3.4 Determinants

Consider the system of two linear equations in two variables,

$$a_1x+b_1y = c_1$$
$$a_2x+b_2y = c_2$$

where $c_1 \neq 0$ or $c_2 \neq 0$.

We have learnt in lower classes that this system has a unique solution or not according as $a_1 b_2 - a_2 b_1$ is not zero or zero. In other words, $a_1b_2 - a_2b_1$ determines whether the system has a unique solution or not and hence it is called the 'determinant' of the system. Hence we associate the value $a_1 b_2 - a_2 b_1$ to the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and call it the determinant (simply determinant) of the matrix.

The determinant of 1×1 matrix is defined as its element.

In this section, we define the determinant of a 3×3 matrix, study its properties and the methods of evaluation of certain determinants.

3.4.1 Definition (Minor of an element)

Consider a square matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

The minor of an element in this matrix is defined as the determinant of the 2×2 matrix, obtained after deleting the row and the column in which the element is present.

For example the minor of a_2 is the det. of $\begin{bmatrix} b_1 & c_1 \\ b_3 & c_3 \end{bmatrix} = b_1 c_3 - b_3 c_1$

and the minor of b_3 is the det. of $\begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} = a_1 c_2 - a_2 c_1.$

3.4.2 Definition (Cofactor of an element)

The ***cofactor*** *of an element in the i^{th} row and the j^{th} column of a 3×3 matrix is defined as its minor multiplied by $(-1)^{i+j}$.*

We denote the cofactor of a_{ij} by A_{ij}.

For example, consider the matrix in 3.4.1.

Since a_2 is in 2^{nd} row and 1^{st} column, we have

$$A_2 = \text{cofactor of } a_2 = (-1)^{2+1}(b_1 c_3 - b_3 c_1)$$
$$= -(b_1 c_3 - b_3 c_1)$$
$$= b_3 c_1 - b_1 c_3$$

Since b_3 is in 3^{rd} row and 2^{nd} column, we have

$$B_3 = \text{cofactor of } b_3$$
$$= (-1)^{3+2}(a_1 c_2 - a_2 c_1)$$
$$= a_2 c_1 - a_1 c_2.$$

3.4.3 Example

In the matrix $\begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$ *we list out here under, the minors and cofactors of all the elements.*

element a_{ij}	element present in row i, column j	Minor of a_{ij}	Cofactor of a_{ij}
1	1 1	$\begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -16$	$(-1)^{1+1}(-16) = -16$
0	1 2	$\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = 10$	$(-1)^{1+2}(10) = -10$
-2	1 3	$\begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} = 19$	$(-1)^{1+3}(19) = 19$
3	2 1	$\begin{vmatrix} 0 & -2 \\ 5 & 6 \end{vmatrix} = 10$	$(-1)^{2+1}(10) = -10$
-1	2 2	$\begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 14$	$(-1)^{2+2}(14) = 14$
2	2 3	$\begin{vmatrix} 1 & 0 \\ 4 & 5 \end{vmatrix} = 5$	$(-1)^{2+3}(5) = -5$
4	3 1	$\begin{vmatrix} 0 & -2 \\ -1 & 2 \end{vmatrix} = -2$	$(-1)^{3+1}(-2) = -2$
5	3 2	$\begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 8$	$(-1)^{3+2}(8) = -8$
6	3 3	$\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = -1$	$(-1)^{3+3}(-1) = -1$

3.4.4 Definition (Determinant)

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. *The sum of the products of elements of the first row with their corresponding cofactors is called the* ***determinant*** *of* A.

The determinant of the matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is written as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

We also denote the determinant of the matrix A by det A or $|A|$.

$$\det A = a_1A_1 + b_1B_1 + c_1C_1$$

So far we have defined the concept of determinant for square matrices of order n for $n = 1, 2, 3$. The concept can be extended to the case $n \geq 4$ also using the principle of mathematical induction. Let $n \geq 4$ and suppose that we know the definition of determinant for square matrices of order $n - 1$. Let $A = [a_{ij}]_{n \times n}$. Then the determinant of A is defined as $\sum_{j=1}^{n} a_{1j} A_{1j}$, where A_{1j} is the cofactor of a_{1j}.

3.4.5 Example

Let us find the determinant of $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ 4 & 5 & 6 \end{bmatrix}$

$\det A$ = sum of the products of elements of the first row with their corresponding cofactors

$= 1\ (\text{cofactor of } 1) + 0\ (\text{cofactor of } 0) + (-2)\ \text{cofactor of } (-2)$

$= 1(-16) + (-2)\ (19)$

$= -16 - 38 = -54.$

3.4.8 Properties of determinants

(i) If each element of a row (or column) of a square matrix is zero, then the determinant of that matrix is zero.

The value of the determinant of such a matrix can be easily found to be zero by expanding it along a row (column) containing zeros.

(ii) If two rows (or columns) of a square matrix are interchanged, then the sign of the determinant changes.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by interchanging first and second rows of A)

$$\det B = a_1(-1)^{2+1}(b_2c_3 - b_3c_2) + b_1(-1)^{2+2}(a_2c_3 - a_3c_2) + c_1(-1)^{2+3}(a_2b_3 - a_3b_2)$$

$$= -[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$

$$= -\det A.$$

(iii) If each element of a row (or column) of a square matrix is multiplied by a number k, then the determinant of the matrix obtained is k times the determinant of the given matrix.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad B = \begin{bmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{bmatrix}$$

(B is obtained by multiplying the elements of first column of A by k)

If the cofactors of a_1, a_2, a_3 in A are A_1, A_2, A_3 then the cofactors of ka_1, ka_2, ka_3 in B are also A_1, A_2, A_3 respectively. Hence

$$\begin{aligned}\det B &= ka_1 A_1 + ka_2 A_2 + ka_3 A_3\\ &= k(a_1 A_1 + a_2 A_2 + a_3 A_3)\\ &= k(\det A).\end{aligned}$$

(iv) If A is square matrix of order 3 and k is a scalar, then $|kA| = k^3|A|$. By applying property (iii), three times, we get the result.

(v) If two rows (or columns) of a square matrix are identical, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \end{bmatrix}$$

identical = same

(second and third rows are identical)

$$\begin{aligned}\text{Then } \det A &= a_1 A_1 + b_1 B_1 + c_1 C_1\\ &= a_1(0) + b_1(0) + c_1(0) = 0.\end{aligned}$$

(vi) If the corresponding elements of two rows (or columns) of a square matrix are in the same ratio, then the determinant of that matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Then

$$\det A = \begin{vmatrix} a_1 & b_1 & c_1 \\ ka_1 & kb_1 & kc_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{by property (iii)}$$

$$= k(0) \quad \text{by property (v)}$$
$$= 0.$$

(vii) If each element in a row (or column) of a square matrix is the sum of two numbers, then its determinant can be expressed as the sum of the determinants of two square matrices as shown below.

$$\text{Let } A = \begin{bmatrix} a_1 + x_1 & b_1 & c_1 \\ a_2 + x_2 & b_2 & c_2 \\ a_3 + x_3 & b_3 & c_3 \end{bmatrix}, B = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, C = \begin{bmatrix} x_1 & b_1 & c_1 \\ x_2 & b_2 & c_2 \\ x_3 & b_3 & c_3 \end{bmatrix}$$

If in A, the cofactors of $a_1+x_1, a_2+x_2, a_3+x_3$ are A_1, A_2, A_3 then the cofactors of a_1, a_2, a_3 in B and of x_1, x_2, x_3 in C are also A_1, A_2, A_3 respectively.

Now,

$$\begin{aligned}\det A &= (a_1+x_1)A_1+(a_2+x_2)A_2+(a_3+x_3)A_3\\ &= (a_1A_1+a_2A_2+a_3A_3)+(x_1A_1+x_2A_2+x_3A_3)\\ &= \det B+\det C.\end{aligned}$$

$$\therefore \quad \begin{vmatrix} a_1+x_1 & b_1 & c_1\\ a_2+x_2 & b_2 & c_2\\ a_3+x_3 & b_3 & c_3\end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} + \begin{vmatrix} x_1 & b_1 & c_1\\ x_2 & b_2 & c_2\\ x_3 & b_3 & c_3\end{vmatrix}.$$

(viii) If each element of a row (or column) of a square matrix is multiplied by a number k and added to the corresponding element of another row (or column) of the matrix, then the determinant of the resultant matrix is equal to the determinant of the given matrix.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{bmatrix} \text{ and } B = \begin{bmatrix} a_1 & b_1 & c_1\\ a_2+ka_1 & b_2+kb_1 & c_2+kc_1\\ a_3 & b_3 & c_3\end{bmatrix}$$

(B is obtained from A by multiplying each element of the 1st row of A by k and then adding them to the corresponding elements of the 2nd row of A)

$$\det B = \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} + \begin{vmatrix} a_1 & b_1 & c_1\\ ka_1 & kb_1 & kc_1\\ a_3 & b_3 & c_3\end{vmatrix} \text{ by property (vii)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} + 0 \quad \text{by property (vi)}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix} = \det A.$$

(ix) The sum of the products of the elements of a row (or column) with the cofactors of the corresponding elements of another row (or column) of a square matrix is zero.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

Consider the sum of the products of the elements of the second row with the cofactors of the corresponding elements of the first row.,

$$\text{i.e., } a_2 A_1 + b_2 B_1 + c_2 C_1$$

$$= a_2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_2 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_2 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{by property (v).}$$

(x) If the elements of a square matrix are polynomials in x and its determinant is zero when $x = a$, then $x - a$ is a factor of the determinant of the matrix.

$$\text{Let } A(x) = \begin{bmatrix} f_1(x) & g_1(x) & h_1(x) \\ f_2(x) & g_2(x) & h_2(x) \\ f_3(x) & g_3(x) & h_3(x) \end{bmatrix}.$$

Now det $[A(x)]$ is a polynomial in x.

If det $[A(a)] = 0$ then by Remainder theorem, $x - a$ is a factor of det $[A(x)]$.

(xi) For any square matrix A, det A = det (A').

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \text{ then } A' = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}.$$

The values of the cofactors of a_1, b_1, c_1, are same in both A and A'.

Hence det A $= a_1 A_1 + b_1 B_1 + c_1 C_1 =$ det A'.

(xii) Det (AB) = (det A) (det B) for matirces A, B of order 2.

$$\text{Consider the matrices } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix},$$

$$\det A = a_{11} a_{22} - a_{21} a_{12}; \ \det B = b_{11} b_{22} - b_{21} b_{12}.$$

$$\text{Now } AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

$$\det(AB) = (a_{11}b_{11}+a_{12}b_{21})(a_{21}b_{12}+a_{22}b_{22})-(a_{21}b_{11}+a_{22}b_{21})(a_{11}b_{12}+a_{12}b_{22})$$

$$= a_{11}a_{21}b_{11}b_{12}+a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}+a_{12}a_{22}b_{21}b_{22}$$

$$-a_{11}a_{21}b_{11}b_{12}-a_{12}a_{21}b_{11}b_{22}-a_{11}a_{22}b_{12}b_{21}-a_{12}a_{22}b_{21}b_{22}$$

$$= a_{11}a_{22}b_{11}b_{22}+a_{12}a_{21}b_{12}b_{21}-a_{12}a_{21}b_{11}b_{22}-a_{11}a_{22}b_{12}b_{21}$$

$$= a_{11}a_{22}(b_{11}b_{22}-b_{12}b_{21})-a_{12}a_{21}(b_{11}b_{22}-b_{12}b_{21})$$

$$= (a_{11}a_{22}-a_{12}a_{21})(b_{11}b_{22}-b_{12}b_{21})$$

$$= (\det A)(\det B).$$

If A and B are matrices of order three then also in a similar manner we can show that

$$\det(AB) = (\det A)(\det B).$$

This is true in general, for all matrices of order n; the proof of this is beyond the scope of this book.

(xiii) For any positive integer n, $\det(A^n) = (\det A)^n$.

(xiv) If A is a triangular matrix (upper or lower), then determinant of A is the product of the diagonal elements.

3.4.9 Notation

While evaluating determinants, we use the following notations.

(i) $R_1 \leftrightarrow R_2$, to mean that the rows R_1 and R_2 are interchanged.

(ii) $R_1 \to kR_1$, to mean that the elements of R_1 are multiplied by k.

(iii) $R_1 \to R_1 + kR_2$ to mean that the elements of R_1 are added with k times the corresponding elements of R_2.

Similar notation is used for other rows and columns.