MATH 603 - Final Assignment

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1 Problems

1. Write a computer program to implement the Fast Fourier Transform (FFT).

2. Using the FFT, write a computer program to solve numerically the initial-value problem (IVP) for the heat equation,

$$\begin{cases} u_t = u_{xx} & (t,x) \in [0,T] \times [0,1] \\ u(0,x) = u_0(x) & x \in [0,1] \end{cases}.$$

2 Problem 1

To implement the FFT, we should first revist the Continuous Fourier Transform of some function $f(x)$,

$$F(\omega) = \hat{f}(\omega) = \mathcal{F}\left[f\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\omega x},$$

where the function can be recovered as

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}.$$

Now, consider discretizing both the original and frequency domains into n equally spaced points, where

$$\begin{cases} \omega_m = 2\pi m/n, & m = 0, 1, \dots, n-1, \\ x_k = x_0 + k\Delta x, & k = 0, 1, \dots, n-1, \end{cases}$$

given that $x_0 = 0$ and $\Delta x = L/(n-1)$. Then, if we let $f_k = f(x_k)$ for $k = 0, 1, \dots, n-1$, we can define the Discrete Fourier Transform (DFT) as

$$f_m^{\#} = \sum_{k=0}^{n-1} f_k e^{-i\omega_m k}, \quad m = 0, 1, \dots, n-1,$$

where the discretization of the function can be recovered as

$$f_k = \frac{1}{n}\sum_{m=0}^{n-1} f_k^{\#} e^{i\omega_m k}, \quad k = 0, 1, \dots, n-1,$$

which we call the Inverse DFT (IDFT). Letting $\xi = e^{i2\pi/n}$, we can instead represent the DFT and IDFT respectively as the following two matrix-vector multiplications,

$$\begin{bmatrix} f_0^{\#} \\ f_1^{\#} \\ f_2^{\#} \\ \vdots \\ f_{n-1}^{\#} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi^{-1} & \xi^{-2} & \dots & \xi^{-(n-1)} \\ 1 & \xi^{-2} & \xi^{-4} & \dots & \xi^{-2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{-(n-1)} & \xi^{-2(n-1)} & \dots & \xi^{-(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix},$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{n-1} \end{bmatrix} = \frac{1}{n} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \xi^{1} & \xi^{2} & \dots & \xi^{(n-1)} \\ 1 & \xi^{2} & \xi^{4} & \dots & \xi^{2(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi^{(n-1)} & \xi^{2(n-1)} & \dots & \xi^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} f_0^{\#} \\ f_1^{\#} \\ f_2^{\#} \\ \vdots \\ f_{n-1}^{\#} \end{bmatrix}.$$

Since both the DFT and IDFT are just $n \times n$ systems, it follows that they both have a computational complexity of $\mathcal{O}(n^2)$. From here, the FFT is derived from noticing redundancies in the computation of the DFT, specifically, from noticing that ξ is periodic and that certain powers of ξ are equal. To illustrate this claim, consider the system of equations generated by the DFT when $n = 4$,

$$\begin{cases} f_0^{\#} &= f_0\xi^0 + f_1\xi^0 + f_2\xi^0 + f_3\xi^0 \\ f_1^{\#} &= f_0\xi^0 + f_1\xi^{-1} + f_2\xi^{-2} + f_3\xi^{-3} \\ f_2^{\#} &= f_0\xi^0 + f_1\xi^{-2} + f_2\xi^{-4} + f_3\xi^{-6} \\ f_3^{\#} &= f_0\xi^0 + f_1\xi^{-3} + f_2\xi^{-6} + f_3\xi^{-9}. \end{cases}$$

If we notice that $\xi^0 = \xi^{-4} = 1$, $\xi^{-2} = \xi^{-6} = -1$, $\xi^{-1} = \xi^{-9} = -i$, and $\xi^{-3} = i$, then we can simplify the above system of equations to,

$$\begin{cases} f_0^{\#} &= (f_0 + f_1) + \xi^0(f_2 + f_3) \\ f_1^{\#} &= (f_0 - f_1) + \xi^{-1}(f_2 - f_3) \\ f_2^{\#} &= (f_0 + f_1) + \xi^{-2}(f_2 + f_3) \\ f_3^{\#} &= (f_0 - f_1) + \xi^{-3}(f_2 - f_3), \end{cases}$$

which has reduced the number of computations from 16 multiplications and 12 additions to 4 multiplications and 12 additions (without and doing any caching or precomputing). This idea can be generalized when n is some positive integer power of 2, i.e. $n = 2^\ell$ where $\ell \in \mathbb{Z}^+$, which instead allows us to represent the DFT as

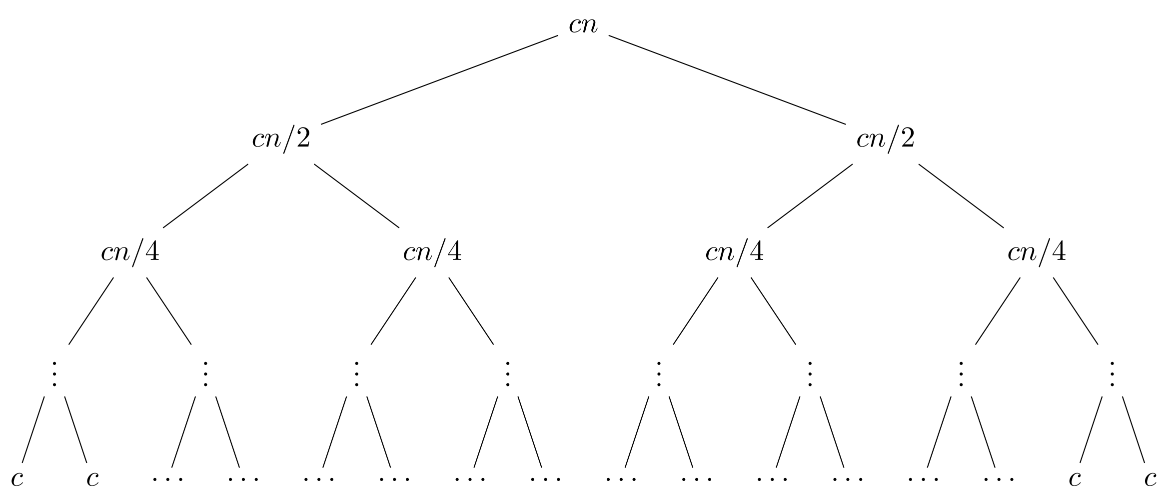
$$f_m^{\#} = \sum_{k=0}^{n-1} f_k \xi^{-mk} = \sum_{k=0}^{\frac{n}{2}-1} f_{2k}\xi^{-m(2k)} + \sum_{k=0}^{\frac{n}{2}-1} f_{2k+1}\xi^{-m(2k+1)}, \tag{1}$$

for $m = 0, 1 \ldots, n-1$, where we are essentially just breaking up the summation into its even and odd indexed summations. If we also notice that

$$\begin{aligned} f_m^{\#} &= \sum_{k=0}^{\frac{n}{2}-1} f_{2k}\xi^{-m(2k)} + \sum_{k=0}^{\frac{n}{2}-1} f_{2k+1}\xi^{-m(2k+1)} \\ &= \sum_{k=0}^{\frac{n}{2}-1} f_{2k}\xi^{-2mk} + \xi^{-m} \sum_{k=0}^{\frac{n}{2}-1} f_{2k+1}\xi^{-2mk} \end{aligned} \tag{2}$$

for $m = 0, 1, \ldots, n-1$, then we can use the idea from (1) on each of the two individual summations in (2). Applying this process recursively until each of the individual summations has just one term, reduces the computational complexity of the DFT from $\mathcal{O}(n^2)$

to $\mathcal{O}(n \log_2 n)$, giving us the FFT. This reduction in complexity can be visualized by the corresponding binary tree generated by the recursive process,



where taking the summation of all values at any level of the tree will equal cn, which roughly describes the number of floating point operations required at each level of recursion, where c is some constant value. Now, the height of the tree is $\log_2 n$ because at the ith level of the tree, the individual nodes represent a summation of $cn/2^i$ terms each, and at the bottom level we know that we should only need

$$cn/2^i = \mathcal{O}(1) = k \tag{3}$$

floating point operations, where k is some constant value. So, if we take the logarithm of both sides of (3), we get

$$\begin{aligned}
&\log_2 (cn/2^i) = \log_2 k \\
&\Rightarrow \log_2 cn - \log_2 2^i = \log_2 k \\
&\Rightarrow \log_2 n + \log_2 c - i \log_2 2 = \log_2 k \\
&\Rightarrow \log_2 n + \log_2 c - \log_2 k = i
\end{aligned}$$

which means the order of the height of the tree is $\mathcal{O}(i) = \mathcal{O}(\log_2 n)$, and since the width of the tree is n, then the computation complexity of the FFT is $\mathcal{O}(n \log_2 n)$. Finally, I will provide an algorithm for the FFT which implements the recursive process that reduces the computational complexity of the DFT:

Algorithm 1 Fast Fourier Transform

Procedure FFT(f, $f^{\#}$, n, ξ)
if $n \leftarrow 1$ **then**
    $f^{\#}[0] \leftarrow f[0]$
else
    $f_e[k] \leftarrow$ empty array of size $\frac{n}{2}$
    $f_o[k] \leftarrow$ empty array of size $\frac{n}{2}$
    for k from 0 to $\frac{n}{2} - 1$ **do**
        $f_e[k] \leftarrow f[2k]$
        $f_o[k] \leftarrow f[2k+1]$
    end for
    $f_e^{\#}[k] \leftarrow$ empty array of size $\frac{n}{2}$
    $f_o^{\#}[k] \leftarrow$ empty array of size $\frac{n}{2}$
    FFT(f_e, $f_e^{\#}$, $\frac{n}{2}$, ξ^2)
    FFT(f_o, $f_o^{\#}$, $\frac{n}{2}$, ξ^2)
    for k from 0 to $n - 1$ **do**
        $f[k] \leftarrow f_e^{\#}[k \mod \frac{n}{2}] + \xi^k f_o^{\#}[k \mod \frac{n}{2}]$
    end for
end if
End Procedure

3 Problem 2