3.6 Consistency and Inconsistency of system of Simultaneous Equations - Rank of a Matrix

We devote this section for the study of the rank of a matrix, existence and the nature of solutions of a system of linear equations - homogeneous and non-homogeneous, in two and three variables.

Consider the following system of simultaneous non-homogeneous linear equations (two equations in two variables):

$$\left.\begin{aligned} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{aligned}\right\} \quad \ldots\ldots \text{ I}$$

These equations can be represented as a matrix equation as A X = D, where

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$ is called the coefficient matrix.

$$X = \begin{bmatrix} x \\ y \end{bmatrix}$$ is called the variable matrix,

and $$D = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$ is called the constant matrix.

AX = D is the matrix representation of the equations given in system I, for

$$AX = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1x+b_1 y \\ a_2x+b_2 y \end{bmatrix}$$

AX = D becomes $$\begin{bmatrix} a_1x+b_1 y \\ a_2x+b_2 y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

and corresponding elements of two equal matrices are equal.

The coefficient matrix augmented with the constant column matrix, is called the Augmented matrix, generally denoted by [A D]. Hence, the augmented matrix of system I is

$$[A\,D] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}.$$

We listed the various systems of equations, along with the corresponding matrix equations and matrices involved in the following tabular form.

Types of systems of simultaneous linear equations

Nature of the system	Equations	Matrix equation	Corresponding Matrices			Augmented matrix [AD]
			Coefficient matrix A	Variables matrix X	Constant matrix D	
I. Non-homogeneous pair of equations in two unknowns	$a_1x+b_1y=c_1$ $a_2x+b_2y=c_2$ $c_1\neq 0$ or $c_2\neq 0$	AX = D	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix}$
II. Non-homogeneous triad of equations in three unknowns	$a_1x+b_1y+c_1z=d_1$ $a_2x+b_2y+c_2z=d_2$ $a_3x+b_3y+c_3z=d_3$ $d_1\neq 0$ or $d_2\neq 0$ or $d_3\neq 0$	AX = D	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$
III. Pair of homogeneous equations in two unknowns	$a_1x+b_1y=0$ $a_2x+b_2y=0$	AX = O	$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$	$\begin{bmatrix} x \\ y \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \end{bmatrix}$
IV. Three homogeneous equations in three unknowns	$a_1x+b_1y+c_1z=0$ $a_2x+b_2y+c_2z=0$ $a_3x+b_3y+c_3z=0$	AX = O	$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$	$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{bmatrix}$

Here we confine to the above types of systems of equations in three variables. Before solving the systems of equations, we first study an important concept namely the rank of a matrix.

3.6.1 Definition (Submatrix)

A matrix obtained by deleting some rows or columns (or both) of a matrix is called a ***submatrix*** *of the given matrix.*

For example, If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ -1 & 2 & 0 \end{bmatrix}$.

Then some submatrices of A are

$\begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$ – obtained by deleting R_2 and C_3 of A

$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ – obtained by deleting R_3 of A

$\begin{bmatrix} 2 & 3 \\ 3 & 1 \\ 2 & 0 \end{bmatrix}$ – obtained by deleting C_1 of A

$[0]$ – obtained by deleting R_1, R_2, C_1 and C_2 of A.

3.6.2 Definition (Rank of a matrix)

Let A be a non-zero matrix. The ***rank*** *of A is defined as the maximum of the orders of the non-singular square submatrices of A. The rank of a null matrix is defined as zero. The rank of a matrix A is denoted by* ***rank (A)***.

3.6.3 Note

If A is a non-zero matrix of order 3, then the rank of A is

(i) 1 if every 2×2 submatrix is singular

(ii) 2 if A is singular and atleast one of its 2×2 submatrices is non-singular

(iii) 3 if A is non-singular.

3.6.4 Examples

1. $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

det $A = -5$. A is non-singular, and hence rank $(A) = 3$.

2. $B = \begin{bmatrix} -1 & -2 & -3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

d. Henc (B) $\neq 3$.

Now $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ is a submatrix of B, whose determinant is 2.

Hence rank $(B) = 2$.

3. $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

C is a matrix of order 3×4.

Let $C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Then C_1 is a square submatrix of C of order 3 and det $C_1 = 1$.

Hence rank of the given matrix is 3.