## 3.5 Adjoint and Inverse of a Matrix

In this section, we define the concepts of invertibility of a matrix and the multiplicative inverse of an invertible matrix and study certain properties of inverses and provide a method of finding the multiplicative inverse of a given invertible matrix.

### 3.5.1 Definition (Singular and Non-singular matrices)

*A square matrix is said to be* ***singular*** *if its determinant is zero. Otherwise it is said to be* ***non-singular.***

For example, $\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$ is a singular matrix while $\begin{bmatrix} 3 & -2 \\ 6 & 4 \end{bmatrix}$ is non-singular.

### 3.5.2 Definition (Adjoint of a matrix)

*The transpose of the matrix formed by replacing the elements of a square matrix A (of order greater than one) with the corresponding cofactors is called the Adjoint of A and is denoted by Adj A.*

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and $A_i, B_i, C_i$ be the cofactors of $a_i, b_i, c_i$ respectively.

$$\text{Then Adj } A = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}^T = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}.$$

### 3.5.3 Definition (Invertible matrix)

*Let A be a square matrix. We say that A is invertible if a matrix B exists such that $AB = BA = I$, where I is the unit matrix of the same order as A and B.*

### 3.5.4 Note

(i) For the products AB and BA to be both defined and equal, it is necessary that A and B are both square matrices of the same order. Thus, non-square matrices are not invertible.

(ii) If $A$ is invertible, then $A$ is non-singular, hence $\det A \neq 0$.

[Let $A$ be invertible. Then there exsists a matrix $B$ such that $AB = I$.

Hence $(\det A)\ (\det B) = \det (AB) = \det I = 1$. Hence $\det A \neq 0$].

(iii) If $B$ exists; such that $AB = BA = I$, then such a $B$ is unique and is denoted by $A^{-1}$ and is called the **multiplicative inverse** or inverse of $A$.

[For, if $B$ and $C$ are inverses of $A$, then by definition $AB = B'A = I$ and $AC = CA = I$. Then $B = BI = B(AC) = (BA)C = IC = C$].

### 3.5.5 Theorem

*Let $A$ and $B$ be invertible matrices. Then $A^{-1}$, $A'$ and $AB$ are invertible. Further*

(i) $(A^{-1})^{-1} = A$

(ii) $(A')^{-1} = (A^{-1})'$

(iii) $(AB)^{-1} = B^{-1}A^{-1}$.