Chapter 0

The essence of mathematics is not to make simple things complicated, but to make complicated things simple.

S. Gudder

Mohammad Azarnoush
Mechanical Engineering
mazarnou@asu.edu

Contents

Set and Numbers

1.1 Set

A set is a collection of objects. For example, the set of days of the week is a set that contains 7 objects: Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday. The things in the collection are called **elements of the set**.

1.1.1 Set Notation

A set is often expressed by listing their elements between commas. enclosed by braces. We often let uppercase letters stand for sets. For example, set S containing numbers 1, 2, 3, 4 and 5 can be written as:

$$S = \{1, 2, 3, 4, 5\}$$

This form of representation, called **the roster method**.
A set can also be written in **set-builder notation**. In this notation, the elements of the set are described but not listed. Here is an example

$$\{\ x\ \mid x\ \text{ is positive number less than 6 }\ \}$$

The set of all x

such that description

The same set using the roster method is

$$\{1, 2, 3, 4, 5\}$$

1.1.2 Infinite and finite sets

Consider set G.

$$G = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$$

Here the dots, called ellipsis, indicate a pattern of numbers that continues forever. A set is called an **infinite set** if it has infinitely many elements; otherwise it is called a **finite set**.

1.1.3 Empty set

There is a special set that, although small, plays a big role. The empty set or Null set is the set $\{\}$ that has no elements. We denote it as ϕ, so $\phi = \{\}$. Whenever you see the symbol ϕ, it stands for $\{\}$.

1.2 Numbers

Some sets are so significant and prevalent that we reserve special symbols for them. The set of natural numbers (i.e., the positive numbers) is denoted by $\mathbb{N}$, that is

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

Set of whole numbers, $\mathbb{W}$, is containing all natural numbers including 0

$$\mathbb{W} = \{0, 1, 2, 3, 4, \ldots\}$$

The set of integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

is another fundamental set. This set not only includes positive and 0, but also negative numbers.
The set of rational numbers is the set of all numbers that can be expressed as a quotient of two integers, with the denominator not 0.

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid \text{a and b are integers and } b \neq 0 \right\}$$

Needles to say, there is a set of irrational numbers $\mathbb{I}$. This set is the set of all numbers whose decimal representations are neither terminating nor repeating. Irrational numbers cannot be expressed as a quotient of integers.

$$\mathbb{I} = \{\, x \mid x \text{ is not a rational number} \,\}$$

Examples of irrational numbers are

$$\begin{aligned} \sqrt{2} &\approx 1.414214 \\ -\sqrt{3} &\approx -1.73205 \\ \pi &\approx 3.14159 \\ e &\approx 2.71828 \end{aligned}$$

Finally, we have a set of real numbers which is the set if numbers that are either rational or irrational; In other words, all numbers we are familiar with them.

$$\mathbb{R} = \{\, x \mid x \text{ is rational or irrational} \,\}$$

Algebra Essentials

2.1 Simplifying

In algebra we will often need to simplify an expression to make it easier to use. There are three basic forms of simplifying which we will review here.

1. Evaluating algebraic expressions
2. Combining like terms
3. Distributive property

2.1.1 Evaluating algebra expressions

The first form of simplifying expressions is used when we know what number each variable in the expression represents. If we know what they represent we can replace each variable with the equivalent number and simplify what remains using order of operations.

Example 2.1. Evaluate $p(q + 4)$ when $p = 2$ and $q = 3$.

Replace p with 2 and q with 3.

$$\begin{aligned} &p(q + 4) \\ &(2)((3) + 4) && \text{Evaluate the parenthesis} \\ &(2)(7) && \text{Multiply} \\ &14 && \text{Our solution} \end{aligned}$$

Note 2.1. Whenever a variable is replaced with something, we will put the new number inside a set of parenthesis. Notice the 2 and 3 in the previous example are in parenthesis. This is to preserve operations that are sometimes lost in a simple replacement. Sometimes the parenthesis won't make a difference, but it is a good habit to always use them to prevent problems later.

Example 2.2. Evaluate $xy + (4 - y)(\frac{x}{3})$ when $x = -3$ and $y = -5$.

Replace all x's with -3 and all y's with -5.

$$\begin{aligned}
xy + (4 - y)(\frac{x}{3}) & \\
(-3)(-5) + (4 - (-5))\left(\frac{(-3)}{3}\right) & \qquad \text{Evaluate parenthesis} \\
15 + (9)(-1) & \qquad \text{Multiply} \\
15 - 9 & \qquad \text{Subtract} \\
6 & \qquad \text{Our solution}
\end{aligned}$$

2.1.2 Combining like terms

It will be more common in our study of algebra that we do not know the value of the variables. In this case, we will have to simplify what we can and leave the variables in our final solution. One way we can simplify expressions is to combine like terms. Like terms are terms where the variables match exactly (exponents included).
If we have like terms we are allowed to add (or subtract) the numbers in front of the variables, then keep the variables the same.

Example 2.3. Simplify.

$$10x - 4y - 19x + 6y$$

$$\begin{aligned}
10x - 4y - 19x + 6y & \qquad \text{Combine like terms } 10x \text{ and } -19x \\
-9x - 4y + 6y & \qquad \text{Combine like terms } -4y \text{ and } 6y \\
-9x + 2y & \qquad \text{Our answer}
\end{aligned}$$

Example 2.4. Simplify.

$$12y^2 - 5y + 4 - 9y^2 + 6y - 1$$

$$\begin{aligned}
12y^2 - 5y + 4 - 9y^2 + 6y - 1 & \qquad \text{Combine like terms } 12y^2 \text{ and } -9y^2 \\
3y^2 - 5y + 4 + 6y - 1 & \qquad \text{Combine like terms } -5y \text{ and } 6y \\
3y^2 + y + 4 - 1 & \qquad \text{Finally subtract } 4 - 1 \\
3y^2 + y + 3 & \qquad \text{Our answer}
\end{aligned}$$

2.1.3 Distributive property

A final method to simplify is known as distributing. Often as we work with problems there will be a set of parenthesis that make solving a problem difficult, if not impossible. To get rid of these unwanted parenthesis we have the distributive property. Using this property we multiply the number in front of the parenthesis by each term inside of the parenthesis.

$$a(b+c) = ab + ac$$

Example 2.5. Simplify $2(3x - 10)$.

$$\begin{aligned}
2(3x-10) &\qquad \text{Distribute 2 over } 3x \text{ and } -10\\
2(3x)+2(-10) &\qquad \text{Multiply}\\
6x-20 &\qquad \text{Our solution}
\end{aligned}$$

Example 2.6. Simplify $-6(-4+6y)$.

$$\begin{aligned}
-6(-4+6y) &\qquad \text{Distribute } -6 \text{ over } -4 \text{ and } 6y\\
-6(-4)-6(6y) &\qquad \text{Multiply}\\
24-36y &\qquad \text{Our solution}
\end{aligned}$$

Note 2.2. The most common error in distributing is a sign error, be very careful with your signs!

Note 2.3. It is possible to distribute just a negative through parenthesis. If we have a negative in front of parenthesis we can think of it like a "-1" in front and distribute the "-1" through. This is shown in the following example.

Example 2.7. Simplify $-(7y - 4x + z - 3)$.

$$\begin{aligned}
-1(7y-4x+z-3) &\qquad \text{Negative can be thought of as } -1\\
-1(7y)-1(-4x)-1(z)-1(-3) &\qquad \text{Multiply each term by } -1\\
-7y+4x-z+3 &\qquad \text{Our solution}
\end{aligned}$$

Example 2.8. Simplify $5 + 2(3x - 8)$.

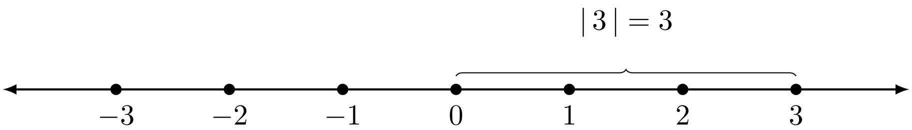
$5 + 2(3x - 8)$	Distribute 2, multiply each term by 2
$5 + 6x - 16$	Combine like terms
$6x - 11$	Our solution

Example 2.9. Simplify $4(6x - 1) - (4x - 19)$.

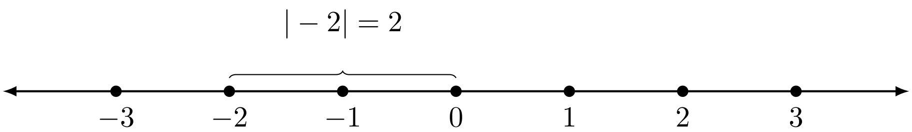
$4(6x - 1) - (4x - 19)$	Negative in middle can be thought of as -1
$4(6x - 1) - 1(4x - 19)$	Distribute
$24x - 4 - 4x + 19$	Combine like terms
$20x + 15$	Our solution

2.2 Absolute Value

The absolute value of a real number a is denoted by $|a|$. it is the distance from a to the origin 0 on the number line. For example, the $|3|$ is a distance from 3 to 0, which is 3.



Another example is $|-2|$.The distance from -2 to 0 is $+2$.



As you can see, because the absolute value represent a distance, therefore it is always positive.

$$\big|number\big| = +$$

We can give a formula for the absolute value of the number, which depends on whether a is positive or negative.

$$|a| = \begin{cases} a & \text{if a is positive} \\ -a & \text{if a is negative} \end{cases}$$

Example 2.10. Evaluate $4 - |-4-5|$.

Begin by evaluating inside the absolute value:

$$\begin{aligned} 4-|-4-5| &\qquad -4-5 \text{ is equal to } -9 \\ 4-|-9| &\qquad \text{The absolute value of } -9 \text{ is } 9 \\ 4-(9) &\qquad \text{Subtract} \\ -5 &\qquad \text{Our solution} \end{aligned}$$

2.2.1 Properties of absolute value

Some additional useful properties are given below.

- $|\,a\,| \geq 0$, Non-negativity
- $|\,a\,| = 0 \Longleftrightarrow a = 0$, Positive-definiteness
- $|\,a\,| = |-a|$, Evenness
- $|\,ab\,| = |a|\,|b|$, Multiplicative
- $\left|\dfrac{a}{b}\right| = \dfrac{|\,a\,|}{|\,b\,|}$, Preservation of division

2.2.2 Absolute value equations

When solving equations with absolute value we can end up with more than one possible answer. This is because what is in the absolute value can be either negative or positive and we must account for both possibilities when solving equations.

Example 2.11. Solve $|x| = 4$

Absolute value can be positive and negative, so our solution is

$$x = 4 \quad \textit{or} \quad x = -7$$

Note 2.4. When we have absolute values in our problem it is important to first isolate the absolute value, then remove the absolute value by considering both the positive and negative solutions.

Example 2.12. Solve $7 + |x| = 10$

Notice that absolute value is not alone.

$$\begin{aligned} 7+|x| &= 10 \qquad \text{Subtract 7 from both sides} \\ |x| &= 3 \end{aligned}$$

$$x = 3 \quad or \quad x = -3 \qquad \text{Our answers}$$

The following example requires two steps to isolate the absolute value. The idea is the same as a two-step equation, add or subtract, then multiply or divide.

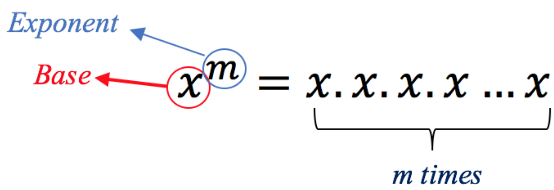
Example 2.13. Solve $-6|x| + 2 = -10$

$$\begin{aligned} -6|x| + 2 &= -10 && \text{Isolate } |x|\text{, Subtract 2 from both sides} \\ -6|x| &= -12 && \text{Divide both sides by -6} \\ |x| &= 2 && \text{Absolute value can be positive or negative} \\ x = 2 \quad or \quad x &= -2 && \text{Our solution} \end{aligned}$$

Exponents and Radicals

3.1 Exponent expressions

An exponent (also called power or degree) tells us how many times a number will be multiplied by itself. For example, consider x^5. The exponent is 5 and x is called the base. This means that the variable x will be multiplied by itself 5 times. You can also think of this as x to the fifth power.



3.1.1 Exponent Rules

All of the exponent rules are summarized in the Table 1. Practice plays an important role in learning exponent rules. Even if you forgot a particular rule, you can prove it by your self using the definition of exponents.
For instance, let's consider $x^3 \cdot x^5$. Here x^3 means that we should multiply x by itself 3 times. Same thing for x^5: multiply the x by itself 5 times.

$$\begin{aligned} x^3 \cdot x^5 &= (x \cdot x \cdot x)(x \cdot x \cdot x \cdot x \cdot x) \\ &= x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \end{aligned}$$

Since x is multiplied by itself 8 times, we can write our final answer like this

$$x^3 \cdot x^5 = x^8$$

As you can see, the summation of the exponents is $3 + 5 = 8$. So that's how we get our final solution when you multiply two numbers with the same bases-just need to add their exponents. Same thing if we have division. For example, $\dfrac{y^3}{y^2}$ is equal to

$$\begin{aligned} \frac{y^3}{y^2} &= \frac{\cancel{y} \cdot \cancel{y} \cdot y}{\cancel{y} \cdot \cancel{y}} \\ &= y \end{aligned}$$

The same procedure can be used to prove for the following rules.

Table 3.1: Exponent Rules

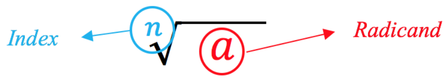
Name and Definition	***General Form***	***Example***
Product Rule Same base, add exponents	$x^m.x^n = x^{m+n}$	$x^2.x^3 = x^5$
Quotient Rule Same base, subtract exponents	$\frac{x^m}{x^n} = x^{m-n}$	$\frac{x^8}{x^5} = x^{8-5} = x^3$
Power Rule I Power raised to a power: Multiply exponent	$(x^m)^n = x^{mn}$	$(x^4)^7 = x^{4.7} = x^{28}$
Power Rule II Product to power: Distribute to each base	$(xy)^n = x^n y^n$	$(ab)^{10} = a^{10}b^{10}$
Power Rule III Division to power: Distribute to each base	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{z}{d}\right)^4 = \frac{z^4}{d^4}$
Negative Exponent I Flip and change sign to positive	$x^{-n} = \frac{1}{x^n}$	$x^{-5} = \frac{1}{x^5}$
Negative Exponent II Swap top and denominator and change sign to positive	$\frac{x^{-n}}{y^{-m}} = \frac{y^m}{x^n}$	$\frac{a^{-4}}{x^{-10}} = \frac{x^{10}}{a^4}$
Zero Exponent Anything to the zero power (except 0) is one	$x^0 = 1$	$(-10x^2)^0 = 1$

3.2 Radical expressions

While square roots are the most common type of radical we work with, we can take higher roots of numbers as well: cube roots, fourth roots, fifth roots, etc. The nth root of a number is a value that when multiplied by itself n times, you will get the original number.

$$\sqrt[n]{a} = b \Longleftrightarrow a = b^n \tag{3.1}$$

The symbol $\sqrt{}$ is called radical, and the small letter n inside the radical is called the index. It tells us which root we are taking, or which power we are "un-doing".



3.2.1 Square and higher-root

Remember the square root of a number is always positive.For square roots the index is 2. As this is the most common root, the two is not usually written.

$$
\begin{aligned}
\sqrt{4} &= 2 && \text{because } 2^2{=}4 \\
\sqrt[4]{81} &= 3 && \text{because } 3^4{=}81 \\
\sqrt[5]{16807} &= 7 && \text{because } 7^5{=}16807 \\
\sqrt{-125} &= \text{✗} && \text{Not defined in real numbers!!} \\
\sqrt[3]{-64} &= -4 && \text{because } (-4)^3{=}\text{-}64 \\
\sqrt[7]{-128} &= -2 && \text{because } (-2)^7{=}\text{-}128
\end{aligned}
$$

We must be careful of a few things as we work with higher roots.

1. It is important not to forget to check the index on the root. $\sqrt{81} = 9$ but $\sqrt[4]{81} = 3$. This is because $9^2 = 81$ and $3^4 = 81$.
2. Another thing to watch out for is negative numbers under roots. We can take an odd root of a negative number, because a negative number raised to an odd power is still negative. However, we cannot take an even root of a negative number. In this case, we will say is undefined.

Note 3.1. Later, we will discuss how to work with roots of negative, but for now we will simply say they are undefined.

Radical in real realm

- If index, n, is an odd number such as 3, 5, 7, 9, ... then

$$\sqrt[n]{+} = + \checkmark$$

$$\sqrt[n]{-} = - \checkmark$$

- If index, n, is an even number such as 2, 4, 6, 8,... then

$$\sqrt[n]{+} = + \checkmark$$

$$\sqrt[n]{-} = \text{ Not defined ✗}$$

Example 3.1. If possible, find the roots. If there is no real number root, say so.

a) $\sqrt[3]{216}$ b) $\sqrt[5]{-32}$ c) $\sqrt[4]{-81}$

a) The index is an odd number, we have an answer $\sqrt[3]{216} = 6$.

b) Although the radicand is a negative number, the index is an odd number, so we still have an answer $\sqrt[5]{-32} = -2$.

c) The index is an even number, 4. Therefore, when the radicand is negative we won't have any real solution. $\sqrt[4]{-81} =$ not defined ✗. The answer to this radical is actually an imaginary number which we learn how to deal with them later.

3.2.2 Exponential Notation

We can write the nth root in two different notations: One in radical form, which we are already familiar with it; Or using the exponential form. The latter form, is very useful especially when we want to simplify a root or logarithms.

Exponential Notation

If n is an integer greater than one and x is an integer, then the nth root of x can be expressed as

$$\sqrt[n]{x} = x^{1/n} \tag{3.2}$$

We can easily prove that $\sqrt[n]{x^m}$ is equal to $x^{m/n}$. We begin with the exponential notation form (3.2):

$$\begin{aligned} \sqrt[n]{x} &= x^{1/n} && \text{Raise both sides to the power of } m \\ \left(\sqrt[n]{x}\right)^m &= \left(x^{1/n}\right)^m && \text{Use power rule I} \\ \left(\sqrt[n]{x}\right)^m &= x^{m/n} && \text{(i)} \end{aligned}$$

The left-hand side means to multiply $\sqrt[n]{x}$, m times so

$$\begin{aligned} \left(\sqrt[n]{x}\right)^m &= \sqrt[n]{x} \cdot \sqrt[n]{x} \ldots \sqrt[n]{x} \\ \left(\sqrt[n]{x}\right)^m &= \sqrt[n]{x^m} && \text{(ii)} \end{aligned}$$

Comparing (i) and (ii), we will get

$$\sqrt[n]{x^m} = x^{m/n}$$

According to this formula when $m = n$, we will get $\sqrt[n]{x^n} = x^{n/n} = x$. In other words, raising the $\sqrt[n]{x}$ to the power of n yields to n. This works very well when the index is an odd number.Examples are

$$\sqrt[3]{10^3} = 3$$
$$\sqrt[5]{(-4)^5} = -4$$

However, when the index is an even number, we know we the answer of a root should be a positive number. For example, $\sqrt[2]{(-3)^2}$ is equal to 3 not -3. To solve this problem, we need to use absolute value.

Summary

$$\sqrt[n]{x^m} = x^{m/n} \tag{3.3}$$

For all real x,

$$\sqrt[n]{x^n} = |x| \quad \text{, if } n \text{ is an even number.} \tag{3.4}$$
$$\sqrt[n]{x^n} = x \quad \text{, if } n \text{ is an odd number.} \tag{3.5}$$

Example 3.2. Change to rational exponents and simplify.

a. $\sqrt[3]{x^3}$

b. $\sqrt[4]{w^4}$

c. $\sqrt[3]{125x^3y^6}$

a.

$\sqrt[3]{x^3}$	Use (3.5)
x	Our solution

b.

$\sqrt[4]{w^4}$	Use (3.4)
$\lvert w \rvert$	Our solution

c.

$\sqrt[3]{125x^3y^6}$	Use (3.3)
$\left(125x^3y^6\right)^{1/3}$	Use power rule II
$125^{(1/3)}x^{(3)(1/3)}y^{(6)(1/3)}$	Simplify
$5xy^2$	Our solution

Polynomials

4.1 Polynomials

A term is defined as an expression containing a number or the product of a number and one or more variables raised to powers. Examples of terms are

$$3x^2,\ -x^2y^3,\ 6ab,\ -10$$

A polynomial is a single term or a finite sum of terms. The powers of the variables in a polynomial must be **a positive integer**. For example, $4x^3 - 15x^2 - x + 2$ is an example of polynomial.

4.2 Monomial, binomial and trinomial

A polynomial with a single term is often called a **monomial**. Examples are,

$$7x, \quad -20, \quad 10y^7$$

A monomial can be a number, a variable, or the product of a number and one or more variables with whole number exponents.
Moreover, if a polynomial has only two terms, then it is often called a **binomial**. Such as

$$10x^4 - 1, \quad a^2 + b, \quad -2x^2 + 9x$$

Finally, a **trinomial** is another special type of polynomial with only three terms. Examples are

$$10x^2 + 5x - 1, \quad -15y^5 + y^4 + 12y^2$$

4.2.1 Degree of a polynomial

The degree of a polynomial is the highest power of the variable in the polynomial. Examples:

$$\begin{aligned} 4x^3 - 15x^2 + x - 2 &\qquad \text{Degree} = 3 \\ 7w - w^2 &\qquad \text{Degree} = 2 \\ 10y^7 - 100y^2 - 1000 &\qquad \text{Degree} = 7 \end{aligned}$$

Note 4.1. The degree of a polynomial consisting of a single number is zero. For instance, 7 is a polynomial with degree of 0.

4.2.2 Leading coefficient and the constant term

The number preceding the variable in each term is called the coefficient of that variable or the coefficient of that term. For example, in $4x^3-15x^2+x-2$

$$\begin{aligned}\text{Coefficient of } x^3 &= 4\\ \text{Coefficient of } x^2 &= -15\\ \text{Coefficient of } x\ &= 1\end{aligned}$$

Note 4.2. In our example, the coefficient of x is 1 because $x = 1.x$. So if there is no number in front of the variable, the coefficient is 1.

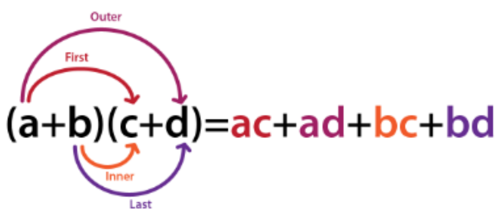
The coefficient of the variable with the highest exponent is called leading coefficient and the number without variable is called the constant term. Considering $-20x^6 + 23x^4 + 2x^3 - 10x + 44$, for example, we have

$$\begin{aligned}\text{Leading coefficient} &= -20\\ \text{Constant term} &= 44\end{aligned}$$

4.3 FOIL method

When you are multiplying two binomials, you need to use FOIL method. The word FOIL is an acronym for the four terms of the product:

- First (”first” terms of each binomial are multiplied together)
- Outer (”outside” terms are multiplied—that is, the first term of the first binomial and the second term of the second)
- Inner (”inside” terms are multiplied—second term of the first binomial and first term of the second)
- Last (”last” terms of each binomial are multiplied)



Example 4.1. Simplify $(2x + 1)(x - 3)$

Using the FOIL method, we get

$$\begin{aligned}(2x + 1)(x - 3) & \\ (2x)(x) + (2x)(-3) + (1)(x) + (1)(-3) & \\ 2x^2 - 6x + x - 3 & \qquad \text{Combine like terms} \\ 2x^2 - 5x - 3 & \qquad \text{Our solution}\end{aligned}$$

Example 4.2. Simplify $(4x + y)(3x - 2y)$.

Use the FOIL method, we'll get

$$\begin{aligned}(4x + y)(3x - 2y) & \\ (4x)(3x) + (4x)(-2y) + (y)(3x) + (y)(-2y) & \\ 12x^2 - 8xy + 3yx - 2y^2 & \qquad \text{Combine like terms} \\ 12x^2 - 5xy - 2y^2 & \qquad \text{Our answer}\end{aligned}$$

In previous example, some students like to think of the FOIL method as distributing the first term $4x$ through the $(3x - 2y)$ and distributing the second term $7y$ through the $(3x - 2y)$. Thinking about FOIL in this way makes it possible to extend this method to problems with more terms.

Example 4.3. Simplify $(2x - 3)(4x^2 - 7x + 1)$.

$$\begin{aligned}(2x - 3)(4x^2 - 7x + 1) & \quad \text{Distribute } 2x \text{ and } -3 \\ (2x)(4x^2)+(2x)(-7x)+(2x)(1)+(-3)(4x^2)+(-3)(-7x)+(-3)(1) & \\ 8x^3 - 14x^2 + 2x - 12x^2 + 21x + 3 & \quad \text{Combine like terms} \\ 8x^3 - 26x^2 + 23x + 3 & \quad \text{Our answer}\end{aligned}$$

Factoring

5.1 Factoring trinomials

In this section, we learn how to factor out trinomial in the forms of ax^2+bx+c where a, b and c are real numbers and $a \neq 0$. This trinomial with degree of 2, is also called quadratic. There are two different cases:

i. $a = 1$

ii. $a \neq 1$.

A trinomial can be factored as $\left(x+\boxed{?}\right)\left(x+\boxed{?}\right)$. We'll find two appropriate numbers to replace the questions' marks.

5.1.1 Factoring trinomial when a=1

if $a = 1$, we have

$$x^2 + bx + c$$

To factor this type of trinomial, we need to follow these steps:

Factoring Trinomial when $a = 1$

1 Factor out *gcf*, if possible

2 Find two numbers that their product is c
(don't forget negative numbers).

3 Choose a pair whose their sum is b.

4 Replace questions marks with numbers you found.

Example 5.1. Factor $x^2 - 10x + 21$.

By comparing $x^2 - 10x + 21$ with $ax^2 + bx + c$, one can find out that $a = 1, b = -10$ and $c = 21$.

Step 1. The *gcf* is 1, so we skip this step.
Step 2. We need to find two numbers whose their product is $c = 21$. Those numbers are

Factors of 21	
1	21
-1	-21
3	7
-3	-7

Step 3. We choose a pair that their sum is $b = -10$. Numbers -3 and -7 work.
Step 4. Replace

$$\left(x + \boxed{?}\right)\left(x + \boxed{?}\right) \qquad \text{Replace boxes with } -3 \text{ and } -7$$
$$(x-3)(x-7) \qquad \text{Our solution}$$

Example 5.2. Factor $x^2 - 13x - 48$

We have $a = 1$, $b = -13$ and $c = -48$.
Step 1. The *gcf* is 1, so we can skip this step.
Step 2. Here, we should find two numbers that their product is $c = -48$.

Factors of -48	
1	-48
-1	48
2	-24
-2	24
3	-16
-3	16
4	-12
-4	12

Step 3. Sum of 3 and -13 yields $b = -13$.
Step 4. Replace

$$\left(x + \boxed{?}\right)\left(x + \boxed{?}\right) \qquad \text{Replace boxes with } 3 \text{ and } -13$$
$$(x+3)(x-13) \qquad \text{Our solution}$$

5.1.2 Factoring trinomial when a is not 1

When factoring trinomials we will use the **ac method** to split the middle term and then factor by grouping. In this method, we must follow these steps:

Factoring Trinomial when $a \neq 1$

1 Factor out *gcf*, if possible

2 Multiply a by c.

3 Find two numbers that their product is ac and their sum is b.

4 Use those factors to write the middle term, bx, as the sum of two term.

5 Factor by grouping.

The ac method is named ac because we multiply ac to find out what we want to multiply to. Other than this step, the process somehow the same as what we had.

Example 5.3. Factor $3x^2 + 11x + 6$.

The *gcf* is 1, so skip that step. Thus,

$$\begin{aligned}
3x^2 + 11x + 6 &\quad \text{Multiply } a = 3 \text{ by } c = 6 \\
&\quad ac = (3)(6) = 18 \\
&\quad \text{Find two numbers, multiply to 18 and add to 11} \\
3x^2 + 2x + 9x + 6 &\quad \text{Numbers are 2 and 9, split the middle term} \\
3x(x+3) + 2(x+3) &\quad \text{Factor by grouping} \\
(x+3)(3x+2) &\quad \text{Our solution}
\end{aligned}$$

Example 5.4. Factor $10x^2 - 27x + 5$.

The *gcf* is 1, so skip that step. Thus,

$$\begin{aligned}
10x^2 - 27x + 5 &\quad \text{Multiply } a = 10 \text{ by } c = 5 \text{, so } ac = 50 \\
&\quad \text{Find two numbers, multiply to 50 and add to -27} \\
10x^2 - 25x - 2x + 5 &\quad \text{Numbers are -25 and -2, split the middle term} \\
5x(2x-5) - 1(2x-5) &\quad \text{Factor by grouping} \\
(2x-5)(5x-1) &\quad \text{Our solution}
\end{aligned}$$

Not all trinomials can be factored in both cases. If there is no combinations that multiply and add correctly then we can say the trinomial is prime and cannot be factored.

Example 5.5. Factor $3x^2 + 2x - 7$.

$3x^2 + 2x - 7$	Multiply $a = 3$ by $c = -7$, so $ac = -21$
	Find two numbers, multiply to -21 and add to 2
	There are no numbers that multiply to -21 and add to 2

Therefore, this trinomial cannot be factored and it is prime.

Rational Expressions

Rational expression is an expression of the form P/Q , where P and Q are polynomials and Q is not zero. Examples are

$$\frac{3x^4 - 56x^2}{4x - 10}, \quad \frac{5}{y+1}$$

6.1 Simplifying

Multiplying and dividing rational expressions is very similar to the process we use to multiply and divide fractions.

Reduce Common factor

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \tag{6.1}$$

To simplify, you need to factor numerator and denominator , if possible. Then find their common factors and cancel them out.

Example 6.1. Simplify each expression.

$\frac{x^2 - 36y^2}{x^2 - 3xy - 18y^2} = ?$	Factor numerator and denominator
$\frac{\cancel{(x-6y)}(x+6y)}{(x+3y)\cancel{(x-6y)}}$	Cancel out common factors
$\frac{x+6y}{x+3y}$	Our solution

$\frac{3xy}{3xy^2 + 6x^2y} = ?$	Factor denominator
$\frac{\cancel{3xy}}{\cancel{(3xy)}(y+2x)}$	Cancel out common factor

$$\frac{1}{y+2x} \qquad \text{Our Solution}$$

$$\frac{x^3-4x^2-5x}{3x^2-30x+75} = ? \qquad \text{Factor GCF}$$
$$\frac{x(x^2-4x-5)}{3(x^2-10x+25)} = ? \qquad \text{Factor out completely}$$
$$\frac{x(x+1)\cancel{(x-5)}}{3\cancel{(x-5)}(x-5)} \qquad \text{Reduce}$$
$$\frac{x(x+1)}{3(x-5)} \qquad \text{Our solution}$$

$$\frac{-3x+6y}{x^2-7xy+10y^2} = ? \qquad \text{Factor GCF}$$
$$\frac{-3(x-2y)}{x^2-7xy+10y^2} = ? \qquad \text{Factor out completely}$$
$$\frac{-3\cancel{(x-2y)}}{(x-5y)\cancel{(x-2y)}} = ? \qquad \text{Reduce}$$
$$\frac{-3}{x-5y} \qquad \text{Our solution}$$

6.2 Multiplying and Dividing

When multiplying with rational expressions we first divide out common factors,(6.1), then multiply straight across, (6.2). The process is identical for division with the extra first step of multiplying by the reciprocal, (6.3).

Multiplication

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd} \tag{6.2}$$
$$(b, d \neq 0)$$

Division

$$\frac{a}{b}\div\frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c} \tag{6.3}$$
$$(b, c, d \neq 0)$$

Example 6.2. Find each multiplication and division.

$$\frac{2x^2+5x+2}{4x^2-1}\cdot\frac{10x^2+5x-5}{x^2+x-2} = ?$$

First, factor each fractions.

$$\frac{(2x+1)(x+2)}{(2x-1)(2x+1)} \cdot \frac{5(2x-1)(x+1)}{(x-1)(x+2)}$$

Remove common factors,

$$\frac{\cancel{(2x+1)}\cancel{(x+2)}}{\cancel{(2x-1)}\cancel{(2x+1)}} \cdot \frac{5\cancel{(2x-1)}(x+1)}{(x-1)\cancel{(x+2)}}$$

now multiply,

$$\frac{5(x+1)}{x-1} \checkmark$$

$$\frac{8x^3+27y^3}{64x^3-y^3} \div \frac{4x^2-9y^2}{16x^2+4xy+y^2} =?$$

First, change the division to multiplication and swap the numerator and denominator of second fraction

$$\frac{8x^3+27y^3}{64x^3-y^3} \cdot \frac{16x^2+4xy+y^2}{4x^2-9y^2}$$

Now, factor each fraction

$$\frac{(2x+3y)(4x^2-6xy+9y^2)}{(4x-y)(16x^2+4xy+y^2)} \cdot \frac{16x^2+4xy+y^2}{(2x-3y)(2x+3y)}$$

Cancel out common factors, then multiply

$$\frac{\cancel{(2x+3y)}(4x^2-6xy+9y^2)}{(4x-y)\cancel{(16x^2+4xy+y^2)}} \cdot \frac{\cancel{16x^2+4xy+y^2}}{(2x-3y)\cancel{(2x+3y)}}$$

$$\frac{4x^2-6xy+9y^2}{(4x-y)(2x-3y)} \checkmark$$

6.3 Least Common Denominator (LCD)

As with fractions, the least common denominator or LCD is very important to working with rational expressions. The process we use to find and LCD is based on the process used to find the LCD of integers. Usually the LCD of a and b are denoted by $LCD(a, b)$. There are many methods to find LCD. Here we will only discuss **listing multiples** method.

6.3.1 Listing multiples

In this method, we list some multiples of each denominator. Then the lowest common multiple will be our answer.

Example 6.3. What is LCD(4,6) = ?

First, list multiples of 4 and 6:

$$\begin{aligned} &\text{Multiples of 4}: \ 4, 8, 12, 16, 20, 24, 28, \cdots \\ &\text{Multiples of 6}: \ 6, 12, 18, 24, 30, \cdots \end{aligned}$$

Then we choose the common multiples:

$$\text{Multiples of 4} : 4, 8, 12, 16, 20, 24, 28, \cdots$$
$$\text{Multiples of 6} : 6, 12, 18, 24, 30, \cdots$$

The lowest common multiple is 12, so LCD(4,6) =12

Note 6.1. When finding the LCD of several monomials we first find the LCD of the coefficients, then use all variables and attach the highest exponent on each variable.

Example 6.4. Find the LCD of $4x^2y^5$ and $6x^4y^3z^6$.

We begin by finding the LCD of coefficients 4 and 6. The LCD(4,6) is 12. Then use all variables with highest exponents on each variable, $x^4y^5z^6$. Finally, multiply them to get the LCD is $12x^4y^5z^6$.

Note 6.2. The same pattern can be used on polynomials that have more than one term. However, we must first factor each polynomial so we can identify all the factors to be used (attaching highest exponent if necessary).

Example 6.5. Find the LCD of $x^2 - x - 12$ and $x - 4$.

Factor each polynomial.

$$x^2 - x - 12 = (x + 3)(x - 4) \qquad (x - 4) = \text{not factorable}$$

Multiply all different factors with highest exponent on each factor.

$$LCD = (x + 3)(x - 4) \qquad \text{Our solution}$$

Note 6.3. Notice we only used $(x-4)$ once in our LCD. This is because it only appears as a factor once in either polynomial. The only time we need to repeat a factor or use an exponent on a factor is if there are exponents when one of the polynomials is factored.

Example 6.6. Find the LCD of $x^2-10x+25$ and $x^2-14x+45$

Begin by factoring each polynomial.

$$x^2-10x+25=(x-5)^2 \qquad x^2-14x+45=(x-9)(x-5)$$

Multiply all different factors. If they are repeated, choose the highest exponent.

$$LCD=(x-5)^2(x-9) \qquad \text{Our solution}$$

6.4 Adding and Subtracting

Adding and subtracting rational expressions is identical to adding and subtracting with integers. Recall that when adding with a common denominator we add the numerators and keep the denominator. This is the same process used with rational expressions. Remember to reduce, if possible, your final answer.

Be very careful with the subtraction. Subtraction with common denominator follows the same pattern, though the subtraction can cause problems if we are not careful with it.To avoid any sign errors, always use parenthesis and then distribute negative.

Example 6.7.

$$(a)\ \frac{2x+1}{x^2-27y}+\frac{6x}{x^2-27y}=?$$

Both are having the same denominator, so just add the numerators.

$$\begin{aligned}\frac{2x+1}{x^2-27y}+\frac{6x}{x^2-27y}&=\frac{2x+1+6x}{x^2-27y}\\&=\frac{8x+1}{x^2-27y}\end{aligned}$$

$$(b)\ \frac{4x}{2x^2+11x-6}-\frac{3x+1}{2x^2+11x-6}=?$$

Their denominator is the same, subtract the numerators.

$$\begin{aligned}\frac{4x}{2x^2+11x-6}-\frac{3x+1}{2x^2+11x-6}&=\frac{4x-(3x+1)}{2x^2+11x-6}\\&=\frac{x-1}{2x^2+11x-6}\end{aligned}$$

When we don't have a common denominator we will have to find the least common denominator (LCD) and build up each fraction so the denominators match. Here are the steps you can use to build up the fraction:

1. Find LCD.
2. Compare the denominator with LCD and find out what factors are missing.
3. multiply the fraction by the missing factors like this $\left(\dfrac{missing\ factors}{missing\ factors}\right)$
For instace, if the missing factor is $x+2$ then you need to build up the fraction by multiplying it by $\left(\dfrac{x+2}{x+2}\right)$

Example 6.8.

$$\frac{8}{(x-4)(x+3)}+\frac{3}{x-4}=?$$

The LCD is $(x-4)(x+3)$. By comparing, we find out the second fraction is missing $x+3$, so build it up.

$$\frac{8}{(x-4)(x+3)}+\frac{3}{x-4}\left(\frac{x+3}{x+3}\right)$$
$$\frac{8}{(x-4)(x+3)}+\frac{3(x+3)}{(x-4)(x+3)}$$

Now their denominator is same, add their numerators

$$\frac{8+3(x+3)}{(x-4)(x+3)}$$
$$\frac{8+3x+9}{(x-4)(x+3)}$$
$$\frac{3x+17}{(x-4)(x+3)}\ \checkmark$$

Example 6.9.

$$\frac{4x+2}{x^2+x-12}-\frac{3x+8}{x^2+6x+8}=?$$

First, we need to factor each denominator

$$x^2+x-12=(x-3)(x+4) \qquad x^2+6x+8=(x+4)(x+2)$$

We can now find the LCD which is $(x-3)(x+4)(x+2)$. Compare each denominator with LCD and build them up:

$$\frac{4x+2}{(x-3)(x+4)}\left(\frac{x+2}{x+2}\right)-\frac{3x+8}{(x+4)(x+2)}\left(\frac{x-3}{x-3}\right)$$

$$\frac{(4x+2)(x+2)}{(x-3)(x+4)(x+2)} - \frac{(3x+8)(x-3)}{(x+4)(x+2)(x-3)}$$

Simplify and subtract

$$\frac{4x^2 + +10x + 4}{(x-3)(x+4)(x+2)} - \frac{3x^2 - x - 24}{(x+4)(x+2)(x-3)}$$
$$\frac{4x^2 + 10x + 4 - (3x^2 - x - 24)}{(x-3)(x+4)(x+2)}$$
$$\frac{x^2 + 11x + 28}{(x-3)(x+4)(x+2)} \checkmark$$

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