Chapter 4

Pure mathematics is on the whole distinctly more useful than applied. For what is useful above all is technique, and mathematical technique is taught mainly through pure mathematics.

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G. H. Hardy

Mohammad Azarnoush
Mechanical Engineering
mazarnou@asu.edu

Contents

Exponential Functions

1.1 Introduction

As our study of algebra gets more advanced we begin to study more involved functions. One pair of inverse functions we will look at are exponential functions and logarithmic functions. Here we will look at exponential functions and then we will consider logarithmic functions in another sections.

1.2 Exponential functions

The exponential function is defined by

$$f(x) = a^x \tag{1.1}$$

where a is a positive number other than 1 ($a > 0$ and $a \neq 1$) and x is an real number. The reason why $a \neq 0, 1$ is clear. When $a = 0$ or 1, we will have a horizontal line.
On the other hand, it is not straightforward why the base cannot be a negative number. Let's consider $a = -2$ so that $f(x) = (-2)^x$. Choosing any integer $x-$values yields to another real numbers. For instance

$$\begin{aligned} f(0) &= 1 & f(-1) &= -\frac{1}{2} \\ f(1) &= -2 & f(-2) &= \frac{1}{4} \\ f(2) &= 4 & f(-3) &= \frac{1}{8} \\ &\vdots & &\vdots \end{aligned}$$

Here, we don't encounter any problems. However, the graph does not just exist as a set of these isolated points. The problem arise when x is a fraction such as 0.5. In this case, we will get

$$f(0.5) = (-2)^{0.5} = 1.41i$$

As it happens, we get a complex number. Likewise, imaginary parts can appear in following x values

$$\begin{aligned} f(0.75) &= -1.19 + 1.19i \\ f(1.25) &= -1.68 - 1.68i \\ f(1.5) &= -2.83i \\ f(1.75) &= 2.38 - 1.38i \\ &\vdots \end{aligned}$$

In order to make sense of this, we need to be able to plot these complex y values. Thus, we need another axis besides the normal x and y axes. Here, we will plot all x values on the x-axis. We then locate the real part of y values on the y-axis and their imaginary part ,if they have any, on the new z-axis. The graph of $f(x) = (-2)^x$ is shown in Figure 1.1.

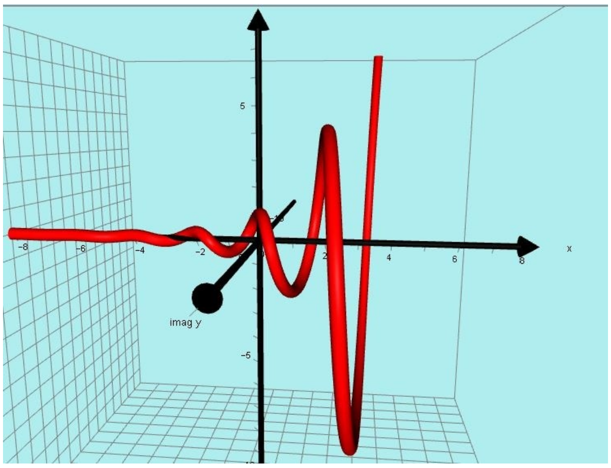

Figure 1.1: The graph of $f(x) = (-2)^x$

As you can observe, the negative bases lie beyond the scope of this course. Therefore, we will only consider positive base other than 1.

1.2.1 Exponent rules

Here are some important algebra rules for exponential functions:

$$a^0 = 1 \quad \text{Any number to the zero power is equal to 1} \tag{1.2}$$

$$a^{m/n} = \sqrt[n]{a^m} \quad \text{Rational notation} \tag{1.3}$$

$$(a^n)^m = a^{nm} \quad \text{product rule} \tag{1.4}$$

$$a^{-x} = \frac{1}{a^x} \quad \text{Negative exponent rule} \tag{1.5}$$

1.2.2 Graphing

It's really important that you know the general shape of the graph of an exponential function. There are two options: either the base is greater than 1, or the base is less than 1 (but still positive).

Base Greater than 1

If a is greater than 1, then the graph of $f(x) = a^x$ grows taller as it moves to the right. Keeping in mind that a^x is positive for any number x, and that $a^0 = 1$, we now have a pretty good idea of what the graph of $f(x) = a^x$ looks like if $a > 1$.

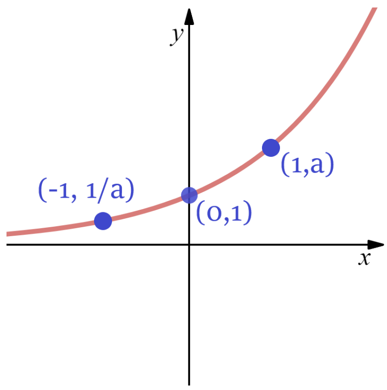

Figure 1.2: The graph of $f(x) = a^x$ when $a > 1$

As you can see in Figure 1.2, when moving to the left, the graph becomes shorter and shorter, shrinking toward, but never touching, the x-axis; In other words, $y = 0$ is a horizontal asymptote.
Not only does the graph grow bigger as it moves to the right, but it gets big in a hurry. For example, if we look at the exponential function whose base is 2, then

$$f(64) = 2^{64} = 18,446,744,073,709,525,000$$

And 2 isn't even a very big number to be using for a base (any positive number can be a base, and plenty of numbers are much, much bigger than 2). The bigger the base of an exponential function, the faster its graph grows as it moves to the right.
Moving to the left, the graph of $f(x) = a^x$ grows small very quickly if $a > 1$. Again, if we look at the exponential function whose base is 2, then

$$f(-10) = 2^{-10} = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$$

The bigger the base, the faster the graph of an exponential function shrinks as it moves to the left.

Base less than 1 (but still positive)

The graph of $f(x) = a^x$ when the base is smaller than 1 slopes down as it moves to the right, but it is always positive. As it moves to the left, the graph grows tall very quickly. Once again, the y-intercept is at 1 because a^0 is always 1.

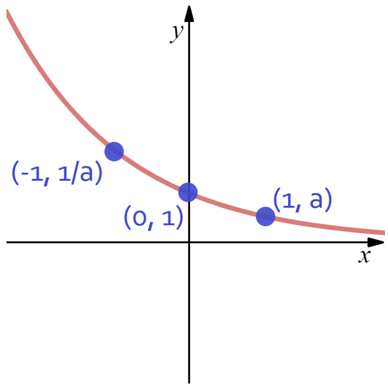

Figure 1.3: The graph of $f(x) = a^x$ when $0 < a < 1$

1.2.3 One-to-One function

For any exponential function in the form of a^x, domain is all real numbers $(-\infty, +\infty)$. To check what the range of $f(x)$ is, we think of compressing the graph of $f(x)$ onto the y-axis. If we did that, we would see that the range of $f(x)$ is the set of all positive numbers, $(0, +\infty)$.
We also see from the graph of $f(x) = a^x$, if either $a > 1$ or $0 < a < 1$, $f(x)$ is one-to-one. Remember that to check if $f(x)$ is one-to-one, we can use the horizontal line test (which $f(x)$ passes the test). This means that exponential function has an inverse function; That inverse function is logarithmic function which we will talk about it in later sections.

Characteristics of $f(x) = a^x$

1. The domain is $(-\infty, +\infty)$.
2. The range is $(0, +\infty)$.
3. The y-intercept is $(0, 1)$.
4. $y = 0$ is a horizontal asymptote.
5. It is a one-to-one function, i.e. it has an inverse.

1.3 e

Some numbers are so important in math that they get their own name. One such number is e. It is a real number, but it is not a rational number. It's very near to – but not equal to – the rational number 2.7. The importance of the number e becomes more apparent after studying calculus, but we can say something about it here.
The approximate value of e to nine decimal places is

$$e \approx 2.718281827$$

Number e is often called Euler's Number after *Leonhard Euler*.It is also called Natural Base.
The discovery of this number is credited to *Jacob Bernouli* in 1683. He discovered this constant by studying a following question about compound interest:
"An account starts with \$1.00 and pays 100 percent interest per year. If the interest is credited once, at the end of the year, the value of the account at year-end will be \$2.00. What happens if the interest is computed and credited more frequently during the year?"
If the interest is credited twice in the year, the interest rate for each 6 months will be 50%, so the accumulated value after one year will be

Figure 1.4: Portrait of *Leonhard Euler*

$$\$1.00 \times (1.5)^2 = \$2.25$$

For other compounding we will get

$$\begin{array}{ll}
\$1.00 \times (1.25)^4 = \$2.4414 & \text{Compounded Quarterly} \\
\$1.00 \times (1 + \frac{1}{12})^{12} = \$2.613035 & \text{Compounded Monthly} \\
\$1.00 \times (1 + \frac{1}{52})^{52} = \$2.692597 & \text{Compounded Weekly} \\
\$1.00 \times (1 + \frac{1}{360})^{360} = \$2.714516 & \text{Compounded Daily} \\
\vdots & \vdots \\
(1+\frac{1}{n})^n & \text{Compounded in } n \text{ intervals}
\end{array}$$

Bernoulli noticed that this sequence approaches a limit with larger n. But it was *Euler* who prove that this expression is approaching a constant number and used the notation e for this number. He also showed that e is an irrational number and it is equal to the following beautiful continued fraction

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cdots}}}}}}}}$$

1.3.1 Natural exponential function

The natural exponential function is defined by

$$f(x) = e^x \tag{1.6}$$

where base is e and x can any real number. Since $e > 1$, therefore its graph is increasing. As you can guess, this function also has a horizontal asymptote $y = 0$. Likewise, $(0, 1)$ is the y-intercept.

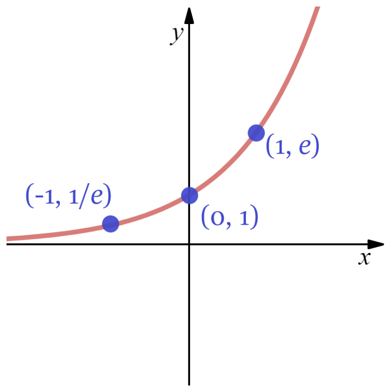

Figure 1.5: The graph of natural exponential function

1.3.2 Compound interest

One of the application of exponential functions is compound interest. When money is invested in an account (or given out on loan) a certain amount is added to the balance. This money added to the balance is called interest. Once that interest is added to the balance, it will earn more interest during the next compounding period. This idea of earning interest on interest is called compound interest.

There are several ways interest can be paid. When interest is compounded, one can calculate the balance after any amount of time using the following formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \tag{1.7}$$

where,

$$\begin{aligned} A &: \text{final amount (accumulated value)} \\ P &: \text{Principal} \\ r &: \text{Interest rate (as a decimal)} \\ n &: \text{the number of compounds per year} \\ t &: \text{time in years} \end{aligned}$$

If interest is compounded

- annually, then $n = 1$
- semi-annually, then $n = 2$
- quarterly, then $n = 4$
- monthly, then $n = 12$
- weekly, then $n = 52$
- daily, then $n = 360$

Some banks use continuous compounding, where the number of compounding periods increases infinitely. When we see the word "continuously" we will know that we cannot use the formula (1.7) . Instead we will use the following formula:

$$A = Pe^{rt} \tag{1.8}$$

Example 1.1. A sum of \$10,000 is invested at annual rate of 8%. Find the balance in the account after 5 years subject to

a. quarterly compounding;

b. and continuous compounding

We have $P = 10,000$, $r = 8\%$ which in decimal is $r = \dfrac{8}{100} = 0.08$, and $t = 5$ years. a. In this part, money is compounded quarterly, so $n = 4$. Plugging all of these values into (1.7) formula, we can find A

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ A &= \$10,000\left(1 + \frac{0.08}{4}\right)^{(4)(5)} \\ A &= \$10,000\,(1.02)^{20} \\ A &= \$10,000\,(1.485947) \\ A &= \$14,859.47 \end{aligned}$$

b. if money is compounding continuously, then we should use (1.8)

$$
\begin{aligned}
&A = Pe^{rt}\\
&A = \$10,000e^{(0.08)(5)}\\
&A = \$10,000e^{0.4}\\
&A = \$10,000(1.4918225)\\
&A = \$14,918.25
\end{aligned}
$$

As you can observe, the accumulated value of continuous compounding is greater than quarterly compounding.

Logarithmic Functions

2.1 What is logarithm?

Logarithm are-at the most basic level-invented to avoid very large numbers. It was first introduced by *John Napier* in 1614. As a matter of fact, a logarithm is an operation that can be applied on a number and return its exponent; In other words, logarithm is always equal to an exponent.

$$Logarithm = Exponent$$

Logarithm is denoted as log. If I asked you what is the 2^4, I am actually asking you to use the exponential function to calculate that statement. We all know that $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot = 16$. Nonetheless, in Logarithm realm, we asked inversely. "What power should I raised base 2 to get 16"? In other words,

$$2^? = 16$$

We know the answer to this question is 4. Using Logarithm, we can rewrite this question into

$$\log_2 16 =?$$

Notice the base 2 is written as a subscript. You can read it as "log base 2 of 16".

Logarithmic function

For $x > 0$ and $b > 0$, $b \neq 1$,

$$\log_b x = y \quad \text{is equivalent to} \quad b^y = x \tag{2.1}$$

The function $f(x) = log_b x$ is the logarithmic function with base b.

Figure 2.1 is showing each component of logarithms:

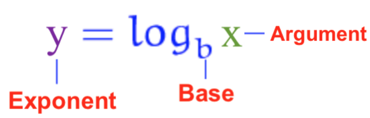

Figure 2.1: Components of a log

A good way to remember how log works, draw an arrow as shown in Figure 2.2. Raise b to the power of y to obtain x.

$$\log_b x = y$$

Figure 2.2: Arrows help us to convert logarithmic form to exponential form

Fact 2.1. Logarithm gives us an exponent whereas exponential function will raise the base to that exponent; In other words, logarithm and exponential functions are undoing each other. Consequently, they are inverse of each other.

Example 2.1. Write each equation in its equivalent exponential form.

a. $3 = \log_7 x$ **b.** $2 = \log_b 25$ **c.** $\log_4 26 = y$

Using the definition of log, equation (2.1), we can rewrite them in their exponential form.

a.

$$\begin{aligned} 3 &= \log_7 x && \text{Use the definition of log} \\ 7^3 &= x && \text{Our solution} \end{aligned}$$

b.

$$\begin{aligned} 2 &= \log_b 25 && \text{Use the definition of log} \\ b^2 &= 25 && \text{Our solution} \end{aligned}$$

c.

$$\begin{aligned} \log_4 26 &= y && \text{Use the definition of log} \\ 26 &= 4^y && \text{Our solution} \end{aligned}$$

Example 2.2. Write each equation in its equivalent logarithmic form.

a. $2^5 = x$ b. $b^3 = 27$ c. $e^y = 33$

a.

$$\begin{aligned} 2^5 &= x && \text{Recall } log = exponent\text{, We know base is 2} \\ \log_2 x &= 5 && \text{Our solution} \end{aligned}$$

b.

$$\begin{aligned} b^3 &= 27 && \text{Using } log = exponent\text{, We know base is b} \\ \log_b 27 &= 3 && \text{Our solution} \end{aligned}$$

c.

$$\begin{aligned} e^y &= 33 && \text{Using } log = exponent\text{, We know base is } e \\ \log_e 33 &= y && \text{Our solution} \end{aligned}$$

2.2 Evaluating logarithm without using calculator

By using the definition of logarithm, we can find the value of some logarithms. For example, to find $log_3 81$, we should ask ourselves, "3 to what power gives 81?" In other words,

$$3^? = 81$$

It is obvious that the answer is 4 because $3^4 = 81$. Therefore $\log_3 81 = 4$. In next section, you will see that product rule makes it much easier to evaluate such logarithms.

Example 2.3. Without using the calculator evaluate each logarithm:

a. $\log_{10} 100$

b. $\log_5 \dfrac{1}{125}$

c. $\log_{36} 6$

d. $\log_3 \sqrt[7]{3}$

a.

$$\begin{aligned} \log_{10} 100 &=? && \text{10 to what power gives 100?} \\ &2 && \text{Our Solution} \end{aligned}$$

b.

$$\begin{aligned} \log_5 \frac{1}{125} &=? && \frac{1}{125} \text{ is equal to } 5^{-3} \\ & && \text{5 to what power gives } 5^{-3}? \\ &-3 && \text{Our Solution} \end{aligned}$$

c.

$$\begin{aligned} \log_{36} 6 &=? && \text{36 to what power gives 6?} \\ &\frac{1}{2} && \text{Our solution} \end{aligned}$$

d.

$$\begin{aligned} \log_3 \sqrt[7]{3} &= ? && \sqrt[7]{3} \text{ is equal to } 3^{1/7} \\ & && 3 \text{ to what power gives } 3^{1/7}? \\ \frac{1}{7} & && \text{Our Solution} \end{aligned}$$

Note 2.1. In next section, you will see that power rule property will help us find this type of logarithms easily.

2.3 Common and natural logarithm

In mathematics, we love number 10 (why?). Euler number also appears in a lot of problems. That's why most of the time, our bases are either 10 or e.

log and ln

- $\log_{10} x$ is written as "$\log x$" and is called common logarithm.
- $\log_e x$ is written as "$\ln x$" and is called natural logarithm.

2.4 Identity and inverse properties

We know that $a^1 = a$. We can rewrite this expression in logarithmic form, using (2.1), and we will get

$$\log_a a = 1 \tag{2.2}$$

Likewise, we must know that $a^0 = 1$, and by using (2.1), we have

$$\log_a 1 = 0 \tag{2.3}$$

These two properties are called *identity properties*-because we are having number 1.

As we discussed earlier, we realized that exponential and logarithmic functions are inverse of each other. Here we can prove it easily. First we know that exponential function is a one-to-one function, so it must have an inverse. Let's consider $f(x) = a^x$,

$$\begin{aligned} y &= a^x && \text{Switch } x \text{ and } y \\ x &= a^y && \text{We need to solve for } x, x \text{ is an exponent} \\ & && \text{Use definition of log, equation (2.1)} \\ \log_a x &= y && \text{replace } y \text{ with } f^{-1}(x) \\ \log_a x &= f^{-1}(x) \end{aligned}$$

Thus, inverse of exponential function is logarithmic function and because they both undo each other, we have

$$f\left(f^{-1}(x)\right) = x \tag{2.4}$$
$$f^{-1}\left(f(x)\right) = x \tag{2.5}$$

Considering $f(x) = a^x$ and $f^{-1}(x) = log_a x$, we will get

$$a^{\log_a x} = x \qquad \text{Using property (2.4)} \tag{2.6}$$
$$\log_a(a^x) = x \qquad \text{Using property (2.5)} \tag{2.7}$$

These simple properties make us to find some logarithm without using calculator. Table 2.1 summarize identity and inverse properties for any positive base, $a > 0$:

Table 2.1: Identity and inverse properties

Identity properties	Inverse properties
$\log_a a = 1$	$a^{\log_a x} = x$
$\log_a 1 = 0$	$\log_a(a^x) = x$

We can also use identity and inverse properties for common and natural logarithm by changing the base to 10 and e, respectively. Table 2.2 shows these properties using log and ln.

Table 2.2: Identity and inverse properties using common and natural log

Common logarithm	Natural logarithm
$\log 10 = 1$	$\ln e = 1$
$\log 1 = 0$	$\ln 1 = 0$
$10^{\log x} = x$	$e^{\ln x} = x$
$\log 10^x = x$	$\ln e^x = x$

Example 2.4. Evaluate each logarithm:

a. $\log_9 9$ **b.** $\log_8 1$

Using identity properties, we will get

$$\log_9 9 = 1 \qquad \text{Identity property (2.2)}$$
$$\log_8 1 = 0 \qquad \text{Identity property (2.3)}$$

2.5 Graph of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$. As you can observe, the $x = 0$ is a vertical asymptote.

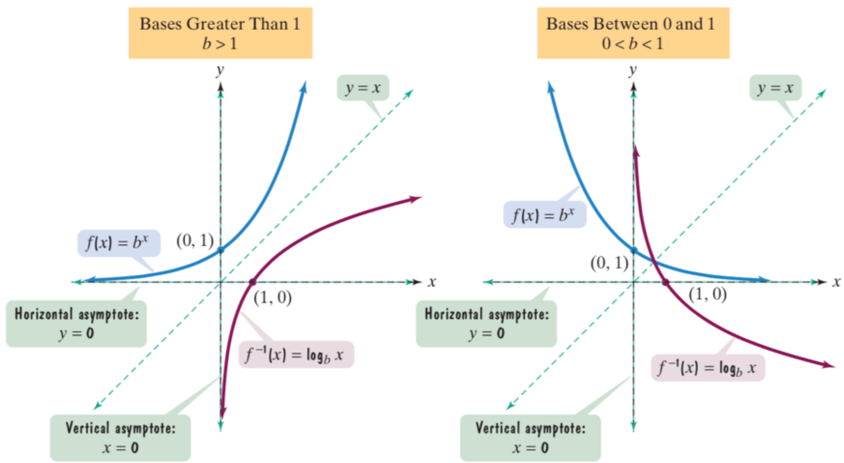

Figure 2.3: The Graph of exponential and logarithmic functions

You can also plot each logarithmic graph by constructing a table of values and connecting points.

Characteristics of $y = log_a x$

- The domain is $(0, +\infty)$
- The range is $(-\infty, +\infty)$
- The x-intercept is $(1, 0)$
- $x = 0$ is a vertical asymptote

2.5.1 Domain of logarithmic function

When determining domain, it is important to determine where the function would not exist. Regarding logarithmic function, we can only take the logarithm of values greater than 0. In other words, if we have $f(x) = log_a(g)$, therefore the domain of this function consists of all x for which $g > 0$.

Example 2.5. Find the domain of each function.

a. $f(x) = \log_4(x-5)$

b. $g(x) = \ln(4-x)$

c. $h(x) = \ln x^2$

a.

$$\begin{aligned} f(x) &= \log_4(x-5) && (x-5) \text{ should be positive} \\ x-5 &> 0 && \text{Solve for } x \\ x &> 5 && \text{Our domain} \\ (5, +\infty) & && \text{In interval notation} \end{aligned}$$

b.

$$\begin{aligned} g(x) &= \ln(4-x) && (4-x) \text{ should be positive} \\ 4-x &> 0 && \text{Solve for } x \\ 4 &> x && \text{Our domain} \\ (-\infty, 4) & && \text{In interval notation} \end{aligned}$$

c.

$$\begin{aligned} h(x) &= \ln x^2 && x^2 \text{ should be positive} \\ x^2 &> 0 && \text{Solve for } x \\ x > 0 \quad orx &< 0 && \text{Our domain} \\ (-\infty, 0) \cup (0, +\infty) & && \text{In interval notation} \end{aligned}$$

Example 2.6. The percentage of adult height attained by a boy who is x years old can be modeled by

$$f(x) = 29 + 48.8\log(x+1)$$

where x represents the boy's age (from 5 to 15) and $f(x)$ represents the percentage of his adult height. Approximately what percentage of his adult height has a boy attained at age ten?

Substitute the boy's age, 10, for x and evaluate the function.

$$f(10) = 29 + 48.8\log(10+1)$$

$$
\begin{aligned}
f(10) &= 29 + 48.8\log(11) \\
f(10) &= 29 + 48.8(1.04139) \\
f(10) &= 79.819832 \approx 80
\end{aligned}
$$

A 10-year-old boy has attained approximately 80% of his adult height.

Properties of Logarithms

3.1 More properties

We will discuss three important properties of logarithms. We can use them either to expand or condense logarithms. All of them can be proven using the exponent rules.

Rules of Logarithms

$$\log_b(M \cdot N) = \log_b M + \log_b N \qquad \text{Product rule} \qquad (3.1)$$

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N \qquad \text{Quotient rule} \qquad (3.2)$$

$$\log_b(M^p) = p \log_b M \qquad \text{Power rule} \qquad (3.3)$$

Note 3.1. The product rule is telling us that the logarithm of a product is the sum of logarithms. However, based on the quotient rule, the log of a quotient is the difference of logs. Regarding power rule, you can think of it as pulling the exponent in the front.

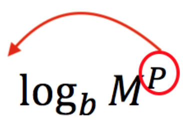

Figure 3.1: Power rule

To **expand** a single logarithm follow these steps:

- First try to use either product rule (3.1) or quotient rule (3.2)
- Then use power rule (3.3)

If you want to **condense** several logarithms as a single log, however, it is convenient to use the guidelines listed below for condensing logarithms:

- First apply power rule to move out the number in front of the logarithm to the exponent of variable
- Apply product or quotient rule to change the addition and subtraction logarithms to the multiplication and division.

Example 3.1. Expand each following logarithmic expression.

a. $\log_6(7 \cdot 11)$

b. $\log(100x)$

c. $\log_8\left(\dfrac{23}{x}\right)$

d. $\ln\left(\dfrac{e^5}{11}\right)$

e. $\log(x+4)^2$

f. $\log_6 3^9$

g. $\ln\sqrt[3]{x}$

h. $\log_6 36x$

a.

$\log_6(7 \cdot 11)$	Use product rule (3.1)
$\log_6 7 + \log_6 11$	Our solution

b.

$\log(100x)$	Use product rule (3.1)
$\log 100 + \log x$	We know $10^2 = 100$ so $\log 100 = 2$
$2 + \log x$	Our solution

c.

$\log_8\left(\dfrac{23}{x}\right)$	Use quotient rule (3.2)
$\log_8 23 - \log_8 x$	Our solution

d.

$\ln\left(\dfrac{e^5}{11}\right)$	Use quotient rule (3.2)
$\ln e^5 - \ln 11$	We are technically done!
	But we can find $\ln e^5$ by
	using the power rule (3.3), first
$5\ln e - \ln 11$	then use identity (2.2)

$$5(1) - \ln 11 \quad \text{Multiply}$$
$$5 - \ln 11 \quad \text{Our solution}$$

e.

$$\log(x+4)^2 \quad \text{Use power rule (3.3)}$$
$$2\log(x+4) \quad \text{Our Solution}$$

f.

$$\log_6 3^9 \quad \text{Use power rule (3.3)}$$
$$9\log_6(3) \quad \text{Our Solution}$$

g.

$$\ln \sqrt[3]{x} \quad \text{Rewrite in rational notation (1.3)}$$
$$\ln x^{1/3} \quad \text{Use power rule (3.3)}$$
$$\frac{1}{3}\ln x \quad \text{Our Solution}$$

h.

$$\log_6 36x \quad \text{Use product rule (3.1)}$$
$$\log_6 36 + \log_6 x \quad \text{6 to power 2 gives us 36, so } \log_6 36 = 2$$
$$2 + \log_6 x \quad \text{Our solution}$$

Note 3.2. As you observed in Example 1, in some cases you can simplify your answer more. You always need to keep asking yourself whether you can simplify it more.

Example 3.2. Write as each expression a single logarithm.

a. $\log 25 + \log 4$

b. $2\ln(x-3) - \ln x$

c. $\frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y$

The question is asking us to condense each expression.
a.

$$\begin{aligned} \log 25 + \log 4 & \quad \text{Use the product rule} \\ \log(25 \cdot 4) & \quad \text{Multiply} \\ \log(100) & \quad \text{Simplify, 100 is } 10^2 \\ \log(10^2) & \quad \text{Apply power rule} \\ 2\log(10) & \quad \text{We know } \log 10 = 1 \\ 2 & \quad \text{Our Solution} \end{aligned}$$

b.

$$\begin{aligned} 2\ln(x-3) - \ln x & \quad \text{Apply power rule} \\ \ln(x-3)^2 - \ln x & \quad \text{Use quotient rule} \\ \ln\frac{(x-3)^2}{x} & \quad \text{Our Solution} \end{aligned}$$

c.

$$\begin{aligned} \frac{1}{4}\log_b x - 2\log_b 5 - 10\log_b y & \quad \text{Apply power rule} \\ \log_b x^{1/4} - \log_b 5^2 - \log_b y^{10} & \quad \text{Use quotient rule on first two logs} \\ \log_b(\frac{x^{1/4}}{25}) - \log_b y^{10} & \quad \text{Apply quotient rule, again} \\ \log_b(\frac{x^{1/4}}{25y^{10}}) & \quad \text{Our Solution} \end{aligned}$$

3.2 Change of base property

We may often need to evaluate logarithms with other bases. In this case, we can change the base of a logarithm to any other base (including 10 and e), using the change-of-base property.For any logarithmic bases a and N, and any positive number M,

$$log_N M = \frac{\log_b M}{\log_b N} \tag{3.4}$$

In other words, the logarithm of M with base N is equal to the logarithm of M with any new base divided by the logarithm of N with that new base.

$$\log_N M = \frac{\log_a M}{\log_a N}$$

Figure 3.2: Arrows to help us remember: M goes up, N goes down.

The change-of-base property is used to write a logarithm in terms of quantities that can be evaluated with a calculator. Because calculators contain keys for common (base 10) and natural (base e) logarithms, we will frequently introduce base 10 or base e.

Table 3.1: Change-of-base property: introducing common and natural log

Common logarithm	Natural logarithm
$log_N M = \dfrac{\log M}{\log N}$	$log_N M = \dfrac{\ln M}{\ln N}$

Example 3.3. Use natural logarithms to evaluate $\log_7 2506$. Round your answer to 2 decimal places.

$$\begin{aligned} \log_7 2506 & \quad \text{Use change-of-base property} \\ \frac{\ln 2506}{\ln 7} & \quad \text{Use calculator to find each ln} \\ \frac{7.82644314}{1.94591015} & \quad \text{Divide} \\ \approx 4.02 & \quad \text{Our solution} \end{aligned}$$

Example 3.4. Use common logarithms to evaluate $\log_3 764$.

$$\begin{aligned} \log_3 764 & \quad \text{Use change-of-base property} \\ \frac{\log 764}{\log 3} & \quad \text{Use calculator to find each log} \\ \frac{2.88309336}{0.47712125} & \quad \text{Divide} \\ \approx 6.04 & \quad \text{Our solution} \end{aligned}$$

Exponential and Logarithmic Equations

In this section, we will learn how to solve the exponential and logarithmic equations An exponential is an equation containing a variable in an exponent. Examples are

$$2^{(4x+1)} = 8 \qquad (12)^{(3x-x^2)} = 144 \qquad (18)^5 = 18^{(y^3-12)}$$

On the other hand, a logarithmic equation is an equation containing a variable in a logarithmic expression. Examples are

$$\log_{12}(y+4) = 4 \qquad \ln(z+4) - \ln(2z-1) = \ln\frac{1}{z} \qquad \log_3 10 = \log_3(x^2+7x)$$

4.1 Exponential equations

There are two different types of exponential equations:

- Exponential equations with the same base on each side;
- Exponential equations with the different bases.

Exponential equations

Exponential equations with the same base on each side

$$\text{if } b^M = b^N, \text{ then } M = N$$

Exponential equations with different bases

1. Isolate the exponential expression.
2. If the base is 10, then take common logarithm (log) on both sides If the base is a number other than 10, take the natural logarithm (ln) on both sides.
3. Apply power rule
4. Solve for the variable

Example 4.1. Solve each equations.

a. $5^{3x-6} = 125$

b. $8^{x+2} = 4^{x-3}$

c. $5^x = 134$

d. $10^x = 8000$

e. $3^{2x-1} = 7^{x+1}$

a.

$$\begin{aligned} 5^{3x-6} &= 125 && \text{125 is } 5^3 \\ 5^{3x-6} &= 5^3 && \text{Same base on both sides} \\ 3x - 6 &= 3 && \text{Solve for } x \\ x &= 3 && \text{Our Solution} \end{aligned}$$

b.

$$\begin{aligned} 8^{x+2} &= 4^{x-3} && \text{8 is } 2^3 \text{ and 4 is } 2^2 \\ (2^3)^{x+2} &= (2^2)^{x-3} && \text{Use rule (1.4)} \\ 2^{3(x+2)} &= 2^{2(x-3)} && \text{Same base on both sides} \\ 3(x+2) &= 2(x-3) && \text{Solve for } x \\ x &= -12 && \text{Our solution} \end{aligned}$$

c.

$$\begin{aligned} 5^x &= 134 && \text{Different bases, take ln} \\ \ln 5^x &= \ln 134 && \text{Apply power rule} \\ x \ln 5 &= \ln 134 && \text{Solve for } x \\ x &= \frac{\ln 134}{\ln 5} && \text{Use calculator} \\ x &\approx 3.04 && \text{Our solution} \end{aligned}$$

d.

$$\begin{aligned} 10^x &= 8000 && \text{Different bases, take log} \\ \log 10^x &= \log 8000 && \text{Apply power rule} \\ x \log 10 &= \log 8000 && \text{We know } \log 10 = 1 \end{aligned}$$

$$\begin{aligned} x &= \log 8000 && \text{Use calculator} \\ x &\approx 3.90 && \text{Our Solution} \end{aligned}$$

e.

$$\begin{aligned} 3^{2x-1} &= 7^{x+1} && \text{Different bases, take ln} \\ \ln 3^{2x-1} &= \ln 7^{x+1} && \text{Apply power rule} \\ (2x-1)\ln 3 &= (x+1)\ln 7 && \text{Distribute} \\ 2x\ln 3 - \ln 3 &= x\ln 7 + \ln 7 && \text{To Isolate } x\text{, add } \ln 3 \\ 2x\ln 3 &= x\ln 7 + \ln 3 + \ln 7 && \text{Subtract } x\ln 7 \\ 2x\ln 3 - x\ln 7 &= \ln 3 + \ln 7 && \text{Factor } x \text{ on LHS} \\ (2\ln 3 - \ln 7)x &= \ln 3 + \ln 7 && \text{Divide by } 2\ln 3 - \ln 7 \\ x &= \frac{\ln 3 + \ln 7}{2\ln 3 - \ln 7} && \text{Use calculator} \\ x &\approx 12.11 && \text{Our solution} \end{aligned}$$

Note 4.1. Don't use a calculator immediately whenever you see a logarithm of a number in your equations. Instead, try to solve for an unknown variable and simplify as much as possible. Then use the calculator and round your answer.

Example 4.2. Solve $e^{2x} - 8e^x + 7 = 0$. To solve these types of problems, first use (1.4) to rewrite $e^{2x} = (e^x)^2$. Then replace all e^x with y,

$$\begin{aligned} (e^x)^2 - 8e^x + 7 &= 0 && \text{Use (1.4)} \\ y^2 - 8y + 7 &= 0 && \text{Substitute } e^x = y \end{aligned}$$

Using this trick, you can no easily see a trinomial which can be factored easily. $(y-7)(y-1) = 0$. Then substitute back $y = e^x$.

$$(e^x - 7)(e^x - 1) = 0 \qquad \text{Set each factor equal to zero}$$

So you will get

$$\begin{cases} e^x - 7 = 0 & \to e^x = 7 \\ e^x - 1 = 0 & \to e^x = 1 \end{cases}$$

Take ln from both sides and recall from (2.2) that ln $e = 1$

$$\begin{cases} \ln e^x = 7 \to x\ln e = 7 \to x = 7 \\ \ln e^x = 1 \to x\ln e = 1 \to x = 1 \end{cases}$$

4.2 Logarithmic equations

Similar to exponential equations, we have two situations:

- Logarithms with the same bases on both sides
- Only one logarithm on one side

Logarithmic equations

Logarithms with the same bases on both sides

$$\text{if } \log_b M = log_b N, \text{ then } M = N$$

Only one logarithm on one side

1. Isolate the log.
2. Use the definition of logarithms (2.1) rewrite the log in exponential form.
3. Solve for variable.

Very important:
Always check your answer(s). The argument of log must be positive.

Example 4.3. Solve the following logarithmic equation:

$$\ln(x-3) = \ln(7x-23) - \ln(x+1)$$

We start by condensing the LHS:

$$\begin{aligned}
\ln(x-3) &= \ln\frac{(7x-23)}{x+1} && \text{Same base on both sides} \\
x-3 &= \frac{(7x-23)}{x+1} && \text{Multiply both sides by}(x+1) \\
(x-3)(x+1) &= 7x-23 && \text{FOIL} \\
x^2-2x-3 &= 7x-23 && \text{subtract } 7x-23 \\
x^2-9x-20 &= 0 && \text{Factor} \\
(x-4)(x-5) &= 0 && \text{Set each equal to zero} \\
x = 4 \text{ and } x &= 5 && \text{Always check the solutions}
\end{aligned}$$

Check.
Plug $x = 4$ and $x = 5$. If you get any negative log, then that answer is not correct. Otherwise, that x-value will be our answer.

For $x = 4$:

$$\begin{aligned}
\ln(4-3) &= \ln(7(4)-23) - \ln(4+1) && \text{Plug it and then simplify}\\
\ln 1 &= \ln(5) - \ln(5) && \text{No negative log appeared}\\
x &= 4 \quad \checkmark && \text{Correct answer}\\
\text{For } x &= 5:\\
\ln(5-3) &= \ln(7(5)-23) - \ln(5+1) && \text{Plug it and then simplify}\\
\ln 2 &= \ln(12) - \ln(6) && \text{No negative log appeared}\\
x &= 5 \quad \checkmark && \text{Correct answer}
\end{aligned}$$

The solution set is $\{4,\ 5\}$.

Example 4.4. Solve the following logarithmic equations.

$$\log_2(x-2) = 3$$

$$\begin{aligned}
\log_2(x-2) &= 3 && \text{One logarithm, use definition of log}\\
2^3 &= x-2 && \text{Solve for } x\\
10 &= x && \text{Check the answer!}
\end{aligned}$$

check.

$$\begin{aligned}
\text{For } x &= 10:\\
\log_2(10-2) &= 3 && \text{Plug it and then simplify}\\
\log_2(8) &= 3 && \text{No negative log appeared}\\
x &= 10 \quad \checkmark && \text{Correct answer}
\end{aligned}$$

The solution set is $\{10\}$.

Example 4.5. Solve the following logarithmic equations.

$$4\ln(3x) = 8$$

We need to isolate ln first,

$$\begin{aligned}
4\ln(3x) &= 8 && \text{Divide both sides by 4}\\
\ln(3x) &= 2 && \text{One logarithm, use definition of log}\\
3x &= e^2 && \text{Solve for } x\\
x &= \frac{e^2}{3} && \text{Check the answer!}
\end{aligned}$$

check.

$$\text{For } x = \frac{e^2}{3}:$$

$$
\begin{aligned}
4\ln\left(3\left(\frac{e^2}{3}\right)\right) &= 8 && \text{Plug it and then simplify}\\
4\ln\left(e^2\right) &= 8 && \text{No negative log appeared}\\
x &= \frac{e^2}{3} \quad \checkmark && \text{Correct answer}
\end{aligned}
$$

The solution set is $\left\{\dfrac{e^2}{3}\right\}$.

Example 4.6. Solve the following logarithmic equations.

$$\log x + \log(x-3) = 1$$

$$
\begin{aligned}
\log x + \log(x-3) &= 1 && \text{Condense logs using (3.1)}\\
\log x(x-3) &= 1 && \text{One logarithm, use definition of log}\\
x(x-3) &= 10^1 && \text{Distribute LHS}\\
x^2 - 3x &= 10 && \text{Subtract 10 from both sides}\\
x^2 - 3x - 10 &= 0 && \text{Factor}\\
(x+2)(x-5) &= 0 && \text{Set each equal to zero}\\
x = -2 \text{ and } x &= 5 && \text{Always check!}
\end{aligned}
$$

check.

$$
\begin{aligned}
\text{For } x &= -2: \\
\log(-2) + \log(-2-3) &= 1 && \text{Plug it and then simplify}\\
\log(-2) + \log(-5) &= 1 && \text{Negative log appeared}\\
x &= -2 \quad ✗ && \text{Wrong answer}
\end{aligned}
$$

$$
\begin{aligned}
\text{For } x &= 5: \\
\log(5) + \log(5-3) &= 1 && \text{Plug it and then simplify}\\
\log(5) + \log(2) &= 1 && \text{No negative log appeared}\\
x &= 5 \quad \checkmark && \text{Correct answer}
\end{aligned}
$$

The solution set is $\{5\}$.

Exponential Growth and Decay: Modeling Data

There are many phenomena in the worlds that the amount of something whether grows or decay fast. For example, compounding money, bacteria growth in a petri dish and radioactive decay of an element.

5.1 The exponential growth

An exponential growth process is one in which the rate of increase of a quantity is proportional to the present value of that quantity. The simplest example is a savings account.

The exponential growth

The model for exponential growth is

$$A = A_0 e^{kt} \tag{5.1}$$

where,

$$\begin{aligned} A_0 &= \text{initial amount or size} \\ t &= \text{time} \\ A &= \text{The amount at time } t \\ k &= \text{a constant.} \end{aligned}$$

In this model, $k > 0$ and it is called the growth rate.

Example 5.1. In 2000, the population of Africa was 807 million and by 2011 it had grown to 1052 million.

a. Use the exponential growth model $A = A_0 e^{kt}$, in which t is the number of years after 2000, to find the exponential growth function that models the data.

b. By which year will Africa's population reach 2000 million, or two billion?

a. We are given

$$\begin{aligned} A_0 &= 807 \quad \text{million} \\ t &= 11 \quad \text{years} \\ A &= 1050 \quad \text{million} \end{aligned}$$

and we are looking for k. Plug them into $A = A_0 e^{kt}$ and solve for k.

$$\begin{aligned}
A &= A_0 e^{kt} && \text{Substitute } A,\ A_0 \text{ and } t \\
1052 &= 807 e^{k(11)} && \text{Divide both sides by 807} \\
\frac{1052}{807} &= e^{k(11)} && \text{Exponential equation, take ln} \\
\ln\left(\frac{1052}{807}\right) &= \ln e^{k(11)} && \text{Apply power rule} \\
\ln\left(\frac{1052}{807}\right) &= k(11)\ln e && \text{We know } \ln e = 1 \\
\ln\left(\frac{1052}{807}\right) &= k(11) && \text{Divide both sides by 11} \\
\frac{\ln\left(\frac{1052}{807}\right)}{11} &= k && \text{Use calculator} \\
\frac{0.2651247}{11} &= k && \\
0.024 &\approx k && \text{Our solution}
\end{aligned}$$

b. We have

$$\begin{aligned} A_0 &= 807 \quad \text{million} \\ A &= 2000 \quad \text{million} \end{aligned}$$

and we are looking for t. Substitute and solve for it.

$$\begin{aligned}
2000 &= 807 e^{0.024t} && \text{Divide both sides by 807} \\
\frac{2000}{807} &= e^{0.024t} && \text{Exponential equation, take ln} \\
\ln\left(\frac{2000}{807}\right) &= \ln e^{0.024t} && \text{Apply power rule} \\
\ln\left(\frac{2000}{807}\right) &= 0.024t\ln e && \text{We know } \ln e = 1 \\
\ln\left(\frac{2000}{807}\right) &= 0.024t && \text{Divide both sides by 0.024} \\
\frac{\ln\left(\frac{2000}{807}\right)}{0.024} &= t && \text{Use calculator} \\
\frac{0.024102248}{0.024} &= t &&
\end{aligned}$$

$$38 \approx t \qquad \text{Our solution}$$

So after 38 years, in 2038, the population of Africa reach 2000 million.

5.2 The exponential decay

When a population decays exponentially, it decreases at a rate that is proportional to its size at any time t.

The exponential decay

The model for exponential decay is

$$A = A_0 e^{kt} \tag{5.2}$$

where,

$$\begin{aligned} A_0 &= \text{initial amount or size} \\ t &= \text{time} \\ A &= \text{The amount at time } t \\ k &= \text{a constant.} \end{aligned}$$

In this model, $k < 0$ and it is called the decay rate.

Note 5.1. Many times, the amount of a substance is expressed in terms of half-life, meaning

$$A = A_0/2$$

the time it takes for half of any given quantity to decay so that only half of its original amount remains.

Example 5.2. Strontium-90 is a waste product from nuclear reactors. As a consequence of fallout from atmospheric nuclear tests, we all have a measurable amount of strontium-90 in our bones.

a. The half-life of strontium-90 is 28 years, meaning that after 28 years a given amount of the substance will have decayed to half the original amount. Find the exponential decay model for strontium-90.

b. Suppose that a nuclear accident occurs and releases 60 grams of strontium-90 into the atmosphere. How long will it take for strontium-90 to decay to a level of 10 grams?

a. We know the half-life of strontium-90 is 28 years. This means after 28 years, the amount of strontium-90 will be $A = \frac{A_0}{2}$. Plug them into the (5.2) and solve for k.

$$
\begin{aligned}
A &= A_0 e^{kt} && \text{Substitute } A \text{, and } t \\
\frac{A_0}{2} &= A_0 e^{k(28)} && \text{Cancel out } A_0 \\
\frac{1}{2} &= e^{k(28)} && \text{Exponential equation, take ln} \\
\ln\left(\frac{1}{2}\right) &= \ln e^{k(28)} && \text{Apply power rule} \\
\ln\left(\frac{1}{2}\right) &= k(28)\ln e && \text{We know } \ln e = 1 \\
\ln\left(\frac{1}{2}\right) &= k(28) && \text{Divide both sides by 28} \\
\frac{\ln\left(\frac{1}{2}\right)}{28} &= k && \text{Use calculator} \\
\frac{-0.6931472}{28} &= k \\
-0.0248 &\approx k && \text{Our solution}
\end{aligned}
$$

b. We are looking for t, when $A_0 = 60$ grams and $A = 10$ grams. Thus,

$$
\begin{aligned}
10 &= 60e^{-0.0248t} && \text{Divide both sides by 10} \\
\frac{10}{60} &= e^{-0.0248t} && \text{Reduce LHS} \\
\frac{1}{6} &= e^{-0.0248t} && \text{Exponential equation, take ln} \\
\ln\frac{1}{6} &= \ln e^{-0.0248t} && \text{Apply power rule} \\
\ln\frac{1}{6} &= -0.0248t\ln e && \text{We know } \ln e = 1 \\
\ln\frac{1}{6} &= -0.0248t && \text{Divide both sides by -0.0248} \\
\frac{\ln\frac{1}{6}}{-0.0248} &= t && \text{Use calculator} \\
\frac{-1.7917595}{-0.0248} &= t \\
72 &\approx t && \text{Our solution}
\end{aligned}
$$

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