3.6.8 Definition (Consistent and Inconsistent systems)

We say that a system of linear equations is

(i) ***consistent*** *if it has a solution.*

(ii) ***inconsistent*** *if it has no solution.*

3.6.9 Solutions of nonhomogeneous system of equations

We consider solving the following system of 3 equations in 3 unknowns

$$a_1x+b_1\ y+c_1z=d_1$$

$$a_2x+b_2\ y+c_2z=d_2$$

$$a_3x+b_3\ y+c_3z=d_3$$

This system can be represented by a matrix equation AX = D where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}_{3\times3}$$ is the coefficient matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$$ is the variable matrix,

$$D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \text{ is the constant matrix,}$$

$$[A\,D] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix} \text{ is the augmented matrix.}$$

We state here a theorem without proof, which indicates the nature of solutions of the system.

3.6.10 Theorem

The system of three equations in three unknowns $AX = D$ *has*

(i) *a unique solution if rank* $(A) =$ *rank* $([A\,D]) = 3$.

(ii) *infinitely many solutions if rank* $(A) =$ *rank* $([A\,D]) < 3$.

(iii) *no solution if rank* $(A) \neq$ *rank* $([A\,D])$.

Note that the system is consistent if and only if rank $(A) =$ rank $([A\,D])$.

The method of solving the equations is illustrated in the following example.

3.6.11 Example

Show that the system of equations given below is not consistent.

$$2x + 6y = -11$$
$$6x + 20y - 6z = -3$$
$$6y - 18z = -1$$

Solution: The given system of equations can be written in the form

$AX = D$, where

$$A = \begin{bmatrix} 2 & 6 & 0 \\ 6 & 20 & -6 \\ 0 & 6 & -18 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad D = \begin{bmatrix} -11 \\ -3 \\ -1 \end{bmatrix}.$$

Consider the augmented matrix

$$[A\,D] = \begin{bmatrix} 2 & 6 & 0 & -11 \\ 6 & 20 & -6 & -3 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $R_2 \to R_2 - 3R_1$, we get

$$[A\,D] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 6 & -18 & -1 \end{bmatrix}.$$

On applying $R_3 \to R_3 - 3R_2$, we get

$$[AD] \sim \begin{bmatrix} 2 & 6 & 0 & -11 \\ 0 & 2 & -6 & 30 \\ 0 & 0 & 0 & -91 \end{bmatrix}.$$

Now rank of $[A\ D] = 3$, since the 3×3 submatrix

$$\begin{bmatrix} 6 & 0 & -11 \\ 2 & -6 & 30 \\ 0 & 0 & -91 \end{bmatrix},$$

is non - singular (its determinant is $-91\,(6)\,(-6) \neq 0$)

But the rank of the coefficient matrix is not 3 because

$$\det \begin{bmatrix} 2 & 6 & 0 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore$ rank of $(A) \neq$ rank $([AD])$.

Hence the given system is inconsistent.

3.7 Solution of Simultaneous Linear Equations

In this section we discuss some methods of solving systems of simultaneous linear equations.

3.7.1. Cramer's Rule

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

where $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is non-singular.

Let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the solution of the equation $AX = D$, where $D = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$\text{Then } x\Delta = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

On applying $C_1 \to C_1 + yC_2 + zC_3$ we get

$$x\Delta = \begin{vmatrix} a_1x+b_1y+c_1z & b_1 & c_1 \\ a_2x+b_2y+c_2z & b_2 & c_2 \\ a_3x+b_3y+c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore x = \frac{\Delta_1}{\Delta} \text{ where } \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.$$

Similarly we get

$$y = \frac{\Delta_2}{\Delta}, \text{ where } \Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } z = \frac{\Delta_3}{\Delta}, \text{ where } \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

$$\therefore \frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}.$$ This is known as **Cramer's Rule.**

3.7.2. Matrix inversion method

Consider the matrix equation $AX = D$, where A is non-singular. Then we can find A^{-1}.

$$\begin{aligned} AX = D &\Leftrightarrow A^{-1}(AX) = A^{-1}D \\ &\Leftrightarrow (A^{-1}A)X = A^{-1}D \\ &\Leftrightarrow IX = A^{-1}D \quad (\text{I is the unit matrix}). \\ &\Leftrightarrow X = A^{-1}D \end{aligned}$$

From this x, y and z are known.

3.7.3 Solved Problems

1. Problem: *Solve the following simultaneous linear equations by using Cramer's rule.*

$$3x + 4y + 5z = 18$$
$$2x - y + 8z = 13$$
$$5x - 2y + 7z = 20$$

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

Then we can write the given equations in the form of matrix equation as $AX = D$.

$$\Delta = \det A = \begin{vmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{vmatrix}$$

$$= 3\begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix}$$

$$= 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 27 + 104 + 5 = 136 \neq 0.$$

Hence we can solve the given equation by using Cramer's rule.

$$\Delta_1 = \begin{vmatrix} 18 & 4 & 5 \\ 13 & -1 & 8 \\ 20 & -2 & 7 \end{vmatrix} = 408$$

$$\Delta_2 = \begin{vmatrix} 3 & 18 & 5 \\ 2 & 13 & 8 \\ 5 & 20 & 7 \end{vmatrix} = 136$$

$$\Delta_3 = \begin{vmatrix} 3 & 4 & 18 \\ 2 & -1 & 13 \\ 5 & -2 & 20 \end{vmatrix} = 136.$$

Hence by Cramer's rule,

$$x = \frac{\Delta_1}{\Delta} = \frac{408}{136} = 3; \quad y = \frac{\Delta_2}{\Delta} = \frac{136}{136} = 1 \text{ and } z = \frac{\Delta_3}{\Delta} = \frac{136}{136} = 1.$$

$\therefore$ The solution of the given system of equations is $x = 3,\ y = 1 = z$.

2. Problem : *Solve* $3x+4y+5z=18; 2x-y+8z=13$ *and* $5x-2y+7z=20$ *by using 'Matrix inversion method'.*

Solution:

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } D = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}.$$

Then we can write the given equations in the form $AX = D$.

$$\det A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix} = 136 \neq 0.$$

Hence we can solve the given equations by 'Matrix inversion method'.

We have $\text{Adj } A = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$.

From matrix inversion method,

$$X = A^{-1}D = \frac{\text{Adj}A}{\det A}.\ D = \frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore\ x = 3,\ y = 1$ and $z = 1$ is the solution of the given system of equations.

Note

Observe that Cramer's Rule and Matrix inversion method can be applied only when the coefficient matrix A is non-singular. The Gauss-Jordan method given in 3.7.4 below can be applied even otherwise, as in 3.6.13.

3.7.7 Solution of a homogeneous system of linear equations

We consider the following homogeneous linear equations

$$a_1x+b_1y+c_1z=0$$

$$a_2x+b_2y+c_2z=0$$

$$a_3x+b_3y+c_3z=0\,.$$

The equivalent matrix equation of the above system is $AX=O$ where

$$A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Clearly the coefficient matrix A and the augmented matrix have the same rank, for they differ by a column of zeros. Thus a system of homogeneous equations is always consistent. In fact, $x = y = z = 0$ is always a solution. We call this the **trivial solution**. We are however, interested in finding whether or not there are non trivial solutions.

We state below a theorem without proof, which indicates the nature of solutions of the system.

3.7.8 Theorem

The system of equations $AX = O$ *has*

(i) *the trivial solution only, if rank (A) is* 3

(ii) *an infinite number of solutions if rank (A) is less than* 3.

The method of solving a system of homogeneous linear equations is similar to that adopted on the examples given in 3.6.13. However, some problems are solved here under.