3.1 Types of matrices

In this section, we define a matrix, its order and various types of matrices.

3.1.1 Definition (Matrix)

An ordered rectangular array of elements is called a ***matrix.***

We confine our discussion to matrices whose elements are real or complex numbers; or real or complex valued functions. Matrices are generally enclosed by brackets.

We denote matrices by capital letters A,B, C...

The following are some examples of matrices.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$\begin{array}{cc} & \\ C = & \begin{bmatrix} x+1 & x^2-1 & 3 \\ -3 & 2 & \sin x \\ 7+\sin x & 4 & 3+\sin 2x \end{bmatrix} \\ & \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 1^{st}\text{ column} & 2^{nd}\text{ column} & 3^{rd}\text{ column} \end{array} \end{array} \begin{array}{l} \rightarrow 1^{st}\text{ row} \\ \rightarrow 2^{nd}\text{ row} \\ \rightarrow 3^{rd}\text{ row} \\ \\ \\ \end{array}$$

In the above examples, the horizontal lines of elements are said to constitute **the rows** of the matrix and the vertical lines of elements are said to constitute **the columns** of the matrix. Thus A has 2 rows and 3 columns, B has 2 rows and 2 columns, while C has 3 rows and 3 columns.

3.1.2 Definition (Order of Matrix)

A matrix having m rows and n columns is said to be of ***order*** $m \times n$*, read as m cross n or m by n.*

In the above examples, A is of order 2×3, B is of order 2×2 and C is of order 3×3.

In general, a matrix having m rows and n columns is represented as follows.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & & & & \\ \cdots & \cdots & & & \cdots & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

In the above matrix every element is specified by its position in terms of the row and column in which the element is present. The first and second suffices of an element indicate respectively the row and column in which the element is present. For example a_{23} is the element present in the second row and the third column.

> In compact form the above matrix is denoted by
> $A = [a_{ij}]_{m \times n}$ where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Throughout this chapter, we generally consider matrices of order $m \times n$, where $m \in \{1, 2, 3\}$ and $n \in \{1, 2, 3, 4\}$.

3.1.3 Types of matrices

1. Square matrix

*A matrix in which the number of rows is equal to the number of columns, is called a **square matrix**.*

$A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$. In this case we say that A is a square matrix of order m. For example,

$[2]$ is a square matrix of order 1.

$\begin{bmatrix} 1 & -1 \\ 0 & 4 \end{bmatrix}$ is a square matrix of order 2,

and $\begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$ is a square matrix of order 3.

If $A = [a_{ij}]$ is a square matrix of order n, the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are said to constitute its *Principal diagonal* or simply the *diagonal*. Hence a_{ij} is an element of the diagonal or non-diagonal according as $i = j$ or $i \neq j$.

The sum of the elements of the diagonal of a square matrix A is called the *trace* of A and is denoted by Tr (A).

If $A = [a_{ij}]$ is a square matrix of order n, then $\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}$.

For example, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & -1 & 2 \\ 7 & 6 & 9 \end{bmatrix}$, then $\text{Tr}(A) = 2 + (-1) + 9 = 10$.

2. Diagonal matrix

*If each non-diagonal element of a square matrix is equal to zero, then the matrix is called a **diagonal matrix**.*

For example, $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are diagonal matrices.

If $A = [a_{ij}]_{n \times n}$ is a diagonal matrix, it is sometimes denoted as diag $[a_{11}, a_{22}, \ldots, a_{nn}]$.

3. Scalar matrix

If each non-diagonal element of a square matrix is zero and all diagonal elements are equal to each other, then it is called a scalar matrix.

For example, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ are all scalar matrices.

4. Unit (Identity) matrix

If each non-diagonal element of a square matrix is equal to zero and each diagonal element is equal to 1, then that matrix is called a Unit matrix **or** *Identity matrix.*

We denote the unit matrix of order n by I_n, or simply by I, when there is no ambiguity about the order.

For example, $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are unit matrices.

$[a_{ij}]_{n \times n}$ is a unit matrix
$\Leftrightarrow a_{ij} = 1$ if $i = j$ and $a_{ij} = 0$ if $i \neq j$

5. Null matrix or Zero matrix

If each element of a matrix is zero, then it is called a Null matrix **or** *Zero matrix. It is denoted by $O_{m \times n}$ or simply by O.*

For example, $O_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are null matrices.

6. Row matrix and Column matrix

A matrix with only one row is called a Row matrix **(or** *row vector) and a matrix with only one column is called a Column matrix* **(or** *column vector).*

For example, $[1 \quad 3 \quad -2]$ is a row matrix (order 1×3),

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a column matrix (order 2×1).

7. Triangular matrices

A square matrix $A = [a_{ij}]$ *is said to be* ***Upper Triangular*** *if* $a_{ij} = 0$ *for all* $i > j$.

A is said to be ***Lower Triangular*** *if* $a_{ij} = 0$ *for all* $i < j$.

For example, $\begin{bmatrix} 2 & -4 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix}$ are upper triangular matrices while

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ are lower triangular matrices.

Observe that I_3 and O_3 are both upper and lower triangular matrices.

> $A = [a_{ij}]_{n \times n}$, is
>
> Upper Triangular if $a_{ij} = 0$ for all $i > j$.
>
> Lower Triangular if $a_{ij} = 0$ for all $i < j$

3.1.4 Definition (Equality of matrices)

Matrices A and B are said to be equal if A and B are of the same order and the corresponding elements of A and B are the same.

Thus $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$

are equal if $a_{ij} = b_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$.

3.1.5 Definition (Sum of two matrices)

Let A and B be matrices of the same order. Then the sum of A and B, denoted by A + B, is defined as the matrix of the same order in which each element is the sum of the corresponding elements of A and B.

> If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$,
>
> then $A + B = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$

For example, if $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -2 & 7 \\ 3 & 2 & -1 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 3+1 & 2+(-2) & -1+7 \\ 4+3 & -3+2 & 1+(-1) \end{bmatrix} = \begin{bmatrix} 4 & 0 & 6 \\ 7 & -1 & 0 \end{bmatrix}.$$

3.1.6 Properties of Addition of matrices

Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ be matrices of the same order. Then the addition of matrices satisfies the following properties :

(i) *Commutative Property*

$$A + B = B + A$$

Now
$$\begin{aligned} A + B &= [a_{ij}] + [b_{ij}] \\ &= [a_{ij} + b_{ij}] \\ &= [b_{ij} + a_{ij}] \\ &= [b_{ij}] + [a_{ij}] \\ &= B + A \end{aligned}$$

Addition of matrices is commutative. i.e., $A + B = B + A$

(ii) *Associative Property*

$$A + (B + C) = (A + B) + C$$

Now
$$\begin{aligned} (A+B)+C &= \left([a_{ij}] + [b_{ij}]\right) + [c_{ij}] \\ &= [a_{ij} + b_{ij}] + [c_{ij}] \\ &= [(a_{ij} + b_{ij}) + c_{ij}] \\ &= [a_{ij} + (b_{ij} + c_{ij})] \quad \text{(why?)} \\ &= [a_{ij}] + [b_{ij} + c_{ij}] \\ &= [a_{ij}] + \left([b_{ij}] + [c_{ij}]\right) \\ &= A + (B + C) \end{aligned}$$

Addition of matrices obeys Associative Property i.e., $A + (B + C) = (A + B) + C$

(iii) *Additive identity*

If A is a $m \times n$ matrix and O is the $(m \times n)$ null matrix,
$A + O = O + A = A$. We call O the additive indentity in the set of all $m \times n$ matrices.

(iv) *Additive inverse*

If A is an $(m \times n)$ matrix then there is a unique $m \times n$ matrix B such that $A + B = B + A = O$, O being the $m \times n$ null matrix.

This B is denoted by $-A$ and is called the additive inverse of A. Infact if $A = [a_{ij}]$, then $B = [-a_{ij}]$.