P(x) \leq 0$ $\longrightarrow$ choose negative intervals.
   - If $P(x) > 0$ or $P(x) \geq 0$ $\longrightarrow$ choose positive intervals.

Example 3.2. Solve $x^2 - 4x < -3$

First, add 3 to both sides to express the inequality in the standard form, $x^2 - 4x + 3 < 0$. Let $P(x) = x^2 - 4x + 3$. We must find the boundary points to create the sign table

$$\begin{aligned} P(x) &= 0 && \text{Solve for } x \\ x^2 - 4x + 3 &= 0 && \text{Factor!} \end{aligned}$$

$$\begin{aligned} (x-1)(x-3) &= 0 && \text{Set each factor equal to 0} \\ x = 1 \text{ or } x &= 3 && \text{Our boundary points} \end{aligned}$$

These two points divide the number line into three sections:

Intervals	$(-\infty, 1)$	$(1, 3)$	$(3, +\infty)$
Chosen number	0	2	4
$P(x)$	$P(0) = 3$	$P(2) = -1$	$P(4) = 3$
Sign	$+$	$-$	$+$

Since we have $<$ in our original inequality, therefore all boundary points are excluded and we use open circles.

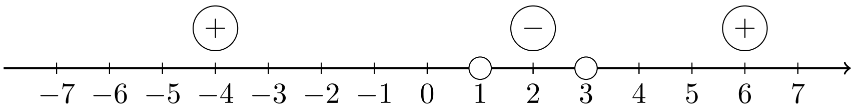

Figure 3.4: The sign table of $P(x) = x^2 - 4x + 3$.

Finally, we have $<$ in the original inequality, thus we should choose negative intervals. So our solution is $(-1, 3)$. Notice -1 and 3 are excluded.

Example 3.3. Solve $x^3 - 16x - x^2 + 16 \geq 0$

The inequality is already in the standard form. Let $P(x) = x^3 - 16x - x^2 + 16$. So, we begin by finding the boundary points:

$$\begin{aligned} P(x) &= 0 && \text{Solve for } x \\ x^3 - 16x - x^2 + 16 &= 0 && \text{Factor } x \text{ from first two terms} \\ & && \text{and } -1 \text{ from two last terms} \\ x(x^2 - 16) - (x^2 - 16) &= 0 && \text{Factor out } x^2 - 16 \\ (x^2 - 16)(x - 1) &= 0 && \text{Set each factor equal to 0} \\ x = \pm 4 \text{ or } x &= 1 && \text{Our boundary points} \end{aligned}$$

We have three boundary points dividing the number line into four intervals:

Intervals	$(-\infty, -4)$	$(-4, 1)$	$(1, 4)$	$(4, +\infty)$
Chosen number	-5	0	2	5
$P(x)$	$P(-5) = -54$	$P(0) = 16$	$P(2) = -12$	$P(5) = 36$
Sign	$-$	$+$	$-$	$+$

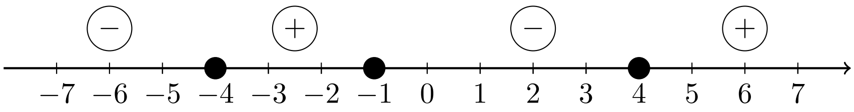

Figure 3.5: The sign table of $P(x) = x^3 - 16x - x^2 + 16$.

Because of $\geq$ in the original inequality, all boundary points are included. Also we must choose, all positive intervals. So the solution is

$$[-4, -1] \cup [4, +\infty)$$

3.3 Rational inequalities

All steps are the same except the fact that zeros of denominator are always excluded. The zeros of numerator, however, depends on the inequality sign. If we have $<$ or $>$ then they are excluded; otherwise, they are included.

Solving a rational inequality

1. **Standard form**: Express the equation in the standard form.

$$R(x) < 0 \quad \textit{or} \quad R(x) > 0 \quad (\text{same for } \leq \text{ or } \geq)$$

2. **Sign table**: Create a sign table

3. **Shaded or open circle**: The boundary points of denominator are always excluded. For numerator:

   - For $<$ or $>$ $\longrightarrow$ excluded and use open circles.
   - For $\leq$ or $\geq$ $\longrightarrow$ included and use shaded circles.

4. **Choosing intervals**: Based on the original inequality, we can choose appropriate intervals:

   - If $R(x) < 0$ or $R(x) \leq 0 \longrightarrow$ choose negative intervals.
   - If $R(x) > 0$ or $R(x) \geq 0 \longrightarrow$ choose positive intervals.

Example 3.4. Solve $\dfrac{x-4}{(x-2)^2} \geq 0$

The inequality is already expressed in standard form. Let $R(x) = \dfrac{x-4}{(x-2)^2}$ and find its boundary points

$$\text{Numerator} = 0 \qquad\qquad \text{Denominator} = 0$$

$$x - 4 = 0 \qquad (x-2)^2 = 0$$
$$x = 4 \qquad x = 2$$

These two boundary points will divide the number line into three intervals:

Intervals	$(-\infty, 2)$	$(2, 4)$	$(4, +\infty)$
Chosen number	1	3	5
$R(x)$	$R(1) = -3$	$R(3) = -1$	$R(5) = 0.11$
Sign	$-$	$-$	$+$

Notice, $x = 2$ is a zero of denominator so it is excluded. $x = 4$ is the zero of numerator and it's included because of $\geq$ in the original inequality.

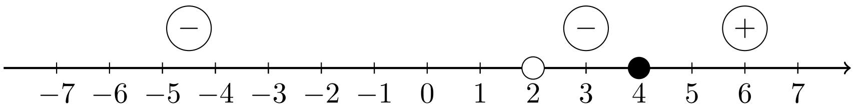

Figure 3.6: The sign table of $R(x) = \dfrac{x-4}{(x-2)^2}$.

Because we have $\geq$ in the original inequality, we must select the positive signs. So our solution is $[4, +\infty)$.

Example 3.5. Solve $\dfrac{x-1}{x+3} < 1$

First, we must express the inequality in the standard form.

$$\begin{aligned} \frac{x-1}{x+3} &< 1 && \text{Subtract 1} \\ \frac{x-1}{x+3} - 1 &< 0 && \text{LCD is } x+3 \\ \frac{x-1}{x+3} - 1\left(\frac{x+3}{x+3}\right) &< 0 && \text{Simplify} \\ \frac{x-1-(x+3)}{x+3} &< 0 && \text{Combine like terms} \\ \frac{-4}{x+3} &< 0 && \text{Standard form} \end{aligned}$$

Let $R(x) = \dfrac{-4}{x+3}$. Since we don't have any variable x in numerator, so only set the denominator equal to 0.

$$\text{Denominator} = 0$$

$$
\begin{aligned}
x + 3 &= 0 \\
x &= -3
\end{aligned}
$$

This boundary point divides the number line into two intervals.

Intervals	$(-\infty, -3)$	$(-3, +\infty)$
Chosen number	-4	0
$R(x)$	$R(-4) = 4$	$R(-3) = -1.33$
Sign	+	−

Don't forget $x = -3$ is the zero of the denominator thus, it is not included in our solution. Here is the sign table of $R(x)$:

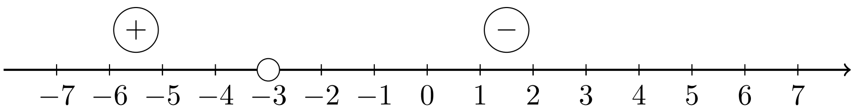

Figure 3.7: The sign table of $R(x) = \dfrac{-4}{x+3}$.

We have $<$ in the original inequality which means we are looking for negative intervals. Our solution is $(-3, +\infty)$.

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