3.2 Scalar multiple of a matrix and multiplication of matrices

This section is devoted to the study of multiplication of a matrix (i) by a scalar and (ii) by a matrix. We also study the properties of multiplication.

3.2.1 Definition (Scalar multiple of a matrix)

Let A be a matrix of order $m \times n$ and k be a scalar (i.e., real or complex number). Then the $m \times n$ matrix obtained by multiplying each element of A by k is called a scalar multiple of A and is denoted by k A.

$$\boxed{\text{If } A = [a_{ij}]_{m\times n} \text{ then } kA = [ka_{ij}]_{m\times n}}$$

For example if $k = 2$ and $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}$ then

$$kA = 2A = \begin{bmatrix} 2\times 3 & 2\times 2 & 2\times(-1) \\ 2\times 4 & 2\times(-3) & 2\times 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -2 \\ 8 & -6 & 2 \end{bmatrix}.$$

3.2.2 Note

$(-1)A = -A$ because $A + (-1)A = O$.

3.2.3 Properties of scalar multiplication of a matrix

Let A and B be matrices of the same order and α, β be scalars. Then

(i) $\alpha(\beta A) = (\alpha\beta)A = \beta(\alpha A)$

(ii) $(\alpha+\beta)A = \alpha A + \beta A$

(iii) $\alpha(A+B) = \alpha A + \alpha B$

(iv) $\alpha O = O$

(v) $0A = O$

Consider (ii) Let $A = [a_{ij}]_{mxn}$

$$\begin{aligned}
(\alpha+\beta)A &= (\alpha+\beta)[a_{ij}] \\
&= [(\alpha+\beta)a_{ij}] \quad \text{..... by definition 3.2.1} \\
&= [\alpha a_{ij} + \beta a_{ij}] \quad \text{..... by distributive law of numbers} \\
&= [\alpha a_{ij}] + [\beta a_{ij}] \\
&= \alpha[a_{ij}] + \beta[a_{ij}] \\
&= \alpha A + \beta A
\end{aligned}$$

Verification of properties (i), (iii), (iv) and (v) is left to the student as an exercise.

3.2.5 Multiplication of matrices

We say that matrices A *and* B *are* ***conformable for multiplication*** *in that order (giving the product* AB*) if the number of columns of* A *is equal to the number of rows of* B.

3.2.6 Definition (Product of two matrices)

Let $A = [a_{ik}]_{m\times n}$ and $B = [b_{kj}]_{n\times p}$, *be two matrices. Then the matrix* $C = [c_{ij}]_{m\times p}$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ *is called the product of A and B and is denoted by AB.*

Observe that when the orders of A and B are $m \times n$ and $n \times p$, the order of the product matrix AB is $m \times p$. Every element of AB is in the form of a sum of products of certain elements of A and of B. For example, in $C = AB = [c_{ij}]_{m\times p}$

$$c_{23} = \sum_{k=1}^{n} a_{2k} b_{k3} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} + ... + a_{2n}b_{n3}$$

= the sum of the products of the elements of second row of A with the corresponding elements of the 3rd column of B

A useful method to understand and to remember matrix multiplication is illustrated in the following example.

Let $A_{2\times3} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ and $B_{3\times4} = \begin{bmatrix} 1 & 3 & -4 & 2 \\ -1 & 0 & 3 & 5 \\ 0 & 4 & 7 & -6 \end{bmatrix}$

Let the rows of A be R_1, R_2 and the columns of B be C_1, C_2, C_3, C_4. When $A_{2\times3}$ is multiplied with $B_{3\times4}$, the order of the product matrix C = AB is 2×4.

$\therefore$ Let $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \end{bmatrix}$.

Then $C = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 & R_1C_4 \\ R_2C_1 & R_2C_2 & R_2C_3 & R_2C_4 \end{bmatrix}$.

$c_{11} = R_1C_1$ = sum of the products of the 1st row elements of A with the corresponding elements of the 1st column of B.
$= 2(1) + 3(-1) + 1(0) = -1.$

$c_{12} = R_1C_2$ = sum of the products of the 1st row elements of A with the corresponding elements of the 2nd column of B
$= 2(3) + 3(0) + 1(4) = 10.$

3.2.7 Examples

1. **Example**

Consider the matrices $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}$.

Clearly A, B as well as B, A are conformable.

Further $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2(0)+3(-1) & 2(4)+3(2) \\ 1(0)+2(-1) & 1(4)+2(2) \end{bmatrix} = \begin{bmatrix} -3 & 14 \\ -2 & 8 \end{bmatrix}$.

Now $BA = \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 0(2)+4(1) & 0(3)+4(2) \\ -1(2)+2(1) & -1(3)+2(2) \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 0 & 1 \end{bmatrix}$.

Hence the products AB and BA are not necessarily equal.

2. Example

A certain bookshop has 10 dozen chemistry books, 8 dozen physics books, 10 dozen economics books. Their selling prices are Rs. 80, Rs. 60 and Rs. 40 each respectively. Using matrix algebra, find the total value of the books in the shop.

Solution: Number of books

$$A = \begin{array}{ccc} \text{Chemistry} & \text{Physics} & \text{Economics} \\ \end{array}$$
$$A = \begin{bmatrix} 10\times12 & 8\times12 & 10\times12 \\ =120 & =96 & =120 \end{bmatrix}$$

Selling price (in rupees)

$$B = \begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix} \begin{array}{l} \text{Chemistry} \\ \text{Physics} \\ \text{Economics} \end{array}$$

Total value of the books in the shop.

$$AB = [120 \quad 96 \quad 120]\begin{bmatrix} 80 \\ 60 \\ 40 \end{bmatrix}$$

$$= [120 \times 80 + 96 \times 60 + 120 \times 40]$$

$$= [9600 + 5760 + 4800]$$

$$= [20160] \text{ (in rupees).}$$

3.2.8 Note

Matrix multiplication is not commutative. If A and B are matrices conformable for multiplication, AB exists, but BA may not exist; even if BA exists, AB and BA may not have the same order and even if they have the same order they may not be equal.

1. If the orders of A and B are 2×3 and 3×4 respectively then the order of AB is 2×4, but BA does not exist. (The number of columns of B is not equal to the number of rows of A, that is B and A are not conformable for multiplication).

2. If the orders A and B are 2×3 and 3×2 respectively, then the order of AB is 2×2, while the order of BA is 3×3. Hence AB and BA can not be equal.
3. For the matrices A and B of example 1, 3.2.7, AB and BA have the same order but $AB \neq BA$.

This does not mean however, that $AB \neq BA$ for every pair of matrices A, B for which AB and BA are defined and are of same order.

For instance, $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$.

Verify whether every pair of diagonal matrices of same order commute or not!

Also, verify by an example whether a pair of square matrices of same order, whose product is a scalar matrix, commute or not!

3.2.9 Note

Let $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$ then $AB = BA = O$.

We know that in case of real numbers a, b if $ab = 0$ then $a = 0$ or $b = 0$. But in matrices, the product of two non-zero matrices could be a zero matrix, as seen from the above example.

3.2.10 Note

If $AB = AC$ and $A \neq O$, then it is not necessary that $B = C$.

For example, if $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 3 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix}$

we have $AB = O = AC$, but $B \neq C$.

3.2.11 Properties of multiplication of matrices

Multiplication of matrices possesses the following properties, which we state without proof.

1. **The Associative Law**

For any three matrices A, B and C, we have $(AB)C = A(BC)$ in the sense that whenever one side of the equality is defined, then the otherside is also defined and the equality holds.

2. **The Distributive Law**

For any three matrices A, B and C, we have

(i) $A(B + C) = AB + AC$ (Left Distributive Law)

(ii) $(A+B)C = AC + BC$ (Right Distributive Law)

in the sense that whenever one side of the equation is defined, then the otherside is also defined and the equality holds.

3. **Existence of multiplicative identity**

If I is the identity matrix of order n, then for every square matrix A of order n

$$IA = AI = A.$$

3.2.12 Note

(i) For any square matrix A, we denote $A \cdot A$ by A^2. In general, for any positive integer n, $n > 1$, the product A. A. A ... A (taken n times) is denoted by A^n.

(ii) If A and B are matrices of orders $m \times n$ and $n \times p$ respectively and α, β are scalars, then

$$(\alpha A).(\beta B) = \alpha\beta(AB) = ((\alpha\beta)A)B = A.((\alpha\beta)B).$$

(iii) If α is a scalar, A is a square matrix and n is a positive integer, then

$$(\alpha A)^n = \alpha^n A^n \text{ and } \alpha A = (\alpha I)A.$$

We now verify all the properties of multiplication, in the following solved problems.

3.3 Transpose of a matrix

In this section we define the Transpose of a martrix and study its properties. We also define symmetric and skew symmetric matrices.

3.3.1 Definition (Transpose of a matrix)

If $A = [a_{ij}]$ is an $m \times n$ matrix, then the matrix obtained by interchanging the rows and columns of A is called the **transpose of A**. *Transpose of the matrix A is denoted by A' or A^T. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ji}]_{n\times m}$.*

For example if

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & \sqrt{7} \end{bmatrix} \quad \text{then } A' = \begin{bmatrix} 3 & 4 & 0 \\ 2 & 1 & \sqrt{7} \end{bmatrix}.$$

3.3.2 Properties of transpose of matrices

We now state the following properties of transpose of matrices without proof. These may be verified by taking suitable examples.

For any two matrices A, B of suitable orders, we have

(i) $(A')' = A$ (ii) $(kA)' = kA'$

(iii) $(A+B)' = A'+B'$ (iv) $(AB)' = B'A'$

3.3.3 Example

If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$

Verify that (i) $(A')' = A$ (ii) $(A+B)' = A'+B'$ (iii) $(5B)' = 5(B)'$

Solution

(i) We have $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$

$$\Rightarrow \quad A' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$\Rightarrow (A')' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} = A.$$

(ii) $$A+B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 8 & 7 \\ 6 & 3 & 7 \end{bmatrix}$$

$$\therefore (A+B)' = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix} = (A+B)'.$$

(iii) We have $$5B = 5\begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 & 0 \\ 20 & -10 & -5 \end{bmatrix}$$

$$(5B)' = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

also $$5B' = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

Thus $(5B)' = 5B'$.

3.3.4 Example

If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ *then verify that* $(AB)' = B'A'$.

Solution

We have $AB = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 15 & 4 \\ -28 & -18 \end{bmatrix}$

$\therefore (AB)' = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$.

Now $A' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix}$ and $B' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$

$\therefore B'A' = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} 15 & -28 \\ 4 & -18 \end{bmatrix}$

Hence $(AB)' = B'A'$.

3.3.5 Definition (Symmetric matrix)

A square matrix A is said to be ***symmetric*** *if* $A' = A$.

3.3.6 Note

(i) The zero matrix $O_{n\times n}$, any diagonal matrix and the unit matrix $I_{n\times n}$ are symmetric.

(ii) If A is a symmetric matrix, then the $(i,j)^{th}$ element of A is the same as the $(j,i)^{th}$ element of A.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -3 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Observe that $a_{12} = a_{21} = 2$, $a_{13} = a_{31} = 0$ and $a_{23} = a_{32} = -1$. So A is symmetric.

(iii) If A is a square matrix, then $A + A'$ is a symmetric matrix.

3.3.7 Definition (Skew-symmetric matrix)

A square matrix A is said to be ***skew-symmetric*** *if* $A' = -A$.

For example, $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$ are skew - symmetric matrices.

3.3.8 Note

(i) The zero matrix $O_{n\times n}$ is skew-symmetric.

(ii) If A is a **skew-symmetric** matrix, then the $(i, j)^{th}$ element of A is the same as the negative of the $(j,i)^{th}$ element of A.

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

Observe that $a_{12} = 1 = -a_{21}$, $a_{13} = -2 = -a_{31}$ and $a_{23} = 4 = -a_{32}$,

since the diagonal elements a_{11}, a_{22} and a_{33} do not change while transposing the given matrix, if $A = \left[a_{ij}\right]_{n\times n}$ is a skew symmetric matrix, then $a_{ii} = -a_{ii}$ so that $a_{ii} = 0 (i = 1, 2, ...n)$.

(iii) If A is a square matrix, then $A - A'$ is a skew-symmetric matrix.

(iv) If A is a symmetric (or skew-symmetric) matrix, then kA is also symmetric (or skew-symmetric) for any scalar k.