Chapter 1

The book of nature is written in the language of Mathematics.

1623
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Contents

Graphs

1.1 Rectangle coordinate system

The purpose of a coordinate system is to uniquely determine the position of an object on a plane.

Rene Descartes devised a simple idea. He intersected two number lines at right angle and the position of a point in a plane can be determined by its distance from each of the lines. This system is called the **Cartesian coordinate system** or **rectangle coordinate system**.

In rectangle coordinate system, the horizontal number line called x-axis, and the vertical number line called y-axis. The origin is the intersection of the x- and y-axes. These two number lines divide, as illustrated in Figure (1.2), the whole plane into 4 region which are called quadrant.

Figure 1.1: Portrait of Rene Descartes.

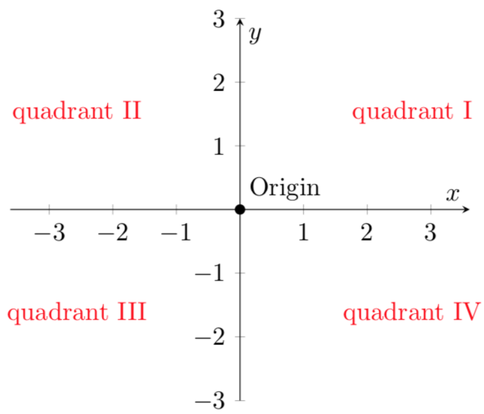


Figure 1.2: Rectangle coordinate system with its quadrants.

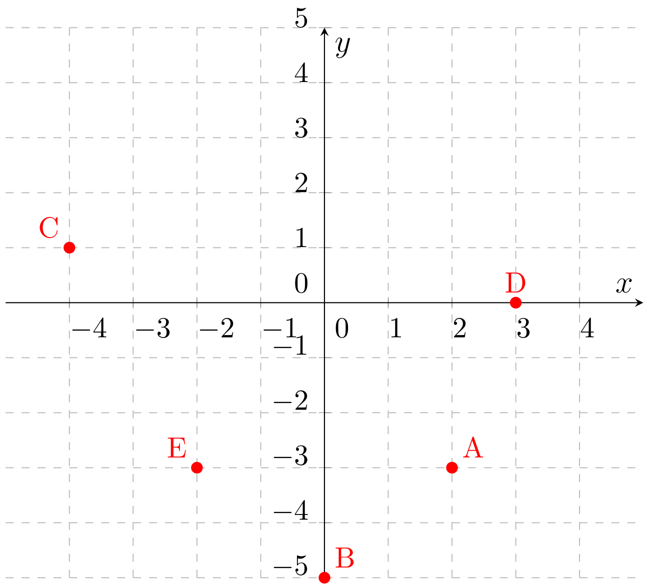
Points are labeled with ordered pairs of real numbers (x, y), called the coordinates of the point, which give the horizontal and vertical distance of the point from the origin, respectively. The origin has a ordered pair (0, 0 Locations of the points in the plane are determined in relationship to (0, 0). All points in the plane are located in one of four quadrants or on the x- or y-axis.

1.2 Plotting points

To plot a point, start at the origin, proceed horizontally the distance and direction indicated by the x-coordinate, then vertically the distance and direction indicated by the y-coordinate. The resulting point is often labeled with its ordered pair coordinates and/or a capital letter. For example, the point 2 units to the right of the origin and 3 units up could be labeled $A(2, 3)$.

Example 1.1. Plot the following points on a rectangle coordinate system. $\mathrm{A}(2, -3),\ \mathrm{B}(0, -5),\ \mathrm{C}(-4, 1),\ \mathrm{D}(3, 0),\ \mathrm{E}(-2, -3)$

Here are the points



1.3 Equations in two variables

A relationship between two quantities can be expressed as an equation in two variable, such as

$$y = 5 - x^2$$

A solution of an equation in two variables, x and y, is an ordered pair (x, y) that makes the equation a true statement.

Example 1.2. For the linear equation $-2x+3y=8$ determine whether the ordered pair is a solution.

(a) $(-4,0)$

(b) $(2,-4)$

(c) $(1,{}^{10}\!/_3)$

(a) the ordered pair $(-4,0)$ indicates that $x=-4$ and $y=0$. Substitute them into the equation:

$$\begin{aligned} -2(-4)+3(0) &\stackrel{?}{=} 8 \\ 8+0 &\stackrel{?}{=} 8 \\ 8 &\stackrel{?}{=} 8 \quad \checkmark \end{aligned}$$

So, $(-4,0)$ satisfies the equations and it is our solution.
(b) We plug $x=2$ and $y=-4$ into equation:

$$\begin{aligned} -2(2)+3(-4) &\stackrel{?}{=} 8 \\ -4-12 &\stackrel{?}{=} 8 \\ -16 &\stackrel{?}{=} 8 \quad ✗ \end{aligned}$$

Thus, $(2,-4)$ is not the solution. (c) In this part, we have $x=1$ and $y=\dfrac{10}{3}$:

$$\begin{aligned} -2(1)+3\left(\frac{10}{3}\right) &\stackrel{?}{=} 8 \\ -2+10 &\stackrel{?}{=} 8 \\ 8 &\stackrel{?}{=} 8 \quad \checkmark \end{aligned}$$

The ordered pair $\left(1,\dfrac{10}{3}\right)$ is actually one of the solution as well.

1.4 Graphs of equations

The main purpose of graphs is not to plot random points, but rather to give a picture of the solutions to an equation. We may have an equation such as $x+y=1$. We may be interested in what type of solution are possible in this equation. We can visualize the solution by making a graph of possible x and y combinations that make this equation a true statement.
We will have to start by finding possible x and y combinations. To find the a point on a line, just choose any number for x, plug it into equation and

solve for y. For example, to find a point on $x + y = 1$, I would choose $x = 0$ and I'd get $y = 1$. So the ordered pair $(0, 1)$ is one of many points on the line.

Fact 1.1. You can choose a number for x and solve for y or vice versa.

Fact 1.2. Many students ask "What number should I choose?" The answer is any number! It is entirely up to you. But it makes sense to choose a small and simple number to avoid any long and hard calculation.

We'll usually draw a table, called the table of values or xy table. We then choose a value, for example $x = 0$ and solve for y. Next, I will choose another value for x and solve for y again.

x	y
$\vdots$	$\vdots$
0	?
1	?
2	?
$\vdots$	$\vdots$

Steps to graph any equations

(1) Find several ordered pairs on the equations.

(2) Plot these ordered pairs as points in the rectangle coordinate system.

(3) Connect the points with a smooth curve or line.

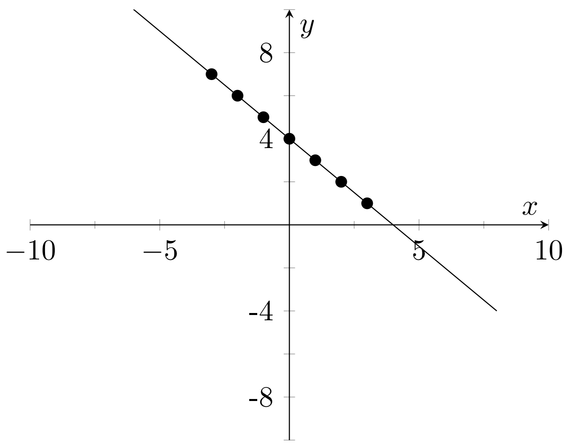
Note 1.1. In this section, we use 4 to 6 ordered pairs to graph an equation. However, you don't always need to find too many ordered pairs. Later, we will discuss how many ordered pairs we need for a specific equation.

Example 1.3. Graph the equation $y = 4 - x$. Select integers for x, starting with -3 and ending with 3.

Our x values are -3, -2, -1, 0, 1, 2, and 3. We plug them into the equation and solve for y to find our ordered pairs.

x	y	
-3	7	$y = 4 - (-3) = 7$
-2	6	$y = 4 - (-2) = 6$
-1	5	$y = 4 - (-1) = 5$
0	4	$y = 4 - (0) = 4$
1	3	$y = 4 - (1) = 3$
2	2	$y = 4 - (2) = 2$
3	1	$y = 4 - (3) = 1$

Next, we plot all ordered pairs and connect them to get our graph.



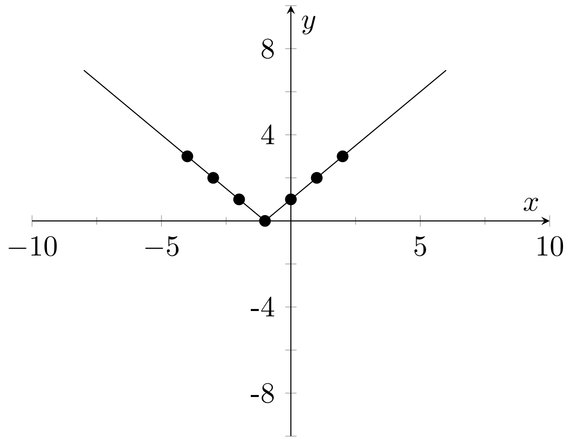
Note 1.2. We can use the same method to graph an equation with absolute values. Recall that absolute values makes the inside number positive.

Example 1.4. Graph the equation $y = |x+1|$. Select integers for x, starting with -4 and ending with 2.

We begin by creating a table of values.

x	y	
-4	3	$y = \|(-4) + 1\| = \|-3\| = 3$
-3	2	$y = \|(-3) + 1\| = \|-2\| = 2$
-2	1	$y = \|(-2) + 1\| = \|-1\| = 1$
-1	0	$y = \|(-1) + 1\| = \|0\| = 0$
0	1	$y = \|(0) + 1\| = \|1\| = 1$
1	2	$y = \|(1) + 1\| = \|2\| = 2$
2	3	$y = \|(2) + 1\| = \|3\| = 3$

Finally, plot all points and connect them to get the graph.



1.4.1 x-intercept and y-intercept

x-intercept and y-intercept are two important point on the line. The x-intercept is a point where a graph intersects the x-axis. These point are actually the roots or zeros of an equation.
Whereas the y-intercept is a point where a graph intersects the y-axis. This point help us a lot and make it easier for us to graph.

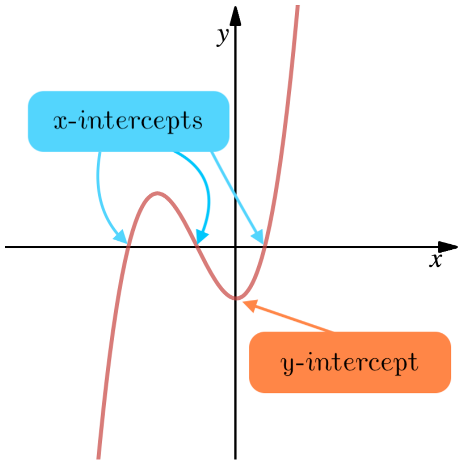
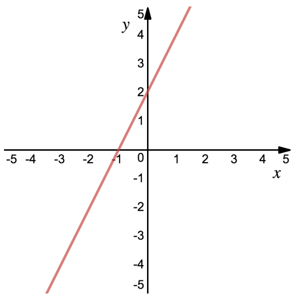


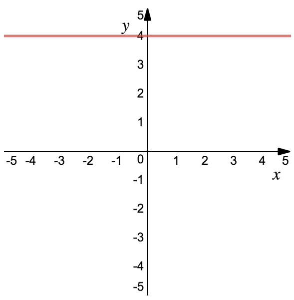
Figure 1.3: x- and y-intercepts.

Since x-intercept lies on x-axis, therefore its y-coordinate is zero. Likewise, y-intercept is on y-axis so its x-coordinate is zero.

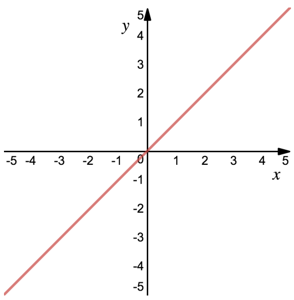
Example 1.5. Identify the x- and y-intercepts.



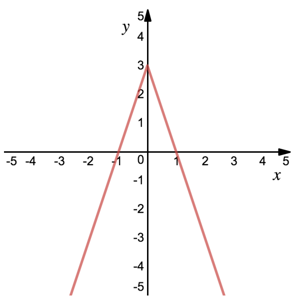
x-intercept: -1
y-intercept: 2



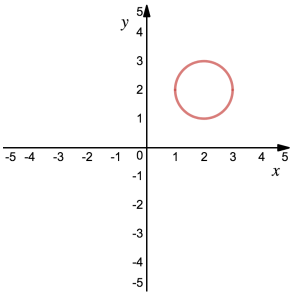
no x-intercept
y-intercept: 4



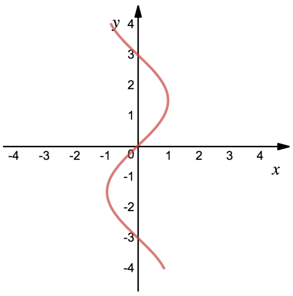
x-intercept: 0
y-intercept: 0



x-intercept: -1 and 1
y-intercept: 3



no x-intercept
no y-intercept



x-intercept: 0
y-intercept: -3, 0 and 3

Linear and Rational Equations

2.1 Linear equations in 1 variable

A linear equation in one variable x is an equation that can be expressed in the form

$$ax + b = 0 \tag{2.1}$$

where $a, b \in \mathbb{R}$ and $a \neq 0$.
x is our unknown and solving an equation in x involves determining all values of x that make our equation a true statement. Such values are solutions, or roots, of the equation.
An example of a linear equation is $5x + 10 = 0$. if we replace x with -2, we obtain

$$\begin{aligned} 5(-2) + 10 &= 0 \\ -10 + 10 &= 0 \quad \text{True statement } \checkmark \end{aligned}$$

Thus, $x = -2$ is a solution of the equation $5x + 10 = 0$. We also say that -3 satisfies the equation $5x + 10 = 0$. However, $x = 1$, for example, does not satisfy the equation

$$\begin{aligned} 5(1) + 10 &= 0 \\ 5 + 10 &= 0 \\ 15 &= 0 \quad \text{False statement } ✗ \end{aligned}$$

Therfore, $x = 1$ is not our solution.

2.1.1 Properties of equality

Two equations with exactly the same solutions are called equivalent equations.
To solve an equation, we try to simplify our equations in a way that the variable stands alone on one side of the equal sign. Here are the properties that we use an equations. In these properties A, B, and C stand for any algebraic expressions, and the symbol $\iff$ mean "is equivalent to".

Properties of Equality

Property	Description
$A = B \iff A + C = B + C$ $A = B \iff A - C = B - C$	Adding or subtracting the same quantity to both sides of an equation gives an equivalent equations.
$A = B \iff CA = CB$ $A = B \iff \dfrac{A}{C} = \dfrac{B}{C}$	Multiplying or dividing both sides of an equation by the same nonzero quantity gives an equivalent equations.

Note 2.1. These properties require that you perform the same operation on both sides of an equation when solving it.

Note 2.2. LHS is just a short way of saying the left-hand side of an equation. Likewise, RHS stands for "right-hand side" of the original equations.

2.2 Solving a linear equation

To solve equations, the general rule is to *isolate* the variable; In other words, changing the equation to an equivalent equation with all terms that have variable x on one side and all constant terms on the other.

Steps to solve an equation

(1) **Simplify** each side separately by clearing parentheses, using the distributive property as needed, and combining like terms.

(2) **Move** all terms with variables on one side and all numbers (constant terms) on the other.

(3) **Isolate** the variable.

(4) **Check** your answer.

We often express our final answer in set notation, called the solution set.

Example 2.1. Solve and check $4x + 5 = 29$.

$$
\begin{aligned}
4x + 5 &= 29 && \text{Each side's been already simplified} \\
4x + 5 - 5 &= 29 - 5 && \text{Subtract 5 from both sides}
\end{aligned}
$$

$$\begin{aligned} 4x &= 24 && \text{Divide both sides by 4} \\ x &= 6 && \text{Our answer} \end{aligned}$$

check. Replace x with 6 in the original equations

$$\begin{aligned} 4 \cdot 6 + 5 &\stackrel{?}{=} 29 \\ 24 + 5 &\stackrel{?}{=} 29 \\ 29 &= 29 \checkmark \end{aligned}$$

The solution set is $\{6\}$

Example 2.2. Solve and check $4(2x+1) = 29 + 3(2x-5)$.

$$\begin{aligned} 4(2x+1) &= 29 + 3(2x-5) && \text{Simplify using distributive} \\ 8x + 4 &= 29 + 6x - 15 && \text{Combine like terms} \\ 8x + 4 &= 6x + 14 && \text{Subtract 6x from both sides} \\ 8x + 4 - 6x &= 6x + 14 - 6x && \text{Simplify} \\ 2x + 4 &= 14 && \text{Subtract 4 from both sides} \\ 2x + 4 - 4 &= 14 - 4 && \text{Simplify} \\ 2x &= 10 && \text{Divide both sides by 2} \\ x &= 5 && \text{Our answer} \end{aligned}$$

check. Substitute $x = 5$ in the original equations

$$\begin{aligned} 4(2(5)+1) &\stackrel{?}{=} 29 + 3(2(5)-5) \\ 44 &\stackrel{?}{=} 29 + 15 \\ 44 &= 44 \checkmark \end{aligned}$$

The solution set is $\{5\}$.

Note 2.3. Sometimes we have fraction in our equation. In this case, we can get rid of all fraction by multiplying both sides by LCD.

Example 2.3. Solve and check $\dfrac{x-3}{4} = \dfrac{5}{14} - \dfrac{x+5}{7}$

$$\begin{aligned} \frac{x-3}{4} &= \frac{5}{14} - \frac{x+5}{7} && \text{The LCD of 4,14 and 7 is 28} \\ 28\left(\frac{x-3}{4} = \frac{5}{14} - \frac{x+5}{7}\right) & && \text{Simplify} \end{aligned}$$

$$\begin{aligned} 7(x-3) &= 2(5) - 4(x+5) && \text{Use distributive property} \\ 7x - 21 &= 10 - 4x - 20 && \text{Combine like terms} \\ 7x - 21 &= -4x - 10 && \text{Add 4x to both sides} \\ 7x - 21 &= -10 && \text{Add 21 to both sides} \\ 11x &= 11 && \text{Divide both sides by 11} \\ x &= 1 && \text{Our solution} \end{aligned}$$

check. Substitute 1 for x in the original equation. You should obtain $-1/2 = -1/2$. This true statement confirms that the solution set is $\{1\}$.

2.3 Rational equations

A rational equation is an equation containing one or more rational expressions, such as

$$\frac{1}{x} = \frac{1}{6} + \frac{5}{7x}$$

Although rational equations are not linear, it can simplify to one when we multiply by the LCD.

Example 2.4. Solve $\dfrac{5}{2x} = \dfrac{17}{18} - \dfrac{1}{3x}$

The LCD of $2x$, 18 and $3x$ is $18x$.

$$\begin{aligned} \frac{5}{2x} &= \frac{17}{18} - \frac{1}{3x} && \text{Multiply both sides by LCD} \\ {\color{red}18x}{\color{red}\left(}\frac{5}{2x} &= \frac{17}{18} - \frac{1}{3x}{\color{red}\right)} && \text{Distribute} \\ 9(5) &= x(17) - 6(1) && \text{Add 6 to both sides} \\ 45{\color{red}+6} &= 17x - 6{\color{red}+6} && \text{Simplify} \\ 51 &= 17x && \text{Divide both side by 17} \\ 3 &= x && \text{Our solution} \end{aligned}$$

check. Substitute $x = 3$. We'll get a true statement $5/6 = 5/6$. Therefore, the solution set is $\{3\}$.

Note 2.4. Dividing any number by 0 cause a problem because such a division is not defined. Thus, When you are checking the solution, those answers which makes the denominator 0 are not acceptable.

Example 2.5. Solve $\dfrac{x}{x-2} = \dfrac{2}{x-2} - \dfrac{2}{3}$.

We begin by finding the LCD of all fractions. The LCD of $x - 2$ and 3 is $3(x - 2)$. Now we multiply both sides by LCD

$$\begin{aligned} {\color{red}3(x-2)}{\color{red}\left(}\frac{x}{x-2} &= \frac{2}{x-2} - \frac{2}{3}{\color{red}\right)} && \text{Distribute} \end{aligned}$$

$$\begin{aligned} 3(x) &= 3(2) - (x-2)(2) && \text{Simplify} \\ 3x &= 6 - 2x + 4 && \text{Combine like terms} \\ 3x &= -2x + 10 && \text{Add } 2x \text{ to both sides} \\ 5x &= 10 && \text{Divide both sides by 5} \\ x &= 2 && \text{Our answer} \end{aligned}$$

check. The most important part is checking. When we plug $x = 2$ into the rational equations, we get

$$\frac{x}{2-2} = \frac{2}{2-2} - \frac{2}{3}$$
$$\frac{x}{0} = \frac{2}{0} - \frac{2}{3}$$

Since the denominator became zero, therefore $x = 2$ is not our answer. Thus, we don't have any solution and our solution set is ϕ, the empty set.

Example 2.6. Solve $\dfrac{1}{x+4} + \dfrac{1}{x-4} = \dfrac{22}{x^2-16}$

To find the LCD, we first need to factor $x^2 - 16$. This denominator can be factored using difference of squares formula,

$$x^2 - 16 = (x-4)(x+4)$$

Therefore the LCD of all denominators is $(x-4)(x+4)$. So multiply LCD by both sides

$$\begin{aligned} (x-4)(x+4)\left(\frac{1}{x+4} + \frac{1}{x-4} = \frac{22}{(x-4)(x+4)}\right) && \text{Distribute} \\ (x-4) + (x+4) = 22 && \text{Combine like terms} \\ 2x = 22 && \text{Divide both sides by 2} \\ x = 11 && \text{Our answer} \end{aligned}$$

check. we need to check our solution.$x = 11$ does not make any denominater 0 and we'll get a true statement $22/105 = 22/105$. Thus, $x = 11$ is our answer and the solution set is $\{11\}$.

2.4 Types of equations

Equations are classified based on their solution sets.

1. **Identity**: An equation that is true for all real numbers. For example $x + 3 = x + 2 + 1$. When solving this type of equations, we'll reach to a true statement $0 = 0$. This indicate that all real numbers can be our answer. The solution set can be expressed as

$$\{x \mid x \in \mathbb{R}\}$$

2. **Conditional equation**: An equation that is not identity, but it is valid for at least one real number. This type of equations that we were solving so far, such as $2x + 3 = 19$.

3. **Inconsistent equation**: An equation that is not true for any real numbers. Often when we are solving this type of equations, we reached to a false statement. For instance, $x = x + 7$ has no solution, because we end up with $0 = 7$ while solving this equation. In this case, the solution set is ϕ

Example 2.7. Solve and determine whether the equation

$$4x - 7 = 4(x - 1) + 3$$

$$\begin{aligned}
4x - 7 &= 4(x - 1) + 3 && \text{Apply distributive property} \\
4x - 7 &= 4x - 4 + 3 && \text{Combine like terms on RHS} \\
4x - 7 &= 4x - 1 && \text{Subtract } 4x \text{ from both sides} \\
-7 &= -1 && \text{False statement } ✗
\end{aligned}$$

Because we got a false statement, this equation is inconsistent equation. Therefore, we have no solution and the solution set is ϕ.

Example 2.8. Solve and determine whether the equation

$$7x + 9 = 9(x + 1) - 2x$$

$$\begin{aligned}
7x + 9 &= 9(x + 1) - 2x && \text{Apply distributive property} \\
7x + 9 &= 9x + 9 - 2x && \text{Combine like terms on RHS} \\
7x + 9 &= 7x + 9 && \text{Subtract } 7x \text{ from both sides} \\
9 &= 9 && \text{True statement}
\end{aligned}$$

Here, we got a true statement which means that this equation is identity equation. Therefore, all real numbers are our solution and the solution set is $\{x \mid x \in \mathbb{R}\}$.

Models and Applications

3.1 Word problems

Word problems can be tricky. Often it takes a bit of practice to translate the English sentence into a mathematical sentence. This is what we will focus on here with some basic number problems, and simple interest problem.
A few important phrases are described below that can give us clues for how to set up a problem.

Table 3.1: Conversion from English to math

Words	*Interpretation in Mathematics*
Is, are, was, will be, has, ...	Equal sign $=$
Of, product, multiplied by, times	Multiplication $\times$
Divided by, per, each, out of, ratio	Division $\div$
Plus, added to, and, sum, combined	Addition $+$
Minus, fewer, decreased by, difference, subtracted from	Subtraction $-$
More than	Often represents addition and is usually built backwards, writing the second part plus the first.
Less than	Often represents subtraction and is usually built backwards as well, writing the second part minus the first.

Using these key phrases we can take a number problem and set up and equation and solve.

Example 3.1. If 25 less than five times a certain number is 235. What is the number?

Let x be the unknown number. Less than built backwards. we will have

$$5\,x - 25 = 235$$

five times less than is

Now we can solve for x:

$$\begin{aligned} 5x - 25 &= 235 && \text{Add 25 to both sides} \\ 5x &= 260 && \text{Divide by 5} \\ x &= 52 && \text{The number is 52} \end{aligned}$$

Example 3.2. The average yearly salary of a woman with a bachelor's degree exceeds that of a woman with an associate's degree by \$14 thousand. The average yearly salary of a woman with a master's degree exceeds that of a woman with an associate's degree by \$26. Combined, three woman with each of these educational attainments earn \$139 thousand. Find the average yearly salary of woman with each of these levels of education.

With no information about a woman with associate's degree, we call it x. Therefore

$$\begin{aligned} &x && \text{A woman with associate's degree} \\ &x + 14 && \text{A woman with bachelor's degree} \\ &x + 26 && \text{A woman with master's degree} \end{aligned}$$

We know three women combined earn \$319, thus

$$\begin{aligned} x + (x + 14) + (x + 26) &= 139 && \text{This is our equation} \\ 3x + 40 &= 139 && \text{Subtract 40 from both sides} \\ 3x &= 99 && \text{Divide both sides by 3} \\ x &= 33 && \text{Associate's degree} \end{aligned}$$

Thus, a bachelor will earn $x + 14 = 33 + 14 = 47$ and one with master's degree will earn $x + 26 = 33 + 26 = 59$.

3.2 Simple interest

When you borrow money from a bank or when a bank "borrows" your money by keeping it for you in a savings account, the borrower in each case must pay for the privilege of using the money. The fee that is paid is called interest. The most basic type of interest is simple interest, which is just an annual percentage of the total amount borrowed or deposited.
The amount of a loan or deposit is called the **principal** P. The annual percentage paid for the use of this money is the **interest rate** r. We will use the variable t to stand for the number of years that the money is on deposit and the variable I to stand for the total interest earned.

Simple Interest

The following simple interest formula gives the amount of interest I earned:

$$I = Prt \tag{3.1}$$

where,

- P: principal (money deposited)
- t: years
- r: interest rate (as a decimal)

Note 3.1. When using this formula, remember to convert r from a percent to a decimal. For instance, in decimal form, 6% is $\frac{6}{100} = 0.06$.

Example 3.3. You inherited \$5000 with the stipulation that for the first year the money had to be invested in two funds paying 9% and 11% annual interest. How much did you invest at each rate if the total interest earned for the year was \$487?

The problem asks for the amount she has invested at each rate. So we let

$$x = \text{the amount invested at } 9\%$$

Therefore, the remaining money invested at 11% is

$$5000 - x = \text{the amount invested at } 11\%$$

The interest for each rate after 1 year can be found using the simple interest formula (3.1)

at 9% rate	at 11% rate
$I = x(0.09)(1)$	$I = (5000 - x)(0.11)(1)$
$= 0.09x$	$= 550 - 0.11x$

Since the total interest was \$487 so

$$\begin{aligned} \text{interest at } 9\% + \text{interest at } 11\% &= \text{total interest} \\ 0.09x + (550 - 0.11x) &= 487 \end{aligned}$$

Now solve for x

$$\begin{aligned} 0.09x + (550 - 0.11x) &= 487 && \text{Combine like terms} \\ 550 - 0.02x &= 487 && \text{Subtract 550 from both sides} \\ -0.02x &= -63 && \text{Divide both sides by } -0.02 \\ x &= 3150 && \text{Invested at } 9\% \end{aligned}$$

For 11%, we invested $5000 - x = 5000 - 3150 = 1850$.

3.3 Solving a formula for on of its variable

Solving formulas is like solving general linear equations except we will have several variables in the problem and we will be attempting to solve for one specific variable.
For example, let's consider *Albert Einstein*'s famous formula $E = mc^2$. We may be interested in solving for the variable m. This means we want to isolate the m so the equation has m on one side, and everything else on the other. So the solution looks like this $m = E/c^2$.
When solving formulas for a variable, we need to focus on the one variable we are trying to solve for, all the others are treated just like numbers.

Example 3.4. Solve the formula $P = 2l + 2w$ for w.

We must isolate w

$$\begin{aligned} P &= 2l + 2w && \text{Subtract } 2l \text{ from both sides} \\ P - 2l &= 2w && \text{Divide both sides by } w \\ \frac{P-2l}{2} &= w && \text{Our answer} \end{aligned}$$

Example 3.5. Solve the formula $P = C + MC$ for C.

We need to isolate C

$$\begin{aligned} P &= C + MC && \text{Factor out } C \text{ on RHS} \\ P &= C(1+M) && \text{Divide both sides by } (1+M) \\ \frac{P}{1+M} &= C && \text{Our answer} \end{aligned}$$

Example 3.6. Solve the formula $\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$ for f.

Whenever you see a fraction, the first thing comes to your mind should be LCD. The LCD of p, q and f is (pqf).

$$\begin{aligned} \frac{1}{p} + \frac{1}{q} &= \frac{1}{f} && \text{Multiply both sides by LCD} \\ {\color{red}pqf\left(\frac{1}{p} + \frac{1}{q} = \frac{1}{f}\right)} & && \text{Distribute} \\ \cancel{p}qf\left(\frac{1}{\cancel{p}}\right) + p\cancel{q}f\left(\frac{1}{\cancel{q}}\right) &= pq\cancel{f}\left(\frac{1}{\cancel{f}}\right) && \text{Simplify} \\ qf + pf &= pq && \text{Factor out } f \text{ on LHS} \\ f(q+p) &= pq && \text{Divide both sides by } q+p \\ f &= \frac{pq}{q+p} && \text{Our answer} \end{aligned}$$

Complex Numbers

4.1 Imaginary unit

When mathematics was first used, the primary purpose was for counting. Thus they did not originally use negatives, zero, fractions or irrational numbers. However, the ancient Egyptians quickly developed the need for “a part” and so they made up a new type of number, the ratio or fraction. The Ancient Greeks did not believe in irrational numbers (people were killed for believing otherwise). The Mayans of Central America later made up the number zero when they found use for it as a placeholder. Ancient Chinese Mathematicians made up negative numbers when they found use for them. The square roots of a negative number created enough of problem to stop many mathematician in their tracks. Square roots ask us to find a number, that when multiplied by itself, yield the number inside the root sign. The square root of nine is three because three times three is nine.

$$\sqrt{9} = 3$$

But what about roots of negative numbers? What is the square root of negative nine? Positive three won't work, and neither will negative three, so we're stuck.

$$\sqrt{-9} = ?$$

Usually mathematicians would interpret this as the problem's way of saying there are no solutions. Like 0 and negative numbers before, $\sqrt{-1}$ was generally regarded with suspicion because it did not correspond to anything people could think of it in the real world. For this reason, $\sqrt{-1}$ was given the terrible names ”imaginary” or ”impossible”.

A century or so later, Euler begin using the simple i to indicate that $\sqrt{-1}$ exist, making the algebra less clunky. Unfortunately, the name imaginary stuck around and that's still we call these numbers today. In response, everything on the original number line gets the name ”real”.

Imaginary unit i

$$i = \sqrt{-1}, \text{ thus } \quad i^2 = -1 \tag{4.1}$$

Using this notation, we can express any negative radicand in terms of i

$$\sqrt{-a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$$

Principle square root of a negative number

$$\sqrt{-a} = i\sqrt{a} \tag{4.2}$$

Example 4.1.

$$\begin{aligned}
&a)\sqrt{-49} = \sqrt{-1}\sqrt{49} = i\sqrt{49} = 7i\\
&b)\sqrt{-31} = \sqrt{-1}\sqrt{31} = i\sqrt{31}\\
&c)\sqrt{-98} = \sqrt{-1}\sqrt{98} = i\sqrt{98} = i\sqrt{2\cdot 7^2} = 7i\sqrt{2}
\end{aligned}$$

Note 4.1. We cannot join two square roots using product rule, $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$, when a and b are both negative numbers. To multiply negative radicands, first use the definition of imaginary number to take out i. Then multiply the radical with positive radicands. For example:

$$\begin{aligned}
\sqrt{-8}\cdot\sqrt{-2} = i\sqrt{8}\cdot i\sqrt{2} &= i^2\sqrt{16}\\
&= (-1)\sqrt{4^2}\\
&= -4
\end{aligned}$$

4.2 Complex Numbers

When we put together a real number and imaginary number, we obtain what we call a complex number.

$$\underbrace{a}_{\text{Real Part}} + \underbrace{bi}_{\text{Imaginary Part}}$$

Complex numbers

A complex number is a number that can be written in the form $a+bi$, where a, b are real numbers. a is called the **real part** and bi is called the **imaginary part**.

Examples of complex numbers include

$$\begin{array}{cc}
3i & -10i\\
4+6i & -1-2\sqrt{5}i\\
\vdots & \vdots
\end{array}$$

4.2.1 Equality of complex numbers

Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$; In other words, their real parts and their imaginary parts should be equal to each other.

Example 4.2. Find the real numbers if $-7 + 2yi\sqrt{3} = x + 6i\sqrt{3}$

$$\begin{aligned} \textit{Real Parts:} \quad & -7 = x \checkmark \\ \textit{Imaginary Parts:} \quad & 2y\sqrt{3} = 6\sqrt{3} \longrightarrow y = 3 \checkmark \end{aligned}$$

Note 4.2. When performing operations (add, subtract, multilpy, divide) we can handle i just like we handle any other variable. This means when adding and subtracting complex numbers we simply add or combine like terms.

Example 4.3. Simplify the following expression: $(3 - 4i) - (-2 - 18i)$

Be careful with negatives!

$$\begin{aligned} (3 - 4i) - (-2 - 18i) \quad & \text{Distribute -1} \\ 3 - 4i + 2 + 18i \quad & \text{Combine like terms} \\ 5 + 14i \quad & \text{Our Solution} \end{aligned}$$

Note 4.3. Multiplying with complex numbers is the same as multiplying with variables with one exception, we will want to simplify our final answer so there are no exponents on i.

Example 4.4. Multiply: $(3i)(7i)$

$$\begin{aligned} (3i)(7i) \quad & \text{Multiply coefficients and } i \\ 21i^2 \quad & \text{Simplify, } i^2 = -1 \\ 21(-1) \quad & \text{Multiply} \\ -21 \quad & \text{Our Solution} \end{aligned}$$

Example 4.5. Multiply: $(5i)(3i - 2)$

$$\begin{aligned} (5i)(3i - 2) \quad & \text{Distribute } 5i \text{ by } 3i - 2 \\ 15i^2 - 10i \quad & \text{Simplify, } i^2 = -1 \\ 15(-1) - 10i \quad & \text{Multiply} \\ -15 - 10i \quad & \text{Our Solution} \end{aligned}$$

Example 4.6. Use the FOIL method to multiply: $(4-2i)(3-7i)$

$$\begin{aligned}
(4-2i)(3-7i) &\quad \text{FOIL} \\
12-28i-6i+14(i^2) &\quad \text{Simplify, } i^2=-1 \\
12-28i-6i+14(-1) &\quad \text{Multiply} \\
12-28i-6i-14 &\quad \text{Combine like terms} \\
-2-34i &\quad \text{Our Solution}
\end{aligned}$$

4.3 Conjugates

Imagine we have a complex number $a+bi$. Then a complex number obtained by changing the sign of imaginary part of the complex number is called the conjugate of $a+bi$.
Conjugates are very important. If we multiply $a+bi$ by its conjugate $a-bi$, we obtain a real number.

$$\begin{aligned}
(a+bi)(a-bi) &= (a)(a)+(a)(-bi)+(bi)(a)+(bi)(-bi) \\
&= a^2 - \cancel{abi} + \cancel{abi} - b^2i^2 \quad (\text{we know} \ \ i^2=-1) \\
&= a^2 - b^2(-1) = a^2+b^2
\end{aligned}$$

Conjugate of a complex number

The complex conjugate of the number $a+bi$ is $a-bi$, and the complex conjugate of $a-bi$ is $a+bi$. The multiplication of complex conjugates yields a real number

$$(a+bi)(a-bi) = a^2+b^2$$

4.3.1 Rationalizing complex numbers

It is considered a bad practice o have a imaginary number in the denominator of a fraction. Because the goal of the division procedure is to obtain a real number in the number. To rationalize a fraction, or write it in a standard form, you should multiply the numerator and denominator of division by the complex conjugate of the denominator.

Example 4.7. Divide and then write it the following expression in the standard form:

$$\frac{4+2i}{3+4i}$$

$$
\begin{aligned}
\frac{4+2i}{3+4i} &\quad \text{Multiply by conjugate of denominator} \\
\frac{4+2i}{3+4i}\left(\frac{3-4i}{3-4i}\right) &\quad \text{Multiply} \\
\frac{12-16i+6i-8i^2}{9+16} &\quad i^2=-1 \\
\frac{12-16i+6i+8}{9+16} &\quad \text{Combine like terms in numerator} \\
\frac{20-10i}{25} &\quad \text{Divide numerator by 25} \\
\frac{20}{25}-\frac{10}{25}i &\quad \text{Simplify} \\
\frac{4}{5}-\frac{2}{5}i &\quad \text{Our solution in the standard form}
\end{aligned}
$$

Example 4.8. Divide and write it the following expression in the standard form:

$$\frac{5i}{7+i}$$

$$
\begin{aligned}
\frac{5i}{7+i} &\quad \text{Multiply by conjugate of denominator} \\
\frac{5i}{7+i}\left(\frac{7-i}{7-i}\right) &\quad \text{FOIL} \\
\frac{35i-5i^2}{49+1} &\quad i^2=-1 \\
\frac{5-35i}{50} &\quad \text{Divide the numerator by 50} \\
\frac{5}{50}-\frac{35}{50}i &\quad \text{Simplify} \\
\frac{1}{10}-\frac{7}{10}i &\quad \text{Our solution in the standard form}
\end{aligned}
$$

Quadratic Equations

The equation in the form $ax^2 + bx + c = 0$ is called quadratic equations $(a \neq 0)$. There are several methods to solve quadratic equations:

- Factoring
- Square Root Property
- Completing the square
- Quadratic Formula

5.1 Zero-product principle

If an equation can be factored, the zero-product principle can be used to find its roots (zeros). This principle states that when the product of two factors equals zero, then at least one of the factors is zero

Zero-product principle

$$\text{if } a \cdot b = 0 \longrightarrow a = 0 \text{ or } b = 0. \tag{5.1}$$

For example, consider the equation $(x + 4)(x - 3) = 0$. Based on the zero-product principle, we can set each factor equal to zero and solve for x.

$$\begin{aligned} x + 4 = 0 &\longrightarrow x = -4 \\ x - 3 = 0 &\longrightarrow x = 3 \end{aligned}$$

Therefore the solution set is $\{-4, 3\}$.

Note 5.1. Be very careful! We can use the zero-product principle if and only if one side of equation is 0. Thus, if the original equation is not set to zero, then first make one side of the equation 0 and then use this principle.

5.2 Factoring

In this method, we factor the quadratic equations completely. Then we apply the zero-factor property principle, setting each factor equal to zero to find the solutions.

Example 5.1. Solve by factoring $3x^2 - 9x = 0$.

$$\begin{aligned} 3x^2 - 9x &= 0 && \text{Factor out} \\ 3x(x-9) &= 0 && \text{Set each factor to zero} \\ 3x = 0 \longrightarrow x &= 0 \\ x - 9 = 0 \longrightarrow x &= 9 \end{aligned}$$

The solution set is $\{0, 9\}$.

Example 5.2. Solve by factoring $2x^2 + x = 1$.

Begin by setting the equation equal to 0.

$$\begin{aligned} 2x^2 + x &= 1 && \text{Subtract 1} \\ 2x^2 + x - 1 &= 0 && \text{Factor} \\ (2x-1)(x+1) &= 0 && \text{Use zero-product principle} \\ 2x - 1 = 0 \longrightarrow x &= \frac{1}{2} \\ x + 1 = 0 \longrightarrow x &= -1 \end{aligned}$$

Thus, the solution set is $\{-1, 1/2\}$.

5.3 Square root property

As you might expect, we can clear squares by using square root. When we are taking square root we have two results: one will be positive, the other will be negative. Let's say we are asked to solve

$$x^2 = 25$$

One of the solution is 5, because $(5)(5) = 25$. The other one is -5, because $(-5)(-5) = 25$.

The square root property

If $x^2 = a$, then $x = \pm\sqrt{a}$ for all real numbers a.

Therefore, if we have square on one side and a number on the other side,we can take a square root from both sides of that equation. Don't forget, taking square root create two answers: one is positive and the other one is negative. In other words, you must always need to add $\pm$ signs.

Example 5.3. Solve using the square root property: $x^2 - 121 = 0$

$x^2 - 121 = 0$	Isolate part with exponent
$x^2 = 121$	Take square root form both sides
$\left(\sqrt{x^2}\right) = \pm\sqrt{121}$	Simplify
$x = \pm 11$	Our Solution
$\{-11, 11\}$ or $\{\pm 11\}$	The solution set

Example 5.4. Solve using the square root property: $x^2 = 18$

$x^2 = 18$	Take square root form both sides
$\left(\sqrt{x^2}\right) = \pm\sqrt{18}$	Simplify, 18 is $2 \cdot 3^2$
$x = \pm\sqrt{2 \cdot 3^2}$	Take out 3
$x = \pm 3\sqrt{2}$	Our Solution
$\left\{-3\sqrt{2}, 3\sqrt{2}\right\}$ or $\left\{\pm 3\sqrt{2}\right\}$	The solution set

Example 5.5. Solve using the following quadratic equation: $5x^2 + 1 = 46$

First, we need to isolate the part with the exponent.

$5x^2 + 1 = 46$	Subtract 1
$5x^2 = 45$	Divide by 5
$x^2 = 9$	Take square root
$\left(\sqrt{x^2}\right) = \pm\sqrt{9}$	Simplify
$x = \pm 3$	Our Solution
$\{-3, 3\}$ or $\{\pm 3\}$	The solution set

Note 5.2. Sometimes we might obtain roots that are complex numbers.

Example 5.6. Solve using the following quadratic equation: $3x^2 = -27$

As always, isolate the x^2 first

$3x^2 = -27$	Divide by 3

$$
\begin{aligned}
\left(\sqrt{x^2}\right) &= \pm\sqrt{-9} && \text{Take square root}\\
x &= \pm\sqrt{-9} && \text{Simplify, } \sqrt{-1} = i\\
x &= \pm 3i && \text{Our Solution}\\
\{-3i,\, 3i\} \text{ or } \{\pm 3i\} & && \text{The solution set}
\end{aligned}
$$

Example 5.7. Solve the quadratic equation using the square root property: $(2x+3)^2 = 7$

$$
\begin{aligned}
(2x+3)^2 &= 7 && \text{Take square root}\\
\left(\sqrt{(2x+3)^2}\right) &= \pm\sqrt{7} && \text{Simplify}\\
2x+3 &= \pm\sqrt{7} && \text{Solve for } x \text{, subtract 3}\\
2x &= -3 \pm \sqrt{7} && \text{Divide both sides by 2}\\
x &= \frac{-3 \pm \sqrt{7}}{2} && \text{Our Solution}\\
\left\{\frac{-3}{2} \pm \frac{\sqrt{7}}{2}\right\} & && \text{The solution set}
\end{aligned}
$$

5.4 Completing the square

Sometimes we have an equations that cannot be factored. Consider the following equation:

$$x^2 - 2x - 7 = 0$$

The equation cannot be factored, however there are two solutions to this equation,

$$1 + 2\sqrt{2} \text{ and } 1 - 2\sqrt{2}$$

To find these two solutions we will use a method known as completing the square.

Once we use completing the square method, we will change the quadratic into a perfect square which can easily be solved with the square root property. Although we might not use this method frequently, it is actually helpful to prove a lot of useful formula such quadratic formula, equation of ellipse, hyperbole and so forth.

The following five steps describe the process used to complete the square.

Completing the square

1. If coefficient of x^2 is not 1, divide both sides by its coefficient. Otherwise, skip this step.
2. Move the constant term to right-hand side.
3. Take the coefficient of x, divide it by 2, then square it. Add the result to both left and right-hand side.
4. Factor the left-hand side. You will have a perfect square.
5. Solve by the square root property.

Example 5.8. Solve $2x^2 + 20x + 48 = 0$ using the completing the square method.

First, we divide both sides by the coefficient of x^2, which is 2.

$$\begin{aligned}\frac{1}{2}\left(2x^2 + 20x + 48\right) &= \frac{1}{2}(0)\\ x^2 + 10x + 24 &= 0\end{aligned}$$

Next step, subtract 24 from both sides

$$x^2 + 10x = -24$$

Now the coefficient of x is 10, so we must add $\left(\frac{1}{2}(10)\right)^2$ to both sides,

$$\begin{aligned}x^2 + 10x + \left(\frac{1}{2}10\right)^2 &= -24 + \left(\frac{1}{2}(10)\right)^2\\ x^2 + 10x + (5)^2 &= -24 + (5)^2\\ x^2 + 10x + 25 &= 1 && \text{Factor the left-hand side}\\ (x+5)^2 &= 1 && \text{Use the square root property}\\ x + 5 &= \pm\sqrt{1}\\ x &= -5 \pm 1 && \text{Our solution}\end{aligned}$$

Thus, our solutions are

$$\begin{cases} x = -5 + 1 = -4 \\ x = -5 - 1 = -6 \end{cases}$$

The solution set is $\{-6, -4\}$

Example 5.9. Solve $x^2 - 3x - 2 = 0$ using the completing the square method.

Coefficient of x^2 is 1, so skip the first step. Then move the constant term to the other side.

$$x^2 - 3x = 2$$

Coefficient of x is -3. So divide -3 by 2 and add square of it to both sides

$$\begin{aligned}
x^2 - 3x + \left(\frac{1}{2}(-3)\right)^2 &= 2 + \left(\frac{1}{2}(-3)\right)^2 \\
x^2 - 3x + \left(\frac{9}{4}\right) &= \frac{17}{4} \\
x^2 - 3x + \left(\frac{9}{4}\right) &= \frac{17}{4} && \text{Factor} \\
(x - \frac{3}{2})^2 &= \frac{17}{4} && \text{Use the square root property} \\
x - \frac{3}{2} &= \pm\sqrt{\frac{17}{4}} \\
x &= \frac{3}{2} \pm \sqrt{\frac{17}{4}}
\end{aligned}$$

The root $\sqrt{\frac{17}{4}}$ can be simplified to $\frac{\sqrt{17}}{2}$. So our solutions are

$$x = \begin{cases} \frac{3}{2} + \frac{\sqrt{17}}{2} = \frac{3+\sqrt{17}}{2} \\ \\ \frac{3}{2} - \frac{\sqrt{17}}{2} = \frac{3-\sqrt{17}}{2} \end{cases}$$

The solution set is $\left\{\frac{3}{2} + \frac{\sqrt{17}}{2}, \frac{3}{2} - \frac{\sqrt{17}}{2}\right\}$.

As several of the examples have shown, when solving by completing the square we will often need to use fractions and be comfortable finding common denominators and adding fractions together. Once we get comfortable solving by completing the square and using the five steps, any quadratic equation can be easily solved.

5.5 Quadratic formula

The general from of a quadratic is $ax^2+bx+c = 0$. By using the completing the square method , we can find a formula to solve any quadratic equations.

To begin, we divide both sides by a,

$$x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0$$

Then, move the constant term $\dfrac{c}{a}$ to the other side

$$x^2 + \left(\frac{b}{a}\right)x = -\frac{c}{a}$$

Add the square of half of the coefficient of x, i.e. $\left(\dfrac{b}{2a}\right)^2$, to both sides

$$x^2 + \left(\frac{b}{a}\right)x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

On the left-hand side, we have a perfect square,

$$\left(x + \frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

Now, simplify the right-hand sides, and then take square root from both sides

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a^2} - \frac{c}{a} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Next step, we subtract $\dfrac{b}{2a}$ from both sides, we can find x

$$\begin{aligned}
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \blacksquare
\end{aligned}$$

Using the above formula, we can find the zeros of any quadratic equations.

Quadratic Formula

The Roots of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5.2}$$

Note 5.3. As we are solving using the quadratic formula, it is important to remember the equation must first be equal to zero.

Example 5.10. Solve $x^2 + 3x = -2$.

First we need to set the equation equal to zero, so add 2 to both sides

$$x^2 + 3x + 2 = 0$$

By comparing the above equation with $ax^2 + bx + c = 0$, we get $a = 1$, $b = 3$ and $c = 2$. Substituting them into quadratic formula (5.2) yields

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\
x &= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(2)}}{2(1)} && \text{Substitute} \\
x &= \frac{-3 \pm \sqrt{9 - 8}}{2} && \text{Simplify} \\
x &= \frac{-3 \pm \sqrt{1}}{2} && \text{We know, } \sqrt{1} = 1 \\
x &= \frac{-3 \pm 1}{2} && \text{Simplify} \\
x &= -1 \text{ or } x = -2 && \text{Our solutions}
\end{aligned}$$

The solution set is $\{-2, -1\}$.

Example 5.11. Solve $x^2 = x + 3$.

$$\begin{aligned}
x^2 &= x + 3 && \text{Set the equation equal to 0} \\
x^2 - x - 3 &= 0 && a = 1,\ b - 1 \text{ and } c = -3 \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{Use the quadratic formula}
\end{aligned}$$

$$
\begin{aligned}
x &= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)} && \text{Simplify} \\
x &= \frac{1 \pm \sqrt{13}}{2} && \text{Our solutions}
\end{aligned}
$$

Often we need to break up the fraction. So our answers will be $\frac{1}{2} \pm \frac{\sqrt{13}}{2}$ and the solution set is $\{\frac{1}{2} - \frac{\sqrt{13}}{2}, \frac{1}{2} + \frac{\sqrt{13}}{2}\}$.

Example 5.12. Solve $3x^2 + 4x + 8 = 2x^2 + 6x - 5$.

First we need to set the quadratic equation equal to zero.

$$
\begin{aligned}
3x^2 + 4x + 8 &= 2x^2 + 6x - 5 && \text{Subtract } 2x^2 \\
x^2 + 4x + 8 &= 6x - 5 && \text{Subtract } 6x \\
x^2 - 2x + 8 &= -5 && \text{Add 5} \\
x^2 - 2x + 13 &= 0 && \text{The standard form}
\end{aligned}
$$

So $a = 1$, $b = -2$ and $c = 13$, thus

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(13)}}{2(1)} && \text{Substitute} \\
x &= \frac{2 \pm \sqrt{4 - 52}}{2} && \text{Simplify} \\
x &= \frac{2 \pm \sqrt{-48}}{2} && \sqrt{-48} = i\sqrt{48} \\
x &= \frac{2 \pm i\sqrt{48}}{2} && \text{48 is } 16 \cdot 3 \text{ (16 is a perfect square)} \\
x &= \frac{2 \pm i\sqrt{16 \cdot 3}}{2} && \text{3 will remain inside, take out 16} \\
x &= \frac{2 \pm 4i\sqrt{3}}{2} && \text{Reduce them by dividing by 2} \\
x &= 1 \pm 2i\sqrt{3} && \text{Our solutions}
\end{aligned}
$$

The solution set is $\left\{1 - 2i\sqrt{3}, 1 + 2i\sqrt{3}\right\}$.

Note 5.4. When we use the quadratic formula we don't necessarily get two unique answers. We can end up with only one solution if the square root simplifies to zero.

Example 5.13. Solve $4x^2 - 12x + 9 = 0$.

a, b and c are 4, -12 and 9, respectively. Substitute them into the quadratic formula (5.2), we will find

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\ x &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)} && \text{Substitute} \\ x &= \frac{12 \pm \sqrt{144 - 144}}{8} && \text{Simplify} \\ x &= \frac{12 \pm \sqrt{0}}{8} && \sqrt{0} \text{ is } 0 \\ x &= \frac{12}{8} && \text{Reduce by dividing by 4} \\ x &= \frac{3}{2} && \text{Our solution} \end{aligned}$$

The solution set is $\left\{\frac{3}{2}\right\}$.

Note 5.5. If a term is missing from the quadratic, we can still solve with the quadratic formula, we simply use zero for that term. The order is important, so if the term with x is missing, we have $b = 0$, if the constant term is missing, we have $c = 0$.

Example 5.14. Solve $3x^2 + 8 = 0$.

Since x is missing that means $b = 0$. a and c are 3 and 8, respectively. Plug them into the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} && \text{The quadratic formula} \\ x &= \frac{-(0) \pm \sqrt{(0)^2 - 4(3)(8)}}{2(3)} && \text{Substitute} \\ x &= \frac{\pm\sqrt{0 - 96}}{6} && \text{Simplify} \\ x &= \frac{\pm\sqrt{-96}}{6} && 96 \text{ is } 16 \cdot 6 \text{ (16 is a perfect square)} \\ x &= \frac{\pm\sqrt{-16 \cdot 6}}{8} && \text{6 will remain inside, take our 16} \\ x &= \frac{\pm 4i\sqrt{6}}{8} && \text{Reduce by dividing by 4} \\ x &= \frac{\pm i\sqrt{6}}{2} && \text{Our Solution} \end{aligned}$$

The solution set is $\left\{ -\frac{i\sqrt{6}}{2}, \frac{i\sqrt{6}}{2} \right\}$.

5.5.1 Discriminant

The square root of $b^2 - 4ac$ in the quadratic formula play an important role in how our zeros will look like. Often we denote it as $\Delta = b^2 - 4ac$ and is called discriminant. So we can rewrite the quadratic formula (5.2) as

$$x = \frac{-b \pm \sqrt{\Delta}}{2a}$$

If $\Delta = 0$, then we will have

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b}{2a}$$

In this case, we will have only one solution. This unique solution is called the *double root*.
When $\Delta > 0$, then $\sqrt{\Delta}$ will also be a positive real number. Thus, we will get two *distinct real zeros*. However, when $\Delta < 0$, $\sqrt{\Delta}$ will be an imaginary number. In this case, we will get *two complex numbers*.

Discriminant Δ

The discriminant of a quadratic function ,$ax^2 + bx + c = 0$ is defined as $\Delta = b^2 - 4ac$. There are 3 possibilities as follows:

Discriminant	Roots
$\Delta = 0$	1 root (double root)
$\Delta < 0$	2 complex roots
$\Delta > 0$	2 real roots

Moreover, we can predict more when Δ is positive.

- If $\Delta > 0$ and Δ is a perfect square, then our two different solutions are rational. Such an equation can be solves by factoring.
- If $\Delta > 0$ and Δ is not a perfect square, our two different solutions are irrational.

Example 5.15. Use the discriminant to determine what the type of solutions each of the following equations has.

(a) $x^2 - 4x + 13 = 0$

(b) $9x^2 + 6x - 7 = 0$

(c) $9x^2 + 12x + 4 = 0$

(a) $a = 1$, $b = -4$ and $c = 13$, so $\Delta = (-4)^2 - 4(1)(13) = -36$. Since $\Delta < 0$ we have two complex solutions.

(b) Here we have $a = 9$, $b = 6$ and $c = -7$, so $\Delta = (6)^2 - 4(9)(-7) = 288$. Because $\Delta > 0$ and it is not a perfect square, so we have two different irrational solutions.

(c) We have $a = 9$, $b = 12$ and $c = 4$, so $\Delta = (12)^2 - 4(9)(4) = 0$. Therefore, we have a double root (one solution).

Other type of equations

Up until now, we learn how to solve linear equations, rational equations and quadratic equations. Rational equations and quadratic equations are two examples of non-linear equations. In this section, we will learn to solve other types of non-linear equations

1. Polynomial equations of degree 3 or higher.
2. Radical equations.
3. Equations with rational exponents.
4. Equations that are quadratic in form.
5. Absolute value equations.

6.1 Polynomial equations of degree 3 or higher

When a polynomial sets to zero, we have a polynomial equations. Examples are $x^3 + 2x - 3 = 0$, $-x^6 + x^4 + x^2 = 0$, and $5x^4 - 7x^3 = 0$.
As you might remember from chapter 0, the degree of a polynomial is the highest exponent of the variable.
If the degree is 1, we have a linear equations such as $2x+8 = 0$. If the degree is 2, then we get a quadratic equations, for example $-7x^2 - 8x + 3 = 0$. In this section, we will focus on solving polynomial equations of degree 3 or higher.

Solving a polynomial equations by factoring

1. **Move** all terms to one side, thereby obtaining 0 on the other side
2. **Factor out** completely.
3. Use **zero-product principle**.

Example 6.1. Solve by factoring $4x^4 = 12x^2$.

$$\begin{aligned} 4x^4 &= 12x^2 && \text{Set equation equal to 0, Subtract } 12x^2 \\ 4x^4 - 12x^2 &= 0 && \text{Common factor is } 4x^2\text{, factor out} \\ 4x^2(x^2 - 3) &= 0 && \text{Use zero-product principle} \end{aligned}$$

Set each factor equal to zero:

$$\begin{aligned} 4x^2 &= 0 \qquad & x^2 - 3 &= 0 \\ x^2 &= 0 \qquad & x^2 &= 3 \\ x &= \pm\sqrt{0} \qquad & x &= \pm\sqrt{3} \\ x &= 0 \end{aligned}$$

If you check them you will see all of the answers working perfectly. Therefore the solution set is $\{-\sqrt{3}, 0, \sqrt{3}\}$.

Note 6.1. Often we have 4 terms and you need to use factoring by grouping. First, group terms with common factors, then factor out *gcf.* Finally, factor out the binomial binomial and use zero-product principle to find the solutions.

Example 6.2. Solve by factoring $2x^3 + 3x^2 = 8x + 12$

$$\begin{aligned} 2x^3 + 3x^2 &= 8x + 12 && \text{Set one side equal to 0} \\ & && \text{Subtract } 8x \text{ and 12 from both sides} \\ 2x^3 + 3x^2 - 8x - 12 &= 0 && \text{Factor } x^2 \text{ from first two terms} \\ x^2(2x+3) - 8x - 12 &= 0 && \text{Factor } -4 \text{ from last two terms} \\ x^2(2x+3) - 4(2x+3) &= 0 && \text{Factor out the common binomial,} \\ & && 2x+3\text{, from each term} \\ (2x+3)(x^2-4) &= 0 && \text{Use zero-product principle} \end{aligned}$$

Set each factor equal to zero:

$$\begin{aligned} 2x + 3 &= 0 \qquad & x^2 - 4 &= 0 \\ 2x &= -3 \qquad & x^2 &= 4 \\ x &= -\frac{3}{2} \qquad & x &= \pm\sqrt{4} \\ & & x &= \pm 2 \end{aligned}$$

Check the three solutions by substituting them into the original equations. It can be verified that all of them satisfy the equations. Thus, the solution set is $\{-2, -3/2, 2\}$.

6.2 Radical equations

Here we look at equations that have roots in the problem. As you might expect, to clear a root we can raise both sides to an exponent. For instance, to clear a square root we can rise both sides to the second power or to clear a cubed root we can raise both sides to a third power.
There is one catch to solving a problem with roots in it. Sometimes we end up with solutions that don't actually work in the equation, particularly when the index on the root is even. So for these problems it will be very important to check our answer(s). If a value does not work it is called an **extraneous solution** and not included in the final solution.

Steps to solve a radical equation

1. Isolate the radical expression.
2. Raise both sides to an exponent (that exponent is actually the index of root. So if we have square root, square each side).
3. Solve for x.
4. Always check your answer if you had a square root (or any root with even index). Otherwise, you don't need to check.

Example 6.3. Solve the following radical equations:

$$\sqrt{3x-8} = x-2$$

$$\begin{aligned}
\sqrt{3x-8} &= x-2 && \text{Even index! We have to check our answer(s)}\\
\left(\sqrt{3x-8}\right)^2 &= (x-2)^2 && \text{Squaring both sides}\\
3x-8 &= x^2+4-4x && \text{Subtract } 3x\\
-8 &= x^2+4-7x && \text{Subtract } -8\\
0 &= x^2-4-7x && \text{Simplify RHS}\\
0 &= x^2-7x+12 && \text{Factor trinomial}\\
0 &= (x-3)(x-4) && \text{Set each factor equal to zero}
\end{aligned}$$

we will get two answers

$$\begin{cases} x-3=0 & \to x=3 \\ x-4=0 & \to x=4 \end{cases}$$

check.

$$\text{For } x=3: \sqrt{3(3)-8} \stackrel{?}{=} 3-2 \qquad \text{For } x=4: \sqrt{3(4)-8} \stackrel{?}{=} 4-2$$

$$\sqrt{1} \stackrel{?}{=} 1 \qquad\qquad \sqrt{4} \stackrel{?}{=} 2$$
$$1 = 1 \checkmark \qquad\qquad 2 = 2 \checkmark$$

Therefore, the solution set is $\{1,\, 2\}$.

Example 6.4. Solve the following radical equations:

$$\sqrt[3]{x-1} = -4$$

$$\begin{aligned}
\sqrt[3]{x-1} &= -4 && \text{Odd index! we don't need to check answer(s)}\\
\left(\sqrt[3]{x-1}\right)^3 &= -4^3 && \text{Cube both sides}\\
x - 1 &= -64 && \text{Add 1}\\
x &= -63 && \text{Our Solution}
\end{aligned}$$

Note 6.2. If the radical is not alone on one side of the equation we will have to solve for the radical before we raise it to an exponent. This is often referred to as *isolating the radical term.*

Example 6.5. Solve the following radical equation:

$$-4 + \sqrt{4+x} = x$$

$$\begin{aligned}
-4 + \sqrt{4+x} &= x && \text{Even index! Need to check our solutions}\\
\sqrt{4+x} &= 4 + x && \text{Isolate radical by adding 4 to both sides}\\
\left(\sqrt{4+x}\right)^2 &= (4+x)^2 && \text{Square both sides}\\
4 + x &= 16 + x^2 + 8x && \text{Subtract } 4+x \text{ from both sides}\\
0 &= 16 + x^2 + 8x - (x+4) && \text{Combine like terms, rearrange terms}\\
0 &= x^2 + 7x + 12 && \text{Factor}\\
0 &= (x+3)(x+4) && \text{Set each factor equal to zero}
\end{aligned}$$

we will get two answers

$$\begin{cases} x + 3 = 0 & \to x = -3\\ x + 4 = 0 & \to x = -4 \end{cases}$$

check.

$$\text{For } x = -3 : -4 + \sqrt{4-3} \stackrel{?}{=} -3 \qquad \text{For } x = -4 : -4 + \sqrt{4-4} \stackrel{?}{=} -4$$
$$-4 + \sqrt{1} \stackrel{?}{=} -3 \qquad\qquad -4 + 0 \stackrel{?}{=} -4$$
$$-3 = -3 \checkmark \qquad\qquad -4 = -4 \checkmark$$

Thus, the solution set is $\{-3,\, -4\}$.

Example 6.6. Solve the following radical equation:

$$x + \sqrt{4x+1} = 5$$

$$\begin{aligned}
x + \sqrt{4x+1} &= 5 && \text{Even index! Need to check our solutions} \\
\sqrt{4x+1} &= 5 - x && \text{Isolate radical} \\
\left(\sqrt{4x+1}\right)^2 &= (5-x)^2 && \text{Square both sides} \\
4x + 1 &= 25 + x^2 - 10x && \text{Subtract } 4x+1 \text{ from both sides} \\
0 &= 25 + x^2 - 10x - (4x+1) && \text{Combine like terms, rearrange terms} \\
0 &= x^2 - 14x + 24 && \text{Factor} \\
0 &= (x-2)(x-12) && \text{Set each factor equal to zero}
\end{aligned}$$

we will get two answers

$$\begin{cases} x - 2 = 0 & \to x = 2 \\ x - 12 = 0 & \to x = 12 \end{cases}$$

check.

$$\begin{aligned}
\text{For } x = 2 : 2 + \sqrt{4(2)+1} &\stackrel{?}{=} 5 & \text{For } x = 12 : 12 + \sqrt{4(12)+1} &\stackrel{?}{=} 5 \\
2 + \sqrt{9} &\stackrel{?}{=} 5 & 12 + \sqrt{49} &\stackrel{?}{=} 5 \\
5 &= 5 \checkmark & 19 &= 5 \text{ ✗}
\end{aligned}$$

The solution set is $\{2\}$.

6.2.1 Squaring each side twice

When there is more than one square root in the problem, after isolating one root and squaring both sides we may still have a root remaining in the problem. In this case we will again isolate the term with the second root and square both sides. When isolating, we will isolate the term with the square root.

Example 6.7. Solve the following equation:

$$\sqrt{x+5} - \sqrt{x-3} = 2$$

$$\begin{aligned}
\sqrt{x+5} - \sqrt{x-3} &= 2 && \text{Even Index! Need to check our answers} \\
\sqrt{x+5} &= \sqrt{x-3} + 2 && \text{Isolate one of the roots}
\end{aligned}$$

$$\begin{aligned}
\left(\sqrt{x+5}\right)^2 &= \left(\sqrt{x-3}+2\right)^2 && \text{Square both sides}\\
x+5 &= (x-3)+4+4\sqrt{x-3} && \text{Combine like terms on RHS}\\
x+5 &= x+1+4\sqrt{x-3} && \text{Isolate the radical, subtract } x\\
5 &= 1+4\sqrt{x-3} && \text{Subtract 1}\\
4 &= 4\sqrt{x-3} && \text{Divide both sides by 4}\\
1 &= \sqrt{x-3} && \text{Square each sides}\\
(1)^2 &= \left(\sqrt{x-3}\right)^2 && \text{Simplify}\\
1 &= x-3 && \text{Add 3 to both sides}\\
4 &= x && \text{Our answer}
\end{aligned}$$

check.
Because we had square root, we must check our answer!

$$\begin{aligned}
\text{For } x=4: \sqrt{(4)+5}-\sqrt{(4)-3} &\stackrel{?}{=} 2\\
\sqrt{9}-\sqrt{1} &\stackrel{?}{=} 2\\
3-1 &= 2 \checkmark
\end{aligned}$$

The solution set is $\{4\}$

Note 6.3. Recall that

$$\boxed{\begin{aligned}(a+b)^2 &= a^2+b^2+2ab \qquad (6.1)\\ (a-b)^2 &= a^2+b^2-2ab \qquad (6.2)\end{aligned}}$$

Most students make a mistake and square a binomial like this

$$\begin{aligned}
(a+b)^2 &= a^2+b^2 \quad ✗\\
(a-b)^2 &= a^2-b^2 \quad ✗
\end{aligned}$$

which is absolutely wrong. If you don't recall the formula, try using the FOIL method. You will get the same answer as (6.1) or (6.2).

6.3 Equations with rational exponents

The equation, $x^{m/n}=k$, is an example of a rational exponent equation. In such equations, we need to convert the rational exponent to its radical form using the following formula:

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m \tag{6.3}$$

In other words, the denominator of a rational exponent becomes the index on our radical. Once we have done this, we clear the exponent m by taking

radical from both sides. Recall, if m is even we must add $\pm$ to one side of the equation; otherwise, we don't do anything.
Next, we will clear the index n by raising both sides to n. Finally, we have our x alone and we can solve for it. As always, don't forget to check your answers when m is an even number.

Steps to solve a equation with rational exponent

To solve a rational equation, $x^{m/n} = k$, follow these steps:

1. Isolate the term with rational exponent.
2. Convert the rational exponent to the radical form.
$$\left(\sqrt[n]{a}\right)^m = k$$
3. Clear m by taking radical from both sides. If $m \in \mathbb{E}$, add $\pm$ to one side of an equation.
4. Clear n by raising both sides to power of n.
5. Solve for x
6. Check your solution(s) if $m \in \mathbb{E}$

Example 6.8. Solve $5x^{\frac{3}{2}} - 25 = 0$.

$$\begin{aligned}
5x^{\frac{3}{2}} - 25 &= 0 && \text{Isolate } x^{\frac{3}{2}}\text{, add 25 to both sides}\\
5x^{\frac{3}{2}} &= 25 && \text{Divide both sides by 5}\\
x^{\frac{3}{2}} &= 5 && \text{Convert it to the radical}\\
\left(\sqrt{x}\right)^3 &= 5 && \text{Take cube root}\\
\sqrt{x} &= \sqrt[3]{5} && \text{Square both sides}\\
x &= \left(\sqrt[3]{5}\right)^2 && \text{Simplify}\\
x &= \sqrt[3]{25} && \text{Our solution}
\end{aligned}$$

Since we didn't take any even roots, we don't need to check our solution. Thus, the solution set is $\{\sqrt[3]{25}\}$.

Example 6.9. Solve $x^{\frac{2}{3}} - 8 = -4$.

$$
\begin{aligned}
x^{\frac{2}{3}} - 8 &= -4 && \text{Isolate } x^{\frac{2}{3}}\text{, add 8 to both sides}\\
x^{\frac{2}{3}} &= 4 && \text{Convert it to the radical form}\\
\left(\sqrt[3]{x}\right)^2 &= 4 && \text{Take square root, add } \pm\\
\sqrt[3]{x} &= \pm\sqrt{4} && \text{Simplify RHS}\\
\sqrt[3]{x} &= \pm 2 && \text{Cube both sides}\\
x &= (\pm 2)^3 && \text{Simplify RHS}\\
x &= \pm 8 && \text{Our solution}
\end{aligned}
$$

Because we took even root, we need to check our answers. One can verify that -8 and 8 are both true solutions by substituting them into the original equation. Therefore, the solution set is $\{-8, 8\}$.

6.4 Equations that are quadratic in form

Some equations are not quadratic but can be written as quadratic using an appropriate substitution.
Consider the equation $x^4 - 8x^2 - 9 = 0$. Notice here that the variable of the first term is nothing more than the variable of the second term squared.

$$x^4 = \left(x^2\right)^2$$

In other words, the exponent on the first term was twice the exponent on the second term. This, along with the fact that third term is a constant, means that this equation is reducible to quadratic in form. We will solve this by first defining

$$u = x^2$$

Using this substitution, we'll get

$$(u)^2 - 8u^2 - 9 = 0$$

The new equation, the one with the u's,is a quadratic equation and we can solve that by factoring.

$$
\begin{aligned}
u^2 - 8u^2 - 9 &= 0\\
(u-9)(u+1) &= 0
\end{aligned}
$$

we will get

$$
\begin{cases}
u - 9 = 0 & \to u = 9\\
u + 1 = 0 & \to u = -1
\end{cases}
$$

These are not the solutions that we're looking for. We want values of x, not values of u. To get values of x, all we need to substitute back $u = x^2$.

$$\begin{cases} u = 9 & \to x^2 = 9 \\ u = 0 & \to x^2 = -1 \end{cases}$$

Now we can find x,

$$\begin{cases} x^2 = 9 & \to x = \pm\sqrt{9} = \pm 3 \\ x^2 = -1 & \to x = \pm\sqrt{-1} = \pm i \end{cases}$$

Therefore we have four solutions. The solution set is $\{-3,\ 3,\ -i,\ i\}$. In most cases to make the check if an equation is reducible to quadratic in form, all that we really need to do is to check that one of the exponents is twice the other.

How to find out an equation is in the quadratic form?

1. There are 3 terms.
2. There is a constant term.
3. The exponent of the first term is **twice** the exponent of middle term.

Steps to solve an equation that is in quadratic form

1. Substitute $u = \textit{middle term}$.
2. Your new equation will be a quadratic in form $au^2 + bu + c = 0$. Solve them using factoring or quadratic formula.
3. Once you found your solutions in terms of u, then substitute back $u = \textit{middle term}$.
4. Solve for x.

Example 6.10. Solve: $x^4 - 5x^2 + 6 = 0$

$$\begin{aligned} x^4 - 5x^2 + 6 &= 0 && \text{4 is twice of 2. We know } x^4 = \left(x^2\right)^2 \\ \left(x^2\right)^2 - 5x^2 + 6 &= 0 && \text{Substitute } u = x^2 \\ u^2 - 5u + 6 &= 0 && \text{Factor or use quadratic formula} \\ (u-3)(u-2) &= 0 && \text{Set each factor equal to 0} \end{aligned}$$

$$u-3=0 \rightarrow \boxed{u=3} \qquad \text{Equation (A)}$$
$$u-2=0 \rightarrow \boxed{u=2} \qquad \text{Equation (B)}$$

Now substitute back $u = x^2$ into both equations (A) and (B):

Equation (A)	Equation (B)
$u=3$	$u=2$
$x^2=3$	$x^2=2$
$x=\pm\sqrt{3}$	$x=\pm\sqrt{2}$

The solution set is $\left\{-\sqrt{3},\ -\sqrt{2},\ \sqrt{2},\ \sqrt{3}\right\}$.

Example 6.11. Solve $3x^{\frac{2}{3}} - 11x^{\frac{1}{3}} - 4 = 0$.

$$\begin{aligned}
3x^{\frac{2}{3}} - 11x^{\frac{1}{3}} - 4 &= 0 && \text{2/3 is twice of 1/3}\\
3(x^{\frac{1}{3}})^2 - 11x^{\frac{1}{3}} - 4 &= 0 && \text{Substitute } u = x^{\frac{1}{3}}\\
3u^2 - 11u - 4 &= 0 && \text{Factor or use quadratic formula}\\
(3u+1)(u-4) &= 0 && \text{Set each factor equal to 0}\\
3u+1=0 \rightarrow &\boxed{u=-\frac{1}{3}} && \text{Equation (A)}\\
u-4=0 \rightarrow &\boxed{u=4} && \text{Equation (B)}
\end{aligned}$$

Next we Substitute back $u = x^{\frac{1}{3}}$ into both equations (A) and (B):

Equation (A)	Equation (B)
$u=-\frac{1}{3}$	$u=4$
$x^{\frac{1}{3}}=-\frac{1}{3}$	$x^{\frac{1}{3}}=4$
$\left(x^{\frac{1}{3}}\right)^3=\left(-\frac{1}{3}\right)^3$	$\left(x^{\frac{1}{3}}\right)^3=4^3$
$x=-\frac{1}{27}$	$x=64$

The solution set is $\left\{-\frac{1}{3},\ 64\right\}$.

Example 6.12. Solve $\left(x^2-4\right)^2 + (x^2-4) - 6 = 0$.

If you look carefully, you will notice the equation contains $x^2 - 4$ and $x^2 - 4$ squared. So

$$\begin{aligned}
\left(x^2-4\right)^2 + (x^2-4) - 6 &= 0 && \text{Let } u = x^2 - 4\\
u^2 + u - 6 &= 0 && \text{Factor}\\
(u+3)(u-2) &= 0 && \text{Set each factor equal to } 0\\
u + 3 = 0 \to \boxed{u = -3} & && \text{Equation (A)}\\
u - 2 = 0 \to \boxed{u = 2} & && \text{Equation (B)}
\end{aligned}$$

Substitute back $u = x^2 - 4$ into both (A) and (B), then solve for x

Equation (A)	Equation (B)
$u = -3$	$u = 2$
$x^2 - 4 = -3$	$x^2 - 4 = 2$
$x^2 = 1$	$x^2 = 6$
$x = \pm 1$	$x = \pm\sqrt{6}$

The solution set is $\left\{-\sqrt{6},\ -1,\ 1,\ \sqrt{6}\right\}$.

6.5 Absolute value equations

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations.

Absolute value rule

If $|x| = a$ then

$$x = a \quad \text{or} \quad x = -a$$

When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions.

Steps to solve an absolute value equation

1. Isolate the absolute value term.
2. Remove the absolute value and add $\pm$.
3. Solve for x.

Example 6.13. Solve $5|x| - 4 = 6$

$$\begin{aligned} 5|x| - 4 &= 6 && \text{Isolate the absolute value,add 4 to both sides} \\ 5|x| &= 10 && \text{Divide both sides by 5} \\ |x| &= 2 && \text{Absolute value can be positive or negative} \\ x &= \pm 2 && \text{Our solution} \end{aligned}$$

The solution set is $\{-2, 2\}$.

Note 6.4. Notice we never combine what is inside the absolute value with what is outside the absolute value. This is very important as it will often change the final result to an incorrect solution.

Often the absolute value will have more than just a variable in it. In this case we will have to solve the resulting equations when we consider the positive and negative possibilities. This is shown in following Example.

Example 6.14. Solve $|2x - 1| = 5$

$$\begin{aligned} |2x - 1| &= 5 && \text{Absolute value can be positive or negative} \\ 2x - 1 &= \pm 5 && \text{We get two equations} \end{aligned}$$

Solve each equation for x

$$\begin{cases} 2x - 1 = 5 & \Longrightarrow x = 3 \\ 2x - 1 = -5 & \Longrightarrow x = -2 \end{cases}$$

We have two solutions,-2 and 3. The solution set is $\{-2, 3\}$.
Again, it is important to remember that the absolute value must be alone first before we consider the positive and negative possibilities.

Example 6.15. Solve $4|3 - 4x| - 20 = 0$.

$$\begin{aligned} 4|3 - 4x| - 20 &= 0 && \text{Isolate absolute value, add 20 to both sides} \\ 4|3 - 4x| &= 20 && \text{Divide both sides by 4} \\ |3 - 4x| &= 5 && \text{Absolute value can be positive or negative} \\ 3 - 4x &= \pm 5 && \text{We get two equation} \end{aligned}$$

We will get

$$\begin{cases} 3 - 4x = 5 & \Longrightarrow x = -\dfrac{1}{2} \\ \\ 3 - 4x = -5 & \Longrightarrow x = 2 \end{cases}$$

Therefore the solution set is $\left\{-\dfrac{1}{2}, 2\right\}$.

Linear inequalities

7.1 Introduction

An equality states that two values are not equal. In mathematics, we read inequality from left to side, using these symbols:

$>$	*Greater than*
$<$	*Less than*
$\geq$	*Greater than or equal to*
$\leq$	*Less than or equal to*

7.2 Linear inequality

In an inequality, we stating both sides of the equation are not equal to each other. It can also be seen as an order relation; that is, it tells us which one of the two expressions is smaller, or larger, than the other one.
For example, $x \geq 3$ means that x can be 3 or any real numbers greater than 3. Another example is $0.2x + 120 \leq 120$.
Solving an inequality is the process of finding the set of numbers that satisfies the inequality. We have 3 ways to represent the solution set of an inequality

1. Set-builder notation
2. Interval notation
3. Graphing

We have discussed set-builder notation in chapter 0. We begin this section by looking at interval notation.

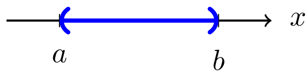
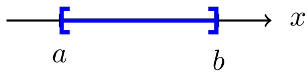
7.2.1 Interval notation

Suppose that a and b are real numbers such that $a < b$. We use two symbols: parentheses **( )** and brackets **[ ]**:

- **( )** is used for less than, $<$, or greater than, $>$. This means that specified values for a or b **are not included**.

- [ ] is used for less than or equal to, $\le$, or greater than or equal to, $\ge$. This means that specified values for a or b **are included**.

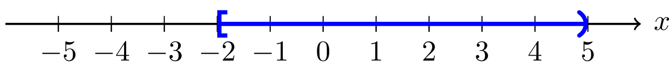
Table 7.1: Intervals on real numbers

Set-builder notation	Interval notation	Graph
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \le x \le b\}$	$[a, b]$	
$\{x \mid a \le x < b\}$	$[a, b)$	
$\{x \mid a < x \le b\}$	$(a, b]$	
$\{x \mid x > a\}$	$(a, +\infty)$	
$\{x \mid x \ge a\}$	$[a, +\infty)$	
$\{x \mid x < b\}$	$(-\infty, b)$	
$\{x \mid x \le b\}$	$(-\infty, b]$	
$\{x \mid x \in \mathbb{R}\}$	$(-\infty, +\infty)$	

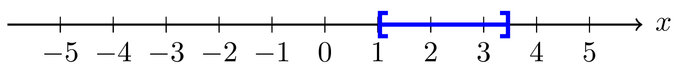
We use $+\infty$ to signify that the values continue getting larger without end (unbounded to the right on the number line). On the other hand, $-\infty$ signify that the values continue getting smaller without end (unbounded to the left on the number line).

Example 7.1. Express each interval in set-builder notation and graph.
a. $[-2, 5)$
b. $[1, 3.5]$
c. $(-\infty, -1)$

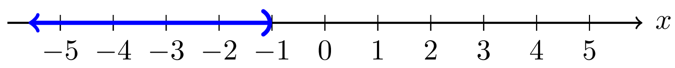
It's get easier if we convert them into set-builder notation first.
a. $[-2, 5) = \{x \mid -2 \le x < 5\}$



b. $[1,\ 3.5] = \{x \mid 1 \le x \le 3.5\}$



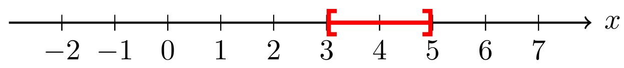
c. $(-\infty,\ -1) = \{x \mid x < -1\}$



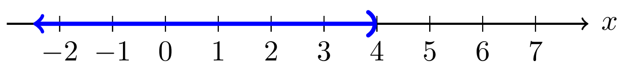
7.2.2 Intersections and unions

Consider two intervals A and B. The common portion of two intervals A and B is called *intersection* of A and B and it's denoted as $A \cap B$.
The merge of two intervals A and B, however, is called the *union* of A and B and it's denoted as $A \cup B$.
For example, let's take a look at two intervals $(-\infty,\ 4)$ and $[3,\ 5]$

$[3,\ 5]$:

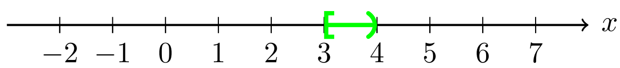


$(-\infty,\ 4)$:

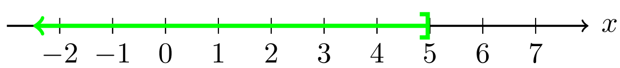


Intersection and union of these two intervals are

$$(-\infty,\ 4) \cap [3,\ 5] = [3,\ 4)$$



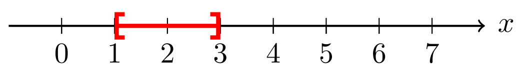
$$(-\infty,\ 4) \cup [3,\ 5] = (-\infty,\ 5]$$



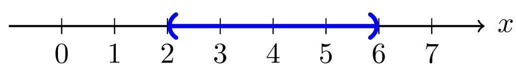
Example 7.2. Use graphs to find each set:
a. $[1,\ 3] \cap (2, 6)$ b. $[1,\ 3] \cup (2, 6)$

a. The intersection of the intervals $[1,\ 3]$ and $(2, 6)$ consists of the numbers that are in both intervals.

Graph of $[1,\ 3]$:

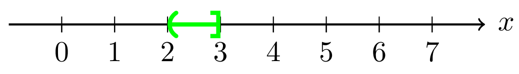


Graph of $(2,\ 6)$:



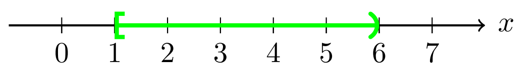
To find $[1,\ 3] \cap (2, 6)$, take the portion of the number lines that two graphs have in common (overlap).

$$[1,\ 3] \cap (2, 6) = (2,\ 3]$$



b. The union of the intervals $[1,\ 3]$ and $(2, 6)$ consists of the numbers that are in either one interval or the other (or both). To find $[1,\ 3] \cup (2, 6)$, take the portion of the number lines representing the total collection of numbers in two graphs (sum).

$$[1,\ 3] \cup (2, 6) = [1,\ 6)$$



7.3 Solving a linear inequality

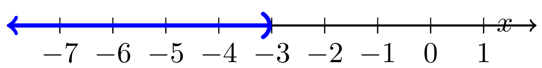
Solving inequalities is very similar to solving equations with one exception. we can add, subtract, multiply, or divide on both sides of the inequality. But if we multiply or divide by a negative number, the symbol will need to be flipped its direction. We will keep that in mind as we solve inequalities.

Example 7.3. Solve and graph the solution set on a number line:

$$-2x - 4 > x + 5$$

$$\begin{aligned}
-2x - 4 &> x + 5 && \text{Add } 2x \text{ to both sides} \\
-4 &> 3x + 5 && \text{Subtract 5 from both sides} \\
-9 &> 3x && \text{Divide both sides by 3} \\
& && \text{3 is positive, no need to flip} \\
-3 &> x && \text{Our solution}
\end{aligned}$$

Thus, our answer in set-builder notation is $\{x \mid x < -3\}$. The interval notation for this solution set is $(-\infty,\ -3)$. The graph of the solution set is shown as follows:

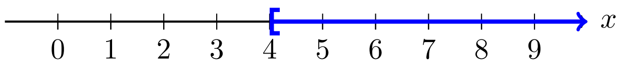


Example 7.4. Solve and graph the solution set on a number line:

$$3x + 1 \leq 7x - 15$$

$$\begin{aligned}
3x+1 &\le 7x-15 && \text{Subtract } 7x \text{ from both sides}\\
-4x+1 &\le -15 && \text{Subtract 1 from both sides}\\
-4x &\le -16 && \text{Divide both sides by } -4\\
& && \text{Divide by a negative number, flip the sign!}\\
x &\ge 4 && \text{Our answer}
\end{aligned}$$

Our final solution can expressed in set-builder notation and interval notation as $\{x \mid x \ge 4\}$ and $[4, +\infty)$, respectively. The graph of solution set is shown as follows

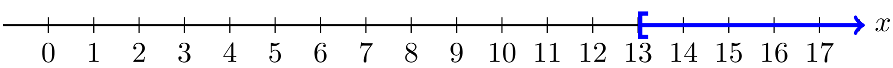


Example 7.5. Solve and graph the solution set on a number line:

$$\frac{x-4}{2} \ge \frac{x-2}{3} + \frac{5}{6}$$

$$\begin{aligned}
\frac{x-4}{2} &\ge \frac{x-2}{3} + \frac{5}{6} && \text{LCD of 2,3 and 6 is 6.}\\
& && \text{Multiply both sides by 6}\\
6\left(\frac{x-4}{2} \right.&\ge \left.\frac{x-2}{3} + \frac{5}{6}\right) && \text{Distribute and simplify}\\
3(x-4) &\ge 2(x-2)+1(5) && \text{Multiply}\\
3x-12 &\ge 2x-4+5 && \text{Combine like terms on LHS}\\
3x-12 &\ge 2x+1 && \text{Subtract } 2x \text{ from both sides}\\
x-12 &\ge 1 && \text{Add 12 to both sides}\\
x &\ge 13 && \text{Our answer}\\
\{x \mid x &\ge 13\} && \text{Set-builder notation}\\
[13,\, &+\infty) && \text{Interval notation}
\end{aligned}$$

The graph of solution is shown below



7.4 Special inequalities

Similar to the linear equations, we can reach to a true a statement, like $0 < 1$, while solving a linear inequality. In this case, the solution set is all real numbers $\{x \mid x \text{ is a real number}\}$ or $(-\infty, +\infty)$.
Likewise, if we get a false statement, like $0 > 1$, the solution set is $\{\,\}$ or ϕ.

Example 7.6. Solve each inequality:

a. $3(x+1) > 3x+2$

b. $x+1 \leq x-1$.

a.

$$\begin{aligned} 3(x+1) &> 3x+2 && \text{Simplify LHS} \\ 3x+3 &> 3x+2 && \text{Subtract } 3x \text{ from both sides} \\ 3 &> 2 && \text{A true statement} \\ \{x \mid x &\in \mathbb{R}\} && \text{Our solution set in set-builder notation} \\ (-\infty&, +\infty) && \text{Or in interval notation} \end{aligned}$$

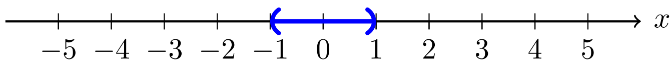
b.

$$\begin{aligned} x+1 &\leq x-1 && \text{Subtract } x \text{ from both sides} \\ 1 &\leq -1 && \text{A false statement} \\ &\phi && \text{Our solution} \end{aligned}$$

7.5 Absolute value inequalities

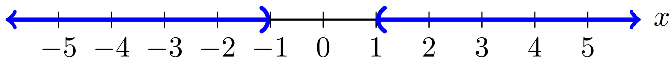
When an inequality has an absolute value we will have to remove the absolute value in order to graph the solution or give interval notation. The way we remove the absolute value depends on the direction of the inequality symbol.

Consider $|x| < 1$:
Absolute value is defined as distance from zero. Another way to read this inequality would be the distance from zero is less than 1. So on a number line we will shade all points that are less than 1 units away from zero.



When the absolute value is less than a number will will remove the absolute value by changing the problem to a three part inequality, with the negative value on the left and the positive value on the right. So $|x| < 1$ becomes $-1 < x < 1$, as the graph above illustrates.

Now consider $|x| > 1$:
Absolute value is defined as distance from zero. Another way to read this inequality would be the distance from zero is greater than 1. So on the number line we shade all points that are more than 1 units away from zero.



When the absolute value is greater than a number we will remove the absolute value and get two inequalities: the first inequality looking just like the problem with no absolute value, the second flipping the inequality symbol and changing the value to a negative. So $|x| > 1$ becomes $x > 1$ or $x < -1$, as the graph above illustrates.

How to solve an absolute value inequalities

First Isolate the absolute value term.

- If $|u| < a$ then $-a < u < a$.
- If $|u| > a$ then $u < -a$ or $u > a$.

These rules are valid if $<$ is replaced by $\leq$ or $>$ is replaced by $\geq$.

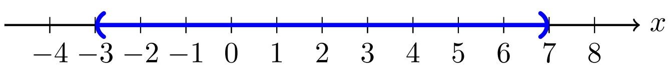
Example 7.7. Solve and graph the solution set on a number line:

$$|x - 2| < 5$$

The absolute value has already been isolated. Thus,

$$\begin{aligned} -5 < x - 2 < 5 &\qquad \text{Add 2 to all sides} \\ -3 < x < 7 &\qquad \text{Our solution} \end{aligned}$$

Thus, the solution set is $\{x \mid -3 < x < 7\}$ or using interval notation we have $(-3, 7)$. The graph of the solution is illustrated as follows



Example 7.8. Solve and graph the solution set on a number line:

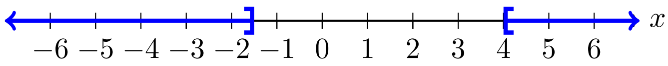
$$|4x - 5| - 5 \geq 6$$

$$\begin{aligned} |4x - 5| - 5 \geq 6 &\qquad \text{Isolate the absolute value} \\ |4x - 5| \geq 11 & \end{aligned}$$

Because of $\geq$ sign we have two inequalities:

$$\begin{aligned} 4x - 5 &\leq -11 &\quad \text{or} \quad && 4x - 5 &\geq 11 \\ 4x &\leq -6 &\quad \text{or} \quad && 4x &\geq 16 \\ x &\leq -\frac{6}{4} &\quad \text{or} \quad && x &\geq \frac{16}{4} \\ x &\leq -\frac{3}{2} &\quad \text{or} \quad && x &\geq 4 \end{aligned}$$

So our solution set is $\left\{x \mid x \leq -\frac{3}{2} \text{ or } x \geq 4\right\}$ or using interval notation $\left(-\infty, -\frac{3}{2}\right] \cup [4, +\infty)$. The graph is shown below



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